ESAIM: COCV 30 (2024) 6
https://doi.org/10.1051/cocv/2024001

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

EXACT INTERNAL CONTROLLABILITY AND EXACT INTERNAL SYNCHRONIZATION FOR A KIND OF FIRST ORDER HYPERBOLIC SYSTEM

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Abstract. In this paper we investigate the exact controllability and exact synchronization in L^2 space for a 1-D first order linear hyperbolic system with internal controls located on some part of the domain. Based on the exact boundary controllability theory, we first use the constructive method to establish the exact controllability by internal controls for a time reversible system with inhomogeneous boundary conditions. The method is then adapted to prove the exact internal null controllability for a general system with homogeneous boundary conditions. These results can be then used to establish the exact internal synchronization for the system, and related subjects are further studied.

Mathematics Subject Classification. 93B05, 93C20, 35L40, 37N35.

Received January 1, 2023. Accepted December 27, 2023.

1. Introduction

First order hyperbolic systems are widely used to model various systems in real life. The exact (null) controllability of 1-D first order hyperbolic systems with boundary controls has been intensively studied even in the quasi-linear case in [1–9], and recently in [10–17], for solutions in L^∞, L^2 or C^1 spaces. However, there are few results for the internal controllability of first order hyperbolic systems. Using an extension method, the authors of [18] established the local exact controllability in C^1 space for 1-D first order quasilinear hyperbolic systems with both boundary and internal controls, and a situation when only internal controls are needed for the controllability was also discussed. While the authors of [19] were more interested in the controllability with a reduced number of controls, which has been a challenging problem for the last decades (see, for example, [20–23]). In [19], firstly, by the method of characteristics, the sufficient and necessary conditions are obtained for the exact controllability by internal controls for the 1-D first order linear hyperbolic system with the same velocity with periodic boundary conditions; then, the local exact controllability is established by only one internal control for a 2×2 1-D first order quasilinear hyperbolic system with different velocities with periodic boundary conditions by means of the fixed point theorem of Gromov based on the notion of algebraic solvability for partial differential

Keywords and phrases: Exact internal controllability, exact internal synchronization, first order linear hyperbolic system.

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operators, but this requires higher regularity property for the initial and final data. Moreover, there are also some results for the internal controllability of other types of systems such as wave equations, parabolic systems and so on ([20, 23–26] *etc.*).

On the other hand, with a reduced number of controls, since the exact controllability cannot be realized in many cases, some weak properties of the system are taken into consideration, such as the exact boundary synchronization, the approximate boundary controllability or the approximate boundary synchronization (see [27] and references therein). The synchronization is a widespread natural phenomenon and has been studied in many applications (see [28–31] *etc.*). In PDEs case it was firstly and then systematically studied for a coupled system of wave equations with different boundary controls in the framework of weak solutions ([27, 32–43] *etc.*). Recently, the study of synchronization has been extended from wave equations to first order hyperbolic systems, which contain a larger range of systems including wave equations. The one-sided exact boundary synchronization has been discussed in [44] for a kind of first order quasilinear hyperbolic system with negative spreading speeds in the framework of classical solutions. In [45] we studied the exact synchronization in L^2 space in the framework of weak solutions by one-sided boundary control for the 1-D first order linear hyperbolic system. We also quote some other works like the optimal control for the exact synchronization of parabolic system in [46], the synchronization of distributed parameter systems on networks in [47] and references therein.

This paper will deal with the exact controllability and exact synchronization in L^2 space for a 1-D first order linear hyperbolic system with general boundary conditions with internal controls that are located on some part of the domain. Based on the one-sided exact boundary controllability established in [16], the constructive method will be applied to construct the internal control to realize the exact controllability of the system, then similarly to the discussion of boundary synchronization ([45]), the internal synchronization can be established for the system, and the corresponding exactly synchronizable states can be equally discussed. In addition, the non-exact internal (null) controllability will also be discussed by the method of duality, and some examples will be given in the end.

We first give some preliminaries in the Introduction. In Section 2 we establish the exact internal controllability for system (1.1)–(1.3), and the exact internal null controllability for the system with homogeneous boundary conditions in Section 3. Corresponding observability will be presented in Section 4, and the non-exact internal (null) controllability for the system will be discussed in Section 5. The exact internal synchronization together with the corresponding exactly synchronizable states will be taken into consideration in Section 6. Finally, some examples of systems that can not realize the exact internal (null) controllability or the exact internal synchronization will be given in Section 7.

Consider the following 1-D first order linear hyperbolic system

$$U_t + \Lambda U_x + AU = B\Theta, \qquad t \in (0, +\infty), \quad x \in (0, L) \tag{1.1}$$

with the boundary conditions

$$U^+(t, 0) = G_0 U^-(t, 0) + H^+(t), \quad t \in (0, +\infty), \tag{1.2}$$
$$U^-(t, L) = G_1 U^+(t, L) + H^-(t), \quad t \in (0, +\infty) \tag{1.3}$$

and the initial data

$$t = 0: \quad U(0, x) = U_0(x), \quad x \in (0, L), \tag{1.4}$$

where $U = U(t, x) : (0, +\infty) \times (0, L) \to \mathbb{R}^N$ denotes the state variable, and $\Theta = \Theta(t, x) : (0, +\infty) \times (0, L) \to \mathbb{R}^M (M \leq N)$ is the internal control. $\Lambda = \mathrm{diag}\{\Lambda^-, \Lambda^+\}$ is a diagonal matrix of order N, such that

$$\Lambda^- := \mathrm{diag}\{\lambda_1, \cdots, \lambda_{N^-}\}, \quad \Lambda^+ := \mathrm{diag}\{\lambda_{N^-+1}, \cdots, \lambda_N\} \tag{1.5}$$

with $\lambda_r < 0(r = 1, \cdots, N^-)$ and $\lambda_s > 0(s = N^- + 1, \cdots, N)$, the coupling matrix $A = (a_{ij})$ is of order N, the internal control matrix B is of order $N \times M$ and full column-rank. Let $N^+ = N - N^-$. The boundary coupling matrices G_0 and G_1 are of order $N^+ \times N^-$ and $N^- \times N^+$, respectively. All the matrices mentioned above are with constant elements. Moreover, $U = (U^-, U^+)^T$ with $U^- = (u_1, \dots, u_{N^-})^T$ and $U^+ = (u_{N^-+1}, \dots, u_N)^T$, $H = (H^-, H^+)^T$ with $H^- = (h_1, \dots, h_{N^-})^T$ and $H^+ = (h_{N^-+1}, \dots, h_N)^T$.

The following well-posedness can be easily obtained from Lemma 3.2 in [12] or Theorem 3.3 in [16] by the characteristic method and a standard approximation procedure.

Lemma 1.1. *For any given $T > 0$, for any given initial data $U_0 \in (L^2(0, L))^N$, any given function $\Theta \in (L^2(0, T; L^2(0, L)))^M$ and any given boundary function $H \in (L^2(0, T))^N$, the mixed problem (1.1)–(1.4) admits a unique weak solution $U = U(t, x) \in (C^0([0, T]; L^2(0, L)))^N$, satisfying*

$$\|U(t, \cdot)\|_{(L^2(0,L))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|\Theta\|_{(L^2(0,T;L^2(0,L)))^M} + \|H\|_{(L^2(0,T))^N}), \quad \forall t \in [0, T], \tag{1.6}$$

$$\|U(\cdot, 0)\|_{(L^2(0,T))^N} \le c(\|U_0\|_{(L^2(0,L))^N} + \|\Theta\|_{(L^2(0,T;L^2(0,L)))^M} + \|H\|_{(L^2(0,T))^N}) \tag{1.7}$$

and

$$\|U(\cdot, L)\|_{(L^2(0,T))^N} \le c(\|U_0\|_{(L^2(0,L))^N} + \|\Theta\|_{(L^2(0,T;L^2(0,L)))^M} + \|H\|_{(L^2(0,T))^N}), \tag{1.8}$$

here and hereafter, c denotes a positive constant.

Similarly to Remark 3.4 in [16], we have

Lemma 1.2. *Assume that the number of positive eigenvalues is equal to that of negative ones, namely, $N^- = N^+ = \frac{N}{2}$. Assume furthermore that $G_i(i = 0, 1)$ are invertible, namely, rank$(G_0) =$ rank$(G_1) = \frac{N}{2}$. Then system (1.1)–(1.3) is time reversible.*

For the homogeneous system

$$U_t + \Lambda U_x + AU = 0, \qquad t \in (0, +\infty), \quad x \in (0, L) \tag{1.9}$$

with (1.2)–(1.3), by Lemma 4.2 in [16], we have the following one-sided exact boundary controllability, which will be used to prove the internal controllability for system (1.1).

