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QUANTITATIVE RAPID AND FINITE TIME STABILIZATION OF THE HEAT EQUATION

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Abstract. The finite time stabilizability of the one dimensional heat equation is proved by Coron–Nguyên [J.-M. Coron and H.-M. Nguyen, *Arch. Ration. Mech. Anal.* **225** (2017) 993–1023], while the same question for multidimensional spaces remained open. Inspired by Coron–Trélat [J.-M. Coron and E. Trélat, *SIAM J. Control Optim.* **43** (2004) 549–569] we introduce a new method to stabilize multidimensional heat equations quantitatively in finite time and call it Frequency Lyapunov method. This method naturally combines spectral inequality [G. Lebeau and L. Robbiano, *Comm. Partial Diff. Equ.* **20** (1995) 335–356] and constructive feedback stabilization. As application this approach also yields a constructive proof for null controllability, which gives sharing optimal cost $Ce^{C/T}$ with explicit controls and works perfectly for related nonlinear models such as Navier–Stokes equations [S. Xiang, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **40** (2023) 1487–1511.].

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1. Introduction

Let Ω be an open domain in $\mathbb{R}^d$ with smooth boundary and $\omega \subset \Omega$ an open subset. We are interested in the quantitative finite time stabilization of the internal controlled heat equation,

$$\begin{aligned} y_t &= \Delta y + \chi_\omega f \ \text{ in } \Omega, \\ y &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

It is well known that in the 90's *the null controllability* of the above system was simultaneously discovered by Lebeau–Robbiano and Fursikov–Imanuvilov *via* different approaches [1, 2], relying on duality arguments [3–5] and Carleman estimates. We refer to [6] for a complete and pedagogical introduction on these different but somehow complementary methods. Later on fruitful results have been proved devoted to the related controllability problems, which include but not limited to [7–11]. In the recent paper dues to Burq–Moyano [12], they have shown that the heat equation is null controllable even if the control only acts on a zero measure spacetime set, where they have adapted the latest breakthrough on propagation of smallness by Logunov–Malinnikova [13–15].

Keywords and phrases: Finite time stabilization, quantitative, spectral estimate, null controllability.

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We also mention the works on optimal costs: there exists $c, C > 0$ such that for every $T \in (0,1)$, for every initial state $y_0 \in L^2(\Omega)$, there exists a control $f \in L^2(0,T;L^2(\Omega))$ such that the unique solution of (1.1) satisfies $y(T) = 0$ and

$$\|f\|_{L^2(0,T;L^2(\Omega))} \leq c\mathrm{e}^{\frac{C}{T}}\|y_0\|_{L^2(\Omega)}.$$

This estimate is "optimal" in the sense that there exist Ω and ω such that for every $\varepsilon > 0$, for every $c, C > 0$, the following estimate is not true:

$$\|f\|_{L^2(0,T;L^2(\Omega))} \leq c\mathrm{e}^{\frac{C}{T^{1-\varepsilon}}}\|y_0\|_{L^2(\Omega)}.$$

This type of optimal costs was first discovered in the 80s' by Seidman [16] and Güichal [17] for the one-dimensional heat equation. The same result was proved by Miller [18] for multi-dimensional cases. See for example the works of Miller [19] for more details on this subject.

1.1. Quantitative rapid and finite time stabilization problems

The study on stabilization of the heat equation is fruitful. The exponential stabilization of the heat equation was first achieved by Lasiecka–Triggiani [20] and Barbu–Wang [21] based on LQ theory. There are many related work on this topic, and most of them rely on the spectrum based method. Roughly speaking, by decomposing the evolution of the heat equation by Fourier series, thanks to the natural fast dissipation of the high frequency modes, the stabilization of the system can be reduced to the problem of stabilizing finitely many low-frequency modes. This idea leads to various methods on *exponential* and even *rapid stabilization*, which include but not limited to the Riccati method [21–24] for which the control is obtained by solving a minimization problem, the impulse control [25, 26] where the control is allowed to act on a zero measure time set, and straightforward construction techniques [27].

If one is able to stabilize a system, then one may ask how robust it is under perturbation. This requires a comprehensive understanding of the costs. Mathematically speaking, assume that we are stabilizing the energy $E(t)$ of the system,

$$E(t) \leq C_\lambda \mathrm{e}^{-\lambda t} E(0), \ \forall t \in (0, +\infty),$$

then we need to characterize the value of C_λ. Knowning this quantitative information allows us to predict the time required to reach the half-value period of the energy, to calculate error estimates due to perturbation including noise and delay of the system, and among others. For instance, suppose that for every λ the value of C_λ behaves as $\mathrm{e}^{c_0\lambda^\alpha}$, thus in the case that $\alpha \geq 1$ the half-value period is bigger than $c_0\lambda^{\alpha-1}$, and in the case that $\alpha < 1$ this period converges to 0 as λ becomes larger and larger. Moreover, such a quantitatve description is also needed if one wants to stabilize a system in finite time, which will be introduced later on. These are the reasons to motivating the subject of *quantitatve rapid stabilization*.

Actually, some of the above introduced techniques on rapid stabilization lead to quantitatve estimates. One shall in particular mention the works on impulse control [25, 26], where the authors have used the spectral inequality [2] to get desired estimates. Indeed, the current framework also relies on the spectral inequality to connect quantitative estimates and rapid stabilization, however, the feedback contructions of these two methods are different. Independent of these spectrum methods, the backstepping method also yields quantitative estimates. This method, first introduced by Krstic and his collaborators [28], corresponds to moving the spectrum with the help of some feedback laws. It has been generalized in [29, 30] by Fredholm transformation, and has turned out to be efficient for various one-dimensional models [31–35]. However, it is still a challenging open problem to introduce the backstepping method on general multidimensional models.

To be compared with the controllability problem, the *finite time stabilization* problem is required to be more constructive. We need to construct feedback laws such that for every initial state the solution of the closed-loop system becomes zero in finite time. In [36] Coron and Nguyên have proved the finite time stabilization of the one-dimensional heat equation with Dirichlet boundary control. The crucial credit of this reference is on the quantitative rapid stabilization of the one-dimensional heat equation using the backstepping method, namely for every λ they have construct feedback law to make the closed-loop system decay with rate λ for which the constant C_λ behaves as $\mathrm{e}^{C\sqrt{\lambda}}$. Later on fruitful results have been proved on finite time stabilization. We refer to the paper by Coron and the author [37], Introduction for a detailed review on this subject.

Before stating the detailed theorems, we briefly explain some *terminologies* that will be used for stabilization problems. A *time-varying feedback law* U is an application

$$\begin{cases} U : \mathbb{R} \times L^2(\Omega) & \to \quad L^2(\Omega) \\ \qquad (t; y) & \mapsto \quad U(t; y). \end{cases}$$

A *stationary* feedback law is such an application only depends on $L^2(\Omega)$, and a T-*periodic* feedback law is a time-varying feedback law such that $U(t + T; y) = U(t; y)$. The *closed-loop system* associated to a feedback law U is the evolution equation

$$\begin{cases} y_t = \Delta y + \chi_\omega U(t; y), \ (t, x) \in (s, +\infty) \times \Omega, \\ y(t, x) = 0, \ (t, x) \in (s, +\infty) \times \partial\Omega. \end{cases} \tag{1.2}$$

Eventually we are interested in T-*periodic proper* feedback laws. Heuristically speaking, a feedback law U is called *proper* if the Cauchy problem associated to the closed-loop system (1.2) admits a unique solution for every $s \in \mathbb{R}$ and for every initial data $y_0 \in L^2(\Omega)$ at time s. Therefore, formally we are allowed to define a "*flow*", $\Phi(t, s; y_0)$, as the state at time t of the solution of (1.2) with initial state $y(s, x) = y_0(x)$, where $y_0 \in L^2(\Omega)$ and $t \geq s$. Please follow Section 4.1 for precise definitions on solutions of closed-loop systems, proper feedback laws, "flow" with respect to systems with proper feedback laws, as well as finite time stabilization.

1.2. The main result

In this paper, we solve the finite time stabilization problem for the multidimensional heat equation, by introducing a new method on quantitative rapid stabilization, see Lemma 1.3 for this rapid stabilization result. This new method shares the advantages of providing simple and physical feedback laws and of achieving quantitative estimates. Armed with this quantitative rapid stabilization lemma, by combining the piecewise feedback law idea introducing by Coron and Nguyên in [36] we solve the finite time stabilization problem.

Theorem 1.1 (Semi-global finite time stabilization)**.** *For any $\Lambda \geq 1$, for any $T > 0$, we construct an explicit T-periodic proper feedback law U satisfying*

$$||\chi_\omega U(t; y)||_{L^2(\Omega)} \leq C||y||_{L^2(\Omega)} + 2||y||^{1/2}_{L^2(\Omega)}, \ \forall\, y \in L^2(\Omega), \ \forall\, t \in \mathbb{R},$$

with some C effectively computable, that stabilizes system (1.2) in finite time:

(i) (2T stabilization) $\Phi(2T + t, t; y_0) = 0, \ \ \forall\, t \in \mathbb{R}, \ \forall\, ||y_0||_{L^2(\Omega)} \leq \Lambda.$

(ii) (Uniform stability) For every $\delta > 0$ there exists an effectively computable $\eta > 0$ such that

$$\left(||(y_0||_{L^2(\Omega)} \leq \eta\right) \Rightarrow \left(||\Phi(t, t'; y_0)||_{L^2(\Omega)} \leq \delta, \ \forall\, t' \in \mathbb{R}, \ \forall\, t \in (t', t' + 2T)\right).$$

Remark 1.2. Let us emphasize that the "uniform stability" condition is one of the essential differences between null controllability and finite time stabilization. Indeed, this condition is crucial for stabilization problems as in reality systems may have errors and exist perturbations, thus the stabilizing system are required to overcome these difficulties. Another main difficulty for closed-loop stabilization compared to open-loop control is that the feedback for closed-loop system only depends on current states, while the open loop control depends on initial states.

1.3. Strategy of the proof of the main result

In this section we briefly illustrate the ideas on achieving the finite time stabilization result, which is composed by three steps.

Step 1. Quantitative rapid stabilization.

Lemma 1.3 (Quantitative rapid stabilization)**.** *There exists an effectively computable constant $C > 0$ such that for any $\lambda > 0$ we construct an explicit stationary feedback law $\mathcal{G}_\lambda : L^2(\Omega) \to L^2(\Omega)$, such that the closed-loop system*

$$\begin{aligned} y_t &= \Delta y + \chi_\omega \mathcal{G}_\lambda y \quad \textit{in } \Omega, \\ y &= 0 \quad \textit{on } \partial\Omega, \end{aligned}$$

is exponentially stable:

$$||\Phi(t, s; y_0)||_{L^2(\Omega)} + ||1_\omega \mathcal{G}_\lambda \Phi(t, s; y_0)||_{L^2(\omega)} \leq C e^{C\sqrt{\lambda}} e^{-\frac{\lambda}{2}(t-s)} ||y_0||_{L^2(\Omega)}, \ \forall\, s \in \mathbb{R}, \forall\, t \in [s, +\infty).$$

The proof of this lemma is based on the Lyapunov method. More precisely, for every λ we have constrcut a feedback law $\mathcal{G}_\lambda$ and a Lyapunov function E_λ satisfying

$$\left(C e^{C\sqrt{\lambda}}\right)^{-1} ||y_0||_{L^2(\Omega)} \leq E_\lambda(y) \leq C e^{C\sqrt{\lambda}} ||y_0||_{L^2(\Omega)}, \forall y \in L^2(\Omega),$$

such that the solution satisfies

$$E_\lambda(y(t)) \leq -\lambda E_\lambda(y(0)).$$

The preceding inequalities implies the quantitative exponential stability.

