ESAIM: COCV 30 (2024) 38
https://doi.org/10.1051/cocv/2024027

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

EXPONENTIAL DECAY OF SOLUTIONS OF DAMPED WAVE EQUATIONS IN ONE DIMENSIONAL SPACE IN THE L^p FRAMEWORK FOR VARIOUS BOUNDARY CONDITIONS

Yacine Chitour[1] and Hoai-Minh Nguyen[2,*]

Abstract. We establish the decay of the solutions of the damped wave equations in one dimensional space for the Dirichlet, Neumann, and dynamic boundary conditions where the damping coefficient is a function of space and time. The analysis is based on the study of the corresponding hyperbolic systems associated with the Riemann invariants. The key ingredient in the study of these systems is the use of the internal dissipation energy to estimate the difference of solutions with their mean values in an average sense.

Mathematics Subject Classification. 35L02, 35L05, 93D20, 93D23.

Received January 18, 2023. Accepted March 19, 2024.

1. Introduction

This paper is devoted to the decay of solution of the damped wave equations in one dimensional space in the L^p-framework for $1 < p < +\infty$ for various boundary conditions where the damping depends on space and time. More precisely, we consider the damped wave equation

$$\begin{cases} \partial_{tt} u - \partial_{xx} u + a \partial_t u = 0 & \text{in } \mathbb{R}_+ \times (0,1), \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 & \text{on } (0,1), \end{cases} \tag{1.1}$$

equipped with the following boundary conditions:

$$\text{Dirichlet boundary condition: } u(t,0) = u(t,1) = 0, \text{ for } t \geq 0, \tag{1.2}$$

$$\text{Neumann boundary condition: } \partial_x u(t,0) = \partial_x u(t,1) = 0, \text{ for } t \geq 0, \tag{1.3}$$

and, for $\kappa > 0$,

$$\text{dynamic boundary condition: } \partial_x u(t,0) - \kappa \partial_t u(t,0) = \partial_x u(t,1) + \kappa \partial_t u(t,1) = 0, \text{ for } t \geq 0. \tag{1.4}$$

Keywords and phrases: Damped wave equations, damped hyperbolic system, exponential decay, Riemann invariants.

[1] Laboratoire des signaux et systèmes, Université Paris Saclay, France.
[2] Laboratoire Jacques Louis Lions, Sorbonne Université, Paris, France.

[*] Corresponding author: hoai-minh.nguyen@sorbonne-universite.fr

Here $u_0 \in W^{1,p}(0,1)$ (with $u_0(0) = u_0(1) = 0$, *i.e.*, $u_0 \in W_0^{1,p}(0,1)$, in the case where the Dirichlet boundary condition is considered), and $u_1 \in L^p(0,1)$ are the initial conditions. Moreover, $a \in L^\infty\big(\mathbb{R}_+ \times (0,1)\big)$ is assumed to verify the following hypothesis:

$$a \geq 0, \text{ and } \exists \lambda, \varepsilon_0 > 0, \ (x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset (0,1) \text{ such that } a \geq \lambda \text{ on } \mathbb{R}_+ \times (x_0 - \varepsilon_0, x_0 + \varepsilon_0), \tag{1.5}$$

i.e, a is non-negative and $a(t,x) \geq \lambda > 0$ for $t \geq 0$ and for x in some open subset of $(0,1)$. The region where $a > 0$ represents the region in which the damping term is active.

The decay of the solutions of (1.1) equipped with either (1.2), or (1.3), or (1.4) has been extensively investigated in the case where a is independent of t, *i.e.*, $a(t,x) = a(x)$ and mainly in the L^2-framework, *i.e.* within an Hilbertian setting. In this case, concerning the Dirichlet boundary condition, under the additional geometric multiplier condition on a, by the multiplier method, see *e.g.*, [1, 2], one can prove that the solution decays exponentially, *i.e.*, there exist positive constants C and γ independent of u such that

$$\|\partial_t u(t,\cdot)\|_{L^2(0,1)} + \|\partial_x u(t,\cdot)\|_{L^2(0,1)} \leq C e^{-\gamma t}\Big(\|\partial_t u(0,\cdot)\|_{L^2(0,1)} + \|\partial_x u(0,\cdot)\|_{L^2(0,1)}\Big), \ t \geq 0. \tag{1.6}$$

The assumption that a satisfies the geometric multiplier condition is equivalent to the requirement that $a(x) \geq \lambda > 0$ on some neighbourhood of 0 or 1. Based on more sophisticate arguments in the seminal work of Bardos, Lebeau, and Rauch on the controllability of the wave equation [3], Lebeau [4] showed that (1.6) also holds *without* the geometric multiplier condition on a, see also the work of Rauch and Taylor [5]. When the damping coefficient a is also time-dependent, similar results have been obtained recently by Le Rousseau *et al.* in [6]. It is worth noticing that strong stabilization, *i.e.*, the energy decay to zero for each trajectory, has been established previously using LaSalle's invariance argument [7, 8]. The analysis of the nonlinear setting associated with (1.1) can be found in [9–15] and the references therein. Similar results holds for the Neumann boundary condition [3, 6, 13, 15]. Concerning the dynamic boundary condition without interior damping effect, *i.e.*, $a \equiv 0$, the analysis for L^2-framework was previously initiated by Quinn and Russell [16]. They proved that the energy exponentially decays in L^2-framework in one dimensional space. The exponential decay for higher dimensional space was proved by Lagnese [17] using the multiplier technique (see also [16]). The decay hence was established for the geometric multiplier condition and this technique was later extended in [18], see also [19] for a nice account on these issues.

Much less is known about the asymptotic stability of (1.1) equipped with either (1.2), or (1.3), or (1.4) in L^p-framework. This is probably due to the fact that for linear wave equations considered in domains of $\mathbb{R}^d$ with $d \geq 2$ is not a well defined bounded operator in general in L^p framework with $p \neq 2$, a result due to Peral [20]. As far as we know, the only work concerning exponential decay in the L^p-framework is due to Kafnemer *et al.* [21], where the Dirichlet boundary condition was considered. For the damping coefficient a being time-independent, they showed that the decay holds under the additional geometric multiplier condition on a for $1 < p < +\infty$. Their analysis is *via* the multiplier technique involving various non-linear test functions. In the case of zero damping and with a dynamic boundary condition, previous results have been obtained in [22] where the problem has been reduced to the study of a discrete time dynamical system over appropriate functional spaces.

The goal of this paper is to give a unified approach to deal with all the boundary considered in (1.2), (1.3), and (1.4) in the L^p-framework for $1 < p < +\infty$ under the condition (1.5). Our results thus hold even in the case where a is a function of time and space. The analysis is based on the study of the corresponding hyperbolic systems associated with the Riemann invariants for which new insights are required.

Concerning the Dirichlet boundary condition, we obtain the following result.

Theorem 1.1. *Let $1 < p < +\infty$, $\varepsilon_0 > 0$, $\lambda > 0$, and let $a \in L^\infty\big(\mathbb{R}_+ \times (0,1)\big)$ be such that $a \geq 0$ and $a \geq \lambda > 0$ in $\mathbb{R}_+ \times (x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset \mathbb{R}_+ \times (0,1)$ for some $x_0 \in (0,1)$. Then there exist positive constants C and γ depending only on p, $\|a\|_{L^\infty\left(\mathbb{R}_+ \times (0,1)\right)}$, ε_0, and λ such that for all $u_0 \in W_0^{1,p}(0,1)$ and $u_1 \in L^p(0,1)$, the unique* *weak solution* $u \in C([0,+\infty); W_0^{1,p}(0,1)) \cap C^1([0,+\infty); L^p(0,1))$ *of* (1.1) *and* (1.2) *satisfies*

$$\|\partial_t u(t,\cdot)\|_{L^p(0,1)} + \|\partial_x u(t,\cdot)\|_{L^p(0,1)} \le C e^{-\gamma t}\Big(\|u_1\|_{L^p(0,1)} + \|\partial_x u_0\|_{L^p(0,1)}\Big), \ t \ge 0. \tag{1.7}$$

The meaning of the (weak) solutions given Theorem 1.1 is given in Section 2 (see Def. 2.1) and their well-posedness is also established there (see Prop. 2.5). Our analysis is *via* the study of the decay of solutions of hyperbolic systems which are associated with (1.1) *via* the Riemann invariants. Such a decay for the hyperbolic system, even in the case $p = 2$, is new to our knowledge. The analysis of these systems has its own interest and is motivated by recent analysis on the controllability of hyperbolic systems in one dimensional space [23–26].

As in [21, 27], we set

$$\rho(t,x) = u_x(t,x) + u_t(t,x) \quad \text{and} \quad \xi(t,x) = u_x(t,x) - u_t(t,x) \text{ for } (t,x) \in \mathbb{R}_+ \times (0,1). \tag{1.8}$$

One can check that for a smooth solution u of (1.1) and (1.2), the pair of functions (ρ,ξ) defined in (1.8) satisfies the system

$$\begin{cases} \rho_t - \rho_x = -\frac{1}{2}a(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \xi_t + \xi_x = \frac{1}{2}a(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \rho(t,0) - \xi(t,0) = \rho(t,1) - \xi(t,1) = 0 & \text{in } \mathbb{R}_+. \end{cases} \tag{1.9}$$

One *cannot* hope the decay of a general solutions of (1.9) since any pair (c,c) where $c \in \mathbb{R}$ is a constant is a solution of (1.9). Nevertheless, for (ρ,ξ) being defined by (1.9) for a solution u of (1.1), one also has the following additional information

$$\int_0^1 \rho(t,x) + \xi(t,x)\, dx = 0 \text{ for } t \ge 0. \tag{1.10}$$

Concerning System (1.9) itself (*i.e.*, without necessarily assuming (1.10)), we prove the following result, which takes into account (1.10).

Theorem 1.2. *Let* $1 < p < +\infty$, $\varepsilon_0 > 0$, $\lambda > 0$, *and* $a \in L^\infty\big(\mathbb{R}_+ \times (0,1)\big)$ *be such that* $a \ge 0$ *and* $a \ge \lambda > 0$ *in* $\mathbb{R}_+ \times (x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset \mathbb{R}_+ \times (0,1)$ *for some* $x_0 \in (0,1)$. *There exist a positive constant* C *and a positive constant* γ *depending only on on* p, $\|a\|_{L^\infty(\mathbb{R}_+\times(0,1))}$, ε_0, *and* λ *such that the unique solution* (ρ,ξ) *of* (1.9) *with the initial condition* $\rho(0,\cdot) = \rho_0$ *and* $\xi(0,\cdot) = \xi_0$ *satisfies*

$$\|(\rho - c_0, \xi - c_0)(t,\cdot)\|_{L^p(0,1)} \le C e^{-\gamma t} \|(\rho(0,\cdot) - c_0, \xi(0,\cdot) - c_0)\|_{L^p(0,1)}, \ t \ge 0, \tag{1.11}$$

where

$$c_0 := \frac{1}{2}\int_0^1 \big(\rho(0,x) + \xi(0,x)\big)\, dx, \tag{1.12}$$

In Theorem 1.2, we consider the broad solutions. It is understood through the broad solution in finite time: for $T > 0$ and $1 \le p < +\infty$, a broad solution u of the system

$$\begin{cases} \rho_t - \rho_x = -\frac{1}{2}a(\rho - \xi) & \text{in } (0,T)\times(0,1), \\ \xi_t + \xi_x = \frac{1}{2}a(\rho - \xi) & \text{in } (0,T)\times(0,1), \\ \rho(t,0) - \xi(t,0) = \rho(t,1) - \xi(t,1) = 0 & \text{in } (0,T), \\ \rho(0,\cdot) = \rho_0, \quad \xi(0,\cdot) = \xi_0 & \text{in } (0,1), \end{cases} \tag{1.13}$$

is a pair of functions $(\rho, \xi) \in C([0,T]; \left[L^p(0,1)\right]^2) \cap C([0,1]; \left[L^p(0,T)\right]^2)$ which obey the characteristic rules, see *e.g.*, [24]. The well-posedness of (1.13) can be found in [24] (see also the appendix of [28]). The analysis there is mainly for the case $p = 2$ but the arguments extend naturally for the case $1 \le p < +\infty$.