Lemma 1.3. *Assume that the number of positive eigenvalues is not larger than that of negative ones, namely, $N^+ \le N^-$ (i.e. $N \le 2N^-$). Assume furthermore that rank$(G_0) = N^+$. Let $T \geqslant \bar{T}_0$, where*

$$\bar{T}_0 = L\Big(\max_{1 \le r \le N^-} \frac{1}{|\lambda_r|} + \max_{N^-+1 \le s \le N} \frac{1}{\lambda_s}\Big) > 0. \tag{1.10}$$

Let $H^+(t) \in (L^2(0, T))^{N^+}$. For any given initial data $U_0(x) \in (L^2(0, L))^N$, there exist a boundary control $H^-(t) \in (L^2(0, T))^{N^-}$, such that problem (1.9) and (1.2)–(1.4) admits a unique weak solution $U = U(t, x) \in (C^0([0, T]; L^2(0, L)))^N$, satisfying exactly

$$t = T: \ U(T, x) = U_T(x), \quad x \in (0, L). \tag{1.11}$$

where $U_T(x) \in (L^2(0, L))^N$ denotes any fixed final data. Moreover, we have

$$\|H^-\|_{(L^2(0,T))^{N^-}} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H^+\|_{(L^2(0,T))^{N^+}}). \tag{1.12}$$

For system with homogenous boundary conditions on the side without controls, the exact boundary null controllability can be easily realized by one-sided boundary controls.

Lemma 1.4. *(Lemma 4.3 in [16]) Let $T \geq \bar{T}_0$, where $\bar{T}_0$ is given by (1.10). Assume that $H^-(t) \equiv 0$ (resp. $H^+(t) \equiv 0$), then for any given initial data $U_0(x) \in (L^2(0,L))^N$, there exists a boundary control $H^+(t) \in (L^2(0,T))^{N^+}$ (resp. $H^-(t) \in (L^2(0,T))^{N^-}$), satisfying*

$$\|H^+\|_{(L^2(0,T))^{N^+}} \leqslant c\|U_0\|_{(L^2(0,L))^N} \quad (\textit{resp.} \ \ \|H^-\|_{(L^2(0,T))^{N^-}} \leqslant c\|U_0\|_{(L^2(0,L))^N}), \tag{1.13}$$

such that problem (1.9) and (1.2)–(1.4) admits a unique weak solution $U = U(t,x) \in (C^0([0,T];L^2(0,L)))^N$, satisfying the null final state at the time $t = T$:

$$t = T: \ U(T,x) = 0, \quad x \in (0,L). \tag{1.14}$$

2. Exact internal controllability

In this section, we will establish the exact internal controllability for system (1.1)–(1.3) by means of the constructive method proposed in [9] and [3], for this purpose, the internal controls will be acting on a part of the domain, and the number of the internal controls should be equal to the number of the state variables. The proof is inspired by Proposition 3.2 in [19], in which the two-sided exact boundary controllability was applied to deal with the periodic boundary conditions of the system, however, we will use the one-sided exact boundary controllability instead of the two-sided exact boundary controllability in the construction of the solution and the control. Moreover, the proof involves solving a backward problem of system (1.9), so that the system is required to be time reversible, thus, by Lemma 1.2, we should assume that

$$N^- = N^+ = \frac{N}{2} \quad \text{and} \quad \text{rank}(G_0) = \text{rank}(G_1) = \frac{N}{2}. \tag{2.1}$$

Theorem 2.1. *Assume that (2.1) is satisfied. Let $0 \leqslant a < b \leqslant L$ and let*

$$T > T_0 \overset{def.}{=} \frac{L-(b-a)+|L-(b+a)|}{2}\big(\max_{1\leqslant r\leqslant N^-}\frac{1}{|\lambda_r|} + \max_{N^-+1\leqslant s\leqslant N}\frac{1}{\lambda_s}\big). \tag{2.2}$$

Let $\delta > 0$ be so small that $b - a > 2\delta$,

$$T - 2\delta \geqslant (a+\delta)\big(\max_{1\leqslant r\leqslant N^-}\frac{1}{|\lambda_r|} + \max_{N^-+1\leqslant s\leqslant N}\frac{1}{\lambda_s}\big) \tag{2.3}$$

and

$$T - 2\delta \geqslant (L-b+\delta)\big(\max_{1\leqslant r\leqslant N^-}\frac{1}{|\lambda_r|} + \max_{N^-+1\leqslant s\leqslant N}\frac{1}{\lambda_s}\big). \tag{2.4}$$

If rank$(B) = N$, then for any given initial data $U_0(x) \in (L^2(0,L))^N$ and final data $U_T(x) \in (L^2(0,L))^N$, for any given boundary function $H(t) \in (L^2(0,T))^N$, there exists a control $\Theta \in (C^0([0,T];L^2(0,L)))^N$ with supp $\Theta \subseteq [\delta, T-\delta] \times [a,b]$, satisfying

$$\|\Theta\|_{(C^0([0,T];L^2(0,L)))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}), \tag{2.5}$$

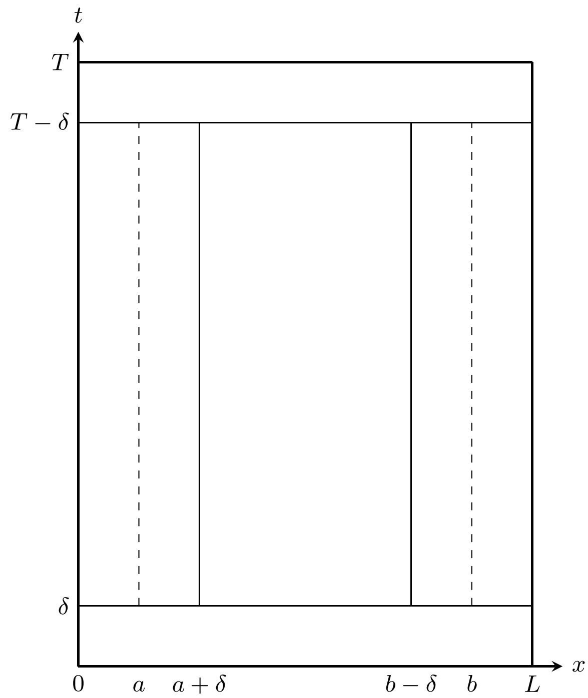

FIGURE 1. Division of the area.

such that the mixed problem (1.1)–(1.4) admits a unique weak solution $U = U(t,x) \in (C^0([0,T]; L^2(0,L)))^N$, *satisfying exactly the final state at the time* $t = T$*:*

$$t = T: \quad U(T,x) = U_T(x), \quad x \in (0,L). \tag{2.6}$$

Proof. We will construct the internal control by the following steps (see Fig. 1):

1. Solving a forward mixed problem (1.9) and (1.2)–(1.4) on the domain $R(T) = \{(t,x)|t \in (0,T), x \in (0,L)\}$, by Lemma 1.1, we obtain the unique solution $U = U_f(t,x) \in (C^0([0,T]; L^2(0,L)))^N$, satisfying

$$\|U_f\|_{(C^0([0,T];L^2(0,L)))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}), \tag{2.7}$$

here and hereafter c denotes a positive constant. Let $w(x) \overset{\text{def.}}{=} U_f(\delta,x) \in (L^2(0,L))^N$. We have

$$\|w\|_{(L^2(0,L))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}). \tag{2.8}$$

2. Noting (2.1), by Lemma 1.2, we can solve a backward mixed problem (1.9), (1.2)–(1.3) and (2.6) on the domain $R(T) = \{(t,x)|t \in (0,T), x \in (0,L)\}$, and obtain the unique solution $U = U_b(t,x) \in (C^0([0,T]; L^2(0,L)))^N$, satisfying

$$\|U_b\|_{(C^0([0,T];L^2(0,L)))^N} \leqslant c(\|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}). \tag{2.9}$$

Let $v(x) \overset{\text{def.}}{=} U_b(T-\delta,x) \in (L^2(0,L))^N$. We have

$$\|v\|_{(L^2(0,L))^N} \leqslant c(\|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}). \tag{2.10}$$

3. On the domain $R_l = \{(t,x)|t \in (\delta, T-\delta), x \in (0, a+\delta)\}$, we consider the mixed problem of system (1.9) with the initial condition

$$t = \delta : \quad U(\delta, x) = w(x), \quad x \in (0, a+\delta) \tag{2.11}$$

and the boundary conditions (1.2) and

$$x = a+\delta : \quad U^- = \varphi(t), \quad t \in (\delta, T-\delta). \tag{2.12}$$

Noting (2.1), (2.3), (2.8) and (2.10), by Lemma 1.3, there exists a boundary control $\varphi(t) \in (L^2(\delta, T-\delta))^N$, satisfying

$$\begin{aligned} \|\varphi\|_{(L^2(\delta,T-\delta))^N} &\leqslant c(\|w\|_{(L^2(0,a+\delta))^N} + \|v\|_{(L^2(0,a+\delta))^N} + \|H^+\|_{(L^2(\delta,T-\delta))^{N^+}}) \\ &\leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}), \end{aligned} \tag{2.13}$$

such that this problem admits a unique solution $U = U_l(t,x) \in (C^0([\delta, T-\delta]; L^2(0, a+\delta)))^N$, which satisfies exactly the final condition

$$t = T-\delta : \quad U(T-\delta, x) = v(x), \quad x \in (0, a+\delta), \tag{2.14}$$

and, noting (1.6), (2.8) and (2.13), we have

$$\begin{aligned} \|U_l\|_{(C^0([\delta,T-\delta];L^2(0,a+\delta)))^N} &\leqslant c(\|w\|_{(L^2(0,a+\delta))^N} + \|H^+\|_{(L^2(\delta,T-\delta))^{N^+}} + \|\varphi\|_{(L^2(\delta,T-\delta))^N}) \\ &\leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}). \end{aligned} \tag{2.15}$$