To be compared with the fruitful stabilization results in the literature commented in Section 1.1, the main novelties of this new quantitative rapid stabilization method are twofold. On the one hand, the feedback law that we designed is simple and explicit, according to equation (2.14) the feedback law is simply the low-frequency projection times a constant. On the other hand, because our Lyapunov stabilization approach is stable for nonlinear perturbation (see Sect. 2.5 and Rem. 2.8), it works perfectly for nonlinear systems to obatin quantitative estimates, such as the nonlinear heat equations and Navier–Stokes equations [38].

Step 2. Piecewise control for null controllability.

Lemma 1.4 (Totally constructive null controllability sharing optimal cost)**.** *There exists an effectively computable constant $C > 0$ such that, for any $T \in (0, 1)$, for any $y_0 \in L^2(\Omega)$, we can construct an explicit control $f|_{[0,T]}(t, x)$ for the control system (1.1) such that the unique solution satisfies*

$$y(0, \cdot) = y_0(\cdot) \ \textit{and} \ y(T, \cdot) = 0,$$

moreover,

$$||\chi_\omega f(t, x)||_{L^\infty(0,T;L^2(\Omega))} \leq e^{C/T} ||y_0||_{L^2(\Omega)}.$$

This part is mainly inspired by Coron and Nguyên's work. We find a sequence of $\{(\lambda_n, T_n)\}_{n\in\mathbb{N}^*}$ satisfying

$$\begin{gathered} 0 = T_0 < T_1 < T_2 < ... < T_n < ..., \\ 1 < \lambda_1 < \lambda_2 < ... < \lambda_n < ..., \\ \lim_{n\to\infty} T_n = T, \ \lim_{n\to\infty} \lambda_n = +\infty, \end{gathered}$$

such that on each time interval $[T_{n-1}, T_n)$ we apply the stationary feedback law $\mathcal{G}_{\lambda_n}$ that is related to the decay rate λ. Therefore, we are investiging the following closed-loop system:

$$\begin{cases} y_t = \Delta y + \chi_\omega \mathcal{G}_{\lambda_n} y, \quad \forall t \in [T_{n-1}, T_n), \forall n \in \mathbb{N}^*, \\ y|_{\partial\Omega} = 0, \\ y(0,\cdot) = y_0(\cdot). \end{cases}$$

By carefully choosing the values of $\{(\lambda_n, T_n)\}_{n\in\mathbb{N}^*}$, we can prove that for any given initial state $y(0)$, the unique solution of the system on $[0, T]$ satisfies

$$\lim_{t\to T^-} y(t) = 0 \text{ in } L^2(\Omega).$$

One such example is given in [36], where

$$\lambda_n = T - 1/n \text{ and } \lambda_n = Cn^4.$$

One can further verify that by adapting these controls, see Proposition 3.1 for details, ths control cost is given by

$$||\chi_\omega f(t,x)||_{L^\infty(0,T;L^2(\Omega))} \leq \mathrm{e}^{C/T^2} ||y_0||_{L^2(\Omega)}.$$

Actually, by a more delicate choice of candidates $\{(\lambda_n, T_n)\}_{n\in\mathbb{N}^*}$, we can recover Miller's result [19] on the optimal control cost

$$||\chi_\omega f(t,x)||_{L^2(0,T;L^2(\Omega))} \leq \mathrm{e}^{C/T} ||y_0||_{L^2(\Omega)}.$$

As a consequence, this result shows that Coron and Nguyên's finite time stabilization approach for null controllability also yields optimal costs. It is noteworthy that the whole control process is constructive and does not rely on duality arguments. Since no functional analysis argument is involved the result is on the space $L^\infty(0,T;L^2(\omega))$ which is slightly stronger than the typical space $L^2(0,T;L^2(\omega))$ in the literature [18].

We also refer to the recent work of Beauchard and Pravda-Starov [39], Theorem 2.1, where, they have provided an abstract characterization on such cost bounds for a large class of degenerate parabolic equations. It will be interesting to know whether the frequency Lyapunov approach can be adapted to such systems.

Step 3. Piecewise feedback law for finite time stabilization.

In this final step we extend the feedback law in period $[0, T]$ designed in *Step 2* periodically to $t \in \mathbb{R}$. A priori, thanks to Lemma 1.4, we know that for every $y \in L^2(\Omega)$, for every $s \in \mathbb{R}$ there is $\Phi(t, s; y) = 0$ for every $t \in [s + T, +\infty)$, which is nearly the property (i) of the main theorem. In order to obtain finite time stabilization, it suffices to modify the feedback law such that the uniform stability property is satisfied. Remark that such a step of modifying the feedback law is necessary, otherwise we can find a sequence of $\{(t_n, y_n)\}_{n\in\mathbb{N}^*} \in (0, T) \times L^2(\Omega)$ satisfying $\|y_n\|_{L^2(\Omega)} \leq 1/n$ but we are not able to demonstrate from direct well-posedness estimates that $\|\Phi(t, t_n; y_n)\|_{L^2(\Omega)} \leq 1, \forall t \in (t_n, t_n + T)$.

1.4. Conclusion

In conclusion we solve the finite time stabilization problem for the multidimensional heat equation. The main novelty of this paper is on the introduction of the "frequency Lyapunov method": benefiting from spectral inequality estimates and constructive Lyapunoc techniques, the simple and physical feedback law rapidly stabizes the system together with quantitative estimates, namely quantitative rapid stabilization. As a consequence of the finite time stabilization, we provide a constructive approach to the null-controllability of linear and nonlinear heat equations. Moreover, this control also gives the optimal cost $e^{C/T}$.

This method for quantitative rapid stabilization and finite time stabilization is robust, and it can be applied to various linear and nonlinear parabolic systems such as diffusive equations, Navier-Stokes equations, harmonic map heat flow and others.

Organization of the paper. The rest of this paper is organized as follows. Successively we prove the lemmas concerning quantitative rapid stabilization and null controllability in Section 2, Section 3. In Section 2 we also comment on the stabilization of nonlinear systems and the stabilization on higher regularity spaces. Finally, in Section 4 we combine these two lemmas to proof the main theorem.

Statement on notations: for readers convenience we summarize some notations and constants that will be defined and used later on. Moreover, once a constant is defined, from then on we will use it directly. Notations (τ_i, e_i) and $N(\lambda)$ about eigenvalues defined in Section 2.2; orthogonal projection $P_N, P_N^{\perp}$ defined after equation (2.12); truncated operator $\mathcal{K}_r$ in (2.3); γ_λ and μ_λ in (2.13); feedback law $\mathcal{F}_\lambda$ in (2.14); r_λ in (2.15); the partition T_n, λ_n, and I_n by (3.1). All the following constants are independent of $\lambda > 0$: C_1 defined in Proposition 2.5; C_2 in equation (2.15); Γ by (3.5) and C_3 by (3.8).

2. Quantitative rapid stabilization

2.1. Well-posedness results

In this preliminary section we briefly review the well-posedness results for the following Cauchy problem

$$\begin{cases} y_t = \Delta y + f(t,x), \ (t,x) \in (t_1,t_2) \times \Omega, \\ y(t,x) = 0, \ \ (t,x) \in (t_1,t_2) \times \partial\Omega, \\ y(t_1,x) = y_0(x), \end{cases} \tag{2.1}$$

as well as the related closed-loop systems with stationary feedback laws *i.e.* $f(t,x) = \mathcal{L}y$, where $\mathcal{L}$ is a bounded operator on $L^2(\Omega)$.

The well-posedness for both open-loop systems and closed-loop systems with stationary feedback laws are well-known, here we adapt the definition of the solution in the transposition sense, for which the well-poseness results are derived from classical Hille–Yosida semi-group theory. Transposition sense solution is introduced by Lions [3], for those who are not familiar with those definitions, we refer to the monograph by Coron [40], Chapters 1 and 2 for an excellent introduction on this subject.

Definition 2.1. Let $t_1, t_2 \in \mathbb{R}$ be such that $t_1 < t_2$. Let $y_0 \in L^2(\Omega)$ and $f(t,x) \in L^2(t_1,t_2; L^2(\Omega))$. A solution to the Cauchy problem (2.1) is a function $y \in C^0([t_1,t_2]; L^2(\Omega)) \cap L^2(t_1,t_2; H_0^1(\Omega))$ such that, for every $\tau \in [t_1,t_2]$ and for every $\phi \in C^0([t_1,\tau]; H^1(\Omega))$ such that

$$\phi_t \in L^2((t_1,t_2)\times\Omega), \ \Delta\phi \in L^2((0,T)\times\Omega), \ \text{and} \ \phi(t,\cdot) \in H_0^1(\Omega) \ \forall\, t \in [t_1,\tau],$$

one has

$$\int_\Omega y(\tau,x)\phi(\tau,x)\mathrm{d}x-\int_\Omega y_0(x)\phi(t_1,x)\mathrm{d}x-\int_{t_1}^{\tau}\int_\Omega f\phi\mathrm{d}x\mathrm{d}t-\int_{t_1}^{\tau}\int_\Omega(\phi_t+\Delta\phi)y\mathrm{d}x\mathrm{d}t=0.$$

For this classical equation one has the following well-posedness result, see for instance [41].

Theorem 2.2. *For any $T\in(0,1]$, for any $y_0\in L^2(\Omega)$, and for any $f\in L^2(0,T;L^2(\Omega))$, the Cauchy problem (2.1) has a unique solution. Moreover, this solution satisfies*

$$\begin{aligned}||y(t)||_{L^2(\Omega)}&\le||y_0||_{L^2(\Omega)}+2||f||_{L^2(0,t;L^2(\Omega))},\ \forall\,t\in(0,T],\\ ||\nabla y||_{L^2(0,t;L^2(\Omega))}&\le||y_0||_{L^2(\Omega)}+2||f||_{L^2(0,t;L^2(\Omega))},\ \forall\,t\in(0,T].\end{aligned}$$

The definition of solutions to closed-loop systems with stationary feedback laws is classical, and it can be regarded as a special case of the more general definition of solutions for time-varying feedback systems, which will be explained in Section 4.1. Concerning closed-loop systems with stationary feedback laws we have the following well-posedness results.

Theorem 2.3. *Let $\varphi_i\in L^2(\Omega),1\le i\le n$ be given functions. Let $l_i:L^2(\Omega)\to\mathbb{R},1\le i\le n$ be given bounded linear operators. For any $y_0\in L^2(\Omega)$ the Cauchy problem*

$$\begin{cases}y_t=\Delta y+\chi_\omega\Big(\sum_{i=1}^n l_i(y)\varphi_i\Big),\ (t,x)\in(0,T)\times\Omega,\\ y(t,x)=0,\ (t,x)\in(0,T)\times\partial\Omega,\\ y(0,x)=y_0(x),\end{cases}$$

has a unique solution.

