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MEAN-VARIANCE PORTFOLIO SELECTION WITH NON-LINEAR WEALTH DYNAMICS AND RANDOM COEFFICIENTS

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Abstract. This paper studies the continuous time mean-variance portfolio selection problem with one kind of non-linear wealth dynamics. To deal with the expectation constraint, an auxiliary stochastic control problem is firstly solved by two new generalized stochastic Riccati equations from which a candidate portfolio in feedback form is constructed, and the corresponding wealth process will never cross the vertex of the parabola. In order to verify the optimality of the candidate portfolio, the convex duality (requires the monotonicity of the cost function) is established to give another more direct expression of the terminal wealth level. The variance-optimal martingale measure and the link between the non-linear financial market and the classical linear market are also provided. Finally, we obtain the efficient frontier in closed form. From our results, people are more likely to invest their money in riskless asset compared with the classical linear market.

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1. Introduction

A mean-variance portfolio selection problem is to find the optimal portfolio strategy which minimizes the variance of its terminal wealth while its expected terminal wealth equals a prescribed level. Markowitz [1], [2] first studied this problem in the single-period setting. Its multi-period and continuous time counterparts have been studied extensively in the literature; see, *e.g.* Bielecki *et al.* [3], Jin *et al.* [4], Li *et al.* [5], Li *et al.* [6], Zhou *et al.* [7] and the references therein. For the general topic of mean variance hedging, please refer to Černý *et al.* [8], Černý and Kallsen [9], Schweizer [10].

Most of the literature on mean-variance portfolio selection stays in a linear market, *i.e.*, the wealth dynamic is a linear equation due to the proper market setting like frictionless trading. While in reality, the wealth dynamic is rare to be linear because of different kinds of friction in trading, and we have to deal with nonlinearity in the market. For example, a large investor's portfolio may affect the return of the stock's price which leads to a

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non-linear wealth dynamic. When some taxes must be paid on the gains made on the stocks, we also encounter nonlinearity in the wealth equation.

As for the continuous time mean-variance portfolio selection problem with non-linear wealth dynamic, Ji [11] obtained a necessary condition for the optimal terminal wealth when the drift of the wealth dynamic is differentiable. He derived a stochastic maximum principle which characterized the optimal terminal wealth. But the stochastic maximum principle in Ji [11] relies heavily on the differentiability assumption of the drift with respect to (X, π). For our non-differentiable case, the key step is to find an appropriate sub-derivative so as to construct the optimal wealth which is not concerned in [11]. Fu *et al.* [12] studied the continuous time mean-variance portfolio selection problem with higher borrowing rate in which the wealth dynamic is non-linear and the coefficient is not smooth. They employed the viscosity solution of the HJB equation to characterize the optimal portfolio strategy.

In this paper, the continuous time mean-variance portfolio selection problem with one kind of non-linear wealth dynamics is studied. The drift is not differentiable with respect to π in the model. When the coefficients are all deterministic continuous functions, Ji and Shi [13] solved this problem *via* the viscosity solution of the corresponding HJB equation. But for non-linear wealth dynamics with random coefficients such as stochastic return rates and stochastic volatilities, the method of HJB equation is no longer applicable.

Compared with classical linear market, the non-linear wealth dynamic brings new challenges. As the terminal expectation constraint $\mathbb{E}X_T^\pi = K$ is no longer linear in π. Whence it is unclear whether the feasible portfolio set is convex or not. Furthermore, the Lagrange strong duality which was widely used in solving mean-variance portfolio selection problem for linear market (see *e.g.* [14], [6]) is absent *a priori*. Instead, by introducing a Lagrange multiplier, we only have weak duality. Fortunately, we can take advantage of the weak duality to fix a lower bound for our problem, then construct a candidate portfolio π, and verify the optimality of π finally. In this procedure, we will in the first place confront a stochastic control problem without state constrain (but with non-linear dynamic and quadratic cost). Inspired by Hu and Zhou [14] in which the mean-variance portfolio selection problem with cone constraints was studied, this stochastic control problem could be solved by a generalized linear quadratic (LQ) approach. We find that our problem can be solved by studying the positive and negative parts of the process $X_t - de^{\int_t^T r_s \mathrm{d}s}$ separately (see Thm. 4.6). This approach leads to two new generalized stochastic Riccati equations. Through an exponential transformation, we prove the global solvability of these two generalized stochastic Riccati equations. Furthermore, we show that the positive or negative of the process $X_t - de^{\int_t^T r_s \mathrm{d}s}$ depends only on the positive or negative of its initial value (see Rem. 4.7). Things become apparently different when there are jumps in the price processes, please see Czichowsky and Schweizer [15], where a coupled system of backward stochastic differential equations (BSDEs) is deduced to characterize the value process. Then by solving a convex optimization problem (2.6), a candidate portfolio in feedback form is obtained.

But when it comes to verify the optimality of the candidate π (mainly $\mathbb{E}X_T^\pi = K$), this feedback form is no longer friendly. So the convex duality method, a theory which was highly developed in utility maximization problems (see *e.g.* Cvitanic and Karatzas [16] and the seminal book [17] for a systematic account on this subject) is applied to give another expression of the candidate portfolio and, especially, its corresponding terminal wealth. The main advantage of this method at this stage is that it can directly identify the optimal terminal wealth by studying the corresponding dual problem. Even though the quadratic function, that one is trying to minimise, lacks monotonicity or Inada condition used in establishing convex duality of utility maximization problem, problem (4.2) (with $\hat{d}$ in place of d) is still rather close to utility maximization because the optimal wealth process X_t never crosses the vertex of the parabola as suggested by Remark 4.7 ex post. Note that this is no longer the case for processes with jumps as in Czichowsky and Schweizer [15].

Except for expressing the optimal terminal wealth more directly by establishing the convex duality, we obtain some new sharp results which was not discovered in the generalized LQ approach. Further, this procedure helps us to understand the non-linear wealth dynamic better. In more detail, we succeed in obtaining the variance-optimal martingale measure, a concept introduced firstly in Schweizer [18], from which we find the links between the non-linear financial market and classical linear market. Actually, these two kind of markets are linked by the equivalent martingale measures, also called risk-neutral measures (see [19–21]). It is worth to point out that the financial market in our setting is incomplete which yields infinitely many equivalent martingale measures. Based on the explicit characterization of the variance optimal martingale measure, we show that our non-linear wealth dynamic is equivalent to a linear wealth dynamic with a appropriately chosen mean excess return rate from the viewpoint of optimization. And this mean excess return rate is exactly the sub-derivative claimed in Corollary 4.4 of Ji [11].

This paper is organized as follows. In section 2, we formulate the problem and sketch the idea to solve it. Section 3 concerns the feasibility of problem (2.2). The generalized LQ approach is employed to solve an auxiliary stochastic control problem without state constraint in section 4. A real valued Lagrange multiplier is found in sections 5. In section 6, we construct and verify the optimality of a candidate portfolio. Finally, the efficient strategy and efficient frontier are obtained in closed forms. Some concluding remarks are given in Section 7. Appendices contain technique proofs of Theorem 6.1.

2. Formulation of the problem

Let $W=(W^1,...,W^n)'$ be a standard n-dimensional Brownian motion defined on a filtered complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the natural filtration associated with the n-dimensional Brownian motion W and augmented.

We introduce the following spaces:

$$\begin{aligned}
&L^2(\Omega,\mathcal{F}_T;\mathbb{R})=\Big\{\xi:\Omega\to\mathbb{R}\big|\xi \text{ is } \mathcal{F}_T\text{-measurable, and } \mathbb{E}|\xi|^2<\infty\Big\},\\
&L^2(0,T;\mathbb{R})=\Big\{\phi:[0,T]\times\Omega\to\mathbb{R}\big|(\phi_t)_{0\leq t\leq T} \text{ is an } \{\mathcal{F}_t\}_{t\geq0} \text{-predictable process,}\\
&\qquad\text{and } \mathbb{E}\textstyle\int_0^T|\phi_t|^2\mathrm{d}t<\infty\Big\},\\
&L^\infty(0,T;\mathbb{R})=\Big\{\phi:[0,T]\times\Omega\to\mathbb{R}\;\Big|\;(\phi_t)_{0\leq t\leq T} \text{ is an } \{\mathcal{F}_t\}_{t\geq0}\text{-predictable essentially bounded process}\Big\}.
\end{aligned}$$

These definitions are generalized in the obvious way to the cases when ϕ is $\mathbb{R}^m$ or $\mathbb{R}^{m\times n}$-valued. We denote the transpose of a matrix M by M', and its norm by $|M|=\sqrt{\mathrm{trace}(MM')}$. In our argument, "almost surely" (a.s.), "almost everywhere" (a.e.) and (t,ω) may be suppressed for notation simplicity in some circumstances when no confusion occurs. Throughout this paper, we take the following notations. For any $x\in\mathbb{R}^m$, denote

$$x^+=(x_1^+,...,x_m^+)',\ x^-=(x_1^-,...,x_m^-)',$$

where

$$x_i^+=\begin{cases} x_i, & \text{if } x_i\geq 0;\\ 0, & \text{if } x_i<0,\end{cases} \quad\text{and } x_i^-=(-x_i)^+,\ i=1,...,m.$$

For any $\underline{x},\ \bar{x}\in\mathbb{R}^m$, we write $\underline{x}\leq\bar{x}$ if $\underline{x}_i\leq\bar{x}_i,\ i=1,...,m$.

Consider a financial market consisting of a riskless asset (the money market instrument or bond) whose price is S^0 and m $(m\leq n)$ risky securities (the stocks) whose prices are $S^1,...,S^m$. An investor decides at time $t\in[0,T]$ what amount π_t^i of his total wealth X_t to invest in the ith stock, $i=1,...,m$. The portfolio $\pi_t=(\pi_t^1,...,\pi_t^m)'$

and $\pi_t^0 := X_t - \sum_{i=1}^m \pi_t^i$ are $\mathcal{F}_t$-adapted. Then consider the following non-linear wealth dynamic:

$$\begin{cases} \mathrm{d}X_t = (r_t X_t + (\pi_t^+)'\underline{\mu}_t - (\pi_t^-)'\bar{\mu}_t)\mathrm{d}t + \pi_t'\sigma_t \mathrm{d}W_t, \\ X_0 = x \in \mathbb{R},\ t \in [0,T] \end{cases} \tag{2.1}$$

where r_t is the interest rate, $\underline{\mu}_t = (\underline{\mu}_t^1, ..., \underline{\mu}_t^m)'$, $\bar{\mu}_t = (\bar{\mu}_t^1, ..., \bar{\mu}_t^m)'$ are mean excess return rates for long positions and short positions, and $\sigma_t = \{\sigma_t^{ij}\}_{1\le i\le m, 1\le j\le n}$ is the volatility rate of risky assets. Note that the drift of the wealth equation (2.1) is Lipschitz but not differentiable with respect to π, which violates assumption (H1) in [11].

Assumption 2.1. r is a deterministic measurable bounded scalar-valued function.

Assumption 2.2. $\underline{\mu},\ \bar{\mu} \in L^\infty(0,T;\mathbb{R}^m)$ and $\underline{\mu}_t \le \bar{\mu}_t,\ i = 1, ..., m$. $\sigma \in L^\infty(0,T;\mathbb{R}^{m\times n})$ and

$$\exists \varepsilon > 0,\ \rho'\sigma_t\sigma_t'\rho \ge \varepsilon|\rho|^2,\ \forall \rho \in \mathbb{R}^m.$$

Indeed, it is the following three examples that motivate us to study the wealth dynamic (2.1). For simplicity, we suppose that there is only one stock in each of these three examples.

Example 2.3 (Short selling is costly)**.** Jouini and Kallal [22, 23] proposed the following model.

Let $\bar{b}_t \ge \underline{b}_t \ge r_t$. When short selling is possible but costly, one has different expected returns for long and short position of the stock. In this case, the asset prices are given by

$$\begin{cases} \mathrm{d}S_t^0 = S_t^0 r_t \mathrm{d}t,\ S_0^0 = s_0; \\ \mathrm{d}S_t^1 = S_t^1 \Big[\big(\underline{b}_t \mathbf{1}_{\{\pi_t \ge 0\}} + \bar{b}_t \mathbf{1}_{\{\pi_t < 0\}}\big)\mathrm{d}t + \sigma_t \mathrm{d}W_t\Big],\ S_0^1 = s_1 > 0. \end{cases}$$

Then the wealth process $X \equiv X^{x,\pi}$ of the self-financed investor who is endowed with initial wealth x is governed by the following stochastic differential equation,

$$\begin{cases} \mathrm{d}X_t = \pi_t \frac{dS_t^1}{S_t^1} + (X_t - \pi_t)\frac{\mathrm{d}S_t^0}{S_t^0} \\ \qquad = (r_t X_t + \pi_t^+ \underline{\mu}_t - \pi_t^- \bar{\mu}_t)\mathrm{d}t + \pi_t\sigma_t \mathrm{d}W_t, \\ X_0 = x, \end{cases}$$

where $\underline{\mu}_t = \underline{b}_t - r_t,\ \bar{\mu}_t = \bar{b}_t - r_t,\ t \in [0,T]$.

Example 2.4 (Price pressure model for large investors)**.** Cuoco and Cvitanic [24] gave the following price pressure model.

Let ε be a small positive number such that $b_t - r_t \ge \varepsilon \ge 0$. The portfolio strategy of a large investor could affect the expected return of the stock and the affection level is small. The asset prices are governed by

$$\begin{cases} \mathrm{d}S_t^0 = S_t^0 r_t \mathrm{d}t,\ S_0^0 = s_0; \\ \mathrm{d}S_t^1 = S_t^1 \Big[\big(b_t - \varepsilon\,\mathrm{sgn}(\pi_t)\big)\mathrm{d}t + \sigma_t \mathrm{d}W_t\Big],\ S_0^1 = s_1 > 0, \end{cases}$$

where

$$\mathrm{sgn}(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \ne 0; \\ 0, & \text{otherwise}. \end{cases}$$

In this specific large investor model, buying the risky security depresses its expected return while shorting it increases its expected return as explained in Cuoco and Cvitanic [24].

The wealth equation can be written

$$\begin{cases} \mathrm{d}X_t = (r_tX_t + (b_t - r_t)\pi_t - \varepsilon|\pi_t|)\mathrm{d}t + \pi_t\sigma_t\mathrm{d}W_t \\ \qquad\;\; = (r_tX_t + \pi_t^+\underline{\mu}_t - \pi_t^-\bar{\mu}_t)\mathrm{d}t + \pi_t\sigma_t\mathrm{d}W_t, \\ X_0 = x, \end{cases}$$

where $\underline{\mu}_t := b_t - r_t - \varepsilon$ and $\bar{\mu}_t := b_t - r_t + \varepsilon,\ t \in [0, T]$.