In the L^p-framework, the Neumann boundary condition and its corresponding hyperbolic systems are discussed in Section 5 and the dynamic boundary condition and its corresponding hyperbolic systems are discussed in Section 6. Concerning the dynamic boundary condition, the decay holds even under the assumption $a \ge 0$. The analysis for the Neumann case shares a large part in common with the one of the Dirichlet boundary condition. The difference in their analysis comes from taking into account differently the boundary condition. The analysis of the dynamic condition is similar but much simpler.

The study of the wave equation in one dimensional space *via* the corresponding hyperbolic system is known. The controllability and stability of hyperbolic systems has been also investigated extensively. This goes back to the work of Russel [29, 30] and Rauch and Taylor [5]. Many important progress has been obtained recently, see, *e.g.*, [31] and the references therein. It is worth noting that many works have been devoted to the L^2-framework. Less is studied in the L^p-scale. In this direction, we want to mention [23] where the exponential stability is studied for dissipative boundary condition.

Concerning the wave equation in one dimensional space, the exponential decay in L^2-setting for the dynamic boundary condition is also established *via* its corresponding hyperbolic systems [16]. However, to our knowledge, the exponential decay for the Dirichlet and Neumann boundary conditions has not been established even in L^2-framework *via* this approach. Our work is new and quite distinct from the one in [16] and has its own interest. First, the analysis in [16] uses essentially the fact that the boundary is strictly dissipative, *i.e.*, $\kappa > 0$ in (1.4). Thus the analysis cannot be used for the Dirichlet and Neumann boundary conditions. Moreover, it is not clear how to extend it to the L^p-framework. Concerning our analysis, the key observation is that the information of the internal energy allows one to control the difference of the solutions and its mean value in the interval of time $(0,T)$ in an average sense. This observation is implemented in two lemmas (Lem. 3.2 and Lem. 3.3) after using a standard result (Lem. 3.1) presented in Section 3. These two lemmas are the main ingredients of our analysis for the Dirichlet and Neumann boundary conditions. The proof of the first lemma is mainly based on the characteristic method while as the proof of the second lemma is inspired from the theory of functions with bounded mean oscillations due to John and Nirenberg [32]. As seen later that, the analysis for the dynamic boundary condition is much simpler for which the use of Lemma 3.1 is sufficient.

An interesting point of our analysis is the fact that these lemmas do not depend on the boundary conditions used. In fact, one can apply it in a setting where a bound of the internal energy is accessible. This allows us to deal with all the boundary conditions considered in this paper by the same way. Another point of our analysis which is helpful to be mentioned is that the asymptotic stability for hyperbolic systems in one dimensional space has been mainly studied for general solutions. This is not the case in the setting of Theorem 1.2 where the asymptotic stability holds under condition (1.10). It is also worth noting that the time-dependent coefficients generally make the phenomena more complex, see [28] for a discussion on the optimal null-controllable time.

The analysis in this paper cannot handle the cases $p = 1$ and $p = +\infty$. Partial results in this direction for the Dirichlet boundary condition can be found in [21] where a is constant and in some range. These cases will be considered elsewhere by different approaches.

The paper is organized as follows. The well-posedness of (1.1) equipped with one of the boundary conditions (1.2) and (1.3) is discussed in Section 2, where a slightly more general context is considered (the boundary condition (1.4) is considered directly in Sect. 6; comments on this point is given in Rem. 6.5). Section 4 is devoted to the proof of Theorem 1.1 and Theorem 1.2. We also relaxed slightly the non-negative assumption on a in Theorem 1.1 and Theorem 1.2 there (see Thm. 4.1 and Thm. 4.2) using a standard perturbative argument. The Neumann boundary condition is studied in Section 5 and the Dynamic boundary condition is considered in Section 6.

2. The well-posedness in L^p-setting

In this section, we give the meaning of the solutions of the damped wave equation (1.1) equipped with either the Dirichlet boundary condition (1.2) or the Neumann boundary condition (1.3) and establish their well-posedness in the L^p-framework with $1 \le p \le +\infty$. We will consider a slightly more general context. More precisely, we consider the system

$$\begin{cases} \partial_{tt} u - \partial_{xx} u + a \partial_t u + b \partial_x u + cu = f & \text{in } (0, T) \times (0, 1), \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 & \text{in } (0, 1), \end{cases} \tag{2.1}$$

equipped with either

$$\text{Dirichlet boundary condition: } u(t, 0) = u(t, 1) = 0 \text{ for } t \in (0, T), \tag{2.2}$$

or

$$\text{Neumann boundary condition: } \partial_x u(t, 0) = \partial_x u(t, 1) = 0 \text{ for } t \in (0, T). \tag{2.3}$$

Here $a, b, c \in L^\infty((0, T) \times (0, 1))$ and $f \in L^p((0, T) \times (0, 1))$.

We begin with the Dirichlet boundary condition.

Definition 2.1. Let $T > 0$, $1 \le p < +\infty$, $a, b, c \in L^\infty((0, T) \times (0, 1))$, $f \in L^p((0, T) \times (0, 1))$, $u_0 \in W^{1,p}_0(0, 1)$, and $u_1 \in L^p(0, 1)$. A function $u \in C([0, T]; W^{1,p}_0(0, 1)) \cap C^1([0, T]; L^p(0, 1))$ is called a (weak) solution of (2.1) and (2.2) (up to time T) if

$$u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 \text{ in } (0, 1), \tag{2.4}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^1 u(t, x) v(x) \,\mathrm{d}x + \int_0^1 u_x(t, x) v_x(x) \,\mathrm{d}x + \int_0^1 a(t, x) u_t(t, x) v(x) \,\mathrm{d}x \\ + \int_0^1 b(t, x) u_x(t, x) v(x) \,\mathrm{d}x + \int_0^1 c(t, x) u(t, x) v(x) \,\mathrm{d}x = \int_0^1 f(t, x) v(x) \,\mathrm{d}x \tag{2.5}$$

in the distributional sense in $(0, T)$ for all $v \in C^1_c(0, 1)$.

Definition 2.1 can be modified to deal with the case $p = +\infty$ as follows.

Definition 2.2. Let $T > 0$, $a, b, c \in L^\infty((0,T)\times(0,1))$, $f \in L^\infty((0,T)\times(0,1))$, $u_0 \in W^{1,\infty}_0(0,1)$, and $u_1 \in L^\infty(0,1)$. A function $u \in L^\infty([0,T]; W^{1,\infty}_0(0,1)) \cap W^{1,\infty}([0,T]; L^\infty(0,1))$ is called a (weak) solution of (2.1) and (2.2) (up to time T) if $u \in C([0,T]; W^{1,2}_0(0,1)) \cap C^1([0,T]; L^2(0,1))$[1] and satisfies (2.4) and (2.5).

Concerning the Neumann boundary condition, we have the following definition.

Definition 2.3. Let $T > 0$, $1 \le p < +\infty$, $a, b, c \in L^\infty((0,T)\times(0,1))$, $f \in L^p((0,T)\times(0,1))$, $u_0 \in W^{1,p}(0,1)$, and $u_1 \in L^p(0,1)$. A function $u \in C([0,T]; W^{1,p}(0,1)) \cap C^1([0,T]; L^p(0,1))$ is called a (weak) solution of (2.1) and (2.3) (up to time T) if (2.4) is valid and

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^1 u(t,x)v(x)\,\mathrm{d}x + \int_0^1 u_x(t,x)v_x(x)\,\mathrm{d}x
$$

$$
+ \int_0^1 b(t,x)u_x(t,x)v(x)\,\mathrm{d}x + \int_0^1 c(t,x)u(t,x)v(x)\,\mathrm{d}x + \int_0^1 a(t,x)u_t(t,x)v(x)\,\mathrm{d}x = \int_0^1 f(t,x)v(x)\,\mathrm{d}x \tag{2.6}
$$

holds in the distributional sense in $(0,T)$ for all $v \in C^1([0,1])$.

Definition 2.1 can be modified to deal with the case $p = +\infty$ as follows.

Definition 2.4. Let $T > 0$, $a, b, c \in L^\infty((0,T)\times(0,1))$, $f \in L^\infty((0,T)\times(0,1))$, $u_0 \in W^{1,\infty}(0,1)$, and $u_1 \in L^\infty(0,1)$. A function $u \in L^\infty([0,T]; W^{1,\infty}(0,1)) \cap W^{1,\infty}([0,T]; L^\infty(0,1))$ is called a (weak) solution of (2.1) and (2.3) (up to time T) if $u \in C([0,T]; W^{1,2}(0,1)) \cap C^1([0,T]; L^2(0,1))$[2], (2.4) is valid, and (2.5) holds in the distributional sense in $(0,T)$ for all $v \in C^1([0,1])$.

Concerning the well-posedness of the Dirichlet system (2.1) and (2.2), we establish the following result.

Proposition 2.5. *Let $T > 0$, $1 \le p \le +\infty$, and $a, b, c \in L^\infty((0,T)\times(0,1))$, and let $u_0 \in W^{1,p}_0(0,1)$, $u_1 \in L^p(0,1)$, and $f \in L^p\big((0,T)\times(0,1)\big)$. Then there exists a unique (weak) solution u of (2.1) and (2.2). Moreover, it holds*

$$
\|\partial_t u(t,\cdot)\|_{L^p(0,1)} + \|\partial_x u(t,\cdot)\|_{L^p(0,1)} \le C\Big(\|u_1\|_{L^p(0,1)} + \|\partial_x u_0\|_{L^p(0,1)} + \|f\|_{L^p\big((0,T)\times(0,1)\big)}\Big), \ t \ge 0 \tag{2.7}
$$

for some positive constant $C = C(p, T, \|a\|_{L^\infty}, \|b\|_{L^\infty}, \|c\|_{L^\infty})$ which is independent of u_0, u_1, and f.

Concerning the well-posedness of the Neumann system (2.1) and (2.3), we prove the following result.

Proposition 2.6. *Let $T > 0$, $1 \le p \le +\infty$, and $a, b, c \in L^\infty((0,T)\times(0,1))$, and let $u_0 \in W^{1,p}(0,1)$, $u_1 \in L^p(0,1)$, and $f \in L^p\big((0,T)\times(0,1)\big)$. Then there exists a unique (weak) solution u of (2.1) and (2.3) and*

$$
\|\partial_t u(t,\cdot)\|_{L^p(0,1)} + \|\partial_x u(t,\cdot)\|_{L^p(0,1)} \le C\Big(\|u_1\|_{L^p(0,1)} + \|\partial_x u_0\|_{L^p(0,1)} + \|f\|_{L^p\big((0,T)\times(0,1)\big)}\Big), \ t \ge 0 \tag{2.8}
$$

for some positive constant $C = C(p, T, \|a\|_{L^\infty}, \|b\|_{L^\infty}, \|c\|_{L^\infty})$ which is independent of u_0, u_1, and f.