Similar results exist for non-linear Lipschitz stationary feedback laws, the proof of which is a simple modification based on fixed point arguments and *a priori* estimates. For $r\in(0,1/2]$ we introduce the cutoff function $\chi_r\in C^\infty(\mathbb{R};[0,1])$ and the Lipschitz operator $\mathcal{K}_r:L^2(\Omega)\to L^2(\Omega)$ satisfying

$$\chi_r(x)=1\text{ for }x\in[0,r],\quad\chi_r(x)=0\text{ for }x\in[2r,+\infty),\tag{2.2}$$

$$\mathcal{K}_r(y)=y\cdot\chi_r\left(||y||_{L^2(\Omega)}\right),\ \forall\,y\in L^2(\Omega).\tag{2.3}$$

Theorem 2.4. *Let $T\in(0,1]$. Let $r\in(0,1/2]$. Let $\varphi_i\in L^2(\Omega),1\le i\le n$ be given functions. Let $l_i:L^2(\Omega)\to\mathbb{R},1\le i\le n$ be given bounded linear operators. For any $y_0\in L^2(\Omega)$ the Cauchy problem*

$$\begin{cases}y_t=\Delta y+\chi_\omega\mathcal{K}_r\Big(\sum_{i=1}^n l_i(y)\varphi_i\Big),\ (t,x)\in(0,T)\times\Omega,\\ y(t,x)=0,\ (t,x)\in(0,T)\times\partial\Omega,\\ y(0,x)=y_0(x),\end{cases}$$

has a unique solution.

Proof. It suffices to prove the local (in time) existence of a unique solution. Moreover, it suffices to present the related *a priori* estimates. For ease of notations we simply denote the feedback law $1_\omega\mathcal{K}_r\Big(\sum_{i=1}^n l_i(y)\varphi_i\Big)$ as $\mathcal{H}(y)$, whose value is bounded by $C||y||_{L^2(\Omega)}$. The feedback law $\mathcal{H}$ is Lipschitz in $L^2(\Omega)$ with Lipschitz constant L.

For $||y_0||_{L^2(\Omega)}=R$, we select some $\widetilde{T}\le\min\{(4C)^{-2},(4L)^{-2}\}$ and denote $\mathcal{X}_{\widetilde{T}}$ by the Banach space $C^0([0,\widetilde{T}];L^2(\Omega))$ with the corresponding norm given by the $C^0([0,\widetilde{T}];L^2(\Omega))$-norm. We further define $\mathcal{X}_{\widetilde{T}}(2R)$

as the $2R$-radius closed ball in $\mathcal{X}_{\widetilde{T}}$. Next, for any $y \in \mathcal{X}_{\widetilde{T}}$ we denote $\mathcal{S}(y)$ as the unique solution of the Cauchy problem (2.1) on $[0, \widetilde{T}]$ with the initial state y_0 and the source term $\mathcal{H}(y)$. Thanks to Theorem 2.2, this solution is also in $L^2(0, \widetilde{T}; H_0^1(\Omega))$. By applying Theorem 2.2 one easily verifies that

$$||\mathcal{S}(y)||_{\mathcal{X}_{\widetilde{T}}} \leq 2R, \ \forall \ y \in \mathcal{X}_{\widetilde{T}}(2R),$$
$$||\mathcal{S}(y_1) - \mathcal{S}(y_2)||_{\mathcal{X}_{\widetilde{T}}} \leq \frac{1}{2}||y_1 - y_2||_{\mathcal{X}_{\widetilde{T}}}, \ \forall \ y_1, y_2 \in \mathcal{X}_{\widetilde{T}}(2R),$$

which, together with Banach fixed point theorem, yields the existence of the unique solution. □

2.2. Spectral inequality

Let us consider the Laplace operator with Dirichlet boundary condition $\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$, there is a Hilbert orthogonal basis of $L^2(\Omega)$:

$$0 < \tau_1 \leq \tau_2 \leq \tau_3 \leq ... \leq \tau_n \leq ...,$$
$$-\Delta e_i = \tau_i e_i \text{ with } e_i|_{\partial\Omega} = 0.$$

Different eigenvalues τ_n may coincide, but each eigenvalue only has finite algebraic multiplicity. For any given positive number $\lambda > 0$, we define $N(\lambda)$ the number of eigenvalues (counting multiplicity) that are not strictly bigger than λ, *i.e.* $\tau_{N(\lambda)} \leq \lambda < \tau_{N(\lambda)+1}$. Moreover, the distribution of $\{\tau_k\}_{k=1}^\infty$ obeys Weyl's law [42]: $N(\lambda) \sim (2\pi)^{-d}\omega_d \operatorname{vol}(\Omega)\lambda^{d/2}$, where ω_d is the volume of the unit ball. For ease of notations, in the following, if there is no confusion sometimes we simply denote N_λ by N.

Concerning the eigenfunctions $\{e_i\}_{i=1}^\infty$ one has the following well-known results:

Proposition 2.5. *The eigenfunctions $\{e_i\}_{i=1}^\infty$ satisfy*

(1) Orthonormal basis: $(e_i, e_j)_{L^2(\Omega)} = \delta_{ij}$.
(2) (Unique continuation) The symmetric matrix J_N given below is invertible,

$$J_N := \left((e_i, e_j)_{L^2(\omega)}\right)_{i,j=1}^N. \tag{2.4}$$

(3) (Spectral inequality) There exist $C_1 \geq 1$ that is independent of $\lambda > 0$ such that

$$||\sum_{i=1}^{N(\lambda)} a_i e_i||^2_{L^2(\omega)} \geq C_1^{-1} \mathrm{e}^{-C_1\sqrt{\lambda}} \sum_{i=1}^{N(\lambda)} a_i^2.$$

Property (1) is a well-known result upon self-adjoint operators with compact resolvent. Property (2) is a direct consequence of the unique continuation property of the Dirichlet operator, we refer to the work by Barbu–Triggiani [23] for more general results. The estimate in Property (3) is also known as Lebeau–Robbiano inequality. This highly non-trivial observation was first discovered by Lebeau and Robbiano in [2], which played a significant role in their paper on the proof of the null controllability of the heat equation. Indeed, the form $\mathrm{e}^{\sqrt{\lambda}}$ is optimal once $\overline{\omega} \neq \Omega$, as illustrated in [6]. However, the optimality of the constant C_1, which clearly depends on the geometry of (Ω, ω), is still open. This kind of spectral inequalities has been extensively studied in the literature for different operators, for example, [43] for nodal sets of Laplace operator, [44] for bi-Laplace operators, [45, 46] for Laplace operators on measurable sets, [25] for degenerate one dimensional elliptic operators, [47] for Stokes operators, [12, 15] for zero measure sets, *etc.*

As a direct consequence of last property in the preceding proposition, we obtain the following quantitative estimate of J_N as quadratic form.

Lemma 2.6. *For $Y_{N(\lambda)} = (a_1, a_2, ..., a_{N(\lambda)})$, we have*

$$Y_{N(\lambda)}^T J_{N(\lambda)} Y_{N(\lambda)} \geq C_1^{-1} \mathrm{e}^{-C_1\sqrt{\lambda}} ||Y_{N(\lambda)}||_2^2.$$

Proof. Actually, letting N be presenting $N(\lambda)$, one directly obtain the following estimate from Proposition 2.5:

$$\begin{aligned} Y_N^T J_N Y_N &= \sum_{1\leq i,j\leq N} a_i \left(e_i, e_j\right)_{L^2(\omega)} a_j \\ &= \left(\sum_{i=1}^N a_i e_i, \sum_{j=1}^N a_j e_j\right)_{L^2(\omega)} \\ &= ||\sum_{i=1}^N a_i e_i||_{L^2(\omega)}^2 \\ &\geq C_1^{-1} \mathrm{e}^{-C_1\sqrt{\lambda}} ||Y_N||_2^2. \end{aligned}$$

□

2.3. Rapid stabilization *via* Lyapunov function approach and explicit feedback law

The following rapid stabilization result is inspired by the Lyapunov function idea introduced by Coron–Trélat [7], where it was used as an intermediate step for their proof of global controllability of steady states of non-linear parabolic equations in one dimensional space. This idea has been adapted to various models, for example [48] on global controllability of one dimensional wave equations and [49] for others. However, though relatively efficient and effectively calculable, no attempt on quantitative estimates has been made. Probably this is because in the proof some general theories as Kalman's rank condition and stabilization matrix are used.

Instead of using abstract stabilizing matrix arguments, here we construct precise Lyapunov functionals and quite surprisingly the spectral estimates by Lebeau–Robbiano are naturally used. That is the reason we get a quantitative rapid stabilization result with $C\mathrm{e}^{C\sqrt{\lambda}}$ estimates.

For any given $\lambda > 0$, we suggest control terms in forms of $\sum_i^{N(\lambda)} e_i|_\omega u_i(t)$ with $u_i(t) \in \mathbb{R}$, thus consider the following controlled problem:

$$y_t = \Delta y + \chi_\omega \left(\sum_{i=1}^{N(\lambda)} e_i u_i(t)\right) \quad \text{in } \Omega, \tag{2.5}$$

$$y = 0 \quad \text{on } \partial\Omega. \tag{2.6}$$

In the rest part of this section, we simply denote N_λ by N. By decomposing

$$y(t,x) = \sum_{i=1}^\infty \left(y(t), e_i\right)_{L^2(\Omega)} e_i = \sum_{i=1}^\infty y_i(t) e_i, \tag{2.7}$$

$$\chi_\omega e_j = \sum_{i=1}^\infty \left(1_\omega e_j, e_i\right)_{L^2(\Omega)} e_i = \sum_{i=1}^\infty \left(e_i, e_j\right)_{L^2(\omega)} e_i, \tag{2.8}$$

and by defining

$$X_N(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \\ ... \\ y_N(t) \end{pmatrix}, \quad U_N(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \\ ... \\ u_N(t) \end{pmatrix}, \quad A_N := \begin{pmatrix} -\tau_1 & & & \\ & -\tau_2 & & \\ & & ... & \\ & & & -\tau_N \end{pmatrix}, \tag{2.9}$$

we know, thanks to the definition of J_N in (2.4), that the finite dimensional system $X_N(t)$ satisfies

$$\dot{X}_N(t) = A_N X_N(t) + J_N U_N(t). \tag{2.10}$$

For any given λ (thus N is given), for $\gamma_\lambda, \mu_\lambda > 0$ that will be fixed later on, we suggest the feedback law

$$U_N(y(t)) := -\gamma_\lambda X_N(t), \tag{2.11}$$

as well as the Lyapunov function (for which we name as "Frequency Lyapunov")

$$V(y) := \mu_\lambda ||X_N||_2^2 + \left(P_N^\perp y, P_N^\perp y\right)_{L^2(\Omega)}, \ \forall y \in L^2(\Omega), \tag{2.12}$$

where $||X_N||_2^2$ is given by $X_N^T X_N = \sum_{i=1}^N y_i^2 = ||P_N y||^2_{L^2(\Omega)}$, P_N is the projection on $\mathrm{Vect}\{e_i\}_{i=1}^N$, and $P_N^\perp$ be its co-projection. Thanks to Theorem 2.3, the closed-loop system (2.5)–(2.11) is well-posed. According to the preceding feedback law, $y(t)$ and $X_N(t)$ satisfy

$$\begin{gathered}
\dot{X}_N(t) = A_N X_N(t) - \gamma_\lambda J_N X_N(t), \\
y_t = \Delta y - \gamma_\lambda \chi_\omega \left(\sum_{i=1}^N e_i y_i(t) \right) \ \text{ in } \Omega, \\
y = 0 \ \text{ on } \partial\Omega.
\end{gathered}$$