Let

$$\widetilde{U}_l(t,x) = \begin{cases} U_l(t,x), & t \in [\delta, T-\delta], \quad x \in (0, a+\delta), \\ 0, & t \in [\delta, T-\delta], \quad x \in (a+\delta, L), \end{cases} \tag{2.16}$$

we have $\widetilde{U}_l = \widetilde{U}_l(t,x) \in (C^0([\delta, T-\delta]; L^2(0,L)))^N$. Noting (2.15), we have

$$\|\widetilde{U}_l\|_{(C^0([\delta,T-\delta];L^2(0,L)))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}). \tag{2.17}$$

4. Similarly, on the domain $R_r = \{(t,x)|t \in (\delta, T-\delta), x \in (b-\delta, L)\}$, we consider the mixed problem of system (1.9) with the initial condition

$$t = \delta : \quad U(\delta, x) = w(x), \quad x \in (b-\delta, L) \tag{2.18}$$

and the boundary conditions (1.3) and

$$x = b-\delta : \quad U^+ = \psi(t), \quad t \in (\delta, T-\delta). \tag{2.19}$$

Noting (2.1), (2.4), (2.8) and (2.10), by Lemma 1.3, there exists a boundary control $\psi(t) \in (L^2(\delta, T-\delta))^N$, satisfying

$$\begin{aligned} \|\psi\|_{(L^2(\delta,T-\delta))^N} &\leqslant c(\|w\|_{(L^2(b-\delta,L))^N} + \|v\|_{(L^2(b-\delta,L))^N} + \|H^-\|_{(L^2(\delta,T-\delta))^{N^-}}) \\ &\leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}), \end{aligned} \tag{2.20}$$

such that this problem admits a unique solution $U = U_r(t,x) \in (C^0([\delta, T-\delta]; L^2(b-\delta, L)))^N$, which satisfies exactly the final condition

$$t = T-\delta: \quad U(T-\delta, x) = v(x), \quad x \in (b-\delta, L), \tag{2.21}$$

and, noting (1.6), (2.8) and (2.20), we have

$$\begin{aligned}\|U_r\|_{(C^0([\delta,T-\delta];L^2(b-\delta,L)))^N} &\leqslant c(\|w\|_{(L^2(b-\delta,L))^N} + \|H^-\|_{(L^2(\delta,T-\delta))^{N^-}} + \|\psi\|_{(L^2(\delta,T-\delta))^N}) \\ &\leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}).\end{aligned} \tag{2.22}$$

Let

$$\widetilde{U}_r(t,x) = \begin{cases} U_r(t,x), & t \in [\delta, T-\delta], \quad x \in (b-\delta, L), \\ 0, & t \in [\delta, T-\delta], \quad x \in (0, b-\delta). \end{cases} \tag{2.23}$$

We have $\widetilde{U}_r = \widetilde{U}_r(t,x) \in (C^0([\delta, T-\delta]; L^2(0,L)))^N$. Noting (2.22), we have

$$\|\widetilde{U}_r\|_{(C^0([\delta,T-\delta];L^2(0,L)))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}). \tag{2.24}$$

5. Let

$$\overline{U} = \overline{U}(t,x) = \eta(t) U_f(t,x) + (1-\eta(t)) U_b(t,x), \quad t \in [\delta, T-\delta], \ x \in (0,L), \tag{2.25}$$

where $\eta = \eta(t) \in C^1[\delta, T-\delta]$ with $0 \leqslant \eta \leqslant 1$, such that

$$\eta(\delta) = 1 \quad \text{and} \quad \eta(T-\delta) = 0. \tag{2.26}$$

It is easy to check that

$$\overline{U}(\delta, x) = w(x), \quad \overline{U}(T-\delta, x) = v(x), \quad x \in (0,L).$$

Moreover, noting (2.7) and (2.9), we have $\overline{U} \in (C^0([\delta, T-\delta]; L^2(0,L)))^N$ and

$$\|\overline{U}\|_{(C^0([\delta,T-\delta];L^2(0,L)))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}). \tag{2.27}$$

Let $\xi = \xi(x) \in C^1[0,L]$ with $0 \leqslant \xi \leqslant 1$, such that

$$\xi(x) = \begin{cases} 1, & x \in [0,a] \cup [b,L], \\ 0, & x \in [a+\delta, b-\delta]. \end{cases} \tag{2.28}$$

Let

$$\widetilde{U} = \widetilde{U}(t,x) = \xi(x)(\widetilde{U}_l(t,x) + \widetilde{U}_r(t,x)) + (1-\xi(x))\overline{U}(t,x), \quad t \in [\delta, T-\delta], \ x \in (0,L). \tag{2.29}$$

It is easy to see that

$$\widetilde{U}(\delta, x) = w(x), \quad \widetilde{U}(T-\delta, x) = v(x), \quad x \in (0,L).$$

Noting (2.17), (2.24) and (2.27), we have $\widetilde{U} \in (C^0([\delta, T-\delta]; L^2(0,L)))^N$ and

$$\|\widetilde{U}\|_{(C^0([\delta,T-\delta];L^2(0,L)))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}). \tag{2.30}$$

6. Define $U = U(t,x) \in (C^0([0,T]; L^2(0,L)))^N$ as

$$U(t,x) = \begin{cases} U_f(t,x), & t \in [0,\delta),\ x \in (0,L), \\ \widetilde{U}(t,x), & t \in [\delta, T-\delta],\ x \in (0,L), \\ U_b(t,x), & t \in (T-\delta, T],\ x \in (0,L). \end{cases} \tag{2.31}$$

Obviously, $U(t,x)$ satisfies the initial condition (1.4), the final condition (2.6) and the boundary conditions (1.2) and (1.3). Noting that B is an invertible square matrix, let

$$\Theta = \Theta(t,x) = B^{-1}(U_t + \Lambda U_x + AU). \tag{2.32}$$

Noting (2.16), (2.23), (2.25), (2.29) and (2.31), since U_f, U_b, U_l, U_r all satisfy (1.9), it is easy to see that $\Theta \in (C^0([0,T]; L^2(0,L)))^N$, and by the construction of U, we have supp $\Theta \subseteq [\delta, T-\delta] \times [a,b]$. Noting (2.7), (2.9) and (2.30), we get

$$\|U\|_{(C^0([0,T];L^2(0,L)))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|U_T\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^N}), \tag{2.33}$$

then, we have (2.5). Thus, we obtain the desired exact internal controllability with the internal control Θ. □

Remark 2.2. The estimate of controllable time (2.2) guarantees the one-sided exact boundary controllability on the left and right domains R_l and R_r, respectively.

Remark 2.3. Noting (2.2), when the controls act on the whole domain $R(T) = \{(t,x) | t \in (0,T), x \in (0,L)\}$, namely, when

$$\delta = 0, \quad a = 0 \quad \text{and} \quad b = L, \tag{2.34}$$

the exact internal controllability can be realized at any time $t = T > 0$. In fact, in this case, the constructive procedure can be simplified: (2.3) and (2.4) are no longer needed, moreover, after above Steps 1 and 2, let

$$U = U(t,x) = \eta(t)U_f(t,x) + (1-\eta(t))U_b(t,x), \quad t \in [0,T],\ x \in (0,L), \tag{2.35}$$

where $\eta(t) \in C^1[0,T]$ is given as above with (2.34). It is easy to check that U satisfies (1.2)–(1.4), (2.6) and (2.33). Defining the internal control $\Theta \in (C^0([0,T]; L^2(0,L)))^N$ as in (2.32), we have (2.5).

3. Exact internal null controllability

Obviously, the exact internal null controllability of system (1.1)–(1.3) can be realized by Theorem 2.1. However, under the assumption

$$H^-(t) = H^+(t) \equiv 0, \quad t \in (0, +\infty), \tag{3.1}$$

the exact internal null controllability can be achieved without condition (2.1). In fact, setting $U_b \equiv 0$ in Step 2, we can avoid solving the backward problem of system (1.9), therefore (2.1) is no longer necessary for realizing the one-sided boundary null controllability given by Lemma 1.4 in R_l and R_r, respectively.

