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ON MEAN-FIELD CONTROL PROBLEMS FOR BACKWARD DOUBLY STOCHASTIC SYSTEMS

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Abstract. This article is concerned with stochastic control problems for backward doubly stochastic differential equations of mean-field type, where the coefficient functions depend on the joint distribution of the state process and the control process. We obtain the stochastic maximum principle which serves as a necessary condition for an optimal control, and we also prove its sufficiency under proper conditions. As a byproduct, we prove the well-posedness for a type of mean-field fully coupled forward-backward doubly stochastic differential equation arising naturally from the control problem, which is of interest in its own right. Some examples are provided to illustrate the applications of our results to control problems in the types of scalar interaction and first order interaction.

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1. Introduction

In this paper, we are concerned with a control problem in which the state process $\{(y_t, z_t), t \in [0, T]\}$ is governed by the following equation

$$\begin{cases} -dy_t = f(t, y_t, z_t, u_t, \mathcal{L}(y_t, z_t, u_t))dt + g(t, y_t, z_t, u_t, \mathcal{L}(y_t, z_t, u_t))d\overleftarrow{B}_t \\ \qquad\quad - z_t dW_t, \quad t \in [0, T], \\ y_T = \xi. \end{cases} \tag{1.1}$$

In the above equation, the control process $\{u_t, t \in [0, T]\}$ is a given stochastic process; $\mathcal{L}(y_t, z_t, u_t)$ stands for the law of the random vector (y_t, z_t, u_t); B and W are two mutually independent Brownian motions; the stochastic integral with respect to B is a backward Itô integral while the one with respect to W is forward. This equation is called a mean-field backward doubly stochastic differential equation (MF-BDSDE) due to its dependence on two Brownian motions as well as on the joint law of state and control processes. The cost functional of the

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control problem is given by

$$J(u) = \mathbb{E}\left[\int_0^T h(t, y_t, z_t, u_t, \mathcal{L}(y_t, z_t, u_t))dt + \Phi(y_0, \mathcal{L}(y_0))\right]. \tag{1.2}$$

Our goal of this paper is to obtain the stochastic maximum principle (SMP), a necessary condition for an optimal control, *i.e.*, a control minimizing $J(u)$. Below we briefly recall some related results, which is by no means complete in the literature.

Stochastic control problems have gained a particular interest due to their broad applications in economics, finance, engineering, etc. The earliest works can be retrospected to Kushner [1] and Bismut [2]. Among others, the theory of general backward stochastic differential equations (BSDEs) introduced in [3] plays an important role in the study of stochastic control problems. As an extension of BSDEs, backward doubly stochastic differential equations (BDSDEs) were introduced by Pardoux and Peng in [4]. We refer to Yong and Zhou [5] and Zhang [6] for more details on stochastic control, BSDEs, and other related topics.

Mean-field models are useful to characterize the asymptotic behavior when the size of the system is getting very large. Mean-field stochastic differential equations (MF-SDEs), also known as equations of McKean-Vlasov type, were first introduced by Kac [7] when investigating physical systems with a large number of interacting particles. The approach of studying large particle systems pioneered by Kac now is called in the literature propagation of chaos and we refer to Sznitman [8] for further reading. In recent years, mean-field theories for BSDEs and BDSDEs were investigated by Buckdahn *et al.* [9] and Li and Xing [10], respectively.

As is well known in the literature of game theory, it is in general hard to construct Nash equilibrium explicitly if the number of players is large. The pioneer work of Lasry and Lions [11] proposed a framework of approximating Nash equilibrium for stochastic games with a large number of players. Huang *et al.* [12] dealt with large games in a similar approach. Later on, Carmona and Delarue [13] provided a probabilistic analysis for large games formulated by Lasry and Lions, in which they resolved the limiting optimal control problem by studying a mean-field forward-backward stochastic differential equation (MF-FBSDE). We refer to [14, 15] and the references therein for more details about mean-field games and related topics.

Mean-field control problems have also attracted considerable attention accompanying with the development of mean-field game theory. At the beginning, the investigation was focused on the control problems which involve the expected values; for instance, Buckdahn *et al.* [16] obtained the global maximum principle for mean-field SDEs (see also [17]). After Lions introduced the notion of derivatives with respect to probability measures in his seminal work [18] (see also [15, 19]), a more general form of mean-field interaction where the law of the solution process is involved has been studied, see *e.g.* [20, 21]. We also refer to [22–24] and the references therein for more development on mean-field control problems.

Motivated by the existing works, in this paper we investigate the mean-field control problem (1.1)–(1.2) for MF-BDSDEs and aim to obtain SMP. We remark that Han *et al.* [25] has obtained SMP for control problems involving such BDSDEs without mean-field terms (see also [26, 27]). In our control problem (1.1)–(1.2), the state process and the cost functional both depend on the joint distribution of the state process and the control process. Note that in our setting, the dependence on the joint distribution is rather general, and in particular, it includes the cases of $\varphi(t, X_t, \mathbb{E}[X_t], u_t)$ and $\widetilde{\mathbb{E}}\big[\varphi(t, X_t, \widetilde{X}_t, u_t)\big]$ which are known as the *scalar interaction* and *first order interaction* of mean-field type, respectively. These two cases will be treated in Section 5 as examples of applying our main result.

In our study of the mean-field control problem for BDSDEs, a combination of existing techniques and methods from BDSDEs, FBSDEs and mean field theory is applied, which is summarized below.

(i) From a modelling perspective, BDSDE is a generalization of BSDE and hence can describe more phenomena in the real world. It is worth mentioning that this generalization is not trivial. For instance, the classical Itô's formula which plays an essential role in the study of stochastic optimal control problems can not be applied directly in our situation due to the appearance of the backward Itô integral in (1.1), and a generalized Itô's formula (see Lem. 3.1, and we refer to [4] for more details) is invoked.

(ii) In the mean-field control problems, the dependence of the coefficient functions on probability measures leads to a failure of the classical calculus, and we employ the concept of L-derivative for functions of probability measures initiated by P. L. Lions [18] (see also [15, 19]). In comparison with the classical situation, this requires additional effort in the analysis, and as a consequence, L-derivatives also appear in the stochastic maximum principle (see Theorem 3.7) and the Hamiltonian system (4.1).

(iii) We prove the well-posedness of the fully coupled mean-field forward backward doubly stochastic differential equations (FBDSDEs) (4.2) which naturally arise when investigating the control problem. This type of equation was first introduced by Peng and Shi [28] and later on was further investigated for instance in [25]. To prove the well-posedness, we adapt the continuation method initiated in [29] where the new product rule Lemma 3.2 is involved (see the proof of Lem. 4.2), also together with a careful treatment of the L-derivatives appearing in our analysis. See Theorem 4.3 and Remark 4.4.

This article is organized as follows. In Section 2, some preliminaries of the L-derivative of functions of probability measures is recalled. In Section 3, we prove our main result of stochastic maximum principle as well as a verification theorem. Section 4 is devoted to the investigation of a type of fully coupled mean-field BDSDE, which is of interest in its own right. Finally, we provide some examples in Section 5.

To conclude this section, we introduce some notations that will be used throughout the article. For two vectors $u, v \in \mathbb{R}^n$, denote by $\langle u, v\rangle$ the scalar product of u and v, by $|v| = \sqrt{\langle v, v\rangle}$ the Euclidean norm of v. For $A, B \in \mathbb{R}^{n\times d}$, we denote the scalar product of A and B by $\langle A, B\rangle = \mathrm{tr}\{AB^{\mathrm{T}}\}$ and the norm of the matrix A by $\|A\| = \sqrt{\mathrm{tr}\{AA^{\mathrm{T}}\}}$, where the superscript T stands for the transpose of vectors or matrices. We also use the notation $\partial_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)^{\mathrm{T}}$ for $x \in \mathbb{R}^n$. Then for $\Psi : \mathbb{R}^n \to \mathbb{R}$, $\partial_x\Psi = \left(\frac{\partial}{\partial x_i}\Psi\right)_{n\times 1}$ is a column vector, and for $\Psi : \mathbb{R}^n \to \mathbb{R}^d$, $\partial_x\Psi = \left(\frac{\partial}{\partial x_i}\Psi_j\right)_{n\times d}$ is a $n \times d$ matrix. Henceforth, we denote by C a generic constant which can be different in different lines.

2. Preliminaries on L-derivative

In this section, we collect some preliminaries on L-differentiability for functions of probability laws which was initiated by P. L. Lions [18]. We refer to [19] and [15] for more details.

For $m \in \mathbb{N}$, let $\mathcal{P}_2(\mathbb{R}^m)$ be the set of probability measures on $\mathbb{R}^m$ with finite second moment. Denote by $W_2(\cdot,\cdot)$ the 2-Wasserstein distance in $\mathcal{P}_2(\mathbb{R}^m)$, *i.e.*,

$$W_2(\mu_1, \mu_2) = \inf\left\{\left(\int_{\mathbb{R}^{2m}} |x-y|^2 \rho(\mathrm{d}x, \mathrm{d}y)\right)^{\frac{1}{2}}\right\},$$

where the infimum is taken over all $\rho \in \mathcal{P}_2(\mathbb{R}^{2m})$ with $\rho(\mathrm{d}x, \mathbb{R}^m) = \mu_1(\mathrm{d}x)$ and $\rho(\mathbb{R}^m, \mathrm{d}y) = \mu_2(\mathrm{d}y)$. Then, $(\mathcal{P}_2(\mathbb{R}^m), W_2)$ is a polish space. It's obvious from the definition that

$$W(\mu_1, \mu_2) \le \left(\mathbb{E}\left[\left|X - Y\right|^2\right]\right)^{\frac{1}{2}},$$

here X and Y are $\mathbb{R}^m$-valued random variables with the distributions μ_1 and μ_2, respectively.

For a function $H : \mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^m) \to \mathbb{R}$, we call $\widetilde{H} : \mathbb{R}^q \times L^2(\Omega; \mathbb{R}^m) \to \mathbb{R}$ a lifting of H if $\widetilde{H}(x, Y) = H(x, \mathcal{L}(Y))$, where $\mathcal{L}(Y)$ means the probability law of Y.

Definition 2.1. A function $H : \mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^m)$ is said to be L-differentiable at $(x_0, \mu_0) \in \mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^m)$ if there exists a random variable $Y_0 \in L^2(\Omega; \mathbb{R}^m)$ with $\mathcal{L}(Y_0) = \mu_0$, such that the lifted function $\widetilde{H}$ is Fréchet differentiable at (x_0, Y_0), *i.e.*, there exists a linear continuous mapping

$$[D\widetilde{H}](x_0, Y_0) : \mathbb{R}^q \times L^2(\Omega; \mathbb{R}^m) \to \mathbb{R}$$

such that

$$\widetilde{H}(x_0 + \Delta x, Y_0 + \Delta Y) - \widetilde{H}(x_0, Y_0) = [D\widetilde{H}](x_0, Y_0)(\Delta x, \Delta Y) + o(|\Delta x| + \|\Delta Y\|_{L^2}). \tag{2.1}$$

Note that $\mathcal{H} := \mathbb{R}^q \times L^2(\Omega; \mathbb{R}^m)$ is a Hilbert space with the inner product

$$\big\langle (x_1, Y_1), (x_2, Y_2) \big\rangle_{\mathcal{H}} = \langle x_1, x_2 \rangle + \mathbb{E}[\langle Y_1, Y_2 \rangle].$$

By Riesz representation theorem, the Fréchet derivative $[D\widetilde{H}](x_0, Y_0)$ can be viewed as an element $D\widetilde{H}(x_0, Y_0)$ in $\mathcal{H}$ in the sense that for all $(x, Y) \in \mathcal{H}$,

$$[D\widetilde{H}](x_0, Y_0)(x, Y) = \big\langle D\widetilde{H}(x_0, Y_0), (x, Y) \big\rangle_{\mathcal{H}}. \tag{2.2}$$

Indeed, there exists a measurable function $g : \mathbb{R}^m \to \mathbb{R}^m$ depending only on μ_0 such that $D\widetilde{H}(x_0, Y) = g(Y)$ a.s. for all Y with $\mathcal{L}(Y) = \mu_0$. Then, we define the *L-derivative of H at (x_0, μ_0) along the random variable Y* by $g(Y)$, which is denoted by $\partial_\mu H(x_0, \mu_0)(Y)$. Thus, we have a.s.

$$\partial_\mu H(x_0, \mathcal{L}(Y))(Y) = g(Y) = D\widetilde{H}(x_0, Y).$$

Example 2.2. Consider the following function H on $\mathcal{P}_2(\mathbb{R}^m)$,

$$H(\mu) = \int_{\mathbb{R}^m} h(y) \mu(dy),$$

where $h : \mathbb{R}^m \to \mathbb{R}$ is twice differentiable with bounded second derivatives. Clearly, the lifted function $\widetilde{H}(Y) = \mathbb{E}\big[h(Y)\big]$ with $\mathcal{L}(Y) = \mu$, and $\partial_\mu H(\mathcal{L}(Y))(Y) = D\widetilde{H}(Y) = \partial_y h(Y)$ by (2.1) and (2.2).

Similarly, for a function $H : \mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k) \to \mathbb{R}$ depending on a vector $x \in \mathbb{R}^q$ and a joint probability law $\mu = (\mu_y, \mu_z, \mu_u) \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k)$, we can define partial L-differentiability. We say that H is joint L-differentiable at (x, μ) if there exists a triple of random variables $(Y, Z, U) \in L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k)$ with $\mathcal{L}(Y, Z, U) = \mu$ such that the lifted function $\widetilde{H}(x, Y, Z, U) = H(x, \mu)$ is Fréchet differentiable at (x, Y, Z, U). Observing $\mathbb{R}^q \times L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k) \cong \mathbb{R}^q \times L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^{n\times l}) \times L^2(\Omega; \mathbb{R}^k)$, the partial L-derivatives $\partial_{\mu_y} H, \partial_{\mu_z} H$ and $\partial_{\mu_u} H$ at (x, μ) along (Y, Z, U) can be defined *via* the following identity

$$D\widetilde{H}(x, Y, Z, U) = \Big(\partial_x H(x, \mu), \partial_{\mu_y} H(x, \mu), \partial_{\mu_z} H(x, \mu), \partial_{\mu_u} H(x, \mu)\Big)(Y, Z, U).$$

We remark that $\partial_x H(x, \mu)(Y, Z, U)$ actually does not depend on (Y, Z, U).

A standard result says that joint continuous differentiability in the two arguments is equivalent to partial differentiability in each of the two arguments and joint continuity of the partial derivatives. Hence, the joint continuity of $\partial_x H(x, \mu)$ means the joint continuity with respect to the Euclidean distance on $\mathbb{R}^q$ and the 2-Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k)$; the joint continuity of $\partial_{\mu_y} H(x, \mu)$ is understood as the joint continuity of the mapping $(x, Y, Z, U) \mapsto \partial_{\mu_y} H(x, \mathcal{L}(Y, Z, U))(Y, Z, U)$ from $\mathbb{R}^q \times L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k)$ to $L^2(\Omega; \mathbb{R}^n)$.

3. Stochastic maximum principle

In this section, we aim to derive our main result of the stochastic maximum principle. First, we fix some mathematical notations, formulate our control problem, and recall Itô's formula for stochastic processes involving backward Itô's integral. Then we present the assumptions which will be used throughout the paper. The maximum principle will be obtained *via* the classical variational method. Assuming proper convexity conditions on the Hamiltonian, we prove a verification theorem, *i.e.*, showing that the stochastic maximum principle is also a sufficient condition for an optimal control.