As we know from Theorem 2.4 that the solution of the above closed-loop system y is indeed in $C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, which further implies that X_N and $\frac{\mathrm{d}}{\mathrm{d}t}(X_N)$ are in $C^0([0,T]; \mathbb{R}^N)$, $\frac{\mathrm{d}}{\mathrm{d}t} y$ and $\frac{\mathrm{d}}{\mathrm{d}t}(P_N^\perp y)$ in $L^2(0,T; H^{-1}(\Omega))$. Moreover, based on [41], Theorem 10.11 and Banach fixed-point theorem, if the initial state y_0 further belongs to $H_0^1(\Omega)$, then the unique solution is indeed in $C^0([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$, which further implies that X_N and $\frac{\mathrm{d}}{\mathrm{d}t}(X_N)$ are in $C^0([0,T]; \mathbb{R}^N)$, $\frac{\mathrm{d}}{\mathrm{d}t} y$ and $\frac{\mathrm{d}}{\mathrm{d}t}(P_N^\perp y)$ in $L^2(0,T; L^2(\Omega))$. Thus, for any initial state $y_0 \in H_0^1(\Omega)$ there is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(y(t)) &= \mu_\lambda \frac{\mathrm{d}}{\mathrm{d}t} ||X_N||_2^2 + \frac{\mathrm{d}}{\mathrm{d}t} \left(P_N^\perp y, P_N^\perp y\right)_{L^2(\Omega)} \\
&= \mu_\lambda \frac{\mathrm{d}}{\mathrm{d}t} ||X_N||_2^2 + 2 \left(P_N^\perp y, \frac{\mathrm{d}}{\mathrm{d}t} P_N^\perp y\right)_{L^2(\Omega)} \\
&= \mu_\lambda \frac{\mathrm{d}}{\mathrm{d}t} ||X_N||_2^2 + 2 \left(P_N^\perp y, \frac{\mathrm{d}}{\mathrm{d}t} y\right)_{L^2(\Omega)} \\
&= \mu_\lambda \frac{\mathrm{d}}{\mathrm{d}t} ||X_N||_2^2 + 2 \left\langle P_N^\perp y, \frac{\mathrm{d}}{\mathrm{d}t} y\right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)}.
\end{aligned}$$

For any initial state $y_0 \in L^2(\Omega)$, by selecting a sequence of initial states $\{y_0^n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ which converges to y_0 in the sense of $L^2(\Omega)$, there is a sequence of solutions $\{y^n\}_{n\in\mathbb{N}} \subset C^0([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$. In addition, the solutions $\{y^n\}_{n\in\mathbb{N}}$, the low-frequency projections $\{X_N^n\}_{n\in\mathbb{N}}$, and high-frequency projections $\{P_N^\perp y^n\}_{n\in\mathbb{N}}$ satisfy

$$\begin{gathered}
\left(y^n, P_N^\perp y^n\right) \to \left(y, P_N^\perp y\right) \ \text{ in } C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)), \\
\left(X_N^n, \frac{\mathrm{d}}{\mathrm{d}t}(X_N^n)\right) \to \left(X^n, \frac{\mathrm{d}}{\mathrm{d}t}(X^n)\right) \ \text{ in } C^0([0,T]; \mathbb{R}^N),
\end{gathered}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}y^n, \frac{\mathrm{d}}{\mathrm{d}t}(P_N^{\perp}y^n)\right) \to \left(\frac{\mathrm{d}}{\mathrm{d}t}y, \frac{\mathrm{d}}{\mathrm{d}t}(P_N^{\perp}y)\right) \text{ in } L^2(0,T;H^{-1}(\Omega)).$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}V\left(y(t)\right) = \mu_\lambda \frac{\mathrm{d}}{\mathrm{d}t}||X_N||_2^2 + 2\left\langle P_N^{\perp}y, \frac{\mathrm{d}}{\mathrm{d}t}y\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)},$$

in the distribution sense in $L^1(0,T)$.
On the one hand we know that

$$\mu_\lambda \frac{\mathrm{d}}{\mathrm{d}t}||X_N||_2^2 = \mu_\lambda X_N^T\left(A_N^T + A_N - 2\gamma_\lambda J_N\right)X_N = 2\mu_\lambda X_N^T\big(A_N - \gamma_\lambda J_N\big)X_N \leq -2\mu_\lambda\gamma_\lambda C_1^{-1}\mathrm{e}^{-C_1\sqrt{\lambda}}||X_N||_2^2.$$

On the other hand we have

$$\begin{aligned}
2\left\langle P_N^{\perp}y, \frac{\mathrm{d}}{\mathrm{d}t}y\right\rangle_{H_0^1(\Omega)\times H^{-1}(\Omega)} &= 2\left\langle P_N^{\perp}y, \Delta y\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - 2\gamma_\lambda\left(P_N^{\perp}y, \chi_\omega\left(\sum_{i=1}^{N} e_i y_i(t)\right)\right)_{L^2(\Omega)}, \\
&= -2\sum_{i=N+1}^{\infty}\tau_i y_i^2 - 2\gamma_\lambda\left(P_N^{\perp}y, \chi_\omega(P_N y)\right)_{L^2(\Omega)}, \\
&\leq -2\lambda||P_N^{\perp}y||_{L^2(\Omega)}^2 + 2\gamma_\lambda||P_N^{\perp}y||_{L^2(\Omega)}||P_N y||_{L^2(\Omega)}, \\
&\leq -2\lambda||P_N^{\perp}y||_{L^2(\Omega)}^2 + \lambda||P_N^{\perp}y||_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda}||P_N y||_{L^2(\Omega)}^2, \\
&= -\lambda||P_N^{\perp}y||_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda}||X_N||_2^2.
\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(y(t)) \leq -2\mu_\lambda\gamma_\lambda C_1^{-1}\mathrm{e}^{-C_1\sqrt{\lambda}}||X_N||_2^2 - \lambda||P_N^{\perp}y||_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda}||X_N||_2^2.$$

Motivated from the above estimate, we choose

$$\gamma_\lambda := C_1\mathrm{e}^{C_1\sqrt{\lambda}}\lambda, \;\; \mu_\lambda := \frac{\gamma_\lambda^2}{\lambda^2} = C_1^2\mathrm{e}^{2C_1\sqrt{\lambda}}, \tag{2.13}$$

which further yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(y(t)) &\leq -2\mu_\lambda\lambda||X_N||_2^2 - \lambda||P_N^{\perp}y||_{L^2(\Omega)}^2 + \mu_\lambda\lambda||X_N||_2^2, \\
&\leq -\mu_\lambda\lambda||X_N||_2^2 - \lambda||P_N^{\perp}y||_{L^2(\Omega)}^2, \\
&= -\lambda\left(\mu_\lambda||X_N||_2^2 + ||P_N^{\perp}y||_{L^2(\Omega)}^2\right) = -\lambda V(y(t)).
\end{aligned}$$

Since $\mu_\lambda \geq 1$ for $C_1 \geq 1$, we know that,

$$||y(t)||_{L^2(\Omega)}^2 \leq V(y(t)) \leq \mathrm{e}^{-\lambda t}V(y(0)) \leq \mathrm{e}^{-\lambda t}\mu_\lambda||y(0)||_{L^2(\Omega)}^2 \leq C_1^2\mathrm{e}^{2C_1\sqrt{\lambda}}\mathrm{e}^{-\lambda t}||y(0)||_{L^2(\Omega)}^2,$$

thus

$$||y(t)||_{L^2(\Omega)} \leq C_1 \mathrm{e}^{C_1\sqrt{\lambda}} \mathrm{e}^{-\frac{\lambda}{2}t} ||y(0)||_{L^2(\Omega)}.$$

Moreover since the control (feedback) is given by

$$1_\omega f(t,x) = -\gamma_\lambda \chi_\omega \left(\sum_{i=1}^{N} e_i X_i(t) \right),$$

we know that

$$||f(t,\cdot)||_{L^2(\Omega)} \leq \gamma_\lambda ||X_{N(\lambda)}||_2 \leq \gamma_\lambda ||y(t)||_{L^2(\Omega)} \leq \lambda C_1^2 \mathrm{e}^{2C_1\sqrt{\lambda}} \mathrm{e}^{-\frac{\lambda}{2}t} ||y(0)||_{L^2(\Omega)}.$$

By applying the above explicit feedback law, we get the following rapid stabilization result. For any $\lambda > 0$ we define an explicit stationary feedback law $\mathcal{F}_\lambda : L^2(\Omega) \to L^2(\Omega)$,

$$\mathcal{F}_\lambda y := -\gamma_\lambda \left(\sum_{i=1}^{N(\lambda)} \Big(y, e_i\Big)_{L^2(\Omega)} e_i \right) = -\gamma_\lambda P_{N(\lambda)} y \text{ with } \gamma_\lambda = C_1 \mathrm{e}^{C_1\sqrt{\lambda}} \lambda, \tag{2.14}$$

where $P_{N(\lambda)}$ is the projection on the sub-space spanned by $\{e_i\}_{i=1}^{N(\lambda)}$, and $N(\lambda)$ is the number of eigenvalues (counting multiplicity) that are not strictly bigger than λ. Clearly, there exists $C_2 \geq 2C_1$ such that for all $\lambda > 0$,

$$\lambda C_1^2 \mathrm{e}^{2C_1\sqrt{\lambda}},\ C_1^2 \mathrm{e}^{2C_1\sqrt{\lambda}},\ \lambda C_1 \mathrm{e}^{C_1\sqrt{\lambda}},\ C_1 \mathrm{e}^{C_1\sqrt{\lambda}} \leq C_2 \mathrm{e}^{C_2\sqrt{\lambda}} =: \frac{1}{r_\lambda}. \tag{2.15}$$

Theorem 2.7. *For any $\lambda > 0$ the closed-loop system*

$$\begin{gathered} y_t = \Delta y + \chi_\omega \mathcal{F}_\lambda y \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

is exponentially stable. More precisely, for any $s \in \mathbb{R}$ the Cauchy problem

$$\begin{gathered} y_t = \Delta y - \gamma_\lambda \chi_\omega \left(\sum_{i=1}^{N(\lambda)} \Big(y(t), e_i\Big)_{L^2(\Omega)} e_i \right),\ \forall (t,x) \in [s,+\infty) \times \Omega, \\ y(t,x) = 0,\ \forall (t,x) \in [s,+\infty) \times \partial\Omega, \\ y(s,x) = y_0(x), \end{gathered}$$

has a unique solution in $C^0([s,+\infty); L^2(\Omega)) \cap L^2_{loc}(s,+\infty; H^1_0(\Omega))$, and this unique solution satisfies

$$||y(t)||_{L^2(\Omega)} \leq C_1 \mathrm{e}^{C_1\sqrt{\lambda}} \mathrm{e}^{-\frac{\lambda}{2}(t-s)} ||y_0||_{L^2(\Omega)},\ \forall\, t \in [s,+\infty), \tag{2.16}$$

$$||\mathcal{F}_\lambda y(t)||_{L^2(\Omega)} \leq C_2 \mathrm{e}^{C_2\sqrt{\lambda}} \mathrm{e}^{-\frac{\lambda}{2}(t-s)} ||y_0||_{L^2(\Omega)},\ \forall\, t \in [s,+\infty). \tag{2.17}$$

Remark 2.8. The choice of our feedback law $\mathcal{F}_\lambda y$ in (2.14) has strong physical meaning, as it is exactly the low frequency projection. Hence, it is easy to be adapted to other models as well as to $\mathbb{R}^n$ spaces.

