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LAGRANGIAN DUAL METHOD FOR SOLVING STOCHASTIC LINEAR QUADRATIC OPTIMAL CONTROL PROBLEMS WITH TERMINAL STATE CONSTRAINTS

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Abstract. A stochastic linear quadratic (LQ) optimal control problem with a pointwise linear equality constraint on the terminal state is considered. A strong Lagrangian duality theorem is proved under a uniform convexity condition on the cost functional and a surjectivity condition on the linear constraint mapping. Based on the Lagrangian duality, two approaches are proposed to solve the constrained stochastic LQ problem. First, a theoretical method is given to construct the closed-form solution by the strong duality. Second, an iterative algorithm, called augmented Lagrangian method (ALM), is proposed. The strong convergence of the iterative sequence generated by ALM is proved. In addition, some sufficient conditions for the surjectivity of the linear constraint mapping are obtained.

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1. Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ (satisfying the usual conditions), on which a one-dimensional standard Wiener process W is defined such that $\mathbb{F}$ is the natural filtration generated by W (augmented by all the $\mathbb{P}$-null sets). Fix $m, n, \ell \in \mathbb{N}$. Denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively the inner product and norm in $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^\ell$, which can be identified from the contexts.

Let us consider the controlled linear stochastic differential equation

$$\begin{cases} \mathrm{d}X^{x,u}(t) = \big(A(t)X^{x,u}(t) + B(t)u(t)\big)\mathrm{d}t + \big(C(t)X^{x,u}(t) + D(t)u(t)\big)\mathrm{d}W(t), \quad t \in [0, T], \\ X(0) = x \end{cases} \tag{1.1}$$

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with the quadratic cost functional

$$J(u) = \frac{1}{2}\mathbb{E}\left[\int_0^T \langle Q(t)X^{x,u}(t), X^{x,u}(t)\rangle + \langle R(t)u(t), u(t)\rangle \mathrm{d}t + \langle GX^{x,u}(T), X^{x,u}(T)\rangle\right] \tag{1.2}$$

and the terminal state constraint

$$MX^{x,u}(T) - b = 0, \quad a.s. \tag{1.3}$$

Here, u is the control valued in $\mathbb{R}^m$ and $X^{x,u}$ is the state valued in $\mathbb{R}^n$ with initial datum x and control u. The coefficients A, B, C, D, Q, R, G, M and b satisfy proper assumptions which shall be given in the next section. Denote by $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ the space of $\mathbb{F}$-progressively measurable stochastic processes u valued in $\mathbb{R}^m$ such that $\mathbb{E}\int_0^T |u(t)|^2 \mathrm{d}t < \infty$. The constrained stochastic linear quadratic optimal control problem considered in this paper is

$$\begin{cases} \min & J(u), \\ \text{s.t.} & u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), \\ & (X^{x,u}, u) \text{ satisfies (1.1) and} \\ & MX^{x,u}(T) - b = 0, \quad a.s. \end{cases} \tag{CSLQ}$$

The (CSLQ) is feasible if there is a control $u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ with corresponding state $X^{x,u}$ such that the state constraint (1.3) is satisfied. Any minimizer $\bar{u}$ of (CSLQ) is called an optimal control, the corresponding state process $\bar{X}^{x,\bar{u}}$ is called an optimal state, and $(\bar{X}^{x,\bar{u}}, \bar{u})$ is called an optimal pair.

Stochastic LQ problem is one of the fundamental problems in stochastic control theory and has wide range of applications in many fields, such as engineering, management science and mathematical finance. The stochastic LQ problem without constraints was initiated by Wonham [1] and studied extensively for both deterministic and random coefficients by many researchers in the past few decades. For instance, Bismut [2] first studied the stochastic LQ problem with random coefficients. Chen, Li and Zhou [3] found for the first time that stochastic LQ problem with indefinite control weight cost may still be well-posed, which is essentially different from its deterministic counterpart. Rami, Moore and Zhou [4] proposed a much general Riccati equation with an additional algebraic equality constraint and proved that the solvability of the generalized Riccati equation is sufficient and necessary for the well-posedness of the indefinite stochastic LQ problem with deterministic coefficients. Tang [5, 6] proved the existence and uniqueness of the solution to the backward stochastic Riccati equation for stochastic LQ problem with random coefficients under the regular case that the wight matrix R is uniformly positive definite. Kohlmann and Tang [7], Hu and Zhou [8] studied the the existence and uniqueness of the solution to the backward stochastic Riccati equation for stochastic LQ problem with random coefficients in some special indefinite cases. Sun, Li and Yong [9], Sun, Xiong and Yong [10], Sun and Yong [11] studied the relationship between the open-loop solvability and closed-loop solvability for stochastic LQ problems.

In many applications, the control or/and the state of the control system should satisfy some constraints. Obviously, solving the constrained stochastic LQ problems is more challenging than solving the problems without constraints. The stochastic LQ problem with cone control constraints and random coefficients was studied by Hu and Zhou [12]. An explicit optimal feedback control was obtained in [12] by introducing two extended stochastic Riccati equations. Chen and Zhou [13] discussed the stochastic LQ problem in infinite time horizon with conic control constraints. Recently, Hu, Shi and Xu studied in [14] and [15] respectively the finite time horizon and infinite time horizon stochastic LQ problem with regime switching, random coefficients and cone control constraints. Lim and Zhou [16] studied the stochastic LQ problem with mixed control-state integral type quadratic inequality constraints. Wu, Gao, Lu and Li [17] discussed the scalar-state stochastic LQ problem with mixed pointwise state-control linear inequality constraints. Feng, Hu and Huang [18] considered a stochastic

LQ problem with a terminal state affine expectation constraint when they studied the backward Stackelberg differential game involving a single leader and single follower. In [16, 18], the Slater's condition and the Lagrange dual approach are used to treat the inequality state constraints.

In contrast with the stochastic LQ problems with inequality state constraints or mixed control-state inequality constraints, less progress has been made on the stochastic LQ problems with equality state constraints. Clearly, the feasibility of stochastic optimal control problems with equality state constraints is more difficult than that of stochastic optimal control problems with inequality state constraints. In addition, when the dual approach is used to handle the state constraints, instead of the Slater's condition for inequality constraints, some surjectivity conditions which are closely related to the controllability of the control system have to be assumed for the equality constraints.

In [19], Lim gave a closed-form expression of the optimal portfolio for a mean-variance portfolio selection problem in which the state $X^{x,u}(\cdot)$ is a real-scalar process and constrained by the expectation type equality constraint

$$\mathbb{E}X^{x,u}(T) = c \tag{1.4}$$

for some constant $c \in \mathbb{R}$. Kohlmann and Tang [7], Section 6.2 studied the general multi-dimensional stochastic LQ problem with state constraint (1.4) (in which $X^{x,u}(T)$ is valued in $\mathbb{R}^n$ and $c \in \mathbb{R}^n$). A feedback solution with parameter for the optimal control was given by the Riccati equation. Zhang and Zhang [20] studied the stochastic LQ problem with state constraint

$$\mathbb{E}(MX^{x,u}(T) - b) = 0 \tag{1.5}$$

under the solvability condition on a stochastic Riccati equation and a surjectivity condition on the linear constraint mapping. The equivalence between the surjectivity condition and a Kalman- type rank condition is proved in [20] for the special case of deterministic coefficients. In both [19] and [20], the Lagrangian duality is the main tool to handle the state constraint and the optimal parameter of the feedback optimal solution is determined explicitly by solving the dual problem.

Compared with the expectation type terminal state constraint (1.5), the stochastic LQ problems with (sample point) pointwise terminal state constraints are more complicated. First, the pointwise type state constraints are more restrictive than the expectation type state constraints and hence some stronger conditions are needed to ensure the feasibility of the correlated state constrained problems. Second, the dual variable for the pointwise type state constraint (1.3) is a random vector and the corresponding dual problem is a stochastic programming problem which is hard to be solved explicitly in general.

In [21], Bi, Sun and Xiong used the backward stochastic differential equation approach to study a stochastic LQ problem with deterministic coefficients, fixed terminal state and a pointwise linear equality constraint on the initial state. A closed-form solution is obtained by solving a Riccati equation and an algebraic matrix equation for the optimal parameter. Besides, there are a few articles on stochastic LQ problems with pointwise equality constraints for the special case of the norm optimal control problem, *i.e.*, $M \equiv I_n$, $Q(t) \equiv 0$, $R(t) \equiv I_m$, $G \equiv 0$ (I_n and I_m are respectively the identity matrices of size n and m). See for instance, Gashi [22], Wang and Zhang [23], Wang, Yang, Yong and Yu [24].

In this paper, we shall discuss the stochastic LQ problem with pointwise linear equality constraint (1.3). Similarly to our previous work [20] for expectation type terminal state constraint (1.5), we prove a strong Lagrangian duality theorem for the constrained stochastic LQ problem (CSLQ) under a uniform convexity condition on the cost functional and a surjectivity condition on the linear constraint mapping. An equivalent characterization of the surjectivity condition is discussed by the controllability theory of linear control systems. In addition, a Kalman-type rank condition, which is sufficient for the surjectivity condition, is derived in the special case of deterministic coefficients. Different from [20], the dual problem for (CSLQ) cannot be solved explicitly by its first-order necessary condition. As a result, the closed-form solution to (CSLQ) cannot be obtained directly by the dual approach. See Section 3 for more details. To overcome that difficulty, we introduce an iterative algorithm, called augmented Lagrangian method (ALM), to solve the constrained stochastic LQ problem (CSLQ).

The ALM is originally proposed independently by Hestenes [25] and Powell [26] for solving finite dimensional constrained optimization problems and has been extensively studied by many scholars in the past few decades. We refer the reader to [27–29] and the references cited therein for its infinite dimensional extensions and applications in numerical solution to partial differential equations and deterministic optimal control problems. Recently, Pfeiffer [30] proposed an ALM for solving non-linear stochastic control problems with inequality type terminal state constraints. The cost functional and the inequality constraints in [30] are functional of the probability distribution of the terminal state.

In this paper, we show that the ALM is effective to solve the constrained stochastic LQ problem (CSLQ). Under proper conditions, we prove that the iterative sequence generated by ALM converges strongly to the optimal control of (CSLQ). The basic idea is from the ALM for the quadratic programming problem with linear equality constraints(see, for instance, Chapter 1 in [28]). Indeed, the cost functional (1.2) can be represented as a quadratic functional of control though introducing some proper operators, for more details we refer the readers to [10], Theorem 3.4. Then, the convergence of the ALM for (CSLQ) can be obtained by verifying the convergence conditions of the ALM for quadratic programming problem with linear equality constraints. Instead of using such an abstract approach, in this paper we prove the convergence result directly by the elementary techniques in stochastic control.

The main contributions of this paper are as follows:

- (i) The Lagrangian dual method is proposed to solve the constrained stochastic LQ problem (CSLQ). First, the closed-form solution with optimal parameter is constructed by the Lagrangian duality principle. Then, the ALM is introduced to solve (CSLQ) and the strong convergence of the iterative sequence is proved in a simple and direct way.
- (ii) Some verifiable sufficient conditions are given to ensure the strong duality between the (CSLQ) and its dual problem. Those sufficient conditions are also the convergence conditions of the ALM for (CSLQ).
- (iii) As a byproduct of the convergence proof of the ALM, a first-order necessary and sufficient condition for the optimal control of (CSLQ) is obtained by the Lagrangian duality theory (see Lem. 4.5).