3.1. Some preliminaries for the control problem

On a probability space $(\Omega, \mathcal{F}, P)$ satisfying usual conditions, let $\{B_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ be two mutually independent Brownian motions, taking values in $\mathbb{R}^d$ and $\mathbb{R}^l$ respectively. Denote by $\mathcal{N}$ the collection of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, denote

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B\,,$$

where $\mathcal{F}_t^W = \sigma\big\{W_r, 0 \leq r \leq t\big\} \vee \mathcal{N}$ is the augmented σ-field generated by W and similarly $\mathcal{F}_{t,T}^B = \sigma\big\{B_r - B_t, t \leq r \leq T\big\} \vee \mathcal{N}$. We stress that $\mathcal{F}_t$ is neither increasing nor decreasing in t and hence does not constitute a filtration. Now let us introduce the following spaces:

$$\begin{aligned}
L^2_{\mathcal{G}}(\mathbb{R}^n) &= \Big\{\xi : \Omega \to \mathbb{R}^n; \xi \in \mathcal{G} \text{ and } \mathbb{E}\left[|\xi|^2\right] < +\infty\Big\} \text{ for any } \sigma\text{-field } \mathcal{G} \subset \mathcal{F};\\
L^2_{\mathcal{F}}([s,r];\mathbb{R}^n) &= \Big\{\phi : [s,r] \times \Omega \to \mathbb{R}^n; \phi_t \in \mathcal{F}_t \text{ for } t \in [s,r] \text{ and } \mathbb{E}\left[\int_s^r |\phi_t|^2 \mathrm{d}t\right] < +\infty\Big\};\\
S^2_{\mathcal{F}}([s,r];\mathbb{R}^n) &= \Big\{\phi : [s,r] \times \Omega \to \mathbb{R}^n; \phi \text{ is continuous a.s., } \phi_t \in \mathcal{F}_t \text{ for } t \in [s,r],\\
&\qquad \text{and } \mathbb{E}\Big[\sup_{s\leq t\leq r} |\phi_t|^2\Big] < +\infty\Big\}.
\end{aligned}$$

The state process $(y_t, z_t)_{0\leq t\leq T}$ is governed by the following BDSDE

$$\begin{cases} -dy_t = f(t, y_t, z_t, u_t, \mathcal{L}(y_t, z_t, u_t))\mathrm{d}t + g(t, y_t, z_t, u_t, \mathcal{L}(y_t, z_t, u_t))d\overleftarrow{B}_t \\ \qquad\quad - z_t dW_t,\ t \in [0, T], \\ y_T = \xi, \end{cases} \tag{3.1}$$

with ξ a given $\mathcal{F}_T$-measurable random variable. We aim to minimize the cost functional given by

$$J(u) = \mathbb{E}\left[\int_0^T h(t, y_t, z_t, u_t, \mathcal{L}(y_t, z_t, u_t))\mathrm{d}t + \Phi(y_0, \mathcal{L}(y_0))\right], \tag{3.2}$$

over the set $\mathcal{U} := L^2_{\mathcal{F}}([0, T]; \mathbb{U})$ of admissible controls, where $\mathbb{U}$ is a closed convex subset of $\mathbb{R}^k$. The functions f, g and h are measurable mappings from $[0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k)$ to $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$ and $\mathbb{R}$, respectively.

We stress that the state process (y, z) and the cost functional $J(u)$ depend on the joint distribution $\mathcal{L}(y_t, z_t, u_t)$ of the state and the control processes.

To end this subsection, we recall Itô's formula obtained in [4], Lemma 1.3, which is a key ingredient in our analysis.

Lemma 3.1. *Let* $\alpha \in S^2_{\mathcal{F}}([0,T];\mathbb{R}^n), \beta \in L^2_{\mathcal{F}}([0,T];\mathbb{R}^n), \gamma \in L^2_{\mathcal{F}}([0,T];\mathbb{R}^{n\times d}), \theta \in L^2_{\mathcal{F}}([0,T];\mathbb{R}^{n\times l})$ *be such that*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s d\overleftarrow{B}_s + \int_0^t \theta_s dW_s, 0 \le t \le T.$$

Then for $\phi \in C^2(\mathbb{R}^n)$, *we have*

$$\begin{aligned}\phi(\alpha_t) =&\phi(\alpha_0) + \int_0^t \langle \partial_\alpha \phi(\alpha_s), \beta_s \rangle \mathrm{d}s + \int_0^t \langle \partial_\alpha \phi(\alpha_s), \gamma_s d\overleftarrow{B}_s \rangle + \int_0^t \langle \partial_\alpha \phi(\alpha_s), \theta_s dW_s \rangle \\ &- \frac{1}{2}\int_0^t tr\Big[\partial_{\alpha\alpha}\phi(\alpha_s)\gamma_s\gamma_s^T\Big]\mathrm{d}s + \frac{1}{2}\int_0^t tr\Big[\partial_{\alpha\alpha}\phi(\alpha_s)\theta_s\theta_s^T\Big]\mathrm{d}s. \end{aligned} \tag{3.3}$$

The following product rule is a direct corollary of Lemma 3.1.

Lemma 3.2. *Consider the processes* y *and* p *given by*

$$\begin{cases} dy_t = f_t dt + g_t d\overleftarrow{B}_t + z_t dW_t, \\ dp_t = F_t dt + G_t dW_t + q_t d\overleftarrow{B}_t, \end{cases}$$

where f, g, z, F, G, q *all belong to* $L^2_{\mathcal{F}}([0,T];\mathbb{R}^m)$ *with proper dimension* m. *We have*

$$d\langle p_t, y_t \rangle = \langle dp_t, y_t \rangle + \langle p_t, dy_t \rangle + \langle G_t, z_t \rangle dt - \langle g_t, q_t \rangle dt. \tag{3.4}$$

3.2. Main assumptions and the variational equation

We assume the following conditions for our control problem (3.1)-(3.2).

(H1) The functions $f(t,0,0,0,\delta_0)$ and $g(t,0,0,0,\delta_0)$ are uniformly bounded, where δ_0 is the Dirac measure at 0. The functions f, g and h are differentiable with respect to $(y,z,u) \in \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k$ for each $t \in [0,T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k)$. Moreover, for $\rho = y, z, u$, the partial derivative $\partial_\rho \varphi$ is continuous and uniformly bounded in (t,y,z,u,μ) for $\varphi = f, g, h$. In particular, we require $\|\partial_z g(t,y,z,u,\mu)\| < \alpha_1 \in (0,1)$.

(H2) The functions f, g and h are L-differentiable with respect to μ. Moreover, for $\nu = \mu_y, \mu_z, \mu_u$, the L-derivative $\partial_\nu \varphi$ is continuous with L^2-norm being uniformly bounded in (t,y,z,u,μ) for $\varphi = f, g, h$. In particular, we require

$$\int_{\mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k} \big\|\partial_{\mu_z} g(t,y,z,u,\mu)(y',z',u')\big\|^2 d\mu(y',z',u') < \alpha_2 \in (0, 1-\alpha_1).$$

(H3) The function Φ is differentiable with respect to y and L-differentiable with respect to μ_y, and moreover $\partial_y \Phi(y,\mu)$ and $\partial_{\mu_y}\Phi(y,\mathcal{L}(Y))(Y)$ are jointly continuous.

Remark 3.3. Note that, if f, g are continuously differentiable with uniformly bounded partial derivatives as assumed in (H1) and (H2), we can deduce that f, g are Lipschitz in (y,z,u) and μ. Precisely, there exists a constant C and $0 < \alpha_1, \alpha_2 < 1$ with $\alpha_1 + \alpha_2 < 1$ such that for all $y, y' \in \mathbb{R}^n$, $z, z' \in \mathbb{R}^{n\times l}$, $u, u' \in \mathbb{R}^k$, $\mu = \mathcal{L}(Y,Z,U), \mu' = \mathcal{L}(Y',Z',U') \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k)$,

$$\begin{aligned}&\big|f(t,y,z,u,\mu) - f(t,y',z',u',\mu')\big|^2 \\ &\le C\Big(|y-y'|^2 + \|z-z'\|^2 + |u-u'|^2 + \mathbb{E}\big[|Y-Y'|^2 + \|Z-Z'\|^2 + |U-U'|^2\big]\Big),\end{aligned}$$

and

$$\begin{aligned}&\big\|g(t,y,z,u,\mu)-g(t,y',z',u',\mu')\big\|^2\\&\le C\Big(|y-y'|^2+|u-u'|^2+\mathbb{E}\big[|Y-Y'|^2+|U-U'|^2\big]\Big)+\Big(\alpha_1\|z-z'\|^2+\alpha_2\mathbb{E}\big[\|Z-Z'\|^2\big]\Big).\end{aligned}$$

The following result borrowed from [10] provides the existence and uniqueness for the solution of (3.1).

Theorem 3.4. *Under the Assumptions (H1) and (H2), for any fixed $u=(u_t)_{0\le t\le T}\in\mathcal{U}$, there exists a unique solution $(y^u,z^u)\in S^2_{\mathcal{F}}([0,T];\mathbb{R}^n)\times L^2_{\mathcal{F}}([0,T];\mathbb{R}^{n\times l})$ to (3.1).*

Let $u^*\in\mathcal{U}$ be an optimal control, *i.e.*, $J(u^*)=\inf\limits_{v\in\mathcal{U}}J(v)$, and (y^*,z^*) be the corresponding state process. We shall introduce some notations that will be used in the sequel.

Recalling that $\mathcal{U}=L^2_{\mathcal{F}}([0,T];\mathbb{U})$ with $\mathbb{U}$ being a convex set of $\mathbb{R}^k$, we have $u^\varepsilon:=u^*+\varepsilon v\in\mathcal{U}$ for $0\le\varepsilon\le1$ and all $v=u-u^*$ with arbitrarily $u\in\mathcal{U}$. Let $(y^\varepsilon,z^\varepsilon)$ denote the solution of (3.1) with $u=u^\varepsilon$. We shall take the following abbreviated notations

$$\begin{aligned}&\theta^*_t=(y^*_t,z^*_t,u^*_t,\mathcal{L}(y^*_t,z^*_t,u^*_t)),\ \ \theta^\varepsilon_t=(y^\varepsilon_t,z^\varepsilon_t,u^\varepsilon_t,\mathcal{L}(y^\varepsilon_t,z^\varepsilon_t,u^\varepsilon_t)),\\&y^\lambda_t=y_t+\lambda(y^\varepsilon_t-y^*_t),\ \ z^\lambda_t=z_t+\lambda(z^\varepsilon_t-z^*_t),\ \ u^\lambda_t=u^*_t+\lambda\varepsilon v_t,\\&\theta^\lambda_t=(y^\lambda_t,z^\lambda_t,u^\lambda_t,\mathcal{L}(y^\lambda_t,z^\lambda_t,u^\lambda_t)).\end{aligned}\tag{3.5}$$

Let $(\widetilde\Omega,\widetilde{\mathcal{F}},\widetilde P)$ be a copy of $(\Omega,\mathcal{F},P)$. For a random variable X defined on $(\Omega,\mathcal{F},P)$, we denote by $\widetilde X$ its copy on $\widetilde\Omega$. For any integrable random variable ξ on the probability space $(\Omega\times\widetilde\Omega,\mathcal{F}\times\widetilde{\mathcal{F}},P\otimes\widetilde P)$, we denote

$$\mathbb{E}\big[\xi(\omega,\widetilde\omega)\big]=\int_\Omega\xi(\omega,\widetilde\omega)P(d\omega)\text{ and }\widetilde{\mathbb{E}}\big[\xi(\omega,\widetilde\omega)\big]=\int_{\widetilde\Omega}\xi(\omega,\widetilde\omega)P(d\widetilde\omega).\tag{3.6}$$

With the above notations in mind, we introduce the following linear backward doubly stochastic differential equation

$$\begin{cases}-dK_t=&\Big\{\partial_yf(t,\theta^*_t)K_t+\partial_zf(t,\theta^*_t)L_t+\partial_uf(t,\theta^*_t)v_t+\widetilde{\mathbb{E}}\big[\partial_{\mu_y}f(t,\theta^*_t)(\widetilde y^*_t,\widetilde z^*_t,\widetilde u^*_t)\widetilde K_t\big]\\&\quad+\widetilde{\mathbb{E}}\big[\partial_{\mu_z}f(t,\theta^*_t)(\widetilde y^*_t,\widetilde z^*_t,\widetilde u^*_t)\widetilde L_t\big]+\widetilde{\mathbb{E}}\big[\partial_{\mu_u}f(t,\theta^*_t)(\widetilde y^*_t,\widetilde z^*_t,\widetilde u^*_t)\widetilde v_t\big]\Big\}dt\\&+\Big\{\partial_yg(t,\theta^*_t)K_t+\partial_zg(t,\theta^*_t)L_t+\partial_ug(t,\theta^*_t)v_t+\widetilde{\mathbb{E}}\big[\partial_{\mu_y}g(t,\theta^*_t)(\widetilde y^*_t,\widetilde z^*_t,\widetilde u^*_t)\widetilde K_t\big]\\&\quad+\widetilde{\mathbb{E}}\big[\partial_{\mu_z}g(t,\theta^*_t)(\widetilde y^*_t,\widetilde z^*_t,\widetilde u^*_t)\widetilde L_t\big]+\widetilde{\mathbb{E}}\big[\partial_{\mu_u}g(t,\theta^*_t)(\widetilde y^*_t,\widetilde z^*_t,\widetilde u^*_t)\widetilde v_t\big]\Big\}d\overleftarrow{B}_t\\&-L_tdW_t,\ \ t\in[0,T],\\K_T=&0,\end{cases}\tag{3.7}$$

to which there exists a unique solution $(K,L)\in S^2_{\mathcal{F}}([0,T];\mathbb{R}^n)\times L^2_{\mathcal{F}}([0,T];\mathbb{R}^{n\times l})$ by Theorem 3.4.

Proposition 3.5. *Let assumptions (H1)–(H3) hold. Then, we have*

$$\lim_{\varepsilon\to0}\mathbb{E}\left[\sup_{0\le t\le T}\Big|\frac{y^\varepsilon_t-y^*_t}{\varepsilon}-K_t\Big|^2\right]=0\ \ \textit{and}\ \ \lim_{\varepsilon\to0}\mathbb{E}\left[\int_0^T\Big\|\frac{z^\varepsilon_t-z^*_t}{\varepsilon}-L_t\Big\|^2dt\right]=0.$$

Proof. Denote

$$\hat{y}_t^\varepsilon = \frac{y_t^\varepsilon - y_t^*}{\varepsilon} - K_t \quad \text{and} \quad \hat{z}_t^\varepsilon = \frac{z_t^\varepsilon - z_t^*}{\varepsilon} - L_t. \tag{3.8}$$

Then, by (3.1) and (3.7), $(\hat{y}_t^\varepsilon, \hat{z}_t^\varepsilon)_{0\le t\le T}$ solves the following equation,

$$\begin{cases} -d\hat{y}_t^\varepsilon = \Big\{ \frac{1}{\varepsilon}\big[f(t,\theta_t^\varepsilon) - f(t,\theta_t^*)\big] - \partial_y f(t,\theta_t^*)K_t - \partial_z f(t,\theta_t^*)L_t - \partial_u f(t,\theta_t^*)v_t \\ \qquad - \widetilde{\mathbb{E}}\big[\partial_{\mu_y} f(t,\theta_t^*)(\widetilde{y}_t^*, \widetilde{z}_t^*, \widetilde{u}_t^*)\widetilde{K}_t\big] - \widetilde{\mathbb{E}}\big[\partial_{\mu_z} f(t,\theta_t^*)(\widetilde{y}_t^*, \widetilde{z}_t^*, \widetilde{u}_t^*)\widetilde{L}_t\big] \\ \qquad - \widetilde{\mathbb{E}}\big[\partial_{\mu_u} f(t,\theta_t^*)(\widetilde{y}_t^*, \widetilde{z}_t^*, \widetilde{u}_t^*)\widetilde{v}_t\big]\Big\} dt \\ \quad + \Big\{ \frac{1}{\varepsilon}\big[g(t,\theta_t^\varepsilon) - g(t,\theta_t^*)\big] - \partial_y g(t,\theta_t^*)K_t - \partial_z g(t,\theta_t^*)L_t - \partial_u g(t,\theta_t^*)v_t \\ \qquad - \widetilde{\mathbb{E}}\big[\partial_{\mu_y} g(t,\theta_t^*)(\widetilde{y}_t^*, \widetilde{z}_t^*, \widetilde{u}_t^*)\widetilde{K}_t\big] - \widetilde{\mathbb{E}}\big[\partial_{\mu_z} g(t,\theta_t^*)(\widetilde{y}_t^*, \widetilde{z}_t^*, \widetilde{u}_t^*)\widetilde{L}_t\big] \\ \qquad - \widetilde{\mathbb{E}}\big[\partial_{\mu_u} g(t,\theta_t^*)(\widetilde{y}_t^*, \widetilde{z}_t^*, \widetilde{u}_t^*)\widetilde{v}_t\big]\Big\} d\overleftarrow{B}_t - \hat{z}_t^\varepsilon dW_t, \ \ t\in[0,T], \\ \hat{y}_T^\varepsilon = 0. \end{cases} \tag{3.9}$$