Remark 2.9. After the first version of this manuscript appeared online, Alphonse and Martin further pointed out that the Lyapunov function (2.12) can be simply written as $\mu_\lambda ||P_N y||^2_{L^2} + ||P_N^\perp y||^2_{L^2}$, then the calculation of $\frac{\mathrm{d}}{\mathrm{d}t}||X_N||_2^2$ is slightly simpler:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(P_N y, P_N y\right)_{L^2(\Omega)} = \left(P_N y, \Delta y - \gamma_\lambda \chi_\omega P_N y\right)_{L^2(\Omega)} \le -\gamma_\lambda C_1^{-1} \mathrm{e}^{-C_1\sqrt{\lambda}} \left(P_N y, P_N y\right)_{L^2(\Omega)}.$$

They have successfully applied this method to rapid stabilization of a large class of diffusive equations from thick control supports [50]. The form that we used as X_N can clearly observe the evolution of the low frequency, while the second form is more direct on the use of the spectral inequality.

Remark 2.10. This quantitative result heavily depends on the spectral inequality Lemma 2.6. Even if we do not have such strong estimate, as mentioned in Proposition 2.5 the unique continuation property can still imply the positive definite of matrix J_N, which also provides some estimate on the quadratic form in Lemma 2.6. By adapting such a (nonquantitative) estimate and applying the same calculation we can still select γ and μ that lead to rapid stabilization.

2.4. Rapid stabilization for higher regularity

Actually the feedback law $\mathcal{F}_\lambda$ presented in (2.14) and Theorem 2.7 also stabilizes the system in $H^1_0(\Omega)$ space, let us briefly comment on this issue without going into details. We refer to Brezis [41], Chapters 9 and 10 and Lions–Magenes [51] for related well-posedness results.

Let the Hilbert space $H^1_0(\Omega)$ be endowed with scalar product $\int_\Omega \nabla u \cdot \nabla v$. The eigenfunctions $\{e_i/\sqrt{\tau_i}\}_{i=1}^\infty$ form an orthonormal basis of $H^1_0(\Omega)$. For any $y_0 \in H^1_0(\Omega)$ and any $f(t,x) \in L^2(0,T;L^2(\Omega))$, the Cauchy problem (2.1) admits a unique solution $y(t)$ in $C^0([0,T];H^1_0(\Omega)) \cap L^2(0,T;H^2(\Omega))$, thanks to the *a priori* estimate,

$$||\Delta y||^2_{L^2(0,t;L^2(\Omega))} + ||\nabla y(t)||^2_{L^2(\Omega)} \le ||\nabla y_0||^2_{L^2(\Omega)} + ||f||^2_{L^2(0,t;L^2(\Omega))},\ \forall\, t \in (0,T].$$

We further adapt the notations in the preceding section, and even the same choice of γ_λ. For any $\lambda > 0$, we consider the Lyapunov functional on $H^1_0(\Omega)$:

$$V_1(y) := \widetilde{\mu}_\lambda ||X_{N(\lambda)}||_2^2 + ||P^T_{N(\lambda)} y||^2_{H^1_0(\Omega)},\ \forall\, y \in H^1_0(\Omega),$$

with $\widetilde{\mu}_\lambda := \gamma_\lambda^2/\lambda$. It is actually equivalent to the $H^1_0(\Omega)$ norm by

$$\frac{1}{1+\lambda}||y||^2_{H^1_0(\Omega)} \le V_1(y) \le \left(1 + \frac{\widetilde{\mu}_\lambda}{\tau_1}\right) ||y||^2_{H^1_0(\Omega)},\ \forall\, y \in H^1_0(\Omega).$$

Then, similar estimation implies that the solution $y(t)$ of the closed-loop system (2.5)–(2.11) with feedback law $\mathcal{F}_\lambda$ verifies, where N means $N(\lambda)$,

$$\begin{aligned}
\dot{V}_1(y(t)) &\le -2\widetilde{\mu}_\lambda \gamma_\lambda C_1^{-1} \mathrm{e}^{-C_1\sqrt{\lambda}} ||X_N||_2^2 - 2\left(P_N^T \Delta y, \Delta y - \gamma_\lambda \chi_\omega (P_N y)\right)_{L^2(\Omega)},\\
&\le -2\lambda\widetilde{\mu}_\lambda ||X_N||_2^2 - \lambda ||P_N^T y||^2_{H^1_0(\Omega)} - ||P_N^T \Delta y||^2_{L^2(\Omega)} + ||P_N^T \Delta y||^2_{L^2(\Omega)} + \gamma_\lambda^2 ||P_N y||^2_{L^2(\Omega)},\\
&\le -2\lambda\widetilde{\mu}_\lambda ||X_N||_2^2 - \lambda ||P_N^T y||^2_{H^1_0(\Omega)} + \lambda \widetilde{\mu}_\lambda ||X_N||_2^2,\\
&\le -\lambda V_1(y(t)).
\end{aligned}$$

The stabilization on $H^1_0(\Omega)$ space becomes more important when it is combined with the Sobolev embedding $H^1(\Omega) \subseteq L^p(\Omega)$ with $p = \frac{2d}{d-2}$. As for stabilization for even higher regularities, $H^2(\Omega) \cap H^1_0(\Omega)$ for example, probably one needs to replace the control setting $\chi_\omega f$ by $a_\omega f$ with some smooth truncated function $a_\omega(x)$ that is supported in ω and equals to 1 in an open subset $\omega_1 \subset \omega$.

2.5. Rapid stabilization for nonlinear systems

Indeed the same feedback $\mathcal{F}_\lambda$ also stabilizes some nonlinear systems provided the initial data is small, we also comment on this issue without going into details. Again we adapt the notations and the choice of γ_λ in Section 2.3.

Let us consider the two dimensional subcritical heat equation with $\omega \subset \Omega \subset \mathbb{R}^2$,

$$\begin{gathered} y_t = \Delta y + y^3 - \gamma_\lambda \chi_\omega \left(P_{N(\lambda)} y\right) \quad \text{in } \Omega, \\ y = 0 \;\; \text{on } \partial\Omega. \end{gathered}$$

By defining

$$Y_{N(\lambda)}(t) := \Big((y^3(t), e_i)_{L^2(\Omega)}\Big)_{i=1}^{N(\lambda)},$$

and by denoting N by $N(\lambda)$, we know that

$$\dot{X}_N(t) = A_N X_N(t) - \gamma_\lambda J_N X_N(t) + Y_N(t).$$

As a consequence

$$\begin{aligned} \mu_\lambda \frac{\mathrm{d}}{\mathrm{d}t} ||X_N||_2^2 &= \mu_\lambda X_N^T \left(A_N^T + A_N - 2\gamma_\lambda J_N\right) X_N + \mu_\lambda \left(Y_N^T X_N + X_N^T Y_N\right) \\ &\le -2\mu_\lambda \lambda ||X_N||_2^2 - 2\mu_\lambda ||\nabla P_N y||^2_{L^2(\Omega)} + 2\mu_\lambda \int_\Omega y^3 \left(P_N y\right) \mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(P_N^\perp y, P_N^\perp y\right)_{L^2(\Omega)} &= 2 \left\langle P_N^\perp y, \Delta y - \gamma_\lambda \chi_\omega (P_N y) + y^3 \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)}, \\ &\le -\frac{3}{2}\lambda ||P_N^\perp y||^2_{L^2(\Omega)} - \frac{1}{2} ||\nabla P_N^\perp y||^2_{L^2(\Omega)} + \lambda ||P_N^\perp y||^2_{L^2(\Omega)} + \frac{\gamma_\lambda^2}{\lambda} ||X_N||_2^2 + 2\int_\Omega y^3 \left(P_N^\perp y\right) \mathrm{d}x, \\ &= -\frac{\lambda}{2} ||P_N^\perp y||^2_{L^2(\Omega)} + \mu_\lambda \lambda ||X_N||_2^2 - \frac{1}{2} ||\nabla P_N^\perp y||^2_{L^2(\Omega)} + 2\int_\Omega y^3 \left(P_N^\perp y\right) \mathrm{d}x. \end{aligned}$$

By recalling Ladyzhenskaya's inequality, which is a special case of the Gagliardo-Nirenberg interpolation inequality (see [41] for this general inequality),

$$||f||_{L^4(\Omega)} \le \sqrt{2} ||f||^{1/2}_{L^2(\Omega)} ||\nabla f||^{1/2}_{L^2(\Omega)}, \; \forall\, f \in H_0^1(\Omega),$$

we further get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(y(t)) &\le \left(-\frac{\lambda}{2}\right) V(y(t)) - \frac{1}{2} ||\nabla y||^2_{L^2(\Omega)} + 2\mu_\lambda ||y||^3_{L^4(\Omega)} ||P_N y||_{L^4(\Omega)} + 2||y||^3_{L^4(\Omega)} ||P_N^\perp y||_{L^4(\Omega)}, \\ &\le \left(-\frac{\lambda}{2}\right) V(y(t)) - \frac{1}{2} ||\nabla y||^2_{L^2(\Omega)} + 16\mu_\lambda ||y||^2_{L^2(\Omega)} ||\nabla y||^2_{L^2(\Omega)}, \\ &\le \left(-\frac{\lambda}{2}\right) V(y(t)) - ||\nabla y||^2_{L^2(\Omega)} \left(\frac{1}{2} - 16\mu_\lambda V(y(t))\right). \end{aligned}$$

Let us define

$$R_\lambda := \frac{r_\lambda}{8} \leq (8\mu_\lambda)^{-1}. \tag{2.18}$$

Then for any initial data y_0 satisfying $||y_0||_{L^2(\Omega)} \leq R_\lambda$ we have

$$V(y(t)) \leq \mathrm{e}^{-\frac{\lambda}{2}t} V(y(0)), \ \forall\, t \geq 0,$$

which further implies,

$$||y(t)||^2_{L^2(\Omega)} \leq V(y(t)) \leq \mathrm{e}^{-\frac{\lambda}{2}t} V(y(0)) \leq \mathrm{e}^{-\frac{\lambda}{2}t}\mu_\lambda ||y(0)||^2_{L^2(\Omega)} \leq C_1^2 \mathrm{e}^{2C_1\sqrt{\lambda}} \mathrm{e}^{-\frac{\lambda}{2}t}||y(0)||^2_{L^2(\Omega)}.$$

Therefore the unique solution of the closed-loop system decays exponentially as

$$||y(t)||_{L^2(\Omega)} \leq C_1 \mathrm{e}^{C_1\sqrt{\lambda}} \mathrm{e}^{-\frac{\lambda}{4}t} ||y_0||_{L^2(\Omega)}, \ \forall\, ||y_0||_{L^2(\Omega)} \leq R_\lambda, \ \forall\, t \in [0, +\infty).$$

In the end, let us remark that without the feedback law the solution may blow up.