Remark 2.7. The definition of weak solutions and the well-posedness are stated for $p = 1$ and $p = +\infty$ as well. The existence and the well-posedness is well-known in the case $p = 2$. The standard analysis in the case $p = 2$ is *via* the Galerkin method.

[1] By interpolation, one can use $C([0,T]; W^{1,2}_0(0,1)) \cap C^1([0,T]; L^2(0,1))$ instead of $C([0,T]; W^{1,2}_0(0,1)) \cap C^1([0,T]; L^2(0,1))$ for any $1 \le q < +\infty$. This condition is used to give the meaning of the initial conditions.

[2] By interpolation, one can use $C([0,T]; W^{1,2}_0(0,1)) \cap C^1([0,T]; L^2(0,1))$ instead of $C([0,T]; W^{1,2}(0,1)) \cap C^1([0,T]; L^2(0,1))$ for any $1 \le q < +\infty$. This condition is used to give the meaning of the initial conditions.

The rest of this section is devoted to the proof of Proposition 2.5 and Proposition 2.6 in Section 2.1 and Section 2.2, respectively.

2.1. Proof of Proposition 2.5

The proof is divided into two steps in which we prove the uniqueness and the existence.

• Step 1: Proof of the uniqueness. Assume that u is a (weak) solution of (2.1) with $f = 0$ in $(0, T) \times (0, 1)$ and $u_0 = u_1 = 0$ in $(0, 1)$. We will show that $u = 0$ in $(0, T) \times (0, 1)$. Set

$$g(t, x) = -a(t, x)\partial_t u(t, x) - b(t, x)\partial_x u(t, x) - c(t, x)u(t, x). \tag{2.9}$$

Then u is a weak solution of the system

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = g & \text{in } (0, T) \times (0, 1), \\ u(t, 0) = u(t, 1) = 0 & \text{for } t \in (0, T), \\ u(0, \cdot) = 0, \quad \partial_t u(0, \cdot) = 0 & \text{in } (0, 1). \end{cases} \tag{2.10}$$

Extend u and g in $(0, T) \times \mathbb{R}$ by appropriate reflection in x first by odd extension in $(-1, 0)$, *i.e.*, $u(t, x) = -u(t, -x)$ and $g(t, x) = -g(t, -x)$ in $(0, T) \times (-1, 0)$ and so on, and still denote the extension by u and g. Then $u \in C([0, T]; W^{1,p}(-k, k)) \cap C^1([0, t]; L^p(-k, k))$ and $g \in L^p\big((0, T) \times (-k, k)\big)$ for $k \geq 1$ and for $1 \leq p < +\infty$, and similar facts holds for $p = +\infty$. We also obtain that $u(0, \cdot) = 0$ and $\partial_t u(0, \cdot) = 0$ in $\mathbb{R}$, and

$$\partial_{tt} u - \partial_{xx} u = g \text{ in } (0, T) \times \mathbb{R} \text{ in the distributional sense.} \tag{2.11}$$

The d'Alembert formula gives, for $t \geq 0$, that

$$u(t, x) = \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} g(\tau, y) \, \mathrm{d}y \, \mathrm{d}\tau. \tag{2.12}$$

We then obtain for $t \geq 0$

$$\partial_t u(t, x) = \frac{1}{2} \int_0^t g(\tau, x + t - \tau) + g(\tau, x - t + \tau) \, \mathrm{d}\tau \tag{2.13}$$

and

$$\partial_x u(t, x) = \frac{1}{2} \int_0^t g(\tau, x + t - \tau) - g(\tau, x - t + \tau) \, \mathrm{d}\tau. \tag{2.14}$$

Using (2.9), we derive from (2.12), (2.13) and (2.14) that, for $1 \leq p < +\infty$ and for $t \geq 0$,

$$\int_0^1 |\partial_t u(t, x)|^p + |\partial_t u(t, x)|^p + |\partial_x u(t, x)|^p \, \mathrm{d}x \\ \leq C \int_0^t \int_0^1 \Big(|\partial_t u(s, y)|^p + |\partial_x u(s, y)|^p + |u(s, y)|^p \Big) \, \mathrm{d}y \, \mathrm{d}s, \tag{2.15}$$

and, for $p = +\infty$,

$$\|u(t,\cdot)\|_{L^\infty(0,1)} + \|\partial_t u(t,\cdot)\|_{L^\infty(0,1)} + \|\partial_x u(t,\cdot)\|_{L^\infty(0,1)}$$
$$\leq Ct\Big(\|\partial_t u(t,\cdot)\|_{L^\infty\big((0,t)\times(0,1)\big)} + \|\partial_x u(t,\cdot)\|_{L^\infty\big((0,t)\times(0,1)\big)} + \|u(t,\cdot)\|_{L^\infty\big((0,t)\times(0,1)\big)}\Big), \quad (2.16)$$

for positive constant C only depending only on $p, T, \|a\|_{L^\infty}, \|b\|_{L^\infty}, \|c\|_{L^\infty}$. In the sequel, such constants will again be denoted by C.

It is immediate to deduce from the above equations that $u = 0$ on $[0, 1/2C] \times (0,1)$ and then $u = 0$ in $(0,T) \times (0,1)$. The proof of the uniqueness is complete.

• Step 2: Proof of the existence. Let (a_n), (b_n), and (c_n) be smooth functions in $[0,T] \times [0,1]$ such that $\operatorname{supp} a_n, \operatorname{supp} b_n, \operatorname{supp} c_n \cap 0 \times [0,1] = \emptyset$,

$$(a_n, b_n, c_n) \rightharpoonup (a,b,c) \text{ weakly star in } \Big(L^\infty\big((0,T)\times(0,1)\big)\Big)^3,$$

and

$$(a_n, b_n, c_n) \to (a,b,c) \text{ in } \Big(L^q\big((0,T)\times(0,1)\big)\Big)^3 \text{ for } 1 \leq q < +\infty.$$

Let $u_{0,n} \in C_c^\infty(0,1)$ and $u_{1,n} \in C_c^\infty(0,1)$ be such that, if $1 \leq p < +\infty$,

$$u_{0,n} \to u_0 \text{ in } W_0^{1,p}(0,1) \quad \text{and} \quad u_{1,n} \to u_1 \text{ in } L^p(0,1),$$

and, if $p = +\infty$ then the following two facts hold

$$u_{0,n} \rightharpoonup u_0 \text{ weakly star in } W_0^{1,\infty}(0,1) \quad \text{and} \quad u_{1,n} \rightharpoonup u_1 \text{ weakly star in } L^\infty(0,1),$$

and, for $1 \leq q < +\infty$,

$$u_{0,n} \to u_0 \text{ in } W_0^{1,q}(0,1) \quad \text{and} \quad u_{1,n} \to u_1 \text{ in } L^q(0,1).$$

The existence of (a_n, b_n, c_n) and the existence of $u_{0,n}$ and $u_{1,n}$ follows from the standard theory of Sobolev spaces, see, *e.g.*, [33].

Let u_n be the weak solution corresponding to (a_n, b_n, c_n) with initial data $(u_{0,n}, u_{1,n})$. Then u_n is smooth in $[0,T] \times [0,1]$. Set

$$g_n(t,x) = -a_n(t,x)\partial_t u_n(t,x) - b_n(t,x)\partial_x u_n(t,x) - c_n(t,x)u_n(t,x) \text{ in } (0,T)\times(0,1).$$

Extend u_n, g_n, and f in $(0,T) \times \mathbb{R}$ by first odd refection in $(-1,0)$ and so on, and still denote the extension by u_n and g_n, and f. We then have

$$\partial_{tt} u_n - \partial_{xx} u_n = g_n + f \text{ in } (0,T) \times \mathbb{R}, \quad (2.17)$$

The d'Alembert formula gives

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} g_n(\tau,y) + f(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau + \frac{1}{2}\Big(u_n(0,x-t) + u_n(0,x+t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} \partial_t u_n(0,y)\,\mathrm{d}y.$$

As in the proof of the uniqueness, we then have, for $1 \le p < +\infty$ and $0 < t < T$,

$$\int_0^1 |u_n(t,x)|^p + |\partial_t u_n(t,x)|^p + |\partial_x u_n(t,x)|^p \, \mathrm{d}x$$
$$\le C \int_0^t \int_0^1 \Big(|\partial_t u(s,y)|^p + |\partial_x u(s,y)|^p \Big) \, \mathrm{d}y \, \mathrm{d}s$$
$$+ C \left(\|u_n(0,\cdot)\|_{W^{1,p}}^p + \|\partial_t u_n(0,\cdot)\|_{L^p}^p + \int_0^t \int_0^1 |f(s,y)|^p \, \mathrm{d}y \, \mathrm{d}s \right), \quad (2.18)$$

and, for $p = +\infty$,

$$\|u(t,\cdot)\|_{L^\infty(0,1)} + \|\partial_t u(t,\cdot)\|_{L^\infty(0,1)} + \|\partial_x u(t,\cdot)\|_{L^\infty(0,1)}$$
$$\le Ct \left(\|\partial_t u(t,\cdot)\|_{L^\infty\big((0,t)\times(0,1)\big)} + \|\partial_x u(t,\cdot)\|_{L^\infty\big((0,t)\times(0,1)\big)} \right)$$
$$+ C \left(\|u_n(0,\cdot)\|_{W^{1,\infty}} + \|\partial_t u_n(0,\cdot)\|_{L^\infty} + \|f\|_{L^\infty\big((0,t)\times(0,1)\big)} \right). \quad (2.19)$$

Letting $n \to +\infty$, we derive (2.8) from (2.18) and (2.19).

To derive that $u \in C([0,T]; W_0^{1,p}(0,1)) \cap C^1([0,T]; L^p(0,1))$ in the case $1 \le p < +\infty$ and $u \in C([0,T]; W_0^{1,2}(0,1)) \cap C^1([0,T]; L^2(0,1))$ otherwise, one just notes that (u_n) is a Cauchy sequence in these spaces correspondingly.

The proof is complete. □

Remark 2.8. Our proof on the well-posedness is quite standard and is based on the d'Alembert formula. This formula was also used previously in [21].

Remark 2.9. There are several ways to give the notion of weak solution even in the case $p = 2$, see, *e.g.*, [34, 35]. The definitions given here is a nature modification of the case $p = 2$ given in [34].

2.2. Proof of Proposition 2.6

The proof of Proposition 2.6 is similar to the one of Proposition 2.5. To apply the d'Alembert formula, one just needs to extend various function appropriately and differently. For example, in the proof of the uniqueness, one extend u and g in $(0,T) \times \mathbb{R}$ by appropriate reflection in x first by even extension in $(-1,0)$, *i.e.*, $u(t,x) = u(t,-x)$ and $g(t,x) = g(t,-x)$ in $(0,T) \times (-1,0)$ and so on. The details are left to the reader. □

3. Some useful lemmas

In this section, we prove three lemmas which will be used through out the rest of the paper. The first one is quite standard and the last two ones are the main ingredients of our analysis for the Dirichlet and Neumann boundary condition. We begin with the following lemma.

