ESAIM: COCV 30 (2024) 15
https://doi.org/10.1051/cocv/2023089

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

NONZERO-SUM STOCHASTIC IMPULSE GAMES WITH AN APPLICATION IN COMPETITIVE RETAIL ENERGY MARKETS

René Aïd[1,*], Lamia Ben Ajmia[2],
M'hamed Gaïgi[2] and Mohamed Mnif[2]

Abstract. We study a nonzero-sum stochastic differential game with both players adopting impulse controls, on a finite time horizon. The objective of each player is to maximize her total expected discounted profits. The resolution methodology relies on the connection between Nash equilibrium and the corresponding system of quasi-variational inequalities (QVIs in short). We prove, by means of the weak dynamic programming principle for the stochastic differential game, that the equilibrium expected payoff of each player is a constrained viscosity solution to the associated QVIs system in the class of linear growth functions. We also introduce a family of equilibrium expected payoffs converging to our equilibrium expected payoff of each player, and which is characterized as the unique constrained viscosity solutions of an approximation of our QVIs system. This convergence result is useful for numerical purpose. We apply a probabilistic numerical scheme which approximates the solution of the QVIs system to the case of the competition between two electricity retailers. We show how our model reproduces the qualitative behavior of electricity retail competition.

Mathematics Subject Classification. 93E20, 49L20, 49L25, 49N70.

Received June 25, 2022. Accepted December 04, 2023.

1. Introduction

In this paper, we study a general two-players nonzero-sum stochastic differential game with impulse controls, on a finite horizon by characterizing the associated Nash Equilibrium. Stochastic differential games have been widely studied. The theory of two-player zero-sum differential games was pioneered by Isaacs [1]. Evans and Souganidis [2] studied differential games by means of the viscosity theory, characterizing the upper and the lower value functions as the unique viscosity solutions to the corresponding Hamilton- Jacobi-Bellman-Isaacs partial differential equations. In the literature, many authors focused on stochastic differential game with continuous controls. The case of impulse games has less been considered. Stochastic impulse games (SIGs) are at the intersection between differential game theory and stochastic impulse control. They are more realistic in modeling finance problems for example. Aid et al. [3] studied a problem of Nash equilibrium in a general nonzero-sum impulse game for two players. They derived the corresponding system of quasi-variational inequalities (QVIs) and provides an application in the case of a one-dimensional state problem of forex trading competition between

Keywords and phrases: Stochastic impulse games, Nash equilibrium, viscosity solution.

[1] LEDa, Université Paris Dauphine, PSL Research University, Paris, France.
[2] ENIT-LAMSIN, University of Tunis El Manar, Tunis, Tunisia.

[*] Corresponding author: rene.aid@dauphine.fr

two players. Thanks to a verification theorem, they identified reasonable candidates for the intervention and continuation regions of both players and their strategies. Cosso [4] studied a two-players zero-sum stochastic differential game with both players adopting impulse controls, on a finite time horizon. The HJBI partial differential equation of the game turns out to be a double-obstacle quasi-variational inequality. The two obstacles are implicitly given. He proved that the upper and lower value functions coincide. He also showed by means of the dynamic programming principle for the stochastic differential game, that the upper and the lower value functions are the unique viscosity solution to the HJBI equation, and so the game admits a value.
Our model is inspired by Aid et al. [3] together with the electricity retail pricing model of Basei [5]. In our model, we assume that retailers buy the energy (electricity or gas) in the wholesale market at the same price. The price of the energy is modeled by the Black-Scholes process. Finally, retailers sell the energy to their final consumers at a fixed price which could be different for each retailer. Both retailers' objective is to maximize their total expected discounted profits. Their instantaneous profits are composed of two parts: sale revenue (market share times retail price) and sourcing cost (market share times wholesale market price). The interaction between the opposing retailers is implicitly considered through the market share. We formulate the problem as a nonzero-sum stochastic impulse control game. The impulse control problem of each player is formulated with three state variables (besides time variable) related to the wholesale price, the price she offers and the proposed price of the other player which evolve in $\mathbb{R}^3_+$. The strategy of the retailers consists on changing the energy price in discrete time by using impulse control. Such model reproduces the fierce competition between the two players which characterizes modern deregulated energy markets.
To tackle our problem, we use the dynamic programming approach. We prove that the two players can act simultaneously and the order of intervention of the two players is not important since we can permute between the two impulse operators of players, which is not the case in the model considered by Aid et al. [3]. We prove a verification theorem in a general setting which shows that under some regularity assumptions, if we solve the system of QVIs, then we provide the Nash equilibrium of the impulse game which characterizes completely the equilibrium expected payoff of each player. Then we prove that the equilibrium expected payoffs of the players are viscosity solutions of the associated system of QVIs.

In our set-up, it is not obvious to prove the continuity of the equilibrium expected payoffs of the two players, and so it is then natural to consider the concept of discontinuous viscosity solutions, which provides by now a well established method for dealing with stochastic control problems. More precisely, we need to consider constrained viscosity solutions since the state process is allowed to be everywhere in $\mathbb{R}^3_+$ and so the system of QVIs is satisfied even in the boundary of the solvency region. For the comparison theorem, we consider a small variation of our original model by adding a fixed cost paid by player i when player j makes an intervention. We prove that the equilibrium expected payoffs of the perturbed problem converge pointwise to the equilibrium expected payoffs of the original problem. We provide a suitable strict super-solution of the system of QVIs solution in the spirit of Crandal et al. [6]. In the comparison theorem, we construct a test function by introducing a penalization function related to the distance from an element of the solvency region to the boundary, so that the optimum associated with the strict super-solution is not attained on the boundary. We adapt the arguments of Barles [7] which needs a smooth boundary. Similar difficulties have been studied in Akian et al. [8] and Ly Vath et al. [9]. It is known that the function distance is continuously differentiable with bounded derivatives in the smooth part of the boundary (see Gilbarg [10]). In our context, the boundary is smooth except the corner lines. We prove that one can compare a sub-solution with a super-solution to the system of QVIs provided that one can compare them at the terminal date but also on the corner lines of the solvency boundary. It is also known that the comparison theorem is crucial to approximate numerically the system of QVIs solution. In this work, we limit our numerical study to present the numerical scheme and some numerical results for the equilibrium intervention strategy. We claim that the equilibrium expected payoffs could be obtained as the limit of an iterative procedure where each step is an optimal stopping problem and the reward function is related to the impulse operator. We use a numerical approximation algorithm based on quantization procedure instead of finite difference methods to approximate the equilibrium expected payoffs, the intervention and the no-intervention regions of each player. The convergence of our numerical scheme, in particular, the monotonicity, the stability and the consistency properties will be postponed in a future research. In Aïd et al. [11], the authors tackled a nonzero-sum stochastic impulse games by implementing a policy iteration algorithm. To the best of our knowledge, it is the only paper which is interested to approximate numerically the solution of a system of QVIs, and it opens up the possibility of finding other numerical schemes to approximate the equilibrium expected payoffs and the associated equilibrium strategies.

The paper is organized as follows. Section 2 is devoted to present the model, to introduce rigorously the stochastic differential game, Nash equilibrium and the associated system of QVIs. Then, we study the properties of the equilibrium expected payoffs. In section 3 , we give some properties of the equilibrium expected payoff. In section 4, we prove a verification theorem which gives a characterization of the Nash equilibrium. In section 5, we prove that the two equilibrium expected payoffs are viscosity solutions to the system of QVIs followed in section 6 by a comparison theorem for an auxiliary family of equilibrium expected payoffs. Finally, in section 7, we explain the numerical scheme and we give some numerical results.

2. The model

This section presents the details of the model.

2.1. The Dynamics of the state process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ supporting a one dimensional Brownian motion W on a finite horizon $[0,T]$, $T < \infty$. We assume that the wholesale energy price is modeled by a Black-Scholes process. For a fixed $t \in [0,T]$ and $x \in \mathbb{R}_+$, the process $X^{t,x}$ satisfies

$$dX_s^{t,x} = X_s^{t,x}(\mu \mathrm{d}s + \sigma dW_s),\ s \in [t,T],\ X_s^{t,x} = x,\ s \in [0,t], \tag{2.1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.
We consider a nonzero-sum impulse game where two players, which are retailers, can intervene on a continuous time. After buying the energy, they sell it to consumers. The retailers price is modeled by a stochastic process $Y := (Y^1, Y^2) \in (0,\infty) \times (0,\infty)$ which is a piecewise-constant process. More precisely, the control strategy of each player, representing her intervention, is represented by an impulse control strategy $\alpha^i := (\tau_n^i, \zeta_n^i)_{n\geq 1}$ for $i \in \{1,2\}$, where for $t \in [0,T]$, $(\tau_n^i)_{n\geq 1}$ is an increasing sequence of $\mathbb{F}$ stopping times representing the intervention times of player i such that $\tau_n^i \in [t,T] \cup \{+\infty\}$ for all $n \geq 1$, and ζ_n^i, $n \geq 1$, are $\mathcal{F}_{\tau_n^i}$-measurable random variables. The sequence $(\tau_n^i, \zeta_n^i)_{n\geq 1}$ may be *a priori* finite or infinite. The process Y^i starts from $Y_{t^-}^i = y^i$ and evolves according to :

$$dY_s^i = 0, \qquad \tau_n^i \leq s < \tau_{n+1}^i \tag{2.2}$$

$$Y_{\tau_{n+1}^i}^i = \Gamma^i(Y_{\tau_{n+1}^{i-}}^i, \zeta_{n+1}^i), \tag{2.3}$$

where $\Gamma^i(y^i, \zeta^i) = y^i \mathrm{e}^{\lambda \zeta^i}$ and $\lambda > 0$. We further assume that, for $i \in \{1,2\}$, $\zeta^i \in [\zeta_{min}, \zeta_{max}]$ where $\zeta_{min} < 0 < \zeta_{max}$ are fixed constants. This assumption could be seen as a restriction imposed on the players by the regulator to avoid over-shifting prices. Moreover, with our specific choice of the function Γ^i, on the one hand we ensure the positivity of the new prices, on the other hand, the players could increase the price with a positive impulse (*i.e.* $\zeta > 0$) and decrease it with a negative impulse (*i.e.* $\zeta < 0$).

We then naturally introduce solvency region: $\mathcal{S} = (0,\infty)^3$, and we denote its closure and its boundary by

$$\bar{\mathcal{S}} = \mathbb{R}_+^3,\ \partial\mathcal{S} = \bar{\mathcal{S}} \setminus \mathcal{S}.$$

In the sequel, we explicit the corner lines of the boundary $\partial\mathcal{S}$ as follows:

$$\partial^x\mathcal{S} = \mathbb{R}^+ \times \{(0,0)\},\ \partial^{y_1}\mathcal{S} = \{0\} \times (0,\infty) \times \{0\},\ \partial^{y_2}\mathcal{S} = \{(0,0)\} \times (0,\infty).$$

We denote by

$$D_0 = \partial^x \mathcal{S} \cup \partial^{y_1} \mathcal{S} \cup \partial^{y_2} \mathcal{S}.$$

Let $Z = (X, Y^1, Y^2)$ the process living in the space $\mathcal{S}$. We denote by $Z^{t,z,\alpha_1,\alpha_2}$ the process Z that starts at $z = (x, y^1, y^2)$ at time t and is controlled by α^1 and α^2.

For the rest of the paper, to simplify the notations and when there is no ambiguity, we will use $Z^{t,z}$ instead of $Z^{t,z,\alpha^1,\alpha^2}$. Hence when we use $Z^{t,z}$, one has to check under which control the process Z evolves. In most cases it can be easily guessed.

2.2. Equilibrium expected payoff

The objective of the player is to maximize the gain function J^i defined as follows:

$$J^i(t,z,\alpha^i,\alpha^j) = \mathbb{E}[\int_t^T \mathrm{e}^{-\rho^i(s-t)} f^i(Z_s^{t,z,\alpha^i,\alpha^j})\mathrm{d}s - \sum_{t\le\tau_k^i<T} \mathrm{e}^{-\rho^i(\tau_k^i-t)}\phi^i(Y^{i,t,y^i,\alpha^i,\alpha^j}_{(\tau_k^i)^-},\zeta_k^i) + \mathrm{e}^{-\rho^i(T-t)} g^i(Z_T^{t,z,\alpha^i,\alpha^j})], \tag{2.4}$$

where, for $i \in \{1, 2\}$, $j \neq i$, ρ^i is a discount factor; $f^i : \mathbb{R}^3_+ \longrightarrow \mathbb{R}$ is the running payoff, $g^i : \mathbb{R}^3_+ \longrightarrow \mathbb{R}$ is the terminal payoff and $\phi^i : \mathbb{R}_+ \times [\zeta_{min}, \zeta_{max}] \longrightarrow \mathbb{R}$ is the intervention cost function. In order to have a realistic model, the running payoff and the terminal payoff of player i depend on the price she asks, the opponent pricing choice and the wholesale price proposed in the market. In our model, the retailer margin of player i, represented by $y^i - x$, is truncated by a linear function of the retail price. This regulatory constraint forces retailers to drop down their prices when the wholesale price falls. In our model f^i and g^i take the form:

$$f^i(x, y^i, y^j) = g^i(x, y^i, y^j) = ((y^i - x)\pi^i(y^i, y^j)) \wedge K^i x, \tag{2.5}$$

here the function π^i represents the market share and is defined by:

$$\pi^i(y^i, y^j) = \begin{cases} 1 & \text{if } y^i - y^j \le -\Delta, \\ -\frac{1}{2\Delta}(y^i - y^j - \Delta) & \text{if } -\Delta < y^i - y^j < \Delta, \\ 0 & \text{if } y^i - y^j \ge \Delta, \end{cases} \tag{2.6}$$

K^i and Δ are positive constants. For $y^i = y^j$, the market is split equally between the two retailers. The following assumption will be needed in our work:

Assumption 2.1. (i) There exist positive constants C_1^ϕ and C_2^ϕ s.t.

$$C_1^\phi \le \phi^i(y^i, \zeta) \le C_2^\phi \ \forall (y^i, \zeta) \in (0, +\infty) \times [\zeta_{min}, \zeta_{max}].$$

(ii)The function ϕ^i is continuous.

Remark 2.2. Thanks to the choice of f^i and g^i, we deduce that for all $z \in \mathcal{S}$, $f^i(z) \le K^i x$ and $g^i(z) \le K^i x$. Moreover $f^i(z) \ge -(C^i + x)$ and $g^i(z) \ge -(C^i + x)$, where C^i is a positive constant.

We define the number of intervention times between t and T and the null strategy as follows:

Definition 2.3. The number of intervention times between t and T is defined by:

$$\mathcal{N}_t(\alpha^i) := Card\{n \ : t \le \tau_n^i < T\}. \tag{2.7}$$

We allow player i to choose $\tau_n^i = \infty$ for all n and this defines the null strategy. Same for player j.

We need to impose a uniform integrability condition on $\mathcal{N}_t(\alpha^i)$.

Definition 2.4. For $i \in \{1,2\}$, we define $\mathcal{D}_t^i$ as follows

$$\mathcal{D}_t^i := \{\alpha^i = (\tau_n^i, \zeta_n^i)_{n\geq 1}, \text{s.t.} \, \mathcal{N}_t(\alpha^i) \text{ is uniformly integrable}\}. \tag{2.8}$$

For technical reason related to the dynamic programming principle, see Remark 5.2 in Bouchard and Touzi [12], we shall restrict the admissible strategies to the following set. For $i \in \{1,2\}$

$$\mathcal{A}_t^i := \{\alpha^i = (\tau_n^i, \zeta_n^i)_{n\geq 1};\ s.t.\ \tau_1^i \geq t,\ \alpha^i \in \mathcal{D}_t^i \text{ and } \alpha^i \text{ is independent of } \mathcal{F}_t\}, \tag{2.9}$$

and therefore, we define the equilibrium expected payoffs as $(v^i)_{i\in\{1,2\}}$ on $[0,T]\times\mathcal{S}$ by

$$v^i(t,z) := \sup_{\alpha^i\in\mathcal{A}_t^i, \alpha^j\in BR_t^j(\alpha^i)} J^i(t,z,\alpha^i,\alpha^j) \quad i \neq j, \tag{2.10}$$

where $BR^j(\alpha^i)$ is the set of strategies which produces the most favorable outcome for player j, taking the strategy of player i as given *i.e.*

$$BR_t^j(\alpha^i) := \{\alpha \in \mathcal{A}_t^j \ s.t.\ J^j(t,z,\alpha^i,\alpha) \geq J^j(t,z,\alpha^i,\alpha^j),\ \forall \alpha^j \in \mathcal{A}_t^j\}$$

In the following, we give the definition of Nash equilibrium of our game.

Definition 2.5. We say that the couple $(\alpha^{i,*},\alpha^{j,*}) \in \mathcal{A}_t^i \times \mathcal{A}_t^j$ is a Nash equilibrium, if we have:

$$J^i(t,z,\alpha^{i,*},\alpha^{j,*}) \geq J^i(t,z,\alpha^i,\alpha^{j,*}) \text{ for all } \alpha^i \in \mathcal{A}_t^i$$
$$J^j(t,z,\alpha^{i,*},\alpha^{j,*}) \geq J^j(t,z,\alpha^{i,*},\alpha^j) \text{ for all } \alpha^j \in \mathcal{A}_t^j.$$

Remark 2.6. Suppose that there exists α^{i*} and α^{j*} such that $v^i(t,z) = J^i(t,z,\alpha^{i*},\alpha^{j*})$ and $v^j(t,z) = J^j(t,z,\alpha^{i*},\alpha^{j*})$. Then $\alpha^{i*} \in BR^i(\alpha^{j*})$ and $\alpha^{j*} \in BR^j(\alpha^{i*})$ which implies that for all $(\alpha^i,\alpha^j) \in \mathcal{A}_t^i \times \mathcal{A}_t^j$, we have

$$J^i(t,z,\alpha^{i,*},\alpha^{j,*}) \geq J^i(t,z,\alpha^i,\alpha^{j,*}) \quad \text{and} \quad J^j(t,z,\alpha^{i,*},\alpha^{j,*}) \geq J^j(t,z,\alpha^{i,*},\alpha^j).$$

The corresponding expected payoffs v^i, v^j are equilibrium payoffs and they are not unique, in the sense that there could be other Nash equilibria whose expected payoffs are different than those in (2.10). It yields that the existence of strategies attaining the suprema in (2.10) is something much stronger than the existence of a Nash equilibrium.

Remark 2.7. It is easy to see that if $\alpha^{i*} = (\tau_n^{i*}, \zeta_n^{i*})$ exists, then, we have $\mathbb{E}[\mathcal{N}_t(\alpha^{i*})] < \infty$. In our case, to obtain the growth property of the equilibrium expected payoff, we need a uniform integrability condition of the number of intervention times which is stronger integrability condition. In fact, for $i \in \{1,2\}$, α^i an admissible strategy and α^j the best response against α^i, from Assumption 2.1(i), we have:

$$\begin{aligned} J^i(t,z,\alpha^i,\alpha^j) &\leq \mathbb{E}[\int_t^T f^i(Z_s^{t,z})\mathrm{d}s - C_1^\phi \mathrm{e}^{-\rho^i T}\mathcal{N}_t(\alpha^i) + g^i(Z_T^{t,z})] \\ &\leq \mathbb{E}[\int_t^T K^i X_s^{t,x}\mathrm{d}s - C_1^\phi \mathrm{e}^{-\rho^i T}\mathcal{N}_t(\alpha^i) + K^i X_T^{t,x}]. \end{aligned} \tag{2.11}$$

where the second inequality is obtained from Remark 2.2. If $\mathbb{E}[\mathcal{N}_t(\alpha^i)] = +\infty$, then from (2.11), we deduce that:

$$J^i(t, z, \alpha^i, \alpha^j) = -\infty$$

On the other hand, by choosing the null strategy by player i, $J^i(t, z, 0, \alpha^j) > -\infty$, where here α^j denotes the best response against the null strategy. Hence, an equilibrium strategy can not have infinite expected number of interventions.

2.3. HJB QVIs

In this subsection, we provide an heuristic derivation of the system of QVIs. We define the non-local operator called intervention operator $\mathcal{M}^i$ for $i \in \{1, 2\}$ by:

$$\mathcal{M}^i h^i(t, z) = \sup_{\zeta^i \in [\zeta_{min}, \zeta_{max}]} \Big\{ h^i(t, x, \Gamma^i(y^i, \zeta^i), y^j) - \phi^i(y^i, \zeta^i) \Big\},$$

for a given couple of locally bounded functions (h^i, h^j) s.t. $h^i : [0, T] \times S \longrightarrow \mathbb{R}$, $\forall i \neq j \in \{1, 2\}$. It is known that if the function h^i is upper semicontinuous, by selection measurable theorem, there exists a Borel-measurable function $\hat{\zeta}^{i*} : [0, T] \times S \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}^i h^i(t, z) = h(t, x, \Gamma^i(y^i, \hat{\zeta}^{i*}(t, y^i)), y^j) - \phi^i(y^i, \hat{\zeta}^{i*}(t, y^i)),$$

(see for example Prop. 7.33, chapter 7, in [13]). Similarly, we define the second intervention operator $\mathcal{H}^i$ for $i \in \{1, 2\}$ by

$$\mathcal{H}^i h^i(t, z) = h^i(t, x, y^i, \Gamma^j(y^j, \zeta^j)),$$

where $\zeta^j = \underset{\zeta \in [\zeta_{min}, \zeta_{max}]}{\operatorname{argmax}} \Big\{ h^j(t, x, y^i, \Gamma^j(y^j, \zeta)) - \phi^j(y^j, \zeta) \Big\}$.

Thanks to the following Proposition, the two players could act together. In fact, we can permute between $\mathcal{M}^i$ and $\mathcal{H}^i$ for $i \in \{1, 2\}$. In Cosso [4] and Aid et al. [3], the authors assume that in the case when a simultaneous intervention of both players might occur, player 1 is prior to player 2. This is due to the fact that when one of the players makes an intervention, the other one's criteria is affected by a quantity ψ. A direct consequence is that they have in their model $\mathcal{M}^i\mathcal{H}^i v^i \neq \mathcal{H}^i\mathcal{M}^i v^i$. This is not the case in our model as we take $\psi = 0$. This allows us to switch between the intervention operators $\mathcal{M}^i$ and $\mathcal{H}^i$ (see the following Proposition), hence, both players can intervene at the same time.

Proposition 2.8. *For all* $(t, z) \in [0, T] \times \mathcal{S}$*, we have*

$$\mathcal{M}^i\mathcal{H}^i v^i(t, z) = \mathcal{H}^i\mathcal{M}^i v^i(t, z), \forall i, j \in \{1, 2\},$$

Proof. From the definition of $\mathcal{M}^i$ and $\mathcal{H}^i$ we have

$$\begin{aligned}
\mathcal{M}^i\mathcal{H}^i v^i(t, z) &= \sup_{\zeta^i \in [\zeta_{min}, \zeta_{max}]} \Big\{ \mathcal{H}^i v^i(t, x, \Gamma^i(y^i, \zeta^i), y^j) - \phi^i(y^i, \zeta^i) \Big\}, \\
&= \sup_{\zeta^i \in [\zeta_{min}, \zeta_{max}]} \Big\{ v^i(t, x, \Gamma^i(y^i, \zeta^i), \Gamma^j(y^j, \zeta^j)) - \phi^i(y^i, \zeta^i) \Big\}, \\
&= \mathcal{M}^i v^i(t, x, y^i, \Gamma^j(y^j, \zeta^j)) \\
&= \mathcal{H}^i\mathcal{M}^i v^i(t, x, y^i, y^j) \\
&= \mathcal{H}^i\mathcal{M}^i v^i(t, z)
\end{aligned}$$

□

We define $\overline{\mathcal{I}}^i$ as the intervention region of player j and $\mathcal{I}^i = [0,T) \times \bar{\mathcal{S}} \setminus \overline{\mathcal{I}}^i$. The associated QVIs of our control problem are as follows:

$$\min\{v^i - \mathcal{M}^i\mathcal{H}^i v^i, v^i - \mathcal{H}^i v^i\} = 0 \quad \text{in} \quad \overline{\mathcal{I}}^i \tag{2.12}$$

$$\min\{-\frac{\partial v^i}{\partial t} - \mathcal{L}v^i - f^i + \rho^i v^i, v^i - \mathcal{M}^i v^i\} = 0 \quad \text{in} \quad \mathcal{I}^i \tag{2.13}$$

$$v^i = g^i \quad \text{in} \quad \{T\} \times \bar{\mathcal{S}}, \tag{2.14}$$

where

$$\mathcal{L}v^i(t,z) = \mu x \frac{\partial v^i}{\partial x}(t,z) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 v^i}{\partial x^2}(t,z), \tag{2.15}$$

$$\mathcal{M}^i v^i(t,z) = \sup_{\zeta^i \in [\zeta_{min}, \zeta_{max}]} \Big\{ v^i(t,x,\Gamma^i(y^i,\zeta^i), y^j) - \phi^i(y^i,\zeta^i) \Big\}, \tag{2.16}$$

$$\mathcal{H}^i v^i(t,z) = v^i(t,x,y^i,\Gamma^j(y^j,\zeta^j)), \tag{2.17}$$

and $\zeta^j = \underset{\zeta \in [\zeta_{min}, \zeta_{max}]}{\operatorname{argmax}} \Big\{ v^j(t,y^i,\Gamma^j(y^j,\zeta)) - \phi^j(y^j,\zeta) \Big\}$.