Theorem 3.1. *Assume that (3.1) is satisfied. Let $0 \leqslant a < b \leqslant L$ and let $T > 0$ satisfy (2.2). Let $\delta > 0$ be so small that $b - a > 2\delta$ and (2.3)–(2.4) are satisfied. If rank$(B) = N$, then for any given initial data $U_0(x) \in (L^2(0,L))^N$, there exists a control $\Theta \in (C^0([0,T]; L^2(0,L)))^N$ with supp $\Theta \subseteq [\delta, T-\delta] \times [a,b]$, satisfying*

$$\|\Theta\|_{(C^0([0,T];L^2(0,L)))^N} \leqslant c\|U_0\|_{(L^2(0,L))^N}, \tag{3.2}$$

such that the mixed problem (1.1)–(1.4) admits a unique weak solution $U = U(t,x) \in (C^0([0,T]; L^2(0,L)))^N$, satisfying exactly the null final state at the time $t = T$:

$$t = T: \quad U(T,x) \equiv 0, \quad x \in (0,L). \tag{3.3}$$

Proof. The proof is similar to that of Theorem 2.1. The only difference is that instead of solving a backward problem of system (1.9) in Step 2, we directly define $U_b(t,x) \equiv 0$, correspondingly, $v(x) \equiv 0$. Then the final states in (2.14) and (2.21) are zero, thus, in Step 3 and 4, instead of Lemma 1.3, we can apply Lemma 1.4 to realize the exact null controllability on R_l and R_r by boundary controls on $x = a+\delta$ and $x = b-\delta$, respectively. The estimate of controllable time (2.2) guarantees the one-sided exact boundary null controllability on the left and right domains $R_l$and R_r, respectively. We omit the details here. □

Remark 3.2. When the controls act on the whole domain $R(T) = \{(t,x)|t \in (0,T), x \in (0,L)\}$, namely, when (2.34) holds, noting (2.2), the exact null controllability can be realized at any time $t = T > 0$. In fact, in this case, the constructive procedure is much simplified: (2.3) and (2.4) are no longer needed, moreover, after Step 1, let

$$U = U(t,x) = \eta(t)U_f(t,x), \quad t \in [0,T],\ x \in (0,L), \tag{3.4}$$

where $\eta(t) \in C^1[0,T]$ is given as above with (2.34). It is easy to check that U satisfies (1.2)–(1.4), (3.3) and

$$\|U\|_{(C^0([0,T];L^2(0,L)))^N} \leqslant c\|U_0\|_{(L^2(0,L))^N}. \tag{3.5}$$

Defining the control $\Theta \in (C^0([0,T]; L^2(0,L)))^N$ as in (2.32), it satisfies (3.2).

Remark 3.3. The internal controls given in Theorems 2.1 and 3.1 are not unique. On the other hand, in general, in order to realize the exact internal (null) controllability, more internal controls than boundary controls are required.

Remark 3.4. When $(b-a)$ is a constant, the controllable times given in Theorems 2.1 and 3.1 reach the minimum when a and b are symmetric about $x = \frac{L}{2}$, namely, $a + b = L$.

4. Observability

We now look at the exact internal (null) controllability from the point of view of duality for the following system with homogenous boundary conditions

$$\begin{cases} U_t + \Lambda U_x + AU = B\Theta, & t \in (0,+\infty), \quad x \in (0,L), \\ U^+(t,0) = G_0 U^-(t,0), & t \in (0,+\infty), \\ U^-(t,L) = G_1 U^+(t,L), & t \in (0,+\infty). \end{cases} \tag{4.1}$$

The relationship between controllability and observability has been studied in [4] in the framework of C^1 solutions for a simpler system under boundary controls and boundary observations. In our situation, we may use the same idea. For any given $\Phi_T \in (C^1[0,L])^N$, let $\Phi = \Phi(t,x) \in (C^1([0,T]\times[0,L]))^N$ ([48]) be the solution to the following adjoint system:

$$\begin{cases} \Phi_t + \Lambda\Phi_x - A^T\Phi = 0, & t \in (0,T), \quad x \in (0,L), \\ \Phi^-(t,0) = \widetilde{G}_0\Phi^+(t,0), & t \in (0,T), \\ \Phi^+(t,L) = \widetilde{G}_1\Phi^-(t,L), & t \in (0,T) \end{cases} \tag{4.2}$$

with the final data

$$t = T: \quad \Phi(T,x) = \Phi_T(x), \quad x \in (0,L), \tag{4.3}$$

where

$$\widetilde{G}_0 = -(\Lambda^-)^{-1}G_0^T\Lambda^+, \quad \widetilde{G}_1 = -(\Lambda^+)^{-1}G_1^T\Lambda^-. \tag{4.4}$$

Multiplying Φ^T on system (4.1) with (1.4) and integrating by parts, we get

$$\int_0^L \Phi_T^T U(T,x)\mathrm{d}x - \int_0^L \Phi^T(0,x)U_0\mathrm{d}x = \int_0^T\int_0^L \Phi^T B\Theta\mathrm{d}x\mathrm{d}t. \tag{4.5}$$

Firstly, by linearity, the exact internal controllability of system (4.1) is equivalent to the fact that system (4.1) with the null initial data

$$t = 0: \quad U(0,x) \equiv 0, \quad x \in (0,L) \tag{4.6}$$

is exactly controllable, namely, for any given final data $U_T(x) \in (L^2(0,L))^N$, there exists a control Θ such that the weak solution $U = U(t,x)$ to the mixed problem (4.1) and (4.6) satisfies exactly the final state (2.6). Taking (2.6) and (4.6) in (4.5) and noting that supp $\Theta \subseteq [\delta, T-\delta]\times[a,b]$, we have

$$\int_0^L \Phi_T^T U_T\mathrm{d}x = \int_\delta^{T-\delta}\int_a^b \Phi^T B\Theta\mathrm{d}x\mathrm{d}t. \tag{4.7}$$

Then we have

Theorem 4.1. *If system (4.1) is exactly internally controllable with controls Θ in $(C^0([0,T];L^2(0,L)))^M (M \leqslant N)$ with supp $\Theta \subseteq [\delta,T-\delta]\times[a,b]$, then the adjoint system (4.2) satisfies the following strong B-observability:*

$$\textit{If} \quad B^T\Phi = 0, \quad (t,x) \in (\delta, T-\delta)\times(a,b), \quad \textit{then} \quad \Phi_T \equiv 0, \quad x \in (0,L). \tag{4.8}$$

Secondly, for the exact internal null controllability, taking (3.3) in (4.5) and noting that supp $\Theta \subseteq [\delta, T-\delta]\times[a,b]$, we have

$$-\int_0^L \Phi^T(0,x)U_0\mathrm{d}x = \int_\delta^{T-\delta}\int_a^b \Phi^T B\Theta\mathrm{d}x\mathrm{d}t. \tag{4.9}$$

Then we have

Theorem 4.2. *If system (4.1) is exactly null controllable under the control* $\Theta \in (C^0([0,T];L^2(0,L)))^M$ $(M \leqslant N)$ *with supp* $\Theta \subseteq [\delta, T-\delta] \times [a,b]$, *then the adjoint system (4.2) satisfies the following weak B-observability:*

$$\text{If} \quad B^T\Phi = 0, \quad (t,x) \in (\delta, T-\delta) \times (a,b), \quad \text{then} \quad \Phi(0,x) \equiv 0, \quad x \in (0,L). \tag{4.10}$$

These results help us to further study the non-controllability for the system in the next section.

5. Non-controllability

The study of non-controllability is rather difficult in general since first order hyperbolic systems have very broad content. In this paper, by the method of duality we will point out some types of systems that are not able to realize the exact internal (null) controllability because of the coupling pattern of the system. The same method can be also applied to study the non-controllability with boundary controls for the system discussed in [45].

5.1. Non-controllability

By Theorem 4.1 for system (4.2), once we can choose a non-trivial decoupled subsystem taking values in $Ker(B^T)$, we will get the lack of exact controllability for any finite time $T \in (0, +\infty)$. More precisely, we have

Theorem 5.1. *Assume that* Λ, A^T *and* G^T *share a non-trivial common invariant subspace* $\mathfrak{E} \neq \{0\}$, *such that* $\mathfrak{E} \subseteq Ker(B^T)$, *then system (4.1) is not exactly controllable in any finite time* $T \in (0, +\infty)$.

Choosing a non-trivial final state $\Phi_T \in \mathfrak{E}$ at the time $t = T$, the solution Φ of the adjoint system (4.2) keeps taking values in $\mathfrak{E}$, and then $B^T\Phi = 0$, hence, by Theorem 4.1, the strong B-observability fails for system (4.2). To show the details, we first choose a basis $\{E_i\}$ for $\mathfrak{E}$. Since $\mathfrak{E}$ is an invariant subspace of $\Lambda = \mathrm{diag}\{\lambda_k\}_{k=1}^N$, without loss of generality, we can assume the each E_i is a vector of the canonical basis of $\mathbb{R}^N$, namely,

$$E_i = e_{k_i} = (0, \ldots, 0, \overset{k_i\text{-th}}{1}, 0, \ldots, 0)^T.$$

Moreover, as introduced in (1.5), we have

$$\Lambda = \begin{pmatrix} \Lambda^- & \mathbf{0} \\ \mathbf{0} & \Lambda^+ \end{pmatrix}$$

and without loss of generality, we may assume p $(0 \leq p \leq N^-)$ eigenvalues corresponding to E_i are negative and other q $(0 \leq q \leq N^+)$ eigenvalues corresponding to E_i are positive. Then, as we always divide matrices into (N^-, N^+) blocks in the previous sections, we will also divide matrices acting on $\mathfrak{E}$ into (p,q) blocks several times in this section. For instance, we denote $E, E^-, E^+, \Lambda^\flat, \Lambda^{\flat,-}, \Lambda^{\flat,+}$ as

$$E = (E_1, \ldots, E_{p+q}) = (e_{k_1}, \ldots, e_{k_{p+q}}) = \begin{matrix} \\ N^- \\ N^+ \end{matrix} \begin{matrix} p \quad\quad q \\ \begin{pmatrix} E^- & \mathbf{0} \\ \mathbf{0} & E^+ \end{pmatrix} \end{matrix} \tag{5.1}$$

and

$$\Lambda^{\flat,-} = \mathrm{diag}\{\lambda_{k_1}, \ldots, \lambda_{k_p}\}, \tag{5.2}$$

$$\Lambda^{\flat,+} = \mathrm{diag}\{\lambda_{k_{p+1}}, \ldots, \lambda_{k_{p+q}}\}, \tag{5.3}$$

$$\Lambda^\flat = \mathrm{diag}\{\lambda_{k_1}, \ldots, \lambda_{k_{p+q}}\} = \mathrm{diag}\{\Lambda^{\flat,-}, \Lambda^{\flat,+}\}. \tag{5.4}$$