The rest of this paper is organized as follows. In Section 2, we introduce some basic notations and assumptions. In Section 3, we first prove the Lagrangian duality between the (CSLQ) and its dual problem under a uniform convexity condition on the cost functional and a surjectivity condition on the linear constraint mapping. Then, we give the closed-form solution to (CSLQ) by the Lagrangian duality. In Section 4, we propose the ALM for (CSLQ) and prove its strong convergence. Finally we give some verifiable sufficient conditions for the surjectivity condition of the linear constraint mapping in Section 5.

2. Preliminaries and assumptions

Throughout this paper, let $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^\ell$ be respectively the n, m and ℓ-dimensional Euclidean spaces. Let $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$ be respectively the spaces of all $n\times n$ and $n\times m$ real matrices. Denoted by $\mathbb{M}^\top$ the transpose of a matrix $\mathbb{M}$, by $\mathbb{S}^n$ the space of all symmetric $n\times n$ real matrices. The identity matrix of size n is denoted by I_n. For $\mathbb{M},\mathbb{N}\in\mathbb{S}^n$, denote $\mathbb{M}\geq\mathbb{N}$ when $\mathbb{M}-\mathbb{N}$ is positive semidefinite.

For a Banach space $\mathbb{X}$ with its norm $|\cdot|_{\mathbb{X}}$, denote by $\mathcal{B}_{\mathbb{X}}(0,1)$ the open unit ball of $\mathbb{X}$. Denote by $L^2_{\mathcal{F}_T}(\Omega;\mathbb{X})$ the space of $\mathbb{X}$-valued, $\mathcal{F}_T$ measurable random vectors ξ such that $\|\xi\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{X})}\triangleq\big(\mathbb{E}|\xi|^2_{\mathbb{X}}\big)^{\frac{1}{2}}<\infty$; by $L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{X})$ the space of $\mathbb{X}$-valued, $\mathcal{F}_T$ measurable random vectors ξ such that $esssup_\omega|\xi|_{\mathbb{X}}<\infty$; by $L^2_{\mathbb{F}}(0,T;\mathbb{X})$ the space of $\mathbb{X}$-valued, $\mathbb{F}$-progressively measurable stochastic processes η such that $\|\eta\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)}\triangleq\big(\mathbb{E}\int_0^T|\eta(t)|^2_{\mathbb{X}}\mathrm{d}t\big)^{\frac{1}{2}}<\infty$; by $L^\infty_{\mathbb{F}}(0,T;\mathbb{X})$ the space of $\mathbb{X}$-valued, $\mathbb{F}$-progressively measurable stochastic processes η such that $esssup_{(t,\omega)}|\eta(t)|_{\mathbb{X}}<\infty$; by $L^2_{\mathbb{F}}\big(\Omega;C\big([0,T];\mathbb{X}\big)\big)$ the space of $\mathbb{X}$-valued, $\mathbb{F}$-progressively measurable continuous stochastic processes η such that $\big[\mathbb{E}\big(\sup_{0\leq t\leq T}|\eta(t)|^2_{\mathbb{X}}\big)\big]^{\frac{1}{2}}<\infty$; by $L^\infty_{\mathbb{F}}\left(\Omega;C\left([0,T],\mathbb{X}\right)\right)$ the space of

$\mathbb{X}$-valued, $\mathbb{F}$-progressively measurable continuous stochastic processes η such that $esssup_\omega \big(\sup_{0\le t\le T}|\eta(t)|_{\mathbb{X}}\big)<\infty$; by $L^\infty_{\mathbb{F}}\big(\Omega;L^2(0,T;\mathbb{X})\big)$ the space of $\mathbb{X}$-valued, $\mathbb{F}$-progressively measurable stochastic processes η such that $esssup_\omega\big(\int_0^T|\eta(t)|^2_{\mathbb{X}}\mathrm{d}t\big)^{\frac12}<\infty$.

Throughout this paper, we make the following assumptions.

(A1) $A(\cdot),C(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times n})$, $B(\cdot),D(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times m})$.
(A2) $Q(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{S}^n)$, $R(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{S}^m)$, $G\in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{S}^n)$, $M\in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^{\ell\times n})$, $b\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$.
(A3) There is a constant $\delta>0$ such that

$$\begin{aligned}J^0(u)&\triangleq\frac12\mathbb{E}\bigg[\int_0^T\langle Q(t)X^{0,u}(t),X^{0,u}(t)\rangle+\langle R(t)u(t),u(t)\rangle\mathrm{d}t+\langle GX^{0,u}(T),X^{0,u}(T)\rangle\bigg]\\&\ge\delta\mathbb{E}\int_0^T|u(t)|^2\mathrm{d}t,\quad\forall\,u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).\end{aligned}\tag{2.1}$$

Here, $X^{0,u}(\cdot)$ is the solution to the control system (1.1) with initial datum 0 and control $u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$.
(A4) For the given matrix-valued random variable $M\in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^{\ell\times n})$ and initial datum $x\in\mathbb{R}^n$, the mapping $u\mapsto MX^{x,u}(T)$ is surjective, *i.e.*,

$$L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)=\Big\{MX^{x,u}(T)\;\Big|\;u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)\Big\}.$$

By condition (A4), we have the set of admissible controls

$$U_{ad}\triangleq\Big\{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)\;\Big|\;MX^{x,u}(T)-b=0,\ a.s.\Big\}\tag{2.2}$$

is nonempty. Then, by (A1), (A2) and (A4), the constrained stochastic LQ problem (CSLQ) is well-defined, *i.e.*, for any $u\in U_{ad}$, state equation (1.1) admits a unique solution $X^{x,u}$ satisfying the state constraint (1.3) and $J(u)<+\infty$. In addition, we shall see that the condition (A3), which is called uniform convexity condition in [10], implies the strong convexity of the cost functional $J(\cdot)$. Then, under conditions (A1)–(A4), the constrained stochastic LQ problem (CSLQ) admits unique optimal solution.

Definition 2.1. Let $\mathbb{X}$ be a Banach space, $f:\mathbb{X}\to\mathbb{R}$ is called a strongly convex functional with constant $\sigma>0$ if

$$f\big(\theta x+(1-\theta)y\big)\le\theta f(x)+(1-\theta)f(y)-\frac{\sigma}{2}\theta(1-\theta)|x-y|^2_{\mathbb{X}},\quad\forall\,x,y\in\mathbb{X},\theta\in[0,1].$$

Lemma 2.2. *Suppose that (A1)–(A4) hold. Then the cost functional $J(\cdot)$ is a strongly convex continuous functional on $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ and the constrained stochastic LQ problem (CSLQ) is uniquely solvable.*

Proof. The continuity of $J(\cdot)$ is obvious. For any $u_1,u_2\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, $\theta\in[0,1]$, by (A1)–(A3), we have

$$\begin{aligned}J\big(\theta u_1+(1-\theta)u_2\big)=&\frac12\mathbb{E}\bigg[\int_0^T\Big\langle Q(t)X^{x,\theta u_1+(1-\theta)u_2}(t),X^{x,\theta u_1+(1-\theta)u_2}(t)\Big\rangle\\&\qquad+\langle R(t)(\theta u_1(t)+(1-\theta)u_2(t)),\theta u_1(t)+(1-\theta)u_2(t)\rangle\mathrm{d}t\\&\qquad+\Big\langle GX^{x,\theta u_1+(1-\theta)u_2}(T),X^{x,\theta u_1+(1-\theta)u_2}(T)\Big\rangle\bigg]\\=&\,\theta J(u_1)+(1-\theta)J(u_2)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\theta(1-\theta)\mathbb{E}\Big[\int_0^T \big\langle Q(t)X^{0,u_1-u_2}(t), X^{0,u_1-u_2}(t)\big\rangle \\
&+\langle R(t)(u_1(t)-u_2(t)), u_1(t)-u_2(t)\rangle \mathrm{d}t \\
&+\big\langle GX^{0,u_1-u_2}(T), X^{0,u_1-u_2}(T)\big\rangle\Big] \\
&=\theta J(u_1)+(1-\theta)J(u_2)-\theta(1-\theta)J^0(u_1-u_2) \\
&\le \theta J(u_1)+(1-\theta)J(u_2)-\delta\theta(1-\theta)\mathbb{E}\int_0^T |u_1(t)-u_2(t)|^2\mathrm{d}t,
\end{aligned}
\tag{2.3}
$$

i.e., $J(\cdot)$ is a strongly convex functional on $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$.

By assumption (A4), U_{ad} is nonempty. Since the control system (1.1) is linear and the terminal state constraint (1.3) is a linear equality constraint, U_{ad} is a closed convex subset of $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$. Then, by the standard existence theory of convex optimization (see, for instance, [31], Thm. 2.31), the problem (CSLQ) is uniquely solvable. □

3. Lagrangian duality

In this section, we shall prove a Lagrangian duality theorem for the constrained stochastic LQ problem (CSLQ) and derive a closed-form solution with optimal parameter to (CSLQ) by dual approach.

Let us first recall some basic notions for the Lagrangian duality in optimization. For more details we refer the readers to [31]. Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces, $C_{\mathbb{X}}\subset\mathbb{X}$, $C_{\mathbb{Y}}\subset\mathbb{Y}$ be arbitrary nonempty sets. Let us associate with a functional $\mathbb{L}: C_{\mathbb{X}}\times C_{\mathbb{Y}}\to\mathbb{R}\cup\{\pm\infty\}$ the primal and dual problems defined respectively as follows

$$\inf_{x\in C_{\mathbb{X}}}\sup_{y\in C_{\mathbb{Y}}}\mathbb{L}(x,y), \tag{P}$$

$$\sup_{y\in C_{\mathbb{Y}}}\inf_{x\in C_{\mathbb{X}}}\mathbb{L}(x,y). \tag{D}$$

Definition 3.1 ([31])**.** It is said that the strong duality holds between the primal problem (P) and the dual problem (D) if both problems have finite optimal values and

$$\sup_{y\in C_{\mathbb{Y}}}\inf_{x\in C_{\mathbb{X}}}\mathbb{L}(x,y)=\inf_{x\in C_{\mathbb{X}}}\sup_{y\in C_{\mathbb{Y}}}\mathbb{L}(x,y).$$

$(\bar{x},\bar{y})\in C_{\mathbb{X}}\times C_{\mathbb{Y}}$ is called a saddle point of the functional $\mathbb{L}$ if $\mathbb{L}(\bar{x},\bar{y})\in\mathbb{R}$ and

$$\mathbb{L}(\bar{x},y)\le\mathbb{L}(\bar{x},\bar{y})\le\mathbb{L}(x,\bar{y}),\quad \forall\ (x,y)\in C_{\mathbb{X}}\times C_{\mathbb{Y}}.$$

Now let us consider the Lagrangian duality theory for the constrained stochastic LQ problem (CSLQ). Define the Lagrangian functional for (CSLQ) by

$$\mathcal{L}(u,\lambda)\triangleq J(u)+\mathbb{E}\langle\lambda, MX^{x,u}(T)-b\rangle,\quad \forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),\ \forall\ \lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell).$$

Here, $J(\cdot)$ is the cost functional defined by (1.2). Clearly,

$$\sup_{\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \mathcal{L}(u,\lambda) = \sup_{\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \Big\{ J(u) + \mathbb{E}\langle \lambda, MX^{x,u}(T) - b\rangle \Big\} = \begin{cases} J(u), & MX^{x,u}(T)-b=0,\ a.s. \\ +\infty, & \text{otherwise.}\end{cases}$$

Therefore, the problem (CSLQ) is equivalent to

$$\inf_{u\in U_{ad}} J(u) = \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \sup_{\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \mathcal{L}(u,\lambda). \tag{3.1}$$

Define the dual functional $d : L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell) \to \mathbb{R}\cup\{\pm\infty\}$ by

$$d(\lambda) \triangleq \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda), \quad \forall\ \lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell), \tag{3.2}$$

and define the dual problem for (CSLQ) by

$$\sup_{\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda) = \sup_{\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} d(\lambda). \tag{3.3}$$

Since the cost functional $J(\cdot)$ is strongly convex under conditions (A1)–(A3), $\mathcal{L}(\cdot,\lambda)$ is also a strongly convex functional for any $\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$. Then, the unconstrained stochastic LQ problem in the definition of $d(\lambda)$ admits unique solution and the dual functional $d(\cdot)$ is well-defined. In what follows, we prove the strong duality between (CSLQ) and its dual problem (3.3).