Remark 2.9. $\mathcal{M}^i v^i(t,z)$ represents the equilibrium expected payoff for player i when she makes the optimal impulse in order to increase her equilibrium expected payoff. $\mathcal{H}^i v^i(t,z)$ represents the equilibrium expected payoff for player i, when player j takes the best intervention.

Remark 2.10. The system of variational inequalities (2.12)-(2.13) is satisfied up to the boundary of $\bar{\mathcal{S}}$. Since the dynamics of the process Z is allowed to go everywhere in $\bar{\mathcal{S}}$. We recall that the wholesale price X is modeled by a geometric Brownian motion and the process (Y^1, Y^2) are non-negative piecewise constant and at the intervention times, the price offered by each player is modeled by the exponential function.

3. Properties of the equilibrium expected payoff

3.1. Bound on equilibrium expected payoffs and Boundary properties

We start this section with the following lemma which gives a sharp upper bound on the equilibrium expected payoff v^i for $i \in \{1,2\}$.

Proposition 3.1. *We fix $i \in \{1,2\}$. Under Assumption 2.1, for all $(t,z) \in [0,T] \times \bar{\mathcal{S}}$, we have*

$$|v^i(t,z)| \leq C(1+x), \tag{3.1}$$

where C is a positive constant independent of t and z.

Proof. We consider the no impulse strategy for player i denoted by 0 and α^j the best response against the null strategy, from the definition of f^i and g^i, Remark 2.2 and the positivity of the process Y^i, we obtain

$$\begin{aligned} v^i(t,z) &\geq J^i(t,z,0,\alpha^j) \\ &= \mathbb{E}[\int_t^T \mathrm{e}^{-\rho^i(s-t)} f^i(Z_s^{t,z,0,\alpha^j})\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t)} g^i(Z_T^{t,z,0,\alpha^j})] \\ &\geq -\mathbb{E}[\int_t^T \mathrm{e}^{-\rho^i(s-t)} X_s^{t,x}\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t)} X_T^{t,x}] \\ &\geq -C(1+x), \end{aligned} \tag{3.2}$$

where the last inequality is obtained from (2.1), and C is a generic positive constant which could change from line to line.

Let (α^i, α^j) be an admissible strategy. Using Assumption 2.1(i), we have

$$\begin{aligned} & J^i(t,z,\alpha^i,\alpha^j) \\ \leq\ & \mathbb{E}[\int_t^T \mathrm{e}^{-\rho^i(s-t)} f^i(Z_s^{t,z})\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t)} g^i(Z_T^{t,z})] \\ \leq\ & \mathbb{E}[\int_t^T K^i X_s^{t,x} ds + K^i X_T^{t,x}] \\ \leq\ & \mathbb{E}[\int_t^T K^i x \mathrm{e}^{\mu(s-t)}\mathrm{d}s + K^i x \mathrm{e}^{\mu(T-t)}]. \end{aligned}$$

Since (α^i, α^j) is arbitrary, we obtain

$$v^i(t,z) \leq Cx. \tag{3.3}$$

From inequalities (3.2) and (3.3), we deduce that (3.1) is proved. □

We turn to the behavior of the values functions on the corner lines of the boundary $\partial\mathcal{S}$. The following proposition determines the equilibrium expected payoff on the corner lines of the boundary $\partial\mathcal{S}$.

Proposition 3.2. *We fix $i \in \{1,2\}$. For all $t \in [0,T)$, We have*

$$v^i(t,z) = 0 \quad \text{for all } z \in \partial^{y^j}\mathcal{S} \tag{3.4}$$

$$v^i(t,z) = 0 \quad \text{for all } z \in \partial^{y^i}\mathcal{S} \tag{3.5}$$

$$v^i(t,z) = -\frac{x}{2}\Big(\frac{\mathrm{e}^{(\mu-\rho^i)(T-t)} - 1}{\mu-\rho^i} + \mathrm{e}^{(\mu-\rho^i)(T-t)}\Big) \quad \text{for all } z \in \partial^x\mathcal{S} \tag{3.6}$$

Proof. ⋆ From the definition of (2.5), we have $f^j(0,0,y^j) = g^j(0,0,y^j) = 0$ for all $y^j > 0$. If player j intervenes, her payoff remains unchanged. On the other hand, as $\Gamma^i(0,\zeta^i) = 0$, any intervention of player i will not change the proposed retailer price, and so it not optimal for the two players to intervene. For the price in wholesale market, it remains equal to zero on $[t,T)$. Then, we deduce (3.4).

⋆ Since $f^i(0,y^i,0) = g^i(0,y^i,0) = 0$ for all $y^i > 0$, the payoff of player i remains equal to 0. As above and since $y^j = 0$, any intervention of player j will not change her proposed retailer price. This shows that (3.5) holds.

⋆ Since $\pi^i(0,0) = \frac{1}{2}$, and for any intervention from one of the two players, their retailer prices remain equal to 0, the market share of each player remains unchanged. From the definition of the equilibrium expected payoff v^i, we deduce that

$$v^i(t,x,0,0) = -\mathbb{E}[\int_t^T \mathrm{e}^{-\rho^i(s-t)}\frac{X_s^{t,x}}{2}\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t)}\frac{X_T^{t,x}}{2}].$$

From the definition of the wholesale price, we have $\mathbb{E}[X_s^{t,x}] = x\mathrm{e}^{\mu(s-t)}$. This shows that

$$v^i(t,x,0,0) = -\frac{x}{2}\Big(\int_t^T \mathrm{e}^{(\mu-\rho^i)(s-t)}\mathrm{d}s + \mathrm{e}^{(\mu-\rho^i)(T-t)}\Big),$$

and so (3.6) is proved. □

3.2. Terminal condition

We determine the right terminal condition of the equilibrium expected payoffs. We recall that in our model, at the terminal date, each player can not do an intervention. We obtain such characterization by considering the lower (resp. upper) semicontinuous envelope of v^i, for $i \in \{1,2\}$. For all $z \in \mathcal{S}$, we set

$$\bar{v}^i(T,z) := \limsup_{\substack{(t,z') \to (T,z) \\ t<T,\, z' \in \mathcal{S}}} v(t,z'), \ \ \underline{v}^i(T,z) := \liminf_{\substack{(t,z') \to (T,z) \\ t<T,\, z' \in \mathcal{S}}} v(t,z')$$

Proposition 3.3. *We fix $i \in \{1,2\}$. We have:*

$$\underline{v}^i(T,z) \ = \ \bar{v}^i(T,z) \ = \ g^i(z), \quad \forall z \in \bar{\mathcal{S}}.$$

Proof. 1) Fix some $z \in \bar{\mathcal{S}}$ and consider some sequence $(t_m, z_m)_m \in [0,T) \times \mathcal{S}$ converging to (T,z) and s.t. $\lim_{m\to\infty} v^i(t_m,z_m) = \underline{v}^i(T,z)$. By taking the no impulse control strategy for player i noted by 0, and α^j the best response against this strategy by player j on $[t_m,T]$, we have

$$v^i(t_m,z_m) \geq \mathbb{E}[\int_{t_m}^T \mathrm{e}^{-\rho^i(s-t_m)} f^i(Z_s^{t_m,z_m,0,\alpha^j})\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t_m)} g^i(Z_T^{t_m,z_m,0,\alpha^j})]. \tag{3.7}$$

where

$$Z^{t_m,z_m,0,\alpha^j} = (X^{t_m,x_m}, y_m^i, Y^{j,t_m,y_m^j,0,\alpha^j}).$$

We fix $\epsilon > 0$. As $(t_m,z_m)_m$ converges to (T,z) when m goes to infinity, there exists m_0, such that for $m \geq m_0$, we have $|z_m - z| \leq \epsilon$. We fix $m \geq m_0$. From (2.1), we have for $s \leq t_m$, $X_s^{t_m,x_m} = x$ and for $s \geq t_m$, $X_s^{t_m,x_m} = x_m \mathrm{e}^{(\mu - \frac{\sigma^2}{2})(s-t_m)+\sigma(W_s - W_{t_m})}$, and so $\lim_{m\to\infty} X_s^{t_m,x_m} = x$ $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ a.e. For the jump term $(Y_s^{j,t_m,y_m^j,\alpha^j})_s$, we have $Y_s^{j,t_m,y_m^j,\alpha^j} = y_j^m$ for $s \leq t_m$ and for $s \geq t_m$, we have

$$Y_s^{j,t_m,y_m^j} = y_j^m \prod_{n=0}^{\mathcal{N}_{t_m}^j(\hat{\alpha})} \mathrm{e}^{\lambda \zeta_n^j},$$

where, we recall that $\mathcal{N}_{t_m}^j(\alpha^j)$ is the number of interventions of player j between t_m and T. As it is not allowed to make an intervention at the terminal date, then $\lim_{m\to\infty} \mathcal{N}_{t_m}^j(\alpha^j) = 0$ a.s. which shows that $\lim_{m\to\infty} Y_s^{j,t_m,y_m^j,\alpha^j} = y^j$ $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ a.e. We fix $p > 1$, from Remark 2.2, for $h^i = f^i, g^i$, we have $|h^i(z)| \leq C^i(1+x)$, which implies

$$\begin{aligned} &\mathbb{E}[\left(\int_{t_m}^T \mathrm{e}^{-\rho^i(s-t_m)} f^i(Z_s^{t_m,z_m,0,\alpha^j})\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t_m)} g^i(Z_T^{t_m,z_m,0,\alpha^j})\right)^p] \\ &\leq C\left(\mathbb{E}[\left(\int_{t_m}^T f^i(Z_s^{t_m,z_m,0,\alpha^j})\mathrm{d}s\right)^p] + \mathbb{E}[g^i(Z_T^{t_m,z_m,0,\alpha^j})^p]\right) \\ &\leq C\mathbb{E}[\int_{t_m}^T f^i(Z_s^{t_m,z_m,0,\alpha^j})^p ds](T-t_m)^{p-1} + \mathbb{E}[g^i(Z_T^{t_m,z_m,0,\alpha^j})^p]) \\ &\leq C(1+T^p)(1+\mathbb{E}[\sup_{s\in[t_m,T]} |X_s^{t_m,x_m}|^p]), \end{aligned} \tag{3.8}$$

where the second inequality is obtained by using Hölder inequality and C is a generic constant independent of m. From the definition of the wholesale price and for $m \geq m_0$, we have

$$\mathbb{E}[\sup_{s\in[t_m,T]} |X_s^{t_m,x_m}|^p] \leq C|x_m|^p \leq C(1+|x|^p). \tag{3.9}$$

From (3.8)-(3.9), we deduce

$$\mathbb{E}[\left(\int_{t_m}^T \mathrm{e}^{-\rho^i(s-t_m)} f^i(Z_s^{t_m,z_m,0,\alpha^j})\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t_m)} g^i(Z_T^{t_m,z_m,0,\alpha^j})\right)^p] \leq C(1+|x|^p),$$

where C is a positive constant independent of m. This shows the boundedness of $\int_{t_m}^T \mathrm{e}^{-\rho^i(s-t_m)} f^i(Z_s^{t_m,z_m,0,\alpha^j})\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t_m)} g^i(Z_T^{t_m,z_m,0,\alpha^j})$ in $L^p(\mathbb{P})$ for $p > 1$, which implies the uniform integrability of $\left(\int_{t_m}^T \mathrm{e}^{-\rho^i(s-t_m)} f^i(Z_s^{t_m,z_m,0,\alpha^j})\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t_m)} g^i(Z_T^{t_m,z_m,0,\alpha^j})\right)_m$. It yields that

$$\underline{v}^i(T,z) = \lim_{m\longrightarrow\infty} v^i(t_m,z_m) \geq g^i(z). \tag{3.10}$$

2) Fix some $z \in \bar{\mathcal{S}}$ and consider some sequence $(t_m,z_m)_m \in [0,T)\times\mathcal{S}$ converging to (T,z) and s.t. $\lim_{m\to\infty} v^i(t_m,z_m) = \bar{v}^i(T,z)$. For any m, one can find $\hat{\alpha}^{m,i} = (\hat{\tau}_k^{m,i},\hat{\zeta}_k^{m,i})_k \in \mathcal{A}_{t_m}^i$ and $\hat{\alpha}^{m,j} = (\hat{\tau}_k^{m,j},\hat{\zeta}_k^{m,j})_k \in \mathcal{A}_{t_m}^j$ the best response against the strategy $\hat{\alpha}^{m,i}$ s.t.

$$v(t_m,z_m) \leq \mathbb{E}\left[\int_{t_m}^T f^i(\hat{Z}_s^m)\mathrm{d}s - \sum_{t_m\leq\tau_k^i<T} \mathrm{e}^{-\rho^i(\tau_k^i-t_m)}\phi^i(\hat{Y}^{i,m}_{(\tau_k^{m,i})^-},\zeta_k^{i,m}) + g^i(\hat{Z}_T^m)\right] + \frac{1}{m},$$

where $\hat{Z}^m = (X^{t_m,x_m},\hat{Y}^{i,m},\hat{Y}^{j,m})$ denotes the state process controlled by $\hat{\alpha}^{m,i}$ and $\hat{\alpha}^{m,j}$. Arguing as above, we have $\lim_{m\to\infty}\mathcal{N}_{t_m}^i(\hat{\alpha}^{m,i}) = 0$ a.s. which shows that $\lim_{m\to\infty} Y_s^{i,t_m,y^{i,m},\hat{\alpha}^{m,i}} = y^i$ $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$ a.e. For $p>1$ and $m \geq m_0$, we have :

$$\begin{aligned}\mathbb{E}[\left(\int_{t_m}^T f^i(\hat{Z}_s^m)\mathrm{d}s + g^i(\hat{Z}_T^m)\right)^p] &\leq C(1+(T-t_m)^p)(1+\mathbb{E}[\sup_{s\in[t_m,T]}|X_s^{t_m,x_m}|^p]),\\ &\leq C(1+|x|^{2p}),\end{aligned}$$

where C is a positive constant independent of m. This shows the boundedness of $\int_{t_m}^T f^i(\hat{Z}_s^m)\mathrm{d}s + g^i(\hat{Z}_T^m)$ in $L^p(\mathbb{P})$ for $p>1$, which implies the uniform integrability of $\left(\int_{t_m}^T f^i(\hat{Z}_s^m)\mathrm{d}s + g^i(\hat{Z}_T^m)\right)_m$. It yields that

$$\underline{v}^i(T,z) = \lim_{m\longrightarrow\infty} v^i(t_m,z_m) \leq g^i(z). \tag{3.11}$$

The result follows from (3.10) and (3.11). □

4. Verification theorem

We provide in this section a verification theorem for the problem formalized previously. It says that if a smooth function v^i is solution to the system of QVIs, then it is associated to a Nash equilibrium. We construct the equilibrium strategy of each player in an inductive way, which is useful later to determine numerically the optimal pricing policy of the retailers.

Proposition 4.1. *Let v^i be a function from $[0,T]\times\bar{\mathcal{S}}$ to $\mathbb{R}$, with $i\in\{1,2\}$. We suppose that Assumption 2.1 holds and, we have:*

- (v^1,v^2) *is a solution to* (2.12) − (2.14);
- $(v^1,v^2)\in C^{1,2}([0,T)\times\bar{\mathcal{S}})\times C^{1,2}([0,T)\times\bar{\mathcal{S}})$ *satisfying the growth condition* (3.1).

Let $(t,z)\in[0,T]\times\bar{\mathcal{S}}$, $(i,j)\in\{1,2\}\times\{1,2\}$, $i\neq j$, and define the continuation region:

$$D^i := \{(s,\vartheta)\in[t,T]\times\mathcal{S};\ v^i(s,\vartheta)>\mathcal{M}^i v^i(s,\vartheta) \text{ or } v^i(s,\vartheta)>\mathcal{M}^i\mathcal{H}^i v^i(s,\vartheta)\}.$$

Define the impulse control $\alpha^{i*}=(\tau_n^{i*},\zeta_n^{i*})_{n\geq 1}$, as follows: $\tau_0^{i*}=t^-$ and inductively

$$\tau_{n+1}^{i*} := \inf\{s>\tau_n^{i*}, (s, Z_s^{t,z,\alpha^{i*},\alpha^{j*}})\notin D^i\} \tag{4.1}$$

$$\zeta_{n+1}^{i*} := \hat{\zeta}^{i*}(\tau_{n+1}^{i*}, Z^{t,z,\alpha^{i*},\alpha^{j*}}_{\tau_{n+1}^{i*-}}) \tag{4.2}$$

Assume that $(\alpha^{i,*},\alpha^{j,*})\in\mathcal{A}_t^i\times\mathcal{A}_t^j$ (i.e. an admissible couple of strategies).
Then, $(\alpha^{i,*},\alpha^{j,*})$ is a Nash equilibrium and $v^i(t,z)=J^i(t,z,\alpha^{i,*},\alpha^{j,*})$ for $i\in\{1,2\}$, $j\neq i$.

Proof. We fix $(t,z)\in[0,T]\times\bar{\mathcal{S}}$. We have to prove that

$$v^i(t,z)=J^i(t,z;\alpha^{i*},\alpha^{j*}) \text{ and } v^i(t,z)\geq J^i(t,z;\alpha^i,\alpha^j), \text{ for } (i,j)\in\{1,2\}\times\{1,2\},\ j\neq i,$$

for every $\alpha^i\in\mathcal{A}_t^i$ and $\alpha^j\in\mathcal{A}_t^j$ the best response against α^i.
Step 1: We prove $v^i(t,z)\geq J^i(t,z;\alpha^i,\alpha^j)$.
We will use the following shortened notation:

$$Z=Z^{t,z;\alpha^i,\alpha^j}, \qquad \alpha^i=(\tau_n^i,\zeta_n^i)_{n\geq 1}.$$

For each $r>0$ and $n\in\mathbb{N}$, we set $\tau_r=\inf\{s>t: X_s\in B(x;r)\}$ is the exit time from the ball with radius r and center x.
We define IT^c (resp. IT^d) the set of intervention times of the two players which coincide (resp. differ) as follows:

$$IT^c := \{\tau_n^i,\ n\geq 1 \text{ such that there exists } m\geq 1 \text{ satisfying } \tau_m^i=\tau_m^j\}$$

$$IT^d := \{\tau_n^i,\ n\geq 1 \text{ s.t. } \forall m\geq 1,\ \tau_m^i\neq\tau_m^j\}\cup\{\tau_n^j,\ n\geq 1 \text{ s.t. } \forall m\geq 1,\ \tau_m^j\neq\tau_m^i\}.$$

We apply Itô's formula to the process $\mathrm{e}^{-\rho^i s}v^i(s,Z_s)$ over the interval $[t,\tau_r\wedge T]$. We take the conditional expectations, the stochastic integral vanishes. Then by taking the expectation, we get:

$$
\begin{aligned}
&\mathrm{e}^{-\rho^i t} v^i(t,z) &(4.3)\\
&= \mathbb{E}\Big[\int_t^{\tau_r\wedge T} \mathrm{e}^{-\rho^i u}\left(-\frac{\partial v^i}{\partial t}(u,Z_u) - \mathcal{L}v^i(u,Z_u) + \rho^i v^i(u,Z_u)\right) du\\
&- \sum_{t\le \tau_n^i<\tau_r\wedge T} \mathrm{e}^{-\rho^i \tau_n^i}(v^i(\tau_n^i, Z_{\tau_n^i}) - v^i(\tau_n^i, Z_{\tau_n^i-}))1_{IT^d}(\tau_n^i)\\
&- \sum_{t\le \tau_n^j<\tau_r\wedge T} \mathrm{e}^{-\rho^i \tau_n^j}(v^i(\tau_n^j, Z_{\tau_n^j}) - v^i(\tau_n^j, Z_{\tau_n^j-}))1_{IT^d}(\tau_n^j)\\
&- \sum_{t\le \tau_n^i<\tau_r\wedge T} \mathrm{e}^{-\rho^i \tau_n^i}(v^i(\tau_n^i, Z_{\tau_n^i}) - v^i(\tau_n^i, Z_{\tau_n^i-}))1_{IT^c}(\tau_n^i) + \mathrm{e}^{-\rho^i(\tau_r\wedge T)} v^i(\tau_r\wedge T, Z_{\tau_r\wedge T})\Big].
\end{aligned}
$$

We now estimate each term in the right-hand side of (4.3). As for the first term, two cases are possible: $(s,Z_s)\in\mathcal{I}^i$ *i.e.* the QVI (2.13) holds or $(s,Z_s)\in\bar{\mathcal{I}}^i$ *i.e.* the second player makes impulses which could be simultaneously with the first player. As $\alpha^i\in\mathcal{D}_t^i$, we are interested only in the strategies whose number of interventions is finite almost surely, it follows that:

$$
-\frac{\partial v^i}{\partial t}(s,Z_s) - \mathcal{L}v^i(s,Z_s) + \rho^i v^i(s,Z_s) \ge f^i(Z_s), \text{ for all } s\in[t,T]\,\mathbb{P}\,a.s. \tag{4.4}
$$

Let us now consider the second term: by the definition of $\mathcal{M}^i v^i$, we have:

$$
\begin{aligned}
v^i(\tau_n^i, Z_{\tau_n^{i-}}) &\ge \mathcal{M}^i v^i(\tau_n^i, X_{\tau_n^{i-}}, Y^i_{\tau_n^{i-}}, Y^j_{\tau_n^{i-}})\\
&= \sup_{\zeta_n^i\in[\zeta_{min},\zeta_{max}]}\Big\{v^i(\tau_n^i, X_{\tau_n^{i-}}, \Gamma^i(Y^i_{\tau^n_{i-}}, \zeta_n^{i-}), Y^j_{\tau_n^{i-}}) - \phi^i(Y^i_{(\tau_n^{i-})}, \zeta_n^i)\Big\}\\
&\ge v^i(\tau_n^i, X_{\tau_n^{i-}}, \Gamma^i(Y^i_{\tau_n^{i-}}, \zeta_n^i), Y^i_{\tau_n^{i-}}) - \phi^i(Y^i_{\tau_n^{i-}}, \zeta_n^i)\\
&= v^i(\tau_n^i, Z_{\tau_n^i}) - \phi^i(Y^i_{\tau_n^{i-}}, \zeta_n^i).
\end{aligned} \tag{4.5}
$$

For the third term, we have $(\mathcal{M}^j v^j - v^j)(\tau_n^j, Z_{\tau_n^{j-}}) = 0$, hence, by the definition of $\mathcal{H}^i v^i$, we have:

$$
\begin{aligned}
v^i(\tau_n^j, Z_{(\tau_n^{j-})}) &= \mathcal{H}^i v^i(\tau_n^j, X_{\tau_n^{j-}}, Y^i_{\tau_n^{j-}}, Y^j_{\tau_n^{j-}})\\
&= v^i(\tau_n^j, X_{\tau_n^{j-}}, Y^i_{\tau_n^{j-}}, \Gamma^j(Y^j_{\tau_n^{j-}}, \zeta_n^j))\\
&= v^i(\tau_n^j, Z_{\tau_n^j}).
\end{aligned} \tag{4.6}
$$

We focus on the fourth term which is related to the simultaneous interventions case of the two players i.e for time interventions which are in IT^c , we have:

$$
\begin{aligned}
&v^i(\tau_n^i, Z_{\tau_n^{i-}})\\
&= \mathcal{H}^i v^i(\tau_n^i, Z_{\tau_n^{i-}})\\
&= \mathcal{M}^i\mathcal{H}^i v^i(\tau_n^i, X_{\tau_n^{i-}}, Y^i_{\tau_n^{i-}}, Y^j_{\tau_n^{i-}})\\
&= \sup_{\zeta_n^i\in[\zeta_{min},\zeta_{max}]}\Big\{v^i(\tau_n^i, X_{\tau_n^{i-}}, \Gamma^i(Y^i_{\tau_n^{i-}}, \zeta_n^{i-}), \Gamma^j(Y^j_{\tau_n^{j-}}, \zeta_n^j)) - \phi^i(Y^i_{(\tau_n^{i-})}, \zeta_n^i)\Big\}\\
&\ge v^i(\tau_n^i, X_{\tau_n^{i-}}, \Gamma^i(Y^i_{\tau_n^{i-}}, \zeta_n^i), \Gamma^i(Y^i_{\tau_n^{i-}}, \zeta_n^j)) - \phi^i(Y^i_{\tau_n^{i-}}, \zeta_n^i)\\
&= v^i(\tau_n^i, Z_{\tau_n^i}) - \phi^i(Y^i_{\tau_n^{i-}}, \zeta_n^i).
\end{aligned} \tag{4.7}
$$

By (4.3), and the inequalities (4.4)-(4.6), it follows that:

$$v^i(t,z) \geq \mathbb{E}\big[\int_t^{\tau_r\wedge T} \mathrm{e}^{-\rho^i(s-t)} f^i(Z_u)du - \sum_{t\leq\tau_n^i<\tau_r\wedge T} \mathrm{e}^{-\rho^i(\tau_n^i-t)}\phi^i(Y^i_{\tau_n^{i-}},\zeta_n^i)$$
$$+ \mathrm{e}^{-\rho^i(\tau_r\wedge T-t)} v^i(\tau_r\wedge T, Z_{\tau_r\wedge T})\big].$$

As $f^i(z) \leq K^i x$ (see Rem. 2.2), ϕ^i is bounded (See Assumption 2.1(i)), v^i satisfies linear growth (see inequality (3.1)), and since $\alpha^i \in \mathcal{A}_t^i$, we can pass to the limit as $r \to \infty$ and use the dominated convergence theorem. it yields that:

$$v^i(t,z) \geq J^i(t,z,\alpha^i,\alpha^j).$$

Step 2: $v^i(t,z) = J^i(t,z,\alpha^{i*},\alpha^{j*})$, where $(\alpha^{i*},\alpha^{j*})$ are defined by (4.1)-(4.2)
We argue as in Step 1, but here all the inequalities are equalities by the properties of α^{i*}.

