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LOCAL ESTIMATES FOR VECTORIAL RUDIN–OSHER–FATEMI TYPE PROBLEMS IN ONE DIMENSION

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Abstract. We consider the Rudin–Osher–Fatemi variational denoising model with general regularizing term and L^2 fidelity in one-dimensional, vector-valued setting. We obtain local estimates on the singular part of the variation measure of the minimizer in terms of the singular part of the variation measure of the datum. In the case of homogeneous regularizer, we prove local estimates on the whole variation measure of the minimizer and deduce an analogous result for the gradient flow of the regularizer. We also discuss the question of extending our results to other fidelities.

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1. Introduction

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary and let $\mathcal{F}\colon L^2(\Omega, \mathbb{R}^n) \to [0, \infty]$ be convex, proper, and lower semicontinuous. We consider the functional $\mathcal{E}\colon L^2(\Omega, \mathbb{R}^n) \to [0, \infty]$ given by

$$\mathcal{E}(w) := \lambda \mathcal{F}(w) + \frac{1}{2} \int_\Omega |w - h|^2 \, \mathrm{d}\mathcal{L}^m, \tag{1.1}$$

where $h \in L^2(\Omega, \mathbb{R}^n)$, $\lambda > 0$. This functional is known as the *Moreau–Yosida regularization* of $\mathcal{F}$. By convexity, it is weakly lower semicontinuous. Moreover, it is proper and its sublevel sets are weakly compact. Therefore it admits a minimizer u, which is unique since $\mathcal{E}$ is strictly convex on its domain (*i. e.*, the subset of $L^2(\Omega, \mathbb{R}^n)$ where $\mathcal{E}$ is finite).

We are interested in the situation where $\mathcal{F}$ is given by

$$\mathcal{F}(w) = \int_\Omega F(w_x), \tag{1.2}$$

for sufficiently regular w. Here w_x denotes the (total) derivative of w, and $F\colon \mathbb{R}^{m\times n} \to [0, \infty[$ is a convex function. In the superlinear case, where $F(\xi) \sim |\xi|^p$, $p > 1$, we can define $\mathcal{F}$ by (1.2) on $L^2(\Omega, \mathbb{R}^n) \cap W^{1,p}(\Omega, \mathbb{R}^n)$ and ∞

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outside this set. The resulting functional is indeed lower semicontinuous, which follows from [1], Theorem 1.3 and the Banach–Alaoglu theorem in conjunction with reflexivity of the space $W^{1,p}(\Omega,\mathbb{R}^n)$ for $p>1$.

However, here we consider F of *linear growth*, *i. e.*,

$$C_F^-(|\xi|-1)\le F(\xi)\le C_F^+(|\xi|+1)\quad \text{for } \xi\in\mathbb{R}^{m\times n} \tag{1.3}$$

with $C_F^\pm>0$. In this case, analogous statement does not hold, which is related to lack of reflexivity of $W^{1,1}(\Omega,\mathbb{R}^n)$, and we need to extend the domain of $\mathcal{F}$ in a nontrivial way to $L^2(\Omega,\mathbb{R}^n)\cap BV(\Omega,\mathbb{R}^n)$. Recall that $BV(\Omega,\mathbb{R}^n)$ is the space of those elements w of $L^1(\Omega,\mathbb{R}^n)$ whose distributional derivative w_x belongs to $M(\Omega,\mathbb{R}^{m\times n})$, the space of finite vector-valued Radon measures. Such an extension was introduced and investigated by Goffman and Serrin in [2]. We describe it in more detail in Section 2. Given a BV mapping w, one can write decomposition of its derivative w_x into absolutely continuous and singular parts with respect to the Lebesgue measure as $\mathrm{d}w_x = w_x^{ac}\,\mathrm{d}\mathcal{L}^m + \mathrm{d}w_x^s$. Then, one can define a measure $F(w_x)\in M(\Omega)$ by

$$\mathrm{d}F(w_x)=F(w_x^{ac})\,\mathrm{d}\mathcal{L}^m + F^\infty\left(\tfrac{w_x^s}{|w_x^s|}\right)\mathrm{d}|w_x^s|,\quad \text{where } F^\infty(\omega)=\lim_{t\to\infty}\tfrac{1}{t}F(t\omega) \text{ for } \omega\in\mathbb{S}_1^{n-1}. \tag{1.4}$$

Then, one can prove that $\mathcal{F}$ given by

$$\mathcal{F}(w):=\begin{cases} F(w_x)(\Omega) & \text{for } w\in BV(\Omega,\mathbb{R}^n),\\ \infty & \text{for } w\in L^2(\Omega,\mathbb{R}^n)\setminus BV(\Omega,\mathbb{R}^n)\end{cases} \tag{1.5}$$

is indeed lower semicontinuous (see Thm. 2.5 in Sect. 2). Thus the functional $\mathcal{E}$ given by (1.1) admits a unique minimizer u.

The assumption of linear (as opposed to superlinear) growth of $\mathcal{F}$ is essential for applications in data processing. This is mainly because it allows jump discontinuities in the minimizer, corresponding to sharp edges, a desirable feature in natural images and signals. In the case $F=|\cdot|$, $\mathcal{F}$ coincides with the standard BV seminorm also known as the total variation (TV), while minimization of $\mathcal{E}$ is equivalent [3] to constrained TV minimization proposed in [4] as a tool for noise removal in images. Thenceforth, numerous variants of $\mathcal{E}$ appeared in the image and signal processing literature, including also anisotropic and inhomogeneous choices of F. In particular, in the vectorial 1D case, $F=|\cdot|_1$ (the ℓ^1 norm) was considered in compressed sensing to promote sparsity, see *e. g.* [5]. Conversely, the ℓ^∞ norm was proposed in [6] to exploit the natural correlation between channels in the context of color images. On the other hand, inhomogeneous version of TV regularized in a neighborhood of zero known as Huber-TV was suggested to avoid the phenomenon of staircasing, see *e. g.* [7].

Having established that we may need to search for the minimizer u of $\mathcal{E}$ outside $W^{1,1}(\Omega,\mathbb{R}^n)$, a natural question arises, what can be said about the singular part of u_x. So far, this question has been mostly investigated in the scalar-valued case $n=1$. Let us report briefly on known results in this direction. We note at this point that minimization of $\mathcal{E}$ coincides with the resolvent problem for an evolutionary equation—the gradient flow of $\mathcal{F}$, see Section 7. Papers that we will mention here are sometimes focused on the evolutionary case. However, in many situations the results can be transferred between the two settings. We first recall the following natural definition: if μ,ν are signed Borel measures on an open set $U\subset\mathbb{R}^m$, we say that

$$\mu\le\nu\quad\text{as Borel measures on } U$$

if $\mu(E)\le\nu(E)$ for every Borel $E\subset U$.

In the one-dimensional, scalar-valued case $m=n=1$, for $F=|\cdot|$, it has been proved in [8, 9] that

$$|u_x|\le|h_x|\quad\text{as Borel measures on } \Omega. \tag{1.6}$$

The proof in [9] follows by analysis of level sets of u, thus depending on the linear order of the range $\mathbb{R}$. On the other hand, the reasoning in [8] is based on explicit description of minimizers for piecewise constant h.

Estimate (1.6) is rather powerful. It implies in particular that

$$|u_x^{ac}(x)| \le |h_x^{ac}(x)| \quad \text{for } \mathcal{L}^m\text{-a. e. } x \in \Omega. \tag{1.7}$$

This is related to the particularly degenerate structure of the corresponding Euler–Lagrange equation in this case, formally

$$u - h = \lambda(\operatorname{sgn} u_x)_x.$$

Observe that the r. h. s. vanishes whenever $u_x \neq 0$. This is no longer the case if F is not homogeneous. In particular, if F is smooth and strongly convex, then the E–L equation is a well-understood locally uniformly elliptic equation, for which (1.7) cannot be expected to hold (this is related to the strong maximum principle in the evolutionary limit, see also [10], Thm. 4). However, it has been proved in [11] for general convex F of linear growth in $m = n = 1$ that if $h \in W^{1,1}(\Omega)$, then $u \in W^{1,1}(\Omega)$.

Estimate (1.7) is also known to fail for $m > 1$, see explicit examples in [12], Section 4. In this case, an estimate for the whole singular part of u_x is not known. However, let us recall that for any $w \in BV(\Omega)$, w_x^s can be further decomposed into the *jump part* w_x^j supported on a $\mathcal{H}^{m-1}$-rectifiable set roughly corresponding to jump discontinuities of u, and the remaining Cantor part w_x^c, see [13]. It has been proved in [14, 15] that if F is a norm smooth outside 0, and $h \in BV(\Omega) \cap L^\infty(\Omega)$, then

$$|u_x^j| \le |h_x^j| \quad \text{as Borel measures on } \Omega. \tag{1.8}$$

This estimate has been generalized to a wider class of functions F in [16]. To our knowledge, it remains an open question whether an analogous estimate holds for u_x^c. Moreover, no similar estimate on the singular or jump part of u_x is known to us in the vector-valued setting $n > 1$, except in the one-dimensional case $m = 1$, see below.

In the case where F is the ℓ^1 norm, (1.8) is known to fail, see *e. g.* [17], Example 4 in the evolutionary case. In fact, one can even construct a non-convex polygon $\Omega \subset \mathbb{R}^2$ and $h \in C^\infty(\overline{\Omega})$ such that u has a jump discontinuity in Ω [18]. On the other hand, under the assumption that Ω is convex, it has been proved in [19] that

$$F^\infty(u_x^s)(\Omega) \le F^\infty(h_x^s)(\Omega).$$

In particular, if $h \in W^{1,1}(\Omega)$, then $u \in W^{1,1}(\Omega)$, generalizing [11].

In all papers listed above, only the scalar case $n = 1$ is considered. The only generalization of the mentioned results to the vector-valued case that we know of is [20], where inequality (1.6) is obtained in the case $m = 1$, $n > 1$, $F = |\cdot|$ using integral estimates. In this paper, our goal is to obtain estimates on u_x or u_x^s in the case $m = 1$, $n > 1$ for possibly general F. We assume that Ω is an open, bounded interval in $\mathbb{R}$ and in this case we denote $\Omega \equiv I$.

We will need a structural assumption formally similar to the one in [21] (note however that in [21] $m > 1$, $n = 1$ and the results are not closely related). We recall the following definition (compare *e. g.* [22]). We say that a function $\varphi\colon \mathbb{R}^n \to [0, +\infty[$ is an *anisotropy* if it is convex and positively 1-homogeneous, *i. e.*,

$$\varphi(tp) = t\varphi(p) \quad \text{for } p \in \mathbb{R}^n,\ t \ge 0.$$

For any anisotropy φ, we denote

$$c_\varphi^+ := \max_{0 \neq p \in \mathbb{R}^n} \frac{\varphi(p)}{|p|}, \quad c_\varphi^- := \min_{0 \neq p \in \mathbb{R}^n} \frac{\varphi(p)}{|p|}. \tag{1.9}$$

An anisotropy φ will be called *coercive* if $c_\varphi^- > 0$. Note that any even anisotropy is a seminorm (and vice versa) and any coercive, even anisotropy is a norm (and vice versa). We prove the following three local estimates on the variation measure of minimizers of $\mathcal{E}$ in terms of the variation measure of the datum h under varying assumptions on F.

Theorem 1.1. *Let $F = \varphi\colon \mathbb{R}^n \to [0,\infty[$ be a coercive anisotropy. Suppose that $h \in BV(U,\mathbb{R}^n)$ for an open interval $U \subset I$. Then the minimizer u of $\mathcal{E}$ satisfies*

$$|u_x| \le |h_x| \text{ as Borel measures on } U. \tag{1.10}$$

Theorem 1.2. *Let $F = f\circ\varphi$, where $f\colon [0,\infty[\to[0,\infty[$ is convex, non-decreasing and of linear growth, and $\varphi\colon \mathbb{R}^n \to [0,\infty[$ is a coercive anisotropy. Suppose that $h \in BV(U,\mathbb{R}^n)$ for an open interval $U \subset I$. Then the minimizer u of $\mathcal{E}$ satisfies*

$$|u_x^s| \le \left(\frac{c_\varphi^+}{c_\varphi^-}\right)^2 |h_x^s| \text{ as Borel measures on } U. \tag{1.11}$$

Theorem 1.3. *Let F be as in Theorem (1.2). Moreover, assume that it is strictly convex and differentiable on $\mathbb{R}^n$. Suppose that $h \in BV(U,\mathbb{R}^n)$ for an open interval $U \subset I$. Then the minimizer u of $\mathcal{E}$ satisfies*

$$|u_x^s| \le |h_x^s| \text{ as Borel measures on } U. \tag{1.12}$$

Note that under the assumptions of Theorems 1.1, 1.2 and 1.3, F is indeed a convex function of linear growth.

Before we proceed to the discussion of proofs of the theorems above, let us explain the notation that we use for derivatives throughout the paper. For a suitably differentiable function w on I (in general, on Ω) we denote by w_x the derivative (in general, total derivative) with respect to the spatial variable x. On the other hand, we use the symbol D to denote derivatives of functions such as F defined on "abstract" spaces $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ *etc.* where w, w_x *etc.* take values. In the case of functions on one-dimensional "abstract" space such as f, we use $'$ instead of D. For example, in the case of equation (1.13) below, $(DF_\eta(u_x^{\varepsilon,\eta}))_x$ means the x-derivative of the composite function $DF_\eta \circ u_x^{\varepsilon,\eta}\colon I \to \mathbb{R}^n$.

The basic structure of the proofs of the three theorems is similar—a suitable two-level approximation $\mathcal{E}_{\varepsilon,\eta}$ of the functional $\mathcal{E}$ is used. On the more regular level, the minimizer $u^{\varepsilon,\eta}$ of $\mathcal{E}_{\varepsilon,\eta}$ satisfies the Euler–Lagrange equation

$$u^{\varepsilon,\eta} - h^\varepsilon = \lambda\,(DF_\eta(u_x^{\varepsilon,\eta}))_x \text{ in } I, \quad DF_\eta(u_x^{\varepsilon,\eta}) = 0 \text{ on } \partial I \tag{1.13}$$

in strong sense. We test (1.13) with a suitable function of the form $(\zeta^2 G(u_x^{\varepsilon,\eta}))_x$, where $\zeta \in C_c^1(I)$ and $G : \mathbb{R}^n \to \mathbb{R}$ will be chosen later, to obtain

$$\int_I (u^{\varepsilon,\eta} - h^\varepsilon)\cdot(\zeta^2 G(u_x^{\varepsilon,\eta}))_x \,\mathrm{d}\mathcal{L}^1 = \lambda \int_I (DF_\eta(u_x^{\varepsilon,\eta}))_x \cdot \left(\zeta^2 G(u_x^{\varepsilon,\eta})\right)_x \mathrm{d}\mathcal{L}^1,$$

which after integration by parts on the l. h. s. becomes

$$\int_I \zeta^2 (u_x^{\varepsilon,\eta} - h_x^\varepsilon)\cdot G(u_x^{\varepsilon,\eta}) \,\mathrm{d}\mathcal{L}^1 = -\lambda \int_I (DF_\eta(u_x^{\varepsilon,\eta}))_x \cdot \left(\zeta^2 G(u_x^{\varepsilon,\eta})\right)_x \mathrm{d}\mathcal{L}^1. \tag{1.14}$$

This integral equation is the starting point for obtaining our estimates. The main difficulty of this approach lies in the choice of the function G.