Theorem 3.2. *Suppose that (A1)–(A4) hold and let $\bar u$ be the unique solution to (CSLQ). Then the following two assertions hold true.*

(i) The strong duality between (CSLQ) and its dual problem (3.3) holds true, i.e.

$$\sup_{\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda) = \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \sup_{\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \mathcal{L}(u,\lambda).$$

(ii) The dual problem is solvable, and, if $\bar\lambda$ is the solution to the dual problem then $(\bar u,\bar\lambda)$ is a saddle point of $\mathcal{L}$, i.e.

$$\mathcal{L}(\bar u,\lambda) \le \mathcal{L}(\bar u,\bar\lambda) \le \mathcal{L}(u,\bar\lambda), \quad \forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),\ \forall\ \lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell).$$

Especially,

$$\mathcal{L}(\bar u,\bar\lambda) = \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\bar\lambda). \tag{3.4}$$

Proof. Define

$$\mathcal{K} = \Big\{ (\alpha,\beta) \in \mathbb{R}\times L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell) \Big|\ \exists\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \text{ s.t. } J(u)-J(\bar u)\le\alpha,\ MX^{x,u}(T)-b=\beta,\ a.s.\Big\},$$

and

$$\mathcal{O}=\Big\{(\alpha^{'},\beta^{'})\in\mathbb{R}\times L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)\Big|\alpha^{'}<0,\ \beta^{'}=0,\ a.s.\Big\}.$$

Clearly, both $\mathcal{K}$ and $\mathcal{O}$ are convex sets. We claim that the interior of $\mathcal{K}$ is nonempty. By condition (A4), $u\mapsto MX^{x,u}(T)$ is a surjection. Then, the linear mapping $\Gamma: u\mapsto MX^{0,u}(T)$ is also a surjection. Meanwhile, there exist $\kappa_1,\kappa_2>0$ satisfying

$$\mathbb{E}|MX^{0,u}(T)|^2\le\mathbb{E}|M|^2\cdot|X^{0,u}(T)|^2\le\kappa_1\mathbb{E}|X^{0,u}(T)|^2\le\kappa_2\mathbb{E}\int_0^T|u(t)|^2\mathrm{d}t.$$

According to the classical open mapping theorem (see, for instance, [32], Thm. 5A.1), we know that Γ is an open mapping and there is $\kappa_3>0$ such that for any $\xi\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$ there exists $u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ satisfying $MX^{0,u}(T)=\xi$, and $\|u\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)}\le\kappa_3\|\xi\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)}$. Especially, for any fixed ε and any $\beta\in\varepsilon\mathcal{B}_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)}(0,1)$, there exists $v\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ such that

$$MX^{0,v}(T)=\beta\text{ and }\|v\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)}\le\kappa_3\varepsilon.$$

Then,

$$MX^{x,\bar{u}+v}(T)-b=M\bar{X}^{x,\bar{u}}(T)-b+MX^{0,v}(T)=\beta\in\varepsilon\mathcal{B}_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)}(0,1).\tag{3.5}$$

Let $\alpha>0$. By the continuity of $J(\cdot)$, there is ε such that

$$J(\bar{u}+v)\le J(\bar{u})+\alpha,\quad\forall\ v\in\kappa_3\varepsilon\mathcal{B}_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)}(0,1).\tag{3.6}$$

Combining (3.5) with (3.6), we obtain that

$$(\alpha,+\infty)\times\varepsilon\mathcal{B}_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)}(0,1)\subset\mathcal{K}.$$

This proves that the interior of $\mathcal{K}$ is nonempty.

By the optimality of $\bar{u}$, we obtain $\mathcal{K}\cap\mathcal{O}=\emptyset$. Then, by separation theorem, there is $(\lambda_0,\lambda)\in\mathbb{R}\times L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell),(\lambda_0,\lambda)\neq0$ such that

$$\inf_{(\alpha,\beta)\in\mathcal{K}}\Big\{\lambda_0\alpha+\mathbb{E}\langle\lambda,\beta\rangle\Big\}\ge\sup_{(\alpha^{'},\beta^{'})\in\mathcal{O}}\Big\{\lambda_0\alpha^{'}+\mathbb{E}\langle\lambda,\beta^{'}\rangle\Big\}=\sup_{\alpha^{'}<0}\lambda_0\alpha^{'}.$$

Clearly, $\lambda_0\ge0$, $\sup\limits_{\alpha^{'}<0}\lambda_0\alpha^{'}=0$, and

$$0\le\inf_{(\alpha,\beta)\in\mathcal{K}}\Big\{\lambda_0\alpha+\mathbb{E}\langle\lambda,\beta\rangle\Big\}\le\inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)}\Big\{\lambda_0\big(J(u)-J(\bar{u})\big)+\mathbb{E}\langle\lambda,MX^{x,u}(T)-b\rangle\Big\}.$$

Assume $\lambda_0=0$, then

$$0\le\mathbb{E}\langle\lambda,MX^{x,u}(T)-b\rangle,\quad\forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).$$

By condition (A4), we must have $\lambda = 0$ which contradicts to $(\lambda_0, \lambda) \neq 0$. Therefore, $\lambda_0 > 0$. Let $\bar{\lambda} = \frac{\lambda}{\lambda_0}$, we obtain that

$$0 \leq J(u) - J(\bar{u}) + \mathbb{E}\langle \bar{\lambda}, MX^{x,u}(T) - b\rangle, \quad \forall\, u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).$$

Since $M\bar{X}^{x,\bar{u}}(T) - b = 0$ a.s., then

$$J(\bar{u}) + \mathbb{E}\langle \bar{\lambda}, M\bar{X}^{x,\bar{u}}(T) - b\rangle \leq J(u) + \mathbb{E}\langle \bar{\lambda}, MX^{x,u}(T) - b\rangle, \quad \forall\, u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m). \tag{3.7}$$

By (3.1)–(3.3) and (3.7), we have

$$\begin{aligned}
\inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \mathcal{L}(u,\lambda) =& J(\bar{u}) \\
=& J(\bar{u}) + \mathbb{E}\langle \bar{\lambda}, M\bar{X}^{x,\bar{u}}(T) - b\rangle \\
\leq& \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\bar{\lambda}) \\
=& d(\bar{\lambda}) \\
\leq& \sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} d(\lambda) \\
=& \sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda).
\end{aligned} \tag{3.8}$$

On the other hand, it is obvious that

$$\sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda) \leq \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \mathcal{L}(u,\lambda).$$

Therefore,

$$\sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda) = \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \mathcal{L}(u,\lambda). \tag{3.9}$$

This proves (i).

In addition, by (3.8) and (3.9),

$$d(\bar{\lambda}) = \sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} d(\lambda), \tag{3.10}$$

i.e., the dual problem is solvable and $\bar{\lambda}$ is an optimal solution to the dual problem. Furthermore, for any solution $\bar{\lambda}$ of the dual problem,

$$\sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda) = \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\bar{\lambda}) \leq \mathcal{L}(u,\bar{\lambda}), \quad \forall\, u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m). \tag{3.11}$$

By (3.1), the optimality of $\bar{u}$ and the fact that $M\bar{X}^{x,\bar{u}} - b = 0$, a.s.,

$$\mathcal{L}(\bar{u},\lambda) = \mathcal{L}(\bar{u},\bar{\lambda}) = J(\bar{u}) = \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \sup_{\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)} \mathcal{L}(u,\lambda), \quad \forall\, \lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell). \tag{3.12}$$

Then, combining (3.9) with (3.11)–(3.12), we obtain that $(\bar{u}, \bar{\lambda})$ is a saddle point of $\mathcal{L}$.

This completes the proof of Theorem 3.2. □

By Theorem 3.2, to solve the constrained stochastic LQ problem (CSLQ), we can first find the optimal solution $\bar{\lambda}$ to its dual problem (3.3). Then, by (3.4), (CSLQ) can be transformed into an unconstrained stochastic LQ problem with optimal parameter $\bar{\lambda}$, and, the optimal solution to problem (CSLQ) can be found by the standard method of unconstrained stochastic LQ problem.

Consider the backward stochastic Riccati equation

$$\begin{cases} dP(t) = -\Big[P(t)A(t) + A(t)^\top P(t) + C(t)^\top P(t)C(t) + Q(t) + \Lambda(t)C(t) + C(t)^\top \Lambda(t) \\ \qquad\qquad -L(t)^\top K(t)^{-1}L(t)\Big]\mathrm{d}t + \Lambda(t)\mathrm{d}W(t), \quad t \in [0,T], \\ P(T) = G, \end{cases} \tag{3.13}$$

and the backward stochastic differential equation

$$\begin{cases} \mathrm{d}\varphi_\lambda(t) = -\Big[\Big(A(t)^\top - L(t)^\top K(t)^{-1}B(t)^\top\Big)\varphi_\lambda(t) \\ \qquad\qquad +\Big(C(t)^\top - L(t)^\top K(t)^{-1}D(t)^\top\Big)\psi_\lambda(t)\Big]\mathrm{d}t + \psi_\lambda(t)\mathrm{d}W(t), \quad t \in [0,T], \\ \varphi_\lambda(T) = M^\top \lambda. \end{cases} \tag{3.14}$$

Here,

$$L(t) \triangleq B(t)^\top P(t) + D(t)^\top P(t)C(t) + D(t)^\top \Lambda(t), \quad K(t) \triangleq R(t) + D(t)^\top P(t)D(t). \tag{3.15}$$

By (A1)–(A3) and [10], Theorem 6.1, the Riccati equation (3.13) admits a unique solution $\big(P(\cdot), \Lambda(\cdot)\big) \in L^\infty_{\mathbb{F}}(\Omega; C([0,T], \mathbb{S}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{S}^n)$ such that $K(t) \geq \gamma I_m$, a.e. $t \in [0,T]$, a.s. for some $\gamma > 0$. Similar to [20], when the solution $\big(P(\cdot), \Lambda(\cdot)\big)$ satisfies the regularity condition

$$K(t)^{-1}L(t) \in L^\infty_{\mathbb{F}}\left(\Omega; L^2\left(0,T;\mathbb{R}^{m\times n}\right)\right), \tag{3.16}$$

the dual functional $d(\cdot)$ has a much simpler expression.