Using notations (3.5) and (3.6), some algebraic work shows that (3.9) can be written as

$$\begin{cases} -d\hat{y}_t^\varepsilon = \Big[\int_0^1 \Big\{ \partial_y f(t,\theta_t^\lambda)\hat{y}_t^\varepsilon + \widetilde{\mathbb{E}}\big[\partial_{\mu_y} f(t,\theta_t^\lambda)(\widetilde{y_t^\lambda}, \widetilde{z_t^\lambda}, \widetilde{u_t^\lambda})\widetilde{\hat{y}}_t^\varepsilon\big] \\ \qquad + \partial_z f(t,\theta_t^\lambda)\hat{z}_t^\varepsilon + \widetilde{\mathbb{E}}\big[\partial_{\mu_z} f(t,\theta_t^\lambda)(\widetilde{y_t^\lambda}, \widetilde{z_t^\lambda}, \widetilde{u_t^\lambda})\widetilde{\hat{z}}_t^\varepsilon\big] + F_t^{\lambda,\varepsilon}\Big\} d\lambda\Big] dt \\ \quad + \Big[\int_0^1 \Big\{ \partial_y g(t,\theta_t^\lambda)\hat{y}_t^\varepsilon + \widetilde{\mathbb{E}}\big[\partial_{\mu_y} g(t,\theta_t^\lambda)(\widetilde{y_t^\lambda}, \widetilde{z_t^\lambda}, \widetilde{u_t^\lambda})\widetilde{\hat{y}}_t^\varepsilon\big] \\ \qquad + \partial_z g(t,\theta_t^\lambda)\hat{z}_t^\varepsilon + \widetilde{\mathbb{E}}\big[\partial_{\mu_z} g(t,\theta_t^\lambda)(\widetilde{y_t^\lambda}, \widetilde{z_t^\lambda}, \widetilde{u_t^\lambda})\widetilde{\hat{z}}_t^\varepsilon\big] + G_t^{\lambda,\varepsilon}\Big\} d\lambda\Big] d\overleftarrow{B}_t \\ \quad - \hat{z}_t^\varepsilon dW_t, \ \ t\in[0,T], \\ \hat{y}_T^\varepsilon = 0. \end{cases} \tag{3.10}$$

where

$$\begin{aligned} F_t^{\lambda,\varepsilon} :=& \big(\partial_y f(t,\theta_t^\lambda) - \partial_y f(t,\theta_t^*)\big)K_t + \widetilde{\mathbb{E}}\Big[\big(\partial_{\mu_y} f(t,\theta_t^\lambda)(\widetilde{y_t^\lambda}, \widetilde{z_t^\lambda}, \widetilde{u_t^\lambda}) - \partial_{\mu_y} f(t,\theta_t^*)(\widetilde{y}_t^*, \widetilde{z}_t^*, \widetilde{u}_t^*)\big)\widetilde{K}_t\Big] \\ &+ \big(\partial_z f(t,\theta_t^\lambda) - \partial_z f(t,\theta_t^*)\big)L_t + \widetilde{\mathbb{E}}\Big[\big(\partial_{\mu_z} f(t,\theta_t^\lambda)(\widetilde{y_t^\lambda}, \widetilde{z_t^\lambda}, \widetilde{u_t^\lambda}) - \partial_{\mu_z} f(t,\theta_t^*)(\widetilde{y}_t^*, \widetilde{z}_t^*, \widetilde{u}_t^*)\big)\widetilde{L}_t\Big] \\ &+ \big(\partial_u f(t,\theta_t^\lambda) - \partial_u f(t,\theta_t^*)\big)v_t + \widetilde{\mathbb{E}}\Big[\big(\partial_{\mu_u} f(t,\theta_t^\lambda)(\widetilde{y_t^\lambda}, \widetilde{z_t^\lambda}, \widetilde{u_t^\lambda}) - \partial_{\mu_u} f(t,\theta_t^*)(\widetilde{y}_t^*, \widetilde{z}_t^*, \widetilde{u}_t^*)\big)\widetilde{v}_t\Big], \end{aligned}$$

and $G_t^{\lambda,\varepsilon}$ is of the same form as $F_t^{\lambda,\varepsilon}$ with f replaced by g.

Applying Itô's formula (3.3) to $|\hat{y}_t^\varepsilon|^2$, we have

$$\begin{aligned} &\mathbb{E}\big[|\hat{y}_t^\varepsilon|^2\big] + \mathbb{E}\Big[\int_t^T \|\hat{z}_s^\varepsilon\|^2 ds\Big] \\ =& 2\mathbb{E}\int_t^T \Big\langle \int_0^1 \big\{\partial_y f(s,\theta_s^\lambda)\hat{y}_s^\varepsilon + \widetilde{\mathbb{E}}\big[\partial_{\mu_y} f(s,\theta_s^\lambda)(\widetilde{y_s^\lambda}, \widetilde{z_s^\lambda}, \widetilde{u_s^\lambda})\widetilde{\hat{y}}_s^\varepsilon\big] \end{aligned}$$

$$+ \partial_z f(s,\theta_s^\lambda)\hat z_s^\varepsilon + \widetilde{\mathbb{E}}\big[\partial_{\mu_z} f(s,\theta_s^\lambda)(\widetilde{y_s^\lambda},\widetilde{z_s^\lambda},\widetilde{u_s^\lambda})\widetilde{\hat z_s^\varepsilon}\big] + F_s^{\lambda,\varepsilon}\big\}d\lambda, \hat y_s^\varepsilon\Big\rangle ds$$

$$+ \mathbb{E}\int_t^T \Big\| \int_0^1 \big\{\partial_y g(s,\theta_s^\lambda)\hat y_s^\varepsilon + \widetilde{\mathbb{E}}\big[\partial_{\mu_y} g(s,\theta_s^\lambda)(\widetilde{y_s^\lambda},\widetilde{z_s^\lambda},\widetilde{u_s^\lambda})\widetilde{\hat y_s^\varepsilon}\big]$$

$$+ \partial_z g(s,\theta_s^\lambda)\hat z_s^\varepsilon + \widetilde{\mathbb{E}}\big[\partial_{\mu_z} g(s,\theta_s^\lambda)(\widetilde{y_s^\lambda},\widetilde{z_s^\lambda},\widetilde{u_s^\lambda})\widetilde{\hat z_s^\varepsilon}\big] + G_s^{\lambda,\varepsilon}\big\}d\lambda\Big\|^2 ds.$$

The uniform boundedness of the partial derivatives of f and g as assumed in (H1) and (H2) yields

$$\mathbb{E}\big[|\hat y_t^\varepsilon|^2\big] + \mathbb{E}\Big[\int_t^T \|\hat z_s^\varepsilon\|^2 ds\Big] \le C\mathbb{E}\Big[\int_t^T |\hat y_s^\varepsilon|^2 ds\Big] + \mathbb{E}\int_t^T\int_0^1 \big\{\big|F_s^{\lambda,\varepsilon}\big|^2 + \big\|G_s^{\lambda,\varepsilon}\big\|^2\big\}d\lambda ds.$$

To get the desired result, in view of Gronwall's lemma, it suffices to show

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\int_0^T\int_0^1 \big|F_t^{\lambda,\varepsilon}\big|^2 d\lambda dt\Big] = 0, \tag{3.11}$$

and

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\int_0^T\int_0^1 \big\|G_t^{\lambda,\varepsilon}\big\|^2 d\lambda dt\Big] = 0. \tag{3.12}$$

We shall prove (3.11) below, and (3.12) can be proved in the same way and thus omitted. By Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T\int_0^1 \big|F_t^{\lambda,\varepsilon}\big|^2 d\lambda dt\Big] \\
&\le C\mathbb{E}\int_0^T \Big\{ \int_0^1 \big\{|\partial_y f(t,\theta_t^\lambda) - \partial_y f(t,\theta_t^*)|^2|K_t|^2 + \|\partial_z f(t,\theta_t^\lambda) - \partial_z f(t,\theta_t^*)\|^2\|L_t\|^2\big\}d\lambda \\
&\qquad + \int_0^1 \widetilde{\mathbb{E}}\big[\big|\partial_{\mu_y} f(t,\theta_t^\lambda)(\widetilde{y_t^\lambda},\widetilde{z_t^\lambda},\widetilde{u_t^\lambda}) - \partial_{\mu_y} f(t,\theta_t^*)(\widetilde y_t^*,\widetilde z_t^*,\widetilde u_t^*)\big|^2\big]\widetilde{\mathbb{E}}\big[|\widetilde K_t|^2\big]d\lambda \\
&\qquad + \int_0^1 \widetilde{\mathbb{E}}\big[\big\|\partial_{\mu_z} f(t,\theta_t^\lambda)(\widetilde{y_t^\lambda},\widetilde{z_t^\lambda},\widetilde{u_t^\lambda}) - \partial_{\mu_z} f(t,\theta_t^*)(\widetilde y_t^*,\widetilde z_t^*,\widetilde u_t^*)\big\|^2\big]\widetilde{\mathbb{E}}\big[\|\widetilde L_t\|^2\big]d\lambda \\
&\qquad + \int_0^1 \widetilde{\mathbb{E}}\big[\big|\partial_{\mu_u} f(t,\theta_t^\lambda)(\widetilde{y_t^\lambda},\widetilde{z_t^\lambda},\widetilde{u_t^\lambda}) - \partial_{\mu_u} f(t,\theta_t^*)(\widetilde y_t^*,\widetilde z_t^*,\widetilde u_t^*)\big|^2\big]\widetilde{\mathbb{E}}\big[|\widetilde v_t|^2\big]d\lambda \\
&\qquad + \int_0^1 \big|\partial_u f(t,\theta_t^\lambda) - \partial_u f(t,\theta_t^*)\big|^2|v_t|^2 d\lambda\Big\}dt.
\end{aligned}$$

Due to the continuity and uniform boundedness assumed in (H1) and (H2) for the partial derivatives, we can apply the dominated convergence theorem to prove (3.11). The proof is concluded. □

The differentiability of the cost functional $J(\cdot)$ proved in the following result will be used in the derivation of the variational inequality in Section 3.3.

Proposition 3.6. *Under conditions (H1)–(H3), the cost functional $J(\cdot)$ defined by (3.2) is Gateaux differentiable, and the derivative at u^* in the direction v is given by*

$$\frac{d}{d\varepsilon}J(u^*+\varepsilon v)\Big|_{\varepsilon=0} = \mathbb{E}\int_0^T \Big\{\big\langle \partial_y h(t,\theta_t^*), K_t\big\rangle + \big\langle \partial_z h(t,\theta_t^*), L_t\big\rangle + \big\langle \partial_u h(t,\theta_t^*), v_t\big\rangle$$

$$
\begin{aligned}
&+\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_y} h(t,\theta_t^*)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*),\widetilde{K}_t\big\rangle\big]+\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_z} h(t,\theta_t^*)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*),\widetilde{L}_t\big\rangle\big]\\
&+\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_u} h(t,\theta_t^*)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*),\widetilde{v}_t\big\rangle\big]\Big\}dt\\
&+\mathbb{E}\Big[\big\langle \partial_y\Phi(y_0^*,\mathcal{L}(y_0^*)),K_0\big\rangle+\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_y}\Phi(y_0^*,\mathcal{L}(y_0^*))(\widetilde{y}_0^*),\widetilde{K}_0\big\rangle\big]\Big]. \qquad (3.13)
\end{aligned}
$$

Proof. By the definition (3.2) of J and the notations (3.5), we have

$$
\begin{aligned}
\frac{J(u^\varepsilon)-J(u^*)}{\varepsilon}&=\frac{1}{\varepsilon}\Big[\mathbb{E}\int_0^T\big\{h(t,\theta_t^\varepsilon)-h(t,\theta_t^*)\big\}dt\Big]+\frac{1}{\varepsilon}\mathbb{E}\Big[\Phi(y_0^\varepsilon,\mathcal{L}(y_0^\varepsilon))-\Phi(y_0^*,\mathcal{L}(y_0^*))\Big]\\
&=: I_1+I_2. \qquad (3.14)
\end{aligned}
$$

For the term I_1, Taylor's first-order expansion yields

$$
\begin{aligned}
I_1=&\mathbb{E}\int_0^T\Big\{\big\langle \partial_y h(t,\theta_t^*),K_t\big\rangle+\big\langle \partial_z^* h(t,\theta_t^*),L_t\big\rangle+\big\langle \partial_u h(t,\theta_t^*),v_t\big\rangle\\
&+\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_y} h(t,\theta_t^*)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*),\widetilde{K}_t\big\rangle\big]+\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_z} h(t,\theta_t^*)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*),\widetilde{L}_t\big\rangle\big]\\
&+\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_u} h(t,\theta_t^*)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*),\widetilde{v}_t\big\rangle\big]\Big\}dt+\mathbb{E}\int_0^T\rho_t^\varepsilon dt,
\end{aligned}
$$

with

$$
\begin{aligned}
\rho_t^\varepsilon:=&\int_0^1\big\{\big\langle \partial_y h(t,\theta_t^\lambda),\hat{y}_t^\varepsilon\big\rangle+\big\langle \partial_y h(t,\theta_t^\lambda)-\partial_y h(t,\theta_t^*),K_t\big\rangle\big\}d\lambda\\
&+\int_0^1\big\{\big\langle \partial_z h(t,\theta_t^\lambda),\hat{z}_t^\varepsilon\big\rangle+\big\langle \partial_z h(t,\theta_t^\lambda)-\partial_z h(t,\theta_t^*),L_t\big\rangle\big\}d\lambda\\
&+\int_0^1\big\{\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_y} h(t,\theta_t^\lambda)(\widetilde{y_t^\lambda},\widetilde{z_t^\lambda},\widetilde{u_t^\lambda}),\widetilde{\hat{y}_t^\varepsilon}\big\rangle\big]+\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_z} h(t,\theta_t^\lambda)(\widetilde{y_t^\lambda},\widetilde{z_t^\lambda},\widetilde{u_t^\lambda}),\widetilde{\hat{z}_t^\varepsilon}\big\rangle\big]\big\}d\lambda\\
&+\int_0^1\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_y} h(t,\theta_t^\lambda)(\widetilde{y_t^\lambda},\widetilde{z_t^\lambda},\widetilde{u_t^\lambda})-\partial_{\mu_y} h(t,\theta_t^*)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*),\widetilde{K}_t\big\rangle\big]d\lambda\\
&+\int_0^1\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_z} h(t,\theta_t^\lambda)(\widetilde{y_t^\lambda},\widetilde{z_t^\lambda},\widetilde{u_t^\lambda})-\partial_{\mu_z} h(t,\theta_t^*)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*),\widetilde{L}_t\big\rangle\big]d\lambda\\
&+\int_0^1\widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_u} h(t,\theta_t^\lambda)(\widetilde{y_t^\lambda},\widetilde{z_t^\lambda},\widetilde{u_t^\lambda})-\partial_{\mu_u} h(t,\theta_t^*)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*),\widetilde{v}_t\big\rangle\big]d\lambda\\
&+\int_0^1\langle \partial_u h(t,\theta_t^\lambda)-\partial_u h(t,\theta_t^*),v_t\rangle d\lambda,
\end{aligned}
$$

where we recall that $\hat{y}_t^\varepsilon,\hat{z}_t^\varepsilon$ are given in (3.8).

Note that by (H1) and (H2), the partial derivatives of h are jointly continuous and uniformly bounded. Combining this fact with Proposition 3.5, we can apply dominated convergence theorem to get

$$
\lim_{\varepsilon\to 0}\mathbb{E}\int_0^T\rho_t^\varepsilon dt=0.
$$

The term I_2 in (3.14) can be analyzed in a similar way. The proof is concluded. □

3.3. On necessity of the condition

In this subsection, we present our main result of stochastic maximum principle, which is a necessary condition for an optimal control.