□

5. Viscosity characterization

It is well known that the theory of viscosity solutions is a powerful tool to characterize the equilibrium expected payoff as a solution in a weaker sense of the associated Hamilton Jacobi Bellman equation. The couple of equilibrium expected payoffs are not known to be continuous *a priori* and so we shall work with the notion of discontinuous viscosity solutions. For a locally bounded function u on $[0,T]\times\bar{\mathcal{S}}$, which is the case of the couple of equilibrium expected payoffs (v^1,v^2), we denote by $\underline{u}$ (resp. $\overline{u}$) the lower semi-continuous (LSC) (resp. upper semi-continuous (USC)) envelope of u. We recall that in general, $\underline{u}\leq u\leq\overline{u}$, and that u is LSC iff $u=\underline{u}$, u is UCS iff $u=\overline{u}$, and u is continuous iff $\underline{u}=\overline{u}$ $(=u)$. We denote by $LSC([0,T)\times\bar{\mathcal{S}})$ (resp. $USC([0,T)\times\bar{\mathcal{S}})$) the set of lsc (resp. usc) functions on $[0,T)\times\bar{\mathcal{S}}$.

We work with the suitable notion of constrained viscosity solutions. The use of constrained viscosity solutions was initiated by [14] for first-order equations and applied in stochastic control problems arising in optimal investment problems in [15]. The sub-solution viscosity is satisfied in $[0,T)\times\bar{\mathcal{S}}$ and the super-solution viscosity is satisfied in $[0,T)\times\mathcal{S}$.

Definition 5.1. (i) Let $\mathcal{O}\subset\bar{\mathcal{S}}$. A couple of locally bounded functions $(u^i)_{i\in\{1,2\}\setminus j}$ on $[0,T]\times\bar{\mathcal{S}}$ is a viscosity sub-solution of (2.12)-(2.13) in $[0,T)\times\mathcal{O}$ if for all $i\in\{1,2\}$, $(\bar{t},\bar{z})\in[0,T)\times\mathcal{O}$ and $\varphi^i\in C^{1,2}([0,T)\times\bar{\mathcal{S}})$ s.t. $(\overline{u}^i-\varphi^i)(\bar{t},\bar{z})=0$ and $(\bar{t},\bar{z})$ is a maximum of $\overline{u}^i-\varphi^i$ on $[0,T)\times\mathcal{O}$, we have

$$\overline{u}^i - \max\{\mathcal{M}^i\mathcal{H}^i\overline{u}^i, \mathcal{H}^i\overline{u}^i\} \leq 0 \quad \text{in} \quad \overline{\mathcal{I}}^i \tag{5.1}$$

$$\min\{-\frac{\partial\varphi^i}{\partial t} - \mathcal{L}\varphi^i + \rho^i\varphi^i - f^i, \overline{u}^i - \mathcal{M}^i\overline{u}^i\} \leq 0 \quad \text{in} \quad \mathcal{I}^i \tag{5.2}$$

(ii) Let $\mathcal{O}\subset\mathcal{S}$. A couple of locally bounded functions $(u^i)_{i\in\{1,2\}\setminus j}$ on $[0,T]\times\bar{\mathcal{S}}$ is a viscosity super-solution of (2.12)-(2.13) in $[0,T)\times\mathcal{O}$ if for all $i\in\{1,2\}$, $(\bar{t},\bar{z})\in[0,T)\times\mathcal{O}$ and $\varphi^i\in C^{1,2}([0,T)\times\bar{\mathcal{S}})$ s.t. $(\underline{u}^i-\varphi^i)(\bar{t},\bar{z})=0$ and $(\bar{t},\bar{z})$ is a minimum of $\underline{u}^i-\varphi^i$ on $[0,T)\times\mathcal{O}$, we have

$$\underline{u}^i - \max\{\mathcal{M}^i\mathcal{H}^i\underline{u}^i, \mathcal{H}^i\underline{u}^i\} \geq 0 \quad \text{in} \quad \overline{\mathcal{I}}^i \tag{5.3}$$

$$\min\{-\frac{\partial\varphi^i}{\partial t} - \mathcal{L}\varphi^i + \rho^i\varphi^i - f^i, \underline{u}^i - \mathcal{M}^i\underline{u}^i\} \geq 0 \quad \text{in} \quad \mathcal{I}^i \tag{5.4}$$

(iii) A couple of locally bounded functions $(u^i)_{i\in\{1,2\}\setminus j}$ on $[0,T]\times\bar{\mathcal{S}}$ is a constrained viscosity solution of (2.12)-(2.13) in $[0,T)\times\bar{\mathcal{S}}$ if it is a viscosity sub-solution of (2.12)-(2.13) in $[0,T)\times\bar{\mathcal{S}}$ and a viscosity super-solution of (2.12)-(2.13) in $[0,T)\times\mathcal{S}$.

Remark 5.2. There is an equivalent formulation of viscosity solutions, which is useful for proving uniqueness results, see [6] :
(i) Let $\mathcal{O}\subset\bar{\mathcal{S}}$. A couple of functions $(u^i)_{i\in\{1,2\}\setminus j}\in USC([0,T)\times\bar{\mathcal{S}})$ is viscosity sub-solution of (2.12)-(2.13) in $[0,T)\times\mathcal{O}$ if

$$\min\{\ \overline{u}^i-\mathcal{M}^i\mathcal{H}^i\overline{u}^i,\overline{u}^i-\mathcal{H}^i\overline{u}^i\}\leq 0\,\text{in}\,[0,T)\times\mathcal{O}\cap\overline{\mathcal{I}}^i, \tag{5.5}$$

$$\min\{\ -s_0-\mu x s_1-\frac{\sigma^2x^2}{2}M_{11}+\rho^iu^i-f^i,u^i-\mathcal{M}^iu^i\}\leq 0\,\text{in}\,[0,T)\times\mathcal{O}\cap\mathcal{I}^i, \tag{5.6}$$

where $(s_0,s=(s_k)_{1\leq k\leq 3},M=(M_{k_1k_2})_{1\leq k_1,k_2\leq 3})\in\bar{J}^{2,+}u^i(t,z)$.
(ii)Let $\mathcal{O}\subset\mathcal{S}$. A couple of functions $(u^i)_{i\in\{1,2\}\setminus j}\in USC([0,T)\times\bar{\mathcal{S}})$ is viscosity super-solution of (2.12)-(2.13) in $[0,T)\times\mathcal{O}$ if

$$\min\{\overline{u}^i-\mathcal{M}^i\mathcal{H}^i\overline{u}^i,\overline{u}^i-\mathcal{H}^i\overline{u}^i\}\geq 0\,\text{in}\,[0,T)\times\mathcal{O}\cap\overline{\mathcal{I}}^i, \tag{5.7}$$

$$\min\{-s_0-\mu x s_1-\frac{\sigma^2x^2}{2}M_{11}+\rho^iu^i-f^i,u^i-\mathcal{M}^iu^i\}\geq 0\,\text{in}\,[0,T)\times\mathcal{O}\cap\mathcal{I}^i, \tag{5.8}$$

where $(s_0,s=(s_k)_{1\leq k\leq 3},M=(M_{k_1k_2})_{1\leq k_1,k_2\leq 3})\in\bar{J}^{2,-}u^i(t,z)$.
(iii) A couple of functions $(u^i)_{i\in\{1,2\}\setminus j}$ on $[0,T)\times\bar{\mathcal{S}}$ is a constrained viscosity solution to (2.12)-(2.13) if $\bar{u}^i$ satisfies (5.5)-(5.6) in $[0,T)\times\bar{\mathcal{S}}$, where $(s_0,s,M)\in\bar{J}^{2,+}\bar{u}^i(t,z)$, and $\underline{u}^i$ satisfies (5.7)-(5.8) in $[0,T)\times\mathcal{S}$, where $(s_0,s,M)\in\bar{J}^{2,-}\underline{u}^i(t,z)$.

Here $J^{2,+}u(t,z)$ is the parabolic second order superjet defined by :

$$\begin{aligned}J^{2,+}u^i(t,z)\ &=\ \{(s_0,s,M)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{S}^3\ :\\ &\limsup_{\substack{(t',z')\to(t,z)\\(t',z')\in[0,T)\times\mathcal{S}}}\frac{u^i(t',z')-u^i(t,z)-s_0(t'-t)-s.(z'-z)-\frac{1}{2}(z'-z).M(z'-z)}{|t'-t|+|z'-z|^2}\ \leq\ 0\},\end{aligned}$$

$\mathbb{S}^3$ is the set of symmetric 3×3 matrices, $\bar{J}^{2,+}\bar{u}^i(t,z)$ is its closure :

$$\begin{aligned}\bar{J}^{2,+}u^i(t,x)=\Big\{(s_0,s,M)\ =\ \lim_{m\to\infty}(s_0^m,s^m,M^m)\quad&\text{with }(s_0^m,s^m,M^m)\in J^{2,+}u^i(t_m,z_m)\\ \text{and }\ \lim_{m\to\infty}(t_m,z_m,u^i(t_m,z_m))\ &=\ (t,z,u^i(t,z))\Big\},\end{aligned}$$

and $J^{2,-}u^i(t,z)=-J^{2,+}(-u^i)(t,z)$, $\bar{J}^{2,-}u(t,z)=-\bar{J}^{2,+}(-u^i)(t,z)$.

The dynamic programming principle (DPP) is the key tool to characterize the couple of equilibrium expected payoffs (v^i,v^j) defined by (2.10) as a viscosity solution of the system (2.12)-(2.13). We will use the weak version of the dynamic programming principle (WDPP) developed by Bouchard and Touzi [12] (see Thm. 3.5 and Cor. 3.6). In fact, the DPP is intuitive and its proof requires that the equilibrium expected payoff is measurable on the first stage and the delicate measurable selection theorem holds. The WDPP is a powerful tool to avoid these technical points. We could easily reformulate our model using the canonical setting in order to be consistent with the formulation in Bouchard and Touzi [12]. In fact, let $\Omega=C_0(\mathbb{R}_+,\mathbb{R})$ be the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ starting from 0. We define on Ω the σ-algebra $\mathcal{F}$ generated by the function $\omega\in\Omega\mapsto\omega_t$, for

$t \in \mathbb{R}_+$, and we endow $(\Omega, \mathcal{F})$ with the Wiener measure $\mathbb{P}$. By abuse of notation, we still denote by $\mathcal{F}$ the $\mathbb{P}$-completed σ-algebra. We define on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the real valued process W by

$$W_t(\omega) \quad = \quad \omega_t \ ,$$

for $t \in [0,T]$ and $\omega = (\omega_t)_{t\in[0,T]}$. We then denote by $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the complete filtration generated by W. The wolesale process $X^{t,x}_{.}$ is given by:

$$dX^{t,x}_s(\omega) = X^{t,x}_s(\omega)(\mu \mathrm{d}s + \sigma dW_s(\omega)),\ s \in [t,T],\ X^{t,x}_s = x,\ s \in [0,t].$$

For player i, given a strategy $\alpha^i(\omega) \in \mathcal{A}^i_t$, the process Y^i is given by:

$$\begin{aligned} dY^i_s(\omega) &= 0, \qquad \tau^i_n(\omega) \leq s < \tau^i_{n+1}(\omega) \\ Y^i_{\tau^i_{n+1}}(\omega) &= \Gamma^i(Y^{i-}_{\tau^i_{n+1}(\omega)}(\omega), \zeta^i_{n+1}(\omega)). \end{aligned}$$

Thanks to the structure of Ω, we consider the stopping operator defined as follows: for $0 \leq t \leq T$, we set $({}^t\omega_s)_{s\in[0,T]} = (\omega_{s\wedge t})_{s\in[0,T]}$ and the translation operator defined by: for $\omega, \omega' \in \Omega$ and $0 \leq t \leq T$, we set $(\omega \oplus_t \omega'_s)_{s\in[0,T]} = (\omega_s \mathbb{1}_{s\leq t} + (\omega'_s - \omega'_t + \omega_t)\mathbb{1}_{s>t})_{s\in[0,T]}$.
For $\tau \in \mathcal{T}_{t,T}$ and $\alpha^i = (\tau^i_n, \zeta^i_n)_{n\geq 1} \in \mathcal{A}^i_t$, we define the shifted (random) strategy $\alpha^{i,\tau}$ by

$$\alpha^{i,\tau}(\omega) = \Big\{ \big((\tau^i_n(\omega \oplus_{\tau(\omega)} \omega') - \tau(\omega)), \zeta^i_n(\omega \oplus_{\tau(\omega)} \omega') \big)_{n \geq \kappa(\tau,\alpha^i)(\omega)} \ ,\ \omega' \in \Omega \Big\}$$

with

$$\kappa(\tau, \alpha^i)(\omega) := \sup \{ n \geq 1 \ ,\ \tau^i_n(\omega) \leq \tau(\omega) \}$$

for all $\omega \in \Omega$.

Under this formulation, assumptions independence, causality, stability under concatenation in Bouchard and Touzi [12] are obviously satisfied and the assumption consistency with deterministic initial data is obtained by arguing as in Proposition 5.4 of the same paper. It yields that we have the following result.

Proposition 5.3. *We fix $i \in \{1,2\}$. Let Assumption 2.1 holds. Then,*
(i) for all $(\alpha^i, \alpha^j) \in \mathcal{A}^i_0 \times \mathcal{A}^j_0$, we have $J^i(.,.,\alpha^i,\alpha^j) \in LSC([0,T) \times \bar{\mathcal{S}})$.
(ii) for any stopping time $\theta \in \mathcal{T}_{t,T}$ (set of stopping times valued in $[t,T]$) we have the following inequalities:

$$\begin{aligned} v^i(t,z) \leq \sup_{\alpha^i \in \mathcal{A}^i_t, \alpha^j \in BR^j_t(\alpha^j)} & \mathbb{E}[\int_t^\theta \mathrm{e}^{-\rho^i(s-t)} f^i(Z^{t,z,\alpha^i,\alpha^j}_s)\mathrm{d}s \\ & - \sum_{t \leq \tau^i_k < \theta} \mathrm{e}^{-\rho^i(\tau^i_k - t)} \phi^i(Y^{i,t,y^i,\alpha^i,\alpha^j}_{(\tau^i_k)^-}, \zeta^i_k) + \mathrm{e}^{-\rho^i(\theta-t)} \bar{v}^i(\theta, Z^{t,z,\alpha^i,\alpha^j}_\theta)], \end{aligned} \tag{5.9}$$

$$\begin{aligned} v^i(t,z) \geq \sup_{\alpha^i \in \mathcal{A}^i_t, \alpha^j \in BR^j_t(\alpha^j)} & \mathbb{E}[\int_t^\theta \mathrm{e}^{-\rho^i(s-t)} f^i(Z^{t,z,\alpha^i,\alpha^j}_s)\mathrm{d}s \\ & - \sum_{t \leq \tau^i_k < \theta} \mathrm{e}^{-\rho^i(\tau^i_k - t)} \phi_i(Y^{i,t,y^i,\alpha^i,\alpha^j}_{(\tau^i_k)^-}, \zeta^i_k) + \mathrm{e}^{-\rho^i(\theta-t)} \underline{v}^i(\theta, Z^{t,z,\alpha^i,\alpha^j}_\theta)]. \end{aligned} \tag{5.10}$$

Proof. (i) Let $(\alpha^i,\alpha^j)\in\mathcal{A}_0^i\times\mathcal{A}_0^j$. From Remark 2.2 and (3.1), the random variable $\int_{t_n}^T \mathrm{e}^{-\rho^i(s-t_n)}f^i(Z_s^{t_n,z_n,\alpha^i,\alpha^j})\mathrm{d}s+\mathrm{e}^{-\rho^i(T-t_n)}g^i(Z_T^{t_n,z_n,\alpha^i,\alpha^j})$ is bounded from below by an integrable random variable. By applying Fatou's lemma, we obtain

$$\liminf_{\substack{(t_n,z_n)\to(t,z)\\ t_n\in[0,T),\, z_n\in(0,+\infty)^3}} \mathbb{E}\,\Big[\int_{t_n}^T \mathrm{e}^{-\rho^i(s-t_n)}f^i(Z_s^{t_n,z_n,\alpha^i,\alpha^j})\mathrm{d}s+\mathrm{e}^{-\rho^i(T-t_n)}g^i(Z_T^{t_n,z_n,\alpha^i,\alpha^j})\Big]$$
$$\geq \mathbb{E}\,\Big[\int_{t}^T \mathrm{e}^{-\rho^i(s-t)}f^i(Z_s^{t,z,\alpha^i,\alpha^j})\mathrm{d}s+\mathrm{e}^{-\rho^i(T-t)}g^i(Z_T^{t,z,\alpha^i,\alpha^j})\Big].$$

Under Assumption 2.1(i), and since $\alpha^i\in\mathcal{D}_0^i$, the sequence $\left(-\sum_{t_n\leq\tau_k^i<T}\mathrm{e}^{-\rho^i(\tau_k^i-t_n)}\phi_i(Y_{(\tau_k^i)^-}^{i,t_n,y_n^i,\alpha^i},\zeta_k^i)\right)_n$ is uniformly integrable, and so we obtain

$$\lim_{n\to\infty}\mathbb{E}[-\sum_{t_n\leq\tau_k^i<T}\mathrm{e}^{-\rho^i(\tau_k^i-t_n)}\phi^i(Y_{(\tau_k^i)^-}^{i,t_n,y_n^i,\alpha^i},\zeta_k^i)]=\mathbb{E}[-\sum_{t\leq\tau_k^i<T}\mathrm{e}^{-\rho^i(\tau_k^i-t)}\phi_i(Y_{(\tau_k^i)^-}^{i,t,y^i,\alpha^i,\alpha^j},\zeta_k^i)].$$

It yields that

$$\underline{J}^i(t,z,\alpha^i,\alpha^j)=\liminf_{\substack{(t_n,z_n)\to(t,z)\\ t_n\in[0,T),\, z_n\in(0,+\infty)^3}} J^i(t_n,z_n,\alpha^i,\alpha^j)\geq J^i(t,z,\alpha^i,\alpha^j).$$

This shows that $J^i(.,.,\alpha^i,\alpha^j)\in LSC([0,T)\times\bar{\mathcal{S}})$.
(ii) Since the strategy α^i is independent of $\mathcal{F}_t$, and following [12] (see their Rem. 5.2), one can prove

$$v^i(t,z)=\sup_{\alpha^i\in\mathcal{A}_0^i,\alpha^j\in BR_0^j(\alpha^i)} J^i(t,z,\alpha^i,\alpha^j)\quad i\neq j,$$

which means that one could get rid of the dependence of time t in the set $\mathcal{A}_t^i$.

The proof of inequality (5.9) is then similar to [12].

We fix $R>0$ and a control $(\alpha^i,\alpha^j)\in\mathcal{A}_0^i\times\mathcal{A}_0^j$. We consider the stopping time $\tau^R:=\inf\{s\geq t \text{ s.t. } |Z_s^{t,z,\alpha^i,\alpha^j}-z|>R\}$. Then $Z^{t,z,\alpha^i,\alpha^j}1_{[t,\theta\wedge\tau^R)}$ is L^∞ bounded. By Corollary 3.6 in [12], we have

$$v^i(t,z)\geq\mathbb{E}[\int_t^{\theta\wedge\tau^R}\mathrm{e}^{-\rho^i(s-t)}f^i(Z_s^{t,z,\alpha^i,\alpha^j})\mathrm{d}s-\sum_{t\leq\tau_k^i<\theta\wedge\tau^R}\mathrm{e}^{-\rho^i(\tau_k^i-t)}\phi^i(Y_{(\tau_k^i)^-}^{i,t,y^i,\alpha^i,\alpha^j},\zeta_k^i)$$
$$+\mathrm{e}^{-\rho^i(\theta\wedge\tau^R-t)}\underline{v}^i(\theta\wedge\tau^R,Z_{\theta\wedge\tau^R}^{t,z,\alpha^i,\alpha^j})].$$

Using Assumption 2.1(i), Remark 2.2, inequality (3.1), and since $(\alpha^i,\alpha^j)\in\mathcal{D}_0^i\times\mathcal{D}_0^j$, Fatou's lemma implies

$$\liminf_{R\longrightarrow\infty}\mathbb{E}[\int_t^{\theta\wedge\tau^R}\mathrm{e}^{-\rho^i(s-t)}f^i(Z_s^{t,z,\alpha^i,\alpha^j})\mathrm{d}s-\sum_{t\leq\tau_k^i<\theta\wedge\tau^R}\mathrm{e}^{-\rho^i(\tau_k^i-t)}\phi^i(Y_{(\tau_k^i)^-}^{i,t,y^i,\alpha^i,\alpha^j},\zeta_k^i)$$
$$+\mathrm{e}^{-\rho^i(\theta\wedge\tau^R-t)}\underline{v}^i(\theta\wedge\tau^R,Z_{\theta\wedge\tau^R}^{t,z,\alpha^i,\alpha^j})]$$

$$\geq \mathbb{E}[\int_t^\theta \mathrm{e}^{-\rho^i(s-t)} f^i(Z_s^{t,z,\alpha^i,\alpha^j})\mathrm{d}s - \sum_{t\leq\tau_k^i<\theta} \mathrm{e}^{-\rho^i(\tau_k^i-t)}\phi^i(Y^{i,t,y^i,\alpha^i,\alpha^j}_{(\tau_k^i)^-},\zeta_k^i)$$
$$+\mathrm{e}^{-\rho^i(\theta-t)}\underline{v}^i(\theta, Z_\theta^{t,z,\alpha^i,\alpha^j})].$$

As (α^i,α^j) is chosen arbitrary, we obtain (5.10). □

In the following lemma, we give some auxiliary results on the intervention operators $\mathcal{M}^i$ and $\mathcal{H}^i$ which will be useful later.

Lemma 5.4. *We fix $i \in \{1,2\}$. Let u^i be a locally bounded function on $[0,T)\times\bar{\mathcal{S}}$.*
(i) If u^i is lsc, then $\mathcal{M}^i u^i$ is lsc.
(ii) If u^i is usc, then $\mathcal{M}^i u^i$ is usc.
(iii) If u^i is lsc (resp. usc), then $\mathcal{H}^i u^i$ is lsc (resp. usc).