Proposition 3.3. *Suppose that (A1)–(A4) hold. Let $\big(P(\cdot), \Lambda(\cdot)\big)$ be the solution to Riccati equation (3.13) satisfying the regularity condition (3.16). Then, for any $\lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$,*

$$d(\lambda) = \frac{1}{2}\langle P(0)x, x\rangle + \langle \varphi_\lambda(0), x\rangle - \mathbb{E}\langle b, \lambda\rangle - \frac{1}{2}\mathbb{E}\int_0^T \Big|K(t)^{-\frac{1}{2}}\left[B(t)^\top \varphi_\lambda(t) + D(t)^\top \psi_\lambda(t)\right]\Big|^2 \mathrm{d}t, \tag{3.17}$$

where $(\varphi_\lambda(\cdot), \psi_\lambda(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ is the solution to (3.14) and $L(\cdot)$ and $K(\cdot)$ are defined by (3.15). In addition,

$$\bar{u}_\lambda(t) \triangleq -K(t)^{-1}\Big[L(t)\bar{X}^{x,\bar{u}_\lambda}(t) + B(t)^\top \varphi_\lambda(t) + D(t)^\top \psi_\lambda(t)\Big] \tag{3.18}$$

is the feedback optimal solution to the unconstrained stochastic LQ problem in (3.2) for the definition of $d(\lambda)$, i.e.,

$$\mathcal{L}(\bar{u}_\lambda, \lambda) = \inf_{u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u, \lambda). \tag{3.19}$$

Proof. By the solvability of Riccati equation (3.13) and Itô's formula, we obtain that

$$\begin{aligned}
&\mathbb{E}\langle P(T)X^{x,u}(T), X^{x,u}(T)\rangle - \langle P(0)x, x\rangle \\
&= \mathbb{E}\int_0^T \Big[\langle (L(t)^\top K(t)^{-1}L(t) - Q(t))X^{x,u}(t), X^{x,u}(t)\rangle \\
&\quad + 2\langle L(t)X^{x,u}(t), u(t)\rangle + \langle D(t)^\top P(t)D(t)u(t), u(t)\rangle\Big]\mathrm{d}t.
\end{aligned} \tag{3.20}$$

Also, applying Itô's formula to $\langle \varphi_\lambda(\cdot), X^{x,u}(\cdot)\rangle$, we get

$$\begin{aligned}
&\mathbb{E}\langle \varphi_\lambda(T), X^{x,u}(T)\rangle - \langle \varphi_\lambda(0), x\rangle \\
&= \mathbb{E}\int_0^T \Big[\langle L(t)^\top K(t)^{-1}B(t)^\top\varphi_\lambda(t) + L(t)^\top K(t)^{-1}D(t)^\top\psi_\lambda(t), X^{x,u}(t)\rangle \\
&\quad + \langle B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t), u(t)\rangle\Big]\mathrm{d}t.
\end{aligned} \tag{3.21}$$

Combining (3.20) with (3.21), we have

$$\begin{aligned}
\mathcal{L}(u,\lambda) &= J(u) + \mathbb{E}\langle \lambda, MX^{x,u}(T) - b\rangle \\
&= \frac{1}{2}\mathbb{E}\int_0^T \langle Q(t)X^{x,u}(t), X^{x,u}(t)\rangle + \langle R(t)u(t), u(t)\rangle \mathrm{d}t \\
&\quad + \frac{1}{2}\mathbb{E}\langle GX^{x,u}(T), X^{x,u}(T)\rangle + \mathbb{E}\langle \lambda, MX^{x,u}(T) - b\rangle \\
&= \frac{1}{2}\langle P(0)x, x\rangle + \langle \varphi_\lambda(0), x\rangle - \mathbb{E}\langle b, \lambda\rangle \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^T \Big[\langle L(t)^\top K(t)^{-1}L(t)X^{x,u}(t), X^{x,u}(t)\rangle + \langle K(t)u(t), u(t)\rangle \\
&\quad + 2\langle L(t)X^{x,u}(t) + B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t), u(t)\rangle \\
&\quad + 2\langle L(t)^\top K(t)^{-1}B(t)^\top\varphi_\lambda(t) + L(t)^\top K(t)^{-1}D(t)^\top\psi_\lambda(t), X^{x,u}(t)\rangle\Big]\mathrm{d}t \\
&= \frac{1}{2}\langle P(0)x, x\rangle + \langle \varphi_\lambda(0), x\rangle - \mathbb{E}\langle b, \lambda\rangle \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^T \left|K(t)^{\frac{1}{2}}\left[u(t) + K(t)^{-1}(L(t)X^{x,u}(t) + B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t))\right]\right|^2\mathrm{d}t \\
&\quad - \frac{1}{2}\mathbb{E}\int_0^T \left|K(t)^{-\frac{1}{2}}\left[B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t)\right]\right|^2\mathrm{d}t.
\end{aligned} \tag{3.22}$$

Therefore,

$$\bar{u}_\lambda(t) = -K(t)^{-1}\Big[L(t)\bar{X}^{x,\bar{u}_\lambda}(t) + B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t))\Big]$$

satisfies (3.19) and

$$\begin{aligned}
d(\lambda) &= \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda) \\
&= \frac{1}{2}\langle P(0)x, x\rangle + \langle \varphi_\lambda(0), x\rangle - \mathbb{E}\langle b, \lambda\rangle - \frac{1}{2}\mathbb{E}\int_0^T \left|K(t)^{-\frac{1}{2}}\left[B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t)\right]\right|^2\mathrm{d}t.
\end{aligned}$$

This completes the proof of Proposition 3.3. $\square$

Theorem 3.4. *Suppose that (A1)–(A4) hold. Let $\big(P(\cdot),\Lambda(\cdot)\big)$ be the solution to Riccati equation (3.13) satisfying the regularity condition (3.16). Then the optimal control of (CSLQ) is*

$$\bar u(t)=-K(t)^{-1}\Big[L(t)\bar{\mathbb{X}}^{x,\bar\lambda}(t)+B(t)^\top\varphi_{\bar\lambda}(t)+D(t)^\top\psi_{\bar\lambda}(t)\Big],\ a.s.,$$

where the optimal parameter $\bar\lambda$ is the solution to the first-order optimality condition for the dual problem (3.3) that

$$M\bar{\mathbb{X}}^{x,\bar\lambda}(T)-b=0,\quad a.s. \tag{3.23}$$

and $\bar{\mathbb{X}}^{x,\bar\lambda}$ is the solution to the equation

$$\begin{cases} d\bar{\mathbb{X}}^{x,\bar\lambda}(t)=\Big[(A(t)-B(t)K(t)^{-1}L(t))\bar{\mathbb{X}}^{x,\bar\lambda}(t)-B(t)K(t)^{-1}(B(t)^\top\varphi_{\bar\lambda}(t)+D(t)^\top\psi_{\bar\lambda}(t))\Big]\mathrm{d}t\\ \qquad\qquad+\Big[(C(t)-D(t)K(t)^{-1}L(t))\bar{\mathbb{X}}^{x,\bar\lambda}(t)-D(t)K(t)^{-1}(B(t)^\top\varphi_{\bar\lambda}(t)+D(t)^\top\psi_{\bar\lambda}(t))\Big]\mathrm{d}W(t),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\in[0,T],\\ \bar{\mathbb{X}}^{x,\bar\lambda}(0)=x.\end{cases} \tag{3.24}$$

Proof. Let $\bar\lambda$ be an optimal solution to the dual problem (3.3). Then, by (3.17) and the optimality of $\bar\lambda$, for any $\mu\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$, we have

$$\begin{aligned}0=&\lim_{\varepsilon\to0^+}\frac{d(\bar\lambda+\varepsilon\mu)-d(\bar\lambda)}{\varepsilon}\\ =&-\mathbb{E}\int_0^T\big\langle K(t)^{-1}(B(t)^\top\varphi_{\bar\lambda}(t)+D(t)^\top\psi_{\bar\lambda}(t)),B(t)^\top\varphi_\mu(t)+D(t)^\top\psi_\mu(t)\big\rangle\mathrm{d}t\\ &-\mathbb{E}\langle b,\mu\rangle+\langle\varphi_\mu(0),x\rangle,\end{aligned} \tag{3.25}$$

where (φ_μ,ψ_μ) is the solution to (3.14) with final datum $M^\top\lambda$ replaced by $M^\top\mu$.

Applying Itô's formula to $\langle\bar{\mathbb{X}}^{x,\bar\lambda}(\cdot),\varphi_\mu(\cdot)\rangle$, we get

$$\begin{aligned}&\mathbb{E}\langle M\bar{\mathbb{X}}^{x,\bar\lambda}(T),\mu\rangle\\ &=\mathbb{E}\langle\bar{\mathbb{X}}^{x,\bar\lambda}(T),M^\top\mu\rangle\\ &=\langle\varphi_\mu(0),x\rangle-\mathbb{E}\int_0^T\big\langle K(t)^{-1}(B(t)^\top\varphi_{\bar\lambda}(t)+D(t)^\top\psi_{\bar\lambda}(t)),B(t)^\top\varphi_\mu(t)+D(t)^\top\psi_\mu(t)\big\rangle\mathrm{d}t.\end{aligned} \tag{3.26}$$

Combining (3.25) with (3.26), we obtain that

$$M\bar{\mathbb{X}}^{x,\bar\lambda}(T)-b=0,\quad a.s. \tag{3.27}$$

is the first-order necessary condition for the optimal solution $\bar\lambda$ to the dual problem (3.3). Since the dual function $d(\cdot)$ is concave, condition (3.27) is also sufficient for $\bar\lambda$ being an optimal solution to the dual problem. Then the conclusion follows from Theorem 3.2 and Proposition 3.3. $\square$

Remark 3.5. By Theorem 3.4, we obtain a closed-form solution to the constrained stochastic LQ problem (CSLQ). However, it is in general difficult to gain the optimal parameter $\bar\lambda$ by solving the first-order necessary condition (3.23).

4. Augmented Lagrangian method

In this section, we propose an augmented Lagrangian method (ALM) for solving (CSLQ) and prove its convergence.

Let ρ be a fixed positive constant. We define the augmented Lagrangian functional for (CSLQ) by

$$\begin{aligned}\mathcal{L}_\rho(u,\lambda) &\triangleq J(u)+\mathbb{E}\langle\lambda, MX^{x,u}(T)-b\rangle+\frac{\rho}{2}\mathbb{E}\big|MX^{x,u}(T)-b\big|^2\\ &=\frac{1}{2}\mathbb{E}\bigg[\int_0^T\langle Q(t)X^{x,u}(t),X^{x,u}(t)\rangle+\langle R(t)u(t),u(t)\rangle\mathrm{d}t+\langle GX^{x,u}(T),X^{x,u}(T)\rangle\bigg]\\ &\quad+\mathbb{E}\langle\lambda, MX^{x,u}(T)-b\rangle+\frac{\rho}{2}\mathbb{E}\big|MX^{x,u}(T)-b\big|^2,\quad \forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),\forall\ \lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell).\end{aligned}\tag{4.1}$$

The ALM for (CSLQ) is defined as follows.

ALM for (CSLQ)

Step 0. Let $k=0$. Choose $\lambda^0\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$, $\{r^k\}_{k=0}^\infty\subset(0,\infty)$.

Step 1. Calculate u^{k+1} such that

$$\mathcal{L}_\rho(u^{k+1},\lambda^k)=\inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)}\mathcal{L}_\rho(u,\lambda^k).\tag{4.2}$$

Step 2. If $MX^{x,u^{k+1}}(T)-b=0$, a.s., STOP. Otherwise, update the multiplier by

$$\lambda^{k+1}=\lambda^k+r^k\big(MX^{x,u^{k+1}}(T)-b\big).\tag{4.3}$$

Let $k:=k+1$ and return to Step 1.