Let $H : [0,T]\times\mathbb{R}^n\times\mathbb{R}^{n\times l}\times\mathbb{R}^k\times\mathcal{P}_2(\mathbb{R}^n\times\mathbb{R}^{n\times l}\times\mathbb{R}^k)\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\to\mathbb{R}$ denote the Hamiltonian given by

$$H(t,y,z,u,\mu,p,q) := \big\langle f(t,y,z,u,\mu),p\big\rangle + \big\langle g(t,y,z,u,\mu),q\big\rangle + h(t,y,z,u,\mu). \tag{3.15}$$

Consider the following adjoint equation

$$\begin{cases} dp_t =\Big[\partial_y H(t,\theta_t^*,p_t,q_t) + \widetilde{\mathbb{E}}\big[\partial_{\mu_y} H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big]\Big]dt \\ \qquad\quad + \Big[\partial_z H(t,\theta_t^*,p_t,q_t) + \widetilde{\mathbb{E}}\big[\partial_{\mu_z} H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big]\Big]dW_t \\ \qquad\quad - q_t d\overleftarrow{B}_t, \quad t\in[0,T], \\ p_0 =\partial_y \Phi(y_0^*,\mathcal{L}(y_0^*)) + \widetilde{\mathbb{E}}\big[\partial_{\mu_y}\Phi(\widetilde{y}_0^*,\mathcal{L}(y_0^*))(y_0^*)\big], \end{cases} \tag{3.16}$$

where we have used notations in (3.5). Recalling the equation (3.7) of (K,L), applying Itô's formula to $\langle p_t, K_t\rangle$ from 0 to T and taking expectation, we can get

$$\begin{aligned}\mathbb{E}\big[\langle p_0,K_0\rangle\big] =\mathbb{E}\int_0^T \Big\{ & \big\langle \partial_u^{\mathrm{T}} f(t,\theta_t^*)p_t + \widetilde{\mathbb{E}}\big[\partial_{\mu_u}^{\mathrm{T}} f(t,\widetilde{\theta}_t^*)(y_t^*,z_t^*,u_t^*)\widetilde{p}_t\big] \\ & + \partial_u^{\mathrm{T}} g(t,\theta_t^*)q_t + \widetilde{\mathbb{E}}\big[\partial_{\mu_u}^{\mathrm{T}} g(t,\widetilde{\theta}_t^*)(y_t^*,z_t^*,u_t^*)\widetilde{q}_t\big], v_t\big\rangle \\ & - \big\langle \partial_y h(t,\theta_t^*) + \widetilde{\mathbb{E}}\big[\partial_{\mu_y} h(t,\widetilde{\theta}_t^*)(y_t^*,z_t^*,u_t^*)\big], K_t\big\rangle \\ & - \big\langle \partial_z h(t,\theta_t^*) + \widetilde{\mathbb{E}}\big[\partial_{\mu_z} h(t,\widetilde{\theta}_t^*)(y_t^*,z_t^*,u_t^*)\big], L_t\big\rangle\Big\}dt.\end{aligned}$$

Note that $\mathbb{E}\big[\langle p_0,K_0\rangle\big]$ is the sum of the last two terms on the right-hand side of (3.13). Plugging this expression into (3.13), we get

$$\begin{aligned}&\frac{d}{d\varepsilon}J(u^*+\varepsilon v)\Big|_{\varepsilon=0} \\ =&\mathbb{E}\int_0^T \Big\langle \partial_u^{\mathrm{T}} f(t,\theta_t^*)p_t + \widetilde{\mathbb{E}}\big[\partial_{\mu_u}^{\mathrm{T}} f(t,\widetilde{\theta}_t^*)(y_t^*,z_t^*,u_t^*)\widetilde{p}_t\big] + \partial_u^{\mathrm{T}} g(t,\theta_t^*)q_t \\ &\qquad + \widetilde{\mathbb{E}}\big[\partial_{\mu_u}^{\mathrm{T}} g(t,\widetilde{\theta}_t^*)(y_t^*,z_t^*,u_t^*)\widetilde{q}_t\big] + \partial_u h(t,\theta_t^*) + \widetilde{\mathbb{E}}\big[\partial_{\mu_u} h(t,\widetilde{\theta}_t^*)(y_t^*,z_t^*,u_t^*)\big], v_t\Big\rangle dt.\end{aligned}$$

Using the Hamiltonian H given by (3.15), we can write

$$\frac{d}{d\varepsilon}J(u^*+\varepsilon v)\Big|_{\varepsilon=0} = \mathbb{E}\int_0^T \Big\langle \partial_u H(t,\theta_t^*,p_t,q_t) + \widetilde{\mathbb{E}}\big[\partial_{\mu_u} H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big], v_t\Big\rangle dt. \tag{3.17}$$

Now we are ready to derive our main result, the stochastic maximum principle.

Theorem 3.7. *We assume conditions (H1)–(H3) for the control problem (3.1)–(3.2). Suppose that $u^* = (u_t^*)_{0\le t\le T}\in\mathcal{U}$ is an optimal control, $(y_t^*,z_t^*)_{0\le t\le T}$ is the associated state process, and $(p_t,q_t)_{0\le t\le T}$ is the adjoint process satisfying (3.16). Then, we have, for all $a\in\mathbb{U}$,*

$$\Big\langle \partial_u H(t,\theta_t^*,p_t,q_t) + \widetilde{\mathbb{E}}\big[\partial_{\mu_u} H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big], a-u_t^*\Big\rangle \ge 0, \quad dt\otimes d\mathbb{P} \text{ a.s.}, \tag{3.18}$$

where H is the Hamiltonian defined by (3.15).

Proof. Given any admissible control $(u_t)_{0\le t\le T}\in\mathcal{U}$, we denote $v_t=u_t-u_t^*$. We use the perturbation $u_t^\varepsilon=u_t^*+\varepsilon v_t$. Since u^* is optimal, *i.e.*, $J(u^*)$ achieves the minimum, we have

$$\frac{d}{d\varepsilon}J(u^*+\varepsilon v)\Big|_{\varepsilon=0}\ge 0.$$

This together with (3.17) implies

$$\mathbb{E}\int_0^T\Big\langle\partial_u H(t,\theta_t^*,p_t,q_t)+\widetilde{\mathbb{E}}\big[\partial_{\mu_u}H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big],u_t-u_t^*\Big\rangle dt\ge 0. \tag{3.19}$$

We set an admissible control $(u_t)_{0\le t\le T}$ as follows

$$u_s=\begin{cases}\alpha_s, & s\in[t,t+\varepsilon],\\ u_s^*, & \text{otherwise},\end{cases}$$

where $(\alpha_t)_{0\le t\le T}\in\mathcal{U}$. From (3.19), we have

$$\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\big\langle\partial_u H(s,\theta_s^*,p_s,q_s)+\widetilde{\mathbb{E}}\big[\partial_{\mu_u}H(s,\widetilde{\theta}_s^*,\widetilde{p}_s,\widetilde{q}_s)(y_s^*,z_s^*,u_s^*)\big],\alpha_s-u_s^*\big\rangle ds\right]\ge 0, \tag{3.20}$$

Letting $\varepsilon\to 0^+$, by Lebesgue differential theorem, we have for almost all t,

$$\mathbb{E}\left[\big\langle\partial_u H(t,\theta_t^*,p_t,q_t)+\widetilde{\mathbb{E}}\big[\partial_{\mu_u}H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big],\alpha_t-u_t^*\big\rangle\right]\ge 0.$$

For $A\in\mathcal{F}_t$, we set $\alpha_t=a\mathbf{1}_A+u_t^*\mathbf{1}_{A^c}$ with $a\in\mathbb{U}$. Thus, we have for almost all t,

$$\mathbb{E}\left[\big\langle\partial_u H(t,\theta_t^*,p_t,q_t)+\widetilde{\mathbb{E}}\big[\partial_{\mu_u}H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big],a-u_t^*\big\rangle\mathbf{1}_A\right]\ge 0.$$

As $A\in\mathcal{F}_t$ is chosen arbitrarily, the definition of conditional expectation leads to, for almost all t,

$$\begin{aligned}&\mathbb{E}\left[\big\langle\partial_u H(t,\theta_t^*,p_t,q_t)+\widetilde{\mathbb{E}}\big[\partial_{\mu_u}H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big],a-u_t^*\big\rangle\big|\mathcal{F}_t\right]\\&=\big\langle\partial_u H(t,\theta_t^*,p_t,q_t)+\widetilde{\mathbb{E}}\big[\partial_{\mu_u}H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big],a-u_t^*\big\rangle\ge 0,\text{a.s.}\end{aligned}$$

This proves the desired result. ☐

3.4. On sufficiency of the condition

In this subsection, we prove a verification theorem which states that under proper conditions, the maximum principle (3.18) obtained in Theorem 3.7 does yield an optimal control.

Theorem 3.8. *Assume (H1)–(H3). We further assume that the Hamiltonian H given in (3.15) and Φ are convex in the sense*

$$\begin{aligned}&H(t,y',z',u',\mu',p,q)-H(t,y,z,u,\mu,p,q)\\&\ge\big\langle\partial_y H(t,y,z,u,\mu,p,q),y'-y\big\rangle+\widetilde{\mathbb{E}}\big[\big\langle\partial_{\mu_y}H(t,y,z,u,\mu,p,q)(\widetilde{Y},\widetilde{Z},\widetilde{U}),\widetilde{Y}'-\widetilde{Y}\big\rangle\big]\\&+\big\langle\partial_z H(t,y,z,u,\mu,p,q),z'-z\big\rangle+\widetilde{\mathbb{E}}\big[\big\langle\partial_{\mu_z}H(t,y,z,u,\mu,p,q)(\widetilde{Y},\widetilde{Z},\widetilde{U}),\widetilde{Z}'-\widetilde{Z}\big\rangle\big]\end{aligned}$$

$$+ \big\langle \partial_u H(t,y,z,u,\mu,p,q), u'-u \big\rangle + \widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_u} H(t,y,z,u,\mu,p,q)(\widetilde{Y},\widetilde{Z},\widetilde{U}), \widetilde{U}'-\widetilde{U} \big\rangle\big],$$

and

$$\Phi(y',\mu_y') - \Phi(y,\mu_y) \geq \big\langle \partial_y \Phi(y,\mu_y), y'-y \big\rangle + \widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_y}\Phi(y,\mu_y)(\widetilde{Y}), \widetilde{Y}'-\widetilde{Y}\big\rangle\big],$$

for all $y,y' \in \mathbb{R}^n$, $z,z' \in \mathbb{R}^{n\times l}$, $u,u' \in \mathbb{R}^k$, $\mu,\mu' \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k)$ *with* $\mu = (\mu_y,\mu_z,\mu_u) = \mathcal{L}(\widetilde{Y},\widetilde{Z},\widetilde{U})$, $\mu' = (\mu_y',\mu_z',\mu_u') = \mathcal{L}(\widetilde{Y}',\widetilde{Z}',\widetilde{U}')$, $p \in \mathbb{R}^n$ *and* $q \in \mathbb{R}^{n\times d}$.

Let $u^* = (u_t^*)_{0\leq t\leq T} \in \mathcal{U}$ *be an admissible control,* $(y_t^*, z_t^*)_{0\leq t\leq T}$ *the state process and* $(p_t,q_t)_{0\leq t\leq T}$ *the adjoint process. Then, if* (3.18) *holds,* u^* *is an optimal control.*

Proof. Recalling the definition of (3.2) of J and the notations (3.5), we have

$$J(u) - J(u^*) = \mathbb{E}\int_0^T \big\{h(t,\theta_t^u) - h(t,\theta_t^*)\big\}dt + \mathbb{E}\big[\Phi(y_0^u,\mathcal{L}(y_0^u)) - \Phi(y_0^*,\mathcal{L}(y_0^*))\big],$$

where we use the superscript u to denote the processes associated to the control process $(u_t)_{0\leq t\leq T} \in \mathcal{U}$. It follows directly from the convexity of H and Φ that

$$\begin{aligned}
&h(t,\theta_t^u) - h(t,\theta_t^*)\\
&= H(t,\theta_t^u,p_t,q_t) - H(t,\theta_t^*,p_t,q_t) - \big\langle f(t,\theta_t^u) - f(t,\theta_t^*), p_t\big\rangle - \big\langle g(t,\theta_t^u) - g(t,\theta_t^*), q_t\big\rangle\\
&\geq \big\langle \partial_y H(t,\theta_t^*,p_t,q_t), y_t^u - y_t^*\big\rangle + \widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_y} H(t,\theta_t^*,p_t,q_t)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*), \widetilde{y}_t^u - \widetilde{y}_t^*\big\rangle\big]\\
&\ + \big\langle \partial_z H(t,\theta_t^*,p_t,q_t), z_t^u - z_t^*\big\rangle + \widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_z} H(t,\theta_t^*,p_t,q_t)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*), \widetilde{z}_t^u - \widetilde{z}_t^*\big\rangle\big]\\
&\ + \big\langle \partial_u H(t,\theta_t^*,p_t,q_t), u_t - u_t^*\big\rangle + \widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_u} H(t,\theta_t^*,p_t,q_t)(\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*), \widetilde{u}_t - \widetilde{u}_t^*\big\rangle\big]\\
&\ - \big\langle f(t,\theta_t^u) - f(t,\theta_t^*), p_t\big\rangle - \big\langle g(t,\theta_t^u) - g(t,\theta_t^*), q_t\big\rangle,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\mathbb{E}\big[\Phi(y_0^u,\mathcal{L}(y_0^u)) - \Phi(y_0^*,\mathcal{L}(y_0^*))\big]\\
&\geq \mathbb{E}\Big[\big\langle \partial_y\Phi(y_0^*,\mathcal{L}(y_0^*)), y_0^u - y_0^*\big\rangle + \widetilde{\mathbb{E}}\big[\big\langle \partial_{\mu_y}\Phi(y_0^*,\mathcal{L}(y_0^*))(\widetilde{y}_0^*), \widetilde{y}_0^u - \widetilde{y}_0^*\big\rangle\big]\Big]\\
&= \mathbb{E}\Big[\big\langle \partial_y\Phi(y_0^*,\mathcal{L}(y_0^*)) + \widetilde{\mathbb{E}}\big[\partial_{\mu_y}\Phi(\widetilde{y}_0^*,\mathcal{L}(y_0^*))(y_0^*)\big], y_0^u - y_0^*\big\rangle\Big].
\end{aligned} \tag{3.22}$$

Applying Itô's formula to $\langle p_t, y_t^u - y_t^*\rangle$ yields that

$$\begin{aligned}
&\mathbb{E}\Big[\big\langle \partial_y\Phi(y_0^*,\mathcal{L}(y_0^*)) + \widetilde{\mathbb{E}}\big[\partial_{\mu_y}\Phi(\widetilde{y}_0^*,\mathcal{L}(y_0^*))(y_0^*)\big], y_0^u - y_0^*\big\rangle\Big]\\
&= \mathbb{E}\int_0^T \Big\{\big\langle f(t,\theta_t^u) - f(t,\theta_t^*), p_t\big\rangle + \big\langle g(t,\theta_t^u) - g(t,\theta_t^*), q_t\big\rangle\\
&\qquad - \big\langle \partial_y H(t,\theta_t^*,p_t,q_t) + \widetilde{\mathbb{E}}\big[\partial_{\mu_y} H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big], y_t^u - y_t^*\big\rangle\\
&\qquad - \big\langle \partial_z H(t,\theta_t^*,p_t,q_t) + \widetilde{\mathbb{E}}\big[\partial_{\mu_z} H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big], z_t^u - z_t^*\big\rangle\Big\}dt.
\end{aligned} \tag{3.23}$$

Combining (3.21)–(3.23), and using Fubini's theorem, we have

$$J(u) - J(u^*) \geq \mathbb{E}\int_0^T \Big\langle \partial_u H(t,\theta_t^*,p_t,q_t) + \widetilde{\mathbb{E}}\big[\partial_{\mu_u} H(t,\widetilde{\theta}_t^*,\widetilde{p}_t,\widetilde{q}_t)(y_t^*,z_t^*,u_t^*)\big], u_t - u_t^*\Big\rangle dt$$

Thus, if we assume (3.18), we get

$$J(u) - J(u^*) \geq 0.$$

Noting that $u \in \mathcal{U}$ is chosen arbitrarily, this implies that u^* is an optimal control. The proof is concluded. $\square$

4. Well-posedness of mean-field forward-backward doubly stochastic differential equations

Using the Hamiltonian H given in (3.15), the state equation (3.1) and the adjoint equation (3.16) can be combined in the following system

$$\begin{cases} -dy_t^* = \partial_p H(t, \theta_t^*, p_t, q_t)dt + \partial_q H(t, \theta_t^*, p_t, q_t)d\overleftarrow{B}_t - z_t^* dW_t, \\ \quad dp_t = \Big[\partial_y H(t, \theta_t^*, p_t, q_t) + \widetilde{\mathbb{E}}\big[\partial_{\mu_y} H(t, \widetilde{\theta}_t^*, \widetilde{p}_t, \widetilde{q}_t)(y_t^*, z_t^*, u_t^*)\big]\Big]dt \\ \qquad\quad + \Big[\partial_z H(t, \theta_t^*, p_t, q_t) + \widetilde{\mathbb{E}}\big[\partial_{\mu_z} H(t, \widetilde{\theta}_t^*, \widetilde{p}_t, \widetilde{q}_t)(y_t^*, z_t^*, u_t^*)\big]\Big]dW_t \\ \qquad\quad - q_t d\overleftarrow{B}_t, \ \ t \in [0,T], \\ \quad y_T^* = \xi, \ p_0 = \partial_y \Phi(y_0^*, \mathcal{L}(y_0^*)) + \widetilde{\mathbb{E}}\big[\partial_{\mu_y}\Phi(\widetilde{y}_0^*, \mathcal{L}(y_0^*))(y_0^*)\big], \end{cases} \tag{4.1}$$

where θ_t is given in (3.5).