Proof. We prove only the first and the second statement.
(i) Fix some $(t,z)\in[0,T)\times\bar{\mathcal{S}}$ and let (t_n,z_n) be a sequence in $[0,T)\times\bar{\mathcal{S}}$ converging to (t,z) and s.t. $\mathcal{M}^i u^i(t_n,z_n)$ converges to $\underline{\mathcal{M}^i u^i}(t,z)$. Then, using also the lower semi-continuity of u^i, we have:

$$\begin{aligned}\mathcal{M}^i u^i(t,z) = \sup_{\zeta\in[\zeta_{min},\zeta_{max}]} u^i(t,x,\Gamma^i(y^i,\zeta),y^j)) &\leq \sup_{\zeta\in[\zeta_{min},\zeta_{max}]} \liminf_{n\to\infty} u^i(t_n,x_n,\Gamma^i(y_n^i,\zeta),y_n^j)\\ \leq \liminf_{n\to\infty} \sup_{\zeta\in[\zeta_{min},\zeta_{max}]} u^i(t_n,x_n,\Gamma^i(y_n^i,\zeta),y_n^j) &\leq \lim_{n\to\infty}\mathcal{M}^i u^i(t_n,z_n) = \underline{\mathcal{M}^i u^i}(t,z),\end{aligned}$$

which shows the lower semi-continuity of $\mathcal{M}^i u^i$.
(ii) Fix some $(t,z)\in[0,T)\times\mathcal{S}$ and let $(t_n,z_n)_{n\geq1}$ be a sequence in $[0,T)\times\mathcal{S}$ converging to (t,z) when n goes to infinity. Since u^i is usc, for each $n\geq 1$, by selection measurable theorem, there exists a sequence $(\hat{\zeta}_n)_{n\geq1}$ with $\hat{\zeta}_n\in[\zeta_{min},\zeta_{max}]$ such that:

$$\mathcal{M}^i u^i(t_n,z_n) = u^i(t_n,x_n,\Gamma^i(y_n^i,\hat{\zeta}_n),y_n^j), \quad \forall n\geq 1.$$

As $[\zeta_{min},\zeta_{max}]$ is a compact, the sequence $(\hat{\zeta}_n)_{n\geq1}$ converges, up to a subsequence, to some $\hat{\zeta}\in[\zeta_{min},\zeta_{max}]$. Therefore, we get :

$$\mathcal{M}^i u^i(t,z) \geq u^i(t,x,\Gamma^i(y^i,\hat{\zeta}),y^j) \geq \limsup_{n\to\infty} u^i(t_n,x_n,\Gamma^i(y_n^i,\hat{\zeta}_n),y_n^j) = \limsup_{n\to\infty}\mathcal{M}^i u^i(t_n,z_n),$$

which proves that $\mathcal{M}^i u^i$ is usc.
(iii) The proof is similar to (i) and (ii).

□

The following technical lemma is needed to prove that the equilibrium expected payoffs (v^1,v^2) are viscosity solutions of the associated QVIs system.

Lemma 5.5. *For all $(t,z)\in\mathcal{I}^i$ and $i\in\{1,2\}$ we have:*

$$\underline{v}^i(t,z) - \mathcal{M}^i\underline{v}^i(t,z) \geq 0. \tag{5.11}$$

For all $(t,z)\in\overline{\mathcal{I}}^i$ and $i\in\{1,2\}$ we have:

$$\min\{\underline{v}^i(t,z)-\mathcal{M}^i\mathcal{H}^i\underline{v}^i(t,z), \underline{v}^i(t,z)-\mathcal{H}^i\underline{v}^i(t,z)\} \geq 0, \tag{5.12}$$
$$\min\{\overline{v}^i(t,z)-\mathcal{M}^i\mathcal{H}^i\overline{v}^i(t,z), \overline{v}^i(t,z)-\mathcal{H}^i\overline{v}^i(t,z)\} \leq 0. \tag{5.13}$$

Proof. We fix $i \in \{1,2\}$. Let α^i an admissible control associated to the player i where she chooses to intervene immediately at time t with an arbitrary size $\zeta^i \in [\zeta_{min}, \zeta_{max}]$. By applying the WDPP (5.10), for all $(t,z) \in [0,T) \times \mathcal{S}$, we have that:

$$v^i(t,z) \geq \underline{v}^i(t,x,\Gamma^i(y^i,\zeta^i),y^j) - \phi^i(y^i,\zeta^i).$$

As ϕ^i is continuous (see Asm. 2.1(ii)), the right hand side of the last inequality is lsc, and so we deduce that:

$$\underline{v}^i(t,z) \geq \underline{v}^i(t,x,\Gamma^i(y^i,\zeta^i),y^j) - \phi^i(y^i,\zeta^i).$$

From the arbitrariness of ζ^i, we deduce

$$\underline{v}^i(t,z) \geq \mathcal{M}^i \underline{v}^i(t,z),$$

and so the inequality (5.11) is proved.

We fix $(t,z) \in \overline{\mathcal{I}}^i$. Two cases are possible: the player j decides to intervene or the two players i and j intervene simultaneously. In the first case, we denote by ζ^j the size of intervention of player j. Then, by applying the WDPP (5.10), we have

$$v^i(t,z) \geq \underline{v}^i(t,x,y^i,\Gamma^j(y^j,\zeta^j)) = \mathcal{H}^i \underline{v}^i(t,z).$$

As the right hand side of the last inequality is lsc, we deduce that

$$\underline{v}^i(t,z) \geq \mathcal{H}^i \underline{v}^i(t,z). \tag{5.14}$$

In the second case, if the players i and j decide to intervene in (t,z), then, by applying the WDPP (5.10), for all $(t,z) \in [0,T) \times \mathcal{S}$, we have that

$$v^i(t,z) \geq \underline{v}^i(t,x,\Gamma^i(y^i,\zeta^i),\Gamma^j(y^j,\zeta^j)) - \phi^i(y^i,\zeta^i).$$

Arguing as above, we deduce

$$\underline{v}^i(t,z) \geq \mathcal{M}^i \mathcal{H}^i \underline{v}^i(t,z). \tag{5.15}$$

From (5.14) and (5.15), we deduce that (5.12). It remains to prove (5.13).

From the WDPP (5.9), if player j decides to intervene, we obtain

$$v^i(t,z) \leq \mathcal{H}^i \overline{v}^i(t,z).$$

As the right hand side of the above inequality is usc, then

$$\overline{v}^i(t,z) \leq \mathcal{H}^i \overline{v}^i(t,z). \tag{5.16}$$

If the two players intervene simultaneously, then by using (5.9) and arguing as above, we obtain

$$\overline{v}^i(t,z) \leq \mathcal{M}^i \mathcal{H}^i \overline{v}^i(t,z). \tag{5.17}$$

From (5.16) and (5.17), we deduce (5.13). □

Our main result of the section is the following.

Theorem 5.6. *The couple of functions $(v^i)_{i\in\{1,2\}}$ defined by (2.10) is a constrained viscosity solution of (2.12)-(2.13) in $[0,T)\times\mathcal{S}$.*

Proof of super-solution property on $[0,T)\times\mathcal{S}$:
We fix $i\in\{1,2\}$. Let $(\bar t,\bar z)\in[0,T)\times\mathcal{S}$ and $\varphi^i\in C^{1,2}([0,T)\times\bar{\mathcal{S}})$ s.t. $(\underline{v}^i-\varphi^i)(\bar t,\bar z)=0$ and $(\bar t,\bar z)$ is a minimum of $\underline{v}^i-\varphi^i$ on $[0,T)\times\mathcal{S}$. Two cases are possible: $(\bar t,\bar z)\in\bar{\mathcal{I}}^i$ or $(\bar t,\bar z)\in\mathcal{I}^i$. If $(\bar t,\bar z)\in\bar{\mathcal{I}}^i$, then from Lemma 5.5, we have $\min\{\underline{v}^i(\bar t,\bar z)-\mathcal{M}^i\mathcal{H}^i\underline{v}^i(\bar t,\bar z),\underline{v}^i(\bar t,\bar z)-\mathcal{H}^i\underline{v}^i(\bar t,\bar z)\}\ge 0$ (See inequality (5.12)). If $(\bar t,\bar z)\in\mathcal{I}^i$, then we have $\underline{v}^i(\bar t,\bar z)-\mathcal{M}^i\underline{v}^i(\bar t,\bar z)\ge 0$ (see inequality (5.11)). It remains to show that

$$-\frac{\partial\varphi^i}{\partial t}(\bar t,\bar z)-\mathcal{L}\varphi^i(\bar t,\bar z)+\rho^i\varphi^i(\bar t,\bar z)-f^i(\bar z)\ge 0, \tag{5.18}$$

From the definition of $\underline{v}^i$, there exists a sequence $(t_m,z_m)_{m\ge1}\in\mathcal{S}$ s.t. (t_m,z_m) and $v^i(t_m,z_m)$ converge respectively to $(\bar t,\bar z)$ and $\underline{v}^i(\bar t,\bar z)$ as m goes to infinity. By continuity of φ^i, we also have that $\gamma_m:=v^i(t_m,z_m)-\varphi^i(t_m,z_m)$ converges to 0 as m goes to infinity. We denote by $\tau^{j*}_{1,m}$ the first optimal intervention of player j when the state process Z starts from z_m at time t_m. Two cases are possible $\tau^{j*}_{1,m}=t_m$ or $\tau^{j*}_{1,m}>t_m$. If the first case occurs, then the player j makes an intervention immediately. By applying the WDPP (5.10), $v^i(t_m,z_m)\ge\mathcal{H}^i\underline{v}^i(t_m,z_m)$. As t_m converges to $\bar t$ as m goes to infinity, and since $\mathcal{H}^i\underline{v}^i$ is lsc (see Lem. 5.4 (iii)), we obtain $\underline{v}^i(\bar t,\bar z)\ge\mathcal{H}^i\underline{v}^i(\bar t,\bar z)$. This means that $(\bar t,\bar z)\in\bar{\mathcal{I}}^i$, which is false, and so we must have $\tau^{j*}_{1,m}>t_m$. As $\bar t<T$ and $\bar z\in\mathcal{S}$, for m large enough, one could have $t_m<T$ and there exists $\delta>0$ such $B(z_m,\delta/2):=\{z \text{ s.t } |z-z_m|\le\frac{\delta}{2}\}\subset\mathcal{S}$. Let us then consider the admissible control in $\mathcal{A}^i_{t_m}$ with no impulse until the first exit time τ_m, from the ball $B(z_m,\delta/2)$ before $T\wedge\tau^{j*}_{1,m}$ of the associated state process Z_s, defined by:

$$\tau_m:=\inf\left\{s\ge t_m\ :|Z^{t_m,z_m}_s-z_m|\ge\delta/2\right\}\wedge T\wedge\tau^{j*}_{1,m}.$$

Consider also a strictly positive sequence $(h_m)_m$ s.t. h_m and γ_m/h_m converge to zero as m goes to infinity. By using the WDPP (5.10) for $v_i(t_m,z_m)$ and $\hat\tau_m:=\tau_m\wedge(t_m+h_m)$, we get :

$$\begin{aligned}v^i(t_m,z_m)&=\gamma_m+\varphi^i(t_m,z_m)\\&\ge\mathbb{E}[\int_{t_m}^{\hat\tau_m}\mathrm{e}^{-\rho^i(s-t_m)}f^i(Z^{t_m,z_m}_s)\mathrm{d}s+\mathrm{e}^{-\rho^i(\hat\tau_m-t_m)}\underline{v}^i(\hat\tau_m,Z^{t_m,z_m}_{\hat\tau_m})]\\&\ge\mathbb{E}[\int_{t_m}^{\hat\tau_m}\mathrm{e}^{-\rho^i(s-t_m)}f^i(Z^{t_m,z_m}_s)\mathrm{d}s+\mathrm{e}^{-\rho^i(\hat\tau_m-t_m)}\varphi^i(\hat\tau_m,Z^{t_m,z_m}_{\hat\tau_m})].\end{aligned}$$

Applying Itô's formula to $\varphi^i(s,Z^{t_m,z_m}_s)$ between t_m and $\hat\tau_m$ and noting that the integrand of the stochastic integral term is bounded, we obtain by taking expectation :

$$\frac{\gamma_m}{h_m}+\mathbb{E}\left[\frac{1}{h_m}\int_{t_m}^{\hat\tau_m}\mathrm{e}^{-\rho^i(s-t_m)}\left(-\frac{\partial\varphi^i}{\partial t}-\mathcal{L}\varphi^i+\rho^i\varphi^i-f^i\right)(s,Z^{t_m,z_m}_s)\mathrm{d}s\right]\ge 0. \tag{5.19}$$

By continuity a.s. of Z^{t_m,z_m}_s, we have for m large enough, $\hat\tau_m=t_m+h_m$, and so by the mean-value theorem, the random variable inside the expectation in (5.19) converges a.s. to $(-\frac{\partial\varphi^i}{\partial t}-\mathcal{L}\varphi^i+\rho^i\varphi^i-f^i)(\bar t,\bar z)$ as m goes to infinity. Since this random variable is also bounded by a constant independent of m, we conclude by the dominated convergence theorem and we obtain (5.18).

Proof of sub-solution property on $[0,T)\times\bar{\mathcal{S}}$:

We fix $i\in\{1,2\}$ and $(\bar t,\bar z)\in[0,T)\times\bar{\mathcal{S}}$. If $(\bar t,\bar z)\in\bar{\mathcal{I}}^i$, then from Lemma 5.5, we have $\min\{\overline{v}^i(\bar t,\bar z)-\mathcal{M}^i\mathcal{H}^i\overline{v}^i(\bar t,\bar z),\overline{v}^i(\bar t,\bar z)-\mathcal{H}^i\overline{v}^i(\bar t,\bar z)\}\le 0$ (see inequality (5.13)). If $(\bar t,\bar z)\in\mathcal{I}^i$, we consider $\varphi^i\in C^{1,2}([0,T)\times\bar{\mathcal{S}})$ s.t. $\overline{v}^i(\bar t,\bar z)=\varphi^i(\bar t,\bar z)$ and $\varphi^i\ge\overline{v}^i$ on $\bar{\mathcal{S}}$. If $\overline{v}^i(\bar t,\bar z)\le\mathcal{M}^i\overline{v}^i(\bar t,\bar z)$ then the sub-solution inequality holds trivially. Consider now the case where $\overline{v}^i(\bar t,\bar z)>\mathcal{M}^i\overline{v}^i(\bar t,\bar z)$ and argue by contradiction by assuming on the contrary that

$$\bar\eta\ :=\ -\frac{\partial\varphi^i}{\partial t}(\bar t,\bar z)-\mathcal{L}\varphi^i(\bar t,\bar z)+\rho^i\varphi^i(\bar t,\bar z)-f^i(\bar z)>0.$$

By continuity of f^i (see equation(2.5)), φ^i and its derivatives, and since $(\bar t,\bar z)\in[0,T)\times\bar{\mathcal{S}}$, there exists some $\delta_0>0$ s.t. $\bar t+\delta_0<T$ and for all $0<\delta\le\delta_0$ and for all $(t,z)\in((\bar t-\delta)_+,\bar t+\delta)\times(B(\bar z,\delta)\cap\bar{\mathcal{S}})$, we have:

$$-\frac{\partial\varphi^i}{\partial t}(t,z)-\mathcal{L}\varphi^i(t,z)+\rho^i\varphi^i(t,z)-f^i(z)>\frac{\bar\eta}{2}. \tag{5.20}$$

From the definition of $\overline{v}_i$, there exists a sequence $(t_m,z_m)_{m\ge1}\in((\bar t-\delta/2)_+,\bar t+\delta/2)\times B(\bar z,\delta/2)\cap\bar{\mathcal{S}}$ s.t. (t_m,z_m) and $v^i(t_m,z_m)$ converge respectively to $(\bar t,\bar z)$ and $\overline{v}_i(\bar t,\bar z)$ as m goes to infinity. By continuity of φ^i, we also have that $\gamma_m:=v^i(t_m,z_m)-\varphi^i(t_m,z_m)$ converges to 0 as m goes to infinity. By the WDPP (5.9), given $m\ge1$, for any stopping time τ valued in $[t_m,T]$, we have

$$\begin{aligned}
v^i(t_m,z_m)\le&\sup_{\alpha^i\in\mathcal{A}^i_{t_m}}\mathbb{E}\Big[\int_{t_m}^{\tau}\mathrm{e}^{-\rho^i(s-t_m)}f^i(Z_s^{t_m,z,\alpha^i,\alpha^j})\mathrm{d}s\\
&-\sum_{t_m\le\tau_k^i<\tau}\mathrm{e}^{-\rho^i(\tau_k^i-t_m)}\phi^i(Y^{i,t_m,y_m^i,\alpha^i}_{(\tau_k^i)^-},\zeta_k^i)+\mathrm{e}^{-\rho^i(\tau-t_m)}\overline{v}^i(\tau,Z_\tau^{t_m,z_m,\alpha^i,\alpha^j})\Big]\\
\le&\ \mathbb{E}\Big[\int_{t_m}^{\tau}\mathrm{e}^{-\rho^i(s-t_m)}f^i(\hat Z_s^m)\mathrm{d}s-\sum_{t_m\le\tau^i_{m,k}<\tau}\mathrm{e}^{-\rho^i(\tau^i_{m,k}-t_m)}\phi^i(\hat Y^{i,m}_{(\tau^i_{m,k})^-},\zeta^i_{m,k})\\
&+\mathrm{e}^{-\rho^i(\tau-t_m)}\overline{v}^i(\tau,\hat Z^m_\tau)\Big]+\frac{1}{m}.
\end{aligned}\tag{5.21}$$

Here $\hat Z^m$ is the state process, starting from z_m at t_m, and controlled by (α_m^i,α_m^j) an $\frac{1}{m}-$optimal strategy. We define $\bar\tau_m:=\tau^i_{m,1}\wedge\tau^m_\delta\wedge\tau^j_{m,1}$ where,

$$\tau_\delta^m=\inf\Big\{s\ge t_m\ :\hat Z_s^m\notin\mathring{B}(z_m,\delta/2)\cap\bar{\mathcal{S}}\Big\}\wedge(t_m+\delta/2)$$

is the first exit time before $t_m+\delta/2$ of $\hat Z^m$ from the intersection between open ball $\mathring{B}(z_m,\delta/2)$ and $\bar{\mathcal{S}}$. We claim that $\tau^i_{m,1}>t_m$. In fact, if $\tau^i_{m,1}=t_m$, the player i makes an intervention immediately, then

$$v^i(t_m,z_m)\le\bar v^i(t_m,x_m,\Gamma^i(y_m^i,\zeta^i_{m,k}),y_m^j)-\phi^i(y_m^i,\zeta^i_{m,k})+\frac{1}{m}.$$

Sending m to infinity and taking the supremum over all admissible interventions, we obtain $\bar v^i(\bar t,\bar z)\le\mathcal{M}^i\bar v^i(\bar t,\bar z)$ which is false in this particular case.

We claim that $\tau^j_{m,1}>t_m$. In fact if $\tau^j_{m,1}=t_m$, the player j makes an intervention immediately, then

$$v^j(t_m,z_m)\le\bar v^j(t_m,x_m,y_m^i,\Gamma^j(y_m^j,\zeta^j_{m,k}))-\phi^j(y_m^j,\zeta^j_{m,k})+\frac{1}{m}.$$

Sending m to infinity, we deduce that $\bar{v}^j(\bar{t},\bar{z}) \leq \mathcal{M}^j\bar{v}^j(\bar{t},\bar{z})$ and so $(\bar{t},\bar{z}) \in \bar{\mathcal{I}}^i$ which is false. As the process $\hat{Z}^m$ is càdlàg, we have $\tau_\delta^m > t_m$ P a.s. Then we deduce that $\bar{\tau}^m > t_m$ P a.s. As the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is a Brownian filtration, the stopping time $\bar{\tau}^m$ is predictable and so there exists a sequence of stopping times $(\tau_{n,m})_n$ s.t. $t_m < \bar{\tau}_{n,m} < \bar{\tau}^m$, $\bar{\tau}_{n,m} \uparrow \bar{\tau}^m$, when n goes to infinity. We fix n and m, we choose $\tau = \bar{\tau}_{n,m}$ in (5.21). Then, we get:

$$v^i(t_m,z_m) \leq \mathbb{E}[\int_{t_m}^{\bar{\tau}_{n,m}} \mathrm{e}^{-\rho^i(s-t_m)} f^i(\hat{Z}_s^m)\mathrm{d}s + \mathrm{e}^{-\rho^i(\bar{\tau}_{n,m}-t_m)} v^i(\bar{\tau}_{n,m},\hat{Z}^m_{\bar{\tau}_{n,m}})] + \frac{1}{m}. \tag{5.22}$$

Now, since $\overline{v}_i \leq \varphi_i$, we obtain:

$$\varphi^i(t_m,z_m) + \gamma_m \leq \mathbb{E}[\int_{t_m}^{\bar{\tau}_{n,m}} \mathrm{e}^{-\rho^i(s-t_m)} f^i(\hat{Z}_s^m)\mathrm{d}s + \mathrm{e}^{-\rho^i(\bar{\tau}_{n,m}-t_m)} \varphi^i(\bar{\tau}_{n,m},\hat{Z}_{\bar{\tau}_{n,m}})] + \frac{1}{m}.$$

By applying Itô's formula to $\varphi^i(s,\hat{Z}_s^m)$ between t_m and $\bar{\tau}_{n,m}$, and using (5.20), we then get:

$$\begin{aligned}\gamma_m &\leq \mathbb{E}\left[\int_{t_m}^{\bar{\tau}_{n,m}} \mathrm{e}^{-\rho^i(s-t_m)} \left(\frac{\partial\varphi^i}{\partial t} + \mathcal{L}\varphi^i - \rho^i\varphi^i + f^i\right)(s,\hat{Z}_s^m)\mathrm{d}s\right] \\ &\leq -\frac{\bar{\eta}}{2\rho^i}(1-\mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_{n,m}-t_m)}]).\end{aligned}$$

This implies that: $\lim_{n,m\to\infty}\mathbb{E}[\mathrm{e}^{-\rho^i\bar{\tau}_{n,m}}] = \mathrm{e}^{-\rho^i\bar{t}}$, and

$$\lim_{m\to\infty}\mathbb{E}[\mathrm{e}^{-\rho^i\bar{\tau}_m}] = \mathrm{e}^{-\rho^i\bar{t}},$$

where the last equality is deduced by using monotone convergence theorem. Then, along a subsequence, one could have $\lim_{m\to\infty}\bar{\tau}_m = \bar{t}$ a.s. Applying the WDPP (5.9) for $\tau = \bar{\tau}_m$, we obtain:

$$\begin{aligned}
&v^i(t_m,z_m) \\
&\leq \mathbb{E}[\int_{t_m}^{\bar{\tau}_m} \mathrm{e}^{-\rho^i(s-t_m)} f^i(\hat{Z}_s^m)\mathrm{d}s] + \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m-t_m)}\bar{v}^i(\bar{\tau}_m,\hat{Z}^m_{\bar{\tau}_m})1_{\tau_\delta^m<\tau^i_{m,1}\wedge\tau^j_{m,1}}] \\
&+ \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m-t_m)}\bar{v}^i(\bar{\tau}_m,\hat{X}^m_{\bar{\tau}_m},\hat{Y}^{i,m}_{\bar{\tau}_m},\Gamma^j(\hat{Y}^{j,m}_{\bar{\tau}_m^-},\zeta^j_{m,1}))1_{\tau^j_{m,1}\leq\tau^i_{m,1}\wedge\tau^m_\delta,\,\tau^j_{m,1}\neq\tau^i_{m,1}}] \\
&+ \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m-t_m)}(\bar{v}^i(\bar{\tau}_m,\hat{X}^m_{\bar{\tau}_m},\Gamma^i(\hat{Y}^{i,m}_{\bar{\tau}_m^-},\zeta^i_{m,1}),\Gamma^j(\hat{Y}^{j,m}_{\bar{\tau}_m^-},\zeta^j_{m,1})) \\
&- \phi^i(\hat{Y}^{i,m}_{(\tau^i_{m,1})^-},\zeta^i_{m,1}))1_{\tau^i_{m,1}=\tau^j_{m,1}\leq\tau^m_\delta}] \\
&+ \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m-t_m)}(\bar{v}^i(\bar{\tau}_m,\hat{X}^m_{\bar{\tau}_m},\Gamma^i(\hat{Z}^m_{\bar{\tau}^{m,-}},\zeta^i_{m,1}),\hat{Y}^{j,m}_{\bar{\tau}_m} - \phi^i(\hat{Y}^{i,m}_{(\tau^i_{m,1})^-},\zeta^i_{m,1})) \\
&\quad 1_{\tau^i_{m,1}\leq\tau^m_\delta\wedge\tau^j_{m,1},\tau^i_{m,1}\neq\tau^j_{m,1}}] + \frac{1}{m} \\
&\leq \mathbb{E}[\int_{t_m}^{\bar{\tau}_m} \mathrm{e}^{-\rho^i(\bar{\tau}_m-t_m)} f^i(\hat{Z}_s^m)\mathrm{d}s] + \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m-t_m)}\bar{v}^i(\bar{\tau}_m,\hat{Z}^m_{\bar{\tau}_m})1_{\tau_\delta^m<\tau^i_{m,1}\wedge\tau^j_{m,1}}] \\
&+ \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m-t_m)}\mathcal{H}^i\bar{v}^i(\bar{\tau}_m,\hat{Z}^m_{\bar{\tau}_m^-})1_{\tau^j_{m,1}\leq\tau^i_{m,1}\wedge\tau^m_\delta,\,\tau^j_{m,1}\neq\tau^i_{m,1}}]
\end{aligned}$$

$$+ \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)}\mathcal{M}^i\mathcal{H}^i\bar{v}^i(\bar{\tau}_m, \hat{Z}^m_{\bar{\tau}_m^-})1_{\tau^i_{m,1}=\tau^j_{m,1}\le\tau^m_\delta}]$$
$$+ \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)}\mathcal{M}^i\bar{v}^i(\bar{\tau}_m, \hat{Z}^m_{\bar{\tau}_m^-})1_{\tau^i_{m,1}\le\tau^m_\delta\wedge\tau^j_{m,1},\tau^i_{m,1}\neq\tau^j_{m,1}}] + \frac{1}{m}, \tag{5.23}$$

which implies,

$$\begin{aligned} v^i(t_m, z_m) \le & \sup_{\substack{|t'-\bar{t}|<\delta \\ |z'-\bar{z}|<\delta}} f^i(z')\frac{(1-\mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)}])}{\rho^i} \\ &+ \sup_{\substack{|t'-\bar{t}|<\delta \\ |z'-\bar{z}|<\delta}} \bar{v}^i(t', z')\mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)}1_{\tau^m_\delta<\tau^i_{m,1}\wedge\tau^j_{m,1}}] \\ &+ \sup_{\substack{|t'-\bar{t}|<\delta \\ |z'-\bar{z}|<\delta}} \mathcal{H}^i\bar{v}^i(t', z')\mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)}1_{\tau^j_{m,1}\le\tau^i_{m,1}\wedge\tau^m_\delta,\,\tau^j_{m,1}\neq\tau^i_{m,1}}] \\ &+ \sup_{\substack{|t'-\bar{t}|<\delta \\ |z'-\bar{z}|<\delta}} \mathcal{M}^i\mathcal{H}^i\bar{v}^i(t', z')\mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)}1_{\tau^j_{m,1}=\tau^i_{m,1}\le\tau^m_\delta}] \\ &+ \sup_{\substack{|t'-\bar{t}|<\delta \\ |z'-\bar{z}|<\delta}} \mathcal{M}^i\bar{v}^i(t', z')\mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)}1_{\tau^i_{m,1}\le\tau^m_\delta\wedge\tau^j_{m,1},\,\tau^i_{m,1}\neq\tau^j_{m,1}}] + \frac{1}{m}. \end{aligned} \tag{5.24}$$