Remark 4.1. The unconstrained stochastic LQ sub-problem (4.2) can be solved by constructing its feedback optimal solution. Let us consider the backward stochastic Riccati equation

$$\begin{cases}dP_\rho(t)=-\big[P_\rho(t)A(t)+A(t)^\top P_\rho(t)+C(t)^\top P_\rho(t)C(t)+Q(t)+\Lambda_\rho(t)C(t)+C(t)^\top\Lambda_\rho(t)\\ \qquad\qquad -L_\rho(t)^\top K_\rho(t)^{-1}L_\rho(t)\big]\mathrm{d}t+\Lambda_\rho(t)\mathrm{d}W(t),\quad t\in[0,T],\\ P_\rho(T)=G+\rho M^\top M\end{cases}\tag{4.4}$$

and the backward stochastic differential equation

$$\begin{cases}d\varphi_{\rho,\lambda^k}(t)=-\Big[\big(A(t)^\top-L_\rho(t)^\top K_\rho(t)^{-1}B(t)^\top\big)\varphi_{\rho,\lambda^k}(t)\\ \qquad\qquad +\big(C(t)^\top-L_\rho(t)^\top K_\rho(t)^{-1}D(t)^\top\big)\psi_{\rho,\lambda^k}(t)\Big]\mathrm{d}t+\psi_{\rho,\lambda^k}(t)\mathrm{d}W(t),\quad t\in[0,T],\\ \varphi_{\rho,\lambda^k}(T)=M^\top\lambda^k-\rho M^\top b,\end{cases}\tag{4.5}$$

where

$$L_\rho(t)\triangleq B(t)^\top P_\rho(t)+D(t)^\top P_\rho(t)C(t)+D(t)^\top\Lambda_\rho(t),\ K_\rho(t)\triangleq R(t)+D(t)^\top P_\rho(t)D(t).\tag{4.6}$$

In addition, let us define the functional

$$J^0_\rho(u) \triangleq \frac{1}{2}\mathbb{E}\bigg[\int_0^T \langle Q(t)X^{0,u}(t), X^{0,u}(t)\rangle + \langle R(t)u(t), u(t)\rangle \mathrm{d}t + \langle (G+\rho M^\top M)X^{0,u}(T), X^{0,u}(T)\rangle\bigg],$$

where $u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ and the state $X^{0,u}(\cdot)$ is the solution to control system (1.1) with initial datum 0 and control $u(\cdot)$. Under condition (A3),

$$J^0_\rho(u) \geq J^0(u) \geq \delta\mathbb{E}\int_0^T |u(t)|^2 \mathrm{d}t, \qquad u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).$$

Similarly to Lemma 2.2, we can prove that $\mathcal{L}_\rho(\cdot,\lambda^k)$ is strongly convex on $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ and, the unconstrained stochastic LQ sub-problem (4.2) admits a unique optimal solution. In addition, by [10], Theorem 6.1, the perturbed Riccati equation (4.4) has unique solution $\big(P_\rho(\cdot),\Lambda_\rho(\cdot)\big) \in L^\infty_{\mathbb{F}}\big(\Omega;C([0,T];\mathbb{S}^n)\big) \times L^2_{\mathbb{F}}(0,T;\mathbb{S}^n)$ such that

$$R(t) + D(t)^\top P_\rho(t)D(t) \geq \gamma' I_m, \quad \text{a.e. } t\in[0,T],\ a.s.$$

for some $\gamma' > 0$. Then, the backward stochastic differential equation (4.5) also admits a unique solution $(\varphi_{\rho,\lambda^k},\psi_{\rho,\lambda^k})$. By the standard theory of unconstrained stochastic LQ problem, it can be shown that the optimal solution u^{k+1} to the unconstrained stochastic LQ sub-problem (4.2) has the feedback form

$$u^{k+1}(t) = -K_\rho(t)^{-1}\Big[L_\rho(t)X^{x,u^{k+1}}(t) + B(t)^\top\varphi_{\rho,\lambda^k}(t) + D(t)^\top\psi_{\rho,\lambda^k}(t)\Big], \quad \text{a.e. } t\in[0,T], \text{a.s.}$$

Therefore, to solve the unconstrained stochastic LQ sub-problem (4.2), we only need to solve the backward stochastic Riccati equation (4.4) and the backward stochastic differential equation (4.5).

Now, let us prove the convergence of the ALM for (CSLQ). To this end, we need some technical lemmas. First, we prove that the saddle points of the Lagrangian functional $\mathcal{L}(\cdot,\cdot)$ coincide with those of the augmented Lagrangian functional $\mathcal{L}_\rho(\cdot,\cdot)$.

Lemma 4.2. *$(\bar u,\bar\lambda)$ is a saddle point of $\mathcal{L}(\cdot,\cdot)$ if and only if it is a saddle point of $\mathcal{L}_\rho(\cdot,\cdot)$.*

Proof. If $(\bar u,\bar\lambda)$ is a saddle point of $\mathcal{L}(\cdot,\cdot)$, then

$$\mathcal{L}(\bar u,\lambda) \leq \mathcal{L}(\bar u,\bar\lambda) \leq \mathcal{L}(u,\bar\lambda), \quad \forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),\ \forall\ \lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell). \tag{4.7}$$

Clearly, (4.7) holds true only if

$$M\bar X^{x,\bar u}(T) - b = 0,\ a.s.$$

Then, we have

$$\mathcal{L}_\rho(\bar u,\lambda) = \mathcal{L}(\bar u,\lambda) + \frac{\rho}{2}\mathbb{E}\big|M\bar X^{x,\bar u}(T) - b\big|^2 = \mathcal{L}(\bar u,\lambda)$$

and

$$\mathcal{L}_\rho(\bar u,\bar\lambda) = \mathcal{L}(\bar u,\bar\lambda).$$

It implies that

$$\mathcal{L}_\rho(\bar{u},\lambda)\le\mathcal{L}_\rho(\bar{u},\bar{\lambda}),\qquad\forall\ \lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell).\tag{4.8}$$

In addition, by

$$\mathcal{L}(\bar{u},\bar{\lambda})\le\mathcal{L}(u,\bar{\lambda}),\qquad\forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$$

and

$$\mathbb{E}\big|M\bar{X}^{x,\bar{u}}(T)-b\big|^2=0\le\mathbb{E}\big|MX^{x,u}(T)-b\big|^2,\quad\forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),$$

we have

$$\mathcal{L}(\bar{u},\bar{\lambda})+\frac{\rho}{2}\mathbb{E}\big|M\bar{X}^{x,\bar{u}}(T)-b\big|^2\le\mathcal{L}(u,\bar{\lambda})+\frac{\rho}{2}\mathbb{E}\big|MX^{x,u}(T)-b\big|^2,\quad\forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).$$

Therefore,

$$\mathcal{L}_\rho(\bar{u},\bar{\lambda})\le\mathcal{L}_\rho(u,\bar{\lambda}),\quad\forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).\tag{4.9}$$

Combining (4.8) with (4.9), we obtain that $(\bar{u},\bar{\lambda})$ is a saddle point of $\mathcal{L}_\rho(\cdot,\cdot)$.

Next, suppose that $(\bar{u},\bar{\lambda})$ is a saddle point of $\mathcal{L}_\rho(\cdot,\cdot)$, *i.e.*,

$$\mathcal{L}_\rho(\bar{u},\lambda)\le\mathcal{L}_\rho(\bar{u},\bar{\lambda})\le\mathcal{L}_\rho(u,\bar{\lambda}),\quad\forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),\ \forall\ \lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell).$$

Then, we can also obtain that $M\bar{X}^{x,\bar{u}}(T)-b=0$, a.s. It implies that

$$\mathcal{L}(\bar{u},\lambda)=\mathcal{L}_\rho(\bar{u},\lambda)\le\mathcal{L}_\rho(\bar{u},\bar{\lambda})=\mathcal{L}(\bar{u},\bar{\lambda}),\quad\forall\ \lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell),\tag{4.10}$$

and

$$J(\bar{u})=\mathcal{L}(\bar{u},\bar{\lambda})=\mathcal{L}_\rho(\bar{u},\bar{\lambda})\le\mathcal{L}_\rho(u,\bar{\lambda}),\quad\forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).$$

Then, by the convexity of $J(\cdot)$, for any $u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ and $\theta\in(0,1)$, we have

$$\begin{aligned}J(\bar{u})&\le\mathcal{L}_\rho(\bar{u}+\theta(u-\bar{u}),\bar{\lambda})\\&=J\big(\bar{u}+\theta(u-\bar{u})\big)+\mathbb{E}\langle\bar{\lambda},MX^{x,\bar{u}+\theta(u-\bar{u})}(T)-b\rangle+\frac{\rho}{2}\mathbb{E}\big|MX^{x,\bar{u}+\theta(u-\bar{u})}(T)-b\big|^2\\&\le\theta J(u)+(1-\theta)J(\bar{u})+\mathbb{E}\langle\bar{\lambda},\theta MX^{x,u}(T)+(1-\theta)M\bar{X}^{x,\bar{u}}(T)-b\rangle\\&\qquad+\frac{\rho}{2}\mathbb{E}\big|\theta MX^{x,u}(T)+(1-\theta)M\bar{X}^{x,\bar{u}}(T)-b\big|^2\\&=\theta J(u)+(1-\theta)J(\bar{u})+\theta\mathbb{E}\langle\bar{\lambda},MX^{x,u}(T)-b\rangle+\frac{\rho\theta^2}{2}\mathbb{E}\big|MX^{x,u}(T)-b\big|^2.\end{aligned}$$

It implies that

$$0\le J(u)-J(\bar{u})+\mathbb{E}\langle\bar{\lambda},MX^{x,u}(T)-b\rangle+\frac{\rho\theta}{2}\mathbb{E}\big|MX^{x,u}(T)-b\big|^2,\quad\forall\ u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).$$

Letting $\theta \to 0^+$, we have

$$J(\bar{u}) \leq J(u) + \mathbb{E}\langle \bar{\lambda}, MX^{x,u}(T) - b\rangle, \quad \forall\, u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),$$

i.e.,

$$\mathcal{L}(\bar{u}, \bar{\lambda}) \leq \mathcal{L}(u, \bar{\lambda}), \quad \forall\, u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m). \tag{4.11}$$

Combining (4.10) with (4.11), we obtain that $(\bar{u}, \bar{\lambda})$ is a saddle point of $\mathcal{L}(\cdot,\cdot)$.

This completes the proof of Lemma 4.2. □

Lemma 4.3. *Let (A1)–(A4) hold. Then, $\bar{u}$ is an optimal control of (CSLQ) if and only if there is $\bar{\lambda}$ such that $(\bar{u}, \bar{\lambda})$ is a saddle point of $\mathcal{L}_\rho(\cdot,\cdot)$.*

Proof. It has been proved in Theorem 3.2 that if $\bar{u}$ is an optimal control of (CSLQ), then there is $\bar{\lambda}$ such that $(\bar{u}, \bar{\lambda})$ is a saddle point of $\mathcal{L}(\cdot,\cdot)$. Then, by Lemma 4.2, $(\bar{u}, \bar{\lambda})$ is a saddle point of $\mathcal{L}_\rho(\cdot,\cdot)$.

On the other hand, if $(\bar{u}, \bar{\lambda})$ is a saddle point of $\mathcal{L}_\rho(\cdot,\cdot)$, then

$$M\bar{X}^{x,\bar{u}}(T) - b = 0, \ \ a.s.$$

and

$$J(\bar{u}) = \mathcal{L}_\rho(\bar{u}, \bar{\lambda}) \leq \inf_{u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}_\rho(u, \bar{\lambda}) \leq \inf_{u \in U_{ad}} \mathcal{L}_\rho(u, \bar{\lambda}) = \inf_{u \in U_{ad}} J(u).$$

Therefore, $\bar{u}$ is an optimal control of (CSLQ).

This completes the proof of Lemma 4.3. □

By Lemma 4.3, solving the constrained stochastic LQ problem (CSLQ) is equivalent to finding the saddle point of $\mathcal{L}_\rho(\cdot,\cdot)$.

For the unconstrained stochastic LQ sub-problem (4.2), the following first-order necessary and sufficient condition holds true.