Firstly, we want the function G to satisfy

$$(G(w_x))_x \cdot (DF_\eta(w_x))_x \geq 0 \quad \text{for } w \in W^{2,2}(I, \mathbb{R}^n), \tag{1.15}$$

which enables us to estimate the r. h. s. of (1.14) by

$$-\lambda \int_I (DF_\eta(u_x^{\varepsilon,\eta}))_x \cdot \left(\zeta^2 G(u_x^{\varepsilon,\eta})\right)_x \,\mathrm{d}\mathcal{L}^1 \leq -\lambda \int_I 2\zeta\zeta_x G(u_x^{\varepsilon,\eta}) \cdot (DF_\eta(u_x^{\varepsilon,\eta}))_x \,\mathrm{d}\mathcal{L}^1. \tag{1.16}$$

Generally, we will aim to show that the r. h. s. of (1.14) is "small". It is essential for our argument to take G of the form $G(\xi) = g(\xi)\xi$ with $g\colon \mathbb{R}^n \to \mathbb{R}$. Due to such a choice, the bulk of terms on the r. h. s. of (1.14) vanishes owing to the identity $\xi \cdot D^2\psi(\xi) = 0$ for positively 1-homogeneous $\psi \in C^2(\mathbb{R}^n \setminus \{0\})$, see Proposition 2.7. This will be evident below, in the discussion of the homogeneous case $F = \varphi$. Nonetheless, we make heavy use of this property in the inhomogeneous case as well.

On the other hand, we need $G(\xi) \cdot \xi \sim |\xi|$ for large $|\xi|$ so that on the l. h. s. of (1.14) a quantity of order $|u_x^{\varepsilon,\eta}|$ appears. Dealing with the l. h. s. of (1.14) is easiest when $G = D\mathfrak{G}$ for some convex function $\mathfrak{G}$, due to the inequality $D\mathfrak{G}(p) \cdot (p - q) \geq \mathfrak{G}(p) - \mathfrak{G}(q)$ for $p, q \in \mathbb{R}^n$. Choosing such G is possible in the homogeneous case, however we were unable to find such a function in the inhomogeneous case.

Let us flesh out this idea on the example of Theorem 1.1. In order to explain the heuristic, we will assume that all functions are regular enough for the calculations (as is in the regularized setting), while $F = \varphi$ is positively 1-homogeneous (as is in the limit), ignoring the singularity at 0. We test the Euler–Lagrange equation (1.13) by $(\zeta^2 u_x/|u_x|)_x$ and integrate by parts on the l. h. s. obtaining

$$\int_I \zeta^2 \frac{u_x}{|u_x|} \cdot (h_x - u_x) = \lambda \int_I \left(\zeta^2 \frac{u_x}{|u_x|}\right)_x \cdot (D\varphi(u_x))_x.$$

By homogeneity and convexity of φ,

$$\begin{aligned}\left(\zeta^2 \frac{u_x}{|u_x|}\right)_x \cdot (D\varphi(u_x))_x &= 2\zeta\zeta_x \frac{u_x}{|u_x|} \cdot D^2\varphi(u_x)\, u_{xx} + \zeta^2 u_{xx} \cdot \frac{1}{|u_x|}\left(\mathrm{Id} - \frac{u_x}{|u_x|} \otimes \frac{u_x}{|u_x|}\right) D^2\varphi(u_x)\, u_{xx} \\ &= \zeta^2 u_{xx} \cdot \frac{1}{|u_x|} D^2\varphi(u_x)\, u_{xx} \geq 0.\end{aligned}$$

On the other hand, by convexity of the Euclidean norm,

$$\frac{u_x}{|u_x|} \cdot (h_x - u_x) \leq |h_x| - |u_x|.$$

Summing up, we obtain

$$\int_I \zeta^2 |u_x| \leq \int_I \zeta^2 |h_x|.$$

Since ζ is arbitrary, it follows that $|u_x| \leq |h_x|$ by virtue of Lemma 2.6.

In the actual proof in Section 4, we need to perform the calculations above on the regularized level, estimating lower order terms and minding singularities at $u_x = 0$. For that last reason, instead of $(\zeta^2 u_x/|u_x|)_x$, we will work with test functions of form

$$(\zeta^2 D\mathfrak{G}_k(u_x))_x, \quad \text{where } \mathfrak{G}_k(\xi) = (|\xi| - k)_+,\ k > 0 \tag{1.17}$$

(which need to be further smoothened), yielding $\mathfrak{G}_k(u_x) \le \mathfrak{G}_k(h_x)$. After relaxing regularization parameters, this leads to an inequality between measures which is of the same form, provided that we understand

$$\mathfrak{G}_k(w_x) = \int (|w_x^{ac}| - k)_+ \,\mathrm{d}\mathcal{L}^1 + |w_x^s| \tag{1.18}$$

for $w \in BV(I, \mathbb{R}^n)$. Passing to the limit $k \to 0^+$ we obtain the desired estimate.

In the inhomogeneous case $F = f \circ \varphi$, the argument becomes more involved and, expectedly, yields a bound only on the singular part of u_x. One of essential difficulties in the proof of such estimates is posed by very weak continuity properties of the operator $w \mapsto w_x^s$ on $BV(I, \mathbb{R}^n)$. Notably, maps such as $w \mapsto |w_x^s|(\Omega)$ fail to be semicontinuous w. r. t. weak* as well as strict or area-strict convergence in $BV(I, \mathbb{R}^n)$. Therefore, we need to work instead with quantities of type (1.18) which tend to $|w_x^s|$ as $k \to \infty$, and obtain asymptotic estimates on them. A natural approach would be to use test functions of form (1.17), however in this case property (1.15) fails. Instead, we need to use more complicated test functions of form $(\zeta^2 G_k(u_x))_x$ with $G_k(\xi) = g_k(\varphi(\xi))\xi$, which is no longer a derivative of a convex function. Thus, our procedure leads to estimates on non-convex functions of u_x which are not necessarily lower semicontinuous. Nonetheless, by exploiting equivalence between φ and $|\cdot|$, we can deduce estimates for a convex function of u_x, but only up to a multiplicative constant.

We note that such a constant is undesirable in the context of image processing, since in applications the minimization procedure tends to be iterated many times (in the spirit of the *minimizing movements scheme*—we recall that notion in Section 7). In the case when F is strictly convex and differentiable, we obtain a better estimate (1.12). In the proof we use spherical compactification of $\mathbb{R}^n$ (see Sect. 2) and the fact that the derivative DF is a homeomorphism to improve the convergence of approximate minimizers. This is one of the reasons why we need additional assumptions on F in this case. However, we do not know whether (1.12) can fail otherwise. Actually, one could expect (1.12) to hold for general F of linear growth without the structural assumption $F = f \circ \varphi$. However, our technique does not allow us to venture outside of this paradigm.

In the next section we recall some facts that will be used in the sequel. In Section 3 approximate functionals are defined and in Sections 4, 5, 6 we prove Theorems 1.1, 1.2 and 1.3, respectively. Finally, in Section 7 we discuss the issues of transferring our results to the gradient flow of $\mathcal{F}$ and extending them to more general fidelity terms of form $\int_\Omega \Phi(w-h)$ in place of $\frac{1}{2}\int_\Omega |w-h|^2$.

2. Preliminaries

Notation. Let d be any positive integer. Throughout the paper, $\varrho \in C_c^\infty(\mathbb{R}^d)$ denotes a positive, radially symmetric function supported on the open unit ball $B_1(0)$ such that $\int_{\mathbb{R}^d} \varrho \,\mathrm{d}\mathcal{L}^d = 1$ and $\varrho_\varepsilon(p) = \varepsilon^{-d}\varrho(p/\varepsilon)$. We will say that measures $\mu_j \in M(\Omega, \mathbb{R}^d)$ converge weakly* to $\mu \in M(\Omega, \mathbb{R}^d)$ and write $\mu_j \overset{*}{\rightharpoonup} \mu$, if $\lim_{j\to\infty} \int_\Omega \zeta \,\mathrm{d}\mu_j = \int_\Omega \zeta \,\mathrm{d}\mu$ for any $\zeta \in C_c(\Omega)$.

Functions of measures

Definition (1.4) from the Introduction can be applied to any Radon measure, not necessarily a derivative of a BV function. Throughout this subsection, G denotes a convex function $G\colon \mathbb{R}^d \to [0, \infty[$ of linear growth and Ω a bounded domain in $\mathbb{R}^m$. Given such a function G and a measure $\mu \in M(\Omega, \mathbb{R}^d)$, we define measure $G(\mu)$ by

$$\mathrm{d}G(\mu) = G(\mu^{ac})\,\mathrm{d}\mathcal{L}^m + G^\infty\left(\tfrac{\mu^s}{|\mu^s|}\right)\mathrm{d}|\mu^s|, \quad \text{where } G^\infty(\nu) = \lim_{t\to\infty} \tfrac{1}{t} G(t\nu) \text{ for } \nu \in \mathbb{S}_1^{n-1} \tag{2.1}$$

and $\mu^s/|\mu^s|$ denotes the Radon–Nikodym derivative. The function G^∞ is often called *the recession function* of G. In the following paragraphs, properties of $G(\mu)$ are collected.

The bulk of the proofs of the main theorems consists of estimating some integral quantities, which is done on the regularized level. Therefore, to learn about the quantity in question, which will be of the form $G(\mu)$ for some measure μ, we need lower semicontinuity and continuity properties of sequences of measures $G(\mu_j)$. In general it is not true that $\mu_j \overset{*}{\rightharpoonup} \mu$ implies $G(\mu_j) \overset{*}{\rightharpoonup} G(\mu)$, not even when $G = |\cdot|$, but Lemma 2.4 shows that it is so if μ_j are mollifications of the measure μ.

Theorem 2.1. *([2], p. 172) Let $\mu_j, \mu \in M(\Omega, \mathbb{R}^d)$ and assume that $\mu_j \overset{*}{\rightharpoonup} \mu$. Then for any open $U \subset \Omega$*

$$G(\mu)(U) \leq \liminf_{j\to\infty} G(\mu_j)(U).$$

Lemma 2.2. *Let $\mu_j, \mu \in M(\Omega, \mathbb{R}^d)$ and assume that $\mu_j \overset{*}{\rightharpoonup} \mu$. Then for any non-negative $\zeta \in C_c(\Omega)$*

$$\int_\Omega \zeta \, \mathrm{d}G(\mu) \leq \liminf_{j\to\infty} \int_\Omega \zeta \, \mathrm{d}G(\mu_j).$$

Proof. Suppose that $\liminf_{j\to\infty} \int_\Omega \zeta \, \mathrm{d}G(\mu_j) < \infty$ (if not, the inequality clearly holds). Choose a subsequence μ_{j_ℓ} such that

$$\lim_{\ell\to\infty} \int_\Omega \zeta \, \mathrm{d}G(\mu_{j_\ell}) = \liminf_{j\to\infty} \int_\Omega \zeta \, \mathrm{d}G(\mu_j).$$

Since G is of linear growth, the sequence of measures $\{G(\mu_{j_\ell})\}_\ell$ is bounded and hence there exists a subsequence (which we do not relabel) weakly* convergent to a measure ν. Then $\nu \geq G(\mu)$ as Borel measures on Ω (see *e. g.* [23], Lem. 2.1) and so, in view of the choice of the subsequence μ_{j_ℓ},

$$\int_\Omega \zeta \, \mathrm{d}G(\mu) \leq \int_\Omega \zeta \, \mathrm{d}\nu = \lim_{\ell\to\infty} \int_\Omega \zeta \, \mathrm{d}G(\mu_{j_\ell}) = \liminf_{j\to\infty} \int_\Omega \zeta \, \mathrm{d}G(\mu_j).$$

□

The Jensen-like inequality from Lemma 2.3 will be needed in the proof of Lemma 2.4.

Lemma 2.3. *([2], p. 171) Let $U \subset \mathbb{R}^m$ be an open set and $b\colon U \to [0, \infty[$ an integrable function with $\int_U b \, \mathrm{d}\mathcal{L}^m \neq 0$. For $\mu \in M(U, \mathbb{R}^d)$, it is true that*

$$G\left(\frac{\int_U b \, \mathrm{d}\mu}{\int_U b \, \mathrm{d}\mathcal{L}^m}\right) \leq \frac{\int_U b \, \mathrm{d}G(\mu)}{\int_U b \, \mathrm{d}\mathcal{L}^m}.$$

Lemma 2.4. *Suppose that $w \in L^1(\Omega) \cap BV(U)$ for some open set $U \subset\subset \Omega$ and let $w^\varepsilon = w * \varrho_\varepsilon$. Then, $G(w^\varepsilon_x) \overset{*}{\rightharpoonup} G(w_x)$ in $M(U)$.*

Proof. The statement follows from a small modification of [13], Theorem 2.2, requiring the use of Lemma 2.3. We provide the proof for the convenience of the reader.

On U, $w^\varepsilon_x = w_x * \varrho_\varepsilon$. Using Lemma 2.3 with $b(y) = \varrho_\varepsilon(x - y)$, we obtain that for any compact set $K \subset U$,

$$G(w^\varepsilon_x)(K) = \int_K G\left(\int_\Omega \varrho_\varepsilon(x-y) \, \mathrm{d}w_x(y)\right) \mathrm{d}x \leq \int_K \int_\Omega \varrho_\varepsilon(x-y) \, \mathrm{d}G(w_x)(y) \, \mathrm{d}x,$$

since $\int_\Omega \varrho_\varepsilon(x-y) \, \mathrm{d}y = 1$. Denote $K_\varepsilon := \{x : \mathrm{dist}(x, K) < \varepsilon\}$. Fubini's theorem yields

$$G(w^\varepsilon_x)(K) \leq \int_{K_\varepsilon} \int_K \varrho_\varepsilon(x-y) \, \mathrm{d}x \, \mathrm{d}G(w_x)(y) \leq G(w_x)(K_\varepsilon).$$

By continuity of measure $G(w_x)$, we get

$$\limsup_{\varepsilon\to 0} G(w_x^\varepsilon)(K) \le G(w_x)(K)$$

for any compact $K \subset U$. On the other hand, since $w_x^\varepsilon \overset{*}{\rightharpoonup} w_x$ on U (see [13], Thm. 2.2), by Theorem 2.1 we have

$$G(\mu)(V) \le \liminf_{j\to\infty} G(\mu_j)(V)$$

for any open $V \subset U$. We then conclude by appealing to the criterion for weak convergence of measures in [24], Section 1.9, Theorem 1. □

The next theorem establishes lower semicontinuity of the functional $\mathcal{F}(w) = F(w_x)(\Omega)$ w. r. t. L^1_{loc} convergence, crucial in ensuring existence of a unique minimizer of $\mathcal{E}$.