Since

$$\lim_{m\to\infty}\bar{\tau}_m = \lim_{m\to\infty}\tau^j_{m,1}\wedge\tau^i_{m,1}\wedge\tau^m_\delta = \bar{t}\ \text{ a.s.}, \tag{5.25}$$

by the dominated convergence theorem, we obtain:

$$\lim_{m\to\infty}\mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)}] = 1 \tag{5.26}$$

For the second term in the right hand side of (5.24), we define $A_m := \{\tau^m_\delta < \tau^i_{m,1}\wedge\tau^j_{m,1}\}$ and $A := \cap_{m\ge0}\cup_{n\ge m} A_n$. We assume that $\mathbb{P}(A) > 0$ and we consider $\omega\in A$. As $\lim\limits_{m\to\infty}\tau^m_\delta(\omega) = \bar{t}$ and $\lim\limits_{m\to\infty} x_m = \bar{x}$, then

$$\lim_{m\to\infty}\hat{X}^m_{\tau^m_\delta(\omega)}(\omega) = \lim_{m\to\infty} x_m\mathrm{e}^{(\mu-\frac{\sigma^2}{2})(\tau^m_\delta - t_m)+\sigma(W_{\tau^m_\delta}(\omega)-W_{t_m}(\omega))} = \bar{x}.$$

As the processes $\hat{Y}^{i,m}$ and $\hat{Y}^{j,m}$ are constant on $[t_m, \tau^m_\delta(\omega)]$ and $\lim\limits_{m\to\infty}(y^i_m, y^j_m) = (\bar{y}^i, \bar{y}^j)$, then $\lim\limits_{m\to\infty}\hat{Y}^{i,m}_{\tau^m_\delta(\omega)}(\omega) = \lim\limits_{m\to\infty} y^i_m = \bar{y}^i$, and $\lim\limits_{m\to\infty}\hat{Y}^{j,m}_{\tau^m_\delta(\omega)}(\omega) = \lim\limits_{m\to\infty} y^j_m = \bar{y}^j$. This shows that $\lim\limits_{m\to\infty}\hat{Z}^m_{\tau^m_\delta(\omega)}(\omega) = \bar{z}$. On the other hand, from the definition of τ^m_δ, we have $\hat{Z}^m_{\tau^m_\delta(\omega)}(\omega)\notin B(z_m, \frac{\delta}{2})$ i.e $|\hat{Z}^m_{\tau^m_\delta(\omega)}(\omega) - z_m| > \frac{\delta}{2}$. Sending m to infinity, we obtain $0 > \frac{\delta}{2}$ which is false, and so $\mathbb{P}(A) = 0$. On the other hand, $0\le\mathbb{P}(A_m)\le\mathbb{P}(\cup_{n\ge m}A_n)$, and $\mathbb{P}(\cup_{n\ge m}A_n)\downarrow P(A)$ when m goes to infinity, then $\mathbb{P}(A_m)$ goes to zero when m goes to infinity. By the dominated convergence theorem, we obtain:

$$\lim_{m\to\infty}\mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)}1_{\tau^m_\delta<\tau^i_{m,1}\wedge\tau^j_{m,1}}] = 0. \tag{5.27}$$

For the third term in the right hand side of (5.24), we consider the set $B_m := \{\tau^j_{m,1}\le\tau^i_{m,1}\wedge\tau^m_\delta,\ \tau^j_{m,1}\neq\tau^i_{m,1}\}$. We define $B := \cap_{m\ge0}\cup_{n\ge m} B_n$. We assume that $\mathbb{P}(B) > 0$ and we consider $\omega\in B$. The player j is the first to make an intervention. As the process $\hat{Z}^m_\cdot$ is right continuous, then $\tau^j_{m,1}(\omega) > \bar{t}$. Since $\lim_{m\to\infty} \tau^j_{m,1}(\omega) = \bar{t}$, and $\hat{Z}^m_\cdot$ is right continuous then $\lim_{m\to\infty} \hat{Z}^m_{\tau^{j-}_{m,1}(\omega)}(\omega) = \bar{z}$. It yields that $(\bar{t}, \bar{z}) \in \bar{\mathcal{I}}^i$ which is false. Then, we have $\mathbb{P}(B) = 0$. Arguing as above, by the dominated convergence theorem, we obtain:

$$\lim_{m\to\infty} \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)} 1_{\tau^j_{m,1} < \tau^i_{m,1} \wedge \tau^m_\delta}] = 0. \tag{5.28}$$

For the fourth term in the right hand side of (5.24), we consider the set $C_m := \{\tau^j_{m,1} = \tau^i_{m,1} \le \tau^m_\delta\}$. We define $C := \cap_{m\ge0} \cup_{n\ge m} C_n$. Arguing as above, one could prove that $\mathbb{P}(C) = 0$ and by the dominated convergence theorem, we obtain:

$$\lim_{m\to\infty} \mathbb{E}[\mathrm{e}^{-\rho^i(\bar{\tau}_m - t_m)} 1_{\tau^j_{m,1} = \tau^i_{m,1} \le \tau^m_\delta}] = 0. \tag{5.29}$$

Sending m to infinity, from (5.24),(5.26)-(5.29) we deduce that

$$\overline{v}^i(\bar{t}, \bar{z}) \le \sup_{\substack{|t' - \bar{t}| < \delta \\ |z' - \bar{z}| < \delta}} \mathcal{M}^i \overline{v}^i(t', z')$$

Sending δ to 0, by Lemma 5.4, we have

$$\overline{v}_i(\bar{t}, \bar{z}) \le \mathcal{M}_i \overline{v}_i(\bar{t}, \bar{z}).$$

As $\overline{v}^i(\bar{t}, \bar{z}) > \mathcal{M}^i \overline{v}^i(\bar{t}, \bar{z})$, we obtain a contradiction. This shows that the sub-solution property is satisfied. □

6. Comparison principle of viscosity solutions

We turn to uniqueness result by proving a comparison principle for discontinuous viscosity solutions to the QVIs (2.12)-(2.13). The comparison theorem is based on the Ishii technique [16] to produce a strict super-solution and we adapt arguments to handle the nonlocal operators. The idea is to build a test function so that the maximum associated with the strict super-solution is not attained on the boundary. Furthermore, to construct a strict super-solution, we consider an auxiliary problem by adding a fixed cost $\kappa > 0$ to the intervention operator $\mathcal{H}^i$. This means that for an intervention of size ζ^j made by player j, the operator $\mathcal{H}^i_\kappa$ is defined by:

$$\mathcal{H}^i_\kappa h(t, z) := h(t, x, y^i, \Gamma^j(y^j, \zeta^j)) - \kappa,$$

for all locally bounded function $h : [0, T] \times S \longrightarrow \mathbb{R}$. We define the equilibrium expected payoff of the perturbed problem v^i_κ by:

$$v^i_\kappa(t, z) := \sup_{\alpha^i \in \mathcal{A}^i_t} J^i_\kappa(t, z, \alpha^i, \alpha^j),$$

where

$$\begin{aligned} J^i_\kappa(t, z, \alpha^i, \alpha^j) := \mathbb{E}\Big[\int_t^T \mathrm{e}^{-\rho^i(s-t)} f^i(Z^{t,z,\alpha^i,\alpha^j}_s) \mathrm{d}s - \sum_{t \le \tau^i_k < T} \mathrm{e}^{-\rho^i(\tau^i_k - t)} \phi^i(Y^{i,t,y^i,\alpha^i,\alpha^j}_{(\tau^i_k)^-}, \zeta^i_k) \\ - \kappa \sum_{t \le \tau^j_k < T} \mathrm{e}^{-\rho^i(\tau^j_k - t)} + \mathrm{e}^{-\rho^i(T-t)} g^i(Z^{t,z,\alpha^i,\alpha^j}_T)\Big],\ (t, z) \in [0, T] \times \bar{\mathcal{S}} \end{aligned}$$

One can prove that the couple of equilibrium expected payoffs $(v^i_\kappa)_{i\in\{1,2\}}$ satisfies the growth condition (3.1), and is a constrained viscosity solution in $[0,T)\times\bar{\mathcal{S}}$ of the following system of QVIs:

$$\min\{v^i_\kappa - \mathcal{M}^i\mathcal{H}^i_\kappa v^i_\kappa, v^i_\kappa - \mathcal{H}^i_\kappa v^i_\kappa\} = 0 \quad \text{in} \quad \overline{\mathcal{I}}^i \tag{6.1}$$

$$\min\{-\frac{\partial v^i_\kappa}{\partial t} - \mathcal{L}v^i_\kappa + \rho^i v^i_\kappa - f^i, v^i_\kappa - \mathcal{M}^i v^i_\kappa\} = 0 \quad \text{in} \quad \mathcal{I}^i, \tag{6.2}$$

with terminal condition:

$$v^i_\kappa(T,z) = g^i(z) \quad \text{in} \quad \{T\}\times\bar{\mathcal{S}}, \tag{6.3}$$

and boundary conditions:

$$v^i_\kappa(t,z) = \begin{cases} 0 & \text{if } (t,z)\in[0,T)\times\partial^{y^1}\mathcal{S}\cup\partial^{y^2}\mathcal{S}, \\ -\frac{x}{2}\big(\frac{\mathrm{e}^{(\mu-\rho^i)(T-t)}-1}{\mu-\rho^i} + \mathrm{e}^{(\mu-\rho^i)(T-t)}\big) & \text{if } (t,z)\in[0,T)\times\partial^x\mathcal{S}. \end{cases} \tag{6.4}$$

Remark 6.1. The proof of viscosity property for v^i_κ, $i\in\{1,2\}$, follows the same lines of arguments as for v^i, $i\in\{1,2\}$. For the growth condition (3.1), similar arguments holds as for the lower bound (see inequality (3.2)) and for the upper bound (see inequality (3.3)), since the number of interventions of each player is uniformly integrable and so satisfies $\sup_{\alpha^i\in\mathcal{D}^i_0}\mathbb{E}[\mathcal{N}_T(\alpha^i)]<\infty$.

The first result of this section is a convergence result.

Proposition 6.2. *For $i\in\{1,2\}$, the sequence $(v^i_\kappa)_\kappa$ is nonincreasing, and converges pointwise on $[0,T]\times\bar{\mathcal{S}}$ towards v^i as κ goes to zero.*

Proof. Notice that for any $0<\kappa_1\le\kappa_2$, $(t,z)\in[0,T)\times\bar{\mathcal{S}}$ and $(\alpha^i,\alpha^j)\in\mathcal{A}^i_t\times\mathcal{A}^j_t$, we have

$$J^i_{\kappa_1}(t,z,\alpha^i,\alpha^j) \ge J^i_{\kappa_2}(t,z,\alpha^i,\alpha^j).$$

As α^i is arbitrary and α^j is the best response against α^i, we deduce that $v^i_{\kappa_1}(t,z)\ge v^i_{\kappa_2}(t,z)$. This shows that the sequence (v^i_κ) is nonincreasing, and is upper-bounded by the equilibrium expected payoff v^i, so that

$$\lim_{\kappa\downarrow 0} v^i_\kappa(t,z) \le v^i(t,z), \quad \forall (t,z)\in[0,T]\times\bar{\mathcal{S}}. \tag{6.5}$$

For the converse inequality, we fix some point $(t,z)\in[0,T]\times\bar{\mathcal{S}}$. From the representation (2.10) of $v^i(t,z)$, there exists for any $n\ge1$, an $1/n$-optimal control denoted by α^i_n and α^j_n the best response against α^i_n such that:

$$J^i(t,z,\alpha^i_n,\alpha^j_n) \ge v^i(t,z) - \frac{1}{n}. \tag{6.6}$$

On the other hand, we have $J^i(t,z,\alpha^i_n,\alpha^j_n) = J^i_\kappa(t,z,\alpha^i_n,\alpha^j_n) + \kappa\mathbb{E}[\sum_{t\le\tau^j_k<T}\mathrm{e}^{-\rho^i(\tau^j_k-t)}]$. From (6.6), we deduce that:

$$J^i_\kappa(t,z,\alpha^i_n,\alpha^j_n) + \kappa\mathbb{E}[\sum_{t\le\tau^j_k<T}\mathrm{e}^{-\rho^i(\tau^j_k-t)}] \ge v^i(t,z) - \frac{1}{n}.$$

Taking the supremum over all admissible strategies, we obtain:

$$v_\kappa^i(t,z) + \kappa \sup_{\alpha^j \in \mathcal{A}_t^j} \mathbb{E}[\sum_{t \le \tau_k^j < T} \mathrm{e}^{-\rho^i(\tau_k^j - t)}] \ge v^i(t,z) - \frac{1}{n}$$

By Sending κ to zero and n to infinity, and since $\alpha^j \in \mathcal{D}_t^j$, we obtain

$$\lim_{\kappa \downarrow 0} v_\kappa^i(t,z) \ge v^i(t,z). \tag{6.7}$$

The result follows from (6.6) and (6.7). □

Thanks to the following proposition, we produce a suitable strict viscosity super-solution, which will allow us to compare a viscosity super-solution and a viscosity sub-solution on $[0,T) \times \mathcal{S}$.

Proposition 6.3. *Let w_κ^i, $i \in \{1,2\}$ be a lower semi-continuous viscosity super-solution on $[0,T) \times \mathcal{S}$ of the QVIs (6.1)-(6.2). For all $(t,z) \in [0,T) \times \mathcal{S}$, we consider*

$$h^i(t,z) = \left(A^i + B^i x^2 + C^i \log(y^i + 1) + D^i \log(y^j + 1)\right) \mathrm{e}^{\rho(T-t)},$$

where

$$\begin{aligned} &\rho > (2\mu + \sigma^2 - \rho^i)_+,\ A^i > \tfrac{K^i}{\rho^i + \rho},\ B^i > \tfrac{K^i}{\rho^i + \rho - 2\mu - \sigma^2}, \\ &\tfrac{C_1^\phi}{\lambda \zeta_{max}} \mathrm{e}^{-\rho T} > C^i > 0,\ \text{and}\ \tfrac{\kappa}{\lambda \zeta_{max}} \mathrm{e}^{-\rho T} > D^i > 0. \end{aligned} \tag{6.8}$$

Then, for each $\gamma \in (0,1)$, $w_\kappa^{i,\gamma} = (1-\gamma) w_\kappa^i + \gamma g^i$ is a viscosity super-solution of

$$\begin{cases} \min\{w_\kappa^{i,\gamma} - \mathcal{M}^i \mathcal{H}_\kappa^i w_\kappa^{i,\gamma}, w_\kappa^{i,\gamma} - \mathcal{H}_\kappa^i w_\kappa^{i,\gamma}\} \ge \gamma \eta' \quad \text{in} \quad \overline{\mathcal{I}}^i, \\ \min\{-\dfrac{\partial w_\kappa^{i,\gamma}}{\partial t} - \mathcal{L} w_\kappa^{i,\gamma} + \rho^i w_\kappa^{i,\gamma} - f^i, w_\kappa^{i,\gamma} - \mathcal{M}^i w_\kappa^{i,\gamma}\} \ge \gamma \eta \quad \text{in} \quad \mathcal{I}^i, \end{cases} \tag{6.9}$$

where η and η' are positive constants.

Proof. Let $(t,z) \in \mathcal{I}^i$ and $\gamma \in (0,1)$. We fix $\zeta^i \in [\zeta_{min}, \zeta_{max}]$, then

$$\begin{aligned} & h^i(t,z) - \left(h^i(t,x,\Gamma^i(z^i,\zeta^i), y^j) - \phi^i(y^i,\zeta^i)\right) \qquad (6.10) \\ &= h^i(t,z) - \Big(A^i + B^i x^2 + C^i \log(y^i \mathrm{e}^{\lambda \zeta^i} + 1) + D^i \log(y^j + 1)\Big) \mathrm{e}^{\rho(T-t)} + \phi^i(y^i,\zeta^i) \\ &\ge h^i(t,z) - \left(A^i + B^i x^2 + C^i \log((y^i \mathrm{e}^{\lambda \zeta_{max}} + \mathrm{e}^{\lambda \zeta_{max}}) + D^i \log(y^j + 1)\right) \mathrm{e}^{\rho(T-t)} + C_1^\phi \\ &\ge h^i(t,z) - \left(A^i + B^i x^2 + C^i \log(\mathrm{e}^{\lambda \zeta_{max}}(y^i + 1)) + D^i \log(y^j + 1)\right) \mathrm{e}^{\rho(T-t)} + C_1^\phi \\ &\ge h^i(t,z) - \left(A^i + B^i x^2 + C^i \log(y^i + 1) + C^i \log(\mathrm{e}^{\lambda \zeta_{max}}) + D^i \log(y^j + 1)\right) \mathrm{e}^{\rho(T-t)} + C_1^\phi \\ &\ge h^i(t,z) - h^i(t,z) - C^i \lambda \zeta_{max} \mathrm{e}^{\rho(T-t)} + C_1^\phi \\ &\ge -C^i \lambda \zeta_{max} \mathrm{e}^{\rho(T-t)} + C_1^\phi, \end{aligned}$$

where the first inequality is obtained since ϕ^i satisfies 2.1(i) and ζ_{max} is positive. From the choice of C^i (see (6.8)) and from the arbitrariness of ζ^i, we deduce that:

$$h^i(t,z) - \mathcal{M}^i h^i(t,z) \ge -C^i \lambda \zeta_{max} \mathrm{e}^{\rho T} + C_1^\phi := \eta_1 > 0. \tag{6.11}$$

Moreover, as w^i is lsc and a super-solution of (2.13), we have

$$w^i_\kappa - \mathcal{M}^i w^i_\kappa \geq 0. \tag{6.12}$$

Combining (6.11), (6.12) and using the convexity of the operator$\mathcal{M}$, we obtain

$$\begin{aligned} w^{i,\gamma}_\kappa(t,z) - \mathcal{M}^i w^{i,\gamma}_\kappa(t,z) &\geq (1-\gamma)(w^i_\kappa(t,z) - \mathcal{M}^i w^i_\kappa(t,z)) + \gamma(h^i(t,z) - \mathcal{M}^i h^i(t,z)) \\ &\geq \gamma\eta_1 > 0 \end{aligned} \tag{6.13}$$

From the definition of h^i, for all $(t,z) \in \mathcal{I}^i$, we have

$$\begin{aligned} &-\frac{\partial h^i}{\partial t}(t,z) - \mathcal{L}h^i(t,z) + \rho^i h^i(t,z) - f^i(z) \\ &= (\rho^i + \rho)h^i(t,z) - \mu x \frac{\partial h^i}{\partial x}(t,z) - \frac{\sigma^2}{2}x^2 \frac{\partial^2 h^i}{\partial x^2}(t,z) - f^i(z) \\ &= (\rho^i + \rho)h^i(t,z) - (2\mu + \sigma^2)B^i x^2 \mathrm{e}^{\rho(T-t)} - f^i(z) \\ &\geq (\rho^i + \rho)(A^i + B^i x^2 + C^i log(y^i + 1) + D^i \log(y^j + 1))\mathrm{e}^{\rho(T-t)} - (2\mu + \sigma^2)B^i x^2 \mathrm{e}^{\rho(T-t)} - K^i x \\ &\geq x^2 \left(B^i \mathrm{e}^{\rho(T-t)}(\rho^i + \rho - 2\mu - \sigma^2) - K^i\right) + \left((\rho^i + \rho)A^i \mathrm{e}^{\rho(T-t)} - K^i\right) \\ &+ C^i \mathrm{e}^{\rho(T-t)}(\rho^i + \rho) log(y^i + 1) + D^i \mathrm{e}^{\rho(T-t)}(\rho^i + \rho) log(y^j + 1) \\ &\geq x^2 \left(B^i(\rho^i + \rho - 2\mu - \sigma^2) - K^i\right) + \left((\rho^i + \rho)A^i - K^i\right) \\ &+ C^i(\rho^i + \rho) log(y^i + 1) + D^i(\rho^i + \rho) log(y^j + 1), \end{aligned}$$

where the first and the second inequalities are obtained since $f^i(z) \leq K^i x \leq K^i(1 + x^2)$. Thanks to the choice of A^i, B^i and ρ (see (6.8)), we have

$$-\frac{\partial h^i}{\partial t}(t,z) - \mathcal{L}h^i(t,z) - \rho^i h^i(t,z) - f^i(z) \geq (\rho^i + \rho)A^i - K^i =: \eta_2 > 0. \tag{6.14}$$

Thus we obtain

$$-\frac{\partial w^{i,\gamma}_\kappa}{\partial t} - \mathcal{L}w^{i,\gamma}_\kappa + \rho^i w^{i,\gamma}_\kappa - f^i \geq \gamma\eta_2 \text{ in the viscosity sense.} \tag{6.15}$$

Combining (6.13) and (6.15), we have

$$min\{-\frac{\partial w^{i,\gamma}_\kappa}{\partial t} - \mathcal{L}w^{i,\gamma}_\kappa + \rho^i w^{i,\gamma}_\kappa - f^i, w^{i,\gamma}_\kappa - \mathcal{M}^i w^{i,\gamma}_\kappa\} \geq \gamma\eta \quad \text{in} \quad \mathcal{I}^i, \tag{6.16}$$

where $\eta := \min\{\eta_1, \eta_2\} > 0$ and the QVI (6.16) should be interpreted in the viscosity sense.
Let $(t,z) \in \bar{\mathcal{I}}^i$ and $\gamma \in (0,1)$. From the definition of h^i, we have:

$$\begin{aligned} &h^i(t,z) - \mathcal{H}^i_\kappa h^i(t,z) \\ &= h^i(t,z) - \left(h^i(t,x,y^i,\Gamma^j(z^j,\zeta^j)) - \kappa\right) \\ &= h^i(t,z) - \left(A^i + B^i x^2 + C^i \log(y^i + 1) + D^i \log(y^j \mathrm{e}^{\lambda\zeta^j} + 1)\right) \mathrm{e}^{\rho(T-t)} + \kappa \end{aligned}$$

From the choice of D^i (see (6.8)), we deduce that:

$$h^i(t,z) - \mathcal{H}^i_\kappa h^i(t,z) \geq -D^i\lambda\zeta_{max}\mathrm{e}^{\rho T} + \kappa := \eta_1^{'} > 0. \tag{6.17}$$

On the other hand, for $(t,z) \in \bar{\mathcal{I}}^i$, we have:

$$\begin{aligned}
&h^i(t,z) - \mathcal{M}^i\mathcal{H}^i_\kappa h^i(t,z)\\
&= h^i(t,z) - \left(h^i(t,x,\Gamma^i(z^i,\zeta^i),\Gamma^j(z^j,\zeta^j)) - \phi^i(y^i,\zeta^i) - \kappa\right)\\
&= h^i(t,z) - \Big(A^i + B^i x^2 + C^i\log(y^i\mathrm{e}^{\lambda\zeta^i}+1) + D^i\log(y^j\mathrm{e}^{\lambda\zeta^j}+1)\Big)\mathrm{e}^{\rho(T-t)} + \phi^i(y^i,\zeta^i) + \kappa\\
&\geq -C^i\lambda\zeta_{max}\mathrm{e}^{\rho(T-t)} + C_1^\phi - D^i\lambda\zeta_{max}\mathrm{e}^{\rho(T-t)} + \kappa,
\end{aligned}$$

From the choices of C^i and D^i (see (6.8)), we deduce that:

$$h^i(t,z) - \mathcal{M}^i\mathcal{H}^i_\kappa h^i(t,z) \geq -C^i\lambda\zeta_{max}\mathrm{e}^{\rho T} + C_1^\phi - D^i\lambda\zeta_{max}\mathrm{e}^{\rho T} + \kappa := \eta_2^{'} > 0. \tag{6.18}$$

From (6.17) and (6.18), we deduce that:

$$\min\{h^i - \mathcal{M}^i\mathcal{H}^i_\kappa h^i, h^i - \mathcal{H}^i_\kappa h^i\} \geq \eta^{'} := \min\{\eta_1^{'},\eta_2^{'}\} \quad \text{in} \quad \overline{\mathcal{I}}^i.$$

As w^i_κ is lsc and a viscosity super-solution of (2.12), then

$$\min\{w^i_\kappa - \mathcal{M}^i\mathcal{H}^i_\kappa w^i_\kappa, w^i_\kappa - \mathcal{H}^i_\kappa w^i_\kappa\} \geq 0 \quad \text{in} \quad \overline{\mathcal{I}}^i.$$

We deduce that:

$$\min\{w^{i,\gamma}_\kappa - \mathcal{M}^i\mathcal{H}^i_\kappa w^{i,\gamma}_\kappa, w^{i,\gamma}_\kappa\mathcal{H}^i_\kappa w^{i,\gamma}_\kappa\} \geq \gamma\eta^{'} \quad \text{in} \quad \overline{\mathcal{I}}^i. \tag{6.19}$$

The result follows from (6.16) and (6.19). □

The following result is a comparison theorem. It states that we can compare a viscosity sub-solution to (6.1)-(6.2) in $[0,T)\times\mathcal{S}$ and a viscosity super-solution to (6.1)-(6.2) in $[0,T)\times\mathcal{S}$, provided that we can compare them at the terminal date but also at the corner lines D_0 of the solvency boundary. We have to handle with the difficulties coming from the boundary. We adapt the argument in Barles [7] (see Thm. 4.5) which needs a smooth boundary. Akian et al. [8] proved a comparison theorem where the Hamilton Jacobi Bellman Variational inequality is satisfied up the boundary. The non regularity at some points of the boundary is studied in Ly Vath et al. [9]. They proved a comparison theorem on the solvency region except the non regular part of the boundary. In our case, we can not compare a sub-solution and a super-solution at the corner lines.