Example 2.5 (Trading with taxes)**.** El Karoui *et al.* [25] studied the following financial model with taxes.

Let $\alpha \in [0, 1)$ be a constant. And $b_t \geq r_t$. The asset prices are given by

$$\begin{cases} dS_t^0 = S_t^0 r_t\mathrm{d}t,\ S_0^0 = s_0; \\ dS_t^1 = S_t^1(b_t\mathrm{d}t + \sigma_t\mathrm{d}W_t),\ S_0^1 = s_1 > 0. \end{cases}$$

And there are some taxes which must be paid on the gains made on the stock. In this case, the wealth equation satisfies

$$\begin{cases} \mathrm{d}X_t = (r_tX_t + (b_t - r_t)\pi_t - \alpha\pi^+(b_t - r_t))\mathrm{d}t + \pi_t\sigma_t\mathrm{d}W_t \\ \qquad\;\; = ((r_tX_t + \pi_t^+\underline{\mu}_t - \pi_t^-\bar{\mu}_t)\mathrm{d}t + \pi_t\sigma_t\mathrm{d}W_t, \\ X_0 = x, \end{cases}$$

where $\underline{\mu}_t = (1 - \alpha)(b_t - r_t)$ and $\bar{\mu}_t = b_t - r_t,\ t \in [0, T]$.

Remark 2.6. When $\underline{\mu}_t = \bar{\mu}_t,\ t \in [0, T],\ a.s.$, the wealth dynamic (2.1) degenerates to the classical linear case.

Definition 2.7. A portfolio π is said to be admissible if $\sigma'\pi \in L^2(0, T; \mathbb{R}^n)$ and (X, π) satisfies (2.1).

Denote by $\mathcal{A}(x)$ the set of admissible portfolio π.

Under Assumption 2.1, for a given expectation level $K \geq x_0 \mathrm{e}^{\int_0^T r_s \mathrm{d}s}$, consider the following continuous time mean-variance portfolio selection problem:

$$\begin{aligned} &\text{Minimize } \mathrm{Var}(X_T) = \mathbb{E}(X_T - K)^2, \\ &s.t. \begin{cases} \mathbb{E}X_T = K, \\ \pi \in \mathcal{A}(x). \end{cases} \end{aligned} \tag{2.2}$$

Denote $\Pi = \{\pi | \pi \in \mathcal{A}(x),\ \text{ and } \mathbb{E}X_T = K\}$. The problem (2.2) is called feasible if Π is non empty. Any $\pi \in \Pi$ is called a feasible portfolio for the problem (2.2). Denote by X^π the wealth process (2.1) whenever it is necessary to indicate its dependence on $\pi \in \mathcal{A}(x)$. An optimal strategy π^* to (2.2) is called an efficient strategy corresponding to K. Then $(\mathrm{Var}(X_T^{\pi^*}), K)$ is called an efficient point. The set of all efficient points $\{(\mathrm{Var}(X_T^{\pi^*}), K) \mid K \in [x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}, +\infty)\}$ is called the efficient frontier.

To deal with the constraint $\mathbb{E}X_T = K$, we introduce a Lagrange multiplier $-2\lambda \in \mathbb{R}$ and obtain the following unconstrained optimization problem:

$$\inf_{\pi \in \mathcal{A}(x)} \Big[\mathbb{E}(X_T - K)^2 - 2\lambda(\mathbb{E}X_T - K)\Big]. \tag{2.3}$$

The problem (2.3) yields a lower bound on our original problem (2.2). To be more precise, we have the following weak duality between problems (2.2) and (2.3):

$$\sup_{\lambda\in\mathbb{R}}\inf_{\pi\in\mathcal{A}(x)}\Big[\mathbb{E}(X_T^\pi-K)^2-2\lambda(\mathbb{E}X_T^\pi-K)\Big]\le\inf_{\pi\in\Pi}\mathbb{E}(X_T^\pi-K)^2. \tag{2.4}$$

In fact, let $\hat\pi\in\Pi$ be any feasible portfolio and $\lambda\in\mathbb{R}$, we have

$$\mathbb{E}(X_T^{\hat\pi}-K)^2-2\lambda(\mathbb{E}X_T^{\hat\pi}-K)=\mathbb{E}(X_T^{\hat\pi}-K)^2.$$

Hence

$$\begin{aligned}\inf_{\pi\in\mathcal{A}(x)}\Big[\mathbb{E}(X_T^\pi-K)^2-2\lambda(\mathbb{E}X_T^\pi-K)\Big]&\le\mathbb{E}(X_T^{\hat\pi}-K)^2-2\lambda(\mathbb{E}X_T^{\hat\pi}-K)\\&=\mathbb{E}(X_T^{\hat\pi}-K)^2,\end{aligned}$$

for any $\lambda\in\mathbb{R}$ and any feasible portfolio $\hat\pi\in\Pi$. Then the weak duality (2.4) follows.

Note that we only have the weak duality (2.4) between problems (2.2) and (2.3). If the inequality becomes equality, we say that strong duality holds. And the problem in the left-hand side (LHS) of (2.4) is more likely to be solved than our original problem (2.2) (equivalently the right-hand side (RHS) of (2.4)). Actually, for any $\lambda\in\mathbb{R}$, the problem (2.3) is a stochastic control problem without state constraint (though with non-linear dynamic), we can solve it by a generalization of linear quadratic control technique. And by denoting

$$\ell(\lambda)=\inf_{\pi\in\mathcal{A}(x)}\Big[\mathbb{E}(X_T^\pi-K)^2-2\lambda(\mathbb{E}X_T^\pi-K)\Big],\ \lambda\in\mathbb{R}, \tag{2.5}$$

then ℓ is a concave function as it is the infimum of a class of linear functions of λ. So it is not hard to solve the convex optimization problem $\sup_{\lambda\in\mathbb{R}}\ell(\lambda)$. But unfortunately, due to the non-linear wealth dynamic (2.1), it is very difficult to establish the strong duality or even to prove the convexity of the set of feasible portfolios Π. Nevertheless, we can still take advantage of the weak duality (2.4) to construct a candidate portfolio π^* for our original problem (2.2), then verify the optimality of π^*. The main idea is as follows:

- Step 1: For any $\lambda\in\mathbb{R}$, find an optimal portfolio π^λ to the problem (2.3).
- Step 2: Find an argument maximum $\hat\lambda\in\mathbb{R}$ of

$$\sup_{\lambda\in\mathbb{R}}\ell(\lambda). \tag{2.6}$$

- Step 3: Set $\pi^*=\pi^{\hat\lambda}$, then

$$\begin{aligned}\sup_{\lambda\in\mathbb{R}}\inf_{\pi\in\mathcal{A}(x)}\Big[\mathbb{E}(X_T^\pi-K)^2-2\lambda(\mathbb{E}X_T^\pi-K)\Big]&=\sup_{\lambda\in\mathbb{R}}\Big[\mathbb{E}(X_T^{\pi^\lambda}-K)^2-2\lambda(\mathbb{E}X_T^{\pi^\lambda}-K)\Big]\\&=\mathbb{E}(X_T^{\pi^*}-K)^2-2\hat\lambda(\mathbb{E}X_T^{\pi^*}-K).\end{aligned}$$

  At this time, if we can show $\pi^*\in\Pi$, *i.e.* $\pi^*\in\mathcal{A}(x)$ and $\mathbb{E}X_T^{\pi^*}=K$, then $\mathbb{E}(X_T^{\pi^*}-K)^2$ attains the lower bound of the original problem (2.2), *i.e.* the LHS of (2.4), which verifies the optimality of π^* for problem (2.2).

3. Feasibility of the problem (2.2)

Let us address ourselves to the feasibility of problem (2.2) first.

Theorem 3.1. *Under Assumptions 2.1 and 2.2, the mean-variance problem (2.2) is feasible for any* $K \in [x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}, +\infty)$ *if and only if*

$$\sum_{i=1}^m \mathbb{E}\left[\int_0^T (\underline{\mu}_t^i)^+ \mathrm{d}t\right] > 0 \ \textit{or} \ \sum_{i=1}^m \mathbb{E}\left[\int_0^T (\bar{\mu}_t^i)^- \mathrm{d}t\right] > 0. \tag{3.1}$$

Proof: (1) We first prove the "if" part.

Define

$$M_i = \{(t,\omega) : \underline{\mu}_t^i > 0\}, \ i = 1, 2, ..., m.$$

If $\sum_{i=1}^m \mathbb{E}\left[\int_0^T (\underline{\mu}_t^i)^+ \mathrm{d}t\right] > 0$, then there exists an $i_0 \in \{1, 2, ..., m\}$ such that the product measure (in terms of $\mathbb{P}$ and the Lebesgue measure) of M_{i_0} is nonzero. Denote the i_0^{th} row of σ_t by $\sigma_t^{i_0} = (\sigma_t^{i_0,1}, \cdots, \sigma_t^{i_0,n})$ and the length of the vector $\sigma_t^{i_0}$ by $|\sigma_t^{i_0}|$. Since σ_t is invertible, it is obvious that $|\sigma_t^{i_0}| > 0$. Set

$$\pi_t^i = \begin{cases} 1/|\sigma_t^{i_0}|, & \text{if } i = i_0 \text{ and } (t,\omega) \in M_{i_0}; \\ 0, & \text{if } i \neq i_0 \text{ or } (t,\omega) \notin M_{i_0}. \end{cases}$$

For any nonnegative real number β, we construct a portfolio $\pi_{\beta,t} := \beta(\pi_t^1, \cdots, \pi_t^m)'$. $\pi_{\beta,t}$ is admissible due to $\sigma_t' \pi_{\beta,t} = \beta(\sigma_t^{i_0})' \mathbf{1}_{(t,\omega)\in M_{i_0}} / |\sigma_t^{i_0}|$ and

$$(\pi_{\beta,t}^+)' \underline{\mu}_t - (\pi_{\beta,t}^-)' \bar{\mu}_t = \beta \frac{\underline{\mu}_t^{i_0}}{|\sigma_t^{i_0}|} \mathbf{1}_{(t,\omega)\in M_{i_0}}.$$

The wealth process corresponding to π_β at time T is

$$\begin{aligned} X_T &= x\mathrm{e}^{\int_0^T r_s \mathrm{d}s} + \int_0^T \mathrm{e}^{\int_t^T r_s \mathrm{d}s} ((\pi_{\beta,t})^+)' \underline{\mu}_t - ((\pi_{\beta,t})^-)' \bar{\mu}_t) \mathrm{d}t + \int_0^T \mathrm{e}^{\int_t^T r_s \mathrm{d}s} \pi_{\beta,t}' \sigma_t \mathrm{d}W_t \\ &= x\mathrm{e}^{\int_0^T r_s \mathrm{d}s} + \beta \int_0^T \mathrm{e}^{\int_t^T r_s \mathrm{d}s} \frac{\underline{\mu}_t^{i_0}}{|\sigma_t^{i_0}|} \mathbf{1}_{(t,\omega)\in M_{i_0}} \mathrm{d}t + \beta \int_0^T \mathrm{e}^{\int_t^T r_s \mathrm{d}s} \frac{\sigma_t^{i_0}}{|\sigma_t^{i_0}|} \mathbf{1}_{(t,\omega)\in M_{i_0}} \mathrm{d}W_t. \end{aligned}$$

Taking expectation on both sides, we get

$$\mathbb{E}X_T = x\mathrm{e}^{\int_0^T r_s \mathrm{d}s} + \beta \mathbb{E}\left[\int_0^T \mathrm{e}^{\int_t^T r_s \mathrm{d}s} \frac{\underline{\mu}_t^{i_0}}{|\sigma_t^{i_0}|} \mathbf{1}_{(t,\omega)\in M_{i_0}} \mathrm{d}t\right].$$

Define

$$k = \mathbb{E}\left[\int_0^T \mathrm{e}^{\int_t^T r_s \mathrm{d}s} \frac{\underline{\mu}_t^{i_0}}{|\sigma_t^{i_0}|} \mathbf{1}_{(t,\omega)\in M_{i_0}} \mathrm{d}t\right].$$

We have $k > 0$ since $|\sigma_t^{i_0}| > 0$ and $\underline{\mu}_t^{i_0} \mathbf{1}_{(t,\omega)\in M_{i_0}} > 0$. Taking $\beta = \frac{K - x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}}{k}$, we obtain $\mathbb{E}X_T = K$ which means that the problem (2.2) is feasible. For the case of $\sum_{i=1}^m \mathbb{E}\left[\int_0^T (\bar{\mu}_t^i)^- \mathrm{d}t\right] > 0$, the proof is similar.

(2) Conversely, if the problem (2.2) is feasible for any $K \geq xe^{\int_0^T r_s \mathrm{d}s}$, then for a given $K_0 > xe^{\int_0^T r_s \mathrm{d}s}$, there exists an admissible portfolio π such that

$$K_0 = \mathbb{E}X_T = xe^{\int_0^T r_s \mathrm{d}s} + \mathbb{E}\left[\int_0^T e^{\int_t^T r_s \mathrm{d}s}((\pi_t^+)'\underline{\mu}_t - (\pi_t^-)'\bar{\mu}_t)\mathrm{d}t\right]$$

which leads to

$$\mathbb{E}\left[\int_0^T e^{\int_t^T r_s \mathrm{d}s}((\pi_t^+)'\underline{\mu}_t - (\pi_t^-)'\bar{\mu}_t)\mathrm{d}t\right] > 0. \tag{3.2}$$

If (3.1) does not hold, then we have that $\underline{\mu}_t < 0$ and $\bar{\mu}_t > 0$ hold simultaneously for $t \in [0, T]$. It yields that

$$\mathbb{E}\left[\int_0^T e^{\int_t^T r_s \mathrm{d}s}((\pi_t^+)'\underline{\mu}_t - (\pi_t^-)'\bar{\mu}_t)\mathrm{d}t\right] \leq 0$$

which contradicts (3.2). This completes the proof. □

Remark 3.2. When $\underline{\mu}_t = \bar{\mu}_t$, (3.1) degenerates to $\mathbb{E}\left[\int_0^T |\underline{\mu}_t|^2 \mathrm{d}t\right] > 0$.

From now on, we will assume (3.1) holds throughout this paper.

4. Solution to the problem (2.3)

For any $\lambda \in \mathbb{R}$, set $d = K + \lambda$, then

$$\mathbb{E}(X_T - K)^2 - 2\lambda(\mathbb{E}X_T - K) = \mathbb{E}(X_T - d)^2 - \lambda^2 = \mathbb{E}(X_T - d)^2 - (d - K)^2. \tag{4.1}$$

Therefore at this step, it suffices to solve

$$\text{Minimize } \mathbb{E}(X_T - d)^2, \ \ s.t. \ \pi \in \mathcal{A}(x), \tag{4.2}$$

for any $d \in \mathbb{R}$.