Theorem 2.5. *([2], Theorem 5) Let $G\colon \mathbb{R}^{m\times n} \to [0,\infty[$ be a convex function of linear growth and $w \in BV(\Omega,\mathbb{R}^n)$. Then,*

$$G(w_x)(\Omega) = \inf\left\{\liminf_{\ell\to\infty}\int_\Omega G(w_x^\ell)\,\mathrm{d}\mathcal{L}^m \colon w^\ell \in C^1(\Omega),\, w^\ell \to w \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\}.$$

In the proofs of the main theorems we obtain integral inequalities, from which we need to deduce information about the measures w. r. t. which we integrate in these integrals. This can be done thanks to Lemma 2.6.

Lemma 2.6. *Let μ, ν be positive Radon measures on an open set $U \subset \mathbb{R}^m$ which satisfy*

$$\int_U \zeta^2\,\mathrm{d}\mu \le \int_U \zeta^2\,\mathrm{d}\nu$$

for any $\zeta \in C^1_c(U)$. Then, $\mu \le \nu$ as Borel measures on U.

Proof. Given any open set $V \subset U$, it is possible to find an increasing family $\{K_i\}_i$ of compact sets contained in V whose union equals V. For each i choose a non-negative function $\zeta_i \in C^1_c(V)$ with $||\zeta_i||_\infty \le 1$ which equals 1 on K_i. The assumption of the lemma yields

$$\mu(K_i) \le \int_U \zeta_i^2\,\mathrm{d}\mu \le \int_U \zeta_i^2\,\mathrm{d}\nu = \int_V \zeta_i^2\,\mathrm{d}\nu \le \nu(V),$$

which by continuity of measure μ means that $\mu(V) \le \nu(V)$ for any open $V \subset U$. Now, if E is an arbitrary Borel set, then

$$\mu(E) = \inf\{\mu(V) \colon V \text{ open},\, E \subset V\} \le \inf\{\nu(V) : V \text{ open},\, E \subset V\} = \nu(E).$$

□

Positively 1-homogeneous and convex functions

A function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is called *positively* 1*-homogeneous* if

$$\varphi(tp) = t\varphi(p) \quad \text{for } p \in \mathbb{R}^n,\ t \ge 0.$$

We recollect several known identities satisfied by derivatives of positively 1-homogeneous functions, which are used throughout the paper.

Proposition 2.7. *Suppose that* $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ *is positively* 1*-homogeneous. If* φ *is differentiable on* $\mathbb{R}^n \setminus \{0\}$*, then*

$$\xi \cdot D\varphi(\xi) = \varphi(\xi) \text{ for } \xi \neq 0 \quad \text{and} \quad D\varphi(t\xi) = D\varphi(\xi) \text{ for } \xi \neq 0,\ t > 0.$$

If $\varphi \in C^2(\mathbb{R}^n \setminus \{0\})$*, then*

$$\xi \cdot D^2\varphi(\xi) = 0 \quad \text{for} \quad \xi \neq 0.$$

Proof. If φ is differentiable on $\mathbb{R}^n \setminus \{0\}$, $\xi \neq 0$ and $t > 0$, then

$$\xi \cdot D\varphi(t\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t\xi) = \frac{\mathrm{d}}{\mathrm{d}t}t\varphi(\xi) = \varphi(\xi).$$

Putting $t = 1$ we get $\xi \cdot D\varphi(\xi) = \varphi(\xi)$. On the other hand,

$$tD\varphi(t\xi) = D_\xi(\varphi(t\xi)) = tD\varphi(\xi).$$

Dividing both sides by t, we get $D\varphi(t\xi) = D\varphi(\xi)$. If $\varphi \in C^2(\mathbb{R}^n \setminus \{0\})$, it follows from this equality that

$$\xi \cdot D^2\varphi(t\xi) = \frac{\mathrm{d}}{\mathrm{d}t}D\varphi(t\xi) = \frac{\mathrm{d}}{\mathrm{d}t}D\varphi(\xi) = 0.$$

Putting $t = 1$ we conclude. □

Definition 2.8. Let $\psi\colon \mathbb{R}^n \to [0, \infty[$ be a positively 1-homogeneous function. It is convex if and only if for any two points $p, q \in \mathbb{R}^n$ it is true that

$$\psi(p + q) \leq \psi(p) + \psi(q). \tag{2.2}$$

Emulating Reshetnyak's seminal paper [25], we will say that a positively 1-homogeneous function ψ is *strictly convex in the sense of Reshetnyak* if and only if inequality (2.2) becomes an equality only if $p = 0$ or $q = tp$ for some $t \geq 0$.

In view of the structural assumption $F = f \circ \varphi$ in Theorems 1.2 and 1.3, the notion of strict convexity in the sense of Reshetnyak is central to understanding when such a composition is strictly convex, as shown by

Lemma 2.9. *Let* $\psi\colon \mathbb{R}^n \to [0, \infty[$ *be a positively* 1*-homogeneous function which vanishes only at the origin and* $g\colon [0, \infty[\to [0, \infty[$ *be strictly convex and increasing. If* ψ *is strictly convex in the sense of Reshetnyak, then* $G := g \circ \psi$ *is strictly convex, i. e.,*

$$G(\lambda p + (1 - \lambda)q) < \lambda G(p) + (1 - \lambda)G(q) \quad \text{for any } \lambda \in]0, 1[\text{ and distinct } p, q \in \mathbb{R}^n.$$

If ψ *is not strictly convex in the sense of Reshetnyak, then* G *is not strictly convex.*

Proof. Choose any distinct $p, q \in \mathbb{R}^n$ and $0 < \lambda < 1$ and assume strict convexity (in the sense of Reshetnyak) of ψ. If $p = 0$, then the statement holds since g is strictly convex and ψ vanishes at the origin (and only there). If $q = tp$ for $t > 0$ and $t \neq 1$, then $\psi(q) = t\psi(p) \neq \psi(p)$ and again it suffices to use strict convexity of g. At last, for points for which inequality (2.2) is strict, one can write

$$\begin{aligned} G\left(\lambda p + (1 - \lambda)q\right) &< g\left(\lambda\psi(p) + (1 - \lambda)\psi(q)\right) \\ &\leq \lambda g(\psi(p)) + (1 - \lambda)g(\psi(q)). \end{aligned}$$

The first inequality is strict as g under our assumptions is strictly increasing whereas the second is not as it might happen that $\psi(p) = \psi(q)$.

Suppose now that ψ is not strictly convex (in the sense of Reshetnyak) so that there exist non-zero points p, q such that $p \neq tq$ for any $t > 0$ for which inequality (2.2) yields equality. The same is true for $p/2$ and $q/2$, which implies that ψ has to be affine on the segment $[p, q]$. As a result, for $\lambda = (1 + \psi(p)/\psi(q))^{-1}$ it is true that

$$\psi\left(p + \frac{\psi(p)}{\psi(q)} q\right) = \lambda^{-1}\psi(\lambda p + (1-\lambda)q) = \psi(p) + \psi\left(\frac{\psi(p)}{\psi(q)} q\right).$$

Therefore, we have found points $\tilde{p} = p$, $\tilde{q} = (\psi(p)/\psi(q))q$ such that $\tilde{p} \neq t\tilde{q}$, whose ψ-norms coincide and for which inequality (2.2) turns into equality. Consequently, for any $\lambda \in (0, 1)$

$$G(\lambda\tilde{p} + (1-\lambda)\tilde{q}) = g(\lambda\psi(\tilde{p}) + (1-\lambda)\psi(\tilde{q})) = g(\psi(\tilde{p})) = \lambda G(\tilde{p}) + (1-\lambda)G(\tilde{q}),$$

which concludes the proof. □

We will next apply Lemma 2.9 to obtain conditions on f and φ equivalent to the additional assumptions of strict convexity and differentiability on $F = f \circ \varphi$ in Theorem 1.3.

Proposition 2.10. *Let $G = g \circ \psi$, where $\psi\colon \mathbb{R}^n \to [0, \infty[$ is a coercive anisotropy and $g\colon [0, \infty[\to [0, \infty[$ is convex and non-decreasing. Then*

(i) G is strictly convex if and only if ψ is strictly convex in the sense of Reshetnyak and g is strictly convex,
(ii) G is differentiable on $\mathbb{R}^n$ if and only if ψ is differentiable on $\mathbb{R}^n \setminus \{0\}$, g is differentiable on $[0, \infty[$ and $g'(0) = 0$.

Before proving the proposition, let us address several points related to differentiability of convex functions in general. First of all, we recall that a convex function on an open convex set is differentiable if and only if it is C^1 [26], Corollary 25.5.1. Thus, in (ii) we could equivalently require $G \in C^1(\mathbb{R}^n)$. By the same token, differentiability of ψ on $\mathbb{R}^n \setminus \{0\}$ is equivalent to $\psi \in C^1(\mathbb{R}^n \setminus \{0\})$, since $\mathbb{R}^n \setminus \{0\}$ is a finite sum of open convex sets. We also clarify that by differentiability of $g\colon [0, \infty[\to [0, \infty[$ at 0 we mean that the right-sided derivative at 0 exists, and $g'(0)$ stands for that derivative. In fact, for g convex and non-decreasing, existence of the (right-sided) derivative at 0 is implied by differentiability on $]0, \infty[$. Moreover, under such assumption g' is continuous at 0 and $g \in C^1([0, \infty[)$.

Proof. If ψ is strictly convex in the sense of Reshetnyak and g is strictly convex, then strict convexity of G follows by Lemma 2.9. Conversely, suppose that G is strictly convex. Given $a, b \in [0, \infty[$, $a < b$, take $x \in \mathbb{R}^n$ and $t \in [0, 1[$ such that $\psi(tx) = a$, $\psi(x) = b$. Then, for $\lambda \in]0, 1[$,

$$\begin{aligned} g(\lambda a + (1-\lambda)b) = g(\lambda\psi(tx) + (1-\lambda)\psi(x)) = g(\psi(\lambda tx + (1-\lambda)x)) = G(\lambda tx + (1-\lambda)x)) \\ < \lambda G(tx) + (1-\lambda)G(x) = \lambda g(a) + (1-\lambda)g(b), \end{aligned}$$

i. e., g is strictly convex. With this settled, it follows by Lemma 2.9 that ψ is strictly convex in the sense of Reshetnyak.

If ψ is differentiable on $\mathbb{R}^n \setminus \{0\}$ and g is differentiable on $[0, \infty[$, then G is differentiable on $\mathbb{R}^n \setminus \{0\}$. If moreover $g'(0) = 0$, one can check directly from the definition that G is differentiable at 0 (and $DG(0) = 0$). Conversely, suppose that G is differentiable. For any fixed $x \in \mathbb{R}^n \setminus \{0\}$, $G^x\colon t \mapsto G(tx) = g(\psi(x)t)$ is differentiable on $[0, \infty[$, and

$$x \cdot DG(0) = (G^x)'(0) = g'(0)\psi(x), \quad -x \cdot DG(0) = (G^{-x})'(0) = g'(0)\psi(-x).$$

Since $\psi(x)$ and $\psi(-x)$ are both positive numbers, we infer that $g'(0) = 0$. Next, by assumptions on g, there exists $t_0 > 0$ such that $0 < g'(t_0) \le g'(t)$ for $t \ge t_0$. Thus g restricted to $[t_0, \infty[$ is invertible and g^{-1} is differentiable on $[g(t_0), \infty[$. Thus we have $\psi = g^{-1} \circ G$ and consequently ψ is differentiable outside of a certain ball. By homogeneity of ψ, we deduce that it is differentiable on $\mathbb{R}^n \setminus \{0\}$. □

Lemmata 2.11 and 2.12 collect facts about convex functions of linear growth on the real line. We will use them further in this subsection to define suitable approximations of an anisotropy and also in the course of the proofs of main theorems. The uniform estimate on f'_η in Lemma 2.12 (ii) will turn out to be especially useful.

Lemma 2.11. *Given a convex, non-decreasing $f\colon [0,\infty[\to [0,\infty[$ of linear growth, set*

$$f^\infty := \lim_{t\to\infty} \frac{f(t)}{t}. \tag{2.3}$$

Then,

(i) the recession function defined in (2.1) equals $f^\infty(t) = f^\infty\, t$;
(ii) f is Lipschitz continuous with Lipschitz constant f^∞.

Additionally, assume that $f \in C^1([0,\infty[)$. Then

(iii) $\sup_{t\in]0,\infty[} f'(t) = \lim_{t\to\infty} f'(t) = f^\infty$ and f' is uniformly continuous on $[0,\infty[$.

Proof. Property (i) follows from the definition (2.1). Let us show validity of (ii). Take any $t > s \ge 0$ and any $r > t$. If follows from convexity of f that

$$0 \le \frac{f(t) - f(s)}{t - s} \le \frac{f(2r) - f(r)}{r}, \tag{2.4}$$

which after taking $r \to \infty$ and multiplying by $(t - s)$ gives $f(t) - f(s) \le f^\infty(t - s)$, as required.

We now assume that $f \in C^1([0,\infty[)$. By a reasoning similar to (2.4), we obtain that for any $t \in [0,\infty[$ and sufficiently small $h > 0$, convexity of f means that

$$\frac{1}{t}(f(t) - f(0)) \le \frac{1}{h}(f(t+h) - f(t)) \le \frac{1}{t}(f(2t) - f(t)).$$

Taking limit with $h \to 0^+$ and then with $t \to \infty$ implies that $\lim_{t\to\infty} f'(t) = f^\infty$. Since f' is non-decreasing, $\sup_{t\in]0,\infty[} f'(t) = \lim_{t\to\infty} f'(t)$, as required. We conclude by recalling that a continuous function on $[0,\infty[$ that has a limit at ∞ is uniformly continuous. □

Lemma 2.12. *Given f as in Lemma 2.11, we define the function $f_\eta : [0,\infty[\to [0,\infty[$ with the formula*

$$f_\eta(t) := \big(f(t)\mathbf{1}_{\{t>\eta\}} + f(\eta)\mathbf{1}_{\{t\le\eta\}}\big) * \varrho_{\eta/2}. \tag{2.5}$$

Then,

(i) f_η is non-decreasing, convex and satisfies

$$f(t) \le f_\eta(t) \le f^\infty t + f(\eta), \tag{2.6}$$

which implies that f_η is of linear growth and $f_\eta^\infty = f^\infty$;
(ii) for any $\eta > 0$, $\sup_{t\in]0,\infty[} f'_\eta(t) = \lim_{t\to\infty} f'_\eta(t) = f^\infty$;
(iii) f_η converge uniformly to f on $[0,\infty[$.

Additionally, assume that f is differentiable with $f'(0) = 0$. Then *(iv)* f_η' *converge uniformly to* f'.