Theorem 6.4. *We fix $i\in\{1,2\}$. Suppose that $(u^i_\kappa)_{i\in\{1,2\}} \in USC([0,T)\times\bar{\mathcal{S}})$ is a viscosity sub-solution to (6.1)-(6.2) in $[0,T)\times\bar{\mathcal{S}}$ and $(w^i_\kappa)_{i\in\{1,2\}} \in LSC([0,T)\times\bar{\mathcal{S}})$ is a viscosity super-solution to (6.1)-(6.2) in $[0,T)\times\mathcal{S}$. We assume that u^i_κ and w^i_κ satisfy the growth condition (3.1) and*

$$u^i_\kappa(t,z) \leq \liminf_{(t^{'},z^{'})\longrightarrow(t,z)} w^i_\kappa(t^{'},z^{'}),\ \forall (t,z)\in[0,T)\times D_0, \tag{6.20}$$

$$u^i_\kappa(T,z) := \limsup_{\substack{(t,z^{'})\to(T,z)\\ t<T,z^{'}\in\mathcal{S}}} u^i_\kappa(t^{'},z^{'}) \leq w^i_\kappa(T,z) := \liminf_{\substack{(t,z^{'})\to(T,z)\\ t<T,z^{'}\in\mathcal{S}}} w^i_\kappa(t^{'},z^{'})\ \forall z\in\bar{\mathcal{S}}. \tag{6.21}$$

Then,

$$u^i_\kappa \le w^i_\kappa \quad on\ [0,T]\times\mathcal{S}.$$

Proof. In order to prove the comparison principle, it suffices to show that for all $\gamma\in(0,1)$:

$$\sup_{z\in\mathcal{S}}(u^i_\kappa - w^{i,\gamma}_\kappa)\le 0,$$

where $w^{i,\gamma}_\kappa$ is a strict super-solution satisfying (6.9). We obtain the required result by letting γ to 0. We argue by contradiction. We suppose that

$$\mu^i := \sup_{z\in\mathcal{S}}(u^i_\kappa - w^{i,\gamma}_\kappa) > 0. \tag{6.22}$$

We define w^i_κ for $i\in\{1,2\}$ on $[0,T)\times\partial\mathcal{S}$ by :

$$w^i_\kappa(t,z) = \liminf_{\substack{(t',z')\to(t,z)\\ (t',z')\in[0,T)\times\mathcal{S}}} w^i_\kappa(t',z'),\quad \forall (t,z)\in[0,T)\times\partial\mathcal{S}. \tag{6.23}$$

As u^i_κ is usc and $w^{i,\gamma}_\kappa$ is lsc, then $u^i_\kappa - w^{i,\gamma}_\kappa$ is usc. By the choice of the strict super-solution, we have $\lim_{|x|,|y^i|,|y^j|\longrightarrow\infty} u^i(t,z) - w^{i,\gamma}(t,z) = -\infty$. Hence, the set $\operatorname{argmax}_{[0,T]\times\bar{\mathcal{S}}}(u^i_\kappa - w^{i,\gamma}_\kappa)$ is non empty. As $u^i_\kappa(T,z) - w^{i,\gamma}_\kappa(T,z)\le 0$ and $u^i_\kappa(t,z) - w^{i,\gamma}_\kappa(t,z)\le 0$ for all $(t,z)\in[0,T)\times D_0$, there exists an open set $\mathcal{K}$ with compact closure $\bar{\mathcal{K}}$ s.t.

$$\operatorname{argmax}_{[0,T]\times\bar{\mathcal{S}}}(u^i_\kappa - w^{i,\gamma}_\kappa)\subset[0,T)\times((\bar{\mathcal{S}}\setminus D_0)\cap\mathcal{K}).$$

We choose $(t_0,z_0)\in[0,T)\times(\bar{\mathcal{S}}\setminus D_0\cap\mathcal{K})$ s.t.

$$\mu^i = u^i_\kappa(t_0,z_0) - w^{i,\gamma}_\kappa(t_0,z_0). \tag{6.24}$$

We distinguish two cases:

• *Case 1.* :$z_0\in\partial\mathcal{S}\setminus D_0\cap\mathcal{K}$. From (6.23), there exists a sequence $(t_n,z_n)_{n\ge1}$ in $[0,T)\times(\mathcal{S}\cap\mathcal{K})$ converging to (t_0,z_0) s.t. $w^{i,\gamma}_\kappa(t_n,z_n)$ tends to $w^{i,\gamma}_\kappa(t_0,z_0)$ when n goes to infinity. We then set $\beta_n := |t_n - t_0|$, $\varepsilon_n := |z_n - z_0|$ and consider the function Φ^i_n defined on $[0,T]^2\times(\bar{\mathcal{S}}\cap\bar{\mathcal{K}})^2$ by :

$$\begin{aligned}\Phi^i_n(t,t',z,z') &= u^i_\kappa(t,z) - w^{i,\gamma}_\kappa(t',z') - \varphi^i_n(t,t',z,z')\\ \varphi^i_n(t,t',z,z') &= |t-t_0|^2 + |z-z_0|^4 + \frac{|t-t'|^2}{2\beta_n} + \frac{|z-z'|^2}{2\varepsilon_n} + \left(\frac{d(z')}{d(z_n)} - 1\right)^4.\end{aligned} \tag{6.25}$$

Here $d(z)$ denotes the distance from z to $\partial\mathcal{S}$. It is known that for $z_0\notin D_0$, there exists an open neighborhood of z_0 in which the distance $d(.)$ is twice differentiable with bounded derivatives. Such property fails when z_0 belongs to the corner lines *i.e.* $z_0\in D_0$ (see [10]).

Since Φ^i_n is usc on the compact set $[0,T]^2\times(\bar{\mathcal{S}}\cap\bar{\mathcal{K}})^2$, there exists $(\hat t_n,\hat{t}'_n,\hat z_n,\hat{z}'_n)\in[0,T]^2\times(\bar{\mathcal{S}}\cap\bar{\mathcal{K}})^2$ that attains its maximum on $[0,T]^2\times(\bar{\mathcal{S}}\cap\bar{\mathcal{K}})^2$. We define

$$\mu^i_n := \sup_{[0,T]^2\times(\bar{\mathcal{S}}\cap\mathcal{K})^2}\Phi^i_n(t,t',z,z') \ = \ \Phi^i_n(\hat t_n,\hat{t}'_n,\hat z_n,\hat{z}'_n).$$

Moreover, there exists a subsequence, also denoted $(\hat{t}_n, \tilde{t}'_n, \hat{z}_n, \hat{z}'_n)_{n\geq 1}$, converging to $(\hat{t}_0, \tilde{t}'_0, \hat{z}_0, \hat{z}'_0) \in [0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$. Since $\Phi^i_n(t_0, t_n, z_0, z_n) \leq \mu^i_n = \Phi^i_n(\hat{t}_n, \tilde{t}'_n, \hat{z}_n, \hat{z}'_n)$, we have :

$$u^i_\kappa(t_0, z_0) - w^{i,\gamma}_\kappa(t_n, z_n) - \frac{1}{2}\left(|t_n - t_0| + |z_n - z_0|\right) \tag{6.26}$$

$$\leq \mu^i_n \; = \; u^i_\kappa(\hat{t}_n, \hat{z}_n) - w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n) - \left(|\hat{t}_n - t_0|^2 + |\hat{z}_n - z_0|^4\right) - R_n \tag{6.27}$$

$$\leq u^i_\kappa(\hat{t}_n, \hat{z}_n) - w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n) - \left(|\hat{t}_n - t_0|^2 + |\hat{z}_n - z_0|^4\right), \tag{6.28}$$

where we set

$$R_n := \frac{|\hat{t}_n - \hat{t}'_n|^2}{2\beta_n} + \frac{|\hat{z}_n - \hat{z}'_n|^2}{2\varepsilon_n} + \left(\frac{d(\hat{z}'_n)}{d(z_n)} - 1\right)^4.$$

As u^i_κ, $w^{i,\gamma}_\kappa$ are bounded on $[0,T] \times \bar{\mathcal{S}} \cap \bar{\mathcal{K}}$, then inequality (6.27) implies the boundedness of $(R_n)_{n\geq 1}$. This yields that, there exists a subsequence, also denoted $(R_n)_{n\geq 1}$, which is convergent. As β_n and ε_n go to 0 when n goes to infinity, we must have

$$\hat{t}_0 \; = \; \hat{t}'_0 \text{ and } \hat{z}_0 \; = \; \hat{z}'_0. \tag{6.29}$$

As u^i is upper semi-continuous and $w^{i,\gamma}_\kappa$ is lower semi-continuous, then, by sending n to infinity into (6.26) and (6.28), we obtain

$$\mu^i = u^i_\kappa(t_0, z_0) - w^{i,\gamma}_\kappa(t_0, z_0) \leq u^i_\kappa(\hat{t}_0, \hat{z}_0) - w^{i,\gamma}_\kappa(\hat{t}_0, \hat{z}_0) - |\hat{t}_0 - t_0|^2 - |\hat{z}_0 - z_0|^4 \leq \mu^i,$$

where the last inequality is deduced from the definition of μ^i. It yields that:

$$\hat{t}_0 \; = \hat{t}'_0 \; = \; t_0, \;\; \hat{z}_0 \; = \hat{z}'_0 \; = \; z_0. \tag{6.30}$$

Using again (6.26)-(6.27)-(6.28) and sending n to infinity, we deduce that:

$$\lim_{n\to\infty} \mu^i_n = \mu^i. \tag{6.31}$$

From equality (6.27), we have $\lim_{n\to\infty} \mu^i_n = \mu^i - \lim_{n\to\infty} R_n$, then we deduce that:

$$\lim_{n\to\infty} R_n = 0 \tag{6.32}$$

In particular, for n large enough , we have $\hat{t}_n$, $\hat{t}'_n < T$ (since $t_0 < T$). Besides as $\frac{d(\hat{z}'_n)}{d(z_n)} \to 1$ when n goes to infinity, we have $d(\hat{z}'_n) \geq \frac{d(z_n)}{2}$ for n large enough. It yields that $\hat{z}'_n \in \mathcal{S}$ and consequently $\hat{z}_n \in \mathcal{S}$.
Applying Ishii's lemma (see Thm. 8.3 in [6]) to $(\hat{t}_n, \tilde{t}'_n, \hat{z}_n, \hat{z}'_n) \in [0,T) \times [0,T) \times (\mathcal{S} \cap \mathcal{V}_0) \times (\mathcal{S} \cap \mathcal{V}_0)$ that attains the maximum of Φ^i_n in (6.25), where $\mathcal{V}_0$ is a neighborhood of z_0. We get the existence of two 3×3 symmetric matrices M and N s.t.:

$$\begin{array}{rcl} (s_0, s, M) & \in & \bar{J}^{2,+} u^i_\kappa(\hat{t}_n, \hat{z}_n), \\ (q_0, q, N) & \in & \bar{J}^{2,-} w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n), \end{array}$$

where

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq D^2_{z,z'}\varphi^i_n(\hat{t}_n, \hat{t}^{'}_n, \hat{z}_n, \hat{z}^{'}_n) + \varepsilon_n (D^2_{z,z'}\varphi^i_n(\hat{t}_n, \hat{t}^{'}_n, \hat{z}_n, \hat{z}^{'}_n))^2 \tag{6.33}$$

$$\begin{aligned}
s_0 &= \frac{\partial \varphi^i_n}{\partial t}(\hat{t}_n, \hat{t}^{'}_n, \hat{z}_n, \hat{z}^{'}_n) = 2(\hat{t}_n - t_0) + \frac{(\hat{t}_n - \hat{t}^{'}_n)}{\beta_n},\\
q_0 &= -\frac{\partial \varphi^i_n}{\partial t'}(\hat{t}_n, \hat{t}^{'}_n, \hat{z}_n, \hat{z}^{'}_n) = \frac{(\hat{t}_n - \hat{t}^{'}_n)}{\beta_n},\\
s = (s_k)_{1\leq k\leq 3} &= D_z\varphi^i_n(\hat{t}_n, \hat{t}^{'}_n, \hat{z}_n, \hat{z}^{'}_n) = 4(\hat{z}_n - z_0)|\hat{z}_n - z_0|^2 + \frac{(\hat{z}_n - \hat{z}^{'}_n)}{\varepsilon_n},\\
q = (q_k)_{1\leq k\leq 3} &= -D_{z'}\varphi^i_n(\hat{t}_n, \hat{t}^{'}_n, \hat{z}_n, \hat{z}^{'}_n) = \frac{(\hat{z}_n - \hat{z}^{'}_n)}{\varepsilon_n} - \frac{4}{d(z_n)}\left(\frac{d(\hat{z}^{'}_n)}{d(z_n)} - 1\right)^3 Dd(\hat{z}^{'}_n),
\end{aligned}$$

and

$$D^2_{z,z'}\varphi^i_n(\hat{t}_n, \hat{t}'_n, \hat{z}_n, \hat{z}'_n) = \begin{pmatrix} \frac{1}{\varepsilon_n}I_3 + P_n & -\frac{1}{\varepsilon_n}I_3 \\ -\frac{1}{\varepsilon_n}I_3 & \frac{1}{\varepsilon_n}I_3 + Q_n \end{pmatrix},$$

where

$$\begin{aligned}
P_n &= 4|\hat{z}_n - z_0|^2 I_3 + 8(\hat{z}_n - z_0)(\hat{z}_n - z_0)^\intercal,\\
Q_n &= 12\left(\frac{d(\hat{z}'_n)}{d(\hat{z}_n)} - 1\right)^2 \frac{1}{d(\hat{z}_n)^2} Dd(\hat{z}'_n)Dd(\hat{z}'_n)^\intercal + \frac{4}{d(z_n)}\left(\frac{d(\hat{z}'_n)}{d(\hat{z}_n)} - 1\right)^3 D^2 d(\hat{z}'_n).
\end{aligned}$$

Here $^\intercal$ denotes the transpose operator. By writing the viscosity sub-solution property of u^i_κ and the viscosity super-solution property of $w^{i,\gamma}_\kappa$, we have the following inequalities:

$$\begin{aligned}
&\min\{-s_0 - \mu\hat{x}_n s_1 - \frac{\sigma^2\hat{x}^2_n}{2}M_{11} - f^i(\hat{z}_n) + \rho^i u^i_\kappa(\hat{t}_n, \hat{z}_n), \\
&\quad u^i_\kappa(\hat{t}_n, \hat{z}_n) - \mathcal{M}^i u^i_\kappa(\hat{t}_n, \hat{z}_n)\} \leq 0 \quad \text{if} \quad (\hat{t}_n, \hat{z}_n) \in \mathcal{I}^i, && (6.34)\\
&\min\{-q_0 - \mu\hat{x}^{'}_n q_1 - \frac{\sigma^2\hat{x}^{'2}_n}{2}N_{11} - f^i(\hat{z}^{'}_n) + \rho^i u^i_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n), \\
&\quad w^{i,\gamma}_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n) - \mathcal{M}^i w^{i,\gamma}_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n)\} \geq \gamma\eta \quad \text{if} \quad (\hat{t}^{'}_n, \hat{z}^{'}_n) \in \mathcal{I}^i, && (6.35)\\
&\min\{u^i_\kappa(\hat{t}_n, \hat{z}_n) - \mathcal{M}^i\mathcal{H}^i_\kappa u^i_\kappa(\hat{t}_n, \hat{z}_n), \\
&\quad u^i_\kappa(\hat{t}_n, \hat{z}_n) - \mathcal{H}^i_\kappa u^i_\kappa(\hat{t}_n, \hat{z}_n)\} \leq 0 \quad \text{if} \quad (\hat{t}_n, \hat{z}_n) \in \overline{\mathcal{I}}^i, && (6.36)\\
&\min\{w^{i,\gamma}_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n) - \mathcal{M}^i\mathcal{H}^i_\kappa w^{i,\gamma}_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n), \\
&\quad w^{i,\gamma}_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n) - \mathcal{H}^i_\kappa w^{i,\gamma}_\kappa\} \geq \gamma\eta^{'} \quad \text{if} \quad (\hat{t}^{'}_n, \hat{z}^{'}_n) \in \overline{\mathcal{I}}^i. && (6.37)
\end{aligned}$$

We then distinguish four subcases:
Subcase 1: $u^i_\kappa(\hat{t}_n, \hat{z}_n) - \mathcal{M}^i u^i_\kappa(\hat{t}_n, \hat{z}_n) \leq 0$ in (6.34).
From the definition of $(\hat{t}_n, \hat{t}'_n, \hat{z}_n, \hat{z}'_n)$, we have:

$$\mu^i_n \leq u^i_\kappa(\hat{t}_n, \hat{z}_n) - w^{i,\gamma}_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n)$$

$$\leq \mathcal{M}^i u^i_\kappa(\hat{t}_n, \hat{z}_n) - \mathcal{M}^i w^{i,\gamma}_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n) - \gamma\eta,$$

where the last inequality is deduced from (6.35). Now, letting n going to ∞, and using (6.31), we obtain: we obtain :

$$\begin{aligned}\mu^i &\leq \limsup_{n\to\infty} \mathcal{M}^i u^i_\kappa(\hat{t}_n, \hat{z}_n) - \liminf_{n\to\infty} \mathcal{M}^i w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n) - \gamma\eta \\ &\leq \mathcal{M}^i u^i_\kappa(t_0, z_0) - \mathcal{M}^i w^{i,\gamma}_\kappa(t_0, z_0) - \gamma\eta,\end{aligned}$$

where we used the upper-semicontinuity of $\mathcal{M}^i u^i_\kappa$ and the lower-semicontinuity of $\mathcal{M}^i w^{i,\gamma}_\kappa$ (see Lem. 5.4). As u^i_κ is usc, there exists $\zeta^i \in [\zeta_{min}, \zeta_{max}]$ s.t.
$\mathcal{M}^i u^i_\kappa(t_0, z_0) = u^i_\kappa(t_0, x_0, \Gamma^i(y^i_0, \zeta^i), y^j_0) - \phi^i(y^i_0, \zeta^i)$. We then get

$$\begin{aligned}\mu^i &\leq \mathcal{M}^i u^i_\kappa(t_0, z_0) - \mathcal{M}^i w^{i,\delta}_\kappa(t_0, z_0) - \gamma\eta \\ &\leq u^i_\kappa(t_0, x_0, \Gamma^i(y^i_0, \zeta^i), y^j_0) - w^{i,\delta}_\kappa(t_0, x_0, \Gamma^i(y^i_0, \zeta^i), y^j_0) - \gamma\eta \ \leq \ \mu^i - \gamma\eta,\end{aligned}$$

which is obviously a contradiction.
Subcase 2: $-s_0 - \mu\hat{x}_n s_1 - \frac{\sigma^2\hat{x}_n^2}{2} M_{11} - f^i(\hat{z}_n) + \rho^i u^i_\kappa(\hat{t}_n, \hat{z}_n) \leq 0$ in (6.34). From (6.35), we have $-q_0 - \mu\hat{x}^{'}_n q_1 - \frac{\sigma^2\hat{x}^{'2}_n}{2} N_{11} - f^i(\hat{z}^{'}_n) + \rho^i u^i_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n) \geq \gamma\eta$, which implies in this case

$$\begin{aligned}&-(s_0 - q_0) - \mu(\hat{x}_n s_1 - \hat{x}^{'}_n q_1) - \frac{\sigma^2}{2}(M_{11}\hat{x}^2_n - N_{11}\hat{x}^{'2}_n) - f^i(\hat{z}_n) + f^i(\hat{z}^{'}_n) \\ &+ \rho^i(u^i_\kappa(\hat{t}_n, \hat{z}_n) - u^i_\kappa(\hat{t}^{'}_n, \hat{z}^{'}_n)) \leq -\gamma\eta.\end{aligned} \tag{6.38}$$

We have that

$$s_0 - q_0 = 2(\hat{t}_n - t_0).$$

Since $\hat{t}_n$ goes to t_0 when n goes to infinity, we deduce that $s_0 - q_0$ goes to zero when n goes to infinity. The second term of (6.38) is expressed as follows:

$$\hat{x}_n s_1 - \hat{x}^{'}_n q_1 = 4\hat{x}_n(\hat{x}_n - x_0)|\hat{z}_n - z_0|^2 + 4\frac{(\hat{x}_n - \hat{x}^{'}_n)^2}{\varepsilon_n} + 4\hat{x}^{'}_n \frac{D_x d(\hat{z}^{'}_n)}{d(\hat{z}_n)}\left(\frac{d(\hat{z}^{'}_n)}{d(z_n)} - 1\right)^3,$$

where $D_x d(\hat{z}^{'}_n)$ is the first component of $Dd(\hat{z}^{'}_n)$, which is known to be continuous on $\bar{\mathcal{S}} \setminus D_0$. Using (6.30) and (6.32), we deduce that $\hat{x}_n s_1 - \hat{x}^{'}_n q_1$ goes to 0 when n goes to infinity. Moreover, from (6.33), we have

$$\frac{\sigma^2}{2}(M_{11}\hat{x}^2_n - N_{11}\hat{x}^{'2}_n) \leq \Upsilon_n$$

where

$$\Upsilon_n = \Lambda_n\left(D^2_{z,z'}\varphi_n(\hat{t}_n, \hat{t}'_n, \hat{z}_n, \hat{z}'_n) + \varepsilon_n\left(D^2_{z,z'}\varphi^i_n(\hat{t}_n, \hat{t}^{'}_n, \hat{z}_n, \hat{z}^{'}_n)\right)^2\right)\Lambda^\intercal_n,$$

and

$$\Lambda_n = (\frac{\sigma}{\sqrt{2}}\hat{x}_n, 0, 0, \frac{\sigma}{\sqrt{2}}\hat{x}^{'}_n, 0, 0)$$