Lemma 4.4. *Let (A1)–(A3) hold. Then, u^{k+1} is the optimal control to the unconstrained stochastic LQ sub-problem (4.2) if and only if*

$$R(t)u^{k+1}(t) - B(t)^\top p_{\rho,\lambda^k}(t) - D(t)^\top q_{\rho,\lambda^k}(t) = 0, \quad a.e.\ t \in [0,T],\ \ a.s.$$

where $(p_{\rho,\lambda^k}, q_{\rho,\lambda^k})$ is the solution to the backward stochastic differential equation

$$\begin{cases} dp_{\rho,\lambda^k}(t) = -\Big[A(t)^\top p_{\rho,\lambda^k}(t) + C(t)^\top q_{\rho,\lambda^k}(t) - Q(t)X^{x,u^{k+1}}(t)\Big]\mathrm{d}t + q_{\rho,\lambda^k}(t)\mathrm{d}W(t),\ \ t \in [0,T], \\ p_{\rho,\lambda^k}(T) = -(G + \rho M^\top M)X^{x,u^{k+1}}(T) - M^\top \lambda^k + \rho M^\top b, \end{cases} \tag{4.12}$$

where $X^{x,u^{k+1}}$ is the solution to the linear control system (1.1) with initial datum x and control u^{k+1}.

Proof. For any $\varepsilon > 0$ and $v \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$,

$$\begin{aligned} &\mathcal{L}_\rho(u^{k+1} + \varepsilon v, \lambda^k) - \mathcal{L}_\rho(u^{k+1}, \lambda^k) \\ &= \varepsilon\Big[\mathbb{E}\int_0^T \big\langle Q(t)X^{x,u^{k+1}}(t), X^{0,v}(t)\big\rangle + \big\langle R(t)u^{k+1}(t), v(t)\big\rangle \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\big\langle (G+\rho M^\top M)X^{x,u^{k+1}}(T)+M^\top\lambda^k-\rho M^\top b, X^{0,v}(T)\big\rangle\Big]\\
&+\frac{\varepsilon^2}{2}\Big[\mathbb{E}\int_0^T \big\langle Q(t)X^{0,v}(t), X^{0,v}(t)\big\rangle+\big\langle R(t)v(t),v(t)\big\rangle \mathrm{d}t\\
&+\mathbb{E}\big\langle (G+\rho M^\top M)X^{0,v}(T), X^{0,v}(T)\big\rangle\Big].
\end{aligned}
$$

Here $X^{0,v}$ is the solution to the linear control system (1.1) with initial datum 0 and control v. If u^{k+1} is an optimal control of the sub-problem (4.2), then

$$
\begin{aligned}
0\le &\lim_{\varepsilon\to 0^+}\frac{\mathcal{L}_\rho(u^{k+1}+\varepsilon v,\lambda^k)-\mathcal{L}_\rho(u^{k+1},\lambda^k)}{\varepsilon}\\
=&\mathbb{E}\int_0^T \big\langle Q(t)X^{x,u^{k+1}}(t), X^{0,v}(t)\big\rangle+\big\langle R(t)u^{k+1}(t),v(t)\big\rangle \mathrm{d}t\\
&+\mathbb{E}\big\langle (G+\rho M^\top M)X^{x,u^{k+1}}(T)+M^\top\lambda^k-\rho M^\top b, X^{0,v}(T)\big\rangle,\quad \forall\, v\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).
\end{aligned}
\tag{4.13}
$$

By Itô's formula, we have

$$
\begin{aligned}
&\mathbb{E}\big\langle (G+\rho M^\top M)X^{x,u^{k+1}}(T)+M^\top\lambda^k-\rho M^\top b, X^{0,v}(T)\big\rangle\\
&=-\mathbb{E}\big\langle p_{\rho,\lambda^k}(T), X^{0,v}(T)\big\rangle\\
&=-\mathbb{E}\int_0^T \big\langle B(t)^\top p_{\rho,\lambda^k}(t)+D(t)^\top q_{\rho,\lambda^k}(t), v(t)\big\rangle \mathrm{d}t-\mathbb{E}\int_0^T \big\langle Q(t)X^{x,u^{k+1}}(t), X^{0,v}(t)\big\rangle \mathrm{d}t.
\end{aligned}
\tag{4.14}
$$

Combining (4.13) with (4.14), we obtain

$$
\mathbb{E}\int_0^T \big\langle R(t)u^{k+1}(t)-B(t)^\top p_{\rho,\lambda^k}(t)-D(t)^\top q_{\rho,\lambda^k}(t), v(t)\big\rangle \mathrm{d}t\ge 0,\quad \forall\, v\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),
$$

which implies

$$
R(t)u^{k+1}(t)-B(t)^\top p_{\rho,\lambda^k}(t)-D(t)^\top q_{\rho,\lambda^k}(t)=0,\quad a.e.\ t\in[0,T], a.s.
$$

This proves the necessity.

Next, assume that u^{k+1} satisfies the condition

$$
R(t)u^{k+1}(t)-B(t)^\top p_{\rho,\lambda^k}(t)-D(t)^\top q_{\rho,\lambda^k}(t)=0,\quad a.e.\ t\in[0,T], a.s.
$$

Then

$$
\mathbb{E}\int_0^T \big\langle R(t)u^{k+1}(t)-B(t)^\top p_{\rho,\lambda^k}(t)-D(t)^\top q_{\rho,\lambda^k}(t), v(t)\big\rangle \mathrm{d}t= 0,\quad \forall\, v\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).
\tag{4.15}
$$

By (4.14) and (4.15), we obtain

$$\begin{aligned}&\mathbb{E}\int_0^T \big\langle Q(t)X^{x,u^{k+1}}(t), X^{0,v}(t)\big\rangle + \big\langle R(t)u^{k+1}(t), v(t)\big\rangle \mathrm{d}t\\ &+\mathbb{E}\big\langle (G+\rho M^\top M)X^{x,u^{k+1}}(T) + M^\top\lambda^k - \rho M^\top b, X^{0,v}(T)\big\rangle = 0.\end{aligned}\tag{4.16}$$

Then, by (4.16) and condition (A3), for any $v\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$

$$\begin{aligned}\mathcal{L}_\rho(u^{k+1}+v,\lambda^k) &= \mathcal{L}_\rho(u^{k+1},\lambda^k) + \frac{1}{2}\mathbb{E}\int_0^T \big\langle Q(t)X^{0,v}(t), X^{0,v}(t)\big\rangle + \big\langle R(t)v(t), v(t)\big\rangle \mathrm{d}t\\ &\quad + \frac{1}{2}\mathbb{E}\big\langle (G+\rho M^\top M)X^{0,v}(T), X^{0,v}(T)\big\rangle\\ &\ge \mathcal{L}_\rho(u^{k+1},\lambda^k) + J^0(v)\\ &\ge \mathcal{L}_\rho(u^{k+1},\lambda^k) + \delta\mathbb{E}\int_0^T |v(t)|^2\mathrm{d}t,\end{aligned}$$

which implies that u^{k+1} is an optimal control to the unconstrained stochastic LQ sub-problem (4.2). This proves the sufficiency. □

We have the following first-order necessary and sufficient condition for the constrained stochastic LQ problem (CSLQ).

Lemma 4.5. *Let (A1)–(A4) hold. Then, $\bar u$ is an optimal control of (CSLQ) if and only if there is $\bar\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$ such that*

$$\begin{cases} M\bar X^{x,\bar u}(T) - b = 0,\ \ a.s.\\ R(t)\bar u(t) - B(t)^\top p_{\rho,\bar\lambda}(t) - D(t)^\top q_{\rho,\bar\lambda}(t) = 0, \quad a.e.\ t\in[0,T], a.s.\end{cases}\tag{4.17}$$

where $\big(p_{\rho,\bar\lambda}(\cdot), q_{\rho,\bar\lambda}(\cdot)\big)$ is the solution to (4.12) with $(X^{x,u^{k+1}}(T),\lambda^k)$ replaced by $(\bar X^{x,\bar u}(T),\bar\lambda)$.

Proof. By Lemma 4.3, $\bar u$ is an optimal control of (CSLQ) if and only if there is $\bar\lambda$ such that $(\bar u,\bar\lambda)$ is a saddle point of $\mathcal{L}_\rho(\cdot,\cdot)$. Clearly, $(\bar u,\bar\lambda)$ is a saddle point of $\mathcal{L}_\rho(\cdot,\cdot)$ if and only if

$$M\bar X^{x,\bar u}(T) - b = 0, \quad a.s.$$

and

$$\mathcal{L}_\rho(\bar u,\bar\lambda) = \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}_\rho(u,\bar\lambda).$$

Then, the conclusion follows by a similar argument in Lemma 4.4. □

We are now in a position to establish the main result of this section, namely the convergence of ALM for (CSLQ).

Theorem 4.6. *Suppose that (A1)–(A4) hold. For any $\{r^k\}$ such that $0<r^0\le r^k\le 2\rho$ and any $\lambda^0\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$, the ALM either terminates finitely with the optimal solution $\bar u$ of (CSLQ), or else generates an infinite sequence $\{u^k\}$ which converges strongly to the optimal solution $\bar u$ of (CSLQ) in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$.*

Proof. By Lemma 4.4, for any k, u^{k+1} satisfies the first-order necessary condition

$$R(t)u^{k+1}(t) - B(t)^\top p_{\rho,\lambda^k}(t) - D(t)^\top q_{\rho,\lambda^k}(t) = 0, \quad a.e.\ t \in [0,T], a.s., \tag{4.18}$$

where $\big(p_{\rho,\lambda^k}(\cdot), q_{\rho,\lambda^k}(\cdot)\big)$ is the solution to (4.12).

If $MX^{x,u^{k+1}}(T) - b = 0$, a.s. for some k, then by Lemma 4.5, u^{k+1} is the optimal solution of (CSLQ) and the iteration is terminated.

Now we assume that for any k, $\mathbb{P}(MX^{x,u^{k+1}}(T) - b \neq 0) > 0$. By (4.18) and Itô's formula, we have for any $v \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$,

$$\begin{aligned}&\mathbb{E}\int_0^T \big\langle Q(t)X^{x,u^{k+1}}(t), X^{0,v}(t)\big\rangle + \big\langle R(t)u^{k+1}(t), v(t)\big\rangle \mathrm{d}t \\ &+ \mathbb{E}\big\langle (G+\rho M^\top M)X^{x,u^{k+1}}(T), X^{0,v}(T)\big\rangle + \mathbb{E}\big\langle M^\top\lambda^k - \rho M^\top b, X^{0,v}(T)\big\rangle = 0.\end{aligned}$$

Especially, for $v = u^{k+1} - \bar{u}$, we have

$$\begin{aligned}&\mathbb{E}\int_0^T \big\langle Q(t)X^{x,u^{k+1}}(t), X^{0,u^{k+1}-\bar{u}}(t)\big\rangle + \big\langle R(t)u^{k+1}(t), u^{k+1}(t)-\bar{u}(t)\big\rangle \mathrm{d}t \\ &+ \mathbb{E}\big\langle (G+\rho M^\top M)X^{x,u^{k+1}}(T), X^{0,u^{k+1}-\bar{u}}(T)\big\rangle + \mathbb{E}\big\langle M^\top\lambda^k - \rho M^\top b, X^{0,u^{k+1}-\bar{u}}(T)\big\rangle = 0.\end{aligned} \tag{4.19}$$