3. Null controllability with optimal cost estimates

Armed with the $C\mathrm{e}^{C\sqrt{\lambda}}$ estimates (2.16)–(2.17), exactly the same procedure proposed in [36, 52, 53] by using piecewise stabilizing controls leads to the null controllability. In this section we construct similar feedback laws while keeping an extra attention on control costs. Two different kind of precise feedback laws (control) are considered with control costs $C\mathrm{e}^{\frac{C}{T^{1+\varepsilon}}}$ and $C\mathrm{e}^{\frac{C}{T}}$ respectively.

We mainly focus on the following weaker result, Theorem 3.1, for which the feedback law (control) is nice and the calculation is easy. Then, we can easily improve this result to the stronger ones, such as Corollary 3.2 and Theorem 3.3.

Proposition 3.1. *There exists $C_3 > 0$ such that, for any $T \in (0, 1)$ and for any $y_0 \in L^2(\Omega)$, we find an explicit control $f(t, x)$ for the control system (1.1)–(1) such that the unique solution with initial data $y(0, x) = y_0(x)$ satisfies $y(T) = 0$. Moreover, the controlling cost is given by,*

$$||\chi_\omega f(t, x)||_{L^\infty(0,T;L^2(\Omega))} \leq \mathrm{e}^{C_3/T^2}||y_0||_{L^2(\Omega)}.$$

Proof of Theorem 3.1. We treat only the case where $1/T$ is integer to simplify the presentation. Let us take some $\Gamma > 0$ independent of $T \in (0, 1)$ that will be fixed later on in equation (3.6).

Control design. *Let $T = \frac{1}{n_T}$ with $n_T \in \mathbb{N}^*$. We define,*

$$S_n := T - \frac{1}{n}, \ I_n := [S_n, S_{n+1}), \ \lambda_n := \Gamma^2 n^4 \text{ for } \forall\, n \geq n_T := \frac{1}{T}, \tag{3.1}$$

for any $n \geq n_T$ we consider the control (feedback law) as $\mathcal{F}_{\lambda_n}$ on interval I_n.

More precisely, first on I_{n_T} we consider the closed-loop system (1.1) with feedback law $\mathcal{F}_{\lambda_{n_T}}$ and $y(0, x) = y_0(x)$. According to Theorem 2.7 this system has a unique solution $\widetilde{y}|_{\bar{I}_{n_T}}$. Next, we consider the closed-loop system with feedback law $\mathcal{F}_{\lambda_{n_T+1}}$ and $y(S_{n_T+1}, x) := \widetilde{y}(S_{n_T+1}, x)$ on I_{n_T+1}, which, again, admit a unique solution $\widetilde{y}|_{\bar{I}_{n_T+1}}$. We continue this procedure on $\{I_n\}_{n=n_T}^\infty$ to find eventually a function $\widetilde{y}|_{[0,T)} \in C^0([0, T); L^2(\Omega))$ such that

$\widetilde{y}(T) := \lim_{t\to T^-} \widetilde{y}(t) = 0$, *and that*

$$\widetilde{y}|_{[0,T]} \text{ is the solution of the Cauchy problem (2.1) with control } 1_\omega f|_{I_n} = 1_\omega \mathcal{F}_{\lambda_n}\widetilde{y}|_{I_n}, \forall n \geq n_T.$$

We denote the above constructed solution by $y(t)$ which, by Theorem 2.7, satisfies

$$||y(t)||_{L^2(\Omega)} \leq C_1 \mathrm{e}^{C_1\Gamma n^2}\mathrm{e}^{-\frac{\Gamma^2 n^4}{2}(t-T_n)}||y(T_n)||_{L^2(\Omega)},\ \forall t\in I_n,\ \forall n\geq n_T, \tag{3.2}$$

$$||\mathcal{F}_{\lambda_n} y(t)||_{L^2(\Omega)} \leq C_2 \mathrm{e}^{C_2\Gamma n^2}\mathrm{e}^{-\frac{\Gamma^2 n^4}{2}(t-T_n)}||y(T_n)||_{L^2(\Omega)},\ \forall t\in I_n,\ \forall n\geq n_T. \tag{3.3}$$

Therefore, for $n \geq n_T + 1$ the value of the solution on S_n is controlled by,

$$||y(S_n)||_{L^2(\Omega)} \leq \left(\prod_{k=n_T}^{n-1} C_1 \mathrm{e}^{C_1\Gamma k^2}\mathrm{e}^{-\frac{\Gamma^2 k^2}{4}}\right)||y_0||_{L^2(\Omega)}. \tag{3.4}$$

Inspired by the preceding estimates, we choose the constant $\Gamma > 0$ be such that

$$C_1\mathrm{e}^{C_1\Gamma n^2}, C_2\mathrm{e}^{C_2\Gamma n^2} \leq \mathrm{e}^{\frac{\Gamma^2}{16}n^2},\ \forall n\in\mathbb{N}^*. \tag{3.5}$$

It suffices to take

$$\Gamma := 32(C_1 + C_2). \tag{3.6}$$

The above choice of Γ, combined with (3.4), lead to

$$||y(S_n)||_{L^2(\Omega)} \leq \left(\prod_{k=n_T}^{n-1} \mathrm{e}^{-\frac{3\Gamma^2}{16}k^2}\right)||y_0||_{L^2(\Omega)},\ \forall\ n\geq n_T+1. \tag{3.7}$$

Essentially, it already implies that $y(T_n)$ is strictly decaying to 0 at time T. Next, we concentrate on its cost, *i.e.* the norm of the control term. From (3.2), (3.3), (3.5), and (3.7) we know that for $n \geq n_T + 1$ and $t \in I_n$,

$$\begin{aligned} ||y(t)||_{L^2(\Omega)}, ||\mathcal{F}_{\lambda_n} y(t)||_{L^2(\Omega)} &\leq \mathrm{e}^{\frac{\Gamma^2}{16}n^2}\left(\prod_{k=n_T}^{n-1}\mathrm{e}^{-\frac{3\Gamma^2}{16}k^2}\right)||y_0||_{L^2(\Omega)}, \\ &\leq \exp\left(-\frac{\Gamma^2}{16}\Big(3\big(\sum_{k=n_T}^{n-1}k^2\big)-n^2\Big)\right)||y_0||_{L^2(\Omega)} \leq ||y_0||_{L^2(\Omega)}. \end{aligned}$$

On the other hand, for $n = n_T$ we know that

$$||y(t)||_{L^2(\Omega)}, ||\mathcal{F}_\lambda y(t)||_{L^2(\Omega)} \leq \mathrm{e}^{\frac{\Gamma^2}{16}n_T^2}||y_0||_{L^2(\Omega)} = \mathrm{e}^{\frac{\Gamma^2}{16T^2}}||y_0||_{L^2(\Omega)} = \mathrm{e}^{\frac{C_3}{T^2}}||y_0||_{L^2(\Omega)}, \tag{3.8}$$

where $C_3 := \frac{\Gamma^2}{16}$.

In conclusion, the constructed solution $y(t,x)$ with control $1_\omega f(t,x)$ satisfies

$$||y(t)||_{L^2(\Omega)} \text{ and } ||\chi_\omega f(t,\cdot)||_{L^2(\Omega)} \longrightarrow 0^+,\ \text{ as } t\to T^-,$$

$$||y(t)||_{L^2(\Omega)} \text{ and } ||\chi_\omega f(t,\cdot)||_{L^2(\Omega)} \leq \mathrm{e}^{\frac{\Gamma^2}{16T^2}}||y_0||_{L^2(\Omega)},\ \forall\ t\in[0,T],$$

which completes the proof. □

Actually Theorem 3.1 can be easily improved to the following one *via* simple modification on the choice of T_n and λ_n.

Corollary 3.2. *For any $\varepsilon \in (0,1)$, there exists $C_3^\varepsilon > 0$ such that, for any $T \in (0,1)$, for any $y_0 \in L^2(\Omega)$, we can find an explicit control $f(t,x)$ for the control system (1.1) such that the unique solution with initial data $y(0,x) = y_0(x)$ satisfies $y(T,x) = 0$. Moreover, the cost is controlled by*

$$||\chi_\omega f(t,x)||_{L^\infty(0,T;L^2(\Omega))} \le \mathrm{e}^{C_3^\varepsilon/T^{1+\varepsilon}}||y_0||_{L^2}.$$

Idea of the proof. Indeed, it suffices to take

$$S_{k,n} := T - \frac{1}{n^k},\ \lambda_{k,n} := \Gamma_\varepsilon^2 n^{2(k+1)} \text{ for } \forall n \ge n_T := \frac{1}{T^{1/k}}, \tag{3.9}$$

for some $k \ge 1/\varepsilon$, and to find some suitable Γ_ε. We observe that the energy decay on interval $I_{k,n}$ is dominated by

$$C_1 \mathrm{e}^{C_1 \Gamma_\varepsilon n^{k+1}} \mathrm{e}^{-c\Gamma_\varepsilon^2 n^{k+1}},$$

which allows us to find some Γ_ε satisfying (3.5) type estimates. □

However, $\mathrm{e}^{C/T^{1+\varepsilon}}$ is the best estimate that we can achieve from partitions of type (3.9), which is slightly weaker than the optimal costs [19]: $\mathrm{e}^{C/T}$. Eventually with another choice of partition we can also get the optimal cost from our constructive stabilization approach.