After some straightforward calculation, we then get :

$$\Upsilon_n = 3\frac{(\hat{x}_n - \hat{x}'_n)^2}{\varepsilon_n} + \Lambda_n \left(\begin{pmatrix} 3P_n & -P_n - Q_n \\ -P_n - Q_n & 3Q_n \end{pmatrix} + \varepsilon_n \begin{pmatrix} P_n^2 & 0 \\ 0 & Q_n^2 \end{pmatrix} \right) \Lambda_n^{\intercal},$$

which converges also to zero from (6.30) and (6.32). We have that Υ_n goes to zero when n goes to infinity. Sending n goes to infinity in inequality (6.38), we obtain the required contradiction: $0 \leq -\gamma\eta < 0$.
Subcase 3: $u^i(\hat{t}_n, \hat{z}_n) - \mathcal{H}^i_\kappa u^i(\hat{t}_n, \hat{z}_n) \leq 0$ in (6.36).
From the definition of $(\hat{t}_n, \hat{t}'_n, \hat{z}_n, \hat{z}'_n)$, we have

$$\begin{aligned} \mu^i_n &\leq u^i_\kappa(\hat{t}_n, \hat{z}_n) - w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n) \\ &\leq \mathcal{H}^i_\kappa u^i_\kappa(\hat{t}_n, \hat{z}_n) - \mathcal{H}^i_\kappa w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n) - \gamma\eta', \end{aligned}$$

where the last inequality is deduced from (6.37). Now, using the upper-semicontinuity of $\mathcal{H}^i_\kappa u^i_\kappa$ and the lower-semicontinuity of $\mathcal{H}^i_\kappa w^{i,\gamma}_\kappa$ (see Lem. 5.4 (iii)), letting n going to ∞, and using (6.31), we obtain :

$$\begin{aligned} \mu^i &\leq \limsup_{n\to\infty} \mathcal{H}^i_\kappa u^i_\kappa(\hat{t}_n, \hat{z}_n) - \liminf_{n\to\infty} \mathcal{H}^i_\kappa w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n) - \gamma\eta' \\ &\leq \mathcal{H}^i_\kappa u^i_\kappa(t_0, z_0) - \mathcal{H}^i_\kappa w^{i,\gamma}_\kappa(t_0, z_0) - \gamma\eta \ \leq \ \mu^i - \gamma\eta', \end{aligned}$$

which is obviously a contradiction.
Subcase 4: $u^i_\kappa(\hat{t}_n, \hat{z}_n) - \mathcal{M}^i\mathcal{H}^i_\kappa u^i_\kappa(\hat{t}_n, \hat{z}_n) \leq 0$ in (6.34).
From the definition of $(\hat{t}_n, \hat{t}'_n, \hat{z}_n, \hat{z}'_n)$, we have

$$\begin{aligned} \mu^i_n &\leq u^i_\kappa(\hat{t}_n, \hat{z}_n) - w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n) \\ &\leq \mathcal{M}^i\mathcal{H}^i_\kappa u^i_\kappa(\hat{t}_n, \hat{z}_n) - \mathcal{M}^i\mathcal{H}^i_\kappa w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n) - \gamma\eta', \end{aligned}$$

where the last inequality is deduced from (6.37). Now, letting n going to ∞, and using (6.31), we obtain:

$$\begin{aligned} \mu^i &\leq \limsup_{n\to\infty} \mathcal{M}^i\mathcal{H}^i_\kappa u^i_\kappa(\hat{t}_n, \hat{z}_n) - \liminf_{n\to\infty} \mathcal{M}^i\mathcal{H}^i_\kappa w^{i,\gamma}_\kappa(\hat{t}'_n, \hat{z}'_n) - \gamma\eta \\ &\leq \mathcal{M}^i\mathcal{H}^i_\kappa u^i_\kappa(t_0, z_0) - \mathcal{M}^i\mathcal{H}^i_\kappa w^{i,\gamma}_\kappa(t_0, z_0) - \gamma\eta', \end{aligned}$$

where we used the upper-semicontinuity of $\mathcal{M}^i\mathcal{H}^i_\kappa u^i_\kappa$ and the lower-semicontinuity of $\mathcal{M}^i\mathcal{H}^i_\kappa w^{i,\gamma}_\kappa$ (see Lem. 5.4).
As u^j_κ is usc, there exists $\zeta^j \in [\zeta_{min}, \zeta_{max}]$ s.t. $\mathcal{M}^j u^i_\kappa(t_0, z_0) =$
$u^i_\kappa(t_0, x_0, y^i_0, \Gamma^j(y^j_0, \zeta^j))$, and then $\mathcal{H}^i_\kappa u^i_\kappa(t_0, z_0) = u^i_\kappa(t_0, x_0, y^i_0, \Gamma^j(y^j_0, \zeta^j)) - \kappa$. As u^i_κ is usc, there exists $\zeta^i \in [\zeta_{min}, \zeta_{max}]$ s.t. $\mathcal{M}^i\mathcal{H}^i_\kappa u^i_\kappa(t_0, z_0) = u^i_\kappa(t_0, x_0, \Gamma^i(y^i_0, \zeta^i), \Gamma^j(y^j_0, \zeta^j)) - \phi(y^i_0, \zeta^i) - \kappa$. We then get

$$\begin{aligned} \mu^i &\leq \mathcal{M}^i\mathcal{H}^i_\kappa u^i_\kappa(t_0, z_0) - \mathcal{M}^i\mathcal{H}^i_\kappa w^{i,\delta}_\kappa(t_0, z_0) - \gamma\eta \\ &\leq u^i_\kappa(t_0, x_0, \Gamma^i(y^i_0, \zeta^i), \Gamma^j(y^j_0, \zeta^j)) - w^{i,\delta}_\kappa(t_0, x_0, \Gamma^i(y^i_0, \zeta^i), \Gamma^j(y^j_0, \zeta^j)) - \gamma\eta' \ \leq \ \mu^i - \gamma\eta, \end{aligned}$$

which is obviously a contradiction.
• *Case 2.* : $z_0 \in \mathcal{S} \cap \mathcal{K}$ We consider the function

$$\begin{aligned} \Phi^i_n(t, z, z') &= u^i(t, z) - w^{i,\gamma}(t, z') - \varphi^{i,\gamma}_n(t, z, z') \\ \varphi^{i,\gamma}_n(t, z, z') &= |t - t_0|^2 + |z - z_0|^4 + \frac{n}{2}|z - z'|^2, \end{aligned}$$

for $n \geq 1$, and to take a maximum $(\tilde{t}_n, \tilde{z}_n, \tilde{z}'_n)$ of Φ^i_n. We then show that the sequence $(\tilde{t}_n, \tilde{z}_n, \tilde{z}'_n)_{n\geq 1}$ converges to (t_0, z_0, z_0) as n goes to infinity and we apply Ishii's lemma to get the required contradiction.

□

The condition (6.20) in the comparison theorem that must be satisfied is not obvious to check for v^i, $i \in \{1,2\}$. To circumvent this difficulty, we introduce the following function:

$$\hat{v}^i(t,z) := \mathbb{E}[\int_t^T \mathrm{e}^{-\rho^i(s-t)} f^i(Z_s^{t,z})\mathrm{d}s + \mathrm{e}^{-\rho^i(T-t)} g^i(Z_T^{t,z})],\ \forall\,(t,z) \in [0,T) \times \bar{\mathcal{S}}, \tag{6.39}$$

where $(Z_s^{t,z})_{t\leq s\leq T}$ is the state process associated with the no impulse strategy. The next result shows that the equilibrium expected payoff $\hat{v}^i$ is continuous on the part $[0,T) \times D_0$ of the solvency region.

Proposition 6.5. *For all $i \in \{1,2\}$, we have:*

$$\lim_{(t^{'},z^{'})\longrightarrow(t,z)} \hat{v}^i(t^{'},z^{'}) = \hat{v}^i(t,z) \ \textit{ for all } \ (t,z) \in [0,T) \times D_0, \tag{6.40}$$

and $\hat{v}^i$ is a classical solution of (6.1)-(6.2) in a neighborhood of $[0,T) \times D_0$.

Proof. We fix $i \in \{1,2\}$ and $(t,z) \in [0,T) \times D_0$. We consider $\mathring{B}((t,z),\delta) \cap \bar{\mathcal{S}}$ a neighborhood of (t,z), where $\mathring{B}((t,z),\delta) := \{(t^{'},z^{'}) \text{ s.t. } |z - z^{'}| + |t - t^{'}| < \delta\}$ and δ is a positive constant.

Step 1: We prove the continuity property (6.40). From the definition of the process $(Z_s^{t^{'},z^{'}})_{t^{'}\leq s\leq T}$ with no impulse strategy, we have:

$$\begin{aligned} Z_s^{t^{'},z^{'}} &= (X_s^{t^{'},x^{'}}, Y_s^{1,t^{'},y^{'1}}, Y_s^{2,t^{'},y^{'2}}) = (X_s^{t^{'},x^{'}}, y^{'1}, y^{'2}) = (x^{'}\mathrm{e}^{(\mu-\frac{\sigma^2}{2})(s-t^{'})+\sigma(W_s-W_{t^{'}})}, y^{'1}, y^{'2}) \\ &\longrightarrow\ (X_s^{t,x}, y^1, y^2)\mathrm{d}t \otimes \mathrm{d}\mathbb{P},\ \text{when } (t^{'},z^{'}) \longrightarrow (t,z). \end{aligned}$$

We fix $p > 1$. From Remark 2.2, for $h^i = f^i, g^i$, we have $|h^i(z)| \leq C^i(1+x)$, which implies

$$\begin{aligned} &\mathbb{E}[\left(\int_{t^{'}}^T f^i(Z_s^{t^{'},z^{'}})\mathrm{d}s + g^i(Z_T^{t^{'},z^{'}})\right)^p] \\ &\leq C\left(\mathbb{E}[\left(\int_{t^{'}}^T f^i(Z_s^{t^{'},z^{'}})\mathrm{d}s\right)^p] + \mathbb{E}[g^i(Z_T^{t^{'},z^{'}})^p]\right) \\ &\leq C\mathbb{E}[\int_{t^{'}}^T f^i(Z_s^{t^{'},z^{'}})^p ds](T-t^{'})^{p-1} + \mathbb{E}[g^i(Z_T^{t^{'},z^{'}})^p]) \\ &\leq C(1+T^p)(1+\mathbb{E}[\sup_{s\in[t^{'},T]} |X_s^{t^{'},x^{'}}|^p]), \end{aligned} \tag{6.41}$$

where the second inequality is obtained by using Hölder inequality and C is a generic constant independent of $(t^{'},z^{'})$. From the definition of the wholesale price and for $(t^{'},z^{'}) \in \mathring{B}((t,z),\delta)$, we have

$$\mathbb{E}[\sup_{s\in[t^{'},T]} |X_s^{t^{'},z^{'}}|^p] \leq C|x^{'}|^p \leq C(1+|x|^p). \tag{6.42}$$

From (6.41)-(6.42), we deduce

$$\mathbb{E}[\left(\int_{t'}^{T} f^i(Z_s^{t',z'})\mathrm{d}s + g^i(Z_T^{t',z'})\right)^p] \leq C(1+|x|^p),$$

where C is a positive constant which depends on δ. This shows the boundedness of $\left(\int_{t'}^{T} f^i(Z_s^{t',z'})\mathrm{d}s + g^i(Z_T^{t',z'})\right)$ in $L^p(\mathbb{P})$ for $p > 1$, which implies the uniform integrability of $\left(\int_{t'}^{T} f^i(Z_s^{t',z'})\mathrm{d}s + g^i(Z_T^{t',z'})\right)_{t',z'}$. It yields that $\hat{v}^i(t',z') \longrightarrow \hat{v}^i(t,z)$ when $(t',z') \longrightarrow (t,z)$.

Step 2: We show that $\hat{v}^i$ is the solution of the IQV (2.12) on $\mathring{B}((t,z),\delta') \cap \bar{\mathcal{S}}$ for some δ'. From equation (6.39), and since the process $Z^{t',z'}$ is Log-normal, then $\hat{v}^i$ is regular and we have:

$$-\frac{\partial \hat{v}^i}{\partial t}(t',z') - \mathcal{L}\hat{v}^i(t',z') + \rho^i \hat{v}^i(t',z') - f^i(z') = 0,$$

on $\mathring{B}((t,z),\delta) \cap \bar{\mathcal{S}}$. It remains to prove that $\hat{v}^i(t',z') > \mathcal{M}^i\hat{v}^i(t',z')$ in a neighborhood of (t,z). We argue by contradiction. We assume that for all $\delta > 0$, there exists $(t'',z'') \in \mathring{B}((t,z),\delta) \cap \bar{\mathcal{S}}$ such that $\hat{v}^i(t'',z'') \leq \mathcal{M}^i\hat{v}^i(t'',z'')$. Sending δ towards 0, using Lemma 5.4, and the continuity of $\hat{v}^i$ w.r.t (t,z), we obtain:

$$\hat{v}^i(t,z) \leq \limsup_{(t'',z'')\longrightarrow(t,z)} \mathcal{M}^i\hat{v}^i(t'',z'') \leq \mathcal{M}^i\hat{v}^i(t,z). \tag{6.43}$$

On the other hand, as $(t,z) \in [0,T) \times D_0$, we have

$$\hat{v}^i(t,z) = \begin{cases} 0 & \text{if } (t,z) \in [0,T) \times \partial^{y^1}\mathcal{S} \cup \partial^{y^2}\mathcal{S}, \\ -\frac{x}{2}\left(\frac{\mathrm{e}^{(\mu-\rho^i)(T-t)}-1}{\mu-\rho^i} + \mathrm{e}^{(\mu-\rho^i)(T-t)}\right) & \text{if } (t,z) \in [0,T) \times \partial^x\mathcal{S}. \end{cases}$$

By straight forward computation, we have $\hat{v}^i(t,z) > \mathcal{M}^i\hat{v}^i(t,z)$, which means that there is no intervention in (t,z), and so the contradiction is obtained. Symmetrically, In a neighborhood of $(t,z) \in [0,T) \times D_0$, we have $\hat{v}^j(t',z') > \mathcal{M}^j\hat{v}^j(t',z')$ for $j \neq i$, which means, against the no intervention strategy of player i, the best response of player j is also not to make an intervention. This shows that there exists a neighborhood of $[0,T) \times D_0$ which is included in the continuation region and $\hat{v}^i$ is a regular solution of (6.2) in this neighborhood. □

Proposition 6.6. *(i)For all $i \in \{1,2\}$, we have:*

$$\overline{v}_\kappa^i(t,z) \leq \liminf_{(t',z')\longrightarrow(t,z)} \underline{v}_\kappa^i(t',z'),\ \forall (t,z) \in [0,T) \times D_0, \tag{6.44}$$

(ii)For all $i \in \{1,2\}$, we have:

$$\overline{v}_\kappa^i(t,z) = \underline{v}_\kappa^i(t,z) = v_\kappa^i(t,z),\ \forall (t,z) \in [0,T) \times D_0, \tag{6.45}$$

Proof. **Step 1:** We proved that the equilibrium expected payoff v_κ^i is a viscosity sub-solution of (6.1)-(6.2) in $[0,T) \times \bar{\mathcal{S}}$, then for all $(t,z) \in [0,T) \times D_0$ and $\varphi^i \in C^{1,2}([0,T) \times \bar{\mathcal{S}})$ s.t. $(\overline{v}_\kappa^i - \varphi^i)(t,z) = 0$ and (t,z) is a maximum of $\overline{v}_\kappa^i - \varphi^i$ on $[0,T) \times D_0$, we have

$$\min\{-\frac{\partial \varphi^i}{\partial t}(t,z) - \mathcal{L}\varphi^i(t,z) + \rho^i\varphi^i(t,z) - f^i(z), \overline{v}_\kappa^i(t,z) - \mathcal{M}^i\overline{v}_\kappa^i(t,z)\} \leq 0 \,\text{in}\, [0,T) \times D_0.$$

As $(t,z) \in [0,T) \times D_0$, by straight forward computation, we have $v^i_\kappa(t,z) > \mathcal{M}^i v^i_\kappa(t,z)$, where v^i_κ is given by equation (6.4), which means that there is no intervention in (t,z). It yields that

$$-\frac{\partial \varphi^i}{\partial t}(t,z) - \mathcal{L}\varphi^i(t,z) + \rho^i \varphi^i(t,z) - f^i(z) \leq 0, \text{ in } [0,T) \times D_0. \tag{6.46}$$

For the terminal condition, we take

$$\varphi^i(T,z) \leq g^i(z). \tag{6.47}$$

Tow cases are possible:
⋆ First case: $z \in \partial^{y^j}\mathcal{S} \cup \partial^{y^i}\mathcal{S}$, inequalities (6.46)-(6.47) become

$$-\frac{\partial \varphi^i}{\partial t}(t,z) + \rho^i \varphi^i(t,z) \leq 0, \text{ in } [0,T) \times \partial^{y^j}\mathcal{S} \cup \partial^{y^i}\mathcal{S},$$

and

$$\varphi^i(T,z) \leq 0 = v^i_\kappa(T,z) = \hat{v}^i(T,z).$$

which implies $\varphi^i(t,z) \leq 0 = \hat{v}^i(t,z)$ for all $(t,z) \in [0,T) \times \partial^{y^j}\mathcal{S} \cup \partial^{y^i}\mathcal{S}$
⋆ Second case: $z \in \partial^x\mathcal{S}$, inequalities (6.46)-(6.47) become

$$-\frac{\partial \varphi^i}{\partial t}(t,z) - \mathcal{L}\varphi^i(t,z) + \rho^i \varphi^i(t,z) - f(t,z) \leq 0, \text{ in } [0,T) \times \partial^x\mathcal{S} \setminus (0,0,0), \tag{6.48}$$

and

$$\varphi^i(T,z) \leq 0. \tag{6.49}$$

For the boundary condition, we take

$$\varphi^i(t,0) \leq 0 = v^i_\kappa(t,0) = \hat{v}^i(t,0), \text{ in } [0,T). \tag{6.50}$$

On the other hand, $\hat{v}^i$ satisfies (6.48)-(6.50) with equalities. By classical comparison theorem, we deduce that

$$\overline{v}^i_\kappa(t,z) \leq \varphi^i(t,z) \leq \hat{v}^i(t,z), \text{ in } [0,T) \times D_0. \tag{6.51}$$

Step 2: By definition of $\hat{v}^i$ and since against the no intervention strategy of player i, the best response of player j is also not to make an intervention, we have $\hat{v}^i(t,z) \leq v^i_\kappa(t,z)$ for all (t,z) in a neighborhood of $[0,T) \times D_0$. From Proposition 6.5, the function $\hat{v}^i$ is continuous. It yields that:

$$\hat{v}^i(t',z') \leq \underline{v}^i_\kappa(t',z') \text{ for all } (t',z') \text{ in a neighborhood of } (t,z) \in [0,T) \times D_0.$$

Using again the continuity property of $\hat{v}^i$ (See Prop. 6.5), and since $\underline{v}^i_\kappa$ is lsc, we obtain:

$$\hat{v}^i(t,z) \leq \liminf_{(t',z') \longrightarrow (t,z)} \underline{v}^i_\kappa(t',z') = \underline{v}^i_\kappa(t,z) \text{ for all } (t,z) \in [0,T) \times D_0. \tag{6.52}$$

From inequalities (6.51), (6.52) and , we deduce inequality (6.44) and the continuity property of v^i_κ in the boundary (6.45). □

Finally, combining the previous results, we obtain the following PDE characterization of the equilibrium expected payoff.

Corollary 6.7. *The equilibrium expected payoff v^i_κ is continuous on $[0,T)\times\mathcal{S}$ and is the unique (in $[0,T)\times\mathcal{S}$) constrained viscosity solution to the system of QVIs (6.1)-(6.2) lying in the class of functions with linear growth in x uniformly in (t,y^i,y^j) and satisfying the boundary condition :*

$$\lim_{(t',z')\to(t,z)} v^i_\kappa(t',z') = \begin{cases} 0 & \text{if } (t,z)\in[0,T)\times\partial^{y^1}\mathcal{S}\cup\partial^{y^2}\mathcal{S},\\ -\frac{x}{2}\big(\frac{\mathrm{e}^{(\mu-\rho^i)(T-t)}-1}{\mu-\rho^i}+\mathrm{e}^{(\mu-\rho^i)(T-t)}\big) & \text{if } (t,z)\in[0,T)\times\partial^{x}\mathcal{S}, \end{cases}$$

and the terminal condition

$$v^i_\kappa(T,z)=g^i(z),\quad \forall z\in\bar{\mathcal{S}}.$$

Proof. We have $\bar{v}^i_\kappa$ is an usc viscosity sub-solution to (6.1)-(6.2) in $[0,T)\times\bar{\mathcal{S}}$ and $\underline{v}^i_\kappa$ is a lsc viscosity super-solution to (6.1)-(6.2) in $[0,T)\times\mathcal{S}$. Moreover, by Proposition 6.6 and Proposition 3.3, we have $\bar{v}^i_\kappa(t,z)\le \liminf_{(t',z')\longrightarrow(t,z)} \underline{v}^i_\kappa(t',z')$, for all $(t,z)\in[0,T)\times D_0$, and $\bar{v}^i_\kappa(T,z)=\underline{v}^i_\kappa(T,z)=g^i(z)$ for all $z\in\bar{\mathcal{S}}$. Then by Theorem 6.4, we deduce $\bar{v}^i_\kappa\le\underline{v}^i_\kappa$ on $[0,T]\times\mathcal{S}$, which proves the continuity of v^i_κ on $[0,T)\times\mathcal{S}$. On the other hand, suppose that w^i_κ is another constrained viscosity solution to (6.1)-(6.2) with

$$\lim_{(t',z')\to(t,z)} w^i_\kappa(t',z') = w^i_\kappa(t,z) = v^i_\kappa(t,z),\quad \text{for all } (t,z)\in[0,T)\times D_0,$$

and $w^i_\kappa(T,z)=g^i(z)$ for $z\in\bar{\mathcal{S}}$. Then, $\bar{w}^i_\kappa(t,z)=\underline{v}^i_\kappa(t,z)=\bar{v}^i_\kappa(t,z)=\underline{w}^i_\kappa(t,z)$ for $(t,z)\in[0,T)\times D_0$ and $\bar{w}^i_\kappa(T,z)=\underline{v}^i_\kappa(T,z)=\bar{v}^i_\kappa(T,z)=\underline{w}^i_\kappa(T,z)$ for z in $\bar{\mathcal{S}}$. We then deduce by Theorem 6.4 that $\bar{v}^i_\kappa\le\underline{w}^i_\kappa\le\bar{w}^i_\kappa\le\underline{v}^i_\kappa$ on $[0,T]\times\mathcal{S}$. This proves $v^i_\kappa=w^i_\kappa$ on $[0,T]\times\mathcal{S}$.

□

7. Numerical illustrations

In this section we provide some numerical results describing the equilibrium expected payoffs of the players and their candidate equilibrium policies. A forward computation of the equilibrium expected payoff and the candidate equilibrium strategy is in our knowledge impossible due to the high dimension of the state process and the complexity of our model, therefore we used a numerical scheme based on a quantization technique (see [17]). The convergence of the numerical solution towards the real solution can be shown using consistency, monotonicity and stability arguments and will be further investigated in a future work. A detailed description of the numerical algorithm can be found in the Appendix.

Numerical tests are performed on the localized and discretized grid $\{0,..,T\}\times\{x_{min},..,x_{max}\}\times\{y^1_{min},..,y^1_{max}\}\times\{y^2_{min},..,y^2_{max}\}$. We used the following values for the parameters of the model: $T=1$, $\mu=0$, $\sigma=0.5$, $\zeta_{min}=-2.2$, $\zeta_{max}=1.8$, $x_{min}=y^1_{min}=y^2_{min}=10$, $x_{max}=y^1_{max}=y^2_{max}=90$, $\lambda=0.1$ and $g^1=f^1$ and $g^2=f^2$, $\rho_1=\rho_2=0$, $\phi_1=5$, $\phi_2=2.5$. Besides, the running costs are $f^1(x,y^1,y^2)=(y^1-x)Q(y^1-y^2)$ and $f^2(x,y^1,y^2)=(y^2-x)Q(y^2-y^1)$ where

$$Q(x)=\mathbb{1}_{]-\infty,-\Delta]}-\frac{x-\Delta}{2\Delta}\mathbb{1}_{[-\Delta,\Delta]},$$

with $\Delta=40$. Further, the terminal payoffs are chosen such that $g^1(x,y^1,y^2)=f^1(x,y^1,y^2)$ and $g^2(x,y^1,y^2)=f^2(x,y^1,y^2)$. We begin by computing the equilibrium expected payoff and the candidate equilibrium strategy of each player by solving the system of QVIs using the Algorithm 1 (see Appendix).