Proof. The function $g_\eta(t) := f(t)\mathbf{1}_{\{t>\eta\}} + f(\eta)\mathbf{1}_{\{-\eta\le t\le\eta\}}$ is convex and non-decreasing, hence its mollification, f_η, is also convex and nondecreasing. Observe that $f \le g_\eta$ so to show the left inequality in (2.6), it suffices to show that $g_\eta \le f_\eta$. Writing $t = 1/2(t-s) + 1/2(t+s)$ and using convexity of g_η yields

$$g_\eta(t) \le \frac{1}{2} g_\eta(t-s) + \frac{1}{2} g_\eta(t+s),$$

which after multiplying by $\varrho_{\eta/2}(s)$ and integrating w. r. t. s becomes

$$g_\eta(t) \le \frac{1}{2}\int_{\mathbb{R}} g_\eta(t-s)\varrho_{\eta/2}(s)\,\mathrm{d}s + \frac{1}{2}\int_{\mathbb{R}} g_\eta(t+s)\varrho_{\eta/2}(s)\,\mathrm{d}s = \int_{\mathbb{R}} g_\eta(t-s)\varrho_{\eta/2}(s)\,\mathrm{d}s = f_\eta(t). \tag{2.7}$$

In the second to last inequality, we used the fact that $\varrho_{\eta/2}$ is radially symmetric and hence changing variables $s \mapsto -s$ in $\int_{\mathbb{R}} g_\eta(t+s)\varrho_{\eta/2}(s)\,\mathrm{d}s$ gives $\int_{\mathbb{R}} g_\eta(t-s)\varrho_{\eta/2}(s)\,\mathrm{d}s$. Let us stress that this reasoning follows from convexity of g_η and radial symmetry of ϱ and would work just as well if g_η were a convex function on $\mathbb{R}^n$ for $n > 1$. All in all, we showed that $f_\eta \ge f$.

We shall now show that the right inequality of (2.6) holds by firstly showing it for g_η in place of f_η. For $t \in [0,\eta]$, $g_\eta(t) = f(\eta)$, so it immediately follows from the fact that $t \ge 0$. For $t > \eta$, $g_\eta(t) = f(t) \le f^\infty t + f(0)$, since f is Lipschitz with constant f^∞ as shown in the Lemma 2.11 (ii). Moreover, $g_\eta^\infty = f^\infty$. Therefore, again by Lemma 2.11 (ii), we conclude that g_η is Lipschitz with constant f^∞. Consequently, for any $t, r \in [0,\infty[$

$$|f_\eta(t) - f_\eta(r)| \le \int_{\mathbb{R}} |g_\eta(t-s) - g_\eta(r-s)|\varrho_{\eta/2}(s)\,\mathrm{d}s \le f^\infty|t-s|,$$

which implies that for any $t \ge 0$, $f_\eta(t) - f_\eta(0) \le f^\infty t$ and finishes the proof of property (i).

We have shown that f_η for any $\eta > 0$ is convex, non-decreasing, of linear growth and $f_\eta^\infty = f^\infty$, so by Lemma 2.11 (iii), property (ii) holds.

To prove (iii), observe that for $t > \frac{3}{2}\eta$, $f_\eta(t) = f * \varrho_{\eta/2}(t)$ so we can compute using the Lipschitz constant f^∞

$$f_\eta(t) - f(t) \le \int_{\mathbb{R}} |f(t-s) - f(s)|\varrho_{\eta/2}(s)\,\mathrm{d}s \le f^\infty \int_{\mathbb{R}} s\,\varrho_{\eta/2}(s)\,\mathrm{d}s \le f^\infty\eta/2. \tag{2.8}$$

On the other hand, for $t \in [0, 3/2\eta]$ we can write

$$f_\eta(t) - f(t) \le f_\eta(2\eta) - f(t) \le (f_\eta(2\eta) - f(2\eta)) + (f(2\eta) - f(t)) \le \frac{5\eta}{2} f^\infty. \tag{2.9}$$

Collecting (2.8) and (2.9) shows that as $\eta \to 0^+$, f_η converges uniformly to f on $[0,\infty[$.

We will now see that uniform convergence of f_η' to f' essentially follows from uniform continuity of f' shown in Lemma 2.11 (iii). Fix $\varepsilon > 0$ and choose $\delta > 0$ so that if $|t-s| < \delta$, then $|f'(t) - f'(s)| < \varepsilon/2$. Choose $\eta_0 < \delta/3$, we claim that for any $\eta < \eta_0$, $t \in \mathbb{R}$,

$$|f_\eta'(t) - f'(t)| < \varepsilon. \tag{2.10}$$

If $t < \eta/2$, $f_\eta'(t) = 0$ and $f'(t) < \varepsilon/2$, hence (2.10) holds. If $t > \frac{3}{2}\eta$, $f_\eta'(t) = f' * \varrho_{\eta/2}$. It follows directly from the definition of convolution that (2.10) is satisfied for $\varepsilon/2$ in place of ε. Lastly, if $t \in [\frac{\eta}{2}, \frac{3}{2}\eta]$, then $f'(t) < \varepsilon$.

Moreover, by monotonicity and uniform continuity of f' and the already discussed case of (2.10) for $t > \frac{3}{2}\eta$, we have

$$f_\eta'(t) \le f_\eta'(2\eta) \le f'(2\eta) + \varepsilon/2 < \varepsilon.$$

Therefore, $|f_\eta'(t) - f'(t)| \le \max\{f_\eta'(t), f'(t)\} < \varepsilon$, which concludes the proof. □

The remaining lemmata in this subsection are devoted to constructing a suitable sequence of regular anisotropies approximating a given anisotropy φ. To begin with, we use a slightly modified mollification procedure which retains homogeneity and which has been described by Schneider in his monograph on convex bodies [27].

Lemma 2.13. *([27], Theorem 3.3.1) Take $\varrho(p) := \rho(|p|)$ with $\rho\colon [0,\infty[\to [0,\infty[$ being a smooth function with support contained in $[1/2, 1]$ such that $\int_{\mathbb{R}^n} \rho(|z|)\,\mathrm{d}z = 1$. For any anisotropy φ, the function*

$$\overline{\varphi}_\eta(p) := \int_{\mathbb{R}^n} \varphi(p + |p|z)\varrho_\eta(z)\,\mathrm{d}z \tag{2.11}$$

is an anisotropy of class $C^\infty(\mathbb{R}^n \setminus \{0\})$.

Remark 2.14. We will need the following two facts about $\overline{\varphi}$ in the proof of Proposition 2.15.

- (i) Observe that for all $\eta > 0$ and $p \in \mathbb{R}^n$, $\overline{\varphi}_\eta(p) \ge \varphi(p)$. Indeed, due to homogeneity of φ and $\overline{\varphi}_\eta$ it suffices to check this inequality for $|p| = 1$. The fact that it holds follows from the same reasoning as in (2.7), as indicated in the proof of Lemma 2.12.
- (ii) Moreover, $\overline{\varphi}_\eta$ converges to φ locally uniformly. By uniform continuity of φ on a given compact set K, for any $\varepsilon > 0$, there is $\eta > 0$ s. t. if $|z| < \eta$, then $|(p + |p|z) - p|$ is sufficiently small to guarantee that

$$|\varphi(p) - \varphi(p + |p|z)| < \varepsilon.$$

Therefore, for any $\varepsilon > 0$, there is $\eta > 0$ so that for all $p \in K$

$$|\overline{\varphi}_\eta(p) - \varphi(p)| \le \int_{\mathbb{R}^n} |\varphi(p) - \varphi(p + |p|z)|\varrho_\eta(z)\,\mathrm{d}z \le \varepsilon.$$

In the next lemma, we modify $\overline{\varphi}_\eta$ to get $\widetilde{\varphi}_\eta$ with the property that $D^2\left(\frac{1}{2}\widetilde{\varphi}_\eta^2\right) \ge \eta \mathrm{Id}$, which will be crucial to obtaining suitable estimates later on.

Proposition 2.15. *Let φ be an anisotropy on $\mathbb{R}^n$. There exists a sequence of anisotropies $\{\widetilde{\varphi}_\eta\}_\eta$ with the following properties:*

- *(i) $\widetilde{\varphi}_\eta \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $D^2\left(\frac{1}{2}\widetilde{\varphi}_\eta^2\right) \ge \eta \mathrm{Id}$ in $\mathbb{R}^n \setminus \{0\}$;*
- *(ii) $\widetilde{\varphi}_\eta \ge \varphi$, $\widetilde{\varphi}_\eta$ converge to φ locally uniformly on $\mathbb{R}^n$ and $c^+_{\widetilde{\varphi}_\eta} \to c^+_\varphi$ as $\eta \to 0$, where $c^+_{\widetilde{\varphi}_\eta}$ and c^+_φ are defined in (1.9).*

Assume additionally that $\varphi \in C^1(\mathbb{R}^n \setminus \{0\})$. Then,

- *(iii) $D\widetilde{\varphi}_\eta$ converges to $D\varphi$ uniformly on $\mathbb{R}^n \setminus \{0\}$.*

If $f\colon [0,\infty[\to [0,\infty[$ is strictly convex, increasing, of linear growth and differentiable with $f'(0) = 0$, then

- *(iv) $f_\eta \circ \widetilde{\varphi}_\eta$ and $f \circ \varphi$ is differentiable on $\mathbb{R}^n$ and $D(f_\eta \circ \widetilde{\varphi}_\eta)$ converges to $D(f \circ \varphi)$ uniformly on $\mathbb{R}^n$, where f_η is defined in (2.5).*

Anisotropies which satisfy condition (i) are often referred to as anisotropies of class C^∞_+ (see [27], [28]).

Proof. We define

$$\widetilde{\varphi}_\eta(p) := \sqrt{\overline{\varphi}_\eta^2(p) + \eta|p|^2},$$

where $\overline{\varphi}_\eta$ is given by (2.11). Condition (i) is clearly satisfied whereas $\widetilde{\varphi}_\eta \geq \varphi$ follows from the fact that $\overline{\varphi}_\eta \geq \varphi$ as shown in Remark 2.14 (i). Local uniform convergence of $\widetilde{\varphi}_\eta$ to φ holds since $\overline{\varphi}_\eta$ converges to φ locally uniformly as well as shown in Remark 2.14 (ii). Therefore, as $\widetilde{\varphi}_\eta$ converge to φ uniformly on the unit sphere $\mathbb{S}^{n-1}_1$, for any $\delta > 0$, there exists η_0 s. t. for all $\eta \leq \eta_0$ and for any $p \in \mathbb{R}^n \setminus \{0\}$, it is true that

$$\widetilde{\varphi}_\eta\left(\frac{p}{|p|}\right) \leq \varphi\left(\frac{p}{|p|}\right) + \delta.$$

In view of the already shown fact that $\widetilde{\varphi}_\eta \geq \varphi_\eta$, we get $c^+_\varphi \leq c^+_{\widetilde{\varphi}_\eta} \leq c^+_\varphi + \delta$, which yields the required claim.

The rest of the proof is devoted to the regular case. As shown in Proposition 2.7, the derivative of a positively 1-homogeneous function is 0-homogeneous, hence it is enough to check convergence over the unit sphere. Passing with the differentiation under the integral sign gives for $|p| = 1$

$$D\overline{\varphi}_\eta(p) = \int_{\mathbb{R}^n} D\varphi(p+z)\,(\mathrm{Id} + z \otimes p)\,\varrho_\eta(|z|)\,\mathrm{d}z.$$

Uniform continuity of $D\varphi$ on compact sets implies that $D\overline{\varphi}_\eta$ converges uniformly to $D\varphi$. Then, simple calculations lead to the conclusion that property (iii) is satisfied.

To prove (iv), we start by showing that $f' \circ \widetilde{\varphi}_\eta$ converges to $f' \circ \varphi$ uniformly on $\mathbb{R}^n$. By Lemma 2.11 (iii) we know that for any $\varepsilon > 0$ there exists an M such that if only $t \geq M$, then $|f^\infty - f'(t)| \leq \varepsilon/2$. When $\varphi(p) \geq M$, then by (ii) $\widetilde{\varphi}_\eta(p) \geq M$ and as a result we get

$$|f'(\widetilde{\varphi}_\eta(p)) - f'(\varphi(p))| \leq |f'(\widetilde{\varphi}_\eta(p)) - f^\infty| + |f'(\varphi(p)) - f^\infty| \leq \varepsilon.$$

On the other hand, on the compact ball $\{p\colon \varphi(p) \leq M\}$ functions $\widetilde{\varphi}_\eta$ converge uniformly to φ. Consequently, there exists η_0 such that for all $\eta \leq \eta_0$ and p with $\varphi(p) \leq M$ we have $|\widetilde{\varphi}_\eta(p) - \varphi(p)| \leq \delta$, for a δ ensuring that $|f'(\widetilde{\varphi}_\eta(p)) - f'(\widetilde{\varphi}_\eta(p))| < \varepsilon$ due to uniform continuity of f' shown in Lemma 2.11 (iii). This finishes the argument that $f' \circ \widetilde{\varphi}_\eta$ converges to $f' \circ \varphi$ uniformly.

Next, let us see that $f'_\eta \circ \widetilde{\varphi}_\eta$ converges uniformly to $f' \circ \varphi$. By adding and subtracting $f'(\widetilde{\varphi}_\eta(p))$, we get

$$|f'_\eta(\widetilde{\varphi}_\eta(p)) - f'(\varphi(p))| \leq |f'_\eta(\widetilde{\varphi}_\eta(p)) - f'(\widetilde{\varphi}_\eta(p))| + |f'(\widetilde{\varphi}_\eta(p)) - f'(\varphi(p))|.$$

Uniform convergence of the first term follows from Lemma 2.12 (iv), whereas the fact that the second term converges uniformly to zero was shown in the paragraph above.

It follows directly from the definition of differentiability that f_η and f cancel out the singularity at the origin that $\widetilde{\varphi}_\eta$ and φ have. Therefore, $f_\eta \circ \widetilde{\varphi}_\eta$ and $f \circ \varphi$ are differentiable at the origin and $D(f_\eta \circ \widetilde{\varphi}_\eta)(0) = D(f \circ \varphi)(0) = 0$. Take any $p \neq 0$ and observe that by adding and subtracting $f'_\eta(\widetilde{\varphi}_\eta(p))D\varphi(p)$

$$|D(f_\eta \circ \widetilde{\varphi}_\eta)(p) - D(f \circ \varphi)(p)| \leq f'_\eta(\widetilde{\varphi}_\eta(p))|D\widetilde{\varphi}_\eta(p) - D\varphi(p)| + |f'_\eta(\widetilde{\varphi}_\eta(p)) - f'(\varphi(p))||D\varphi(p)|.$$

By Lemma 2.12 (ii), $f'_\eta(\widetilde{\varphi}_\eta(p)) \leq f^\infty$, hence by property (iii) of this proposition, the first term converges uniformly to zero. Since $|D\varphi|$ is bounded and we have shown in the previous paragraph that $f'_\eta \circ \widetilde{\varphi}_\eta$ converges uniformly to $f' \circ \varphi$, the second term also converges uniformly to zero, which concludes the proof. ☐

Since functions $\widetilde{\varphi}_\eta^2$ may not be C^2 near the origin, while defining the approximate functionals in Section 3, we will replace them with functions $\widetilde{\widetilde{\varphi}}_\eta$ which are C^2 on the whole space and coincide with $\widetilde{\varphi}_\eta^2$ outside a small ball centered at the origin.