If u_t^* is a function of y_t^*, z_t^*, p_t, q_t and their joint distribution (see, *e.g.*, the LQ case in Sect. 5.3), the above system (4.1) can be written as a time-symmetric FBDSDE introduced in Peng and Shi [28] of mean-field type,

$$\begin{cases} -dy_t = f(t, y_t, p_t, z_t, q_t, \mathcal{L}(y_t, p_t, z_t, q_t))dt + g(t, y_t, p_t, z_t, q_t, \mathcal{L}(y_t, p_t, z_t, q_t))d\overleftarrow{B}_t - z_t dW_t, \\ \quad dp_t = F(t, y_t, p_t, z_t, q_t, \mathcal{L}(y_t, p_t, z_t, q_t))dt + G(t, y_t, p_t, z_t, q_t, \mathcal{L}(y_t, p_t, z_t, q_t))dW_t - q_t d\overleftarrow{B}_t, \\ \quad y_T = \xi, \ p_0 = \Psi(y_0, \mathcal{L}(y_0)), \end{cases} \tag{4.2}$$

where ξ is an $\mathcal{F}_T$-measurable random variable, $\Psi : \Omega \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^n$, and f, g, F, G are functions from $\Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d} \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$ to $\mathbb{R}^n, \mathbb{R}^{n\times d}, \mathbb{R}^n, \mathbb{R}^{n\times l}$, respectively.

Definition 4.1. A quadruple of processes (y, p, z, q) is called a solution of (4.2) if $(y, p, z, q) \in L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$ and satisfies (4.2).

Let $\mathcal{A}(t, \zeta, \mu) = (-F, f, -G, g)(t, \zeta, \mu)$ where $\zeta = (y, p, z, q)$ and μ stands for a generic element in $\mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$. Assume that for each $(\zeta, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d} \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$, $\mathcal{A}(\cdot, \zeta, \mu) \in L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$ and that for each $(y, \mu_y) \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$, $\Psi(y, \mu_y) \in L^2_{\mathcal{F}_0}(\mathbb{R}^n)$. For almost all $(t, \omega) \in [0,T] \times \Omega$, we assume the following conditions.

(A1) There exists $k_1 > 0$ such that

$$|\mathcal{A}(t, \zeta, \mu) - \mathcal{A}(t, \zeta', \mu')| \leq k_1\big(|\zeta - \zeta'| + W_2(\mu, \mu')\big),$$
$$|\Psi(y, \mu_y) - \Psi(y', \mu_y')| \leq k_1\big(|y - y'| + W_2(\mu_y, \mu_y')\big),$$

hold for $\zeta, \zeta' \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d}$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$, $y, y' \in \mathbb{R}^n$, and $\mu_y, \mu_y' \in \mathcal{P}_2(\mathbb{R}^n)$.

(A2) Let $\boldsymbol{\zeta} = (Y, P, Z, Q), \boldsymbol{\zeta}' = (Y', P', Z', Q') \in L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d}),\ \mu = \mathcal{L}(\boldsymbol{\zeta}), \mu' = \mathcal{L}(\boldsymbol{\zeta}')$. There exist constants $k_2, k_3, k_4 \geq 0$ with $k_2 + k_3 > 0$ and $k_3 + k_4 > 0$ such that

$$\begin{aligned}&\mathbb{E}\big[\big\langle \mathcal{A}(t,\boldsymbol{\zeta},\mu) - \mathcal{A}(t,\boldsymbol{\zeta}',\mu'), \boldsymbol{\zeta} - \boldsymbol{\zeta}'\big\rangle\big] \\ &\leq -k_2\big(\mathbb{E}[|Y-Y'|^2 + \|Z-Z'\|^2]\big) - k_3\big(\mathbb{E}[|P-P'|^2 + \|Q-Q'\|^2]\big),\end{aligned}$$

and

$$\mathbb{E}\big[\big\langle \Psi(Y,\mathcal{L}(Y)) - \Psi(Y',\mathcal{L}(Y')), Y - Y'\big\rangle\big] \geq k_4 \mathbb{E}[|Y-Y'|^2].$$

Moreover, we make some further assumptions if k_2 or k_3 is zero: we assume

$$\begin{aligned}\|g(t,\boldsymbol{\zeta},\mu) - g(t,\boldsymbol{\zeta}',\mu')\| &\leq k_1(|Y-Y'| + |P-P'| + \|Q-Q'\|) + \lambda_1\|Z-Z'\| \\ &\quad + k_1(\mathbb{E}[|Y-Y'| + |P-P'| + \|Q-Q'\|]) + \lambda_2\mathbb{E}[\|Z-Z'\|],\end{aligned}$$

if $k_2 = 0$, and

$$\begin{aligned}\|G(t,\boldsymbol{\zeta},\mu) - G(t,\boldsymbol{\zeta}',\mu')\| &\leq k_1(|Y-Y'| + \|Z-Z'\| + |P-P'|) + \lambda_1\|Q-Q'\| \\ &\quad + k_1(\mathbb{E}[|Y-Y'| + \|Z-Z'\| + |P-P'|]) + \lambda_2\mathbb{E}[\|Q-Q'\|],\end{aligned}$$

if $k_3 = 0$, where λ_1, λ_2 are nonnegative constants satisfying $\lambda_1 + \lambda_2 < 1$.

We shall employ the method of continuation introduced in [30] (see also [29, 31]) to establish the existence of solution of (4.2). Consider a family of mean-field FBDSDEs parameterized by $\alpha \in [0,1]$,

$$\begin{cases}-dy_t = \big[f^\alpha(t,\zeta_t,\mu_t) + f_0(t)\big]dt + \big[g^\alpha(t,\zeta_t,\mu_t) + g_0(t)\big]d\overleftarrow{B}_t - z_t dW_t,\\ dp_t = \big[F^\alpha(t,\zeta_t,\mu_t) + F_0(t)\big]dt + \big[G^\alpha(t,\zeta_t,\mu_t) + G_0(t)\big]dW_t - q_t d\overleftarrow{B}_t,\\ y_T = \xi,\ p_0 = \Psi^\alpha(y_0,\mathcal{L}(y_0)) + \Psi_0,\end{cases} \tag{4.3}$$

where $\zeta_t = (y_t, p_t, z_t, q_t),\ \mu_t = \mathcal{L}(y_t, p_t, z_t, q_t),\ (F_0, f_0, G_0, g_0) \in L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d}),\ \Psi_0 \in L^2_{\mathcal{F}_0}(\mathbb{R}^n)$ and for any given $\alpha \in [0,1]$,

$$\begin{aligned}&f^\alpha(t,\zeta_t,\mu_t) = \alpha f(t,\zeta_t,\mu_t) - (1-\alpha)k_3 p_t,\ \ g^\alpha(t,\zeta_t,\mu_t) = \alpha g(t,\zeta_t,\mu_t) - (1-\alpha)k_3 q_t,\\ &F^\alpha(t,\zeta_t,\mu_t) = \alpha F(t,\zeta_t,\mu_t) + (1-\alpha)k_2 y_t,\ \ G^\alpha(t,\zeta_t,\mu_t) = \alpha G(t,\zeta_t,\mu_t) + (1-\alpha)k_2 z_t,\\ &\Psi^\alpha(y_0,\mathcal{L}(y_0)) = \alpha\Psi(y_0,\mathcal{L}(y_0)) + (1-\alpha)y_0.\end{aligned}$$

When $\alpha = 0$, equation (4.3) is reduced to

$$\begin{cases}-dy_t = [-k_3 p_t + f_0(t)]dt + [-k_3 q_t + g_0(t)]d\overleftarrow{B}_t - z_t dW_t,\\ dp_t = [k_2 y_t + F_0(t)]dt + [k_2 z_t + G_0(t)]dW_t - q_t d\overleftarrow{B}_t,\\ y_T = \xi,\ p_0 = y_0 + \Psi_0.\end{cases} \tag{4.4}$$

The existence and uniqueness of the solution of equation (4.4) have been obtained in [25], Proposition 3.6. The following lemma is the key ingredient of the continuation method, which says that, if (4.3) has a solution for some $\alpha_0 \in [0,1)$, it also has a solution for $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$, where δ_0 is a constant independent of α_0.

Lemma 4.2. *Under (A1)-(A2), we assume that there exists a constant* $\alpha_0 \in [0,1)$ *such that given any* $(F_0, f_0, G_0, g_0) \in L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$ *and* $\Psi_0 \in L^2_{\mathcal{F}_0}(\mathbb{R}^n)$, $\xi \in L^2_{\mathcal{F}_T}(\mathbb{R}^n)$, *equation* (4.3) *with* $\alpha = \alpha_0$ *has a unique solution. Then, there exists a constant* $\delta_0 \in (0,1)$ *which only depends on* $k_1, k_2, k_3, k_4, \lambda_1, \lambda_2$ *and* T, *such that for any* $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$, *equation* (4.3) *has a unique solution.*

Proof. By the assumption, for each $\overline{\zeta} = (\overline{y}, \overline{p}, \overline{z}, \overline{q}) \in L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$ with law $\overline{\mu} \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$, there exists a unique quadruple $\zeta = (y, p, z, q) \in L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$ satisfying, for $\delta > 0$,

$$\begin{cases} -dy_t = \big[f^{\alpha_0}(t,\zeta_t,\mu_t) + \delta(f(t,\overline{\zeta}_t,\overline{\mu}_t) + k_3\overline{p}_t) + f_0(t)\big]dt \\ \qquad\qquad + \big[g^{\alpha_0}(t,\zeta_t,\mu_t) + \delta(g(t,\overline{\zeta}_t,\overline{\mu}_t) + k_3\overline{q}_t) + g_0(t)\big]d\overleftarrow{B}_t - z_t dW_t, \\ \ \ dp_t = \big[F^{\alpha_0}(t,\zeta_t,\mu_t) + \delta(F(t,\overline{\zeta}_t,\overline{\mu}_t) - k_2\overline{y}_t) + F_0(t)\big]dt \\ \qquad\qquad + \big[G^{\alpha_0}(t,\zeta_t,\mu_t) + \delta(G(t,\overline{\zeta}_t,\overline{\mu}_t) - k_2\overline{z}_t) + G_0(t)\big]dW_t - q_t d\overleftarrow{B}_t, \\ \ \ y_T = \xi,\ p_0 = \Psi^{\alpha_0}(y_0, \mathcal{L}(y_0)) + \delta(\Psi(\overline{y}_0, \mathcal{L}(\overline{y}_0)) - \overline{y}_0) + \Psi_0. \end{cases} \tag{4.5}$$

In order to prove that (4.3) with $\alpha = \alpha_0 + \delta$ has a solution (for sufficiently small δ), it suffices to show that the mapping $I_{\alpha_0,\delta}(\overline{\zeta}) = \zeta$ defined *via* (4.5) is a contraction mapping on $L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d})$. For this purpose, we will obtain some estimations first.

Denote

$$\widehat{\zeta} = (\widehat{y}, \widehat{p}, \widehat{z}, \widehat{q}) = (y - y', p - p', z - z', q - q'),$$
$$\widehat{\overline{\zeta}} = (\widehat{\overline{y}}, \widehat{\overline{p}}, \widehat{\overline{z}}, \widehat{\overline{q}}) = (\overline{y} - \overline{y}', \overline{p} - \overline{p}', \overline{z} - \overline{z}', \overline{q} - \overline{q}').$$

Applying the product rule (3.4) to $\langle \widehat{y}_t, \widehat{p}_t \rangle$ yields

$$\begin{aligned} &\alpha_0 \mathbb{E}\big[\langle \widehat{y}_0, \Psi(y_0, \mathcal{L}(y_0)) - \Psi(y_0', \mathcal{L}(y_0'))\rangle\big] + (1-\alpha_0)\mathbb{E}[|\widehat{y}_0|^2] \\ &\quad + \delta \mathbb{E}\big[\langle \widehat{y}_0, -\widehat{\overline{y}}_0 + \Psi(\overline{y}_0, \mathcal{L}(\overline{y}_0)) - \Psi(\overline{y}_0', \mathcal{L}(\overline{y}_0'))\rangle\big] \\ &= \mathbb{E}\int_0^T \big\{\alpha_0 \langle \mathcal{A}(t,\zeta_t,\mu_t) - \mathcal{A}(t,\zeta_t',\mu_t'), \widehat{\zeta}_t\rangle + \delta\langle \mathcal{A}(t,\overline{\zeta}_t,\overline{\mu}_t) - \mathcal{A}(t,\overline{\zeta}_t',\overline{\mu}_t'), \widehat{\zeta}_t\rangle\big\}dt \\ &\quad - (1-\alpha_0)\mathbb{E}\int_0^T \big\{k_3|\widehat{p}_t|^2 + k_3\|\widehat{q}_t\|^2 + k_2|\widehat{y}_t|^2 + k_2\|\widehat{z}_t\|^2\big\}dt \\ &\quad + \delta\mathbb{E}\int_0^T \big\{k_3\langle \widehat{\overline{p}}_t, \widehat{p}_t\rangle + k_3\langle \widehat{\overline{q}}_t, \widehat{q}_t\rangle + k_2\langle \widehat{\overline{y}}_t, \widehat{y}_t\rangle + k_2\langle \widehat{\overline{z}}_t, \widehat{z}_t\rangle\big\}dt. \end{aligned}$$

By (A1)-(A2), we have

$$\begin{aligned} &(\alpha_0 k_4 + 1 - \alpha_0)\mathbb{E}\big[|\widehat{y}_0|^2\big] + \mathbb{E}\int_0^T \big\{k_3\big(|\widehat{p}_t|^2 + \|\widehat{q}_t\|^2\big) + k_2\big(|\widehat{y}_t|^2 + \|\widehat{z}_t\|^2\big)\big\}dt \\ &\le \delta\Bigg\{\mathbb{E}\int_0^T \big\{k_3\big(\tfrac{1}{2}|\widehat{\overline{p}}_t|^2 + \tfrac{1}{2}|\widehat{p}_t|^2\big) + k_3\big(\tfrac{1}{2}\|\widehat{\overline{q}}_t\|^2 + \tfrac{1}{2}\|\widehat{q}_t\|^2\big) + k_2\big(\tfrac{1}{2}|\widehat{\overline{y}}_t|^2 + \tfrac{1}{2}|\widehat{y}_t|^2\big) \\ &\qquad + k_2\big(\tfrac{1}{2}\|\widehat{\overline{z}}_t\|^2 + \tfrac{1}{2}\|\widehat{z}_t\|^2\big) + |\widehat{\zeta}_t||\mathcal{A}(t,\overline{\zeta}_t,\overline{\mu}_t) - \mathcal{A}(t,\overline{\zeta}_t',\overline{\mu}_t')|\big\}dt \\ &\quad + \mathbb{E}\big[\big(\tfrac{1}{2}|\widehat{y}_0|^2 + \tfrac{1}{2}|\widehat{\overline{y}}_0|^2\big) + |\Psi(\overline{y}_0, \mathcal{L}(\overline{y}_0)) - \Psi(\overline{y}_0', \mathcal{L}(\overline{y}_0'))||\widehat{y}_0|\big]\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&\le \delta\Bigg\{\mathbb{E}\int_0^T \big\{k_3\big(\tfrac{1}{2}|\widehat{\overline{p}}_t|^2+\tfrac{1}{2}|\widehat{p}_t|^2\big)+k_3\big(\tfrac{1}{2}\|\widehat{\overline{q}}_t\|^2+\tfrac{1}{2}\|\widehat{q}_t\|^2\big)+k_2\big(\tfrac{1}{2}|\widehat{\overline{y}}_t|^2+\tfrac{1}{2}|\widehat{y}_t|^2\big)\\
&\qquad\qquad +k_2\big(\tfrac{1}{2}\|\widehat{\overline{z}}_t\|^2+\tfrac{1}{2}\|\widehat{z}_t\|^2\big)+k_1\big(|\widehat{\zeta}_t|^2+\tfrac{1}{2}|\widehat{\overline{\zeta}}_t|^2+\tfrac{1}{2}W_2^2(\overline{\mu}_t,\overline{\mu}'_t)\big)\big\}dt\\
&\qquad +\mathbb{E}\big[\big(\tfrac{1}{2}|\widehat{y}_0|^2+\tfrac{1}{2}|\widehat{\overline{y}}_0|^2\big)+k_1\big(|\widehat{y}_0|^2+\tfrac{1}{2}|\widehat{\overline{y}}_0|^2+\tfrac{1}{2}W_2^2(\mathcal{L}(\overline{y}_0),\mathcal{L}(\overline{y}'_0))\big)\big]\Bigg\}.
\end{aligned}
$$