Theorem 3.3 (Optimal costs)**.** *There exists $C_3^0 > 0$ such that, for any $T \in (0,1)$ and for any $y_0 \in L^2(\Omega)$, we find an explicit control $f(t,x)$ for the control system (1.1) such that the unique solution with initial data $y(0,x) = y_0(x)$ satisfies $y(T,x) = 0$. Moreover, the controlling cost is given by*

$$||\chi_\omega f(t,x)||_{L^\infty(0,T;L^2(\Omega))} \le \mathrm{e}^{C_3^0/T}||y_0||_{L^2}.$$

Proof. As illustrated above we adapt another type of construction to get this optimal result. For the ease of presentation, we only consider the case $1/T = 2^{n_0}$ with $n_0 \in N^*$. More precisely, we consider the following partition as well as the piecewise controlling method explained in the proof of Proposition 3.1 (see Control design),

$$S_n^0 := 2^{-n_0}\left(1 - \frac{1}{2^n}\right),\ I_n^0 := [S_n^0, S_{n+1}^0),\ \lambda_n^0 := Q^2 2^{2(n_0+n)} \text{ for } \forall\, n \ge 0,$$

where $Q > 0$ is a given constant satisfying

$$C_1 \mathrm{e}^{C_1 Q m}, C_2 \mathrm{e}^{C_2 Q m} \le \mathrm{e}^{\frac{Q^2}{16} m},\ \forall\, m \ge 1.$$

Suppose that $y(t)$ is the unique solution satisfying the designed control, then for $n \ge 1$ we are able to estimate $y(S_n)$ by

$$||y(S_n)||_{L^2(\Omega)} \le \left(\prod_{k=0}^{n-1} C_1 \mathrm{e}^{C_1\sqrt{\lambda_k^0}} \mathrm{e}^{-\frac{\lambda_k^0}{2} 2^{-(n_0+k+1)}}\right)||y_0||_{L^2(\Omega)},$$

$$\leq \left(\prod_{k=0}^{n-1} C_1 \mathrm{e}^{C_1 Q 2^{n_0+k}} \mathrm{e}^{-\frac{Q^2}{4} 2^{n_0+k}}\right) ||y_0||_{L^2(\Omega)},$$
$$\leq \left(\prod_{k=0}^{n-1} \mathrm{e}^{-\frac{Q^2}{8} 2^{n_0+k}}\right) ||y_0||_{L^2(\Omega)}.$$

For $n \geq 1$, the preceding estimate further implies that the control term on $t \in I_n^0$ satisfies,

$$||\chi_\omega \mathcal{F}_{\lambda_n^0} y(t)||_{L^2(\Omega)} \leq C_2 \mathrm{e}^{C_2 Q 2^{n_0+n}} ||y(S_n)||_{L^2(\Omega)} \leq \mathrm{e}^{\frac{Q^2}{16} 2^{n_0+n}} \left(\prod_{k=0}^{n-1} \mathrm{e}^{-\frac{Q^2}{8} 2^{n_0+k}}\right) ||y_0||_{L^2(\Omega)} \leq ||y_0||_{L^2(\Omega)}.$$

Therefore, the $L^\infty(0, T; L^2(\Omega))$ norm of the control term $1_\omega f$ is dominated by its $L^\infty(S_0^0, S_1^0; L^2(\Omega))$ norm. As a consequence, we know that for any $t \in [0, T]$,

$$||\chi_\omega f(t,x)||_{L^2(\Omega)} \leq C_2 \mathrm{e}^{C_2 Q 2^{n_0}} ||y_0||_{L^2(\Omega)} \leq \mathrm{e}^{\frac{C_3^0}{T}} ||y_0||_{L^2(\Omega)} \text{ with } C_3^0 = \frac{Q^2}{16}.$$

□

Remark 3.4. It is noteworthy that the $\mathrm{e}^{C/T}$ type cost is optimal to many other systems, for example the Stokes system [47] where similar spectral inequality is proved.

4. Finite time stabilization

In this section, we construct T-periodic proper feedback laws that stabilize system (1.1) in finite time: Theorem 4.4.

4.1. Time-varying feedback laws and finite time stabilization

We are interested in time-varying feedback laws, more precisely proper feedback laws. The following definition of time-varying feedback laws that allows the closed-loop system admit a unique solution borrows directly from the paper [37].

First, we recall the closed-loop system associated to a time-varying feedback law U.

$$\begin{cases} y_t = \Delta y + \chi_\omega U(t; y), \ \forall\, (t,x) \in (s, +\infty) \times \Omega, \\ y = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

Definition 4.1. Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. Let

$$\begin{array}{rcl} U : [s_1, s_2] \times L^2(\Omega) & \to & L^2(\Omega) \\ (t; y) & \mapsto & U(t; y). \end{array}$$

Let $t_1 \in [s_1, s_2]$, $t_2 \in (t_1, s_2]$, and $y_0 \in L^2(\Omega)$. A solution on $[t_1, t_2]$ to the Cauchy problem associated to the closed-loop system (4.1) with initial data y_0 at time t_1 is some $y : [t_1, t_2] \to L^2(\Omega)$ such that

$$t \in (t_1, t_2) \mapsto f(t,x) := U(t; y(t)) \in L^2(t_1, t_2; L^2(\Omega)),$$
y is a solution (see Def. 2.1) of (2.1) with initial data y_0 at time t_1 and the above $1_\omega f(t,x)$.

The so-called *proper* feedback laws is a time-varying feedback law such that the closed-loop system always admit a unique solution.

Definition 4.2. Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. A proper feedback law on $[s_1, s_2]$ is an application

$$\begin{array}{rcl} U : [s_1, s_2] \times L^2(\Omega) & \to & L^2(\Omega) \\ (t; y) & \mapsto & U(t; y). \end{array}$$

such that, for every $t_1 \in [s_1, s_2]$, for every $t_2 \in (t_1, s_2]$, and for every $y_0 \in L^2(\Omega)$, there exists a unique solution on $[t_1, t_2]$ to the Cauchy problem associated to the closed-loop system (4.1) with initial data y_0 at time t_1 according to Definition 4.1.

A proper feedback law is an application U

$$\begin{array}{rcl} U : \mathbb{R} \times L^2(\Omega) & \to & L^2(\Omega) \\ (t; y) & \mapsto & U(t; y). \end{array}$$

such that, for every $s_1 \in \mathbb{R}$ and for every $s_2 \in \mathbb{R}$ satisfying $s_1 < s_2$, the feedback law restricted to $[s_1, s_2] \times L^2(\Omega)$ is a proper feedback law on $[s_1, s_2]$.

For a **proper** feedback law, one can define the **flow** $\Phi : \Delta \times L^2(\Omega) \to L^2(\Omega)$, with $\Delta := \{(t, s);\, t > s\}$ associated to this feedback law: $\Phi(t, s; y_0)$ is the value at time t of the solution y to the closed-loop system (4.1) which is equal to y_0 at time s.

Finally we state the exact definition of the finite time stabilization.

Definition 4.3 (Finite time stabilization of the heat equation)**.** Let $T > 0$. A T-periodic proper feedback law U stabilizes the heat equation in finite time, if the flow Φ of the closed-loop system (4.1) satisfies,

(i) ($2T$ stabilization) $\Phi(2T + t, t; y_0) = 0,\ \ \forall t \in \mathbb{R},\ \forall\, y_0 \in L^2(\Omega)$.
(ii) (Uniform stability) For every $\delta > 0$, there exists $\eta > 0$ such that

$$\left(||(y_0||_{L^2(\Omega)} \leq \eta\right) \Rightarrow \left(||\Phi(t, t'; y_0)||_{L^2(\Omega)} \leq \delta,\ \forall\, t' \in \mathbb{R},\ \forall\, t \in (t', t' + 2T)\right).$$

4.2. Finite time stabilization of the multidimensional heat equation

We focus on the case when $1/T$ be an integer, as the other cases can be trivially treated using time transition.

Different from null controllability we do not pay extra attention to the "stabilizing cost" with respect to T. Indeed, we directly apply the feedback law constructed in the preceding section as combination of stationary feedback laws $\mathcal{F}_{\lambda_n}$ on interval I_n. Then, $\mathcal{F}_{\lambda_n}$ can be regarded as "λ_n frequency" feedback, which is sensible with respect to the states for large n. For example, for some given $y_0 \in L^2(\Omega)$ we consider the Cauchy problem of the closed-loop system with $\mathcal{F}_{\lambda_n}$ and $y(T_n) = y_0$. Thanks to Theorem 2.7, $||y(t)||$ is uniformly bounded by $C_1 e^{C_1\sqrt{\lambda}}||y_0||$ on I_n. By letting n tends to ∞, we are not allowed to get "uniform stability", as commented in Remark 1.2.

Therefore, we introduce some truncated operator on feedback laws, especially for high frequencies λ, to guarantee "uniform stability". However, in this case the natural *a priori* bound for the Cauchy problem that can be expected is $C_\varepsilon||y_0|| + \varepsilon, \forall \varepsilon > 0$. As a result the cost cannot be bounded by $C||y_0||$, that explains why we do not characterize the stabilizing cost in details with respect to T. However, thanks to the precise construction of the feedback laws that will be presented in this section, for any given T an effectively computable stabilizing cost depending on "starting time" and "initial state" can be obtained.

The construction of the T-periodic feedback law.

Before stating the detailed stabilizing theorem, we first recall the following notations and facts:

(D1) we have defined in (2.3) a truncated operator

$$\mathcal{K}_r : L^2(\Omega) \to L^2(\Omega),$$
$$||\mathcal{K}_r(y)||_{L^2(\Omega)} \leq \min\{1, ||y||_{L^2(\Omega)}\};$$

(D2) for every given λ, we have defined in (2.14) a stationary feedback law

$$\mathcal{F}_\lambda : L^2(\Omega) \to L^2(\Omega),$$
$$||\mathcal{F}_\lambda|| \leq C_2 \mathrm{e}^{C_2\sqrt{\lambda}} = (r_\lambda)^{-1}$$

(D3) for every $T \in (0,1)$ satisfying $1/T = n_T \in \mathbb{N}$, we have defined in (3.1) and (3.6) a sequence of $\{\lambda_n, S_n\}_{n\geq n_T}$:

$$S_n := T - \frac{1}{n}, \quad I_n := [S_n, S_{n+1}), \quad \lambda_n := \Gamma^2 n^4.$$

Now, we show that the above constructed feedback law, $\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n} y)$, satisfies

$$||\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n} y)||_{L^2(\Omega)} \leq \min\{1, \sqrt{2||y||_{L^2(\Omega)}}\}. \tag{4.2}$$

Indeed, by the choice of $\mathcal{K}_{r_{\lambda_n}}$ and $\mathcal{F}_{\lambda_n}$ we prove the preceding inequality by two steps.

If $||\mathcal{F}_{\lambda_n} y||_{L^2(\Omega)} \leq 2r_{\lambda_n}$, then

$$||\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n} y)||_{L^2(\Omega)} \leq ||\mathcal{F}_{\lambda_n} y||_{L^2(\Omega)} \leq \sqrt{2r_{\lambda_n}||\mathcal{F}_{\lambda_n} y||_{L^2(\Omega)}} \leq \sqrt{2||y||_{L^2(\Omega)}},$$

moreover, it is also bounded by 1 as $2r_{\lambda_n} \leq 1$.

If $||\mathcal{F}_{\lambda_n} y||_{L^2(\Omega)} > 2r_{\lambda_n}$, then by the choice of $\mathcal{K}_{r_{\lambda_n}}$ we know that $\mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n} y) = 0$.

Theorem 4.4 (Semi-global finite time stabilization of the heat equation)**.** *Let $T = 1/n_T \in (0,1)$ with $n_T \in \mathbb{N}^*$. Let $\Lambda \geq 1$. For any integer $N_T > n_T$, the T-periodic feedback law $U(t;y) : \mathbb{R} \times L^2(\Omega) \to L^2(\Omega)$ given by*

$$U\big|_{[0,T)\times L^2(\Omega)}(t;y) := \begin{cases} \mathcal{F}_{\lambda_n} y, & \forall\, y \in L^2(\Omega), \forall\, t \in I_n, \forall\, n_T \leq n \leq N_T, \\ \mathcal{K}_{r_{\lambda_n}}(\mathcal{F}_{\lambda_n} y), & \forall\, y \in L^2(\Omega), \forall\, t \in I_n, \forall\, n \geq N_T + 1, \end{cases} \tag{4.3}$$

is a proper feedback law for system (4.1).