Proposition 2.16. *There exists a convex function $\widetilde{\widetilde{\varphi}}_\eta \in C^2(\mathbb{R}^n)$ such that*

(i) $\widetilde{\widetilde{\varphi}}_\eta = \widetilde{\varphi}_\eta^2$ *outside* $B_\eta(0)$,
(ii) $D^2\left(\frac{1}{2}\widetilde{\widetilde{\varphi}}_\eta\right) \geq \frac{\eta}{2}\mathrm{Id}$,
(iii) $\widetilde{\widetilde{\varphi}}_\eta(p) \leq C(1+|p|^2)$ *for sufficiently small* $\eta > 0$ *with* $C > 0$ *independent of* η.

If moreover $\varphi \in C^1(\mathbb{R}^n \setminus \{0\})$, *then*

(iv) $|D\widetilde{\widetilde{\varphi}}_\eta(p)| \leq C(1+|p|)$ *for sufficiently small* $\eta > 0$ *with* $C > 0$ *independent of* η.

The last part of the assertion can be showed to hold also without the differentiability assumption on φ, however we will only use it in the regular case.

Proof. Existence of such a function follows from [29], Thoerem 2.1. For completeness, we prove this result in our setting and obtain the additional estimates (iii) and (iv).

Firstly, let us stress that $\widetilde{\varphi}_\eta^2$ is convex, as a composition of an anisotropy and quadratic function. Set $g_\varepsilon := \widetilde{\varphi}_\eta^2 * \varrho_\varepsilon$ for a sufficiently small ε chosen in the course of the proof. As convolutions of convex functions remain convex, g_ε is convex. Recall that by Proposition 2.15 (i), $D^2(\frac{1}{2}\widetilde{\varphi}_\eta^2) \geq \eta\mathrm{Id}$. Since $D^2 g_\varepsilon$ converges uniformly to $D^2\widetilde{\varphi}_\eta^2$ on $\overline{B}_{\eta/2}(0)$, by choosing sufficiently small ε, we can guarantee that $D^2 g_\varepsilon \geq \eta/2\,\mathrm{Id}$ on $\overline{B}_{\eta/2}(0)$. To smooth $\widetilde{\varphi}_\eta^2$ near zero, we set $\widetilde{\widetilde{\varphi}}_\eta := g_\varepsilon$ on $B_{\eta/2}(0)$.

Then, we will glue g_ε with $\widetilde{\varphi}_\eta^2$ along the annulus $B_\eta(0)\setminus B_{\eta/2}(0)$. To this end, we take a function $\lambda \in C_c^\infty(\mathbb{R}^n)$ s. t. $\lambda = 1$ on $B_{\eta/2}(0)$, $\lambda = 0$ outside $B_\eta(0)$ and $\sup_{p\in\overline{B}_\eta(0)}|D\lambda(p)| \leq 4/\eta$ and set

$$\widetilde{\widetilde{\varphi}}_\eta(p) := \begin{cases} \widetilde{\varphi}_\eta^2(p) & \text{for } p \in \mathbb{R}^n \setminus B_\eta(0),\\ \lambda(p)g_\varepsilon(p) + (1-\lambda(p))\widetilde{\varphi}_\eta^2 & \text{for } p \in B_\eta(0)\setminus B_{\eta/2}(0),\\ g_\varepsilon(p) & \text{for } p \in B_{\eta/2}(0). \end{cases}$$

We compute

$$D\widetilde{\widetilde{\varphi}}_\eta = (g_\varepsilon - \widetilde{\varphi}_\eta^2)D\lambda + \lambda Dg_\varepsilon + (1-\lambda)D\widetilde{\varphi}_\eta^2$$

and

$$D^2\widetilde{\widetilde{\varphi}}_\eta = D^2\widetilde{\varphi}_\eta^2 + \lambda\left(D^2 g_\varepsilon - D^2\widetilde{\varphi}_\eta^2\right) + D\lambda \otimes \left(Dg_\varepsilon - D\widetilde{\varphi}_\eta^2\right) + \left(Dg_\varepsilon - D\widetilde{\varphi}_\eta^2\right)\otimes D\lambda + (g_\varepsilon - \widetilde{\varphi}_\eta^2)D^2\lambda.$$

On the annulus $\overline{B}_\eta(0)\setminus B_{\eta/2}(0)$, $D^2\widetilde{\varphi}_\eta^2 \geq 2\eta\,\mathrm{Id}$. Since $(D^2g_\varepsilon - D^2\widetilde{\varphi}_\eta^2)$, $(Dg_\varepsilon - D\widetilde{\varphi}_\eta^2)$, $(g_\varepsilon - \widetilde{\varphi}_\eta^2)$ converge uniformly to 0 as $\varepsilon \to 0$, it is possible to choose a sufficiently small ε to guarantee that

$$\sup_{p\in\overline{B}_\eta(0)} |g_\varepsilon(p) - \widetilde{\varphi}_\eta^2(p)| \leq \eta/4 \quad \text{and} \quad D^2\widetilde{\widetilde{\varphi}}_\eta \geq \eta\mathrm{Id}. \tag{2.12}$$

The second condition implies that $\widetilde{\widetilde{\varphi}}_\eta$ is convex on $\mathbb{R}^n$. It follows from the construction that $\widetilde{\widetilde{\varphi}}_\eta$ is C^2 and $\widetilde{\widetilde{\varphi}}_\eta = \widetilde{\varphi}_\eta^2$ outside $B_\eta(0)$.

Therefore, for $p \in \mathbb{R}^n \setminus B_\eta(0)$, $\widetilde{\widetilde{\varphi}}_\eta(p) \le (c^+_{\widetilde{\varphi}_\eta})^2|p|^2$. By Proposition 2.15 (ii), we can assume that $\widetilde{\varphi}_\eta(p) \le 2c^+_\varphi|p|$ for sufficiently small $\eta > 0$. Consequently,

$$\widetilde{\widetilde{\varphi}}_\eta(p) \le 4(c^+_\varphi)^2|p|^2 \quad \text{for } p \in \mathbb{R}^n \setminus B_\eta(0).$$

It follows from locally uniform convergence of $\widetilde{\varphi}^2_\eta$, see Proposition 2.15, and (2.12) that $\widetilde{\widetilde{\varphi}}_\eta$ converge locally uniformly to φ^2. Consequently, for sufficiently small $\eta > 0$ for all $p \in \overline{B}_\eta(0)$,

$$\widetilde{\widetilde{\varphi}}_\eta(p) \le \varphi^2(p) + 1 \le (c^+_\varphi)^2(|p|^2 + 1).$$

This shows that property (iii) holds for $C = (2c^+_\varphi)^2$.

Next, suppose that $\varphi \in C^1(\mathbb{R}^n \setminus \{0\})$. For all $p \in \mathbb{R}^n \setminus \{0\}$, $D\widetilde{\varphi}^2_\eta(p) = 2\widetilde{\varphi}_\eta(p)D\widetilde{\varphi}_\eta(p)$. By Proposition 2.15 (ii) and (iii), for sufficiently small $\eta > 0$ we can assume that $|D\widetilde{\varphi}_\eta(p)| \le 2|D\varphi(p)|$ and $\widetilde{\varphi}_\eta(p) \le 2c^+_\varphi|p|$. This yields

$$|D\widetilde{\varphi}^2_\eta(p)| \le C|p|,$$

where $C := 8c^+_\varphi \sup_{p\in\mathbb{R}^n} |D\varphi(p)|$ is finite by continuity and 0-homogeneity of $D\varphi$ (see Prop. 2.7). As $\widetilde{\widetilde{\varphi}}_\eta$ coincides with $\widetilde{\varphi}^2_\eta$ outside $\overline{B}_\eta(0)$, it remains to estimate $D\widetilde{\widetilde{\varphi}}_\eta$ on $\overline{B}_\eta(0)$. As Dg_ε converges uniformly to $D\widetilde{\varphi}^2_\eta$ on $\overline{B}_\eta(0)$, Dg_ε is also bounded by a constant independent of η. On $B_{\eta/2}(0)$, $D\widetilde{\widetilde{\varphi}} = Dg_\varepsilon$ and therefore $D\widetilde{\widetilde{\varphi}}$ is bounded independently of η. On the annulus $\overline{B}_\eta(0) \setminus B_{\eta/2}(0)$, we can estimate using (2.12) and the fact that $|D\lambda| \le 4/\eta$:

$$\sup_{p\in\overline{B}_\eta(0)} |D\widetilde{\widetilde{\varphi}}_\eta(p)| \le 1 + \sup_{p\in\overline{B}_\eta(0)} |\lambda(p)Dg_\varepsilon(p) + (1-\lambda(p))D\widetilde{\varphi}^2_\eta(p)|,$$

which is uniformly bounded for small $\eta > 0$. Thus, increasing C if needed, we conclude. □

Spherical compactification

The following construction of $\overline{\mathbb{R}^n}$, compactification of $\mathbb{R}^n$, will be used in Section 6 only. We shall see there that functions which are not necessarily continuous as functions from $I \subset \mathbb{R}$ to $\mathbb{R}^n$ may be continuous as functions from $I \subset \mathbb{R}$ to $\overline{\mathbb{R}^n}$. This will enable us to perform a more subtle limit passage in the regular case of Theorem 1.3 than in the general case of Theoem 1.2.

Definition 2.17. Let $\mathbb{S}^{n-1}_\infty$ denote the *sphere at infinity*, *i. e.*, a set of elements denoted by $\infty\,\omega$ for $\omega \in \mathbb{S}^{n-1}_1$; that is $\mathbb{S}^{n-1}_\infty := \{\infty\,\omega \colon \omega \in \mathbb{S}^{n-1}_1\}$. Then, let $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \mathbb{S}^{n-1}_\infty$. Furthermore, define $\Phi\colon \overline{\mathbb{R}^n} \to \overline{B}_1(0)$ with the formula

$$\Phi(p) := \begin{cases} \frac{p}{1+|p|} & \text{if } p \in \mathbb{R}^n, \\ \omega & \text{if } p = \infty\,\omega \in \mathbb{S}^{n-1}_\infty. \end{cases}$$

We call $\overline{\mathbb{R}^n}$ equipped with the metric

$$\overline{d}(p,q) := |\Phi(p) - \Phi(q)|$$

the *spherical compactification of* $\mathbb{R}^n$.

The space $\overline{\mathbb{R}^n}$ with topology induced by $\overline{d}$ is indeed compact since Φ is a homeomorphism between this space and the closed unit ball with the Euclidean subspace topology. Clearly, any point $p \in \mathbb{R}^n \setminus \{0\}$ can be uniquely represented as $p = r\omega$ with $\omega \in \mathbb{S}^{n-1}$ and $r > 0$. If one looks at a sequence of points $p_j = r_j\omega_j$, it is evident that it converges to a point $p = \infty\omega \in \mathbb{S}^{n-1}_\infty$ if and only if $r_j \to \infty$ and $\omega_j \to \omega$. On the other hand, if a sequence of points p_j converges in $\overline{d}$ to a point $p \in \mathbb{R}^n$, then for all but a finite number of j, p_j lies in $\mathbb{R}^n$ as well. Moreover, uniform continuity of Φ^{-1} implies that in such case $|p_j - p| \to 0$. Additionally, since Φ is 1-Lipschitz, a sequence of points $p_j \in \mathbb{R}^n$ which converges in Euclidean distance to $p \in \mathbb{R}^n$, is also convergent to p in $\overline{\mathbb{R}^n}$ with metric $\overline{d}$.

The purpose of the next two lemmata is to show that given $F = f \circ \varphi$ as in Theorem 1.3, it is possible to extend, in a natural way, its derivative DF to $\overline{\mathbb{R}^n}$ and that this extension, $\overline{DF}$, is invertible. Here we will use that $f \in C^1([0,\infty[)$, that $\varphi \in C^1(\mathbb{R}^n \setminus \{0\})$ and that φ is strictly convex in the sense of Reshetnyak, as shown in Proposition 2.10. Owing to invertibility of $\overline{DF}$, we will retrieve some information about a sequence u^ε_x approximating the derivative u_x of the minimizer u of $\mathcal{E}$ from estimates involving $DF(u^\varepsilon_x)$. Such estimates arise naturally thanks to Euler-Lagrange equation for regularized functionals.

Lemma 2.18. *Let $\varphi\colon \mathbb{R}^n \to [0,\infty[$ be a $C^1(\mathbb{R}^n \setminus \{0\})$ strictly convex (in the sense of Reshetnyak) anisotropy. Then for any $r > 0$ the mapping $D\varphi$ restricted to $\{p \in \mathbb{R}^n \colon \varphi(p) = r\}$ is injective.*

Proof. The function φ^2 is C^1 on the whole $\mathbb{R}^n$ and is strictly convex as shown in Lemma 2.9. Therefore, $D\varphi^2$ is strictly monotone and hence injective. For any $p \neq q$ such that $\varphi(p) = \varphi(q) = r$ for some positive $r > 0$ it is then true that

$$2rD\varphi(p) \neq 2rD\varphi(q),$$

which means that $D\varphi(p) \neq D\varphi(q)$. □

Lemma 2.19. *Let $F = f \circ \varphi$ be as in Theorem 1.3. The mapping $\overline{DF}\colon \overline{\mathbb{R}^n} \to \mathbb{R}^n$ defined as*

$$\overline{DF} := \begin{cases} DF(p) & \text{if } p \in \mathbb{R}^n, \\ f^\infty D\varphi(\omega) & \text{if } p = \infty\omega \in \mathbb{S}^{n-1}_\infty \end{cases}$$

is a homeomorphism onto its image.

Proof. Since $\overline{\mathbb{R}^n}$ is compact, it suffices to prove that $\overline{DF}$ is continuous and injective. Continuity is easily established by considering three cases. Firstly, take a point $p_0 \in \mathbb{R}^n$ and any sequence p_j converging to p_0 in metric $\overline{d}$. Then, necessarily, almost all p_j lie in $\mathbb{R}^n$ as well, so continuity at such a point follows from continuity of DF. Secondly, if one approaches $p_0 = \infty\omega_0 \in \mathbb{S}^{n-1}_\infty$ with a sequence of points $\{p_j\}_j \subset \mathbb{R}^n$, then writing $p_j = r_j\omega_j$ with $r_j \geq 0$, $\omega_j \in \mathbb{S}^{n-1}_1$ gives

$$DF(p_j) = f'(r_j\varphi(\omega_j))D\varphi(\omega_j) \to f^\infty D\varphi(\omega_0) = \overline{DF}(p_0),$$

as $\{\varphi(\omega_j)\}_j$ is bounded away from 0 and $r_j \to \infty$, and $D\varphi$ is continuous on the unit sphere. Lastly, continuity of $\overline{DF}$ restricted to $\mathbb{S}^{n-1}_\infty$ follows directly from continuity of $D\varphi$ on the unit sphere.