Define the following mappings:

$$\begin{aligned} H_{1,t}^*(\pi, P, \Lambda) &:= P\pi'\sigma_t\sigma_t'\pi + 2[P((\pi^+)'\underline{\mu}_t - (\pi^-)'\bar{\mu}_t) + \pi'\sigma_t\Lambda], \\ H_{2,t}^*(\pi, P, \Lambda) &:= P\pi'\sigma_t\sigma_t'\pi - 2[P((\pi^+)'\underline{\mu}_t - (\pi^-)'\bar{\mu}_t) + \pi'\sigma_t\Lambda], \ (t, \pi, P, \Lambda) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n, \end{aligned}$$

and

$$\begin{aligned} H_{1,t}(P, \Lambda) &:= \inf_{\pi \in \mathbb{R}^m} H_{1,t}^*(\pi, P, \Lambda), \\ H_{2,t}(P, \Lambda) &:= \inf_{\pi \in \mathbb{R}^m} H_{2,t}^*(\pi, P, \Lambda), \ (t, P, \Lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n. \end{aligned}$$

Under Assumption 2.2, for any $P > 0$, $\Lambda \in \mathbb{R}^n$, there exists $C_1(P, \Lambda) > 0$

$$H_{1,t}^*(\pi, P, \Lambda) \geq \varepsilon P|\pi|^2 - C_1(P + |\Lambda|)|\pi| = \varepsilon P|\pi|(|\pi| - \frac{C_1(P + |\Lambda|)}{\varepsilon P}).$$

If $|\pi| > \frac{C_1(P+|\Lambda|)}{\varepsilon P}$, then $H^*_{1,t}(\pi, P, \Lambda) > 0$. Notice that $\inf_{\pi\in\mathbb{R}^m} H^*_{1,t}(\pi, P, \Lambda) \leq H^*_{1,t}(0, P, \Lambda) = 0$, this implies that

$$H_{1,t}(P, \Lambda) = \inf_{|\pi|\leq \frac{C_1(P+|\Lambda|)}{\varepsilon P}} H^*_{1,t}(\pi, P, \Lambda) > -\infty.$$

Therefore $H_{1,t}(P, \Lambda)$ is finite. The same conclusion can be drawn for $H_{2,t}(P, \Lambda)$.

In order to solve the sub-problem (4.2), we introduce the following two stochastic Riccati equations:

$$\begin{cases} \mathrm{d}P_{1,t} = -[2r_t P_{1,t} + H_{1,t}(P_{1,t}, \Lambda_{1,t})]\mathrm{d}t + \Lambda'_{1,t}\mathrm{d}W_t, \\ P_{1,T} = 1, \\ P_{1,t} > 0; \end{cases} \tag{4.3}$$

$$\begin{cases} \mathrm{d}P_{2,t} = -[2r_t P_{2,t} + H_{2,t}(P_{2,t}, \Lambda_{2,t})]\mathrm{d}t + \Lambda'_{2,t}\mathrm{d}W_t, \\ P_{2,T} = 1, \\ P_{2,t} > 0. \end{cases} \tag{4.4}$$

These are two BSDEs whose solutions happen to be in the class of martingales of bounded mean oscillation, briefly called BMO martingales. Here we recall some facts about this theory, see Kazamaki [26]. The process $\int_0^\cdot \Lambda'_s \mathrm{d}W_s$ is a BMO martingale if and only if there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\int_\tau^T |\Lambda_s|^2 ds \,\middle|\, \mathcal{F}_\tau\right] \leq C$$

for all stopping times $\tau \leq T$. The stochastic exponential $\mathcal{E}(\int_0^\cdot \Lambda'_s \mathrm{d}W_s)$ of a BMO martingale $\int_0^\cdot \Lambda'_s \mathrm{d}W_s$ is a uniformly integrable martingale. Moreover, if $\int_0^\cdot \Lambda'_s \mathrm{d}W_s$ and $\int_0^\cdot Z'_s \mathrm{d}W_s$ are both BMO martingales, then under the probability measure $\widetilde{\mathbb{P}}$ defined by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}(\int_0^T Z'_s \mathrm{d}W_s)$, $\widetilde{W}_t = W_t - \int_0^t Z_s ds$ is a standard Brownian motion, and $\int_0^\cdot \Lambda'_s d\widetilde{W}_s$ is a BMO martingale. Set

$$\mathcal{BMO} = \{\Lambda \in L^2(0, T; \mathbb{R}^n) \big| \int_0^\cdot \Lambda'_s \mathrm{d}W_s \text{ is a BMO martingale}\}.$$

Definition 4.1. A pair of processes $(P_1, \Lambda_1) \in L^\infty(0, T; \mathbb{R}) \times \mathcal{BMO}$ (resp. (P_2, Λ_2)) is called a solution to the Riccati equation (4.3) (resp. (4.4)) if it satisfies (4.3) (resp. (4.4)).

The Riccati equations (4.3) and (4.4) are highly non-linear BSDEs which violate both the standard Lipschitz conditions and the quadratic growth conditions. There are several results on the solvability of stochastic Riccati equations (see for example Hu and Zhou [14], Kohlmann and Tang [27]). But up to our knowledge, no results can be directly applied to (4.3) and (4.4).

We first give the boundedness results of the solutions to (4.3) and (4.4), which is useful in Corollary (4.5).

Proposition 4.2. *Under Assumptions 2.1 and 2.2, if (P, Λ) is a solution to equation (4.3) (or (4.4)), then*

$$P_t \leq \mathrm{e}^{2\int_t^T r_s \mathrm{d}s}.$$

Proof. We only prove the claim for (4.3) and the proof for (4.4) is similar.

Set

$$\bar{P}_t = P_t \mathrm{e}^{2\int_0^t r_s \mathrm{d}s} \text{ and } \bar{\Lambda}_t = \Lambda_t \mathrm{e}^{2\int_0^t r_s \mathrm{d}s}.$$

Then $(\bar{P}, \bar{\Lambda})$ is a solution to the BSDE

$$\begin{cases} \mathrm{d}\bar{P}_t = -\mathrm{e}^{2\int_0^t r_s \mathrm{d}s} H_{1,t}(\mathrm{e}^{-2\int_0^t r_s \mathrm{d}s}\bar{P}_t, \mathrm{e}^{-2\int_0^t r_s \mathrm{d}s}\bar{\Lambda}_t)\mathrm{d}t + \bar{\Lambda}_t' \mathrm{d}W_t, \\ \bar{P}_T = \mathrm{e}^{2\int_0^T r_s \mathrm{d}s}, \\ \bar{P}_t > 0. \end{cases}$$

Since $H_{1,t} \leq 0$, $\bar{P}_t$ is a sub-martingale. Thus, $\bar{P}_t \leq \mathbb{E}[\bar{P}_T|\mathcal{F}_t] = \bar{P}_T$ which leads to $P_t \leq \mathrm{e}^{2\int_t^T r_s \mathrm{d}s}$. □

Now we prove the existence and uniqueness of solutions to (4.3) and (4.4). Hereafter, we shall use C to represent a generic positive constant which can be different from line to line.

Theorem 4.3. *Suppose $r \in L^\infty(0,T;\mathbb{R})$ and Assumption 2.2 holds. Then, there exists a unique solution (P_1, Λ_1) (resp. (P_2, Λ_2)) to (4.3) (resp. (4.4)), such that $P_1 \geq C$ (resp. $P_2 \geq C$) for some $C > 0$.*

Proof. We only prove the assertion for (4.4) and the arguments for (4.3) are analogous or obvious. The idea is to turn the stochastic Riccati equation (4.4) to a quadratic BSDE (through an exponential transformation) whose existence and uniqueness are known.

Set

$$\mathcal{B} = \{v : [0,T] \times \Omega \to \mathbb{R}^m \ | v \in L^\infty(0,T;\mathbb{R}^m) \text{ and } \underline{\mu}_t \leq v_t \leq \bar{\mu}_t\}. \tag{4.5}$$

Recall the definition of $H_2(P,\Lambda)$, we have, for $P > 0, \ \Lambda \in \mathbb{R}^n$,

$$\begin{aligned} H_{2,t}(P,\Lambda) &= \inf_{\pi\in\mathbb{R}^m} \left[P\pi'\sigma_t\sigma_t'\pi - 2[P((\pi^+)'\underline{\mu}_t - (\pi^-)'\bar{\mu}_t) + \pi'\sigma_t\Lambda]\right] \\ &= \inf_{\pi\in\mathbb{R}^m} \sup_{v\in\mathcal{B}} \left[P\pi'\sigma_t\sigma_t'\pi - 2\pi'(Pv + \sigma_t\Lambda)\right] \\ &= \sup_{v\in\mathcal{B}} \inf_{\pi\in\mathbb{R}^m} \left[P\pi'\sigma_t\sigma_t'\pi - 2\pi'(Pv + \sigma_t\Lambda)\right] \\ &= \sup_{v\in\mathcal{B}} \left[-P(v + \frac{\sigma_t\Lambda}{P})'(\sigma_t\sigma_t')^{-1}(v + \frac{\sigma_t\Lambda}{P})\right] \\ &= -\inf_{v\in\mathcal{B}} \left[P(v + \frac{\sigma_t\Lambda}{P})'(\sigma_t\sigma_t')^{-1}(v + \frac{\sigma_t\Lambda}{P})\right], \end{aligned} \tag{4.6}$$

where we use the min-max theorem in the third equality.

Consider the BSDE with quadratic growth

$$\tilde{Y}_t = \int_t^T g_s(\tilde{Z}_s)\mathrm{d}s - \int_t^T \tilde{Z}_s' \mathrm{d}W_s, \tag{4.7}$$

where

$$g_t(Z) := \inf_{v\in\mathcal{B}} \left|\sigma_t'(\sigma_t\sigma_t')^{-1}v - Z\right|^2 - Z'(I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)Z - \frac{1}{2}|Z|^2 - 2r_t. \tag{4.8}$$

According to Theorem 9.6.3 and Theorem 9.6.4 in [28], BSDE (4.7) has a unique solution $(\tilde{Y}, \tilde{Z}) \in L^\infty(0,T;\mathbb{R}) \times \mathcal{BMO}$.

Set

$$(P_t, \Lambda_t) = (\mathrm{e}^{-\tilde{Y}_t}, -\tilde{Z}_t \mathrm{e}^{-\tilde{Y}_t}), \tag{4.9}$$

then $P_T = \mathrm{e}^{-\tilde{Y}_T} = 1$. And from the boundedness of $\tilde{Y}_t$, we know $\Lambda \in \mathcal{BMO}$. Applying Itô's formula to $\mathrm{e}^{-\tilde{Y}_t}$,

$$\begin{aligned}
\mathrm{d}P_t &= d\mathrm{e}^{-\tilde{Y}_t} \\
&= -\mathrm{e}^{-\tilde{Y}_t}\Big[-\inf_{v\in\mathcal{B}}\Big|\sigma_t'(\sigma_t\sigma_t')^{-1}v - \tilde{Z}_t\Big|^2 + \tilde{Z}_t'(I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)\tilde{Z}_t + 2r_t\Big]\mathrm{d}t - \mathrm{e}^{-\tilde{Y}_t}\tilde{Z}_t'\mathrm{d}W_t \\
&= -\Big[2r_tP_t - P_t\inf_{v\in\mathcal{B}}\Big|\sigma_t'(\sigma_t\sigma_t')^{-1}v + \frac{\Lambda_t}{P_t}\Big|^2 + \frac{1}{P_t}\Lambda_t'(I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)\Lambda_t\Big]\mathrm{d}t + \Lambda_t'\mathrm{d}W_t \\
&= -\Big[2r_tP_t - P_t\inf_{v\in\mathcal{B}}\Big|\sigma_t'(\sigma_t\sigma_t')^{-1}(v + \sigma_t\frac{\Lambda_t}{P_t}) + (I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)\frac{\Lambda_t}{P_t}\Big|^2 \\
&\qquad + \frac{1}{P_t}\Lambda_t'(I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)\Lambda_t\Big]\mathrm{d}t + \Lambda_t'\mathrm{d}W_t \\
&= -\Big[2r_tP_t - P_t\inf_{v\in\mathcal{B}}\Big|\sigma_t'(\sigma_t\sigma_t')^{-1}(v + \sigma_t\frac{\Lambda_t}{P_t})\Big|^2 - P_t\big|(I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)\frac{\Lambda_t}{P_t}\big|^2 \\
&\quad + \frac{1}{P_t}\Lambda_t'(I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)\Lambda_t\Big]\mathrm{d}t + \Lambda_t'\mathrm{d}W_t \\
&= -\Big[2r_tP_t - P_t\inf_{v\in\mathcal{B}}\Big|\sigma_t'(\sigma_t\sigma_t')^{-1}(v + \sigma_t\frac{\Lambda_t}{P_t})\Big|^2\Big]\mathrm{d}t + \Lambda_t'\mathrm{d}W_t \\
&= -\Big[2r_tP_t + H_{2,t}(P_t, \Lambda_t)\Big]\mathrm{d}t + \Lambda_t'\mathrm{d}W_t,
\end{aligned}$$

where we have used the orthogonality of $\sigma_t'(\sigma_t\sigma_t')^{-1}(v + \sigma_t\frac{\Lambda_t}{P_t})$ and $(I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)\frac{\Lambda_t}{P_t}$ in the fifth equality, the idempotency of $I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t$ in the sixth equality and (4.6) in the last equality.

Note that $\tilde{Y}$ is bounded, thus there exists a constant $C > 0$ such that $P_t = \mathrm{e}^{-\tilde{Y}_t} \geq C$. This shows that (P_t, Λ_t) is actually a solution to (4.4).

Let us now prove the uniqueness. Suppose (P, Λ) and $(\tilde{P}, \tilde{\Lambda})$ are two solutions of (4.4), such that $P \geq C,\ \tilde{P} \geq C$ for some $C > 0$. Define the processes

$$(U, V) = \left(\ln P, \frac{\Lambda}{P}\right),\ (\tilde{U}, \tilde{V}) = \left(\ln \tilde{P}, \frac{\tilde{\Lambda}}{\tilde{P}}\right).$$

Then $(U, V),\ (\tilde{U}, \tilde{V}) \in L^\infty(0, T; \mathbb{R}) \times \mathcal{BMO}$. By Itô's formula and similar analysis as in the proof of the existence, it's not hard to show that both (U, V) and $(\tilde{U}, \tilde{V})$ are solutions of (4.7). From the uniqueness of solution to (4.7), we have $U = \tilde{U}$. Hence $P = \tilde{P}$, which gives the uniqueness of solution to (4.4). This completes the proof. □

Remark 4.4. If $m = n$, then $I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t = 0$, and (4.8) becomes

$$g_t(Z) := \inf_{v\in\mathcal{B}}\big|\sigma_t'(\sigma_t\sigma_t')^{-1}v - Z\big|^2 - \frac{1}{2}|Z|^2 - 2r_t.$$

The following corollary is useful in determining the Lagrange multiplier.