Lemma 3.1. *Let $1 < p < +\infty$, $0 < T < \hat{T}_0$, and $a \in L^\infty((0,T) \times (0,1))$ be such that $a \ge 0$ in $(0,T) \times (0,1)$. There exists a positive constant C depending only on p, $\hat{T}_0$, and $\|a\|_{L^\infty}$ such that, for $(\rho, \xi) \in$*

$\big[L^p\big((0,T)\times(0,1)\big)\big]^2$,

$$\int_0^T\int_0^1 a|\rho-\xi|^p(t,x)\,\mathrm{d}x\,\mathrm{d}t \le \begin{cases} Cm_p & \textit{if } p\ge 2,\\ C(m_p+m_p^{2/p}) & \textit{if } 1<p<2,\end{cases} \tag{3.1}$$

where

$$m_p=\int_0^1\int_0^T a(\rho-\xi)(\rho|\rho|^{p-2}-\xi|\xi|^{p-2})(t,x)\,\mathrm{d}t\,\mathrm{d}x. \tag{3.2}$$

Proof. The proof of Lemma 3.1 is quite standard. For the convenience of the reader, we present its proof. There exists a positive constant C_p depending only on p such that

- for $2\le p<+\infty$, it holds, for $\alpha,\beta\in\mathbb{R}$,
$$(\alpha-\beta)(\alpha|\alpha|^{p-2}-\beta|\beta|^{p-2})\ge C_p|\alpha-\beta|^p;$$
- for $1<p<2$, it holds, for $\alpha,\beta\in\mathbb{R}$[3]
$$(\alpha-\beta)(\alpha|\alpha|^{p-2}-\beta|\beta|^{p-2})\ge C_p\min\big\{|\alpha-\beta|^p,|\alpha-\beta|^2\big\}.$$

Using this, we derive that

$$\int_0^T\int_{\substack{0\\|\rho-\xi|\ge 1}}^1 a|\rho-\xi|^p\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\substack{0\\|\rho-\xi|<1}}^1 a|\rho-\xi|^{\max\{p,2\}}\,\mathrm{d}x\,\mathrm{d}t\le m_p.$$

This yields

$$\int_0^T\int_0^1 a|\rho-\xi|^p(t,x)\,\mathrm{d}x\,\mathrm{d}t\le Cm_p \text{ if } p\ge 2, \tag{3.3}$$

and, using Hölder's inequality, one gets

$$\int_0^T\int_0^1 a|\rho-\xi|^p(t,x)\,\mathrm{d}x\,\mathrm{d}t\le C(m_p+m_p^{2/p}) \text{ if } 1<p\le 2, \tag{3.4}$$

The conclusion follows from (3.3) and (3.4). □

The following lemma is one of the main ingredients in the analysis of the Dirichlet and Neumann boundary conditions.

Lemma 3.2. *Let $1<p<+\infty$, $0<T<\hat{T}_0$, $\varepsilon_0>0$, $\lambda>0$, and $a\in L^\infty((0,T)\times(0,1))$ be such that $T>4\varepsilon_0$, $a\ge 0$ and $a\ge\lambda>0$ in $(0,T)\times(x_0-\varepsilon_0,x_0+\varepsilon_0)\subset(0,T)\times(0,1)$ for some $x_0\in(0,1)$. Let (ρ,ξ) be a broad solution of the system*

$$\begin{cases}\rho_t-\rho_x=-\frac{1}{2}a(\rho-\xi) & \textit{in } (0,T)\times(0,1),\\ \xi_t+\xi_x=\frac{1}{2}a(\rho-\xi) & \textit{in } (0,T)\times(0,1).\end{cases} \tag{3.5}$$

[3] Using the symmetry between α and β, one can assume $|\alpha|\ge|\beta|$ and by considering $\beta/|\alpha|$, it is enough to prove these inequalities for $\alpha=1$ and $\beta\in(-1,1)$. One finally reduces the analysis for $\beta\in(0,1)$ and even β close to one. The conclusion follows by performing a Taylor expansion with respect to $1-\beta$.

Set

$$m_p = \int_0^1 \int_0^T a(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t, x)\, dt\, dx. \tag{3.6}$$

Then there exists $z \in (x_0 - \varepsilon_0/2, x_0 + \varepsilon_0/2)$ *such that, with* $T_1 = T - 4\varepsilon_0$,

$$\int_0^{\varepsilon_0/2} \int_0^{T_1} |\rho(t+s, z) - \rho(t, z)|^p\, dt\, ds + \int_0^{\varepsilon_0/2} \int_0^{T_1} |\xi(t+s, z) - \xi(t, z)|^p\, dt\, ds$$
$$+ \int_0^T |\rho(t, z) - \xi(t, z)|^p\, dt + \int_0^T \int_0^1 a|\rho - \xi|^p(t, x)\, dx\, dt$$
$$\leq \begin{cases} Cm_p & \text{if } p \geq 2, \\ C(m_p + m_p^{2/p}) & \text{if } 1 \leq p < 2. \end{cases} \tag{3.7}$$

for some positive constant C *depending only on* $\varepsilon_0, \lambda, p$, $\hat{T}_0$, *and* $\|a\|_{L^\infty}$.

Proof. Set

$$T_2 = T - 2\varepsilon_0.$$

Then $T > T_2 > T_1 > 0$.

We have, for $s \in (-\varepsilon_0/2, \varepsilon_0/2)$ and $y \in (x_0 - \varepsilon_0/2, x_0 + \varepsilon_0/2)$,

$$\rho(t+2s, y) - \rho(t, y) = \Big(\rho(t+2s, y) - \rho(t+s, y+s)\Big) + \Big(\rho(t+s, y+s) - \xi(t+s, y+s)\Big)$$
$$+ \Big(\xi(t+s, y+s) - \xi(t, y)\Big) + \Big(\xi(t, y) - \rho(t, y)\Big). \tag{3.8}$$

By the characteristics method, we obtain

$$\xi(t+s, y+s) - \xi(t, y) = \frac{1}{2} \int_0^s a(t+\tau, y+\tau)\Big(\rho(t+\tau, y+\tau) - \xi(t+\tau, y+\tau)\Big)\, d\tau \tag{3.9}$$

and

$$\rho(t+2s, y) - \rho(t+s, y+s)$$
$$= \frac{1}{2} \int_s^{2s} a(t+\tau, y+2s-\tau)\Big(\rho(t+\tau, y+2s-\tau) - \xi(t+\tau, y+2s-\tau)\Big)\, d\tau. \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), after integrating with respect to t from 0 to T_1, we obtain, for $0 \leq s \leq \varepsilon_0/2$,

$$\int_0^{T_1} |\rho(t+2s, y) - \rho(t, y)|^p\, dt \leq 4^{p-1} \Bigg(\int_0^{T_2} |\rho(t, y+s) - \xi(t, y+s)|^p\, dt$$
$$+2 \int_0^{T_2} \int_0^1 a^p |\rho - \xi|^p(t, x)\, dt\, dx + \int_0^T |\rho(t, y) - \xi(t, y)|^p\, dt\Bigg).$$

Integrating the above inequality with respect to s from 0 to $\varepsilon_0/2$, we obtain

$$\int_0^{\varepsilon_0/2}\int_0^{T_1} |\rho(t+2s,y)-\rho(t,y)|^p \,\mathrm{d}t\,\mathrm{d}s \\ \le 4^p\Bigg(\int_{x_0-\varepsilon_0}^{x_0+\varepsilon_0}\int_0^T |\rho(t,x)-\xi(t,x)|^p \,\mathrm{d}t\,\mathrm{d}x \\ +\varepsilon_0\int_0^T |\rho(t,y)-\xi(t,y)|^p\,\mathrm{d}t + \varepsilon_0\int_0^1\int_0^T a^p|\rho-\xi|^p(t,x)\,\mathrm{d}t\,\mathrm{d}x\Bigg). \quad (3.11)$$

Similarly, we have

$$\int_0^{\varepsilon_0/2}\int_0^{T_1} |\xi(t+2s,y)-\xi(t,y)|^p \,\mathrm{d}t\,\mathrm{d}s \\ \le 4^p\Bigg(\int_{x_0-\varepsilon_0}^{x_0+\varepsilon_0}\int_0^T |\rho(t,x)-\xi(t,x)|^p \,\mathrm{d}t\,\mathrm{d}x \\ +\varepsilon_0\int_0^T |\rho(t,y)-\xi(t,y)|^p\,\mathrm{d}t + \varepsilon_0\int_0^1\int_0^T a^p|\rho-\xi|^p(t,x)\,\mathrm{d}t\,\mathrm{d}x\Bigg). \quad (3.12)$$

Take $y = z \in (x_0-\varepsilon_0/2, x_0+\varepsilon_0/2)$ such that

$$\int_0^T |\rho(t,z)-\xi(t,z)|^p\,\mathrm{d}t \le \frac{1}{\varepsilon_0}\int_{x_0-\varepsilon_0}^{x_0+\varepsilon_0}\int_0^T |\rho-\xi|^p(t,x)\,\mathrm{d}x\,\mathrm{d}t. \quad (3.13)$$

By choosing $y = z$ in (3.11) and (3.12), then by using (3.13) and the fact that (itself consequence of (1.5))

$$\int_{x_0-\varepsilon_0}^{x_0+\varepsilon_0}\int_0^T |\rho(t,x)-\xi(t,x)|^p\,\mathrm{d}t\,\mathrm{d}x \le C(a,p)\int_0^1\int_0^T a^p|\rho-\xi|^p(t,x)\,\mathrm{d}t\,\mathrm{d}x,$$

for some positive constant $C(a,p)$ only depending on a, p, one gets the conclusion. □

The next lemma is also a main ingredient of our analysis for the Dirichlet and Neumann boundary conditions.

Lemma 3.3. *Let $1 \le p < +\infty$ and $L > l > 0$, and let $u \in L^p(0, L+l)$. Then there exists a positive constant C depending only on p, L, and l such that*

$$\int_0^L |u(x) - ⨍_0^L u(y)\,\mathrm{d}y|^p\,\mathrm{d}x \le C\int_0^l\int_0^L |u(x+s)-u(x)|^p\,\mathrm{d}x\,\mathrm{d}s. \quad (3.14)$$

Here and in what follows, $⨍_a^b$ means $\frac{1}{b-a}\int_a^b$ for $b > a$.