To prove injectivity, observe that F being strictly convex implies $DF\colon \mathbb{R}^n \to \mathbb{R}^n$ being strictly monotone, so DF is one-to-one. Therefore, $\overline{DF}(p) \neq \overline{DF}(q)$ for any distinct $p, q \in \mathbb{R}^n$. In view of Lemma 2.18, it suffices to check that $\overline{DF}(p) \neq \overline{DF}(q)$ for $p \in \mathbb{R}^n$ and $q \in \mathbb{S}^{n-1}_\infty$. To this end, let us recall that for any $p \in \mathbb{R}^n \setminus \{0\}$ it is true that

$$\varphi^*\left(D\varphi(p)\right) = 1, \text{ where } \varphi^*(p) := \sup\left\{p \cdot q \colon \varphi(q) \leq 1\right\}. \tag{2.13}$$

Indeed, in view of Proposition 2.7, taking $q = p/\varphi(p)$ gives $\varphi^*\left(D\varphi(p)\right) \geq 1$. On the other hand, the product $q \cdot D\varphi(p)$ equals the derivative of φ at point p in direction q and hence, by triangle inequality,

$$\varphi^*\left(D\varphi(p)\right) = \sup_{\varphi(q)\leq 1} \lim_{t\to 0^+} \frac{1}{t}\left(\varphi(p+tq) - \varphi(p)\right) \leq \sup_{\varphi(q)\leq 1} \varphi(q) = 1.$$

Assume now to the contrary that $DF(p) = f^\infty D\varphi(\omega)$ for some $p \in \mathbb{R}^n \setminus \{0\}$ and $\omega \in \mathbb{S}_1^{n-1}$. Applying φ^* to both sides and using its positive homogeneity results in

$$f'(\varphi(p))\varphi^*(D\varphi(p)) = f^\infty \varphi^*(D\varphi(\omega)).$$

Given (2.13), one gets $f'(\varphi(p)) = f^\infty$, which contradicts strict convexity of f. Therefore, $\overline{DF}$ is injective. □

3. Approximate functionals

Recall that the functional $\mathcal{E}$ defined in (1.1) and (1.5) has the following form

$$\mathcal{E}(w) = \begin{cases} \lambda F(w_x)(I) + \frac{1}{2}\int_I |w-h|^2 \,\mathrm{d}\mathcal{L}^1 & \text{for } w \in BV(I,\mathbb{R}^n), \\ \infty & \text{for } w \in L^2(I,\mathbb{R}^n) \setminus BV(I,\mathbb{R}^n). \end{cases}$$

We denote its unique minimizer with u. Clearly, $u \in BV(I,\mathbb{R}^n)$. In this section we introduce approximate functionals $\mathcal{E}_{\varepsilon,\eta}$ and $\mathcal{E}_\varepsilon$ which will be central to the proofs of the main theorems. The choice of approximation in the proof of Theorem 1.1 is simpler than in the proofs of Theorems 1.2 and 1.3. Nonetheless, the Γ-convergence properties remain the same and therefore are presented together.

Suppose $h \in L^2(I,\mathbb{R}^n) \cap BV(U,\mathbb{R}^n)$ for an open interval $U \subset I$ and extend h by zero beyond I in order to properly define mollifications of h. Set

$$h^\varepsilon := h * \varrho_\varepsilon. \tag{3.1}$$

Given a convex, non-decreasing $f\colon [0,\infty[\to [0,\infty[$ of linear growth and anisotropy $\varphi : \mathbb{R}^n \to [0,\infty[$, recall functions f_η, $\overline{\varphi}_\eta$, $\widetilde{\varphi}_\eta$ and $\widetilde{\widetilde{\varphi}}_\eta$ defined in Lemma 2.12, Lemma 2.13, Proposition 2.15 and Proposition 2.16, respectively. Specific f and φ will be chosen later.

We suppose that F_η is given either by

$$F_\eta := f_\eta \circ \overline{\varphi}_\eta + \frac{\eta}{2}|\cdot|^2 \tag{3.2}$$

or

$$F_\eta := f_\eta \circ \widetilde{\varphi}_\eta + \frac{\eta}{2}\widetilde{\widetilde{\varphi}}_\eta. \tag{3.3}$$

Observe that $F_\eta \in C^2(\mathbb{R}^n)$ since $f'_\eta = 0$ near zero, which cancels out the singularity at the origin that any homogeneous function has. Moreover, $f_\eta \circ \overline{\varphi}_\eta$ and $f_\eta \circ \widetilde{\varphi}_\eta$ are convex, which implies that if F_η is given by (3.2), then $D^2F_\eta \geq \eta\,\mathrm{Id}$. If F_η is given by (3.3), then by Proposition 2.16 $D^2F_\eta \geq \eta^2/2\,\mathrm{Id}$. Therefore, in this section we will assume that

$$D^2F_\eta \geq \frac{\eta^2}{2}\,\mathrm{Id}. \tag{3.4}$$

Furthermore, since $f_\eta \geq f$, $\overline{\varphi} \geq \varphi$, $\widetilde{\varphi}_\eta \geq \varphi$ as shown in Lemma 2.12 (i), Remark 2.14 (i) and Proposition 2.15 (ii), respectively, it is true that

$$F_\eta \geq F = f \circ \varphi. \tag{3.5}$$

We will use a two-layer approximation of $\mathcal{E}$:

$$\mathcal{E}_{\varepsilon,\eta}(w) := \lambda \int_I F_\eta(w_x)\,\mathrm{d}\mathcal{L}^1 + \frac{1}{2}\int_I |w-h^\varepsilon|^2\,\mathrm{d}\mathcal{L}^1 \text{ for } w \in W^{1,2}(I,\mathbb{R}^n) \text{ with mimimizer } u^{\varepsilon,\eta}, \tag{3.6}$$

$$\mathcal{E}_\varepsilon(w) := \begin{cases} \lambda\, F(w_x)(I) + \frac{1}{2}\int_I |w-h^\varepsilon|^2\,\mathrm{d}\mathcal{L}^1 & \text{for } w \in BV(I,\mathbb{R}^n), \\ \infty & \text{for } w \in L^2(I,\mathbb{R}^n)\setminus BV(I,\mathbb{R}^n) \end{cases} \text{ with minimizer } u^\varepsilon. \tag{3.7}$$

Observe that $\mathcal{E}_{\varepsilon,\eta}$ and $\mathcal{E}_\varepsilon$ are strictly convex, non-negative, weakly lower semicontinuous on $W^{1,2}$ and L^2, respectively. Moreover the sublevels of $\mathcal{E}_{\varepsilon,\eta}$, $\mathcal{E}_\varepsilon$ are weakly compact on $W^{1,2}$ and L^2, respectively. Consequently, there indeed exist unique minimizers of these functionals.

Remark 3.1. The minimizer $u^{\varepsilon,\eta}$ of $\mathcal{E}_{\varepsilon,\eta}$ is *a priori* in $W^{1,2}(I,\mathbb{R}^n)$ but it follows by a standard method that $u^{\varepsilon,\eta} \in W^{2,2}(I,\mathbb{R}^n)$, see [19], Appendix. This implies in particular that $u^{\varepsilon,\eta} \in C^1(\overline{I},\mathbb{R}^n)$. Moreover, it satisfies the following Euler–Lagrange equation in the strong sense

$$u^{\varepsilon,\eta} - h^\varepsilon = \lambda\,(DF_\eta(u^{\varepsilon,\eta}_x))_x \text{ in } I, \quad DF_\eta(u^{\varepsilon,\eta}_x) = 0 \text{ on } \partial I. \tag{3.8}$$

In the case of F_η given by (3.2), the boundary condition $DF_\eta(u^{\varepsilon,\eta}_x) = 0$ is equivalent to $u^{\varepsilon,\eta}_x = 0$. If F_η is given by (3.2), this is not necessarily true. However, we can show

Lemma 3.2. *Let $F_\eta = f_\eta \circ \widetilde{\varphi}_\eta + \frac{\eta}{2}\widetilde{\widetilde{\varphi}}_\eta$. Then $DF_\eta^{-1}(0) \in \overline{B}_\eta(0)$.*

Proof. Outside of $B_\eta(0)$, F_η coincides with $f_\eta \circ \widetilde{\varphi}_\eta + \frac{\eta}{2}\widetilde{\varphi}_\eta^2$. Note that both of these functions are coercive, C^1 and (by Lem. 2.9) strictly convex on $\mathbb{R}^n$, so each one has exactly one critical point (the minimizer). The latter one attains its minimum at 0. Thus, by strict convexity,

$$DF_\eta(\xi) = D\left(f_\eta \circ \widetilde{\varphi}_\eta + \tfrac{\eta}{2}\widetilde{\varphi}_\eta^2\right)(\xi) \neq 0 \quad \text{if } \xi \notin \overline{B}_\eta(0).$$

Therefore the critical point (minimizer) of F_η has to belong to $\overline{B}_\eta(0)$. □

Lemma 3.3. *Functionals $\mathcal{E}_{\varepsilon,\eta}$ Γ-converge to $\mathcal{E}_\varepsilon$ w. r. t. weak $W^{1,2}(I,\mathbb{R}^n)$ convergence.*

Proof. In view of (3.5), $\mathcal{E}_{\varepsilon,\eta} \geq \mathcal{E}_\varepsilon$, hence lower limit inequality follows from lower semicontinuity of $\mathcal{E}_\varepsilon$, see *e. g.* [1], Theorem 1.3. As for the recovery sequence, choose $w^\eta := w$ for any $w \in W^{1,2}(I,\mathbb{R}^n)$, then one only needs to show that

$$\lim_{\eta\to 0} \int_I F_\eta(w_x)\,\mathrm{d}\mathcal{L}^1 = \int_I F(w_x)\,\mathrm{d}\mathcal{L}^1. \tag{3.9}$$

For sufficiently small η, $c^+_{\widetilde{\varphi}_\eta} < 2c^+_\varphi$ according to Proposition 2.15 (ii), and $f(\eta) < f(1)$. By the fact that $\overline{\varphi}_\eta \leq \widetilde{\varphi}_\eta$ and Lemma 2.12 (i),

$$f_\eta \circ \overline{\varphi}_\eta(w_x) \leq f_\eta \circ \widetilde{\varphi}_\eta(w_x) \leq 2f^\infty c^+_\varphi |w_x| + f(1).$$

Consequently, dominated convergence theorem yields

$$\lim_{\eta\to 0} \int_I f_\eta \circ \overline{\varphi}_\eta(w_x)\,\mathrm{d}\mathcal{L}^1 = \int_I f\circ\varphi(w_x)\,\mathrm{d}\mathcal{L}^1 \quad \text{and} \quad \lim_{\eta\to 0} \int_I f_\eta \circ \widetilde{\varphi}_\eta(w_x)\,\mathrm{d}\mathcal{L}^1 = \int_I f\circ\varphi(w_x)\,\mathrm{d}\mathcal{L}^1. \tag{3.10}$$

To check validity of (3.9), it remains to see that for any $w \in W^{1,2}(I, \mathbb{R}^n)$

$$\lim_{\eta \to 0} \frac{\eta}{2} \int_I |w_x|^2 \, \mathrm{d}\mathcal{L}^1 = 0 \quad \text{and} \quad \lim_{\eta \to 0} \frac{\eta}{2} \int_I \widetilde{\widetilde{\varphi}}_\eta(w_x) \, \mathrm{d}\mathcal{L}^1 = 0. \tag{3.11}$$

Clearly, the first equality is true, which finishes the proof when F_η is defined as in (3.2). The second equality corresponds to the case (3.3). By Proposition 2.16 (iii), for sufficiently small η and $C > 0$ independent of η,

$$\frac{\eta}{2} \int_I \widetilde{\widetilde{\varphi}}_\eta(w_x) \, \mathrm{d}\mathcal{L}^1 \leq \frac{C\,\eta}{2} \int_I (1 + |w_x|^2) \, \mathrm{d}\mathcal{L}^1,$$

which converges to zero for any $w \in W^{1,2}(I, \mathbb{R}^n)$. □

Lemma 3.4. *Functionals $\mathcal{E}_\varepsilon$ Γ-converge to $\mathcal{E}$ w. r. t. weak convergence in $L^2(I, \mathbb{R}^n)$.*

Proof. Since h^ε converge strongly to h in $L^2(I, \mathbb{R}^n)$, for each $w \in BV(I, \mathbb{R}^n)$ there exists a trivial recovery sequence $w^\varepsilon = w$.

Then, take $w \in BV(I, \mathbb{R}^n)$ and a sequence w^ε converging to w weakly in L^2. Since the norm is lower semicontinuous w. r. t. weak convergence,

$$\liminf_{\varepsilon \to 0} \int_I |w^\varepsilon - h^\varepsilon|^2 \, \mathrm{d}\mathcal{L}^1 \geq \int_I |w - h|^2 \, \mathrm{d}\mathcal{L}^1.$$

Recall that $\mathcal{F}(w) = F(w_x)(I)$. Lower semicontinuity of $\mathcal{F}$ with respect to strong $L^2(I, \mathbb{R}^n)$ convergence follows from Theorem 2.5, as L^2 convergence is stronger than L^1_{loc} convergence. Since $\mathcal{F}$ is convex, it is also lower semicontinuous with respect to weak L^2 convergence. Eventually,

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(w^\varepsilon) = \liminf_{\varepsilon \to 0} \, \mathcal{F}(w^\varepsilon) + \int_I |w^\varepsilon - h^\varepsilon|^2 \, \mathrm{d}\mathcal{L}^1 \geq \mathcal{F}(w) + \int_I |w - h|^2 \, \mathrm{d}\mathcal{L}^1 = \mathcal{E}(w),$$

as required. □

Lemma 3.5. *The sequence $u^{\varepsilon,\eta}$ of minimizers of $\mathcal{E}_{\varepsilon,\eta}$ satisfies*

$$\int_I |u^{\varepsilon,\eta}|^2 \, \mathrm{d}\mathcal{L}^1 + \int_I |u^{\varepsilon,\eta}_x| \, \mathrm{d}\mathcal{L}^1 \leq C \quad \textit{and} \quad \int_I |u^{\varepsilon,\eta}_x|^2 \, \mathrm{d}\mathcal{L}^1 \leq C_\varepsilon \eta + \int_I |h^\varepsilon_x|^2 \, \mathrm{d}\mathcal{L}^1, \tag{3.12}$$

where $C > 0$ does not depend on ε or η and $C_\varepsilon > 0$ does not depend on η. Moreover, as $\eta \to 0^+$, the sequence of minimizers $u^{\varepsilon,\eta}$ of $\mathcal{E}_{\varepsilon,\eta}$ converges strongly in $L^2(I, \mathbb{R}^n)$ and weakly in $W^{1,2}(I, \mathbb{R}^n)$ to the minimizer u^ε of $\mathcal{E}_\varepsilon$.