Corollary 4.5. *Suppose Assumptions 2.1 and 2.2 hold. Let $(P_{1,t},\Lambda_{1,t})$ and $(P_{2,t},\Lambda_{2,t})$ be the unique solutions to (4.3) and (4.4) respectively. Then we have*

$$P_{1,0}\mathrm{e}^{-2\int_0^T r_s\mathrm{d}s} \leq 1 \text{ and } P_{2,0}\mathrm{e}^{-2\int_0^T r_s\mathrm{d}s} < 1.$$

Proof. By Proposition 4.2, we have $P_{1,0}\mathrm{e}^{-2\int_0^T r_s\mathrm{d}s} \leq 1$ and $P_{2,0}\mathrm{e}^{-2\int_0^T r_s\mathrm{d}s} \leq 1$.

If $P_{2,0}\mathrm{e}^{-2\int_0^T r_s\mathrm{d}s} = 1$, then $H_{2,t}(P_{2,t},\Lambda_{2,t}) \equiv 0$ for $t\in[0,T]$. Then $(P_{2,t},\Lambda_{2,t}) = (\mathrm{e}^{2\int_t^T r_s\mathrm{d}s}, 0)$, which leads to

$$H_{2,t}(P_{2,t},0) = P_{2,t}\inf_{\pi\in\mathbb{R}^m}\Big[\pi'\sigma_t\sigma_t'\pi - 2((\pi^+)'\underline{\mu}_t - (\pi^-)'\bar{\mu}_t)\Big] = 0.$$

Note that (3.1) implies that either one of the following two statements hold:

(1) there is at least one of $\underline{\mu}^i,\ i=1,...,m$ strictly greater than 0 on a set of (t,ω) with strictly positive measure;

(2) there is at least one of $\bar{\mu}^i,\ i=1,...,m$ strictly lesser than 0 on a set of (t,ω) with strictly positive measure.

Without loss of generality, we suppose that $\underline{\mu}^1\mathbf{1}_{(t,\omega)\in M} > 0$. Then for a.e. $(t,\omega)\in M$,

$$\begin{aligned}
&\inf_{\pi\in\mathbb{R}^m}\Big[\pi'\sigma_t\sigma_t'\pi - 2((\pi^+)'\underline{\mu}_t - (\pi^-)'\bar{\mu}_t)\Big]\\
\leq& \inf_{\pi\in\mathbb{R}^m_+}\Big[\pi'\sigma_t\sigma_t'\pi - 2\pi'\underline{\mu}_t\Big]\\
\leq& \inf_{\pi\in\mathbb{R}^m_+}\Big[C\pi'\pi - 2\pi'\underline{\mu}_t\Big]\\
\leq& C(\frac{\underline{\mu}_t^1}{C},0,...,0)(\frac{\underline{\mu}_t^1}{C},0,...,0)' - 2(\frac{\underline{\mu}_t^1}{C},0,...,0)(\underline{\mu}_t^1,\underline{\mu}_t^2,...,\underline{\mu}_t^m)'\\
=& -\frac{1}{C}(\underline{\mu}_t^1)^2 < 0,
\end{aligned}$$

where C is a strictly positive constant. Thus we deduce a contradiction. This completes the proof. □

For any $P>0,\ \Lambda\in\mathbb{R}^n$, $H^*_{1,t}(\pi,P,\Lambda)$ is not necessarily convex with respect to π, so it may admit more than one arguments minimum. Let $\widetilde{\Pi}_t(P,\Lambda)$ be the set of arguments minimum of $H^*_{1,t}(\pi,P,\Lambda)$, *i.e.*

$$\widetilde{\Pi}_t(P,\Lambda) = \{\pi_{1,t}(P,\Lambda)|H^*_{1,t}(\pi_{1,t}(P,\Lambda),P,\Lambda) = \inf_{\pi\in\mathbb{R}^m} H^*_{1,t}(\pi,P,\Lambda)\}.$$

Notice that $H^*_{1,t}(\pi,P,\Lambda)$ is continuous with respect to π, by a measurable selection theorem (see *e.g.* Cor. 18.14 in [29] or Prop. 2.4 in [18]), there exists a predictable process $\pi_{1,t}(P,\Lambda)\in\widetilde{\Pi}_t(P,\Lambda)$. While for any $P>0,\ \Lambda\in\mathbb{R}^n$, $H^*_{2,t}(\pi,P,\Lambda)$ is strictly convex with respect to π. So by a measurable selection theorem, it admits a unique predictable argument minimum $\pi_{2,t}(P,\Lambda)$, such that

$$\pi_{2,t}(P,\Lambda) = \operatorname*{argmin}_{\pi\in\mathbb{R}^m}\big[P\pi'\sigma_t\sigma_t'\pi - 2[P((\pi^+)'\underline{\mu}_t - (\pi^-)'\bar{\mu}_t) + \pi'\sigma_t\Lambda]\big]. \tag{4.10}$$

Theorem 4.6. *Suppose Assumptions 2.1 and 2.2 hold. Let $(P_{1,t},\Lambda_{1,t})$ and $(P_{2,t},\Lambda_{2,t})$ be the unique solutions of (4.3) and (4.4) respectively. For any predictable $\pi_{1,t}\in\widetilde{\Pi}_t(P_{1,t},\Lambda_{1,t})$, $\pi_{2,t}$ defined in (4.10), the state feedback control*

$$\pi^d_t = \pi_{1,t}(P_{1,t},\Lambda_{1,t})\big(X_t - d\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}\big)^+ + \pi_{2,t}(P_{2,t},\Lambda_{2,t})\big(X_t - d\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}\big)^- \tag{4.11}$$

is optimal for the problem (4.2). *Moreover, the optimal value is*

$$\inf_{\pi\in\mathcal{A}(x)} \mathbb{E}(X_T-d)^2=\begin{cases}P_{1,0}(x-d\mathrm{e}^{-\int_0^T r_s\mathrm{d}s})^2, & if\ x\geq d\mathrm{e}^{-\int_0^T r_s\mathrm{d}s},\\ P_{2,0}(x-d\mathrm{e}^{-\int_0^T r_s\mathrm{d}s})^2, & if\ x\leq d\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}.\end{cases} \tag{4.12}$$

Proof. For any $\pi\in\mathcal{A}(x)$ with the wealth process X, define

$$Y_t=X_t-d\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}.$$

By Tanaka's formula,

$$\mathrm{d}Y_t^+=\mathbf{1}_{\{Y_t>0\}}(r_tY_t+(\pi_t^+)'\underline{\mu}_t-(\pi_t^-)'\bar{\mu}_t)\mathrm{d}t+\mathbf{1}_{\{Y_t>0\}}\pi_t'\sigma_t\mathrm{d}W_t+\frac{1}{2}\mathrm{d}L_t,$$

where L_t is the local time of Y_t at 0.

Applying Itô's formula to $(Y_t^+)^2$, we have

$$\begin{aligned}&d(Y_t^+)^2\\ &=2Y_t^+\Big\{\mathbf{1}_{\{Y_t>0\}}(r_tY_t+(\pi_t^+)'\underline{\mu}_t-(\pi_t^-)'\bar{\mu}_t)\mathrm{d}t+\mathbf{1}_{\{Y_t>0\}}\pi_t'\sigma_t\mathrm{d}W_t+\frac{1}{2}dL_t\Big\}+\mathbf{1}_{\{Y_t>0\}}\pi_t'\sigma_t\sigma_t'\pi_t\mathrm{d}t\\ &=\Big\{2r_t(Y_t^+)^2+2Y_t^+((\pi_t^+)'\underline{\mu}_t-(\pi_t^-)'\bar{\mu}_t)+\mathbf{1}_{\{Y_t>0\}}\pi_t'\sigma_t\sigma_t'\pi_t\Big\}\mathrm{d}t+2Y_t^+\pi_t'\sigma_t\mathrm{d}W_t,\end{aligned}$$

where we have used the fact $\int_0^t\mid Y_t\mid\mathrm{d}L_t=0$. Then applying Itô's formula to $P_{1,t}(Y_t^+)^2$,

$$\begin{aligned}&\mathrm{d}P_{1,t}(Y_t^+)^2\\ &=\Big\{\mathbf{1}_{\{Y_t>0\}}P_{1,t}\pi_t'\sigma_t\sigma_t'\pi_t+2(Y_t^+)\big[P_{1,t}((\pi_t^+)'\underline{\mu}_t-(\pi_t^-)'\bar{\mu}_t)+\pi_t'\sigma_t\Lambda_{1,t}\big]-(Y_t^+)^2H_{1,t}(P_{1,t},\Lambda_{1,t})\Big\}\mathrm{d}t\\ &\quad+\Big\{2P_{1,t}Y_t^+\pi_t'\sigma_t+(Y_t^+)^2\Lambda_{1,t}'\Big\}\mathrm{d}W_t.\end{aligned} \tag{4.13}$$

Similarly,

$$\begin{aligned}&\mathrm{d}P_{2,t}(Y_t^-)^2\\ &=\Big\{\mathbf{1}_{\{Y_t\leq0\}}P_{2,t}\pi_t'\sigma_t\sigma_t'\pi_t-2(Y_t^-)\big[P_{2,t}((\pi_t^+)'\underline{\mu}_t-(\pi_t^-)'\bar{\mu}_t)+\pi_t'\sigma_t\Lambda_{2,t}\big]-(Y_t^-)^2H_{2,t}(P_{2,t},\Lambda_{2,t})\Big\}\mathrm{d}t\\ &\quad+\Big\{-2P_{2,t}Y_t^-\pi_t'\sigma_t+(Y_t^-)^2\Lambda_{2,t}'\Big\}\mathrm{d}W_t.\end{aligned} \tag{4.14}$$

For $n\geq1$, define a stopping time τ_n as follows:

$$\tau_n=\inf\{t>0\big|\int_0^t|2P_{1,s}Y_s^+\sigma_s'\pi_s+(Y_s^+)^2\Lambda_{1,s}|^2\mathrm{d}s+\int_0^t|-2P_{2,s}Y_s^-\sigma_s'\pi_s+(Y_s^-)^2\Lambda_{2,s}|^2\mathrm{d}s\geq n\}\wedge T, \tag{4.15}$$

where $\inf\varnothing:=+\infty$. It is obvious that $\{\tau_n\}_{n\geq1}$ is an increasing sequence and converges to T. Adding and integrating (4.13) and (4.14) from 0 to τ_n, we get

$$
\begin{aligned}
&\mathbb{E}\big[P_{1,\tau_n}(Y_{\tau_n}^+)^2+P_{2,\tau_n}(Y_{\tau_n}^-)^2\big]\\
&=P_{1,0}(Y_0^+)^2+P_{2,0}(Y_0^-)^2\\
&+\mathbb{E}\int_0^{\tau_n}\Big\{\mathbf{1}_{\{Y_t>0\}}P_{1,t}\pi_t'\sigma_t\sigma_t'\pi_t+2(Y_t^+)\big[P_{1,t}((\pi_t^+)'\underline{\mu}_t-(\pi_t^-)'\bar{\mu}_t)+\pi_t'\sigma_t\Lambda_{1,t}\big]-(Y_t^+)^2H_{1,t}(P_{1,t},\Lambda_{1,t})\\
&\quad+\mathbf{1}_{\{Y_t\le 0\}}P_{2,t}\pi_t'\sigma_t\sigma_t'\pi_t-2(Y_t^-)\big[P_{2,t}((\pi_t^+)'\underline{\mu}_t-(\pi_t^-)'\bar{\mu}_t)+\pi_t'\sigma_t\Lambda_2\big]-(Y_t^-)^2H_{2,t}(P_{2,t},\Lambda_{2,t})\Big\}\mathrm{d}t. \qquad (4.16)
\end{aligned}
$$

For $t\in[0,T]$, denote by $\phi(Y_t,\pi_t)$ the integrand on the RHS of the above equation (4.16). For any $\pi\in\mathcal{A}(x)$ with the wealth process X, define a $\mathbb{R}^m$-valued process u_t by

$$
u_t=\begin{cases}\frac{\pi_t}{|Y_t|}, & \text{if} \quad Y_t\neq 0;\\ 0, & \text{if} \quad Y_t=0.\end{cases}
$$

When $Y_t>0$, the drift term on the RHS of (4.13) becomes

$$
\begin{aligned}
&P_{1,t}\pi_t'\sigma_t\sigma_t'\pi_t+2Y_t\big[P_{1,t}((\pi_t^+)'\underline{\mu}_t-(\pi_t^-)'\bar{\mu}_t)+\pi_t'\sigma_t\Lambda_{1,t}\big]-Y_t^2H_{1,t}(P_{1,t},\Lambda_{1,t})\\
&=Y_t^2\big\{P_{1,t}u_t'\sigma_t\sigma_t'u_t+2\big[P_{1,t}((u_t^+)'\underline{\mu}_t-(u_t^-)'\bar{\mu}_t)+\pi_t'\sigma_t\Lambda_{1,t}\big]-H_{1,t}(P_{1,t},\Lambda_{1,t})\big\}\\
&\ge 0
\end{aligned}
$$

by the definition of $H_{1,t}(P,\Lambda)$. By the definition of $H_{2,t}(P,\Lambda)$, we can show $\phi(Y_t,\pi_t)\ge 0$ if $Y_t<0$. Thus, we obtain that $\phi(Y_t,\pi_t)$ is nonnegative.

For any $\pi\in\mathcal{A}(x)$, it's easy to verify $\mathbb{E}\Big[\sup\limits_{t\in[0,T]}|Y_t|^2\Big]<\infty$. Let $n\to\infty$, and by the dominated convergence theorem, we have

$$
\begin{aligned}
\mathbb{E}(X_T-d)^2=\mathbb{E}(Y_T)^2&=\mathbb{E}\big[P_{1,T}(T)(Y_T^+)^2+P_{2,T}(Y_T^-)^2\big]\\
&=P_{1,0}(Y_0^+)^2+P_{2,0}(Y_0^-)^2+\mathbb{E}\Big[\int_0^T\phi(Y_t,\pi_t)\mathrm{d}t\Big]\\
&\ge P_{1,0}(Y_0^+)^2+P_{2,0}(Y_0^-)^2,
\end{aligned}
$$

where the equality holds at

$$
\pi_t^d=\pi_{1,t}(P_{1,t},\Lambda_{1,t})\big(X_t-d\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}\big)^++\pi_{2,t}(P_{2,t},\Lambda_{2,t})\big(X_t-d\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}\big)^-,
$$

which is (4.11). As a consequence, (4.12) is proved.