Noting

$$
W_2^2(\overline{\mu}_t,\overline{\mu}'_t)\le \mathbb{E}\big[|\widehat{\overline{\zeta}}_t|^2\big],\quad \text{and}\quad W_2^2(\mathcal{L}(\overline{y}_0),\mathcal{L}(\overline{y}'_0))\le \mathbb{E}\big[|\widehat{\overline{y}}_0|^2\big],
$$

we can find a constant $K_1>0$ depending only on k_1,k_2,k_3 such that

$$
\begin{aligned}
&(\alpha_0k_4+1-\alpha_0)\mathbb{E}\big[|\widehat{y}_0|^2\big]+\mathbb{E}\int_0^T \big\{k_2\big(|\widehat{y}_t|^2+\|\widehat{z}_t\|^2\big)+k_3\big(|\widehat{p}_t|^2+\|\widehat{q}_t\|^2\big)\big\}dt\\
&\le \delta K_1\left\{\mathbb{E}\int_0^T \big(|\widehat{\zeta}_t|^2+|\widehat{\overline{\zeta}}_t|^2\big)dt+\mathbb{E}\big[|\widehat{y}_0|^2+|\widehat{\overline{y}}_0|^2\big]\right\}.
\end{aligned}
$$

Noting $(\alpha_0k_4+1-\alpha_0)\ge\min\{1,k_4\}$, we have

$$
\begin{aligned}
&\min\{1,k_4\}\mathbb{E}\big[|\widehat{y}_0|^2\big]+\mathbb{E}\int_0^T \big\{k_2\big(|\widehat{y}_t|^2+\|\widehat{z}_t\|^2\big)+k_3\big(|\widehat{p}_t|^2+\|\widehat{q}_t\|^2\big)\big\}dt\\
&\le \delta K_1\left\{\mathbb{E}\int_0^T \big(|\widehat{\zeta}_t|^2+|\widehat{\overline{\zeta}}_t|^2\big)dt+\mathbb{E}\big[|\widehat{y}_0|^2+|\widehat{\overline{y}}_0|^2\big]\right\}.
\end{aligned}
\tag{4.6}
$$

Applying Itô's formula (3.3) to $|\widehat{y}_t|^2$, we have

$$
\begin{aligned}
&\mathbb{E}\big[|\widehat{y}_t|^2\big]+\mathbb{E}\int_t^T\|\widehat{z}_s\|^2ds\\
&=2\mathbb{E}\int_t^T\big\{\big\langle\widehat{y}_s,\alpha_0(f(s,\zeta_s,\mu_s)-f(s,\zeta'_s,\mu'_s))-(1-\alpha_0)k_3\widehat{p}_s\big\rangle\\
&\qquad\qquad +\big\langle\widehat{y}_s,\delta(f(s,\overline{\zeta}_s,\overline{\mu}_s)-f(s,\overline{\zeta}'_s,\overline{\mu}'_s))+\delta k_3\widehat{\overline{p}}_s\big\rangle\big\}ds\\
&\quad+\mathbb{E}\int_t^T\big\|\alpha_0(g(s,\zeta_s,\mu_s)-g(s,\zeta'_s,\mu'_s))-(1-\alpha_0)k_3\widehat{q}_s\\
&\qquad\qquad +\delta(g(s,\overline{\zeta}_s,\overline{\mu}_s)-g(s,\overline{\zeta}'_s,\overline{\mu}'_s))+\delta k_3\widehat{\overline{q}}_s\big\|^2ds.
\end{aligned}
$$

By the Lipschitz conditions (A1) and the Gronwall's inequality, we can find a constant K_2 depending on only $k_1,k_2,k_3,\lambda_1,\lambda_2$ such that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}\big[|\widehat{y}_t|^2\big]\le K_2\left\{\delta\mathbb{E}\int_0^T|\widehat{\overline{\zeta}}_t|^2dt+\mathbb{E}\int_0^T\big\{|\widehat{p}_t|^2+\|\widehat{q}_t\|^2\big\}dt\right\},\\
&\mathbb{E}\int_0^T\big\{|\widehat{y}_t|^2+\|\widehat{z}_t\|^2\big\}dt\le K_2(T\vee 1)\left\{\delta\mathbb{E}\int_0^T|\widehat{\overline{\zeta}}_t|^2dt+\mathbb{E}\int_0^T\big\{|\widehat{p}_t|^2+\|\widehat{q}_t\|^2\big\}dt\right\}.
\end{aligned}
\tag{4.7}
$$

Similarly, the application of Itô's formula (3.3) to $|\widehat{p}_t|^2$ yields

$$\begin{aligned}
&\mathbb{E}\big[|\widehat{p}_t|^2\big] + \mathbb{E}\int_0^t \|\widehat{q}_s\|^2 ds \\
&= \mathbb{E}\Big[\big|\alpha\big\{\Psi(y_0,\mathcal{L}(y_0)) - \Psi(y_0',\mathcal{L}(y_0'))\big\} + (1-\alpha_0)\widehat{y}_0 \\
&\qquad + \delta\big\{\Psi(\overline{y}_0,\mathcal{L}(\overline{y}_0)) - \Psi(\overline{y}_0',\mathcal{L}(\overline{y}_0')) - \widehat{\overline{y}}_0\big\}\big|^2\Big] \\
&\quad + 2\mathbb{E}\int_0^t \big\{\big\langle \widehat{p}_s, \alpha_0(F(s,\zeta_s,\mu_s) - F(s,\zeta_s',\mu_s')) + (1-\alpha_0)k_2\widehat{y}_s\big\rangle \\
&\qquad\quad + \big\langle \widehat{p}_s, \delta(F(s,\overline{\zeta}_s,\overline{\mu}_s) - F(s,\overline{\zeta}_s',\overline{\mu}_s')) - \delta k_2\widehat{\overline{y}}_s\big\rangle\big\} ds \\
&\quad + \mathbb{E}\int_0^t \big\|\alpha_0(G(s,\zeta_s,\mu_s) - G(s,\zeta_s',\mu_s')) + (1-\alpha_0)k_2\widehat{z}_s \\
&\qquad\quad + \delta(G(s,\overline{\zeta}_s,\overline{\mu}_s) - G(s,\overline{\zeta}_s',\overline{\mu}_s')) - \delta k_2\widehat{\overline{z}}_s\big\|^2 ds
\end{aligned}$$

By the Lipschitz conditions (A1) and the Gronwall's inequality, we can deduce that there exists K_3 depending only on $k_1,k_2,k_3,\lambda_1,\lambda_2$ such that

$$\mathbb{E}\int_0^T \{|\widehat{p}_t|^2 + \|\widehat{q}_t\|^2\} dt \le K_3\left\{\delta\mathbb{E}\left(\int_0^T |\widehat{\overline{\zeta}}_t|^2 dt + |\widehat{\overline{y}}_0|^2\right) + \mathbb{E}\left(\int_0^T \{|\widehat{y}_t|^2 + \|\widehat{z}_t\|^2\} dt + |\widehat{y}_0|^2\right)\right\}. \tag{4.8}$$

Now, in order to obtain the contraction of the mapping $I_{\alpha_0,\delta}$ for small δ, it suffices to prove the following estimation from (4.6)–(4.8),

$$\mathbb{E}\big[|\widehat{y}_0|^2\big] + \mathbb{E}\Big[\int_0^T \big|\widehat{\zeta}_t\big|^2 dt\Big] \le \delta K\left(\mathbb{E}\big[|\widehat{\overline{y}}_0|^2\big] + \mathbb{E}\Big[\int_0^T \big|\widehat{\overline{\zeta}}_t\big|^2 dt\Big]\right), \tag{4.9}$$

for some positive constant K depending on $k_1,k_2,k_3,k_4,\lambda_1,\lambda_2$ and T. The proof is split in three cases according to the positiveness of k_2,k_3,k_4. When $k_2,k_3,k_4>0$, (4.9) is a direct consequence of (4.6), when $k_2>0,k_3=0,k_4>0$, (4.9) follows from (4.6) and (4.8), and when $k_2=0,k_3>0,k_4\ge 0$, it follows from (4.6) and (4.7).

The proof is concluded. □

We are ready to present our main result in this section.

Theorem 4.3. *Under the assumptions (A1)–(A2), there exists a unique solution* $(y,p,z,q)\in L^2_{\mathcal{F}}([0,T];\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times l}\times\mathbb{R}^{n\times d})$ *to equation* (4.2).

Proof. The existence and uniqueness of the solution follows from the well-posedness of equation (4.4) by [25], Proposition 3.6 and Lemma 4.2. We also provide a direct proof for the uniqueness as follows.

Let $\zeta^1 = (y^1, p^1, z^1, q^1)$ and $\zeta^2 = (y^2, p^2, z^2, q^2)$ be two solutions of (4.2). Denote $\widehat{\zeta} = (\widehat{y}, \widehat{p}, \widehat{z}, \widehat{q}) = (y^1 - y^2, p^1 - p^2, z^1 - z^2, q^1 - q^2)$. Applying Itô's formula to $\langle \widehat{y}_t, \widehat{p}_t \rangle$, yields that

$$\begin{aligned}&\mathbb{E}\left[\left\langle \Psi(y_0^1, \mathcal{L}(y_0^1)) - \Psi(y_0^2, \mathcal{L}(y_0^2)), \widehat{y}_0 \right\rangle\right] \\ &= \mathbb{E}\int_0^T \left\langle \mathcal{A}(t, \zeta_t^1, \mu_t^1) - \mathcal{A}(t, \zeta_t^2, \mu_t^2), \widehat{\zeta}_t \right\rangle dt,\end{aligned}$$

where we recall that $\mathcal{A} = (-F, f, -G, g)$. By the monotonicity condition (A2), we have that

$$k_4\mathbb{E}\big[|\widehat{y}_0|^2\big] \le -k_2\mathbb{E}\int_0^T \big\{|\widehat{y}_t|^2 + \|\widehat{z}_t\|^2\big\}dt - k_3\mathbb{E}\int_0^T \big\{|\widehat{p}_t|^2 + \|\widehat{q}_t\|^2\big\}dt \le 0. \tag{4.10}$$

If $k_2, k_3 > 0$, it yields directly $\zeta^1 = \zeta^2$. If k_2 or k_3 is 0, say, $k_2 = 0$ and $k_3 > 0$, we have $p_1 = p_2$ and $q_1 = q_2$ by (4.10), and the uniqueness of (y, z) follows from the classical result of BDSDEs (see [4]). The proof is completed. □

Remark 4.4. When the mean-field FBDSDE (4.2) is reduced to classical FBDSDE (without mean field), Theorem 4.3 recovers the existence and uniqueness result obtained in [28], Theorem 2.2. Our result is also compatible with [29], Theorem 2.6 when (4.2) degenerates to classical FBSDE.

5. Examples

In this section, we apply our results obtained in preceding sections to some special cases. For simplicity, we assume $n = l = d = k = 1$ throughout this section unless otherwise specified.

5.1. Scalar interaction

In this subsection, we consider the scalar interaction type control problem, in which the dependence upon probability measure is through the moments of the probability measure.

More precisely, we assume that the coefficients in the state equation (3.1) take the following form,

$$\begin{aligned}&f(t, y, z, u, \mu) = \hat{f}\big(t, y, z, u, \int \varphi d\mu\big), \quad g(t, y, z, u, \mu) = \hat{g}\big(t, y, z, u, \int \phi d\mu\big), \\ &h(t, y, z, u, \mu) = \hat{h}\big(t, y, z, u, \int \psi d\mu\big), \quad \Phi(y, \mu_y) = \hat{\Phi}\big(y, \int \gamma d\mu_y\big),\end{aligned}$$

for functions $\varphi, \phi, \psi : \mathbb{R}^3 \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ with at most quadratic growth, and functions $\hat{f}$, $\hat{g}$, $\hat{h} : [0, T] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, $\hat{\Phi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying proper regularity conditions. Here $\int \varphi d\mu := \int_{\mathbb{R}^3} \varphi(y, z, u) d\mu(y, z, u) = \mathbb{E}[\varphi(Y, Z, U)]$ where (Y, Z, U) is a random vector with $\mathcal{L}(Y, Z, U) = \mu$.

Similar to Example 2.2 in Section 2, the L-derivatives of f, g, h and Φ can be calculated *via* $\hat{f}, \hat{g}, \hat{h}$ and $\hat{\Phi}$ respectively. For instance,

$$\partial_{\mu_y} f\big(t, y, z, u, \mathcal{L}(Y, Z, U)\big)(Y, Z, U) = \partial_r \hat{f}\big(t, y, z, u, \mathbb{E}[\varphi(Y, Z, U)]\big)\partial_y \varphi(Y, Z, U),$$

where $\partial_r \hat{f}$ denotes the partial derivative with respect to the term $\mathbb{E}[\varphi(Y, Z, U)]$.