Then we have

$$\Lambda E = E\Lambda^{\flat}, \quad \Lambda^- E^- = E^- \lambda^{\flat,-}, \quad \Lambda^+ E^+ = E^+ \Lambda^{\flat,+} \tag{5.5}$$

and

$$(\Lambda^-)^{-1} E^- = E^- (\lambda^{\flat,-})^{-1}, \quad (\Lambda^+)^{-1} E^+ = E^+ (\Lambda^{\flat,+})^{-1}. \tag{5.6}$$

Furthermore, since $\mathfrak{E}$ is an invariant subspace of A^T, there exists a $(p+q)\times(p+q)$ matrix $A^{\flat}$ such that

$$A^T E = E A^{\flat}. \tag{5.7}$$

Similarly, for

$$G = \begin{array}{c} \\ N^- \\ N^+ \end{array} \begin{array}{c} \begin{array}{cc} N^- & N^+ \end{array} \\ \begin{pmatrix} \mathbf{0} & G_1 \\ G_0 & \mathbf{0} \end{pmatrix} \end{array},$$

there exists a $(p+q)\times(p+q)$ matrix

$$G^{\flat} = \begin{array}{c} \\ p \\ q \end{array} \begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \begin{pmatrix} \mathbf{0} & G_1^{\flat} \\ G_0^{\flat} & \mathbf{0} \end{pmatrix} \end{array} \tag{5.8}$$

such that

$$G^T E = E G^{\flat} \tag{5.9}$$

with

$$G_0^T E^+ = E^- G_1^{\flat}, \quad G_1^T E^- = E^+ G_0^{\flat}. \tag{5.10}$$

Then, for

$$\widetilde{G}_0^{\flat} = -(\Lambda^{\flat,-})^{-1} G_0^{\flat} \Lambda^{\flat,+} \tag{5.11}$$

and $\widetilde{G}_0$ defined in (4.2), we have

$$\begin{aligned} \widetilde{G}_0 E^+ &= -(\Lambda^-)^{-1} G_0^T \Lambda^+ E^+ = -(\Lambda^-)^{-1} G_0^T E^+ \Lambda^{\flat,+} \\ &= -(\Lambda^-)^{-1} E^- G_0^{\flat} \Lambda^{\flat,+} = -E^- - (\Lambda^{\flat,-})^{-1} G_0^{\flat} \Lambda^{\flat,+} = E^- \widetilde{G}_0^{\flat}. \end{aligned} \tag{5.12}$$

Similarly, we have

$$\widetilde{G}_1 E^- = E^+ \widetilde{G}_1^{\flat}. \tag{5.13}$$

Thus, for

$$\widetilde{G}^\flat = \begin{matrix} & \begin{matrix} p & q \end{matrix} \\ \begin{matrix} p \\ q \end{matrix} & \begin{pmatrix} \mathbf{0} & \widetilde{G}_0^\flat \\ \widetilde{G}_1^\flat & \mathbf{0} \end{pmatrix} \end{matrix} \quad \text{and} \quad \widetilde{G} = \begin{matrix} & \begin{matrix} N^- & N^+ \end{matrix} \\ \begin{matrix} N^- \\ N^+ \end{matrix} & \begin{pmatrix} \mathbf{0} & \widetilde{G}_0 \\ \widetilde{G}_1 & \mathbf{0} \end{pmatrix} \end{matrix} \tag{5.14}$$

with $\widetilde{G}_0, \widetilde{G}_1$ defined in (4.4), we have

$$\widetilde{G}E = E\widetilde{G}^\flat. \tag{5.15}$$

Finally, noting $\mathfrak{E} \subseteq Ker(B^T)$, we have $B^T E = 0$.

Now, for the projected system with $(p+q)$ equations for

$$V = \begin{pmatrix} V^- \\ V^+ \end{pmatrix}$$

with $V^- \in \mathbb{R}^p, V^+ \in \mathbb{R}^q$ as

$$\begin{cases} V_t + \Lambda^\flat V_x - A^\flat V = 0, & t \in (0,T), x \in (0,L), \\ V^-(t,0) = \widetilde{G}_0^\flat V^+(t,0), & t \in (0,T), \\ V^+(t,L) = \widetilde{G}_1^\flat V^-(t,L), & t \in (0,T), \end{cases} \tag{5.16}$$

we consider its backward problem for the final data

$$t = T: \; V(T,x) = V_T(x), \quad x \in (0,L) \tag{5.17}$$

with $V_T(x) \neq 0 \in (L^2(0,L))^{p+q}$ to get a solution $V \not\equiv 0$ on $(0,T) \times (0,L)$.

Let

$$\Phi = EV = \begin{pmatrix} E^- V^- \\ E^+ V^+ \end{pmatrix} \tag{5.18}$$

and denote $\Phi^- = E^- V^-, \Phi^+ = E^+ V^+$, by (5.5), (5.7), (5.10) and (5.16) we have

$$\begin{aligned} \Phi_t + \Lambda \Phi_x - A^T \Phi &= EV_t + \Lambda E V_x - A^T EV \\ &= EV_t + E\Lambda^\flat V_x - EA^\flat V = E(V_t + \Lambda^\flat V_x - A^\flat V) = 0, \qquad t \in (0,T), x \in (0,L) \end{aligned} \tag{5.19}$$

and

$$\Phi^-(t,0) = E^- V^-(t,0) = E^- \widetilde{G}_0^\flat V^+(t,0) = \widetilde{G}_0 E^+ V^+(t,0) = \widetilde{G}_0 \Phi^+(t,0), \quad t \in (0,T), \tag{5.20}$$

$$\Phi^+(t,L) = E^+ V^+(t,L) = E^+ \widetilde{G}_1^\flat V^-(t,L) = \widetilde{G}_1 E^- V^-(t,L) = \widetilde{G}_1 \Phi^-(t,L), \quad t \in (0,T). \tag{5.21}$$

Then $\Phi(t,x)$ solves system (4.2) with

$$t = T: \; \Phi = EV_T(x) \neq 0,$$

but

$$B^T\Phi = B^T EV \equiv 0, \quad t \in (0,T), x \in (0,L),$$

which shows the failure of strong B-observability and finishes the proof.

5.2. Non-null controllability

In this subsection, we will show that for system (4.1) with $rank(B) < N$, the null controllability may not be achieved by internal controls under certain cases. By Theorem 4.2, we should present some sufficient conditions that lead to the failure of the weak B-observability of the adjoint system (4.2).

As in §5.1, we first need a non-trivial common invariant subspace $\mathfrak{E} \subseteq Ker(B^T)$ shared by Λ, A^T, G^T and thus $\widetilde{G}$. We would then work on this invariant subspace $\Phi \in \mathfrak{E}$, since $B^T\Phi = 0$, the non-trivial solutions Φ in it cannot be observed by B. But the existence of non-trivial solutions is far from being sufficient to show the failure of the weak B-observability in this case. In fact, since we do not need the assumption of the time reversibility in this case, a projected system of V with specifically chosen boundary conditions can always have zero initial data $V(0,x) \equiv 0$ for any final data V_T for T large enough. For instance, when $A^\flat = 0, \widetilde{G}^\flat = 0$, all the solutions to

$$\begin{cases} V_t + \Lambda^\flat V_x = 0, & t \in (0,T), x \in (0,L), \\ V^-(t,0) = 0, & t \in (0,T), \\ V^+(t,L) = 0, & t \in (0,T) \end{cases}$$

would have $V(0,x) \equiv 0$, if $T \geq \max\{|\frac{L}{\lambda_{k_i}}| : \ 1 \leq i \leq p+q\}$.

Inspired by this observation, we have the following two results, the first one shows that the weak B-observability may fail for the time-reversible projected systems, while the second one shows that the weak B-observability would fail for short time.

Theorem 5.2. *Assume that Λ, A^T, G^T share a non-trivial common invariant subspace $\mathfrak{E} \subseteq Ker(B^T)$ with $p = q$ and $Ker(G^T) \bigcap \mathfrak{E} = \{0\}$, then system (4.1) is not exactly null controllable.*

In this case, the projected system (5.16) for V is time-reversible, and by the uniqueness of solutions to the forward problem, for any given $V(0,x) \not\equiv 0 \in (L^2(0,L))^{p+q}$, we can get a non-trivial solution $V(t,x)$. Then, setting $\Phi(t,x) = EV(t,x)$ as in the previous subsection, we get the failure of weak B-observability and thus the lack of null controllability of the original system. See also Theorem 7.1 in [45].