The proof of $\sigma'\pi^d\in L^2(0,T;\mathbb{R}^n)$ is relegated to the appendix, and can be skipped in a first reading. This completes the proof. □

Remark 4.7. From (A.4) and (A.5), we can see that if initial wealth $x\le d^{-\int_0^T r_s ds}$, the optimal state process of problem (4.2) will never exceed $d^{-\int_t^T r_s ds}$. The case $x\ge d^{-\int_0^T r_s ds}$ is parallel.

5. Solution to the problem (2.6)

As $d=\lambda+K$, with a slight abuse of notation, both λ and d are called Lagrange multipliers in the following. From (4.1) and the definition of ℓ in (2.5),

$$
\begin{aligned}
\sup_{\lambda\in\mathbb{R}}\ell(\lambda)&=\sup_{\lambda\in\mathbb{R}}\inf_{\pi\in\mathcal{A}(x)}\Big[\mathbb{E}(X_T^\pi-K)^2-2\lambda(\mathbb{E}X_T^\pi-K)\Big]\\
&=\sup_{d\in\mathbb{R}}\inf_{\pi\in\mathcal{A}(x)}\Big[\mathbb{E}(X_T^\pi-d)^2-(d-K)^2\Big].
\end{aligned}
$$

Therefore, it suffices to determine an argument maximum $\hat{d} \in \mathbb{R}$ of

$$\sup_{d\in\mathbb{R}} \inf_{\pi\in\mathcal{A}(x)} \Big[\mathbb{E}(X_T^\pi - d)^2 - (d-K)^2\Big].$$

From Theorem 4.6,

$$\begin{aligned}
&\inf_{\pi\in\mathcal{A}(x)} \mathbb{E}(X_T - d)^2 - (d-K)^2 \\
&= \begin{cases} P_{1,0}(x - d\mathrm{e}^{-\int_0^T r_s \mathrm{d}s})^2 - (d-K)^2, \ \text{if } x \geq d\mathrm{e}^{-\int_0^T r_s \mathrm{d}s}; \\ P_{2,0}(x - d\mathrm{e}^{-\int_0^T r_s \mathrm{d}s})^2 - (d-K)^2, \ \text{if } x \leq d\mathrm{e}^{-\int_0^T r_s \mathrm{d}s}, \end{cases} \\
&= \begin{cases} \big(P_{1,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s} - 1\big)d^2 - \big(2xP_{1,0}\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} - 2K\big)d + P_{1,0}x^2 - K^2, \ \text{if } d \leq x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}; \\ \big(P_{2,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s} - 1\big)d^2 - \big(2xP_{2,0}\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} - 2K\big)d + P_{2,0}x^2 - K^2, \ \text{if } d \geq x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}, \end{cases}
\end{aligned}$$

Define

$$\begin{aligned}
f(d) &= \big(P_{1,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s} - 1\big)d^2 - \big(2xP_{1,0}\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} - 2K\big)d + P_{1,0}x^2 - K^2; \\
h(d) &= \big(P_{2,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s} - 1\big)d^2 - \big(2xP_{2,0}\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} - 2K\big)d + P_{2,0}x^2 - K^2.
\end{aligned}$$

According to Corollary 4.5, $P_{1,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s} - 1 \leq 0$, $P_{2,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s} - 1 < 0$. Then we obtain

$$\begin{aligned}
f(x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}) &= \max_{d \leq x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}} f(d) = -(x\mathrm{e}^{\int_0^T r_s \mathrm{d}s} - K)^2 \leq 0, \\
h(\hat{d}) &= \max_{d \geq x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}} h(d) = \frac{P_{2,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s}}{1 - P_{2,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s}} \Big(K - x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}\Big)^2 \geq 0, \qquad (5.1)
\end{aligned}$$

where

$$\hat{d} = \frac{xP_{2,0}\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} - K}{P_{2,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s} - 1}. \qquad (5.2)$$

Since $K \geq x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}$, we have

$$\hat{d} \geq x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}, \qquad (5.3)$$

and

$$h(\hat{d}) \geq 0 \geq h(x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}) = -(x\mathrm{e}^{\int_0^T r_s \mathrm{d}s} - K)^2.$$

Thus $\hat{d}$ defined in (5.2) is an argument maximum of

$$\sup_{d\in\mathbb{R}} \inf_{\pi\in\mathcal{A}(x)} \Big[\mathbb{E}(X_T^\pi - d)^2 - (d-K)^2\Big].$$

6. Efficient strategy and efficient frontier

For π^d and $\hat{d}$ defined in (4.11) and (5.2) respectively, set $\pi^* = \pi^{\hat{d}}$, then $\pi^* \in \mathcal{A}(x)$ by Theorem (4.6), and

$$\begin{aligned}\mathbb{E}(X_T^{\pi^*} - K)^2 - 2\hat{\lambda}(\mathbb{E}X_T^{\pi^*} - K) &= \sup_{\lambda\in\mathbb{R}}\Big[\mathbb{E}(X_T^{\pi^\lambda} - K)^2 - 2\lambda(\mathbb{E}X_T^{\pi^\lambda} - K)\Big] \\ &= \sup_{\lambda\in\mathbb{R}}\inf_{\pi\in\mathcal{A}(x)}\Big[\mathbb{E}(X_T^{\pi} - K)^2 - 2\lambda(\mathbb{E}X_T^{\pi} - K)\Big]. \end{aligned} \tag{6.1}$$

is a lower bound of our original problem (2.2), noting (2.4). If we can show $\mathbb{E}X_T^{\pi^*} = K$, then $\pi^* \in \Pi$, and $\mathbb{E}(X_T^{\pi^*} - K)^2 = \mathbb{E}(X_T^{\pi^*} - K)^2 - 2\hat{\lambda}(\mathbb{E}X_T^{\pi^*} - K)$ attains the lower bound (6.1) (the LHS of (2.4)) which verifies the optimality of π^* for problem (2.2). Thus, it remains to prove $\mathbb{E}X_T^{\pi^*} = K$. Put $\pi^* = \pi^{\hat{d}}$ into the wealth equation (2.1), and notice that (A.1), (A.5) and (5.3), we have

$$\begin{aligned}X_t^{\pi^*} = (x - \hat{d}\mathrm{e}^{-\int_0^T r_s\mathrm{d}s})\exp\Big\{\int_0^t \big(r_s - (\pi_2^+)'\underline{\mu}_s + (\pi_2^-)'\bar{\mu}_s - \frac{1}{2}\pi_2'\sigma_s\sigma_s'\pi_2\big)\mathrm{d}s \\ + \int_0^t \pi_2'\sigma_s\mathrm{d}W_s\Big\} + \hat{d}\mathrm{e}^{-\int_0^t r_s\mathrm{d}s}, \end{aligned} \tag{6.2}$$

where π_2 is given in (4.10). As we do not have an explicit expression of π_2, it is difficult to verify $\mathbb{E}X_T^{\pi^*} = K$ with the expression (6.2).

Therefore a more direct expression of the terminal wealth level under π^* is appealing. Noting the convex duality method developed in [16] for utility maximization problem is efficient in finding the optimal terminal wealth directly. In the following, with $\hat{d}$ given in (5.2) and (5.3), we will solve the problem (4.2) through convex duality method. As some byproducts in this procedure, we obtain the variance-optimal martingale measure, a concept firstly introduced in [18], from which the links between the non-linear financial market and classical linear market are obtained. And we find the sub-derivative of the drift in the wealth equation (2.1) with respect to π claimed in Corollary 4.4 of Ji [11].

For any $v \in \mathcal{B}$ (see (4.5) for the definition of $\mathcal{B}$), $\theta \in \mathcal{BMO}$, let $N_t^{v,\theta}$ be the solution of the following stochastic differential equation,

$$\begin{cases}\mathrm{d}N_t^{v,\theta} = -N_t^{v,\theta}\Big[r_t\mathrm{d}t + \big(\sigma_t'(\sigma_t\sigma_t')^{-1}v_t + (I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)\theta_t\big)'\mathrm{d}W_t\Big], \\ N_0^{v,\theta} = 1.\end{cases}$$

Then $N_t^{v,\theta}\mathrm{e}^{\int_0^t r_s\mathrm{d}s}$ is a uniformly integrable martingale on $[0,T]$. Moreover, the equivalent martingale measures $\{\mathbb{Q}^{v,\theta}\}_{(v,\theta)\in\mathcal{B}\times\mathcal{BMO}}$ in this incomplete market could be constructed by $N_t^{v,\theta}$, *i.e.*

$$\frac{d\mathbb{Q}^{v,\theta}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = N_T^{v,\theta}\mathrm{e}^{\int_0^T r_s\mathrm{d}s}.$$

Note that stochastic exponentials of BMO martingales has been applied to characterize the equivalent martingale measures in Delbaen *et al.* [30], Choulli *et al.* [31].

Applying Itô's formula to $X_sN_s^{v,\theta}$ on $[0,t]$, we have

$$\begin{aligned}X_tN_t^{v,\theta} = x + \int_0^t N_s^{v,\theta}\Big[(\pi_s^+)'\underline{\mu}_s - (\pi_s^-)'\bar{\mu}_s - \pi_s'v_s\Big]\mathrm{d}s \\ + \int_0^t N_s^{v,\theta}\Big[\pi_s'\sigma_s - X_sv_s'(\sigma_s\sigma_s')^{-1}\sigma_s - X_s\theta_s'(I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\Big]\mathrm{d}W_s. \end{aligned} \tag{6.3}$$

Set

$$\mathcal{B}_1 = \{(v,\theta) \in \mathcal{B} \times \mathcal{BMO} \mid \text{the stochastic integral in (6.3) is a martingale for any } \pi \in \mathcal{A}(x)\}.$$

Taking expectation of (6.3) and notice that $\underline{\mu}_t \leq v_t \leq \bar{\mu}_t$, we have

$$\mathbb{E}[X_T N_T^{v,\theta}] \leq x, \text{ for any } \ (v,\theta) \in \mathcal{B}_1, \ \pi \in \mathcal{A}(x).$$

Theorem 6.1. *Suppose Assumptions 2.1 and 2.2 hold. Let $(\tilde{Y}, \tilde{Z})$ be the unique solution of (4.7), $\hat{d}$ defined in* (5.2) *and set*

$$\hat{\zeta} = -2\mathrm{e}^{-\tilde{Y}_0}(x - \hat{d}\mathrm{e}^{-\int_0^T r_s \mathrm{d}s}) \tag{6.4}$$

Then the variance-optimal martingale measure $\mathbb{Q}$ is defined through $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_T} = N_T^{\hat{v},\hat{\theta}} \mathrm{e}^{\int_0^T r_s \mathrm{d}s}$, where

$$\hat{v}_t = \operatorname*{argmin}_{v \in \mathcal{B}} |\sigma_t'(\sigma_t\sigma_t')^{-1} v - \tilde{Z}_t|^2, \ \hat{\theta}_t = \tilde{Z}_t, \ t \in [0,T], \ a.s. \tag{6.5}$$

Moreover, the optimal portfolio of the problem (4.2) *could be represented as*

$$\hat{\pi}_t = -\frac{\hat{\zeta}}{2} N_t^{\hat{v},\hat{\theta}} \mathrm{e}^{\tilde{Y}_t} (\sigma_t\sigma_t')^{-1}(\sigma_t \tilde{Z}_t - \hat{v}_t), \tag{6.6}$$

and optimal terminal wealth of the problem (4.2) has the following expression

$$\hat{X}_T = \hat{d} - \frac{\hat{\zeta}}{2} N_T^{\hat{v},\hat{\theta}}.$$

Proof. Step 1: Convex duality. Note that $\hat{d} \geq x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}$ in (5.3) and Remark 4.7, the terminal wealth $X_T^{\pi^{\hat{d}}}$ will never exceed $\hat{d}$. For $0 < \zeta < \infty$, define

$$u(\zeta) = \inf_{x \leq \hat{d}} [(x - \hat{d})^2 + \zeta x] = \hat{d}\zeta - \frac{\zeta^2}{4}.$$

Then $\forall \pi \in \mathcal{A}(x), \ \forall \zeta > 0, \ \forall (v,\theta) \in \mathcal{B}_1$, we have

$$\begin{aligned}
\mathbb{E}(X_T^\pi - d)^2 &\geq \mathbb{E}[u(\zeta N_T^{v,\theta}) - \zeta X_T^\pi N_T^{v,\theta}] \\
&= \mathbb{E}[\hat{d}\zeta N_T^{v,\theta} - \frac{\zeta^2}{4}(N_T^{v,\theta})^2 - \zeta X_T^\pi N_T^{v,\theta}] \\
&\geq \hat{d}\zeta \mathrm{e}^{-\int_0^T r_s \mathrm{d}s} - \frac{\zeta^2}{4}\mathbb{E}(N_T^{v,\theta})^2 - x\zeta
\end{aligned}$$

and the equalities hold if and only if there exists $\hat{\pi} \in \mathcal{A}(x), \ \hat{\zeta} > 0$, and $(\hat{v},\hat{\theta}) \in \mathcal{B}_1$, such that

$$X_T^{\hat{\pi}} = \hat{X}_T := \hat{d} - \frac{\hat{\zeta}}{2} N_T^{\hat{v},\hat{\theta}},$$

is the terminal wealth under the portfolio $\hat{\pi}$, and

$$\mathbb{E}[\hat{X}_T N_T^{\hat{v},\hat{\theta}}] = x \tag{6.7}$$

holds simultaneously. So we introduce the dual problem

$$\begin{aligned}
&\sup_{\substack{\zeta>0\\(v,\theta)\in\mathcal{B}_1}} \Big[\hat{d}\zeta \mathrm{e}^{-\int_0^T r_s \mathrm{d}s} - \frac{\zeta^2}{4}\mathbb{E}(N_T^{v,\theta})^2 - x\zeta\Big]\\
=& -\inf_{\substack{\zeta>0\\(v,\theta)\in\mathcal{B}_1}} \Big[-\hat{d}\zeta \mathrm{e}^{-\int_0^T r_s \mathrm{d}s} + \frac{\zeta^2}{4}\mathbb{E}(N_T^{v,\theta})^2 + x\zeta\Big]\\
=& -\inf_{\zeta>0}\Big[\frac{\zeta^2}{4}\inf_{(v,\theta)\in\mathcal{B}_1}\mathbb{E}(N_T^{v,\theta})^2 + \zeta(x - \hat{d}\mathrm{e}^{-\int_0^T r_s \mathrm{d}s})\Big].
\end{aligned} \tag{6.8}$$