Similarly, by Lemma 4.5 and Itô's formula, we have

$$\begin{aligned}&\mathbb{E}\int_0^T \big\langle Q(t)\bar{X}^{x,\bar{u}}(t), X^{0,u^{k+1}-\bar{u}}(t)\big\rangle + \big\langle R(t)\bar{u}(t), u^{k+1}(t)-\bar{u}(t)\big\rangle \mathrm{d}t \\ &+ \mathbb{E}\big\langle (G+\rho M^\top M)\bar{X}^{x,\bar{u}}(T), X^{0,u^{k+1}-\bar{u}}(T)\big\rangle + \mathbb{E}\big\langle M^\top\bar{\lambda} - \rho M^\top b, X^{0,u^{k+1}-\bar{u}}(T)\big\rangle = 0.\end{aligned} \tag{4.20}$$

By (4.19)–(4.20) and the linearity of control system (1.1), we obtain that

$$\begin{aligned}&\mathbb{E}\int_0^T \big\langle Q(t)X^{0,u^{k+1}-\bar{u}}(t), X^{0,u^{k+1}-\bar{u}}(t)\big\rangle + \big\langle R(t)(u^{k+1}(t)-\bar{u}(t)), u^{k+1}(t)-\bar{u}(t)\big\rangle \mathrm{d}t \\ &+ \mathbb{E}\big\langle GX^{0,u^{k+1}-\bar{u}}(T), X^{0,u^{k+1}-\bar{u}}(T)\big\rangle \\ &+ \mathbb{E}\big\langle M^\top(\lambda^k-\bar{\lambda}), X^{0,u^{k+1}-\bar{u}}(T)\big\rangle + \rho\mathbb{E}\big\langle M^\top M X^{0,u^{k+1}-\bar{u}}(T), X^{0,u^{k+1}-\bar{u}}(T)\big\rangle = 0.\end{aligned}$$

Denoting $v^{k+1} = u^{k+1} - \bar{u}$, we have

$$\begin{aligned}&\mathbb{E}\big\langle \lambda^k - \bar{\lambda}, MX^{0,v^{k+1}}(T)\big\rangle \\ &= -\mathbb{E}\int_0^T \big\langle Q(t)X^{0,v^{k+1}}(t), X^{0,v^{k+1}}(t)\big\rangle + \big\langle R(t)v^{k+1}(t), v^{k+1}(t)\big\rangle \mathrm{d}t \\ &\quad - \mathbb{E}\big\langle GX^{0,v^{k+1}}(T), X^{0,v^{k+1}}(T)\big\rangle - \rho\mathbb{E}\big\langle M^\top M X^{0,v^{k+1}}(T), X^{0,v^{k+1}}(T)\big\rangle.\end{aligned} \tag{4.21}$$

By (4.3), (4.21), condition (A3) and the fact that $M\bar{X}^{x,\bar{u}}(T) - b = 0$, a.s., we have

$$\begin{aligned}\mathbb{E}\big|\lambda^{k+1} - \bar{\lambda}\big|^2 &= \mathbb{E}\big|\lambda^k + r^k(MX^{x,u^{k+1}}(T) - b) - \bar{\lambda}\big|^2 \\ &= \mathbb{E}\big|\lambda^k - \bar{\lambda}\big|^2 + 2r^k\mathbb{E}\big\langle \lambda^k - \bar{\lambda}, MX^{0,v^{k+1}}(T)\big\rangle + (r^k)^2\mathbb{E}\big|MX^{0,v^{k+1}}(T)\big|^2\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\big|\lambda^k-\bar\lambda\big|^2 - 4r^kJ^0(v^{k+1}) - 2r^k\rho\mathbb{E}\big|MX^{0,v^{k+1}}(T)\big|^2 + (r^k)^2\mathbb{E}\big|MX^{0,v^{k+1}}(T)\big|^2\\
&\le \mathbb{E}\big|\lambda^k-\bar\lambda\big|^2 - 4r^k\delta\mathbb{E}\int_0^T|v^{k+1}(t)|^2\mathrm{d}t - r^k(2\rho-r^k)\mathbb{E}\big|MX^{0,v^{k+1}}(T)\big|^2. \qquad (4.22)
\end{aligned}
$$

This proves that the sequence $\left\{\mathbb{E}\big|\lambda^k-\bar\lambda\big|^2\right\}$ is decreasing and bounded below by 0, hence it is convergent. In addition, by (4.22), we have

$$
0 \le 4r^0\delta\mathbb{E}\int_0^T|v^{k+1}(t)|^2\mathrm{d}t \le \mathbb{E}\big|\lambda^k-\bar\lambda\big|^2 - \mathbb{E}\big|\lambda^{k+1}-\bar\lambda\big|^2.
$$

Letting $k\to+\infty$, we have

$$
\|u^{k+1}-\bar u\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} = \|v^{k+1}\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \to 0.
$$

This completes the proof of Theorem 4.6. □

5. The characterization of condition (A4)

In this section, we shall give a sufficient and necessary condition and some sufficient conditions for condition (A4). Some basic ideas are from the fundamental controllability arguments of [33, 34].

In order to characterize the condition (A4), let us consider the following norm optimal control problem:

$$
\begin{cases}
\min & \dfrac{1}{2}\mathbb{E}\displaystyle\int_0^T|u(t)|^2\mathrm{d}t,\\
\text{s.t.} & u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),\\
 & (X^{x,u},u) \text{ satisfies (1.1) and}\\
 & MX^{x,u}(T)-b=0,\ a.s.
\end{cases} \qquad (5.1)
$$

Clearly, the norm optimal control problem (5.1) is a special case of (CSLQ) with $Q(\cdot)\equiv 0$, $G\equiv 0$ and $R(\cdot)\equiv I_m$. Furthermore, $(P(t),\Lambda(t))\equiv 0$ is the solution to its Riccati equation

$$
\begin{cases}
dP(t) = -\big[P(t)A(t)+A(t)^\top P(t)+C(t)^\top P(t)C(t)+\Lambda(t)C(t)+C(t)^\top\Lambda(t)\\
\qquad\qquad -L(t)^\top K(t)^{-1}L(t)\big]\mathrm{d}t+\Lambda(t)\mathrm{d}W(t), \quad t\in[0,T],\\
P(T)=0.
\end{cases}
$$

Here $L(\cdot)$ and $K(\cdot)$ are defined by (3.15).

Define the Lagrangian functional of (5.1) by

$$
\mathcal{L}(u,\lambda) \triangleq \frac{1}{2}\mathbb{E}\int_0^T|u(t)|^2\mathrm{d}t + \mathbb{E}\langle\lambda, MX^{x,u}(T)-b\rangle, \quad \forall\, u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), \forall\,\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell).
$$

For any $\lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$, the equation (3.14) reduces to

$$
\begin{cases}
d\varphi_\lambda(t) = -\Big[A(t)^\top\varphi_\lambda(t)+C(t)^\top\psi_\lambda(t)\Big]\mathrm{d}t+\psi_\lambda(t)\mathrm{d}W(t), \quad t\in[0,T],\\
\varphi_\lambda(T)=M^\top\lambda.
\end{cases} \qquad (5.2)
$$

By (5.2) and Itô's formula, we have

$$\begin{aligned}\mathcal{L}(u,\lambda) &= \frac{1}{2}\mathbb{E}\int_0^T |u(t)|^2\mathrm{d}t + \mathbb{E}\langle \lambda, MX^{x,u}(T) - b\rangle \\ &= \frac{1}{2}\mathbb{E}\int_0^T |u(t)|^2\mathrm{d}t + \langle \varphi_\lambda(0), x\rangle - \mathbb{E}\langle \lambda, b\rangle + \mathbb{E}\int_0^T \langle B(t)^\top \varphi_\lambda(t) + D(t)^\top \psi_\lambda(t), u(t)\rangle \mathrm{d}t \\ &= \frac{1}{2}\mathbb{E}\int_0^T |u(t) + B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t)|^2\mathrm{d}t + \langle \varphi_\lambda(0), x\rangle \\ &\quad - \mathbb{E}\langle \lambda, b\rangle - \frac{1}{2}\mathbb{E}\int_0^T |B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t)|^2\mathrm{d}t.\end{aligned}$$

Then, the Lagrangian dual functional for (5.1) is

$$d(\lambda) \triangleq \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda) = -\frac{1}{2}\mathbb{E}\int_0^T |B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t)|^2\mathrm{d}t + \langle \varphi_\lambda(0), x\rangle - \mathbb{E}\langle \lambda, b\rangle, \tag{5.3}$$

and

$$\bar{u}_\lambda(t) \triangleq -B(t)^\top\varphi_\lambda(t) - D(t)^\top\psi_\lambda(t)$$

satisfies

$$\mathcal{L}(\bar{u}_\lambda,\lambda) = \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \mathcal{L}(u,\lambda).$$

We define the dual problem of (5.1) as follows:

$$\begin{cases} \max \quad d(\lambda) = -\frac{1}{2}\mathbb{E}\int_0^T |B(t)^\top\varphi_\lambda(t) + D(t)^\top\psi_\lambda(t)|^2\mathrm{d}t + \langle \varphi_\lambda(0), x\rangle - \mathbb{E}\langle \lambda, b\rangle, \\ \text{s.t.} \quad \lambda \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell). \end{cases} \tag{5.4}$$

The following result illustrates that the minimal norm control can be constructed by the optimal solution to the dual problem (5.4). It is proved by the same method in [20], Proposition 4.1. The only difference is that, here the dual variable λ belongs to $L^2_{\mathcal{F}_T}(\Omega;R^\ell)$ while in [20], Proposition 4.1 $\lambda \in R^\ell$. A detailed proof is given below for the sake of readability and completeness.

Proposition 5.1. *If $\bar{\lambda}$ is the optimal solution to (5.4), then*

$$\bar{u}(t) = -B(t)^\top\varphi_{\bar{\lambda}}(t) - D(t)^\top\psi_{\bar{\lambda}}(t), \quad a.e.\ t\in[0,T],\ a.s. \tag{5.5}$$

is the optimal solution to (5.1), where $(\varphi_{\bar{\lambda}}, \psi_{\bar{\lambda}})$ is the solution to (5.2) with final datum $M^\top\lambda$ replaced by $M^\top\bar{\lambda}$.

Proof. Let $\bar{\lambda}, \mu \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$, $(\varphi_{\bar{\lambda}}, \psi_{\bar{\lambda}})$ and (φ_μ, ψ_μ) be the solutions to (5.2) with final datum $M^\top\bar{\lambda}$ and $M^\top\mu$, respectively. If $\bar{\lambda}$ is an optimal solution to (5.4), then, for any $\mu \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$,

$$\begin{aligned} 0 &= \mathbb{E}\langle \nabla d(\bar{\lambda}), \mu\rangle \\ &= \langle \varphi_\mu(0), x\rangle - \mathbb{E}\langle b, \mu\rangle - \mathbb{E}\int_0^T \langle B(t)^\top\varphi_{\bar{\lambda}}(t) + D(t)^\top\psi_{\bar{\lambda}}(t), B(t)^\top\varphi_\mu(t) + D(t)^\top\psi_\mu(t)\rangle\mathrm{d}t. \end{aligned} \tag{5.6}$$

Let $\bar{X}^{x,\bar{u}}$ be the solution to the controlled system (1.1) with initial datum x and control $\bar{u}$ defined by (5.5). By (5.5),(5.6) and Itô's formula, for any $\mu \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$,

$$\begin{aligned}
&\mathbb{E}\langle M\bar{X}^{x,\bar{u}}(T)-b,\mu\rangle\\
&=\mathbb{E}\langle M^\top\mu,\bar{X}^{x,\bar{u}}(T)\rangle-\mathbb{E}\langle b,\mu\rangle\\
&=\langle\varphi_\mu(0),x\rangle+\mathbb{E}\int_0^T\langle\bar{u}(t),B(t)^\top\varphi_\mu(t)+D(t)^\top\psi_\mu(t)\rangle\mathrm{d}t-\mathbb{E}\langle b,\mu\rangle\\
&=0.
\end{aligned}\tag{5.7}$$

Due to the arbitrariness of μ, we obtain $M\bar{X}^{x,\bar{u}}(T)-b=0$, a.s. This proves that the $\bar{u}$ defined by (5.5) is a feasible control.