Proof. By scaling, one can assume that $L = 1$. Fix $n \ge 2$ such that $2/n \le l \le 2/(n-1)$.
One first notes that, for $x \in [0,1]$,

$$\fint_x^{x+1/n} \left| u(x) - \fint_x^{x+1/n} u(y)\,\mathrm{d}y \right|^p \mathrm{d}x \overset{\text{Jensen}}{\le} \fint_x^{x+1/n} \fint_x^{x+1/n} |u(x) - u(y)|^p \,\mathrm{d}x\,\mathrm{d}y$$
$$\le n^2 \int_0^{2/n} \int_0^1 |u(x+s) - u(x)|^p \,\mathrm{d}x\,\mathrm{d}s \quad (3.15)$$

and

$$\left| \fint_x^{x+1/n} u(s)\,\mathrm{d}s - \fint_{x+1/n}^{x+2/n} u(t)\,\mathrm{d}t \right|^p \overset{\text{Jensen}}{\le} \fint_x^{x+1/n} \fint_{x+1/n}^{x+2/n} |u(s) - u(t)|^p \,\mathrm{d}t\,\mathrm{d}s$$
$$\le n^2 \int_0^{2/n} \int_0^1 |u(x+s) - u(x)|^p \,\mathrm{d}x\,\mathrm{d}s. \quad (3.16)$$

For $0 \le k \le n-1$, set

$$a_k = \fint_{k/n}^{k/n+1/n} u(s)\,\mathrm{d}s.$$

We then derive from (3.16) that, for $0 \le i < j \le n-1$,

$$|a_j - a_i|^p \le (|a_{i+1} - a_i| + \cdots + |a_j - a_{j-1}|)^p$$
$$\le n^{p-1}(|a_{i+1} - a_i|^p + \cdots + |a_j - a_{j-1}|^p)$$
$$\le n^{p+1} \int_0^{2/n} \int_0^1 |u(x+s) - u(x)|^p \,\mathrm{d}x\,\mathrm{d}s.$$

This implies, for $0 \le k \le n-1$,

$$\left| a_k - \int_0^1 u(t)\,\mathrm{d}t \right|^p \le \left| \frac{1}{n} \sum_{i=0}^{n-1} |a_k - a_i| \right|^p$$
$$\le \frac{1}{n} \sum_{i=0}^{n-1} |a_k - a_i|^p \le n^{p+1} \int_0^{2/n} \int_0^1 |u(x+s) - u(x)|^p \,\mathrm{d}x\,\mathrm{d}s. \quad (3.17)$$

We have

$$\int_0^1 \left| u(x) - \int_0^1 u(y)\,\mathrm{d}y \right|^p \mathrm{d}x = \sum_{k=0}^{n-1} \int_{k/n}^{k/n+1/n} \left| u(x) - \int_0^1 u(y)\,\mathrm{d}y \right|^p \mathrm{d}x$$
$$\le 2^{p-1} \sum_{k=0}^{n-1} \int_{k/n}^{k/n+1/n} |u(x) - a_k|^p \,\mathrm{d}x + 2^{p-1} \sum_{k=0}^{n-1} \left| a_k - \int_0^1 u(y)\,\mathrm{d}y \right|^p \quad (3.18)$$

The conclusion with $C = 2^p n^{p+1}$ now follows from (3.15), (3.17), and (3.18) after noting that $L = 1$ and $2/n \le l$. □

Remark 3.4. Related ideas used in the proof of Lemma 3.3 was implemented in the proof of Caffarelli-Kohn-Nirenberg inequality for fractional Sobolev spaces [36].

4. Exponential decay in L^p-framework for the Dirichlet boundary condition

In this section, we prove Theorem 1.1 and Theorem 1.2. We begin with the proof Theorem 1.2 in the first section, and then use it to prove Theorem 1.1 in the second section. We finally extend these results for a which might be negative in some regions using a standard perturbation argument in the third section.

4.1. Proof of Theorem 1.2

We will only consider smooth solutions (ρ, ξ)[4]. The general case will follow by regularizing arguments. Moreover, replacing (ρ, ξ) by $(\rho - c_0, \xi - c_0)$, where the constant c_0 is defined in (1.12), we can assume that

$$\int_0^1 (\rho_0 + \xi_0) \, \mathrm{d}x = 0.$$

Multiplying the equation of ρ with $\rho|\rho|^{p-2}$, the equation of ξ with $\xi|\xi|^{p-2}$, and integrating the expressions with respect to x, after using the boundary conditions, we obtain, for $t > 0$,

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (|\rho(t,x)|^p + |\xi(t,x)|^p) \, \mathrm{d}x + \frac{1}{2}\int_0^1 a(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x) \, \mathrm{d}x = 0. \tag{4.1}$$

This implies

$$\frac{1}{p}\|(\rho,\xi)(t,\cdot)\|^p_{L^p(0,1)} + \frac{1}{2}\int_0^t\int_0^1 a(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x) \, \mathrm{d}x \, \mathrm{d}t = \frac{1}{p}\|(\rho_0,\xi_0)\|^p_{L^p(0,1)}. \tag{4.2}$$

Integrating the equations of ρ and ξ, summing them up and using the boundary conditions, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(\rho(t,x) + \xi(t,x)\Big) \, \mathrm{d}x = 0 \text{ for } t > 0.$$

It follows that

$$\int_0^1 \Big(\rho(t,x) + \xi(t,x)\Big) \, \mathrm{d}x = \int_0^1 \Big(\rho(0,x) + \xi(0,x)\Big) \, \mathrm{d}x = 0 \text{ for } t \ge 0. \tag{4.3}$$

By (4.2) and (4.3), to derive (1.11), it suffices to prove that there exists a constant $c > 0$ depending only on $\|a\|_{L^\infty(\mathbb{R}_+\times(0,1))}$, ε_0, γ, and p such that for any $T > 2$, there exists $c_T > 0$ only depending on p, T, a so that

$$\int_0^T\int_0^1 a(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x) \, \mathrm{d}x \, \mathrm{d}t \ge c_T\|(\rho_0,\xi_0)\|^p_{L^p(0,1)}. \tag{4.4}$$

By scaling, without loss of generality, one might assume that

$$\|(\rho_0,\xi_0)\|_{L^p(0,1)} = 1 \tag{4.5}$$

[4]We thus assume that a is smooth. Nevertheless, the constants in the estimates which will be derived in the proof depend only on p, $\|a\|_{L^\infty}$, λ, and ε_0.

Set

$$m_p := \int_0^T \int_0^1 a(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Applying Lemma 3.1, we have

$$\int_0^T \int_0^1 a|\rho - \xi|^p(t,x)\,\mathrm{d}x\,\mathrm{d}t \le C(m_p + m_p^{2/p}). \tag{4.6}$$

By Lemma 3.2 there exists $z \in (x_0 - \varepsilon_0/2, x_0 + \varepsilon_0/2)$ such that

$$\int_0^{\varepsilon_0/2} \int_0^T |\rho(t+s,z) - \rho(s,z)|^p\,\mathrm{d}t\,\mathrm{d}s + \int_0^{\varepsilon_0/2} \int_0^T |\xi(t+s,z) - \xi(s,z)|^p\,\mathrm{d}t\,\mathrm{d}s$$
$$+ \int_0^T |\rho(t,z) - \xi(t,z)|^p\,\mathrm{d}t \le C(m_p + m_p^{2/p}). \tag{4.7}$$

By Lemma 3.3, we have

$$\int_0^T |\rho(t,z) - A_\rho|^p\,\mathrm{d}t \le C \int_0^{\varepsilon_0/2} \int_0^T |\rho(t+s,z) - \rho(s,z)|^p\,\mathrm{d}t\,\mathrm{d}s \tag{4.8}$$

and

$$\int_0^T |\xi(t,z) - A_\xi|^p\,\mathrm{d}t \le C \int_0^{\varepsilon_0/2} \int_0^T |\xi(t+s,z) - \xi(s,z)|^p\,\mathrm{d}t\,\mathrm{d}s. \tag{4.9}$$

where we have set

$$A_\rho := ⨍_0^T \rho(s,z)\,\mathrm{d}s, \quad A_\xi := ⨍_0^T \xi(s,z)\,\mathrm{d}s. \tag{4.10}$$

Combining (4.7), (4.8), and (4.9) yields

$$\int_0^T |\rho(t,z) - A_\rho|^p\,\mathrm{d}t + \int_0^T |\xi(t,z) - A_\xi|^p\,\mathrm{d}t + \int_0^T |\rho(t,z) - \xi(t,z)|^p\,\mathrm{d}t \le C(m_p + m_p^{2/p}). \tag{4.11}$$

We next prove the following estimates

$$\int_0^1 |\rho(0,x) - A_\xi|^p\,\mathrm{d}x \le C(m_p + m_p^{2/p}) \tag{4.12}$$

and

$$\int_0^1 |\xi(0,x) - A_\rho|^p\,\mathrm{d}x \le C(m_p + m_p^{2/p}). \tag{4.13}$$

The arguments being similar, we only provide that of (4.12). For $x \in (0,1)$, one has, by using the boundary condition at $x = 0$, *i.e.*, $\rho(\cdot, 0) = \xi(\cdot, 0)$,

$$\begin{aligned}\rho(0,x) &= \Big(\rho(0,x) - \rho(x,0)\Big) + \rho(x,0)\\ &= \Big(\rho(0,x) - \rho(x,0)\Big) + \xi(x,0)\\ &= \Big(\rho(0,x) - \rho(x,0)\Big) + \Big(\xi(x,0) - \xi(x+z,z)\Big) + \xi(x+z,z),\end{aligned}$$

which yields, after substracting A_ξ to both sides of the above equality,

$$\int_0^1 |\rho(0,x) - A_\xi|^p \, \mathrm{d}x \le 3^{p-1} \left(\int_0^1 |\rho(0,x) - \rho(x,0)|^p \, \mathrm{d}x + \int_0^1 |\xi(x,0) - \xi(x+z,z)|^p \, \mathrm{d}x \right.$$
$$\left. + \int_0^1 |\xi(x+z,z) - A_\xi|^p \, \mathrm{d}x \right). \quad (4.14)$$

We use the characteristics method, and (3.9) and (3.10) to upper bound the first two integrals in the right-hand side of (4.14) by $C(m_p + m_p^{2/p})$ thanks to Lemma 3.1. Indeed, we have, as in (3.10),

$$\rho(0,x) - \rho(x,0) = -\frac{1}{2} \int_0^x a(t, x-t)\Big(\rho(t,x-t) - \xi(t,x-t)\Big) \, \mathrm{d}t.$$

Using Hölder's inequality, one derives that, since a is bounded and non-negative,

$$\int_0^1 |\rho(0,x) - \rho(x,0)|^p \, \mathrm{d}x \le C \int_0^1 \int_0^x a(t,x-t) |\rho(t,x-t) - \xi(t,x-t)|^p \, \mathrm{d}t \, \mathrm{d}x$$
$$\le C \int_0^1 \int_0^1 a |\rho - \xi|^p(t,x) \, \mathrm{d}t \, \mathrm{d}x.$$

By Lemma 3.1, one obtains the upper bound for the first integral. Similarly, one reaches the upper bound for the second integral.

As for the third integral in the right-hand side of (4.14), we perform the change of variables $t = x + z$ to obtain

$$\begin{aligned}\int_0^1 |\xi(x+z,z) - A_\xi|^p \, \mathrm{d}x &= \int_z^{z+1} |\xi(t,z) - A_\xi|^p \, \mathrm{d}t\\ &\le \int_0^T |\xi(t,z) - A_\xi|^p \, \mathrm{d}t,\end{aligned}$$

which is upper bounded by $C(m_p + m_p^{2/p})$ according to (4.11) (The condition $T > 2$ is used here.). The proof of (4.12) is complete.