We first deal with the term $\inf_{(v,\theta)\in\mathcal{B}_1}\mathbb{E}(N_T^{v,\theta})^2$. From the definitions of $N_t^{v,\theta}$ and $\tilde{Y}_t$ (see (4.7)),

$$\begin{aligned}
(N_t^{v,\theta})^2\mathrm{e}^{\tilde{Y}_t} =& \mathrm{e}^{\tilde{Y}_0}\exp\Big\{\int_0^t \Big[\tilde{Z} - 2\sigma'(\sigma\sigma')^{-1}v - 2(I_n - \sigma'(\sigma\sigma')^{-1}\sigma)\theta\Big]'\mathrm{d}W_s\\
&\qquad - \frac{1}{2}\int_0^t \big|\tilde{Z} - 2\sigma'(\sigma\sigma')^{-1}v - 2(I_n - \sigma'(\sigma\sigma')^{-1}\sigma)\theta\big|^2\mathrm{d}s\Big\}\\
&\cdot\exp\Big\{\frac{1}{2}\int_0^t \big|\tilde{Z} - 2\sigma'(\sigma\sigma')^{-1}v - 2(I_n - \sigma'(\sigma\sigma')^{-1}\sigma)\theta\big|^2\mathrm{d}s\Big\}\\
&\cdot\exp\Big\{\int_0^t \Big[-\big|\sigma'(\sigma\sigma')^{-1}v + (I_n - \sigma'(\sigma\sigma')^{-1}\sigma)\theta\big|^2 - 2r - g(\tilde{Z})\Big]\mathrm{d}s\Big\}\\
=& \mathrm{e}^{\tilde{Y}_0}\exp\Big\{\int_0^t \Big[\tilde{Z} - 2\sigma'(\sigma\sigma')^{-1}v - 2(I_n - \sigma'(\sigma\sigma')^{-1}\sigma)\theta\Big]'\mathrm{d}W_s\\
&\qquad - \frac{1}{2}\int_0^t \big|\tilde{Z} - 2\sigma'(\sigma\sigma')^{-1}v - 2(I_n - \sigma'(\sigma\sigma')^{-1}\sigma)\theta\big|^2\mathrm{d}s\Big\}\\
&\cdot\exp\Big\{\int_0^t \Big[\big|\sigma'(\sigma\sigma')^{-1}v - \tilde{Z}\big|^2 + (\theta - \tilde{Z})'(I_n - \sigma'(\sigma\sigma')^{-1}\sigma)(\theta - \tilde{Z})\Big]\mathrm{d}s\Big\}\\
&\cdot\exp\Big\{\int_0^t \Big[-\tilde{Z}'(I_n - \sigma'(\sigma\sigma')^{-1}\sigma)\tilde{Z} - \frac{1}{2}|\tilde{Z}|^2 - 2r - g(\tilde{Z})\Big]\mathrm{d}s\Big\}.
\end{aligned}$$

From the definition of g (4.8), $(N_t^{v,\theta})^2\mathrm{e}^{\tilde{Y}_t}$ is a submartingale for any $(v,\theta)\in\mathcal{B}_1$. By the martingale principle [32], $(\hat{v},\hat{\theta})\in\mathcal{B}_1$ is an optimal solution of $\inf_{(v,\theta)\in\mathcal{B}_1}\mathbb{E}(N_T^{v,\theta})^2$ if and only if $(N_t^{\hat{v},\hat{\theta}})^2\mathrm{e}^{\tilde{Y}_t}$ is a martingale. Then we get the representation of $(\hat{v},\hat{\theta})$ in (6.5):

$$\hat{v}_t = \operatorname*{argmin}_{v\in\mathcal{B}}|\sigma_t'(\sigma_t\sigma_t')^{-1}v - \tilde{Z}_t|^2,\ \hat{\theta}_t = \tilde{Z}_t,\ t\in[0,T],\ a.s.$$

and

$$\inf_{(v,\theta)\in\mathcal{B}_1}\mathbb{E}(N_T^{v,\theta})^2 = \mathrm{e}^{\tilde{Y}_0}.$$

By simple calculation, the first infimum in (6.8) is attained at

$$\hat{\zeta} = -2\mathrm{e}^{-\tilde{Y}_0}(x - \hat{d}\mathrm{e}^{-\int_0^T r_s \mathrm{d}s}) > 0.$$

Clearly $\hat{X}_T = \hat{d} - \frac{\hat{\zeta}}{2}N_T^{\hat{v},\hat{\theta}}$ satisfies (6.7).

Step 2: We will show that there exists a portfolio $\hat{\pi} \in \mathcal{A}(x)$ such that $X_T^{\hat{\pi}} = \hat{X}_T$.

Step 3: We will show that $(\hat{v}, \hat{\theta}) \in \mathcal{B}_1$, *i.e.* the stochastic integral in (6.3) is a martingale for any $\pi \in \mathcal{A}(x)$. We postpone the proofs of Step 2 and Step 3 to the appendix, which can be skipped in a first reading.

Step 4: We need to show $\hat{\pi} \in L^2(0, T; \mathbb{R}^m)$. And this can be guaranteed by similar method as in the proof of theorem 4.6 after noticing that $N_s^{\hat{v},\hat{\theta}}\mathrm{e}^{\tilde{Y}_s}$ satisfying the following equation

$$\begin{cases} d(-\frac{\hat{\zeta}}{2}N_s^{\hat{v},\hat{\theta}}\mathrm{e}^{\tilde{Y}_s}) = \Big[-\frac{\hat{\zeta}}{2}r_s N_s^{\hat{v},\hat{\theta}}\mathrm{e}^{\tilde{Y}_s} + \hat{\pi}_s'\hat{v}_s\Big]\mathrm{d}s + \hat{\pi}_s'\sigma_s \mathrm{d}W_s, \\ -\frac{\hat{\zeta}}{2}N_0^{\hat{v},\hat{\theta}}\mathrm{e}^{\tilde{Y}_0} = -\frac{\hat{\zeta}}{2}\mathrm{e}^{\tilde{Y}_0}. \end{cases}$$

Step 5: Combine (5.2) and (6.4), and notice that $P_{2,t} = \mathrm{e}^{-\tilde{Y}_t}$, we have

$$\mathbb{E}\hat{X}_T = \mathbb{E}\Big[\hat{d} - \frac{\hat{\zeta}}{2}N_T^{\hat{v},\hat{\theta}}\Big] = \hat{d} - \frac{\hat{\zeta}}{2}\mathrm{e}^{-\int_0^T r_s ds} = K.$$

This completes the proof. □

Remark 6.2. From (A.13), if $\underline{\mu}^i < \hat{v}^i < \bar{\mu}^i$, then $((\sigma_t\sigma_t')^{-1}(\hat{v} - \sigma_t\tilde{Z}_t))^i = 0$, and $\hat{\pi}_t^i = 0$ by (6.6), *i.e.* the investor should not invest in the ith stock.

Remark 6.3. Both $\pi^* = \pi^{\hat{d}}$ and $\hat{\pi}$ defined in (6.6) are solutions of the problem (4.2) (with $\hat{d}$). They are identical, *i.e.* $\pi^* = \hat{\pi}$, the reason is left to the interested readers.

So far, we achieve the three steps in solving our original problem (2.2). Therefore we have

Theorem 6.4. *Suppose Assumptions 2.1 and 2.2 hold. Let $(P_{2,t}, \Lambda_{2,t})$ be the unique solutions to (4.4), $\pi_2, \hat{d}$ defined in (4.10), (5.2). The efficient strategy of the problem (2.2) can be written as a function of time t and the wealth X_t:*

$$\pi^*(t, X) = -\pi_{2,t}(P_{2,t}, \Lambda_{2,t})\big(X_t - \hat{d}\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\big), \tag{6.9}$$

or equivalently expressed by (6.6). Moreover, the efficient frontier is

$$\mathrm{Var}(X_T) = \frac{P_{2,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s}}{1 - P_{2,0}\mathrm{e}^{-2\int_0^T r_s \mathrm{d}s}}\Big(\mathbb{E}X_T - x\mathrm{e}^{\int_0^T r_s \mathrm{d}s}\Big)^2. \tag{6.10}$$

Proof. The efficient frontier (6.10) comes from (5.1). □

Remark 6.5. When $m = n = 1$ and $\sigma_t > 0$, we have

$$\begin{aligned} H_{2,t}(P, \Lambda) &= \inf_{\pi \in \mathbb{R}} \big[P\sigma_t^2\pi^2 - 2[P(\pi^+\underline{\mu}_t - \pi^-\bar{\mu}_t) + \pi\sigma_t\Lambda]\big] \\ &= \begin{cases} -\frac{(P\underline{\mu}_t + \sigma_t\Lambda)^2}{P\sigma_t^2}, & \text{if } \frac{\sigma_t\Lambda}{P} \geq -\underline{\mu}_t, \\ 0, & \text{if } -\bar{\mu}_t \leq \frac{\sigma_t\Lambda}{P} \leq -\underline{\mu}_t, \\ -\frac{(P\bar{\mu}_t + \sigma_t\Lambda)^2}{P\sigma_t^2}, & \text{if } \frac{\sigma_t\Lambda}{P} \leq -\bar{\mu}_t. \end{cases} \end{aligned}$$

and

$$\pi_{2,t}(P,\Lambda)=\begin{cases}\frac{P\underline{\mu}_t+\sigma_t\Lambda}{P\sigma_t^2}, & \text{if } \frac{\sigma_t\Lambda}{P}\geq -\underline{\mu}_t,\\ 0, & \text{if } -\bar{\mu}_t\leq\frac{\sigma_t\Lambda}{P}\leq -\underline{\mu}_t,\\ \frac{P\bar{\mu}_t+\sigma_t\Lambda}{P\sigma_t^2}, & \text{if } \frac{\sigma_t\Lambda}{P}\leq -\bar{\mu}_t.\end{cases} \tag{6.11}$$

A significant departure from the deterministic coefficients case is the involvement of Λ in (6.11), where Λ serves to capture the effect of the randomness of the coefficients $\underline{\mu},\bar{\mu},\sigma$. When these coefficients are all deterministic, then $\Lambda\equiv 0$. In the linear financial market, $\underline{\mu}=\bar{\mu}=\mu$, then $\pi^*(t,X)=0$ if and only if $\frac{\sigma_t\Lambda_t}{P_t}=-\mu_t$. See *e.g.* Lim and Zhou Proposition 3.2 of [33] for similar results and note that, in the notations of [33], if the interest rate r is deterministic, $H\equiv 1$, then η in (17) of [33] equals 0. While in our non-linear market, $\pi^*(t,X)=0$ if and only if $-\bar{\mu}_t\leq\frac{\sigma_t\Lambda_t}{P_t}\leq -\underline{\mu}_t$. That is to say, the no-trading region becomes larger.

Remark 6.6. If $m=n=1$, and $\underline{\mu}_t$, $\bar{\mu}_t$, σ_t are deterministic continuous functions on $[0,T]$, $0\leq\underline{\mu}_t\leq\bar{\mu}_t$ and $\sigma_t>0$, then the unique solutions of (4.3) and (4.4) are given by

$$(P_{1,t},\Lambda_{1,t})=(\mathrm{e}^{\int_t^T(2r_s-\frac{\bar{\mu}_s}{\sigma_s})\mathrm{d}s},0);\ (P_{2,t},\Lambda_{2,t})=(\mathrm{e}^{\int_t^T(2r_s-\frac{\underline{\mu}_s}{\sigma_s})\mathrm{d}s},0).$$

Thus the optimal value (4.12) of the problem (4.2) degenerates to [13], (3.6), and π_2 in (6.11) becomes $\pi_{2,t}(P,\Lambda)=\underline{\mu}_t/\sigma_t^2$. Accordingly, the efficient portfolio (6.9) degenerates to [13], (3.15). Please note that, similar in spirit to Remark 4.7, [13], (3.15) should be condensed to

$$\pi^*(t,X)=-\frac{\underline{\theta}_t}{\sigma_t}(X-d^*\mathrm{e}^{-\int_t^T r_s ds}),$$ [1]

since $x_0-d^*\mathrm{e}^{-\int_0^T r_s ds}\leq 0$.

Remark 6.7. For any $v\in\mathcal{B}$, the following BSDE (6.12) admits a unique solution $(P_t^v,\Lambda_t^v)\in L^\infty(0,T;\mathbb{R})\times\mathcal{BMO}$, such that $P_t^v\geq C$ for some positive constant C by Theorem 2.2 of [27].

$$\begin{cases}\mathrm{d}P_t^v=-\Big\{rP_t^v-P_t^v(v+\frac{\sigma_t\Lambda^v}{P_t^v})'(\sigma_t\sigma_t')^{-1}(v+\frac{\sigma_t\Lambda_t^v}{P_t^v})\Big\}\mathrm{d}t+(\Lambda_t^v)'\mathrm{d}W_t,\\ P_T^v=1.\end{cases} \tag{6.12}$$

Actually, (6.12) is the Riccati equation associated with mean-variance problem under the linear wealth equation:

$$\begin{cases}\mathrm{d}X_t=(r_tX_t+\pi_t'v_t)\mathrm{d}t+\pi_t'\sigma_t\mathrm{d}W_t,\\ X_0=x.\end{cases}$$

Notice that the solution $(P_{2,t},\Lambda_{2,t})$ of (4.4) is uniformly positive, by Theorem 9.6.7 in [28], we have $P_t^v\leq P_{2,t}$ for any $v\in\mathcal{B}$, thus $\operatorname{ess\,sup}_{v\in\mathcal{B}}P_t^v\leq P_{2,t},\ t\in[0,T],\ a.s.$ Then for $\hat{v}\in\mathcal{B}$ defined in (6.5), we have

$$P_{2,t}=P_t^{\hat{v}}=\operatorname*{ess\,sup}_{v\in\mathcal{B}}P_t^v,\ \Lambda_{2,t}=\Lambda_t^{\hat{v}},\ t\in[0,T],\ a.s. \tag{6.13}$$

by the uniqueness of (4.4). Similarly, we can prove

$$P_{1,t}=P_t^{\tilde{v}}=\operatorname*{ess\,inf}_{v\in\mathcal{B}}P_t^v,\ \Lambda_{1,t}=\Lambda_t^{\tilde{v}},\ t\in[0,T],\ a.s.,$$

[1] Here we use the notations of [13].

where

$$\tilde{v} = \operatorname*{argmin}_{v\in\mathcal{B}} \big[-(v+\frac{\sigma_t\Lambda_{1,t}}{P_{1,t}})'(\sigma_t\sigma_t')^{-1}(v+\frac{\sigma_t\Lambda_{1,t}}{P_{1,t}})\big].$$

Remark 6.8. Let $\hat{v}$ be defined in (6.5). Consider the following mean-variance problem with a linear wealth dynamic:

$$\begin{aligned}&\text{Minimize } \mathbb{E}(X_T-K)^2,\\ &s.t. \begin{cases}\mathbb{E}X_T=K,\\ \sigma'\pi\in L^2(0,T;\mathbb{R}^n),\\ \mathrm{d}X_t=(r_tX_t+\pi_t'\hat{v}_t)\mathrm{d}t+\pi_t'\sigma_t\mathrm{d}W_t,\\ X_0=x.\end{cases}\end{aligned} \tag{6.14}$$

According to Theorem 6.4, the above mean-variance problem admits the optimal portfolio

$$\breve{\pi}(t,X) = -\breve{\pi}_{2,t}(P_t^{\hat{v}},\Lambda_t^{\hat{v}})\big(X_t-\breve{d}\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}\big),$$

where

$$\breve{\pi}_{2,t}(P_t^{\hat{v}},\Lambda_t^{\hat{v}}) = \operatorname*{argmin}_{\pi\in\mathbb{R}^m}\big[P_t^{\hat{v}}\pi'\sigma_t\sigma_t'\pi-2(P_t^{\hat{v}}\pi'\hat{v}_t+\pi'\sigma_t\Lambda_t^{\hat{v}})\big],$$

and

$$\breve{d} = \frac{xP_0^{\hat{v}}\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}-K}{P_0^{\hat{v}}\mathrm{e}^{-2\int_0^T r_s\mathrm{d}s}-1}. \tag{6.15}$$

We claim that both the problems (2.2) and (6.14) share the same optimal portfolio, *i.e.* $\tilde{\pi}(t,X)=\pi^*(t,X)$. Consequently, $\hat{v}$ could be chosen as a proper sub-derivative of $(\pi^+)'\underline{\mu}-(\pi^-)'\bar{\mu}$ with respect to π used in Corollary 4.4 of Ji [11].