Set $\Theta_t^* = (y_t^*, z_t^*, u_t^*)$. Then, the adjoint equation (3.16) can be written as

$$\begin{cases} dp_t = \Big\{ \partial_y \hat{f}\big(t, \Theta_t^*, \mathbb{E}\big[\varphi(\Theta_t^*)\big]\big) p_t + \widetilde{\mathbb{E}}\big[\widetilde{p}_t \partial_r \hat{f}\big(t, \widetilde{\Theta}_t^*, \mathbb{E}\big[\varphi(\Theta_t^*)\big]\big) \partial_y \varphi(\Theta_t^*)\big] \\ \qquad + \partial_y \hat{g}\big(t, \Theta_t^*, \mathbb{E}\big[\phi(\Theta_t^*)\big]\big) q_t + \widetilde{\mathbb{E}}\big[\widetilde{q}_t \partial_r \hat{g}\big(t, \widetilde{\Theta}_t^*, \mathbb{E}\big[\phi(\Theta_t^*)\big] \partial_y \phi(\Theta_t^*)\big)\big] \\ \qquad + \partial_y \hat{h}\big(t, \Theta_t^*, \mathbb{E}\big[\psi(\Theta_t^*)\big]\big) + \widetilde{\mathbb{E}}\big[\partial_r \hat{h}\big(t, \widetilde{\Theta}_t^*, \mathbb{E}\big[\psi(\Theta_t^*)\big]\big) \partial_y \psi(\Theta_t^*)\big] \Big\} dt \\ \qquad + \Big\{ \partial_z \hat{f}\big(t, \Theta_t^*, \mathbb{E}\big[\varphi(\Theta_t^*)\big]\big) p_t + \widetilde{\mathbb{E}}\big[\widetilde{p}_t \partial_r \hat{f}\big(t, \widetilde{\Theta}_t^*, \mathbb{E}\big[\varphi(\Theta_t^*)\big]\big) \partial_z \varphi(\Theta_t^*)\big] \\ \qquad + \partial_z \hat{g}\big(t, \Theta_t^*, \mathbb{E}\big[\phi(\Theta_t^*)\big]\big) q_t + \widetilde{\mathbb{E}}\big[\widetilde{q}_t \partial_r \hat{g}\big(t, \widetilde{\Theta}_t^*, \mathbb{E}\big[\phi(\Theta_t^*)\big] \partial_z \phi(\Theta_t^*)\big)\big] \\ \qquad + \partial_z \hat{h}\big(t, \Theta_t^*, \mathbb{E}\big[\psi(\Theta_t^*)\big]\big) + \widetilde{\mathbb{E}}\big[\partial_r \hat{h}\big(t, \widetilde{\Theta}_t^*, \mathbb{E}\big[\psi(\Theta_t^*)\big]\big) \partial_z \psi(\Theta_t^*)\big] \Big\} dW_t \\ \qquad - q_t d\overleftarrow{B}_t, \ \ t \in [0, T], \\ p_0 = \partial_y \hat{\Phi}\big(y_0^*, \mathbb{E}[\gamma(y_0^*)]\big) + \widetilde{\mathbb{E}}\big[\partial_r \hat{\Phi}\big(\widetilde{y}_0^*, \mathbb{E}[\gamma(y_0^*)]\big) \partial_y \gamma(y_0^*)\big]. \end{cases} \tag{5.1}$$

The stochastic maximum principle (3.18) obtained in Theorem 3.7 becomes

$$\begin{aligned} &\Big\{ \partial_u \hat{f}\big(t, \Theta_t^*, \mathbb{E}\big[\varphi(\Theta_t^*)\big]\big) p_t + \partial_u \hat{g}\big(t, \Theta_t^*, \mathbb{E}\big[\phi(\Theta_t^*)\big]\big) q_t + \partial_u \hat{h}\big(t, \Theta_t^*, \mathbb{E}\big[\psi(\Theta_t^*)\big]\big) \\ &\quad + \widetilde{\mathbb{E}}\big[\widetilde{p}_t \partial_r \hat{f}\big(t, \widetilde{\Theta}_t^*, \mathbb{E}\big[\varphi(\Theta_t^*)\big]\big) \partial_u \varphi(\Theta_t^*)\big] + \widetilde{\mathbb{E}}\big[\widetilde{q}_t \partial_r \hat{g}\big(t, \widetilde{\Theta}_t^*, \mathbb{E}\big[\phi(\Theta_t^*)\big]\big) \partial_u \phi(\Theta_t^*)\big] \\ &\quad + \widetilde{\mathbb{E}}\big[\partial_r \hat{h}\big(t, \widetilde{\Theta}_t^*, \mathbb{E}\big[\psi(\Theta_t^*)\big]\big) \partial_u \psi(\Theta_t^*)\big] \big(a - u_t^*\big) \geq 0, \quad \text{for all } a \in \mathbb{U}. \end{aligned}$$

5.2. First order interaction

In this example, we consider the case of first order interaction where the dependence of the coefficients on the probability measure is linear in the following sense

$$\begin{aligned} f(t, y, z, u, \mu) &= \int_{\mathbb{R}^3} \hat{f}(t, y, z, u, y', z', u') d\mu(y', z', u') = \widetilde{\mathbb{E}}[\hat{f}(t, y, z, u, \widetilde{Y}, \widetilde{Z}, \widetilde{U})], \\ g(t, y, z, u, \mu) &= \int_{\mathbb{R}^3} \hat{g}(t, y, z, u, y', z', u') d\mu(y', z', u') = \widetilde{\mathbb{E}}[\hat{g}(t, y, z, u, \widetilde{Y}, \widetilde{Z}, \widetilde{U})], \\ h(t, y, z, u, \mu) &= \int_{\mathbb{R}^3} \hat{h}(t, y, z, u, y', z', u') d\mu(y', z', u') = \widetilde{\mathbb{E}}[\hat{h}(t, y, z, u, \widetilde{Y}, \widetilde{Z}, \widetilde{U})], \\ \Phi(y, \mu_y) &= \int_{\mathbb{R}} \hat{\Phi}(y, y') d\mu_y(y') = \widetilde{\mathbb{E}}[\hat{\Phi}(y, \widetilde{Y})], \end{aligned}$$

for some functions $\hat{f}$, $\hat{g}$, $\hat{h}$ defined on $\mathbb{R}^3 \times \mathbb{R}^3$ and $\hat{\Phi}$ defined $\mathbb{R} \times \mathbb{R}$ with values on $\mathbb{R}$, where $(\widetilde{Y}, \widetilde{Z}, \widetilde{U})$ is a random vector with the law μ.

The state equation (3.1) with first order interaction corresponds to a type of mean-field BDSDE which may arise naturally in economics, finance and game theorem, etc. We refer to [9] for a study of mean-field BSDEs with first order interaction *via* a limit approach.

Actually, when considering the N-players game where each individual state is governed by

$$dX_t^i = \frac{1}{N} \sum_{j=1}^N \hat{b}(t, X_t^i, X_t^j, u_t^i) dt + \sigma dW_t, \ \ i = 1, \cdots, N,$$

u_t^i denotes the strategy of i-th player. The equation can be rewritten as

$$dX_t^i = b(t, X_t^i, \overline{\mu}_t^N, u_t^i)dt + \sigma dW_t,$$

where $\overline{\mu}_t^N = \frac{1}{N}\sum_{j=1}^{N}\delta_{X_t^j}$ and $b(t,x,\mu,u) = \int_{\mathbb{R}^n}\hat{b}(t,x,x',u)d\mu(x')$. Interaction given by functions of the form is called first order or linear.

Now, we come back to our control problem in the case of first order interaction. From Example 2.2, f, g, h are linear with respect to μ, and Φ is linear in μ_y. For Φ, $\partial_{\mu_y}\Phi(y,\mu_y)(y') = \partial_{y'}\hat{\Phi}(y,y')$ and similarly, $\partial_{\mu_y} f(t,y,z,u,\mu)(y',z',u') = \partial_{y'}\hat{f}(t,y,z,u,y',z',u')$. The adjoint equation is

$$\begin{cases} dp_t = \Big\{ \widetilde{\mathbb{E}}\big[\partial_y \hat{f}(t,\Theta_t^*,\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*)p_t + \partial_y \hat{g}(t,\Theta_t^*,\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*)q_t + \partial_y \hat{h}(t,\Theta_t^*,\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*)\big] \\ \qquad + \widetilde{\mathbb{E}}\big[\partial_{y'}\hat{f}(t,\widetilde{\Theta}_t^*,y_t^*,z_t^*,u_t^*)\widetilde{p}_t + \partial_{y'}\hat{g}(t,\widetilde{\Theta}_t^*,y_t^*,z_t^*,u_t^*)\widetilde{q}_t + \partial_{y'}\hat{h}(t,\widetilde{\Theta}_t^*,y_t^*,z_t^*,u_t^*)\big]\Big\}dt \\ \qquad + \Big\{\widetilde{\mathbb{E}}\big[\partial_z \hat{f}(t,\Theta_t^*,\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*)p_t + \partial_z \hat{g}(t,\Theta_t^*,\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*)q_t + \partial_z \hat{h}(t,\Theta_t^*,\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*)\big] \\ \qquad + \widetilde{\mathbb{E}}\big[\partial_{z'}\hat{f}(t,\widetilde{\Theta}_t^*,y_t^*,z_t^*,u_t^*)\widetilde{p}_t + \partial_{z'}\hat{g}(t,\widetilde{\Theta}_t^*,y_t^*,z_t^*,u_t^*)\widetilde{q}_t + \partial_{z'}\hat{h}(t,\widetilde{\Theta}_t^*,y_t^*,z_t^*,u_t^*)\big]\Big\}dW_t \\ \qquad - q_t d\overleftarrow{B}_t, \ \ t\in[0,T], \\ p_0 = \widetilde{\mathbb{E}}\big[\partial_y\hat{\Phi}(y_0^*,\widetilde{y}_0^*) + \partial_{y'}\hat{\Phi}(\widetilde{y}_0^*,y_0^*)\big]. \end{cases} \tag{5.2}$$

Similarly, it follows from applying stochastic maximum principle in Theorem 3.7 that for $\forall a\in\mathbb{U}$,

$$\Big\{\widetilde{\mathbb{E}}\Big[\partial_u \hat{f}(t,\Theta_t^*,\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*)p_t + \partial_u \hat{g}(t,\Theta_t^*,\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*)q_t + \partial_u \hat{h}(t,\Theta_t^*,\widetilde{y}_t^*,\widetilde{z}_t^*,\widetilde{u}_t^*) \\ + \partial_{u'}\hat{f}(t,\widetilde{\Theta}_t^*,y_t^*,z_t^*,u_t^*)\widetilde{p}_t + \partial_{u'}\hat{g}(t,\widetilde{\Theta}_t^*,y_t^*,z_t^*,u_t^*)\widetilde{q}_t + \partial_{u'}\hat{h}(t,\widetilde{\Theta}_t^*,y_t^*,z_t^*,u_t^*)\Big]\Big\}\big(a-u_t^*\big)\geq 0.$$

5.3. LQ problem

In this subsection, we will apply the stochastic maximum principle derived in Section 3 to a kind of mean-field stochastic linear quadratic control problem with scalar interaction. In such an LQ model, the drift and the volatility in (3.1) are of the form

$$f(t,y_t,z_t,u_t,\mathcal{L}(y_t,z_t,u_t)) = f_1y_t + f_2z_t + f_3u_t + \overline{f}_1\mathbb{E}[y_t] + \overline{f}_2\mathbb{E}[z_t] + \overline{f}_3\mathbb{E}[u_t],$$
$$g(t,y_t,z_t,u_t,\mathcal{L}(y_t,z_t,u_t)) = g_1y_t + g_2z_t + g_3u_t + \overline{g}_1\mathbb{E}[y_t] + \overline{g}_2\mathbb{E}[z_t] + \overline{g}_3\mathbb{E}[u_t],$$

and the cost functional is assumed to be

$$J(u) = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \big\{h_1y_t^2 + h_2z_t^2 + h_3u_t^2 + \overline{h}_1(\mathbb{E}[y_t])^2 + \overline{h}_2(\mathbb{E}[z_t])^2 + \overline{h}_3(\mathbb{E}[u_t])^2\big\}dt \\ + \Phi y_0^2 + \overline{\Phi}(\mathbb{E}[y_0])^2\bigg],$$

where $f_i,\overline{f}_i,\ g_i,\overline{g}_i,\ h_i,\overline{h}_i$ for $i=1,2,3$ and $\Phi,\overline{\Phi}$ are given constants satisfying $h_1,h_2,\overline{h}_1,\overline{h}_2,\Phi,\overline{\Phi}\geq 0$ and

$h_3, \overline{h}_3 > 0$, $|g_2| + |\overline{g}_2| < 1$. In this setting, the Hamiltonian H given by (3.15) is

$$\begin{aligned}H(t,y,z,u,\mu,p,q) =& \big\{f_1 y + f_2 z + f_3 u + \overline{f}_1\mathbb{E}[y] + \overline{f}_2\mathbb{E}[z] + \overline{f}_3\mathbb{E}[u]\big\}p\\ &+\big\{g_1 y + g_2 z + g_3 u + \overline{g}_1\mathbb{E}[y] + \overline{g}_2\mathbb{E}[z] + \overline{g}_3\mathbb{E}[u]\big\}q\\ &+\tfrac{1}{2}\big\{h_1 y^2 + h_2 z^2 + h_3 u^2 + \overline{h}_1(\mathbb{E}[y])^2 + \overline{h}_2(\mathbb{E}[z])^2 + \overline{h}_3(\mathbb{E}[u])^2\big\},\end{aligned} \tag{5.3}$$

and the adjoint equation (3.16) is

$$\begin{cases} dp_t =\big\{f_1 p_t + \overline{f}_1\mathbb{E}[p_t] + g_1 q_t + \overline{g}_1\mathbb{E}[q_t] + h_1 y_t^* + \overline{h}_1\mathbb{E}[y_t^*]\big\}dt\\ \qquad +\big\{f_2 p_t + \overline{f}_2\mathbb{E}[p_t] + g_2 q_t + \overline{g}_2\mathbb{E}[q_t] + h_2 z_t^* + \overline{h}_2\mathbb{E}[z_t^*]\big\}dW_t\\ \qquad -q_t d\overleftarrow{B}_t, \ \ t\in[0,T],\\ p_0 =\Phi y_0^* + \overline{\Phi}\mathbb{E}[y_0^*].\end{cases} \tag{5.4}$$

If we further assume that the control domain $\mathbb{U}$ is the whole space $\mathbb{R}$, the stochastic maximum principle (3.18) yields

$$f_3 p_t + \overline{f}_3\mathbb{E}[p_t] + g_3 q_t + \overline{g}_3\mathbb{E}[q_t] + h_3 u_t^* + \overline{h}_3\mathbb{E}[u_t^*] = 0. \tag{5.5}$$

Taking expectation, we have

$$\mathbb{E}[u_t^*] = -\frac{1}{h_3+\overline{h}_3}\Big\{(f_3+\overline{f}_3)\mathbb{E}[p_t] + (g_3+\overline{g}_3)\mathbb{E}[q_t]\Big\}. \tag{5.6}$$

Plugging this into (5.5), we obtain that

$$u_t^* = -\frac{1}{h_3}\Big\{f_3 p_t + \tfrac{1}{h_3+\overline{h}_3}\big(h_3\overline{f}_3 - \overline{h}_3 f_3\big)\mathbb{E}[p_t] + g_3 q_t + \tfrac{1}{h_3+\overline{h}_3}\big(h_3\overline{g}_3 - \overline{h}_3 g_3\big)\mathbb{E}[q_t]\Big\}. \tag{5.7}$$

If the following stochastic Hamiltonian system

$$\begin{cases} -dy_t^* =\Big\{f_1 y_t^* + f_2 z_t^* + f_3 u_t^* + \overline{f}_1\mathbb{E}[y_t^*] + \overline{f}_2\mathbb{E}[z_t^*] + \overline{f}_3\mathbb{E}[u_t^*]\Big\}dt\\ \qquad\quad + \Big\{g_1 y_t^* + g_2 z_t^* + g_3 u_t^* + \overline{g}_1\mathbb{E}[y_t^*] + \overline{g}_2\mathbb{E}[z_t^*] + \overline{g}_3\mathbb{E}[u_t^*]\Big\}d\overleftarrow{B}_t - z_t^* dW_t,\\ dp_t =\Big\{f_1 p_t + \overline{f}_1\mathbb{E}[p_t] + g_1 q_t + \overline{g}_1\mathbb{E}[q_t] + h_1 y_t^* + \overline{h}_1\mathbb{E}[y_t^*]\Big\}dt\\ \qquad\quad + \Big\{f_2 p_t + \overline{f}_2\mathbb{E}[p_t] + g_2 q_t + \overline{g}_2\mathbb{E}[q_t] + h_2 z_t^* + \overline{h}_2\mathbb{E}[z_t^*]\Big\}dW_t - q_t d\overleftarrow{B}_t,\\ y_T^* =\xi, \ p_0 = \Phi y_0^* + \overline{\Phi}\mathbb{E}[y_0^*].\end{cases} \tag{5.8}$$

with u_t^* being given by (5.7) admits a solution, by the verification theorem in Section 3.4, the control process (5.7) is indeed the unique optimal control. Substituting (5.6) and (5.7) for $\mathbb{E}[u_t^*]$ and u_t^* respectively leads to a strong coupling between the forward and backward equations in (5.8), and we cannot apply Theorem 4.3 due to the lack of monotonicity assumed in condition (A2). In the rest of this subsection, we shall prove the existence and uniqueness of the solution under some weaker conditions which are satisfied by (5.8) without the terms of $\mathbb{E}[u_t^*]$.