Proof. As $u^{\varepsilon,\eta}$ is the minimizer of $\mathcal{E}_{\varepsilon,\eta}$, there holds

$$\mathcal{E}_{\varepsilon,\eta}(u^{\varepsilon,\eta}) \leq \mathcal{E}_{\varepsilon,\eta}(0) = \int_I F_\eta(0) \, \mathrm{d}\mathcal{L}^1 + \int_I |h^\varepsilon|^2 \, \mathrm{d}\mathcal{L}^1 \leq \left(f_\eta(0) + \widetilde{\widetilde{\varphi}}_\eta(0)\right) |I| + \int_I |h|^2 \, \mathrm{d}\mathcal{L}^1. \tag{3.13}$$

Since $f_\eta(0) = f(\eta) \leq f(1)$ and by Proposition 2.16 (iii), for sufficiently small $\eta > 0$, $\widetilde{\widetilde{\varphi}}_\eta(0) \leq C$ for some $C > 0$ independent of η, we get

$$\mathcal{E}_{\varepsilon,\eta}(u^{\varepsilon,\eta}) \leq (f(1) + C) \, |I| + \int_I |h|^2 \, \mathrm{d}\mathcal{L}^1.$$

Therefore, the r. h. s. is uniformly bounded independently of ε and η. On the other hand, by (3.5), linear growth of F, and the inequality $|u^{\varepsilon,\eta}|^2 \le (|u^{\varepsilon,\eta} - h^\varepsilon| + |h^\varepsilon|)^2 \le 2|u^{\varepsilon,\eta} - h^\varepsilon|^2 + 2|h^\varepsilon|^2$,

$$\begin{aligned} &\mathcal{E}_{\varepsilon,\eta}(u^{\varepsilon,\eta}) = \lambda \int_I F_\eta(u_x^{\varepsilon,\eta}) \,\mathrm{d}\mathcal{L}^1 + \frac{1}{2}\int_I |u^{\varepsilon,\eta} - h^\varepsilon|^2 \,\mathrm{d}\mathcal{L}^1 \\ &\ge \lambda \int_I F(u_x^{\varepsilon,\eta}) \,\mathrm{d}\mathcal{L}^1 + \frac{1}{4}\int_I |u^{\varepsilon,\eta}|^2 \,\mathrm{d}\mathcal{L}^1 - \frac{1}{2}\int_I |h^\varepsilon|^2 \,\mathrm{d}\mathcal{L}^1 \ge \lambda \int_I C_F^-(|u_x^{\varepsilon,\eta}| - 1) \,\mathrm{d}\mathcal{L}^1 + \frac{1}{4}\int_I |u^{\varepsilon,\eta}|^2 \,\mathrm{d}\mathcal{L}^1 - \frac{1}{2}\int_I |h|^2 \,\mathrm{d}\mathcal{L}^1, \end{aligned} \tag{3.14}$$

which implies that $\int_I |u^{\varepsilon,\eta}|^2 \,\mathrm{d}\mathcal{L}^1 + \int_I |u_x^{\varepsilon,\eta}| \,\mathrm{d}\mathcal{L}^1$ is indeed bounded independently of ε and η.

Let us now prove that the sequence of minimizers $u^{\varepsilon,\eta}$ is bounded in $W^{1,2}$. We have already showed the estimate in L^2. For the boundedness of the derivatives, let us test the Euler–Lagrange equation (3.8) with $u_{xx}^{\varepsilon,\eta}$ to obtain

$$\int_I (u^{\varepsilon,\eta} - h^\varepsilon) \cdot u_{xx}^{\varepsilon,\eta} \,\mathrm{d}\mathcal{L}^1 = \lambda \int_I (DF_\eta(u_x^{\varepsilon,\eta}))_x \cdot u_{xx}^{\varepsilon,\eta} \,\mathrm{d}\mathcal{L}^1, \tag{3.15}$$

and integrate the l. h. s. of (3.15) by parts over I.

After integration by parts on the r. h. s.,

$$\int_I (h_x^\varepsilon - u_x^{\varepsilon,\eta}) \cdot u_x^{\varepsilon,\eta} \,\mathrm{d}\mathcal{L}^1 + \int_{\partial I} (u^{\varepsilon,\eta} - h^\varepsilon) \cdot u_x^{\varepsilon,\eta} = \lambda \int_I (DF_\eta(u_x^{\varepsilon,\eta}))_x \cdot u_{xx}^{\varepsilon,\eta} \,\mathrm{d}\mathcal{L}^1. \tag{3.16}$$

In the case of F_η given by (3.2), the boundary term vanishes due to the boundary condition $u_x^{\varepsilon,\eta} = 0$ on ∂I. On the other hand, if F_η is given by (3.3), we estimate the boundary term using Lemma 3.2:

$$\left| \int_{\partial I} (u^{\varepsilon,\eta} - h^\varepsilon) \cdot u_x^{\varepsilon,\eta} \right| \le \eta \int_{\partial I} |u^{\varepsilon,\eta}| + \eta \int_{\partial I} |h^\varepsilon|. \tag{3.17}$$

Since $u^{\varepsilon,\eta}$ is uniformly bounded in $W^{1,1}(I, \mathbb{R}^n) \subset C(\overline{I}, \mathbb{R}^n)$, the r. h. s. of (3.17) is bounded by $\frac{1}{2} C_\varepsilon \eta$, where $C_\varepsilon > 0$ is independent of η. The r. h. s. of (3.16) is non-negative due to convexity of F_η therefore, applying Cauchy's inequality,

$$\int_I |u_x^{\varepsilon,\eta}|^2 \,\mathrm{d}\mathcal{L}^1 \le \int_I u_x^{\varepsilon,\eta} \cdot h_x^\varepsilon \,\mathrm{d}\mathcal{L}^1 + \frac{1}{2} C_\varepsilon \eta \le \frac{1}{2}\int_I |u_x^{\varepsilon,\eta}|^2 + \frac{1}{2}\int_I |h_x^\varepsilon|^2 + \frac{1}{2} C_\varepsilon \eta.$$

This concludes the proof of (3.12).

Let us now fix $\varepsilon > 0$. By (3.12), from any subsequence of $\{u^{\varepsilon,\eta}\}_\eta$ it is possible to choose a subsequence $\{u^{\varepsilon,\eta_j}\}_j$ weakly convergent in $W^{1,2}(I, \mathbb{R}^n)$. In view of the Γ-convergence proved in Lemma 3.3, the weak limit of $\{u^{\varepsilon,\eta_j}\}_j$ must coincide with the minimizer u^ε. A standard application of the Rellich–Kondrashov theorem allows us to choose a further subsequence (which we do not relabel) which additionally converges strongly in L^2 to u^ε. Therefore, a properly convergent subsequence can be chosen from arbitrary subsequence of $\{u^{\varepsilon,\eta}\}_\eta$, which shows the desired convergence of the whole sequence. □

Lemma 3.6. *The sequence u^ε of minimizers of $\mathcal{E}_\varepsilon$ satisfies*

$$\int_I |u^\varepsilon|^2 \,\mathrm{d}\mathcal{L}^1 + \int_I |u_x^\varepsilon| \,\mathrm{d}\mathcal{L}^1 \le C, \tag{3.18}$$

where $C > 0$ does not depend on ε. Moreover, as $\varepsilon \to 0^+$, the sequence of minimizers u^ε of $\mathcal{E}_\varepsilon$ converges strongly in $L^2(I, \mathbb{R}^n)$ and weakly in $BV(I, \mathbb{R}^n)$ to the minimizer u of $\mathcal{E}$.*

Proof. We deduce the bound (3.18) from inequality $\mathcal{E}_\varepsilon(u^\varepsilon) \le \mathcal{E}_\varepsilon(0)$, estimating from above and below as in (3.13), (3.14). Then, we show the convergence assertion as in the final part of the proof of Lemma 3.5, with weak* convergence in BV in place of weak convergence in $W^{1,2}$. □

4. The homogeneous case

We will now show that if F is a coercive anisotropy, then the strong estimate $|u_x| \le |h_x|$ holds.

Proof of Theorem 1.1. In this section we choose F_η defined as in (3.2) for $f(t) = t$, *i. e.*,

$$f_\eta(t) := \big(t\mathbf{1}_{\{t>\eta\}} + f(\eta)\mathbf{1}_{\{t\le\eta\}}\big) * \varrho_{\eta/2} \quad \text{and} \quad F_\eta = f_\eta \circ \overline{\varphi}_\eta + \frac{\eta}{2}|\cdot|^2,$$

where $\overline{\varphi}_\eta$ is defined in Lemma 2.13. Recall that $u^{\varepsilon,\eta}$ is the minimizer of $\mathcal{E}_{\varepsilon,\eta}$ as defined in (3.17) and that $u^{\varepsilon,\eta} \in W^{2,2}(I,\mathbb{R}^n)$ according to Remark 3.1. Moreover, $h^\varepsilon = h * \varrho_\varepsilon$ as defined in (3.1).

Firstly, we multiply the Euler–Lagrange equation (3.8) by $u^{\varepsilon,\eta}_{xx}$ to obtain

$$(u^{\varepsilon,\eta} - h^\varepsilon)\cdot u^{\varepsilon,\eta}_{xx} = \lambda\left(DF_\eta(u^{\varepsilon,\eta}_x)\right)_x \cdot u^{\varepsilon,\eta}_{xx} \text{ in } I, \quad u^{\varepsilon,\eta}_x = 0 \text{ on } \partial I. \tag{4.1}$$

After integrating both sides of (4.1) over I, integrating by parts on the l. h. s. and then on the r. h. s. using the observation that $D^2F_\eta \ge \eta\,\mathrm{Id}$ due to convexity of $f_\eta \circ \overline{\varphi}_\eta$, we get

$$\lambda\eta\int_I |u^{\varepsilon,\eta}_{xx}|^2\,\mathrm{d}\mathcal{L}^1 \le \int_I (h^\varepsilon_x - u^{\varepsilon,\eta}_x)\cdot u^{\varepsilon,\eta}_x\,\mathrm{d}\mathcal{L}^1.$$

By using Cauchy's inequality on the r. h. s. we obtain a uniform estimate

$$\lambda\eta\int_I |u^{\varepsilon,\eta}_{xx}|^2\,\mathrm{d}\mathcal{L}^1 \le \frac{1}{2}\int_I |h^\varepsilon_x|^2\,\mathrm{d}\mathcal{L}^1 + \frac{1}{2}\int_I |u^{\varepsilon,\eta}_x|^2\,\mathrm{d}\mathcal{L}^1 - \int_I |u^{\varepsilon,\eta}_x|^2\,\mathrm{d}\mathcal{L}^1 \le \frac{1}{2}\int_I |h^\varepsilon_x|^2\,\mathrm{d}\mathcal{L}^1. \tag{4.2}$$

As already described in the Introduction, we would like to test the Euler–Lagrange equation (3.8) with a function of form

$$(\zeta^2 D\mathfrak{G}_k(u_x))_x, \quad \text{where } \mathfrak{G}_k(\xi) = (|\xi| - k)_+,\ k > 0 \text{ and } \zeta \in C^1_c(I).$$

The role of the function ζ is to localize our estimates, while the parameter k serves to fence off the singularity that $|\cdot|$ and F have at 0. However, to be able to perform our calculations, we also need to smooth out the singularity that $G_k(\xi)$ has at $|\xi| = k$. Thus, for $k > 0$, $\delta > 0$ we define

$$\mathfrak{G}_{k,\delta}(p) = \sqrt{(|p| - k)_+^2 + \delta^2}.$$

We will test the Euler–Lagrange equation (3.8) with $\big(D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\zeta^2\big)_x$, where $\zeta \in C^1_c(U)$ and U is an open subinterval of I as in the statement of Theorem 1.1, which leads to

$$\int_I (u^{\varepsilon,\eta} - h^\varepsilon)\cdot\big(D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\zeta^2\big)_x\,\mathrm{d}\mathcal{L}^1 = \lambda\int_I (DF_\eta(u^{\varepsilon,\eta}_x))_x\cdot\big(D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\zeta^2\big)_x\,\mathrm{d}\mathcal{L}^1. \tag{4.3}$$

Let us firstly estimate the r. h. s. of (4.3) from below. We compute

$$D\mathfrak{G}_{k,\delta}(p) = \frac{(|p| - k)_+}{\sqrt{(|p| - k)_+^2 + \delta^2}}\frac{p}{|p|}$$

and, recalling that $\zeta \in C^1_c(U)$,

$$\big(D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\zeta^2\big)_x = \frac{(|u^{\varepsilon,\eta}_x|-k)_+}{\sqrt{(|u^{\varepsilon,\eta}_x|-k)^2_+ + \delta^2}}\frac{1}{|u^{\varepsilon,\eta}_x|}\left(\mathrm{Id} - \frac{u^{\varepsilon,\eta}_x}{|u^{\varepsilon,\eta}_x|}\otimes\frac{u^{\varepsilon,\eta}_x}{|u^{\varepsilon,\eta}_x|}\right) u^{\varepsilon,\eta}_{xx}\zeta^2 + \frac{\delta^2 \mathbf{1}_{|u^{\varepsilon,\eta}_x|>k}}{\sqrt{(|u^{\varepsilon,\eta}_x|-k)^2_+ + \delta^2}^3}\frac{u^{\varepsilon,\eta}_x}{|u^{\varepsilon,\eta}_x|}\otimes\frac{u^{\varepsilon,\eta}_x}{|u^{\varepsilon,\eta}_x|} u^{\varepsilon,\eta}_{xx}\zeta^2 + \frac{(|u^{\varepsilon,\eta}_x|-k)_+}{\sqrt{(|u^{\varepsilon,\eta}_x|-k)^2_+ + \delta^2}}\frac{u^{\varepsilon,\eta}_x}{|u^{\varepsilon,\eta}_x|}2\zeta\zeta_x. \tag{4.4}$$

For any $p \neq 0$, matrices $\mathrm{Id} - \frac{p}{|p|}\otimes\frac{p}{|p|}$ and $\frac{p}{|p|}\otimes\frac{p}{|p|}$ are non-negative definite. Therefore, after multiplying identity (4.4) by $u^{\varepsilon,\eta}_{xx}$, the first two terms on the r. h. s. are non-negative, which yields

$$\int_I \big(D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\zeta^2\big)_x \cdot u^{\varepsilon,\eta}_{xx}\,\mathrm{d}\mathcal{L}^1 \geq \int_I 2\zeta\zeta_x \underbrace{\frac{(|u^{\varepsilon,\eta}_x|-k)_+}{\sqrt{(|u^{\varepsilon,\eta}_x|-k)^2_+ + \delta^2}}\frac{u^{\varepsilon,\eta}_x}{|u^{\varepsilon,\eta}_x|}}_{D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)} \cdot u^{\varepsilon,\eta}_{xx}\,\mathrm{d}\mathcal{L}^1. \tag{4.5}$$