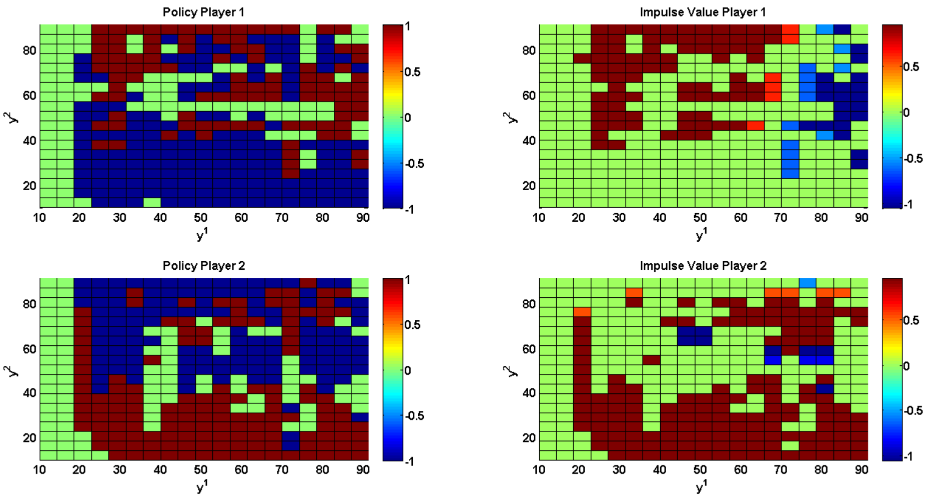

FIGURE 1. *The optimal policies for a fixed $(t, x) = (,)$ for the first player (First Line) and the second player (Second Line). Color code: red: concerned player intervenes, green: concerned player waits, blue: concerned player endures the intervention of the other player.*

First, the Figure 1 presents the candidate equilibrium transaction policy for the two players, *i.e.* the different regions of interventions and continuations in the plane (y^1, y^2) for $t = 0.5$ and $x = 50$ €MWh^{-1}. The first line (resp. second line) of Figure 1 corresponds to the candidate equilibrium policy regions and the corresponding interventions of the player 1 (resp. player 2). In the first column we can distinguish, for both of the players, three different regions, represented by three different colors, corresponding to the candidate equilibrium action given a state (y^1, y^2). Indeed, the blue region represents the states (y^1, y^2) where a player is subject to the intervention of the other player, the green regions represents the states where a player chooses to not intervene and the red region represents the states where the player makes an intervention. The second column represents, whenever a player decides to intervene, the size of the intervention. If the quantity is positive it means that the price is increased and if it is negative it means that the price is lowered.

We can see that, as expected, both the players tend to keep the price spread $|y^1 - y^2|$ as low as possible in order to avoid market share losses. In fact, for instance, at the state $(y^1 = 85, y^2 = 60)$, player 1 chooses to push down her price to keep an acceptable market share position. On the other hand, at the state $(y^1 = 30, y^2 = 70)$, player 1 chooses to push up her price which allows her to make benefits whilst keeping a reasonable market share position.

Second, the Figure 2 (Left) gives an example of a trajectory of the wholesale electricity price X together with the corresponding retail prices trajectories Y^1 and Y^2 of the two players, where the initial state is $(X_0 = 30, Y_0^1 = 40, Y_0^2 = 35)$. Formulas (4.1)-(4.2) proved in the verification theorem (See Prop. 4.1) are crucial to obtain the prices proposed by the retailors. As a matter of comparison, Figure 2 (Right) shows the trajectories of the wholesale price of electricity and retail prices of the six largest energy providers in the UK from January 2004 to March, 2010. We observe several comparable features of the candidate equilibrium retailers price resulting from our impulse game and the real-life experience. Increases in the wholesale price is not immediately followed by an increase in retail prices. There is a delay given by the optimal time to reach the boundary of the action region. Further, even if our model only involves two players, we observe that they do not intervene at the same time, as it is the case in the UK market example. However, they appear to follow an almost synchronized behavior: an increase by a first player is mostly to be followed by an increase of the second player and not by a decrease. Further, the equilibrium trajectories of the retail prices can be increasing while the wholesale price is decreasing (from 0.2 to 0.3 for instance), a phenomenon which is also observed in the UK case (from April, 2006 to March 2007, for instance). The equilibrium trajectories can also decrease, even if these decreases are limited compared to the same reference case of the UK market. Thus, contrary to the belief of the UK energy regulator, the

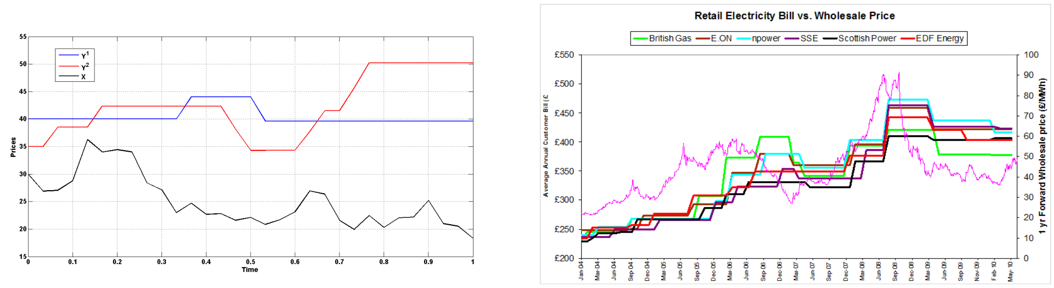

FIGURE 2. (Left) One path-scenario of the wholesale market price and the players' retail prices. (Right) Retail electricity bill compared to wholesale price in the UK (source Ofgem).

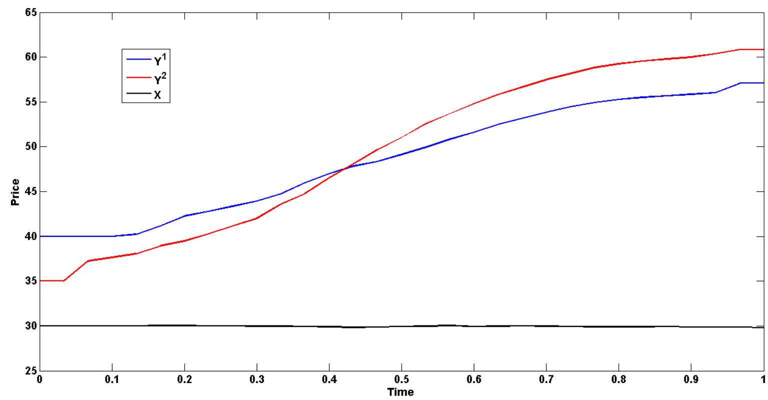

FIGURE 3. *The average trajectory of the market price and the players' prices.*

Ofgem[1], the observed behavior of almost synchronized increase and decrease of retailers prices might not be the result of a tacit collusion mechanism, but is simply the result of decisions in a Nash equilibrium. Finally, the Figure 3 shows the average trajectories of the market price and the players' candidate equilibrium retail price processes over ten thousand simulated trajectories of X, Y^1 and Y^2 on the horizon $[0, T]$. The initial state is the same as in the Figure 2. We notice that, although the wholesale price X is a martingale, the retail prices offered by the two players are increasing. In addition to this observation, we note that the players have almost the same tendency as they try to keep a balanced market share configuration until the maturity. With our choice of parameters, we observe that player 2 starts with a price lower than the player 1's price and attains the maturity with a higher price. This is because the interventions for the player 2 are less expensive making her more dynamic. We can also observe that, throughout the time period, the price spread between the two players is quite small preventing the market share to be imbalanced.

Our model suggests that the players would rather propose increasing prices to maximize their profit. This result might be surprising as one would expect that the players would stick to the wholesale price tendency and would propose a mean constant prices. But, in our model the market shares are split between the two players only according to the difference in the price they offer: consumers do not have an outside option to switch to another energy and no market entry of a competitor may threaten the two players for practicing increasing

[1]The British energy regulator launched an inquiry on energy retailers in 2014. The headline findings of the assessment were: *(...) Possible tacit co-ordination: The assessment has not found evidence of explicit collusion between suppliers. However, there is evidence of possible tacit coordination reflected in the timing and size of price announcements and new evidence that prices rise faster when costs rise than they reduce when costs fall. Although tacit coordination is not a breach of competition law, it reduces competition and worsens outcomes for consumers.* Published on Ofgem website on June 26th, 2014, at the address: www.ofgem.gov.uk/press-releases/ofgem-refers-energy-market-full-competition-investigation.

prices. The thing we find remarkable in this result is that without setting any potential communication device between the two players, we observe on average a behavior that looks like tacit collusion.

Appendix

In the following, we give a detailed description of the numerical procedure used to compute the equilibrium expected payoff and the candidate equilibrium policies associated to the impulse game problem. We used a numerical scheme based on a quantization technique (see [17]) mixed with an iterative procedure. The convergence of the numerical solution towards the real solution can be shown using consistency, monotonicity and stability arguments and will be further investigated in a future work.
For a time step $h > 0$ on the interval $[0, T]$, we introduce a numerical backward Algorithm that approximates the solution of the HJB-QVI system by computing the couple of functions $v_h^i, i = 1, 2$ through:

$$\begin{cases} v_h^i(t,z) \ = \max(\mathcal{M}^i\mathcal{H}_\kappa^i v_h^i(t,z), \mathcal{H}_\kappa^i v_h^i(t,z)) \quad \text{in} \quad \overline{\mathcal{I}^i} \\ v_h^i(t,z) \ = \max\left[\mathbb{E}[v_h^i(t+h, Z_{t+h}^{0,t,z})] + \Sigma_i(t,z), \mathcal{M}^i v_h^i(t,z)\right] \text{in} \ \ \mathcal{I}^i \\ v_h^i(T,z) = g^i(z), \quad \text{in} \ \ \mathcal{S}, \end{cases} \tag{A.1}$$

where

$$\Sigma_i(t,z) = \int_t^{t+h} f^i(Z_s^{t,z})\mathrm{d}s.$$

This approximation scheme seems a priori implicit due to the nonlocal obstacle terms $\mathcal{M}^i$ and $\mathcal{H}^i$. This is typically the case in impulse control problems, and the usual way to circumvent this problem is to iterate the scheme by considering a sequence of optimal stopping problems (see [18], [19] or [20]) which gives us the following explicit procedure:

$$\begin{cases} v_{h,n+1}^i(t,z) \ = \max(\mathcal{M}^i\mathcal{H}_\kappa^i v_{h,n}^i(t,z), \mathcal{H}_\kappa^i v_{h,n}^i(t,z)) \quad \text{in} \quad \overline{\mathcal{I}_i} \\ v_{h,n+1}^i(t,z) \ = \max\left[\mathbb{E}[v_{h,n+1}^i(t+h, Z_{t+h}^{0,t,z})] + \Sigma_i(t,z), \mathcal{M}^i v_{h,n}^i(t,z)\right] \quad \text{in} \ \ \mathcal{I}^i \\ v_{h,n+1}^i(T,z) = g^i(z) \quad \text{in} \ \ \mathcal{S}. \end{cases} \tag{A.2}$$

with

$$v_{h,0}^i(t,z) = \mathbb{E}[\int_t^T f^i(Z_s^{0,t,z})\mathrm{d}s + g^i(Z_T^{0,t,z})]$$

A.1 Time and Space discretization

- Now let us consider the time grid $\mathbb{T} := \{t_k = kh; \quad k = 0,..,m; \quad h = \frac{T}{m}\}$ and $m \in \mathbb{N} \setminus \{0\}$.
- Let $\mathbb{X}$ the uniform grid on $[x_{min}, x_{max}]$ of step $\mathrm{d}x = \frac{x_{max} - x_{min}}{(N_x - 1)}$, where $0 < x_{min} < x_{max}$ and $N_x > 0$. For $j = 0, ..., N_x$, we denote $x_j := x_{min} + j\mathrm{d}x$.
- For $i \in \{1, 2\}$, let $\mathbb{Y}_i$ the discrete uniform grid on $[y_{min}^i, y_{max}^i]$ of step $\mathrm{d}y_i = \frac{y_{max}^i - y_{min}^i}{(N_y - 1)}$, where $0 < y_{min}^i < y_{max}^i$. For $j = 0, ..., N_y$, we denote $y_j^i := y_{min}^i + jdy_i$.

Let $z_j = (x_j, y_j^1, y_j^2) \in \mathbb{G} := \mathbb{X} \times \mathbb{Y}_1 \times \mathbb{Y}_2$, starting from a pair (v_0^1, v_0^2) two fixed vectors we define the following problem:

$$\begin{cases} v_{h,n+1}^i(t_k, z_j) = max(\mathcal{M}^i\mathcal{H}_\kappa^i v_{h,n}^i(t_k, z_j), \mathcal{H}_\kappa^i v_{h,n}^i(t_k, z_j)) \quad \text{in} \ \overline{\mathcal{I}^i} \cap \mathbb{T} \times \mathbb{G} \\ v_{h,n+1}^i(t_k, z_j) = max\left[\mathbb{E}[v_{h,n+1}^i(t_{k+1}, Z_{t_{k+1}}^{0,t_k,z_j})] + \Sigma_i(t_k, z_j), \mathcal{M}^i v_{h,n}^i(t_k, z_j)\right] \ \text{in} \ \mathcal{I}^i \cap \mathbb{T} \times \mathbb{G} \\ v_{h,n+1}^i(T, z_j) \ = g^i(z_j) \ \text{in} \ \mathcal{S} \cap \mathbb{G}. \end{cases} \tag{A.3}$$

A.2 Localization

We define the global localization bound R as follows

$$R = \min(\frac{1}{x_{min}}, \frac{1}{y^1_{min}}, \frac{1}{y^1_{min}}, x_{max}, y^1_{max}, y^2_{max}).$$

Similarly, for $i \in \{1, 2\}$, we define the discrete uniform grid of the admissible interventions:

$$C^i_{M,R}(z) = \{\zeta_k = \zeta_{min} + \frac{k}{M}(\zeta_{max} - \zeta_{min}); 0 \le k \le M;\ s.t.\ \Gamma^i(y^i, \zeta_k) \in \mathcal{S}_{loc}\}, .$$

where $M \in \mathbb{N}^*$ is a fixed constant and

$$\mathcal{S}_{loc} := \mathcal{S} \cap ([x_{min}, x_{max}] \times [y^1_{min}, y^1_{max}] \times [y^2_{min}, y^2_{max}]).$$

is the localized space domain.

We also consider the following projection

$$\begin{aligned} \Pi_R : \mathbb{R}_+ &\to [x_{min}, x_{max}] \\ x &\mapsto x_{min}\mathbb{1}_{[0,x_{min})} + x\mathbb{1}_{[x_{min},x_{max}]} + x_{max}\mathbb{1}_{(x_{max},+\infty)} \end{aligned}$$

Remark A.1. • The projection operator Π_R is crucial to avoid the case where the space state could be outside the localized domain $\mathcal{S}_{loc}$.

• We consider the following discrete and localized impulse operator

$$\begin{aligned} \mathcal{M}^{M,R}_i v^i(t, z) &:= \sup_{\zeta^i \in C^i_{M,R}(z)} \Big\{ v^i(t, x, \Gamma^i(y^i, \zeta^i), y^j) - \phi^i(z, \zeta^i) \Big\}, \\ \mathcal{H}^{M,R}_i v^i(t, z) &:= v^i(t, x, y^i, \Gamma^j(y^j, \zeta^j)) - \kappa, \end{aligned}$$

and $\zeta^j = \mathrm{argmax}_{\zeta \in C^j_{M,R}(z)} \Big\{ v^j(t, y^i, \Gamma^j(y^j, \zeta)) - \phi^j(z, \zeta) \Big\}$.

A.3 Quantization of the Brownian Motion

To compute the conditional expectations arising in the numerical backward scheme, we use the optimal quantization method. The main idea is to use the quantization theory to construct a suitable approximation of the Brownian motion.

It is known that there exists a unique strong solution for the SDE, $\frac{dX^{t,x}_s}{X^{t,x}_s} = \mu \mathrm{d}s + \sigma dW_s$. So it suffices to consider a quantization of the Brownian motion itself.

Recall that the optimal quantization technique consists in approximating the expectation $\mathbb{E}[f(Z)]$, where Z is a normal distributed variable and f is a given real function, by

$$\mathcal{E}[f(\xi)] = \sum_{k \in \xi(\Omega)} f(k)\mathbb{P}(\xi = k)\ .$$

The distribution of the discrete variable ξ is known for a fixed $N := card(\xi(\Omega))$ and the approximation is optimal as the L^2-error between ξ and Z is of order $1/N$ (see [17]). The quantized dynamic programming backward scheme is defined as follows:

$$\begin{cases} v^i_{h,n+1}(t_k, z_j) = \max(\mathcal{M}^{M,R}_i \mathcal{H}^{M,R}_i v^i_{h,n}(t_k, z_j), \mathcal{H}^{M,R}_i v^i_{h,n}(t_k, z_j)) \quad \text{in } \overline{\mathcal{I}^i} \cap \mathbb{T} \times \mathbb{G} \\ v^i_{h,n+1}(t_k, z_j) = \max\Big[\mathcal{E}^{N,R}[v^i_{h,n+1}(t_{k+1}, Z^{0,t_k,z_j}_{t_{k+1}}))] + \Sigma_i(t_k, z_j), \mathcal{M}^{M,R}_i v^i_{h,n}(t_k, z_j)\Big],\ \text{in } \mathcal{I}^i \cap \mathbb{T} \times \mathbb{G} \\ v^i_{h,n+1}(T, z_j) = g^i(z_j) \ \text{ in } \mathcal{S} \cap \mathbb{G}. \end{cases} \quad (A.4)$$

with

$$\begin{aligned}\mathcal{E}^{N,R}[v^i_{h,n+1}(t+h, Z^{0,t,z}_{t+h})] &:= \mathbb{E}[v^i_{h,n+1}(t+h, \Pi_R(x\mathrm{e}^{(\mu-\frac{\sigma^2}{2})h+\sigma \hat{W}^N(h)}), y^i, y^j) \\ &= \sum_{l=1}^{N} v^i_{h,n+1}(t+h, \Pi_R(x\mathrm{e}^{(\mu-\frac{\sigma^2}{2})h+\sigma\sqrt{h}\hat{u}_l}), y^i, y^j)P_l \ ,\end{aligned}$$

where $\hat{W}^N(h) := \sqrt{h}\,\hat{U}$ such that $\hat{U}$ is the N-quantizer of the standard normal distribution, $(\hat{u}_l)_{1\le l\le N}$ are its realizations and $P_l = \mathbb{P}(\hat{U} = \hat{u}_l)$, $1 \le l \le N$ are the weights associated to this quantizer. The optimal grid $(\hat{u}_l)_{1\le l\le N}$ and the associated weights $(P_l)_{1\le l\le N}$ are downloaded from the website: http://www.quantize.maths-fi.com/downloads.
Finally, to approximate the integral Σ_i, we may use the rectangle rule and we obtain:

$$\Sigma_i(t_k, z_j) = \int_{t_k}^{t_{k+1}} f^i(Z^{0,t_k,z_j}_s)\mathrm{d}s \simeq hf^i(z_j).$$

A.4 Final Numerical Algorithm

The final backward scheme for $(t_k, z_j) \in \mathbb{T}\times\mathbb{G}$ is as follows:

Algorithm 1 Policy iteration for system of QVIs (two players)

Set $\varepsilon > 0$ (numerical tolerance) and $n_{max} \in \mathbb{N}$ (maximum iterations).
Initialization: For $i = 1, 2$ and $(t_k, z_j) \in \mathbb{T}\times\mathbb{G}$:
$v^i_{h,0}(t_k, z_j) = \mathcal{E}^{N,R}[\int_{t_k}^{T} f^i(Z^{0,t_k,z_j}_s)\mathrm{d}s + g(Z^{0,t_k,z_j}_T)]$.
Let $n = 0$ (iteration counter) and $PR^0 = +\infty$.
while $PR^n > \varepsilon$ and $n \le n_{max}$ **do**
for $i = 1, 2$ (player i) **do**
$l = 3 - i$ (player l.)
$\mathcal{C}^n_l := \{\mathcal{M}^l v^l_n - v^l_n < 0\} \cap \mathbb{T}\times\mathbb{G}$.
For $(t_k, z_j) \notin \mathcal{C}^n_l$, let

$$v^{n+1}_i(t_k, z_j) = \max(\mathcal{M}^{M,R}_i \mathcal{H}^{M,R}_i v^i_{h,n}(t_k, z_j), \mathcal{H}^{M,R}_i v^i_{h,n}(t_k, z_j)).$$

For $(t_k, z_j) \in \mathcal{C}^n_l$, solve:

$$\begin{aligned}&v^i_{h,n+1}(T, z_j) = g^i(z_j),\\ &v^i_{h,n+1}(t_k, z_j) = \max\left[\mathcal{E}^{N,R}[v^i_{h,n+1}(t_{k+1}, Z^{0,t_k,z_j}_{t_{k+1}})] + \Sigma_i(t_k, z_j), \mathcal{M}^{M,R}_i v^i_{h,n}(t_k, z_j)\right].\end{aligned}$$

end for.
Let PR^{n+1} be the largest pointwise residual to the system of QVIs, *i.e.*
$PR^{n+1} = \max(||v^1_{h,n+1} - v^1_{h,n}||, ||v^2_{h,n+1} - v^2_{h,n}||)$.
Let $n = n + 1$.
end while.

A.5 Numerical Values

In the following we give the values of the numerical parameters introduced in this Appendix and that we used to compute the equilibrium expected payoffs and the associated candidate equilibrium policies:
$m = 30,\ N_x = 30,\ N_y = 20,\ N = 100,\ M = 100,$

References

[1] R. Isaacs, Differential Games. A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization. John Wiley and Sons, Inc., New York-London-Sydney (1965).

[2] L.C. Evans and P.E. Souganidis, Differential games and representation formulas for solutions of Hamilton–Jacobi–Isaacs equations. *Indiana Univ. Math. J.* **33** (1984) 773–797.

[3] R. Aïd, M. Basei, G. Callegaro, L. Campi and T. Vargiolu, Nonzero-sum stochastic differential games with impulse controls: a verification theorem with applications. *Math. Oper. Res.* (2019) 1–28.

[4] A. Cosso, Stochastic differential games involving impulse controls and double-obstacle quasi-variational inequalities. *SIAM J. Control Optim.* **51** (2013) 2102-–2131.

[5] M. Basei, Optimal price management in retail energy markets: an impulse control problem with asymptotic estimates. *Math. Methods Oper. Res.* **89** (2019) 355–383.

[6] M. Crandall, H. Ishii and P.L. Lions, User's guide to viscosity solutions of second order partial differential equations. *Bul. Am. Math. Soc.* **27** (1992) 1–67.

[7] G. Barles, Solutions de viscosité des équations d'Hamilton–Jacobi. *Math. Appl.* (1994).

[8] M. Akian, A. Sulem and M. Taksar, Dynamic optimization of long term growth rate for a portfolio with transaction costs and logarithm utility. *Math. Finance* **11** (2001) 153–188.

[9] V. Ly Vath, M. Mnif and H. Pham, A model of optimal portfolio selection under liquidity risk and price impact. *Finance Stochast.* **11** (2007) 51–90.

[10] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order. Springer Verlag, Berlin (1977).

[11] R. Aïd, F. Bernal, M. Mnif, D. Zabaljauregui and J.P. Zubelli, A policy iteration algorithm for nonzero-sum stochastic impulse games. *ESAIM* **65** (2019) 27–45.

[12] B. Bouchard and N. Touzi, Weak dynamic programming principle for viscosity solutions. *SIAM J. Control Optim.* **49** (2011) 948-–962.

[13] D. Bertsekas and S. Shreve, Stochastic optimal control; the discrete-time case. Mathematics in Science and Engineering. Academic Press (1978).

[14] H. Soner, Optimal control with state-space constraint, I and II. *SIAM J. Cont. Optim.* **24** (1986) 552–561, and 1110–1122.

[15] T. Zariphopoulou, *Optimal Investment-Consumption Models with Constraints.* Ph.D. thesis, Brown University (1988).

[16] K. Ishii, Viscosity solutions of nonlinear second order elliptic PDEs associated with impulse control problems. *Funkcial. Ekvac.* **36** (1993) 123–141.

[17] G. Pagès, H. Pham and J. Printems, Optimal quantization methods and applications to numerical problems in finance, edited by S. Rachev. Handbook on Numerical Methods in Finance (2004) 253–298.

[18] R. Korn, Portfolio optimization with strictly positive transaction costs and impulse control. *Finance Stochast.* **2** (1998) 85–114.

[19] J.P. Chancelier, B. Oksendal and A. Sulem, Combined stochastic control and optimal stopping, and application to numerical approximation of combined stochastic and impulse control. Edited by A. Shiryaev. *Stochastic Financial Mathematics, Proc. Steklov Math. Inst. Moscou.* (2001) 149–175.

[20] M. Gaïgi, V. Ly Vath, M. Mnif and S. Toumi, Numerical approximation for a portfolio optimization problem under liquidity risk and costs. *Appl. Math. Optim.* **74** (2016) 163–195.

[21] B. Øksendal, Stochastic Differential Equations. Springer-Verlag, Berlin (2003).

[22] G. Pagès and H. Luschgy, Functional quantization of a class of Brownian diffusions: a constructive approach. *Stoch. Process. Appl.* **116** (2006) 310-–336.

[23] G. Pagès and H. Luschgy, Functional quantization of Gaussian Process. *J. Funct. Anal.* **196** (2002) 486-–531.