Indeed, combining (4.6), (A.12) with (4.9), we know that $(\hat{v},\pi_2(P_2,\Lambda_2))$ defined in (6.5) and (4.10) is a saddle point of

$$\inf_{\pi\in\mathbb{R}^m}\sup_{v\in\mathcal{B}}\big[P_2\pi'\sigma_t\sigma_t'\pi-2\pi'(P_2v+\sigma_t\Lambda_2)\big] = \sup_{v\in\mathcal{B}}\inf_{\pi\in\mathbb{R}^m}\big[P_2\pi'\sigma_t\sigma_t'\pi-2\pi'(P_2v+\sigma_t\Lambda_2)\big],$$

especially

$$\begin{aligned}&P_2\pi_2(P_2,\Lambda_2)'\sigma_t\sigma_t'\pi_2(P_2,\Lambda_2)-2\pi_2(P_2,\Lambda_2)'(P_2\hat{v}+\sigma_t\Lambda_2)\\ &\le P_2\pi_2(P_2,\Lambda_2)'\sigma_t\sigma_t'\pi_2(P_2,\Lambda_2)-2\pi_2(P_2,\Lambda_2)'(P_2v+\sigma_t\Lambda_2),\ \forall v\in\mathcal{B}.\end{aligned}$$

Therefore, using (6.13), we have

$$\begin{aligned}\breve{\pi}_{2,t}(P_t^{\hat{v}},\Lambda_t^{\hat{v}}) &= \operatorname*{argmin}_{\pi\in\mathbb{R}^m}\big[P_t^{\hat{v}}\pi'\sigma_t\sigma_t'\pi-2[P_t^{\hat{v}}\pi'\hat{v}_t+\pi'\sigma_t\Lambda_t^{\hat{v}}]\big]\\ &= \operatorname*{argmin}_{\pi\in\mathbb{R}^m}\big[P_{2,t}\pi'\sigma_t\sigma_t'\pi-2[P_{2,t}\pi'\hat{v}_t+\pi'\sigma_t\Lambda_{2,t}]\big]\\ &= \pi_2(P_2,\Lambda_2).\end{aligned}$$

On the other hand, comparison of (6.15) and (5.2) shows that, using (6.13) again,

$$\check{d} = \hat{d}.$$

7. Concluding remarks

In this paper, we study mean-variance portfolio selection under non-linear wealth dynamics. Different from the linear wealth case, by introducing a Lagrange multiplier, we only have the weak duality (2.4). Therefore, solutions of the LHS of (2.4) only provide a lower bound for our original problem. After constructing a candidate portfolio $\pi^{\hat{d}}$ from the LHS of (2.4), we need to verify that $\pi^{\hat{d}} \in \Pi$, *i.e.* π is feasible (hence optimal) for our original problem (2.2) (or equivalently the RHS of (2.4)). This is achieved by the convex duality method which gives a more direct expression of the corresponding terminal wealth. Note that the quadratic cost function is not monotone, a property which is usually required for establishing convex duality. Fortunately, Remark 4.7 and (5.3) render the corresponding terminal wealth $X_T^{\pi^{\hat{d}}}$ of the candidate portfolio stay below $\hat{d}$. That is to say, without the analysis in Sections 4 and 5, the convex duality could not be established in Section 6. Finally, the optimal portfolio, efficient frontier and the variance-optimal martingale measure are given in closed forms. And we find the links between the non-linear financial market and classical linear market.

Extensions in other directions can be interesting as well. For instance: (1) How to characterize the optimal portfolio of problems (2.2) or (4.2) when the interest rate r is a stochastic process? (2) Recently, the general form of the mean-variance efficient frontier has been recently established in Černý, Czichowsky and Kallsen [8] with stochastic interest rates and even only risky assets. Can we generalize the results in [8] to the present setting with non-linear wealth dynamics? (3) Mean-variance portfolio selection when the diffusion term is also non-linear with respect to π.

Appendix A. Proofs of some technical results

A.1 Proof of $\sigma'\pi^d \in L^2(0,T;\mathbb{R}^n)$ of Theorem 4.6

It is sufficient to prove $\pi \in L^2(0,T;\mathbb{R}^m)$ since $\sigma \in L^\infty(0,T;\mathbb{R}^{m\times n})$. Note that

$$(\pi^d)^+ = \pi_1^+ Y^+ + \pi_2^+ Y^-, \text{ and } (\pi^d)^- = \pi_1^- Y^+ + \pi_2^- Y^-.$$

We next prove that the following equation (A.1) has a unique continuous $\mathcal{F}_t$-adapted solution.

$$\begin{cases} \mathrm{d}Y_t &= (r_t Y_t + ((\pi_t^d)^+)'\underline{\mu}_t - ((\pi_t^d)^-)'\bar{\mu}_t)\mathrm{d}t + (\pi_t^d)'\sigma_t \mathrm{d}W_t \\ &= (r_t Y_t + Y_t^+(\pi_1^+)'\underline{\mu}_t - Y_t^+(\pi_1^-)'\bar{\mu}_t + Y_t^-(\pi_2^+)'\underline{\mu}_t - Y_t^-(\pi_2^-)'\bar{\mu}_t)\mathrm{d}t + (Y_t^+\pi_1'\sigma_t + Y_t^-\pi_2'\sigma_t)\mathrm{d}W_t, \\ Y_0 &= x - d\mathrm{e}^{-\int_0^T r_s \mathrm{d}s}, \ t\in[0,T]. \end{cases} \tag{A.1}$$

Consider the following two equations:

$$\begin{cases} \mathrm{d}\bar{Y}_t = (r_t\bar{Y}_t + (\pi_1^+)'\underline{\mu}_t\bar{Y}_t - (\pi_1^-)'\bar{\mu}_t\bar{Y}_t)\mathrm{d}t + \bar{Y}_t\pi_1'\sigma_t\mathrm{d}W_t, \\ \bar{Y}_0 = (x - d\mathrm{e}^{-\int_0^T r_s\mathrm{d}s})^+, \ t\in[0,T], \end{cases} \tag{A.2}$$

and

$$\begin{cases} \mathrm{d}\tilde{Y}_t = (r_t\tilde{Y}_t - (\pi_2^+)'\underline{\mu}_t\tilde{Y}_t + (\pi_2^-)'\bar{\mu}_t\tilde{Y}_t)\mathrm{d}t - \tilde{Y}_t\pi_2'\sigma_t\mathrm{d}W_t, \\ \tilde{Y}_0 = (x - d\mathrm{e}^{-\int_0^T r_s\mathrm{d}s})^-, \ t\in[0,T]. \end{cases} \tag{A.3}$$

Then

$$\bar{Y}_t = (x - de^{-\int_0^T r_s \mathrm{d}s})^+ \exp\Big\{\int_0^t \big(r_s + (\pi_1^+)'\underline{\mu}_s - (\pi_1^-)'\bar{\mu}_s - \frac{1}{2}\pi_1'\sigma_s\sigma_s'\pi_1\big)\mathrm{d}s + \int_0^t \pi_1'\sigma_s \mathrm{d}W_s\Big\}, \tag{A.4}$$

and

$$\tilde{Y}_t = (x - de^{-\int_0^T r_s \mathrm{d}s})^- \exp\Big\{\int_0^t \big(r_s - (\pi_2^+)'\underline{\mu}_s + (\pi_2^-)'\bar{\mu}_s - \frac{1}{2}\pi_2'\sigma_s\sigma_s'\pi_2\big)\mathrm{d}s + \int_0^t \pi_2'\sigma_s \mathrm{d}W_s\Big\}. \tag{A.5}$$

It's easy to verify that $Y = \bar{Y} - \tilde{Y}$ is a solution of (A.1). To prove the uniqueness, let Y and $\dot{Y}$ be two solutions of (A.1). Set

$$\hat{Y}_t = Y_t - \dot{Y}_t,\ a_t = \frac{Y_t^+ - \dot{Y}_t^+}{Y_t - \dot{Y}_t}\mathbf{1}_{\{Y_t \neq \dot{Y}_t\}},\ b_t = \frac{Y_t^- - \dot{Y}_t^-}{Y_t - \dot{Y}_t}\mathbf{1}_{\{Y_t \neq \dot{Y}_t\}}.$$

Then $\hat{Y}$ solves the following linear SDE

$$\begin{cases} \mathrm{d}\hat{Y}_t = \hat{Y}_t(r_t + a_t(\pi_1^+)'\underline{\mu}_t - a_t(\pi_1^-)'\bar{\mu}_t + b_t(\pi_2^+)'\underline{\mu}_t - b_t(\pi_2^-)'\bar{\mu}_t)\mathrm{d}t + \hat{Y}_t(a_t\pi_1'\sigma_t + b_t\pi_2'\sigma_t)\mathrm{d}W_t, \\ \hat{Y}_0 = 0,\ t \in [0,T], \end{cases}$$

which has a unique solution $\hat{Y} = 0$.

Thus, (A.1) has a unique solution. We denote it by Y^d. Then

$$\pi_t^d = \pi_{1,t}(P_{1,t}, \Lambda_{1,t})(Y_t^d)^+ + \pi_{2,t}(P_{2,t}, \Lambda_{2,t})(Y_t^d)^-.$$

Denote by τ_n^d the stopping time defined in (4.15) for (Y_t^d, π_t^d). It follows from (4.16) that

$$\mathbb{E}\big[P_{1,\tau_n^d}(Y_{\tau_n^d}^+)^2 + P_{2,\tau_n^d}(Y_{\tau_n^d}^-)^2\big] = P_{1,0}(Y_0^+)^2 + P_{2,0}(Y_0^-)^2. \tag{A.6}$$

Recall that from Theorem 4.3, there exists a constant $C > 0$ such that

$$P_{1,t} \geq C,\ P_{2,t} \geq C,\ t \in [0,T].$$

Then by (A.6), we know

$$C\mathbb{E}(Y_{\tau_n^d \wedge \iota}^d)^2 \leq P_{1,0}(Y_0^+)^2 + P_{2,0}(Y_0^-)^2$$

for any stopping time ι valued in $[0,T]$. Fatou's lemma gives $\mathbb{E}(Y_\iota^d)^2 \leq C$. By Itô's formula, we have

$$(Y_t^d)^2 = y^2 + \int_0^t (2r_s(Y_s^d)^2 + 2Y_s^d(((\pi_s^d)^+)'\underline{\mu}_s - ((\pi_s^d)^-)'\bar{\mu}_s) + |\sigma_s'\pi_s^d|^2)\mathrm{d}s + \int_0^t 2Y_s^d(\pi_s^d)'\sigma_s \mathrm{d}W_s.$$

By the definitions of π_1 and π_2, for each $(t,\omega)\in[0,T]\times\Omega$, π_1^i and π_2^i take values from $\{0,(-(\sigma\sigma')^{-1}(\mu^{I_1}+\frac{\sigma\Lambda_1}{P_1}))^i,(\sigma\sigma')^{-1}(\mu^{I_2}+\frac{\sigma\Lambda_2}{P_2})^i : I_1\subset\{1,2...,m\}, I_2\subset\{1,2...,m\}\}$. Thus, there exists a constant C such that

$$\int_0^T|\sigma_t'\pi_t^d|^2\mathrm{d}t\le C\sup_{0\le t\le T}|Y_t^d|^2\sum_{\substack{I_1\subset\{1,2...,m\}\\ I_2\subset\{1,2...,m\}}}\int_0^T(|(\sigma\sigma')^{-1}(\mu^{I_1}+\frac{\sigma\Lambda_1}{P_1})|^2+|(\sigma\sigma')^{-1}(\mu^{I_2}+\frac{\sigma\Lambda_2}{P_2})|^2)\mathrm{d}t<+\infty,\ a.s. \tag{A.7}$$

since Λ_1, Λ_2 are square integrable and the other terms are bounded. For $n\ge1$, define a stopping time

$$\delta_n=\inf\{t>0\big|\int_0^t|Y_s^d\sigma_s'\pi_s^d|^2\mathrm{d}s\ge n\}\wedge T.$$

Then it converges to T almost surely due to (A.7). So

$$y^2+\mathbb{E}\int_0^{\tau_n^d\wedge\delta_n}|\sigma_s'\pi_s^d|^2ds=\mathbb{E}(Y^d_{\tau_n^d\wedge\delta_n})^2-\mathbb{E}\int_0^{\tau_n^d\wedge\delta_n}(2r_s(Y_s^d)^2+2Y_s^d(((\pi_s^d)^+)'\underline{\mu}_s-((\pi_s^d)^-)'\bar{\mu}_s))\mathrm{d}s.$$

Let $\varepsilon>0$ be the constant in Assumption 2.2, then we have

$$\begin{aligned}\varepsilon\mathbb{E}\int_0^{\tau_n^d\wedge\delta_n}|\pi_s^d|^2ds&\le C+C\mathbb{E}\int_0^{\tau_n^d\wedge\delta_n}2|Y_s^d||\pi_s^d|\mathrm{d}s\\&\le C+\frac{\varepsilon}{2}\mathbb{E}\int_0^{\tau_n^d\wedge\delta_n}|\pi_s^d|^2ds+\frac{2C^2}{\varepsilon}\mathbb{E}\int_0^{\tau_n^d\wedge\delta_n}|Y_s^d|^2ds.\end{aligned}$$