We now resume the argument for (4.4). We start by noticing that, for every $t \in (0,T)$

$$|A_\rho - A_\xi| \le |A_\rho - \rho(t,z)| + |A_\xi - \xi(t,z)| + |\rho(t,z) - \xi(t,z)|.$$

Taking the p-th power, integrating over $t \in (0,T)$ and using (4.11), one gets that

$$|A_\rho - A_\xi|^p \le C(m_p + m_p^{2/p}). \tag{4.15}$$

Similarly, for every $x \in (0,1)$,

$$A_\rho + A_\xi = \big(A_\rho - \xi(0,x)\big) + \big(A_\xi - \rho(0,x)\big) + \big(\rho(0,x) + \xi(0,x)\big).$$

Integrating over $x \in (0,1)$ and using (4.3), then taking the p-th power and using (4.12) and (4.13) yield

$$|A_\rho + A_\xi|^p \le C(m_p + m_p^{2/p}). \tag{4.16}$$

Still, for $x \in (0,1)$, it holds

$$|\rho(0,x)|^p + |\xi(0,x)|^p \le C\Big(|A_\rho - \rho(0,x)|^p + |A_\xi - \xi(0,x)|^p\Big) + C(|A_\rho|^p + |A_\xi|^p).$$

Integrating over $x \in (0,1)$ and using (4.5), one gets

$$1 \le C(|A_\rho|^p + |A_\xi|^p) + C(m_p + m_p^{2/p}). \tag{4.17}$$

Since it holds $|a|^p + |b|^p \le |a+b|^p + |a-b|^p$ for every real numbers a, b, one deduces from (4.15), (4.16) and (4.17) that

$$1 \le C(m_p + m_p^{2/p})$$

and hence $m_p \ge c_3$ for some positive constant depending only on $\|a\|_{L^\infty(\mathbb{R}_+\times(0,1))}$, ε_0, γ, and p (after fixing for instance $T = 3$). The proof of the theorem is complete. □

4.2. Proof of Theorem 1.1

Using Theorem 1.2, we obtain the conclusion of Theorem 1.1 for smooth solutions. The proof in the general case follows from the smooth case by density arguments. □

4.3. On the case a not being non-negative

In this section, we first consider the following perturbed system of (1.9):

$$\begin{cases} \rho_t - \rho_x = -\frac{1}{2}a(\rho - \xi) - b(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \xi_t + \xi_x = \frac{1}{2}a(\rho - \xi) + b(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \rho(t,0) - \xi(t,0) = \rho(t,1) - \xi(t,1) = 0 \text{ in } \mathbb{R}_+. \end{cases} \tag{4.18}$$

We establish the following result.

Theorem 4.1. *Let $1 < p < +\infty$, $\varepsilon_0 > 0$, $\lambda > 0$, and $a, b \in L^\infty\big(\mathbb{R}_+ \times (0,1)\big)$ be such that $a \ge 0$ and $a \ge \lambda > 0$ in $\mathbb{R}_+ \times (x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset \mathbb{R}_+ \times (0,1)$ for some $x_0 \in (0,1)$. There exists a positive constant α depending only on p, $\|a\|_{L^\infty}$, ε_0, and λ such that if*

$$\|b\|_{L^\infty} \le \alpha, \tag{4.19}$$

then there exist constants $C, \gamma > 0$ depending only on p, $\|a\|_{L^\infty}$, ε_0, and λ such that, if $\int_0^1 \rho_0 + \xi_0 \, \mathrm{d}x = 0$, then the solution (ρ, ξ) of (4.18) satisfies

$$\|(\rho, \xi)(t, \cdot)\|_{L^p(0,1)} \le C e^{-\gamma t} \|(\rho_0, \xi_0)\|_{L^p(0,1)}, \; t \ge 0. \tag{4.20}$$

Proof. Multiplying the equation of ρ with $\rho|\rho|^{p-2}$, the equation of ξ with $\xi|\xi|^{p-2}$, and integrating the expressions with respect to x, after using the boundary conditions, we obtain

$$\frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (|\rho(t,x)|^p + |\xi(t,x)|^p) \, \mathrm{d}x + \frac{1}{2} \int_0^1 a(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x) \, \mathrm{d}x$$
$$+ \int_0^1 b(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x) \, \mathrm{d}x = 0.$$

This implies

$$\frac{1}{p} \|(\rho, \xi)(t, \cdot)\|^p_{L^p(0,1)} + \frac{1}{2} \int_0^t \int_0^1 a(\rho - \xi)(\rho|\rho|^{p-1} - \xi|\xi|^{p-1})(t,x) \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_0^t \int_0^1 b(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x) \, \mathrm{d}x = \frac{1}{p} \|(\rho_0, \xi_0)\|^p_{L^p(0,1)}. \tag{4.21}$$

Integrating the equation of ρ and ξ and using the boundary condition, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \Big(\rho(t,x) + \xi(t,x) \Big) \, \mathrm{d}x = 0, \text{ for } t > 0.$$

It follows that

$$\int_0^1 \Big(\rho(t,x) + \xi(t,x) \Big) \, \mathrm{d}x = \int_0^1 \Big(\rho(0,x) + \xi(0,x) \Big) \, \mathrm{d}x = 0, \text{ for } t > 0. \tag{4.22}$$

By (4.21) and (4.22), to derive (4.20), it suffices to prove that there exists a constant $c > 0$ depending only on $\|a\|_{L^\infty(\mathbb{R}_+ \times (0,1))}$, ε_0, γ, and p such that for $T = 3$[5], it holds

$$\int_0^T \int_0^1 a(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x) \, \mathrm{d}x \, \mathrm{d}t \ge c \|(\rho_0, \xi_0)\|^p_{L^p(0,1)}. \tag{4.23}$$

Using the facts that $a \ge 0$ and b is bounded, a simple application of Gronwall's lemma to (4.21) yields the existence of $\alpha > 0$ depending only on $\|b\|_{L^\infty(\mathbb{R}_+ \times (0,1))}$ so that

$$\|(\rho, \xi)(t, \cdot)\|^p_{L^p(0,1)} \le e^{p \alpha t} \|(\rho, \xi)(0, \cdot)\|^p_{L^p(0,1)} \text{ for } t \in [0, T]. \tag{4.24}$$

[5] It holds for $T > 2$ with $c = c_T$.

Let (ρ_1, ξ_1) be the unique solution of the system

$$\begin{cases} \rho_{1,t} - \rho_{1,x} = -\frac{1}{2}a(\rho_1 - \xi_1) - b(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \xi_{1,t} + \xi_{1,x} = \frac{1}{2}a(\rho_1 - \xi_1) + b(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \rho_1(t,0) - \xi_1(t,0) = \rho_1(t,1) - \xi_1(t,1) = 0 & \text{in } \mathbb{R}_+, \\ \rho_1(0,\cdot) = \xi_1(0,\cdot) = 0 & \text{in } (0,1). \end{cases} \tag{4.25}$$

Thus $-b(\rho - \xi)$ and $b(\rho - \xi)$ can be considered as source terms for the system of (ρ_1, ξ_1). We then derive from (4.24) that

$$\|(\rho_1, \xi_1)\|_{L^p(T,\cdot)} \le C\alpha \|(\rho, \xi)(0,\cdot)\|^p_{L^p(0,1)}. \tag{4.26}$$

Set

$$\widetilde{\rho} = \rho - \rho_1 \quad \text{and} \quad \widetilde{\xi} = \xi - \xi_1.$$

Then

$$\begin{cases} \widetilde{\rho}_t - \widetilde{\rho}_x = -\frac{1}{2}a(t,x)(\widetilde{\rho} - \widetilde{\xi}) & \text{in } \mathbb{R}_+ \times (0,1), \\ \widetilde{\xi}_t + \widetilde{\xi}_x = \frac{1}{2}a(t,x)(\widetilde{\rho} - \widetilde{\xi}) & \text{in } \mathbb{R}_+ \times (0,1), \\ \widetilde{\rho}(t,0) - \widetilde{\xi}(t,0) = \widetilde{\rho}(t,1) - \widetilde{\xi}(t,1) = 0 & \text{in } \mathbb{R}_+, \\ \widetilde{\rho}(0,\cdot) = \rho_0, \quad \widetilde{\xi}(0,\cdot) = \xi_0 & \text{in } (0,1). \end{cases} \tag{4.27}$$

Applying Theorem 1.2, we have

$$\|(\widetilde{\rho}, \widetilde{\xi})(T,\cdot)\|_{L^p} \le c\|(\widetilde{\rho}, \widetilde{\xi})(0,\cdot)\|_{L^p} \tag{4.28}$$

for some positive constant c depending only on $\|a\|_{L^\infty}$, ε_0, and λ. The conclusion now follows from (4.26) and (4.27). □

Regarding the wave equation, we have

Theorem 4.2. *Let $1 < p < +\infty$, $\varepsilon_0 > 0$, $\lambda > 0$, and $a, b \in L^\infty\big(\mathbb{R}_+ \times (0,1)\big)$ be such that $a \ge 0$ and $a \ge \lambda > 0$ in $\mathbb{R}_+ \times (x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset \mathbb{R}_+ \times (0,1)$. There exists a positive constant α depending only on p, $\|a\|_{L^\infty}$, ε_0, and λ such that if*

$$\|b\|_{L^\infty} \le \alpha, \tag{4.29}$$

then there exist positive constants C and γ depending on p, $\|a\|_{L^\infty\big(\mathbb{R}_+ \times (0,1)\big)}$, ε_0, and λ such that for all $u_0 \in W^{1,p}_0(0,1)$ and $u_1 \in L^p(0,1)$, the unique weak solution $u \in C([0,+\infty); W^{1,p}_0(0,1)) \cap C^1([0,+\infty); L^p(0,1))$ of

$$\begin{cases} \partial_{tt} u - \partial_{xx} u + \Big(a(t,x) + b(t,x)\Big)\partial_t u = 0 & \text{in } \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = 0 & \text{in } \mathbb{R}_+, \\ u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = u_1 & \text{in } (0,1), \end{cases} \tag{4.30}$$

satisfies

$$\|\partial_t u(t,\cdot)\|^p_{L^p(0,1)} + \|\partial_x u(t,\cdot)\|^p_{L^p(0,1)} \le C e^{-\gamma t}\Big(\|u_1\|^p_{L^p(0,1)} + \|\partial_x u_0\|^p_{L^p(0,1)}\Big), \ t \ge 0. \tag{4.31}$$

Proof. The proof of Theorem 4.2 is similar to that of Theorem 1.1 however instead of using Theorem 1.2 one apply Theorem 4.1. The details are left to the reader. □

5. Exponential decay in L^p-framework for the Neuman boundary condition

In this section, we study the decay of the solutions of the damped wave equation equipped the Neumann boundary condition and the solutions of the corresponding hyperbolic systems. Here is the first main result of this section concerning the wave equation.

Theorem 5.1. *Let $1 < p < +\infty$, $\varepsilon_0 > 0$, $\lambda > 0$, and let $a \in L^\infty\big(\mathbb{R}_+ \times (0,1)\big)$ be such that $a \ge 0$ and $a \ge \lambda > 0$ in $\mathbb{R}_+ \times (x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset \mathbb{R}_+ \times (0,1)$ for some $x_0 \in (0,1)$. There exist positive constants C and γ depending only on p, $\|a\|_{L^\infty\left(\mathbb{R}_+\times(0,1)\right)}$, ε_0, and λ such that for all $u_0 \in W^{1,p}(0,1)$ and $u_1 \in L^p(0,1)$, the unique weak solution $u \in C([0,+\infty); W^{1,p}(0,1)) \cap C^1([0,+\infty); L^p(0,1))$ of (1.1) and (1.3) satisfies*

$$\|\partial_t u(t,\cdot)\|_{L^p(0,1)} + \|\partial_x u(t,\cdot)\|_{L^p(0,1)} \le C e^{-\gamma t}\Big(\|u_1\|_{L^p(0,1)} + \|\partial_x u_0\|_{L^p(0,1)}\Big), \ t \ge 0. \tag{5.1}$$

As in the case where the Dirichlet condition is considered, we use the Riemann invariants to transform (1.1) with Neumann boundary condition into a hyperbolic system. Set

$$\rho(t,x) = u_x(t,x) + u_t(t,x) \quad \text{and} \quad \xi(t,x) = u_x(t,x) - u_t(t,x), \text{ for } (t,x) \in \mathbb{R}_+ \times (0,1). \tag{5.2}$$

One can check that for smooth solutions u of (1.1), the pair of functions (ρ,ξ) defined in (1.8) satisfies the system

$$\begin{cases} \rho_t - \rho_x = -\frac{1}{2}a(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \xi_t + \xi_x = \frac{1}{2}a(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \rho(t,0) + \xi(t,0) = \rho(t,1) + \xi(t,1) = 0 & \text{in } \mathbb{R}_+. \end{cases} \tag{5.3}$$

Concerning (5.3), we prove the following result.