Next, we prove the optimality of $\bar{u}$. Replacing μ by $\bar{\lambda}$ in (5.7), we obtain that

$$0=\langle\varphi_{\bar{\lambda}}(0),x\rangle-\mathbb{E}\int_0^T|\bar{u}(t)|^2\mathrm{d}t-\mathbb{E}\langle b,\bar{\lambda}\rangle.\tag{5.8}$$

For any $u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ with the corresponding state $X^{x,u}$ such that $MX^{x,u}(T)-b=0$, a.s., by Itô's formula,

$$\mathbb{E}\langle X^{x,u}(T),M^\top\bar{\lambda}\rangle=\langle\varphi_{\bar{\lambda}}(0),x\rangle+\mathbb{E}\int_0^T\langle u(t),B(t)^\top\varphi_{\bar{\lambda}}(t)+D(t)^\top\psi_{\bar{\lambda}}(t)\rangle\mathrm{d}t.\tag{5.9}$$

Combining (5.9) with (5.8), we obtain that

$$\begin{aligned}
\mathbb{E}\int_0^T|\bar{u}(t)|^2\mathrm{d}t&=\mathbb{E}\langle MX^{x,u}(T)-b,\bar{\lambda}\rangle-\mathbb{E}\int_0^T\langle u(t),B(t)^\top\varphi_{\bar{\lambda}}(t)+D(t)^\top\psi_{\bar{\lambda}}(t)\rangle\mathrm{d}t\\
&=-\mathbb{E}\int_0^T\langle u(t),B(t)^\top\varphi_{\bar{\lambda}}(t)+D(t)^\top\psi_{\bar{\lambda}}(t)\rangle\mathrm{d}t\\
&\le\Big[\mathbb{E}\int_0^T|u(t)|^2\mathrm{d}t\Big]^{\frac12}\Big[\mathbb{E}\int_0^T|\bar{u}(t)|^2\mathrm{d}t\Big]^{\frac12}.
\end{aligned}$$

This proves the optimality of $\bar{u}$. □

The following theorem gives a necessary and sufficient condition for $u\mapsto MX^{x,u}(T)$ being a surjection.

Theorem 5.2. *Suppose that (A1) holds true. Then, $u\mapsto MX^{x,u}(T)$ is a surjection if and only if there is $c>0$ such that*

$$\mathbb{E}\int_0^T|B(t)^\top\varphi_\lambda(t)+D(t)^\top\psi_\lambda(t)|^2\mathrm{d}t\ge c\mathbb{E}|\lambda|^2,\quad\forall\ \lambda\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell),\tag{5.10}$$

where $(\varphi_\lambda(\cdot),\psi_\lambda(\cdot))$ is the solution to (5.2).

Proof. Let us fix arbitrarily $\alpha\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$ and define

$$\hat{d}_\alpha(\lambda)\triangleq-\frac12\mathbb{E}\int_0^T|B(t)^\top\varphi_\lambda(t)+D(t)^\top\psi_\lambda(t)|^2\mathrm{d}t+\langle\varphi_\lambda(0),x\rangle-\mathbb{E}\langle\lambda,\alpha\rangle.$$

By the well-posedness theory of linear backward stochastic differential equations (see for instance [35], Chap. 7), both $\lambda\mapsto B(\cdot)^\top\varphi_\lambda(\cdot)+D(\cdot)^\top\psi_\lambda(\cdot)$ and $\lambda\mapsto\varphi_\lambda(0)$ are bounded linear mapping. If inequality (5.10) holds true, then we have $-\hat{d}_\alpha(\cdot)$ is a continuous strongly convex functional on $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$. It implies that (5.4) (with b replaced by α) has a unique optimal solution $\bar{\lambda}_\alpha$. Similarly to Proposition 5.1, we conclude that

$$\bar{u}_\alpha(t) = -B(t)^\top \varphi_{\bar{\lambda}_\alpha}(t) + D(t)^\top \psi_{\bar{\lambda}_\alpha}(t), \quad a.e.\ t\in[0,T],\ a.s.$$

is a minimal norm control satisfying

$$M\bar{X}^{x,\bar{u}_\alpha}(T) = \alpha, \quad a.s.$$

This proves the sufficiency.

Next, let us prove the necessity. Suppose by contradiction that $u\mapsto MX^{x,u}(T)$ is surjective, but (5.10) does not hold true. Then, there is $\{\lambda_n\}\subset L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$ such that

$$\mathbb{E}\int_0^T \left|B(t)^\top\varphi_{\lambda_n}(t) + D(t)^\top\psi_{\lambda_n}(t)\right|^2 \mathrm{d}t < \frac{\mathbb{E}|\lambda_n|^2}{n^2}.$$

Set $\widehat{\lambda}_n = \frac{\sqrt{n}\lambda_n}{\left[\mathbb{E}|\lambda_n|^2\right]^{\frac{1}{2}}}$. Then $\mathbb{E}|\widehat{\lambda}_n|^2\to\infty$ and

$$\mathbb{E}\int_0^T \left|B(t)^\top\varphi_{\widehat{\lambda}_n}(t) + D(t)^\top\psi_{\widehat{\lambda}_n}(t)\right|^2 \mathrm{d}t < \frac{n}{\mathbb{E}|\lambda_n|^2}\cdot\frac{\mathbb{E}|\lambda_n|^2}{n^2} = \frac{1}{n}\to 0.$$

Since $u\mapsto MX^{x,u}(T)$ is surjective, for any $\alpha\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^\ell)$, there exists $u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ such that

$$0 = MX^{x,u}(T) - \alpha = MX^{0,u}(T) + MX^{x,0}(T) - \alpha.$$

By Itô's formula,

$$\begin{aligned}
\mathbb{E}\langle\widehat{\lambda}_n, \alpha - MX^{x,0}(T)\rangle &= \mathbb{E}\langle\widehat{\lambda}_n, MX^{0,u}(T)\rangle\\
&= \mathbb{E}\langle M^\top\widehat{\lambda}_n, X^{0,u}(T)\rangle\\
&= \mathbb{E}\int_0^T \langle u(t), B(t)^\top\varphi_{\widehat{\lambda}_n}(t) + D(t)^\top\psi_{\widehat{\lambda}_n}(t)\rangle \mathrm{d}t\\
&\to 0 \quad (n\to\infty).
\end{aligned}$$

By the arbitrariness of α, we obtain that $\widehat{\lambda}_n$ converges weakly to 0. It implies that $\{\widehat{\lambda}_n\}$ is bounded, which contradicts to $\mathbb{E}|\widehat{\lambda}_n|^2\to\infty$ as $n\to\infty$. This proves (5.10). □

In the rest of this section, let us discuss the special case that A, B, C, D and M are deterministic matrices.

Lemma 5.3. *If A,B,C,D and M are deterministic matrices, then the mapping $u\mapsto MX^{x,u}(T)$ is surjective only if $m\geq\ell$ and* $\mathrm{Rank}(MD)=\ell$.

Proof. The proof is similar to that of [34], Proposition 6.3, so we omit it. □

By Lemma 5.3, there are $K_1\in\mathbb{R}^{m\times m}, K_2\in\mathbb{R}^{m\times n}$ such that

$$MDK_1 = \left(I_\ell, 0\right), \qquad MDK_2 = -MC.$$

Fix arbitrarily $z \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $v \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m-\ell})$ and define

$$u^{z,v} \triangleq K_1 \begin{pmatrix} Mz \\ v \end{pmatrix} + K_2 X^{x,u^{z,v}}. \tag{5.11}$$

Substituting (5.11) into the control system (1.1), we have

$$\begin{aligned} dMX^{x,u^{z,v}}(t) &= \left[MAX^{x,u^{z,v}}(t) + MB\left(K_1 \begin{pmatrix} Mz(t) \\ v(t) \end{pmatrix} + K_2 X^{x,u^{z,v}}(t) \right) \right] \mathrm{d}t \\ &\quad + \left[MCX^{x,u^{z,v}}(t) + MD\left(K_1 \begin{pmatrix} Mz(t) \\ v(t) \end{pmatrix} + K_2 X^{x,u^{z,v}}(t) \right) \right] \mathrm{d}W(t) \\ &= \left[M(A+BK_2)X^{x,u^{z,v}}(t) + MBK_1 \begin{pmatrix} Mz(t) \\ v(t) \end{pmatrix} \right] \mathrm{d}t + Mz(t)\mathrm{d}W(t), \quad t \in [0,T]. \end{aligned}$$

Setting $A_1 = A + BK_2$ and letting $A_2,\ B_1$ be the matrices such that

$$BK_1 \begin{pmatrix} Mz(t) \\ v(t) \end{pmatrix} = A_2 z(t) + B_1 v(t),$$

we have

$$\begin{cases} \mathrm{d}MX^{x,u^{z,v}}(t) = \big(MA_1 X^{x,u^{z,v}}(t) + MA_2 z(t) + MB_1 v(t) \big)\mathrm{d}t + Mz(t)\mathrm{d}W(t), \quad t \in [0,T] \\ MX^{x,u^{z,v}}(0) = Mx. \end{cases} \tag{5.12}$$

From (5.12) we obtain that $u \mapsto MX^{x,u}(T)$ is surjective if $(z,v) \mapsto MX^{x,u^{z,v}}(T)$ is surjective.

Consider the backward stochastic control system

$$\begin{cases} \mathrm{d}Y(t) = \big(A_1 Y(t) + A_2 z(t) + B_1 v(t) \big)\mathrm{d}t + z(t)\mathrm{d}W(t), \quad t \in [0,T], \\ Y(T) = \eta_T. \end{cases} \tag{5.13}$$

Clearly, $(z,v) \mapsto MX^{x,u^{z,v}}(T)$ is surjective if $\mathrm{Rank}(M) = \ell$ and (5.13) is exactly controllable in the sense that for any $\eta_T \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, there is $v \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m-\ell})$ such that the solution $(Y(\cdot;\eta_T,v), z(\cdot;\eta_T,v))$ of (5.13) satisfies $Y(0;\eta_T,v) = x$. Note that $\mathrm{Rank}(MD) = \ell$ only if $\mathrm{Rank}(M) = \ell$. We have the following result.

Theorem 5.4. *If A, B, C, D and M are deterministic matrices, then, $u \mapsto MX^{x,u}(T)$ is surjective if*

(i) $\mathrm{Rank}(MD) = \ell$;
(ii) (5.13) is exactly controllable.

Furthermore, by [34], Theorem 6.10, (5.13) is exactly controllable if and only if

$$\mathrm{Rank}\big([B_1, A_1B_1, A_2B_1, A_1^2B_1, A_1A_2B_1, A_2^2B_1, A_2A_1B_1, \ldots] \big) = n.$$

Then, we obtain the following rank condition for the subjectivity of $u \mapsto MX^{x,u}(T)$.

Theorem 5.5. *If A, B, C, D and M are deterministic matrices, then, $u \mapsto MX^{x,u}(T)$ is surjective if*

(i) $\mathrm{Rank}(MD) = \ell$;
(ii) $\mathrm{Rank}\big([B_1, A_1B_1, A_2B_1, A_1^2B_1, A_1A_2B_1, A_2^2B_1, A_2A_1B_1, \ldots] \big) = n$.

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