Theorem 5.3. *Assume that Λ, A^T, G^T share a non-trivial common invariant subspace $\mathfrak{E} \subseteq Ker(B^T)$, then system (4.1) is not exactly null controllable for any given short control time T satisfying*

$$0 < T < \min\left\{\left|\frac{L}{2\lambda_{k_i}}\right| : \ 1 \leq i \leq p+q\right\}.$$

In fact, for the backward problem (5.16)–(5.17), we just set

$$t = T : \ V_i(T,x) \equiv V_{i,T},$$

where $V_{i,T}$ are independent of x. Then, for $\lambda_M = \max\{|\lambda_{k_i}| : \ 1 \leq i \leq p+q\}$, in the space-time set $D_{\text{back}} = \{(t,x) : \ 0 \leq t \leq T, \lambda_M(T-t) \leq x \leq L - \lambda_M(T-t)\}$ which is a subset of the determinant domain of the final

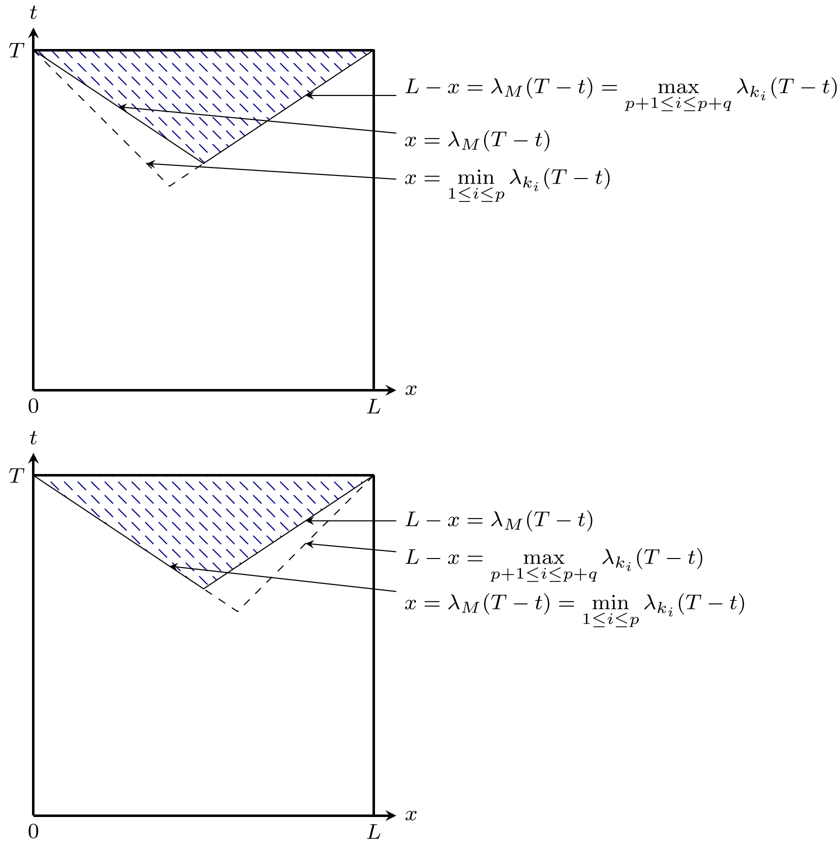

FIGURE 2. D_{back} (shaded domain) and the determinant domain of the final condition (with dashed boundary).

condition (see Fig. 2), $V_i(t,x) = V_i(t)$ will also be independent of x, and then satisfy an ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t} V_i - \sum_{j=1}^{p+q} A^\flat_{ij} V_j = 0,$$

which is time-reversible, thus any given non-trivial final condition $V_{i,T}$ will lead to a non-trivial initial condition at least on a part of the interval $\{t = 0\}$. This yields the failure of the weak B-observability of system (4.2) and then the lack of null controllability of system (4.1).

6. Exact internal synchronization

We now study the exact internal synchronization for system (4.1). Since the methods and results are quite similar to the exact boundary synchronization (see [45]), we only give a brief presentation.

6.1. Exact internal synchronization

We first quickly remind the definition of the exact synchronization. For any given positive integers $N_i(\geqslant 2, i=1,\cdots,n)$, let

$$N=\sum_{i=1}^{n} N_i, \quad N^-=\sum_{i=1}^{m} N_i, \quad N^+=\sum_{i=m+1}^{n} N_i \quad \text{and} \quad \bar{m}=n-m \quad (0<m<n). \tag{6.1}$$

The matrix $\Lambda=\text{diag}\{\Lambda^-,\Lambda^+\}$ is required to be in the form of

$$\Lambda^-=\text{diag}\{\lambda_1 I_{N_1},\cdots,\lambda_m I_{N_m}\}, \quad \Lambda^+=\text{diag}\{\lambda_{m+1} I_{N_{m+1}},\cdots,\lambda_n I_{N_n}\}, \tag{6.2}$$

where $\lambda_r<0(r=1,\cdots,m)$ and $\lambda_s>0(s=m+1,\cdots,n)$, and I_{N_i} denotes the unit matrix of order $N_i(i=1,\cdots,n)$. Correspondingly, the state variable U of system (4.1) is divided into n groups $U=(U^-,U^+)^T$ with $U^-=(U_1,\ldots,U_m)^T$, $U^+=(U_{m+1},\ldots,U_n)^T$ and $U_i=(u_i^{(1)},u_i^{(2)},\cdots,u_i^{(N_i)})^T(i=1,\cdots,n)$. When

$$t\geqslant T: \quad u_i^{(1)}(t,x)\equiv u_i^{(2)}(t,x)\equiv\cdots\equiv u_i^{(N_i)}(t,x)\overset{\text{def.}}{=}\tilde{u}_i(t,x), \quad i=1,\cdots,n, \tag{6.3}$$

system (4.1) is called to be exactly synchronizable at the time $t=T$, where $\tilde{u}=(\tilde{u}_1(t,x),\cdots,\tilde{u}_n(t,x))^T$ is the exactly synchronizable state, which is *a priori* unknown.

Although the control is applied on the domain, the strategy adopted in the case of boundary control ([45]) can still be used to establish the exact internal synchronization. Briefly speaking, by introducing the full row-rank $(N-n)\times N$ matrix of synchronization $C_1=\begin{pmatrix} C_1^- & \\ & C_1^+ \end{pmatrix}$ with

$$C_1^-=\begin{pmatrix} \tilde{C}_1 & & \\ & \ddots & \\ & & \tilde{C}_m \end{pmatrix}, \qquad C_1^+=\begin{pmatrix} \tilde{C}_{m+1} & & \\ & \ddots & \\ & & \tilde{C}_n \end{pmatrix} \tag{6.4}$$

and

$$\tilde{C}_i=\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \end{pmatrix}_{(N_i-1)\times N_i} \qquad (i=1,\cdots,n), \tag{6.5}$$

we define the conditions of C_1-compatibility for A and G by

$$A\text{Ker}(C_1)\subseteq\text{Ker}(C_1) \quad \text{and} \quad G\text{Ker}(C_1)\subseteq\text{Ker}(C_1), \tag{6.6}$$

which (by Lem. 3.2, Cor. 3.9, Lem. 3.5 and Cor. 3.9 in [45]) are equivalent to

$$C_1 A=\overline{A}C_1 \tag{6.7}$$

and

$$C_1^+ G_0=\overline{G}_0 C_1^-, \tag{6.8}$$
$$C_1^- G_1=\overline{G}_1 C_1^+, \tag{6.9}$$

respectively, with matrices $\overline{A}$ of order $(N-n)$, $\overline{G}_0$ of order $(N^+-\bar{m})\times(N^--m)$ and $\overline{G}_1$ of order $(N^--m)\times(N^+-\bar{m})$. Under the conditions of C_1-compatibility (6.6), setting $W=C_1U$, $W^-=C_1^-U^-$ and $W^+=C_1^+U^+$, we get the following reduced system:

$$\begin{cases} W_t+\overline{\Lambda}W_x+\overline{A}W=C_1B\Theta, & t\in(0,+\infty), \quad x\in(0,L), \\ W^+(t,0)=\overline{G}_0W^-(t,0), & t\in(0,+\infty), \\ W^-(t,L)=\overline{G}_1W^+(t,L), & t\in(0,+\infty) \end{cases} \tag{6.10}$$

with the initial data

$$t=0: \quad W(0,x)=C_1U_0(x), \quad x\in(0,L), \tag{6.11}$$

where $\overline{\Lambda}=\mathrm{diag}\{\lambda_1I_{N_1-1},\lambda_2I_{N_2-1},\cdots,\lambda_nI_{N_n-1}\}$, and $\overline{A},\overline{G}_0$ and $\overline{G}_1$ are given by (6.7)–(6.9). Since the exact internal synchronization of system (4.1) is equivalent to the exact internal null controllability of the reduced system (6.10), by Theorem 3.1 we have

Theorem 6.1. *Assume that A and G satisfy the conditions of C_1-compatibility* (6.6)*. If rank$(C_1B)=N-n$, then system* (4.1) *is exactly synchronizable by an internal control $\Theta\in(C^0([0,T];L^2(0,L)))^M(M\leqslant N)$, satisfying*

$$\|\Theta\|_{(C^0([0,T];L^2(0,L)))^M}\leqslant c\|C_1U_0\|_{(L^2(0,L))^{N-n}}\leqslant c\|U_0\|_{(L^2(0,L))^N} \tag{6.12}$$

with supp $\Theta\subseteq[\delta,T-\delta]\times[a,b]$, where $\delta>0$ is small enough and $0\leqslant a<b\leqslant L$.

The proof of Theorem 6.1 can refer to Sections 3 and 4 in [45].