Theorem 5.2. *Let $1 < p < +\infty$, $\varepsilon_0 > 0$, $\lambda > 0$, and $a \in L^\infty\big(\mathbb{R}_+ \times (0,1)\big)$ be such that $a \ge 0$ and $a \ge \lambda > 0$ in $\mathbb{R}_+ \times (x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset \mathbb{R}_+ \times (0,1)$ for some $x_0 \in (0,1)$. Then there exist positive constants C, γ depending only on on p, $\|a\|_{L^\infty\left(\mathbb{R}_+\times(0,1)\right)}$, ε_0, and λ such that the unique solution u of (5.3) with the initial condition $\rho(0,\cdot) = \rho_0$ and $\xi(0,\cdot) = \xi_0$ satisfies*

$$\|(\rho,\xi)(t,\cdot)\|_{L^p(0,1)} \le C e^{-\gamma t}\|(\rho_0,\xi_0)\|_{L^p(0,1)}. \tag{5.4}$$

The rest of this section is organized as follows. The first subsection is devoted to the proof of Theorem 5.2 and the second subsection is devoted to the proof of Theorem 5.1.

5.1. Proof of Theorem 5.2

The argument is in the spirit of that of Theorem 1.2. As in there, we will only consider smooth solutions (ρ,ξ). Multiplying the equation of ρ with $\rho|\rho|^{p-2}$, the equation of ξ with $\xi|\xi|^{p-2}$, and integrating the expressions with respect to x, after using the boundary conditions, we obtain, for $t > 0$,

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (|\rho(t,x)|^p + |\xi(t,x)|^p)\,\mathrm{d}x + \frac{1}{2}\int_0^1 a(\rho-\xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x)\,\mathrm{d}x = 0. \tag{5.5}$$

This implies

$$\frac{1}{p}\|(\rho,\xi)(t,\cdot)\|^p_{L^p(0,1)} + \frac{1}{2}\int_0^t\int_0^1 a(\rho-\xi)(\rho|\rho|^{p-1} - \xi|\xi|^{p-1})(t,x)\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{p}\|(\rho_0,\xi_0)\|^p_{L^p(0,1)}. \tag{5.6}$$

By (5.6), to derive (5.4), it suffices to prove that there exists a constant $c > 0$ depending only on $\|a\|_{L^\infty(\mathbb{R}_+\times(0,1))}$, ε_0, γ, and p such that for any $T > 2$, there exists $c_T > 0$ only depending on p, T, a so that

$$\int_0^T\int_0^1 a(\rho-\xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x)\,\mathrm{d}x\,\mathrm{d}t \geq c_T\|(\rho_0,\xi_0)\|^p_{L^p(0,1)}. \tag{5.7}$$

By scaling, without loss of generality, one might assume that

$$\|(\rho_0,\xi_0)\|_{L^p(0,1)} = 1 \tag{5.8}$$

Set

$$m_p := \int_0^T\int_0^1 a(\rho-\xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Applying Lemma 3.1, we have

$$\int_0^T\int_0^1 a|\rho-\xi|^p(t,x)\,\mathrm{d}x\,\mathrm{d}t \leq C(m_p + m_p^{2/p}). \tag{5.9}$$

By Lemma 3.2 there exists $z \in (x_0 - \varepsilon_0/2, x_0 + \varepsilon_0/2)$ such that

$$\int_0^{\varepsilon_0/2}\int_0^T |\rho(t+s,z) - \rho(t,z)|^p\,\mathrm{d}t\,\mathrm{d}s + \int_0^{\varepsilon_0/2}\int_0^T |\xi(t+s,z) - \xi(t,z)|^p\,\mathrm{d}t\,\mathrm{d}s \\ + \int_0^T |\rho(t,z) - \xi(t,z)|^p\,\mathrm{d}t \leq C(m_p + m_p^{2/p}). \tag{5.10}$$

Applying Lemma 3.3, we obtain

$$\int_0^T \Big|\rho(t,z) - \fint_0^T \rho(s,z)\,\mathrm{d}s\Big|^p\,\mathrm{d}t \leq C\int_0^{\varepsilon_0/2}\int_0^T |\rho(t+s,z) - \rho(t,z)|^p\,\mathrm{d}t\,\mathrm{d}s \tag{5.11}$$

and

$$\int_0^T \Big|\xi(t,z) - \fint_0^T \xi(s,z)\,\mathrm{d}s\Big|^p\,\mathrm{d}t \leq C\int_0^{\varepsilon_0/2}\int_0^T |\xi(t+s,z) - \xi(t,z)|^p\,\mathrm{d}t\,\mathrm{d}s. \tag{5.12}$$

Combining (5.10), (5.11), and (5.12) yields

$$\int_0^T |\rho(t,z) - \fint_0^T \rho(\tau, z)\, \mathrm{d}\tau|^p \, \mathrm{d}t + \int_0^T |\xi(t,z) - \fint_0^T \xi(\tau, z)\, \mathrm{d}\tau|^p \, \mathrm{d}t \\ + \int_0^T |\rho(t,z) - \xi(t,z)|^p \, \mathrm{d}t \le C(m_p + m_p^{2/p}). \quad (5.13)$$

Using the characteristics method to estimate $\rho(\tau, 0)$ by $\rho(\tau - z, z)$ and $\xi(\tau, 0)$ by $\xi(\tau + z, z)$ after using the boundary condition at 0 and choosing appropriately τ, we derive from (5.9) that (5.13) that

$$\left| \fint_0^T \rho(t,z)\, \mathrm{d}t + \fint_0^T \xi(t,z) \mathrm{d}t \right|^p \le C(m_p + m_p^{2/p}). \quad (5.14)$$

As done to obtain (4.12) and (4.13), we use the characteristic methods to estimate $\rho(0, \cdot)$ *via* $\xi(t,z)$ and $\xi(0,\cdot)$ *via* $\rho(t,z)$ after taking into account the boundary conditions (at $x = 0$ for $\rho(0,\cdot)$ and at $x = 1$ for $\xi(0,\cdot)$), we derive from (5.9) and (5.13) that

$$\left| \fint_0^T \rho(t,z)\, \mathrm{d}t \right|^p + \left| \fint_0^T \xi(t,z) \mathrm{d}t \right|^p \ge 1 - C(m_p + m_p^{2/p}). \quad (5.15)$$

Combining (5.14) and (5.15), we derive (after choosing $T = 3$) that there exists a postive constant c_3 only depending on $\|a\|_{L^\infty(\mathbb{R}_+ \times (0,1))}$, ε_0, γ, and p such that $m_p \ge c$. The proof of the theorem is complete. □

5.2. Proof of Theorem 5.1

The proof of Theorem 5.1 is in the same spirit of Theorem 1.1. However, instead of using Theorem 1.2, we apply Theorem 5.2. In fact, as in the proof of Theorem 1.1, we have

$$\int_0^1 |\partial_t u(t,x) - \partial_x u(t,x)|^p + |\partial_t u(t,x) + \partial_x u(t,x)|^p \, \mathrm{d}x \\ \le C e^{-\gamma t} \int_0^1 |\partial_t u(0,x) - \partial_x u(0,x)|^p + |\partial_t u(0,x) + \partial_x u(0,x)|^p \, \mathrm{d}x.$$

Assertion (5.1) follows with two different appropriate positive constants C and γ. □

Remark 5.3. We can also consider the setting similar to the one in Section 4.3 and establish similar results. This allows one to deal with a class of a for which a is not necessary to be non-negative. The analysis for this is almost the same lines as in Section 4.3 and is not pursued here.

6. Exponential decay in L^p-framework for the dynamic boundary condition

In this section, we study the decay of the solution of the damped wave equation equipped the dynamic boundary condition and of the solutions of the corresponding hyperbolic systems. Here is the first main result of this section concerning the wave equation.

Theorem 6.1. *Let $1 < p < +\infty$, $\kappa > 0$, and $a \in L^\infty\big(\mathbb{R}_+ \times (0,1)\big)$ non negative. Then there exist positive constants C, γ depending only on p, κ, and $\|a\|_{L^\infty\big(\mathbb{R}_+ \times (0,1)\big)}$ such that for all $u_0 \in W^{1,p}(0,1)$ and $u_1 \in L^p(0,1)$,* *there exists a unique weak solution* $u \in C([0,+\infty); W^{1,p}(0,1)) \cap C^1([0,+\infty); L^p(0,1))$ *such that* $\partial_t u, \partial_x u \in C([0,1]; L^p(0,T))$ *for all* $T > 0$ *of*

$$\begin{cases} \partial_{tt} u - \partial_{xx} u + a\partial_t u = 0 & \text{in } \mathbb{R}_+ \times (0,1), \\ \partial_x u(t,0) - \kappa \partial_t u(t,0) = \partial_x u(t,1) + \kappa \partial_t u(t,1) = 0 & \text{in } \mathbb{R}_+, \\ u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = u_1 & \text{in } (0,1), \end{cases} \tag{6.1}$$

satisfies

$$\|\partial_t u(t,\cdot)\|_{L^p(0,1)} + \|\partial_x u(t,\cdot)\|_{L^p(0,1)} \le C e^{-\gamma t}\Big(\|u_1\|_{L^p(0,1)} + \|\partial_x u_0\|_{L^p(0,1)}\Big), \ t \ge 0. \tag{6.2}$$

Remark 6.2. In Theorem 6.1, a weak considered solution of (6.1) means that $\partial_{tt} u(t,x) - \partial_{xx} u(t,x) + a(t,x)\partial_t u = 0$ holds in the distributional sense, and the boundary and the initial conditions are understood as usual thanks to the regularity imposing condition on the solutions.

As previously, we use the Riemann invariants to transform the wave equation into a hyperbolic system. Set

$$\rho(t,x) = u_x(t,x) + u_t(t,x) \quad \text{and} \quad \xi(t,x) = u_x(t,x) - u_t(t,x) \text{ for } (t,x) \in \mathbb{R}_+ \times (0,1). \tag{6.3}$$

One can check that for smooth solutions u of (1.1), the pair of functions (ρ, ξ) defined in (1.8) satisfies the system

$$\begin{cases} \rho_t - \rho_x = -\frac{1}{2} a(t,x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \xi_t + \xi_x = \frac{1}{2} a(t,x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \xi(t,0) = c_0 \rho(t,0), \quad \rho(t,1) = c_1 \xi(t,1) & \text{in } \mathbb{R}_+, \end{cases} \tag{6.4}$$

where $c_0 = c_1 = (\kappa - 1)/(\kappa + 1)$.

Regarding System (6.4) with c_0, c_1 not necessarily equal, we prove the following result.

Theorem 6.3. *Let* $1 < p < +\infty$*,* $c_0, c_1 \in (-1,1)$*, and* $a \in L^\infty\big(\mathbb{R}_+ \times (0,1)\big)$ *non negative. Then there exist positive constants* C, γ *depending only on* c_0*,* c_1*, and* $\|a\|_{L^\infty\left(\mathbb{R}_+ \times (0,1)\right)}$ *such that the unique solution* u *of* (6.4) *with the initial condition* $\rho(0,\cdot) = \rho_0$ *and* $\xi(0,\cdot) = \xi_0$ *satisfies*

$$\|(\rho,\xi)(t,\cdot)\|_{L^p(0,1)} \le C e^{-\gamma t} \|(\rho_0, \xi_0)\|_{L^p(0,1)}, \ t \ge 0. \tag{6.5}$$

The rest of this section is organized as follows. The proof of Theorem 6.3 is given in the first section and the proof of Theorem 6.1 is given in the second section.