Moreover, for an effectively computable large N_T the feedback law (4.3) stabilizes system (4.1) in finite time:

(i) (2T stabilization) $\Phi(2T+t, t; y_0) = 0, \ \forall\, t \in \mathbb{R}, \ \forall\, ||y_0||_{L^2(\Omega)} \leq \Lambda$.
(ii) (Uniform stability) For every $\delta > 0$, there exists an effectively computable $\eta > 0$ such that

$$\left(||(y_0||_{L^2(\Omega)} \leq \eta\right) \Rightarrow \left(||\Phi(t, t'; y_0)||_{L^2(\Omega)} \leq \delta, \ \forall\, t' \in \mathbb{R}, \ \forall\, t \in (t', t'+2T)\right).$$

Proof of Theorem 4.3. Thanks to the $C\mathrm{e}^{C\sqrt{\lambda}}$ estimate, the proof of Theorem 4.4 is rather standard. Here we mimic the treatment for similar results on one dimensional parabolic equations [36]. The proof is followed by three steps: the feedback law is proper; condition (i); and condition (ii).

Step 1. First, we show that the feedback law given by (4.3) is proper. Without loss of generality, we only need to prove that for any $s \in [0, T)$ and for any $y_0 \in L^2(\Omega)$ the Cauchy problem

$$\begin{cases} y_t = \Delta y + \chi_\omega U(t; y), \ (t, x) \in (s, T) \times \Omega, \\ y(t, x) = 0, \ (t, x) \in (s, T) \times \partial\Omega, \\ y(s, x) = y_0(x), \end{cases}$$

has a unique solution y, and $\lim_{t \to T^-} y(t) \in L^2(\Omega)$. Actually, the existence of a unique solution on each interval I_n follows directly from Theorem 2.3 for $n \le N_T$ and from Theorem 2.4 for $n > N_T$. Hence,

$$y|_{[s,T)}(t) \in C^0([s, T); L^2(\Omega)).$$

Moreover, $||y(t)||_{L^2(\Omega)}$ is uniformly bounded on $[s, T)$ thanks to Theorem 2.7 and Theorem 2.2. Therefore, the control term on time interval $[s, T)$ is uniformly bounded in $L^2(\Omega)$, thus by applying Theorem 2.2 again we know that

$$y|_{[s,T]}(t) \in C^0([s, T]; L^2(\Omega)) \cap L^2(s, T; H_0^1(\Omega)).$$

Or equivalently, $\lim_{t \to T^-} y(t) \in L^2(\Omega)$ can be proved by the Cauchy sequence argument suggested in [36], p. 1018 for (4.42). Therefore, the flow $\Phi(s, t; y)$ is well-defined on $\Delta \times L^2(\Omega)$.

Step 2. Next, we need to find a suitable integer N_T such that the proper feedback law (4.3) stabilize system (4.1) in finite time, mainly focus on condition (i).

Lemma 4.5. *The following energy estimate concerning the flow of the closed-loop system holds,*

$$||\Phi(T, s; y_0)||_{L^2(\Omega)} \le 2 + e^{\frac{\Gamma^2}{16} N_T^2} ||y_0||_{L^2(\Omega)}, \ \forall \ ||y_0||_{L^2(\Omega)} \le \Lambda, \ \forall \ s \in [0, T). \tag{4.4}$$

Proof of Lemma 4.5. For any given $||y_0||_{L^2(\Omega)} \le \Lambda$.

If $s \in [S_{N_T+1}, T)$, then since the feedback is bounded by 1, Theorem 2.2 yields

$$||\Phi(T, s; y_0)||_{L^2(\Omega)} \le ||y_0||_{L^2(\Omega)} + 2.$$

If $s \in [0, S_{N_T+1})$, then we estimate $||\Phi(T_{N_T+1}, s; y_0)||_{L^2(\Omega)}$. Suppose that $s \in I_n$ with $n_T \le n \le N_T$, then direct calculation shows that (recalling some estimates from Sect. 3, especially (3.2)–(3.7)),

$$\begin{aligned} ||\Phi(S_{N_T+1}, s; y_0)||_{L^2(\Omega)} &\le C_1 e^{C_1\sqrt{\lambda_n}} \left(\prod_{k=n+1}^{N_T} C_1 e^{C_1\sqrt{\lambda_k}} e^{-\frac{\lambda_k}{2}(S_{k+1}-S_k)} \right) ||y_0||_{L^2(\Omega)}, \\ &\le C_1 e^{C_1 \Gamma n^2} \left(\prod_{k=n+1}^{N_T} C_1 e^{C_1 \Gamma k^2} e^{-\frac{\Gamma^2}{4} k^2} \right) ||y_0||_{L^2(\Omega)}, \\ &\le e^{\frac{\Gamma^2}{16} n^2} \left(\prod_{k=n+1}^{N_T} e^{-\frac{3\Gamma^2}{16} k^2} \right) ||y_0||_{L^2(\Omega)}, \\ &\le e^{\frac{\Gamma^2}{16} N_T^2} ||y_0||_{L^2(\Omega)}. \end{aligned}$$

Next, for $\widetilde{y}(S_{N_T+1}) := \Phi(S_{N_T+1}, s; y_0)$ we adapt the case that $s \in [S_{N_T+1}, T)$ to get the required result. $\square$

By applying Lemma 4.5 we know that for any $s\in[0,T)$ and for any $||y_0||_{L^2(\Omega)}\leq \Lambda$,

$$||\Phi(T,s;y_0)||_{L^2(\Omega)}\leq 2+\mathrm{e}^{\frac{\Gamma^2}{16}N_T^2}\Lambda.$$

Let us define $\widetilde{y}(T):=\Phi(T,s;y_0)$. The next step is to show that $\Phi(2T,T;\widetilde{y}(T))=0$, which requires us to seek for suitable N_T such that for every $n\geq N_T+1$ we have,

$$K_{r_{\lambda_n}}\left(\mathcal{F}_{\lambda_n}\Phi(t,T;\widetilde{y}(T))\right)=\mathcal{F}_{\lambda_n}\Phi(t,T;\widetilde{y}(T)),\forall\, t\in I_n+T. \tag{4.5}$$

For ease of notations we simply denote the unique solution of the closed-loop system by

$$\widetilde{y}(t):=\Phi(t,T;\widetilde{y}(T)),\forall\, t\in[T,2T].$$

Thus, in order to prove (4.5) it suffices to show that

$$||\mathcal{F}_{\lambda_n}\widetilde{y}(t)||_{L^2(\Omega)}\leq r_{\lambda_n},\forall\, t\in I_n+T,\ \forall\, n\geq N_T+1.$$

As we know from the proof of Lemma 4.5, or from (3.7), that

$$||\widetilde{y}(S_{N_T+1})||_{L^2(\Omega)}\leq\left(\prod_{k=n_T}^{N_T}\mathrm{e}^{-\frac{3\Gamma^2}{16}k^2}\right)||\widetilde{y}(T)||_{L^2(\Omega)}\leq 2\mathrm{e}^{\frac{\Gamma^2}{16}N_T^2}\Lambda\left(\prod_{k=n_T}^{N_T}\mathrm{e}^{-\frac{3\Gamma^2}{16}k^2}\right),$$

and that for $n\geq N_T+1$,

$$||\widetilde{y}(S_n)||_{L^2(\Omega)}\leq 2\mathrm{e}^{\frac{\Gamma^2}{16}N_T^2}\Lambda\left(\prod_{k=n_T}^{N_T}\mathrm{e}^{-\frac{3\Gamma^2}{16}k^2}\right)\left(\prod_{k=N_T+1}^{n-1}\mathrm{e}^{-\frac{3\Gamma^2}{16}k^2}\right),$$

and that for $n\geq N_T+1$ and $t\in I_n+T$,

$$||\mathcal{F}_{\lambda_n}\widetilde{y}(t)||_{L^2(\Omega)}\leq||\widetilde{y}(S_n)||_{L^2(\Omega)}C_2\mathrm{e}^{C_2\Gamma n^2}\leq||\widetilde{y}(S_n)||_{L^2(\Omega)}\mathrm{e}^{\frac{\Gamma^2}{16}n^2}.$$

Therefore, it suffices to find N_T such that for every $n\geq N_T+1$ we have

$$\left(2\mathrm{e}^{\frac{\Gamma^2}{16}N_T^2}\Lambda\left(\prod_{k=n_T}^{N_T}\mathrm{e}^{-\frac{3\Gamma^2}{16}k^2}\right)\right)\left(\prod_{k=N_T+1}^{n-1}\mathrm{e}^{-\frac{3\Gamma^2}{16}k^2}\right)\mathrm{e}^{\frac{\Gamma^2}{16}n^2}\leq\mathrm{e}^{-\frac{\Gamma^2}{16}n^2}.$$

Thus one only needs to find the existence of N_T such that

$$2\mathrm{e}^{\frac{\Gamma^2}{16}N_T^2}\Lambda\left(\prod_{k=n_T}^{N_T}\mathrm{e}^{-\frac{3\Gamma^2}{16}k^2}\right)\mathrm{e}^{\frac{\Gamma^2}{8}(N_T+1)^2}\leq 1,$$

which is obviously possible for any given $\Lambda>1$.

Step 3. Finally, in order to complete the proof of finite time stabilization, it only remains to prove that the proper feedback law given by (4.3) satisfies condition (ii): uniform stability.

Thanks to Step 2 we know the existence of C such that

$$||\Phi(t,T;y_0)||_{L^2(\Omega)}\leq C||y_0||_{L^2(\Omega)},\forall\,||y_0||_{L^2(\Omega)}\leq 1,\forall\, t\in[T,+\infty).$$

As a consequence for any $\delta > 0$ there exists $\widetilde{\eta} \in (0, \delta)$ such that

$$||\Phi(t, T; y_0)||_{L^2(\Omega)} \leq \delta, \ \forall \, ||y_0||_{L^2(\Omega)} \leq \widetilde{\eta}, \ \forall \, t \in [T, +\infty). \tag{4.6}$$

Moreover, there exists some $\varepsilon > 0$ such that

$$||\Phi(t, s; y_0)||_{L^2(\Omega)} \leq \widetilde{\eta}, \ \forall \, ||y_0||_{L^2(\Omega)} \leq \varepsilon, \ \forall \, s \in [0, T), \ \forall \, t \in [s, T]. \tag{4.7}$$

Indeed, thanks to Theorem 2.2 and the fact that $||\mathcal{K}_{r_{\lambda_n}}(y)|| \leq 1$, there exists $\widetilde{T} \in (0, T)$ such that

$$||\Phi(t, s; y_0)||_{L^2(\Omega)} \leq \widetilde{\eta}, \ \forall \, ||y_0||_{L^2(\Omega)} \leq \widetilde{\eta}/2, \ \forall \, s \in [\widetilde{T}, T), \ \forall \, t \in [s, T]. \tag{4.8}$$

Because the time-varying feedback law U on $[0, \widetilde{T})$ is given by finitely many stationary feedback laws, there exists some $\varepsilon \in (0, \widetilde{\eta}/2)$ such that

$$||\Phi(t, s; y_0)||_{L^2(\Omega)} \leq \widetilde{\eta}/2, \ \forall \, ||y_0||_{L^2(\Omega)} \leq \varepsilon, \ \forall \, s \in [0, \widetilde{T}), \ \forall \, t \in [s, \widetilde{T}]. \tag{4.9}$$

In conclusion, inequalities (4.8)–(4.9) yields (4.7); then estimates (4.6)–(4.7), as well as the fact that $\Phi(2T, s; y_0) = 0$, imply the uniform stability condition (ii). $\square$

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