Remark 6.2. One may check the Appendix A of this paper for discussions on the necessity of the conditions of C_1-compatibility for the coupling matrices A and G. Since the compatibility conditions describe a system's synchronization pattern, which do not depend on how we execute the controls (inside the domain or on the boundary), one may also refer to Section 6 of [16] for the results.

Moreover, since the reduced system (6.10) is exactly null controllable, by Theorem 4.2 we immediately get the following weak observability for synchronization:

Theorem 6.3. *Assume that A and G satisfy the conditions of C_1-compatibility* (6.6)*. If system* (4.1) *is exactly synchronizable under the control $\Theta\in(C^0([0,T];L^2(0,L)))^M(M\leqslant N)$ with supp $\Theta\subseteq[\delta,T-\delta]\times[a,b]$, then the reduced adjoint system*

$$\begin{cases} \Psi_t+\overline{\Lambda}\Psi_x-\overline{A}^T\Psi=0, & t\in(0,T), \quad x\in(0,L), \\ \Psi^-(t,0)=-(\overline{\Lambda}^-)^{-1}\overline{G}_0^T\overline{\Lambda}^+\Psi^+(t,0), & t\in(0,T), \\ \Psi^+(t,L)=-(\overline{\Lambda}^+)^{-1}\overline{G}_1^T\overline{\Lambda}^-\Psi^-(t,L), & t\in(0,T) \end{cases} \tag{6.13}$$

with the final data

$$t=T: \quad \Psi(T,x)=\Psi_T(x), \quad x\in(0,L) \tag{6.14}$$

satisfies the following weak C_1B-observability:

$$\textit{If} \quad (C_1B)^T\Psi=0, \quad (t,x)\in(\delta,T-\delta)\times(a,b), \quad \textit{then} \quad \Psi(0,x)\equiv0, \quad x\in(0,L), \tag{6.15}$$

where $\overline{\Lambda}, \overline{A}, \overline{G}_0$ *and* $\overline{G}_1$ *are given by* (6.7)–(6.9) *and* (6.10)*, respectively.*

6.2. Exactly synchronizable states and their determination

As $t \geqslant T$, the trajectory of the synchronization state is determined only by the system itself without any control. The system that describes the trajectory of the exactly synchronizable state $(\tilde{u}_1(t,x), \cdots, \tilde{u}_n(t,x))^T$ is already given in Section 4.1 in [45], from which we will also see that under certain conditions the attainable set of the exactly synchronizable state at the time $t = T$ will be the whole space $(L^2(0,L))^n$.

In general, the exactly synchronizable states are *a priori* unknown, and depend on the initial data and applied controls. In order to determine the exactly synchronizable states, we first study the simple case that they are independent of internal controls. Let

$$E_i = \begin{pmatrix} \varepsilon_i \\ 0 \end{pmatrix} \in \mathbb{R}^N \ (i = 1, \cdots, m), \qquad E_i = \begin{pmatrix} 0 \\ \varepsilon_i \end{pmatrix} \in \mathbb{R}^N \ (i = m+1, \cdots, n) \tag{6.16}$$

be linearly independent, and $\varepsilon_i \in \mathbb{R}^{N^-} (i = 1, \cdots, m)$, $\varepsilon_i \in \mathbb{R}^{N^+} (i = m+1, \cdots, n)$. We take the idea from the case of exact boundary synchronization (see [45]): Projecting system (4.1) to the subspace $V \overset{def.}{=} \text{Span}\{E_1, \cdots, E_n\}$ which is bi-orthonormal to $\text{Ker}(C_1)$, and setting $\text{Ker}(B^T) = V$, the solution to the projective system independent of applied internal controls consists with the exactly synchronizable states as $t \geqslant T$, then can be used to determine the exactly synchronizable states. This projective system does not depend on applied controls as in the boundary control case. The following results are similar to that in the exact boundary synchronization case with only small adaptations, readers may refer to Section 4.2 of [45].

Theorem 6.4. *Assume that* A *and* G *satisfy the conditions of* C_1*-compatibility* (6.6)*. Assume furthermore that* $E_i (i = 1, \cdots, n)$ *given by* (6.16) *are eigenvectors of* Λ*, and* $V = Span\{E_1, \cdots, E_n\}$ *is a common invariant subspace of* A^T *and* G^T*. Let* $Ker(B^T) = V$*. We have*

$$rank(C_1 B) = rank(B) = N - n, \tag{6.17}$$

thus system (4.1) *is exactly synchronizable, and the exactly synchronizable state* $(\tilde{u}_1(t,x), \cdots, \tilde{u}_n(t,x))^T$ *is independent of applied control* Θ*.*

If system (4.1) is exactly synchronizable under the assumptions of Theorem 6.1, in general, the exactly synchronizable states depend on applied controls, the solution to the projective system mentioned above can be used to estimate the norm of the exactly synchronizable states by the standard method of estimation for PDEs, the result and its proof are similar to that for the exact boundary synchronization (see Sect. 5 of [45]), we omit the details.

7. Examples: non-exact controllability and synchronization

Since the synchronization is so closely related to the controllability, the minimum number of internal controls necessary for the exact internal synchronization is equally challenging as the internal controllability with a reduced number of controls.

In this section we will give some examples of systems to show that the exact internal (null) controllability and the exact internal synchronization can not be realized because of the algebraic properties of the coupling matrices.

Example 7.1. Consider the following system:

$$\begin{cases} u_{1t}+\lambda_1 u_{1x}+u_1+2u_2+4v=0, & t\in(0,+\infty),\quad x\in(0,L),\\ u_{2t}+\lambda_1 u_{2x}+2u_1-u_2+2v=0, & t\in(0,+\infty),\quad x\in(0,L),\\ u_{3t}+\lambda_2 u_{3x}-3u_1+3u_2-2u_3=\Theta, & t\in(0,+\infty),\quad x\in(0,L),\\ v_t+\lambda_3 v_x+u_1+4u_2+3v=0, & t\in(0,+\infty),\quad x\in(0,L),\\ x=0:\quad v=2u_1+4u_2, & t\in(0,+\infty),\\ x=L:\quad u_1=-v,\quad u_2=v,\quad u_3=3v, & t\in(0,+\infty), \end{cases} \tag{7.1}$$

in which $\lambda_1,\ \lambda_2<0$ and $\lambda_3>0$. Let $\Lambda=\mathrm{diag}\{\lambda_1,\lambda_1,\lambda_2,\lambda_3\}$,

$$A=\begin{pmatrix}1&2&0&4\\2&-1&0&2\\-3&3&-2&0\\1&4&0&3\end{pmatrix},\ B=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ G=\begin{pmatrix}0&0&0&-1\\0&0&0&1\\0&0&0&3\\2&4&0&0\end{pmatrix},\ E_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ E_2=\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\ E_3=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

It is easy to check that E_1,E_2 and E_3 are eigenvectors of Λ, $\mathfrak{C}=\mathrm{Span}\{E_1,E_2,E_3\}$ is a common invariant subspace of A^T and G^T, and $\mathfrak{C}\subseteq\mathrm{Ker}(B^T)$. By Theorem 5.1, system (7.1) is not exactly controllable by any given control Θ. In fact, system (7.1) can be divided into two subsystems: one corresponding to u_3 is controllable by control Θ; while the other one corresponding to u_1,u_2 and v is a closed system independent of u_3 and Θ, then is not controllable by any given control Θ.

Example 7.2. Consider the following system:

$$\begin{cases} u_{1t}+\lambda_1 u_{1x}+u_1+5v=0, & t\in(0,+\infty),\quad x\in(0,L),\\ u_{2t}+\lambda_2 u_{2x}+3u_1+4u_2-v=\Theta, & t\in(0,+\infty),\quad x\in(0,L),\\ v_t+\lambda_3 v_x+2u_1+v=0, & t\in(0,+\infty),\quad x\in(0,L),\\ x=0:\quad v=2u_1, & t\in(0,+\infty),\\ x=L:\quad u_1=v,\quad u_2=2v, & t\in(0,+\infty), \end{cases} \tag{7.2}$$

in which $\lambda_1,\ \lambda_2<0$ and $\lambda_3>0$. Let $\Lambda=\mathrm{diag}\{\lambda_1,\lambda_2,\lambda_3\}$,

$$A=\begin{pmatrix}1&0&5\\3&4&-1\\2&0&1\end{pmatrix},\ B=\begin{pmatrix}0\\1\\0\end{pmatrix},\ G=\begin{pmatrix}0&0&1\\0&0&2\\2&0&0\end{pmatrix},\ E_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\ E_2=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

It is easy to check that E_1 and E_2 are eigenvectors of Λ, $\mathfrak{C}=\mathrm{Span}\{E_1,E_2\}$ is a common invariant subspace of A^T and G^T, $\mathfrak{C}\subseteq\mathrm{Ker}(B^T)$, and $\mathrm{Ker}(G^T)\cap\mathfrak{C}=\{0\}$. By Theorem 5.2, system (7.2) is not exactly null controllable by any given control Θ. In fact, system (7.2) can be divided into two subsystems: one corresponding to u_2 is controllable by control Θ; while the other one corresponding to u_1 and v is a closed system independent of u_2 and Θ, then is not null controllable by any given control Θ.