6.1. Proof of Theorem 6.3

We will only consider smooth solutions (ρ, ξ). Multiplying the equation of ρ with ρ, the equation of ξ with ξ, and integrating the expressions with respect to x, after using the boundary conditions, we obtain, for $t > 0$,

$$\frac{1}{p}\frac{d}{dt}\int_0^1 (|\rho(t,x)|^p + |\xi(t,x)|^p)\,dx + \frac{1}{2}\int_0^1 a(\rho - \xi)(\rho|\rho|^{p-2} - \xi|\xi|^{p-2})(t,x)\,dx$$
$$\frac{1}{p}\Big((1 - |c_1|^p)|\xi(t,1)|^p + (1 - |c_0|^p)|\rho(t,0)|^p\Big) = 0. \tag{6.6}$$

This implies

$$\frac{1}{p}\|(\rho,\xi)(t,\cdot)\|^p_{L^p(0,1)} + \frac{1}{2}\int_0^T\int_0^1 a(\rho-\xi)(\rho|\rho|^{p-2}-\xi|\xi|^{p-2})(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{1}{p}\int_0^T \Big((1-|c_1|^p)|\xi(t,1)|^p + (1-|c_0|^p)|\rho(t,0)|^p\Big)\,\mathrm{d}t = \frac{1}{2}\|(\rho_0,\xi_0)\|^2_{L^2(0,1)}. \quad (6.7)$$

To derive (6.5) from (6.7), it suffices to prove that there exists a constant $c > 0$ depending only on $\|a\|_{L^\infty(\mathbb{R}_+\times(0,1))}$, c_0, c_1, ε_0, γ, and p such that for for $T = 3$[6], it holds

$$\int_0^T\int_0^1 a(\rho-\xi)(\rho|\rho|^{p-2}-\xi|\xi|^{p-2})(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\Big(|\xi(t,1)|^p + |\rho(t,0)|^p\Big)\,\mathrm{d}t \geq c\|(\rho_0,\xi_0)\|^p_{L^p(0,1)}. \quad (6.8)$$

After scaling, one might assume without loss of generality that

$$\|(\rho_0,\xi_0)\|_{L^p(0,1)} = 1 \quad (6.9)$$

Applying Lemma 3.1, we have

$$\int_0^T\int_0^1 a|\rho-\xi|^p(t,x)\,\mathrm{d}x\,\mathrm{d}t \leq C(m_p + m_p^{2/p}), \quad (6.10)$$

where

$$m_p := \int_0^T\int_0^1 a(\rho-\xi)(\rho|\rho|^{p-2}-\xi|\xi|^{p-2})(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Using the characteristics method (in particular Eqs. (3.9), (3.10)), we derive that

$$\|(\rho,\xi)(T,\cdot)\|^p_{L^p(0,1)} \leq C\int_0^T\Big(|\xi(t,1)|^p + |\rho(t,0)|^p\Big)\,\mathrm{d}t + C\int_0^T\int_0^1 a^p|\rho-\xi|^p(t,x)\,\mathrm{d}x\,\mathrm{d}t. \quad (6.11)$$

As a consequence of (6.7), (6.9), (6.10), and (6.11), we have

$$\int_0^T\int_0^1 a(\rho-\xi)(\rho|\rho|^{p-2}-\xi|\xi|^{p-2})(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\Big(|\xi(t,1)|^p + |\rho(t,0)|^p\Big)\,\mathrm{d}t \geq c.$$

The proof of the theorem is complete. □

Remark 6.4. In the case $a \equiv 0$, one can show that the exponential stability for $1 \leq p \leq +\infty$ by noting that

$$\|\big(\rho(t+1,0),\rho(t+1,1)\big)\| \leq \max\{|c_0|,|c_1|\}\|\big(\rho(t,0),\rho(t,1)\big)\|.$$

The conclusion then follows using the characteristics method.

[6] It holds for $T > 2$ with $c = c_T$.

6.2. Proof of Theorem 6.1

We first deal with the well-posedness of the system. The uniqueness follows as in the proof of Proposition 2.5 *via* the d'Alembert formula. The existence can be proved by approximation arguments. First deal with smooth solutions (with smooth a) using Theorem 6.3 and then pass to the limit. The details are omitted.

The proof of (6.5) is in the same spirit of (1.7). However, instead of using Theorem 1.2, we apply Theorem 6.3. The details are left to the reader. □

Remark 6.5. One can prove the well-posedness of (1.1) and (1.4) directly in L^p-framework. Nevertheless, to make the sense for the boundary condition, one needs to consider regular solutions and then a is required to be more regular than just L^∞. We here take advantage of the fact that such a system has a hyperbolic structure as given in (6.4). This give us the way to give sense for the solution by imposing the fact $\partial_t u, \partial_x u \in C([0,1]; L^p(0,T))$ for all $T > 0$.

References

[1] V. Komornik, Exact controllability and stabilization. RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester (1994).

[2] J.-L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], Vol. 8. Masson, Paris (1988), Contrôlabilité exacte [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[3] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.* **30** (1992) 1024–1065.

[4] G. Lebeau, Damped wave equation. Algebraic and geometric methods in mathematical physics. *Proceedings of the 1st Ukrainian–French–Romanian Summer School, Kaciveli, Ukraine, September 1–14, 1993.* Kluwer Academic Publishers, Dordrecht (1996) 73–109 (in French).

[5] J. Rauch and M. Taylor, Exponential decay of solutions to hyperbolic equations in bounded domains. *Indiana Univ. Math. J.* **24** (1974) 79–86.

[6] J. Le Rousseau, G. Lebeau, P. Terpolilli, and E. Trélat, Geometric control condition for the wave equation with a time-dependent observation domain. *Anal. PDE* **10** (2017) 983–1015.

[7] C.M. Dafermos, Asymptotic behavior of solutions of evolution equations, Nonlinear evolution equations (Proc. Sympos., Univ. Wisconsin, Madison, Wis.), Publ. Math. Res. Center Univ. Wisconsin, Vol. 40. Academic Press, New York-London (1978) 103–123.

[8] A. Haraux, Comportement à l'infini pour une équation d'ondes non linéaire dissipative. *C. R. Acad. Sci. Paris Sér. A-B* **287** (1978) A507–A509.

[9] F. Alabau-Boussouira, Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems. *Appl. Math. Optim.* **51** (2005) 61–105.

[10] F. Alabau-Boussouira, New trends towards lower energy estimates and optimality for nonlinearly damped vibrating systems. *J. Differ. Equ.* **249** (2010) 1145–1178.

[11] Y. Chitour, S. Marx and C. Prieur, L^p-asymptotic stability analysis of a 1D wave equation with a nonlinear damping. *J. Differ. Equ.* **269** (2020) 8107–8131.

[12] A. Haraux and E. Zuazua, Decay estimates for some semilinear damped hyperbolic problems. *Arch. Ration. Mech. Anal.* **100** (1988) 191–206 (in English).

[13] P. Martinez, Stabilization for the wave equation with Neumann boundary condition by a locally distributed damping, Contrôle des systèmes gouvernés par des équations aux dérivées partielles (Nancy, 1999). ESAIM Proc., Vol. 8. Soc. Math. Appl. Indust., Paris (2000) 119–136.

[14] P. Martinez and J. Vancostenoble, Exponential stability for the wave equation with weak nonmonotone damping. *Portugal. Math.* **57** (2000) 285–310.

[15] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping. *Commun. Partial Differ. Equ.* **15** (1990) 205–235.

[16] J.P. Quinn and D.L. Russell, Asymptotic stability and energy decay rates for solutions of hyperbolic equations with boundary damping. *Proc. Roy. Soc. Edinburgh Sect. A* **77** (1977) 97–127.

[17] J. Lagnese, Decay of solutions of wave equations in a bounded region with boundary dissipation. *J. Differ. Equ.* **50** (1983) 163–182.

[18] K. Liu, Locally distributed control and damping for the conservative systems. *SIAM J. Control Optim.* **35** (1997) 09.

[19] F. Alabau-Boussouira, On some recent advances on stabilization for hyperbolic equations, in Control of Partial Differential Equations. Vol. 2048 of *Lecture Notes in Mathematics*, edited by Cannarsa, Piermarco, Coron and Jean-Michel. Springer (2012) 1–100.

[20] J.C. Peral, L^p estimates for the wave equation. *J. Funct. Anal.* **36** (1980) 114–145.

[21] M. Kafnemer, B. Mebkhout, F. Jean and Y. Chitour, L^p-asymptotic stability of 1D damped wave equations with localized and linear damping. *ESAIM Control Optim. Calc. Var.* **28** (2022) 32.

[22] Y. Chitour, S. Marx and G. Mazanti, One-dimensional wave equation with set-valued boundary damping: well-posedness, asymptotic stability, and decay rates. *ESAIM Control Optim. Calc. Var.* **27** (2021) 62.

[23] J.-M. Coron and H.-M. Nguyen, Dissipative boundary conditions for nonlinear 1-D hyperbolic systems: sharp conditions through an approach via time-delay systems. *SIAM J. Math. Anal.* **47** (2015) 2220–2240.

[24] J.-M. Coron and H.-M. Nguyen, Optimal time for the controllability of linear hyperbolic systems in one-dimensional space. *SIAM J. Control Optim.* **57** (2019) 1127–1156.

[25] J.-M. Coron and H.-M. Nguyen, Finite-time stabilization in optimal time of homogeneous quasilinear hyperbolic systems in one dimensional space. *ESAIM Control Optim. Calc. Var.* **26** (2020) 24

[26] J.-M. Coron and H.-M. Nguyen, Null-controllability of linear hyperbolic systems in one dimensional space. *Syst. Control Lett.* **148** (2021) 104851.

[27] A. Haraux, l^p estimates of solutions to some non-linear wave equations in one space dimension. *Int. J. Math. Model. Numer. Optim.* **1** (2009) 148–152.

[28] J.-M. Coron and H.-M. Nguyen, On the optimal controllability time for linear hyperbolic systems with time-dependent coefficients, https://arxiv.org/abs/2103.02653 (2021).

[29] D.L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations (1973).

[30] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Rev.* **20** (1978) 639–739.

[31] G. Bastin and J.-M. Coron, Stability and boundary stabilization of 1-D hyperbolic systems, in Progress in Nonlinear Differential Equations and their Applications, Vol. 88. Birkhäuser/Springer, Cham (2016).

[32] F. John and L. Nirenberg, On functions of bounded mean oscillation. *Commun. Pure Appl. Math.* **14** (1961) 415–426.

[33] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York (2011).

[34] G. Allaire, Analyse Numérique et Optimisation, 2nd edn. Édition de l'École Polytechnique (2005).

[35] J.-M. Coron, Control and Nonlinearity, Mathematical Surveys and Monographs, Vol. 136. American Mathematical Society, Providence, R, (2007).

[36] H.-M. Nguyen and M. Squassina, Fractional Caffarelli–Kohn–Nirenberg inequalities. *J. Funct. Anal.* **274** (2018) 2661–2672.