Consider the following mean-field FBDSDE

$$\begin{cases} -dy_t = f(t,y_t,Cp_t,z_t,Dq_t,\mathcal{L}(y_t,Cp_t,z_t,Dq_t))dt \\ \qquad\qquad + g(t,y_t,Cp_t,z_t,Dq_t,\mathcal{L}(y_t,Cp_t,z_t,Dq_t))d\overleftarrow{B}_t - z_t dW_t, \\ dp_t = F(t,y_t,p_t,z_t,q_t,\mathcal{L}(y_t,p_t,z_t,q_t))dt \\ \qquad\qquad + G(t,y_t,p_t,z_t,q_t,\mathcal{L}(y_t,p_t,z_t,q_t))dW_t - q_t d\overleftarrow{B}_t. \\ y_T = \xi,\ p_0 = \Psi(y_0,\mathcal{L}(y_0)), \end{cases} \tag{5.9}$$

where C and D are matrices of dimension $n\times n$. For simplicity, here we set $l=d=1$. We introduce below conditions (B1)-(B2) which are parallel to but weaker than (A1)–(A2) imposed in Section 4. As in Section 4, we use the notations $\zeta=(y,p,z,q)$ and $\mathcal{A}(t,\zeta,\mu)=(-F,\breve{f},-G,\breve{g})(t,\zeta,\mu)$, where $\breve{f}(t,\zeta,\mu)=f(t,y,Cp,z,Dq,\mathcal{L}(y,Cp,z,Dq))$ and similarly for $\breve{g}$.

(B1) There exist constants $c_1\geq 0, c_2>0$ such that

$$\begin{aligned} \mathbb{E}\big[\big\langle \mathcal{A}(t,\zeta,\mu)-\mathcal{A}(t,\zeta',\mu'),\widehat{\zeta}\big\rangle\big] &\leq -c_1\mathbb{E}\big[|\widehat{y}|^2+|\widehat{z}|^2\big]-c_2\mathbb{E}\big[|C\widehat{p}+D\widehat{q}|^2\big], \\ \mathbb{E}\big[\big\langle \Psi(y,\mu_y)-\Psi(y',\mu_y'),\widehat{y}\big\rangle\big] &\geq 0, \end{aligned}$$

for $\zeta,\zeta'\in L^2_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n), \mu,\mu'\in\mathcal{P}_2(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n), \mu_y,\mu_y'\in\mathcal{P}_2(\mathbb{R}^n),\ \widehat{\zeta}=(\widehat{y},\widehat{p},\widehat{z},\widehat{q})=(y-y',p-p',z-z',q-q')$.

(B2) There exists a constant $c_3>0$ such that

$$\begin{aligned} |\mathcal{A}(t,\zeta,\mu)-\mathcal{A}(t,\zeta',\mu')| &\leq c_3\big(|\zeta-\zeta'|+W_2(\mu,\mu')\big), \\ |\Psi(y,\mu_y)-\Psi(y',\mu_y')| &\leq c_3\big(|y-y'|+W_2(\mu_y,\mu_y')\big). \end{aligned}$$

Moreover, we assume that there exist $\lambda_1,\lambda_2>0$ with $\lambda_1+\lambda_2<1$ such that for $\forall t\in[0,T]$,

$$\begin{aligned} &\big|f(t,y,Cp,z,Dq,\mathcal{L}(y,Cp,z,Dq))-f(t,y',Cp',z',Dq',\mathcal{L}(y',Cp',z',Dq'))\big|^2 \\ &\leq c_3\big(|\widehat{y}|^2+|\widehat{z}|^2+|C\widehat{p}+D\widehat{q}|^2+\mathbb{E}\big[|\widehat{y}|^2+|\widehat{z}|^2+|C\widehat{p}+D\widehat{q}|^2\big]\big), \\ &\big|g(t,y,Cp,z,Dq,\mathcal{L}(y,Cp,z,Dq))-g(t,y',Cp',z',Dq',\mathcal{L}(y',Cp',z',Dq'))\big|^2 \\ &\leq c_3\big(|\widehat{y}|^2+|C\widehat{p}+D\widehat{q}|^2+\mathbb{E}\big[|\widehat{y}|^2+|C\widehat{p}+D\widehat{q}|^2\big]\big)+\lambda_1|\widehat{z}|^2+\lambda_2\mathbb{E}\big[|\widehat{z}|^2\big], \\ &\big|F(t,y,p,z,q,\mu)-F(t,y',p',z',q',\mu')\big|^2 \leq c_3\big(|\widehat{\zeta}|^2+W_2^2(\mu,\mu')\big), \\ &\big|G(t,y,p,z,q,\mu)-G(t,y',p',z',q',\mu')\big|^2 \\ &\leq c_3\big(|\widehat{y}|^2+|\widehat{z}|^2+|\widehat{p}|^2+\mathbb{E}\big[|\widehat{y}|^2+|\widehat{z}|^2+|\widehat{p}|^2\big]\big)+\lambda_1|\widehat{q}|^2+\lambda_2\mathbb{E}\big[|\widehat{q}|^2\big]. \end{aligned}$$

Theorem 5.1. *Under conditions (B1)–(B2), equation (5.9) admits a unique solution.*

Proof. First we prove the uniqueness. Let $\zeta=(y,z,p,q)$ and $\zeta'=(y',p',z',q')$ be two solutions of (5.9). Applying Itô's formula to $\langle\widehat{y}_t,\widehat{p}_t\rangle$ and taking expectation yield

$$\mathbb{E}\big[\big\langle\widehat{y}_0,\Psi(y_0,\mathcal{L}(y_0))-\Psi(y_0',\mathcal{L}(y_0'))\big\rangle\big]=\mathbb{E}\int_0^T\big\langle\mathcal{A}(t,\zeta_t,\mu_t)-\mathcal{A}(t,\zeta_t',\mu_t'),\widehat{\zeta}_t\big\rangle dt.$$

This together with condition (B1) implies

$$c_2\mathbb{E}\int_0^T \left|C\widehat{p}_t + D\widehat{q}_t\right|^2 dt \le 0,$$

and recalling that $c_2 > 0$, we have

$$|C\widehat{p}_t + D\widehat{q}_t|^2 = 0, \text{ for almost all } t \in [0,T]. \tag{5.10}$$

Now we deal with $|\widehat{y}_t|^2$ in a similar way. Using Lipschitz conditions on f and g in (B2) and taking (5.10) into account, we can get

$$\mathbb{E}[|\widehat{y}_t|^2] + \frac{1}{2}\mathbb{E}\int_t^T |\widehat{z}_s|^2 ds \le c_0\mathbb{E}\int_t^T |\widehat{y}_s|^2 ds,$$

for some positive constant c_0. This implies $\widehat{y} \equiv 0$ by Gronwall's inequality and hence $\widehat{z} \equiv 0$. The uniqueness of (p,q) then follows directly from classical result for BDSDEs.

To obtain the existence of the solution, we consider the following equation:

$$\begin{cases} -dy_t = \Big\{\alpha f(t,y_t,Cp_t,z_t,Dq_t,\mathcal{L}(y_t,Cp_t,z_t,Dq_t)) - (1-\alpha)(C^{\mathrm{T}}Cp_t + C^{\mathrm{T}}Dq_t) + f_0(t)\Big\}dt \\ \qquad\qquad + \Big\{\alpha g(t,y_t,Cp_t,z_t,Dq_t,\mathcal{L}(y_t,Cp_t,z_t,Dq_t)) - (1-\alpha)(D^{\mathrm{T}}Cp_t + D^{\mathrm{T}}Dq_t) + g_0(t)\Big\}d\overleftarrow{B}_t \\ \qquad\qquad - z_t dW_t, \\ \quad dp_t = \Big\{\alpha F(t,y_t,p_t,z_t,q_t,\mathcal{L}(y_t,p_t,z_t,q_t)) + F_0(t)\Big\}dt \\ \qquad\qquad + \Big\{\alpha G(t,y_t,p_t,z_t,q_t,\mathcal{L}(y_t,p_t,z_t,q_t)) + G_0(t)\Big\}dW_t - q_t d\overleftarrow{B}_t, \\ \quad y_T = \xi,\ p_0 = \alpha\Psi(y_0,\mathcal{L}(y_0)) + \Psi_0. \end{cases} \tag{5.11}$$

Clearly, (5.11) with $\alpha = 1$ coincides with (5.9), and when $\alpha = 0$, the existence and uniqueness follows directly from [4]. As in Section 4, we shall take the method of continuation and prove the result of Lemma 4.2 under conditions (B1)-(B2). More precisely, given $\alpha_0 \in [0,1)$ and $\overline{\zeta} = (\overline{y},\overline{p},\overline{z},\overline{q}) \in L^2_{\mathcal{F}}([0,T];\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$, we consider

$$\begin{cases} -dy_t = \Big\{\alpha_0 f(t,y_t,Cp_t,z_t,Dq_t,\mathcal{L}(y_t,Cp_t,z_t,Dq_t)) - (1-\alpha_0)(C^{\mathrm{T}}Cp_t + C^{\mathrm{T}}Dq_t) \\ \qquad\qquad + \delta f(t,\overline{y}_t,C\overline{p}_t,\overline{z}_t,D\overline{q}_t,\mathcal{L}(\overline{y}_t,C\overline{p}_t,\overline{z}_t,D\overline{q}_t)) + \delta(C^{\mathrm{T}}C\overline{p}_t + C^{\mathrm{T}}D\overline{q}_t) + f_0(t)\Big\}dt \\ \qquad\qquad + \Big\{\alpha_0 g(t,y_t,Cp_t,z_t,Dq_t,\mathcal{L}(y_t,Cp_t,z_t,Dq_t)) - (1-\alpha_0)(D^{\mathrm{T}}Cp_t + D^{\mathrm{T}}Dq_t) \\ \qquad\qquad + \delta g(t,\overline{y}_t,C\overline{p}_t,\overline{z}_t,D\overline{q}_t,\mathcal{L}(\overline{y}_t,C\overline{p}_t,\overline{z}_t,D\overline{q}_t)) + \delta(D^{\mathrm{T}}C\overline{p}_t + D^{\mathrm{T}}D\overline{q}_t) + g_0(t)\Big\}d\overleftarrow{B}_t \\ \qquad\qquad - z_t dW_t, \\ \quad dp_t = \Big\{\alpha_0 F(t,y_t,p_t,z_t,q_t,\mathcal{L}(y_t,p_t,z_t,q_t)) + \delta F(t,\overline{y}_t,\overline{p}_t,\overline{z}_t,\overline{q}_t,\mathcal{L}(\overline{y}_t,\overline{p}_t,\overline{z}_t,\overline{q}_t)) + F_0(t)\Big\}dt \\ \qquad\qquad + \Big\{\alpha_0 G(t,y_t,p_t,z_t,q_t,\mathcal{L}(y_t,p_t,z_t,q_t)) + \delta G(t,\overline{y}_t,\overline{p}_t,\overline{z}_t,\overline{q}_t,\mathcal{L}(\overline{y}_t,\overline{p}_t,\overline{z}_t,\overline{q}_t)) + G_0(t)\Big\}dW_t \\ \qquad\qquad - q_t d\overleftarrow{B}_t, \\ \quad y_T = \xi,\ p_0 = \alpha_0\Psi(y_0,\mathcal{L}(y_0)) + \delta\Psi(\overline{y}_0,\mathcal{L}(\overline{y}_0)) + \Psi_0, \end{cases} \tag{5.12}$$

and shall prove that the mapping $\mathbb{I}_{\alpha_0,\delta}(\overline{\zeta}) = (\zeta)$ defined by (5.12) is contractive for δ which is small but independent of α_0.

Applying the product rule (3.4) to $\langle \widehat{y}_t, \widehat{p}_t \rangle$, taking expectation and using (B1)-(B2), we can get the following estimation which is parallel to (4.6): there exists a constant C_1 only depending on c_1, c_2, c_3 such that

$$\mathbb{E}\int_0^T |C\widehat{p}_t + D\widehat{q}_t|^2 dt \le \delta C_1 \left\{ \mathbb{E}\int_0^T \{|\widehat{\overline{\zeta}}_t|^2 + |\widehat{\zeta}_t|^2\} dt + \mathbb{E}\big[|\widehat{\overline{y}}_0|^2 + |\widehat{y}_0|^2\big] \right\}.$$

Applying Itô's formula to $|\widehat{y}_t|^2$ and taking expectation, we can get the following estimates:

$$\mathbb{E}\big[|\widehat{y}_t|^2\big] \le C_2 \mathbb{E}\int_0^T |C\widehat{p}_t + D\widehat{q}_t|^2 dt + \delta C_2 \mathbb{E}\int_0^T |\widehat{\overline{\zeta}}_t|^2 dt,$$

$$\mathbb{E}\int_0^T \{|\widehat{y}_t|^2 + |\widehat{z}_t|^2\} dt \le C_3 \mathbb{E}\int_0^T |C\widehat{p}_t + D\widehat{q}_t|^2 dt + \delta C_3 \mathbb{E}\int_0^T |\widehat{\overline{\zeta}}_t|^2 dt.$$

Similarly, we can also get

$$\mathbb{E}\int_0^T \{|\widehat{p}_t|^2 + |\widehat{q}_t|^2\} dt \le C_4 \mathbb{E}\int_0^T \{|\widehat{y}_t|^2 + |\widehat{z}_t|^2\} dt + \delta C_4 \mathbb{E}\int_0^T |\widehat{\overline{\zeta}}_t|^2 dt + C_4 \mathbb{E}\big[|\widehat{y}_0|^2\big] + \delta C_4 \mathbb{E}\big[|\widehat{\overline{y}}_0|^2\big].$$

Combining the above estimates, we can find a constant L only dependent on $c_1, c_2, c_3, \lambda_1, \lambda_2$ and T, such that

$$\mathbb{E}\int_0^T |\widehat{\zeta}_t|^2 dt + \mathbb{E}\big[|\widehat{y}_0|^2\big] \le \delta L \left(\mathbb{E}\int_0^T |\widehat{\overline{\zeta}}_t|^2 dt + \mathbb{E}\big[|\widehat{\overline{y}}_0|^2\big] \right).$$

Hence, if we choose $\delta = \frac{1}{2L}$, $\mathbb{I}_{\alpha_0,\delta}$ is a contraction mapping and thus equation (5.11) admits a solution for $\alpha = \alpha_0 + \delta$. Noting that the choice of δ is independent of α_0, one can repeat this procedure and show that (5.11) has a solution for all $\alpha \in [0, 1]$. In particular, this implies the existence of solution to (5.9).

The proof is concluded. □

Now, we reconsider the stochastic linear quadratic problem that does not depend on the distribution of the control process, *i.e.*, $\overline{f}_3, \overline{g}_3, \overline{h}_3 = 0$. In such situation, the optimal control u given by (5.7) becomes

$$u_t = -\tfrac{1}{h_3}(f_3 p_t + g_3 q_t), \tag{5.13}$$

and the Hamiltonian system (5.8) now is

$$\begin{cases} -dy_t = \Big\{ f_1 y_t + f_2 z_t - \frac{f_3}{h_3}(f_3 p_t + g_3 q_t) + \overline{f}_1 \mathbb{E}[y_t] + \overline{f}_2 \mathbb{E}[z_t] \Big\} dt \\ \qquad\qquad + \Big\{ g_1 y_t + g_2 z_t - \frac{g_3}{h_3}(f_3 p_t + g_3 q_t) + \overline{g}_1 \mathbb{E}[y_t] + \overline{g}_2 \mathbb{E}[z_t] \Big\} d\overleftarrow{B}_t - z_t dW_t, \\ dp_t = \Big\{ f_1 p_t + \overline{f}_1 \mathbb{E}[p_t] + g_1 q_t + \overline{g}_1 \mathbb{E}[q_t] + h_1 y_t + \overline{h}_1 \mathbb{E}[y_t] \Big\} dt \\ \qquad\qquad + \Big\{ f_2 p_t + \overline{f}_2 \mathbb{E}[p_t] + g_2 q_t + \overline{g}_2 \mathbb{E}[q_t] + h_2 z_t + \overline{h}_2 \mathbb{E}[z_t] \Big\} dW_t - q_t d\overleftarrow{B}_t, \\ y_T = \xi, p_0 = \Phi y_0 + \overline{\Phi} \mathbb{E}[y_0]. \end{cases} \tag{5.14}$$

It can be easily checked that the coefficients in (5.14) satisfy (B1)-(B2) (we remark that the monotonicity condition in (A2) is not satisfied, though). By Theorem 5.1, there exists a unique solution to (5.14). Thus, equations (5.13) together with (5.14) provides a unique optimal control for the mean-field backward doubly stochastic LQ problem without involving the distribution of control.

Remark 5.2. When the mean-field FBDSDE (5.9) is reduced to classical FBDSDE (without mean field), Theorem 5.1 recovers the existence and uniqueness result obtained in [25], Theorem 3.8.

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