By a similar proof as in Theorem 7.3 in [45], we have the following

Proposition 7.3. *Assume that $p=q$, and $\mathfrak{C}$ is a common invariant subspace of Λ,A^T and G^T. Assume furthermore that $\mathfrak{C}\subseteq Ker(B^T)\cap Im(C_1^T)$ and $Ker(G^T)\cap\mathfrak{C}=\{0\}$. Then system (4.1) is not exactly synchronizable.*

Example 7.4. Consider the following system:

$$\begin{cases} u_{1t} + \lambda_1 u_{1x} + u_6 = \Theta_1, & t \in (0,+\infty), \quad x \in (0,L), \\ u_{2t} + \lambda_1 u_{2x} + u_7 = \Theta_2, & t \in (0,+\infty), \quad x \in (0,L), \\ u_{3t} + \lambda_1 u_{3x} - u_4 + u_5 - u_6 + 2u_7 = \Theta_3, & t \in (0,+\infty), \quad x \in (0,L), \\ u_{4t} + \lambda_2 u_{4x} + u_4 + u_5 - 2u_6 + u_7 = -\Theta_2 + 3\Theta_3 - 6\Theta_5, & t \in (0,+\infty), \quad x \in (0,L), \\ u_{5t} + \lambda_2 u_{5x} + 2u_5 - 5u_6 + 4u_7 = -\Theta_2 + 3\Theta_3 - 6\Theta_5, & t \in (0,+\infty), \quad x \in (0,L), \\ u_{6t} + \lambda_3 u_{6x} + u_5 + u_6 + u_7 = \Theta_1 + 2\Theta_4 + \Theta_5, & t \in (0,+\infty), \quad x \in (0,L), \\ u_{7t} + \lambda_3 u_{7x} + u_4 + 2u_6 = \Theta_1 + 2\Theta_4 + \Theta_5, & t \in (0,+\infty), \quad x \in (0,L), \\ x = 0: \ u_6 = 2u_4 + u_5, \ u_7 = 3u_4, & t \in (0,+\infty), \\ x = L: \ u_1 = u_7, \ u_2 = u_6, \ u_3 = 2u_6 - u_7, \ u_4 = 2u_6, \ u_5 = u_6 + u_7, & t \in (0,+\infty), \end{cases} \tag{7.3}$$

in which $\lambda_1, \lambda_2 < 0$ and $\lambda_3 > 0$. Let $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_3\}$, $E_1 = (0,0,0,1,-1,0,0)^T, E_1 = (0,0,0,0,0,1,-1)^T$,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & -6 \\ 0 & -1 & 3 & 0 & -6 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

It is easy to check that A and G satisfy the conditions of C_1-compatibility (6.6). $E_1, E_2 \in \mathrm{Im}(C_1^T)$, E_1 and E_2 are eigenvectors of Λ, $\mathfrak{E} = \mathrm{Span}\{E_1, E_2\}$ is a common invariant subspace of A^T and G^T, $\mathfrak{E} \subseteq \mathrm{Ker}(B^T)$, and $\mathrm{Ker}(G^T) \cap \mathfrak{E} = \{0\}$. By Proposition 7.3, system (7.3) is not exactly synchronizable by any given control $\Theta = (\Theta_1, \Theta_2, \Theta_3, \Theta_3, \Theta_5)^T$. In fact, by a direct calculation it is easy to see that the corresponding reduced system (6.10) falls into the same situation as in Example 7.2, thus system (7.3) is not exactly synchronizable.

Appendix A.

In this appendix, we discuss the necessity of the C_1-compatibility condition (6.6), and by Section 5 of [45], its equivalency (6.7)–(6.9) for the coupling matrices A and G. Noting (6.3), for necessity, such condition should guarantee that once we achieve the exact synchronization state at the time $t = T$, we can release the control for $t > T$, and the system would stay to be synchronized afterwards. For convenience, we recall conditions (6.7)–(6.9) here:

$$C_1 A = \overline{A} C_1 \tag{A.1}$$

$$C_1^+ G_0 = \overline{G}_0 C_1^-, \tag{A.2}$$

$$C_1^- G_1 = \overline{G}_1 C_1^+. \tag{A.3}$$

First, for (6.7), we would show that it is necessary even if we only require that each exact synchronization state, being independent of the x variable, is kept after we release the control. In this case, we can confine our analysis to the triangular space-time domain

$$D = \{(t,x):\ t \geq T, \lambda_M(t-T) \leq x \leq L - \lambda_M(t-T)\},$$

where $\lambda_M = max_i\{|\lambda_i|\}$. Since D is a part of the determinant domain of the interval $\{T\} \times (0,L)$ for the forward problem, and synchronized states at the time T are supposed to be independent of the x variable, $\tilde{u}_i(t,x) = \tilde{u}_i(t)$, in D, system (4.1) degenerates to an ODE system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}U + AU = 0, \quad (t,x) \in D, \\ u_i^{(j)}(T) = \tilde{u}_i(T), \end{cases}$$

which as in [49], directly leads to (6.7).

Next, we show the necessity of (6.8). Suppose (6.3) holds for $t \geq T$, we would check the boundary condition on $x = 0$:

$$U^+(t,0) = G_0 U^-(t,0).$$

For clearness, we denote this condition as

$$u_j^{(h)}(t,0) = \sum_{i=1}^{m}\sum_{l=1}^{N_i} G_{0,j,h,i,l}\, u_i^{(l)}(t,0), \quad h = 1,\dots,N_j; j = m+1,\dots,n.$$

As proved in [2], (6.8) is equivalent to the condition that each blocked row of G_0 have a uniform sum, namely,

$$\sum_{l=1}^{N_i} G_{0,j,h,i,l} = \sum_{l=1}^{N_i} G_{0,j,\tilde{h},i,l}, \quad \forall 1 \leq h, \tilde{h} \leq N_j, \forall i, j. \tag{A.4}$$

Without loss of generality, we suppose by contradiction that (A.4) fails for the first two rows ($j = m+1, h = 1, \tilde{h} = 2$). Then we have

$$\sum_i (\beta_{m+1,1,i} - \beta_{m+1,2,i})^2 \neq 0,$$

where, we denote

$$\beta_{j,h,i} = \sum_{l=1}^{N_i} G_{0,j,h,i,l}.$$

Then, for the first two boundary conditions with synchronized states $\tilde{u}_i$ on $x = 0$, we have

$$\tilde{u}_{m+1}(t,0) = u_{m+1}^{(1)}(t,0) = \sum_{i=1}^{m}\sum_{l=1}^{N_i} G_{0,m+1,1,i,l}\, u_i^{(l)}(t,0) = \sum_{i=1}^{m} \beta_{m+1,1,i}\tilde{u}_i(t,0),$$
$$\tilde{u}_{m+1}(t,0) = u_{m+1}^{(2)}(t,0) = \sum_{i=1}^{m}\sum_{l=1}^{N_i} G_{0,m+1,2,i,l}\, u_i^{(l)}(t,0) = \sum_{i=1}^{m} \beta_{m+1,2,i}\tilde{u}_i(t,0),$$

from which we get

$$0=\sum_{i=1}^{m}(\beta_{m+1,1,i}-\beta_{m+1,2,i})\tilde{u}_i(t,0). \tag{A.5}$$

Differently from $\tilde{u}_i(T,x)$ which can be chosen arbitrarily, we cannot set the value of $\tilde{u}_i(t,0)=(\beta_{m+1,1,i}-\beta_{m+1,2,i})$ as desired. But it is easy to see that $(\tilde{u}_1(t,0),\dots,\tilde{u}_m(t,0))^T$ can be a vector near $(\beta_{m+1,1,1}-\beta_{m+1,2,1},\dots,\beta_{m+1,1,m}-\beta_{m+1,2,m})^T$. In fact, we set smooth synchronized data at $t=T$ as

$$\tilde{u}_i(T,x)=\begin{cases}(\beta_{m+1,1,i}-\beta_{m+1,2,i}), & x\in(0,L/3),\\ 0, & x\in(2L/3,L),\end{cases}\qquad i=1,\dots,m$$

and

$$\tilde{u}_i(T,x)\equiv 0,\quad i=m+1,\dots,n,$$

which satisfies the C^0 compatibility condition at $(T,0)$ and (T,L). Then, for this linear system, we have a unique piecewise C^1 solution (see for instance [50]). For $0<\epsilon\ll 1$, at $x=0$ and $t\in(T,T+\epsilon)$, by the method of characteristics, we have

$$|\tilde{u}_i(t,0)-\tilde{u}_i(T,x)|\le\int_0^{\epsilon}\|A\|_\infty\max_{s,x}|\tilde{u}_k(s,x)|ds\ll 1.$$

Thus we get

$$\sum_{i=1}^{m}(\beta_{m+1,1,i}-\beta_{m+1,2,i})\tilde{u}_i(t,0)>0,\quad\forall t\in(T,T+\epsilon),$$

which contradicts (A.5) and finishes the proof of (6.8). Similarly, we can get (6.9).

Acknowledgements. The authors would like to thank the referee for the careful reading and valuable comments which helped to improve the paper.

Funding

Project supported by the National Natural Science Foundation of China under grants 11831011, 11901082 and 12122104, the Natural Science Foundation of Jiangsu Province under grant BK20190323, the Fundamental Research Funds for the Central Universities, and Shanghai Science and Technology Programs 21ZR1406000, 21JC1400600.

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