After rearrangement, it follows from Fatou's lemma that

$$\mathbb{E}\int_0^T|\pi_s^d|^2ds\le C.$$

A.2 Proof of Step 2 of Theorem 6.1

Define a $\mathcal{F}_t$-adapted process $\hat{X}$ *via*

$$\begin{aligned}\hat{X}_tN_t^{\hat{v},\hat{\theta}}=\mathbb{E}[\hat{X}_TN_T^{\hat{v},\hat{\theta}}|\mathcal{F}_t]&=\mathbb{E}[(\hat{d}-\frac{\hat{\zeta}}{2}N_T^{\hat{v},\hat{\theta}})N_T^{\hat{v},\hat{\theta}}|\mathcal{F}_t]\\&=\hat{d}\mathrm{e}^{-\int_0^Tr_sds}\mathrm{e}^{\int_0^tr_sds}N_t^{\hat{v},\hat{\theta}}-\frac{\hat{\zeta}}{2}(N_t^{\hat{v},\hat{\theta}})^2\mathrm{e}^{\tilde{Y}_t},\ t\in[0,T],\end{aligned}$$

i.e.

$$\hat{X}_t=\hat{d}\mathrm{e}^{-\int_t^Tr_sds}-\frac{\hat{\zeta}}{2}N_t^{\hat{v},\hat{\theta}}\mathrm{e}^{\tilde{Y}_t},\ t\in[0,T]. \tag{A.8}$$

Clearly we have $\hat{X}_0=x$, and

$$\begin{aligned}\mathbb{E}[\hat{d}N_T^{\hat{v},\hat{\theta}}|\mathcal{F}_t]&=\hat{d}\mathrm{e}^{-\int_0^Tr_sds}\mathrm{e}^{\int_0^tr_sds}N_t^{\hat{v},\hat{\theta}}\\&=\hat{d}\mathrm{e}^{-\int_0^Tr_sds}\Big[1-\int_0^t\mathrm{e}^{\int_0^sr_\alpha d\alpha}N_s^{\hat{v},\hat{\theta}}\Big(\sigma_s'(\sigma_s\sigma_s')^{-1}\hat{v}_s+(I_n-\sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\hat{\theta}_s\Big)'\mathrm{d}W_s\Big],\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[\frac{\hat{\zeta}}{2}(N_T^{\hat{v},\hat{\theta}})^2|\mathcal{F}_t] &= \frac{\hat{\zeta}}{2}(N_t^{\hat{v},\hat{\theta}})^2\mathrm{e}^{\tilde{Y}_t} \\
&= \frac{\hat{\zeta}}{2}\Big[\mathrm{e}^{\tilde{Y}_0} + \int_0^t (N_s^{\hat{v},\hat{\theta}})^2\mathrm{e}^{\tilde{Y}_s}\Big(\tilde{Z}_s - 2\sigma_s'(\sigma_s\sigma_s')^{-1}\hat{v}_s - 2(I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\hat{\theta}_s\Big)'\mathrm{d}W_s\Big].
\end{aligned}
$$

From (6.3) and the above two equations, it suffices to prove that there exists $\hat{\pi} \in \mathcal{A}(x)$ such that

$$
\begin{aligned}
&N_s^{\hat{v},\hat{\theta}}\Big[\hat{\pi}_s'\sigma_s - \hat{X}_s\hat{v}_s'(\sigma_s\sigma_s')^{-1}\sigma_s - \hat{X}_s\hat{\theta}_s'(I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\Big] \\
&= -\hat{d}\mathrm{e}^{-\int_0^T r_s ds}\mathrm{e}^{\int_0^s r_\alpha d\alpha}N_s^{\hat{v},\hat{\theta}}\Big(\sigma_s'(\sigma_s\sigma_s')^{-1}\hat{v}_s + (I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\hat{\theta}_s\Big)' \\
&\quad - \frac{\hat{\zeta}}{2}(N_s^{\hat{v},\hat{\theta}})^2\mathrm{e}^{\tilde{Y}_s}\Big(\tilde{Z}_s - 2\sigma_s'(\sigma_s\sigma_s')^{-1}\hat{v}_s - 2(I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\hat{\theta}_s\Big)',
\end{aligned} \tag{A.9}
$$

and

$$
(\hat{\pi}_s^+)'\underline{\mu}_s - (\hat{\pi}_s^-)'\bar{\mu}_s - \hat{\pi}_s'\hat{v}_s = 0 \tag{A.10}
$$

hold simultaneously. Noting (A.8) and $\hat{\theta} = \tilde{Z}$, we have

$$
\begin{aligned}
- N_s^{\hat{v},\hat{\theta}}\hat{X}_s\hat{\theta}_s'(I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s) &= -\hat{d}\mathrm{e}^{-\int_0^T r_s ds}\mathrm{e}^{\int_0^s r_\alpha d\alpha}N_s^{\hat{v},\hat{\theta}}\hat{\theta}_s'(I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s) \\
&\quad - \frac{\hat{\zeta}}{2}(N_s^{\hat{v},\hat{\theta}})^2\mathrm{e}^{\tilde{Y}_s}\Big((I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\tilde{Z}_s - 2(I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\hat{\theta}_s\Big)'.
\end{aligned}
$$

Therefore (A.9) is equivalent to

$$
\begin{aligned}
N_s^{\hat{v},\hat{\theta}}\Big[\hat{\pi}_s'\sigma_s - \hat{X}_s\hat{v}_s'(\sigma_s\sigma_s')^{-1}\sigma_s\Big] &= -\hat{d}\mathrm{e}^{-\int_0^T r_s ds}\mathrm{e}^{\int_0^s r_\alpha d\alpha}N_s^{\hat{v},\hat{\theta}}\Big(\sigma_s'(\sigma_s\sigma_s')^{-1}\hat{v}_s\Big)' \\
&\quad - \frac{\hat{\zeta}}{2}(N_s^{\hat{v},\hat{\theta}})^2\mathrm{e}^{\tilde{Y}_s}\Big(\sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s\tilde{Z}_s - 2\sigma_s'(\sigma_s\sigma_s')^{-1}\hat{v}_s\Big)'.
\end{aligned} \tag{A.11}
$$

Noting (A.8), $\hat{\pi}$ defined in (6.6) satisfies (A.11), hence (A.9). Moreover, we claim that $\hat{\pi}$ satisfies (A.10).

Actually, note that

$$
\begin{aligned}
\hat{v}_t &= \underset{v\in\mathcal{B}}{\operatorname{argmin}}\,|\sigma_t'(\sigma_t\sigma_t')^{-1}v - \tilde{Z}_t|^2 \\
&= \underset{v\in\mathcal{B}}{\operatorname{argmin}}\,|\sigma_t'(\sigma_t\sigma_t')^{-1}v - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t\tilde{Z}_t - (I_n - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)\tilde{Z}_t|^2 \\
&= \underset{v\in\mathcal{B}}{\operatorname{argmin}}\,|\sigma_t'(\sigma_t\sigma_t')^{-1}v - \sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t\tilde{Z}_t|^2 \\
&= \underset{v\in\mathcal{B}}{\operatorname{argmin}}\,|\sigma_t'(\sigma_t\sigma_t')^{-1}(v - \sigma_t\tilde{Z}_t)|^2.
\end{aligned} \tag{A.12}
$$

For any $u \in \mathcal{B}$ and $\varepsilon \in (0,1]$, we have $\hat{v} + \varepsilon(u - \hat{v}) \in \mathcal{B}$ because $\mathcal{B}$ is convex, and

$$
\frac{1}{\varepsilon}\Big[|\sigma_t'(\sigma_t\sigma_t')^{-1}(\hat{v} - \sigma_t\tilde{Z}_t)|^2 - |\sigma_t'(\sigma_t\sigma_t')^{-1}(\hat{v} + \varepsilon(u - \hat{v}) - \sigma_t\tilde{Z}_t)|^2\Big] \le 0.
$$

Sending $\varepsilon \downarrow 0$, we get

$$(u-\hat{v})'(\sigma_t\sigma_t')^{-1}(\hat{v}-\sigma_t\tilde{Z}_t) \geq 0, \ \forall u \in \mathcal{B}. \tag{A.13}$$

Denote the i^{th} component of $(\sigma_t\sigma_t')^{-1}(\hat{v}-\sigma_t\tilde{Z}_t)$ by $((\sigma_t\sigma_t')^{-1}(\hat{v}-\sigma_t\tilde{Z}_t))^i, \ i=1,...,m$. Then there must be

$$\hat{v}^i = \begin{cases} \underline{\mu}^i, \ \text{if } ((\sigma_t\sigma_t')^{-1}(\hat{v}-\sigma_t\tilde{Z}_t))^i \geq 0, \\ \bar{\mu}^i, \ \text{if } ((\sigma_t\sigma_t')^{-1}(\hat{v}-\sigma_t\tilde{Z}_t))^i \leq 0. \end{cases}$$

Recall the presentation (6.6), this implies (A.10).

A.3 Proof of Step 3 of Theorem 6.1

According to (6.3), it suffices to prove that

$$\int_0^t N_s^{\hat{v},\hat{\theta}}\Big[\pi_s'\sigma_s - X_s\hat{v}_s'(\sigma_s\sigma_s')^{-1}\sigma_s - X_s\hat{\theta}_s'(I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\Big]\mathrm{d}W_s$$

is a uniformly integrable martingale for any $\pi \in \mathcal{A}(x)$. Recall that $\mathrm{e}^{\int_0^t r_s ds}N_t^{\hat{v},\hat{\theta}}$ and $(N_t^{\hat{v},\hat{\theta}})^2\mathrm{e}^{\tilde{Y}_t}$ are two uniformly integrable martingales and r is bounded, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|N_t^{\hat{v},\hat{\theta}}|^2\Big] \leq C\mathbb{E}\Big[\sup_{t\in[0,T]}\mathrm{e}^{2\int_0^t r_s ds}|N_t^{\hat{v},\hat{\theta}}|^2\Big] \leq C\mathrm{e}^{2\int_0^T r_s ds}\mathbb{E}\Big[|N_T^{\hat{v},\hat{\theta}}|^2\Big] = C\mathrm{e}^{2\int_0^T r_s ds}\mathrm{e}^{\tilde{Y}_0} < \infty,$$

where the second inequality is due to Doob's inequality. Thus we have

$$\mathbb{E}\Big[\Big(\int_0^T |N_t^{\hat{v},\hat{\theta}}\pi_t'\sigma_t|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big] \leq \frac{1}{2}\mathbb{E}\Big[\sup_{t\in[0,T]}|N_t^{\hat{v},\hat{\theta}}|^2 + \int_0^T |\pi_t'\sigma_t|^2\mathrm{d}t\Big] < \infty,$$

and

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^T |N_t^{\hat{v},\hat{\theta}}X_t\hat{v}_t'(\sigma_t\sigma_t')^{-1}\sigma_t|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big] &\leq C\mathbb{E}\Big[\Big(\int_0^T |N_t^{\hat{v},\hat{\theta}}X_t|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big] \\ &\leq C\sqrt{T}\mathbb{E}\Big[\sup_{t\in[0,T]}|N_t^{\hat{v},\hat{\theta}}X_t|\Big] \\ &\leq \frac{C\sqrt{T}}{2}\mathbb{E}\Big[\sup_{t\in[0,T]}|N_t^{\hat{v},\hat{\theta}}|^2 + \sup_{t\in[0,T]}|X_t|^2\Big] < \infty.\end{aligned}$$

From the definition of $N_t^{\hat{v},\hat{\theta}}$, we know

$$\int_0^t N_s^{\hat{v},\hat{\theta}}\hat{\theta}_s'(I_n - \sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\mathrm{d}W_s = 1 - N_t^{\hat{v},\hat{\theta}} - \int_0^t r_sN_s^{\hat{v},\hat{\theta}}\mathrm{d}s - \int_0^t N_s^{\hat{v},\hat{\theta}}\hat{v}_s'(\sigma_s\sigma_s')^{-1}\sigma_s\mathrm{d}W_s.$$

By the BDG inequality, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T |N_t^{\hat{v},\hat{\theta}}\hat{\theta}_t'(I_n-\sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)|^2\mathrm{d}t \\
\le\, & C\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t N_s^{\hat{v},\hat{\theta}}\hat{\theta}_s'(I_n-\sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\mathrm{d}W_s\Big|^2\Big] \\
\le\, & C+C\mathbb{E}\Big[\sup_{t\in[0,T]}|N_t^{\hat{v},\hat{\theta}}|^2+\Big|\int_0^T N_s^{\hat{v},\hat{\theta}}\mathrm{d}s\Big|^2+\sup_{t\in[0,T]}\Big|\int_0^t N_s^{\hat{v},\hat{\theta}}\hat{v}_s'(\sigma_s\sigma_s')^{-1}\sigma_s\mathrm{d}W_s|^2\Big] \\
\le\, & C+C\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t N_s^{\hat{v},\hat{\theta}}\hat{v}_s'(\sigma_s\sigma_s')^{-1}\sigma_s\mathrm{d}W_s\Big|^2\Big] \\
\le\, & C+C\mathbb{E}\Big[\int_0^T\Big|N_s^{\hat{v},\hat{\theta}}\hat{v}_s'(\sigma_s\sigma_s')^{-1}\sigma_s\Big|^2\mathrm{d}s\Big]<\infty.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T |N_t^{\hat{v},\hat{\theta}}X_t\hat{\theta}_t'(I_n-\sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big] \\
\le\, & \mathbb{E}\Big[\Big(\sup_{t\in[0,T]}X_t\Big)\Big(\int_0^T |N_t^{\hat{v},\hat{\theta}}\hat{\theta}_t'(I_n-\sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big] \\
\le\, & \frac{1}{2}\mathbb{E}\Big[\Big(\sup_{t\in[0,T]}X_t\Big)^2+\int_0^T |N_t^{\hat{v},\hat{\theta}}\hat{\theta}_t'(I_n-\sigma_t'(\sigma_t\sigma_t')^{-1}\sigma_t)|^2\mathrm{d}t\Big]<\infty.
\end{aligned}$$

From the BDG inequality,

$$\int_0^t N_s^{\hat{v},\hat{\theta}}\Big[\pi_s'\sigma_s-X_s\hat{v}_s'(\sigma_s\sigma_s')^{-1}\sigma_s-X_s\hat{\theta}_s'(I_n-\sigma_s'(\sigma_s\sigma_s')^{-1}\sigma_s)\Big]\mathrm{d}W_s$$

is actually a uniformly integrable martingale for any $\pi\in\mathcal{A}(x)$.

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