By the fact that $D\mathfrak{G}_{k,\delta} \leq 1$, Hölder's inequality and, eventually, estimate (4.2), we get

$$\left|\int_I 2\zeta\zeta_x\, D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\cdot u^{\varepsilon,\eta}_{xx}\,\mathrm{d}\mathcal{L}^1\right| \leq 2\max|\zeta\zeta_x||I|^{\frac12}\left(\int_I |u^{\varepsilon,\eta}_{xx}|^2\,\mathrm{d}\mathcal{L}^1\right)^{\frac12} \leq \frac{\sqrt{2}}{\sqrt{\lambda\,\eta}}\max|\zeta\zeta_x||I|^{\frac12}\left(\int_I |h^\varepsilon_x|^2\,\mathrm{d}\mathcal{L}^1\right)^{\frac12}. \tag{4.6}$$

For any real numbers a, b it is true that if $|a| \leq b$, then $a \geq -b$. Therefore, plugging (4.6) into (4.5) yields

$$\int_I \big(D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\zeta^2\big)_x \cdot u^{\varepsilon,\eta}_{xx}\,\mathrm{d}\mathcal{L}^1 \geq -\frac{\sqrt{2}}{\sqrt{\lambda\,\eta}}\max|\zeta\zeta_x||I|^{\frac12}\left(\int_I |h^\varepsilon_x|^2\,\mathrm{d}\mathcal{L}^1\right)^{\frac12}. \tag{4.7}$$

We shall now see that, largely due to the homogeneity of φ,

$$\int_I (DF_\eta(u^{\varepsilon,\eta}_x))_x \cdot \big(D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\zeta^2\big)_x\,\mathrm{d}\mathcal{L}^1 \geq \eta\int_I \big(D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\zeta^2\big)_x \cdot u^{\varepsilon,\eta}_{xx}\,\mathrm{d}\mathcal{L}^1. \tag{4.8}$$

We calculate

$$DF_\eta(p) = f'_\eta(\overline{\varphi}_\eta(p))D\overline{\varphi}_\eta(p) + \eta p,$$

$$(DF_\eta(u^{\varepsilon,\eta}_x))_x = \underbrace{f'_\eta(\overline{\varphi}_\eta(u^{\varepsilon,\eta}_x))\,D^2\overline{\varphi}_\eta(u^{\varepsilon,\eta}_x)u^{\varepsilon,\eta}_{xx}}_{\alpha^{\varepsilon,\eta}} + \underbrace{f''_\eta(\overline{\varphi}_\eta(u^{\varepsilon,\eta}_x))\,(\overline{\varphi}_\eta(u^{\varepsilon,\eta}_x))_x\,D\overline{\varphi}_\eta(u^{\varepsilon,\eta}_x)}_{\beta^{\varepsilon,\eta}} + \eta\,u^{\varepsilon,\eta}_{xx}.$$

When we multiply $\alpha^{\varepsilon,\eta}$ by $\big(D\mathfrak{G}_{k,\delta}(u^{\varepsilon,\eta}_x)\zeta^2\big)_x$ as computed in (4.4), we see that the second and third term vanish due to the identity $p \cdot D^2\overline{\varphi}_\eta(p) = 0$ following from homogeneity of $\overline{\varphi}_\eta$, see Proposition 2.7. By the same identity, the first term in this product simplifies to

$$f'_\eta(\overline{\varphi}_\eta(u^{\varepsilon,\eta}_x))\frac{(|u^{\varepsilon,\eta}_x|-k)_+}{\sqrt{(|u^{\varepsilon,\eta}_x|-k)^2_+ + \delta^2}}\frac{1}{|u^{\varepsilon,\eta}_x|}u^{\varepsilon,\eta}_{xx}\cdot D^2\overline{\varphi}_\eta(u^{\varepsilon,\eta}_x)\,u^{\varepsilon,\eta}_{xx}. \tag{4.9}$$

Since $D^2\overline{\varphi}_\eta(u^{\varepsilon,\eta})$ is non-negative definite and f'_η is non-negative, quantity (4.9) is non-negative. As a result,

$$\alpha^{\varepsilon,\eta} \cdot \left(D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta})\zeta^2\right)_x \geq 0. \tag{4.10}$$

We will now deal with $\beta^{\varepsilon,\eta} \cdot \left(D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta})\zeta^2\right)_x$. Assuming that $k > \frac{2}{c_{\overline{\varphi}}}\eta$, we have $|p| < k$ or $\overline{\varphi}_\eta(p) > 2\eta$ for all $p \in \mathbb{R}^n$. On the set $\{x : |u_x^{\varepsilon,\eta}(x)| < k\}$, we have $\left(D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta})\zeta^2\right)_x = 0$, which means that both sides of (4.8) are equal zero. Since $f(t) = t$ for $t > 2\eta$, we have $f'(\overline{\varphi}_\eta(p)) = 1$ and $f''(\overline{\varphi}_\eta(p)) = 0$ if $\overline{\varphi}_\eta(p) > 2\eta$. Therefore, on the set $\{x : |u_x^{\varepsilon,\eta}(x)| \geq k\}$, the quantity $\beta^{\varepsilon,\eta} \cdot \left(D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta})\zeta^2\right)_x$ equals zero.

Consequently, $(DF_\eta(u_x^{\varepsilon,\eta}))_x \cdot \left(D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta})\zeta^2\right)_x \geq \eta \left(D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta})\zeta^2\right)_x \cdot u_{xx}^{\varepsilon,\eta}$, in particular inequality (4.8) holds. Let us stress here the importance of homogeneity of $F = \varphi$, thanks to which some terms of $(DF_\eta(u_x^{\varepsilon,\eta}))_x \cdot \left(D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta})\zeta^2\right)_x$ disappear. If we assumed that φ was only convex, estimate (4.10) would not necessarily hold since it would involve terms like $v \cdot A\,B\,v$ for some non-zero vector $v \in \mathbb{R}^n$ and two positive semi-definite matrices A, B, which are not necessarily positive. All in all, collecting (4.8) and (4.7) implies that the r. h. s. of (4.3) satisfies

$$\lambda \int_I (DF_\eta(u_x^{\varepsilon,\eta}))_x \cdot \left(D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta})\zeta^2\right)_x \,\mathrm{d}\mathcal{L}^1 \geq -\sqrt{2\lambda\,\eta}\max|\zeta\zeta_x||I|^{\frac{1}{2}} \left(\int_I |h_x^\varepsilon|^2 \,\mathrm{d}\mathcal{L}^1\right)^{\frac{1}{2}}. \tag{4.11}$$

Estimation of the l. h. s. of (4.3) is more straightforward. We begin by integrating by parts, observing that no boundary term appears since $\zeta = 0$ on ∂I, and proceed to use the convexity of $\mathfrak{G}_{k,\delta}$:

$$\begin{aligned}\int_I \left(D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta})\zeta^2\right)_x \cdot (u^{\varepsilon,\eta} - h^\varepsilon) \,\mathrm{d}\mathcal{L}^1 &= \int_I \zeta^2 D\mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta}) \cdot (h_x^\varepsilon - u_x^{\varepsilon,\eta}) \,\mathrm{d}\mathcal{L}^1 \\ &\leq \int_I \zeta^2 \mathfrak{G}_{k,\delta}(h_x^\varepsilon) \,\mathrm{d}\mathcal{L}^1 - \int_I \zeta^2 \mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta}) \,\mathrm{d}\mathcal{L}^1.\end{aligned} \tag{4.12}$$

Eventually, using (4.11) and (4.12) to estimate both sides of (4.3), we arrive at

$$\int_I \zeta^2 \mathfrak{G}_{k,\delta}(h_x^\varepsilon) \,\mathrm{d}\mathcal{L}^1 - \int_I \zeta^2 \mathfrak{G}_{k,\delta}(u_x^{\varepsilon,\eta}) \,\mathrm{d}\mathcal{L}^1 \geq -\sqrt{2\lambda\eta}\max|\zeta\zeta_x||I|^{\frac{1}{2}} \left(\int_I |h_x^\varepsilon|^2 \,\mathrm{d}\mathcal{L}^1\right)^{\frac{1}{2}}.$$

Since for $\delta < 1$, $\mathfrak{G}_{k,\delta}(p) \leq |p| + 1$ and both $u_x^{\varepsilon,\eta}$ and h_x^ε are integrable over I, passing to the limit $\delta \to 0^+$ using dominated convergence yields

$$\int_I \zeta^2 \mathfrak{G}_k(u_x^{\varepsilon,\eta}) \,\mathrm{d}\mathcal{L}^1 \leq \int_I \zeta^2 \mathfrak{G}_k(h_x^\varepsilon) \,\mathrm{d}\mathcal{L}^1 + \sqrt{2\lambda\eta}\max|\zeta\zeta_x||I|^{\frac{1}{2}} \left(\int_I |h_x^\varepsilon|^2 \,\mathrm{d}\mathcal{L}^1\right)^{\frac{1}{2}}. \tag{4.13}$$

As shown in Lemma 3.5, $u_x^{\varepsilon,\eta}$ converge weakly in L^2 to u_x^ε as $\eta \to 0^+$. Therefore, by lower semicontinuity of $w \mapsto \int \mathfrak{G}_k(w) \,\mathrm{d}\mathcal{L}^1$, inequality (4.13) yields

$$\int_I \zeta^2 \mathfrak{G}_k(u_x^\varepsilon) \,\mathrm{d}\mathcal{L}^1 \leq \int_I \zeta^2 \mathfrak{G}_k(h_x^\varepsilon) \,\mathrm{d}\mathcal{L}^1. \tag{4.14}$$

By Lemma 3.6, we know that $u_x^\varepsilon \overset{*}{\rightharpoonup} u_x$. As a result, Lemma 2.2 applied to the l. h. s. of (4.14) and Lemma 2.4 applied to its r. h. s. imply

$$\int_U \zeta^2 \,\mathrm{d}\mathfrak{G}_k(u_x) \leq \int_U \zeta^2 \,\mathrm{d}\mathfrak{G}_k(h_x).$$

Employing Lemma 2.6, we show that

$$\mathfrak{G}_k(u_x) \le \mathfrak{G}_k(h_x)$$

as measures or, in other words,

$$|u_x^s|(V) + \int_V (|u_x^{ac}| - k)_+ \, \mathrm{d}\mathcal{L}^1 \le |h_x^s|(V) + \int_V (|h_x^{ac}| - k)_+ \, \mathrm{d}\mathcal{L}^1$$

for any Borel $V \subset U$. Passing to the limit $k \to 0^+$ we obtain the desired assertion. □

5. The general case

From now on, we choose F_η to be defined as in (3.3). For $k > 0$, let us define

$$G_k(p) = g_k \circ \widetilde{\varphi}_\eta(p) p \quad \text{and} \quad g_k(\sigma) = \frac{(\sigma - k)_+}{\sigma^2} f_\eta'(\sigma). \tag{5.1}$$

Observe that $G_k(p)$ is bounded since f_η is of linear growth, as shown in Lemma 2.12 (i), and $\widetilde{\varphi}_\eta$ is a coercive anisotropy. We test the Euler–Lagrange equation (3.8) with $(\zeta^2 G_k(u_x^{\varepsilon,\eta}))_x$, where $\zeta \in C_c^1(U)$, thus obtaining

$$\frac{1}{\lambda} \int_I \left(\zeta^2 G_k(u_x^{\varepsilon,\eta})\right)_x \cdot (u^{\varepsilon,\eta} - h^\varepsilon) \, \mathrm{d}\mathcal{L}^1 = \int_I \left(\zeta^2 G_k(u_x^{\varepsilon,\eta})\right)_x \cdot \left(DF_\eta(u_x^{\varepsilon,\eta})\right)_x \, \mathrm{d}\mathcal{L}^1. \tag{5.2}$$

After integrating by parts on the l. h. s. of (5.2) and observing that the boundary term vanishes as $\zeta = 0$ on ∂I, we get

$$\frac{1}{\lambda} \int_I \zeta^2 G_k(u_x^{\varepsilon,\eta}) \cdot (u_x^{\varepsilon,\eta} - h_x^\varepsilon) \, \mathrm{d}\mathcal{L}^1 = -\int_I (\zeta^2 G_k(u_x^{\varepsilon,\eta}))_x \cdot \left(DF_\eta(u_x^{\varepsilon,\eta})\right)_x \, \mathrm{d}\mathcal{L}^1. \tag{5.3}$$

Since G_k is not a derivative of a convex function, we cannot estimate l. h. s. as easily as in the homogeneous case. Instead, we obtain the following estimate.

Proposition 5.1. *It holds that*

$$\liminf_{\varepsilon \to 0} \liminf_{\eta \to 0} \int_I \zeta^2 G_k(u_x^{\varepsilon,\eta}) \cdot (u_x^{\varepsilon,\eta} - h_x^\varepsilon) \, \mathrm{d}\mathcal{L}^1 \ge \frac{r(k)}{(c_\varphi^+)^3} \int_I \zeta^2 \, \mathrm{d}A_k(|u_x|) - \frac{f^\infty}{c_\varphi^- c_\varphi^+} \int_I \zeta^2 \, \mathrm{d}(c_\varphi^+ |h_x| - k)_+,$$

where $A_k \colon [0,\infty[\to [0,\infty[$ *is given by*

$$A_k(s) := (c_\varphi^- s - k)_+ (c_\varphi^+ s - k)_+ s^{-1} \text{ for } s > 0, \quad A_k(0) := 0$$

and $r(k)$ *is such that* $\lim_{k\to\infty} r(k) = f^\infty$.

Proof. Let us observe that the function A_k is of linear growth, non-decreasing and convex. Linear growth follows easily from the formula. To check the latter properties, let us consider $s > k/c_\varphi^-$ for the time being. Then A_k is twice differentiable and $A_k'(s) = c_\varphi^- c_\varphi^+ - ks^{-2}$, $A_k''(s) = 2ks^{-3}$. Without loss of generality, it can be assumed that $k > 1$ since we will be interested in passing to the limit with k to infinity. Consequently, $A_k'(s) > 0$, which implies that A_k is increasing. Since $A_k''(s) > 0$, A_k is also convex. Since A_k is continuous, equals zero on $[0, k/c_\varphi^-]$ and is non-decreasing and convex on $]k/c_\varphi^-, \infty[$, it is non-decreasing and convex on $[0,\infty[$. Note that since composition of a convex function with a convex and non-decreasing one remains convex, the function $p \mapsto A_k(|p|)$ is convex.

We will firstly show that for a fixed ε and a suitable choice of $r(k)$,

$$\liminf_{\eta\to 0}\int_I \zeta^2 G_k(u_x^{\varepsilon,\eta})\cdot(u_x^{\varepsilon,\eta}-h_x^\varepsilon)\,\mathrm{d}\mathcal{L}^1 \ge \frac{r(k)}{\left(c_\varphi^+\right)^3}\int_I \zeta^2 A_k(|u_x^\varepsilon|)\,\mathrm{d}\mathcal{L}^1 - \frac{f^\infty}{c_\varphi^- c_\varphi^+}\int_I \zeta^2 (c_\varphi^+|h_x^\varepsilon|-k)_+\,\mathrm{d}\mathcal{L}^1. \tag{5.4}$$

By Proposition 2.15 (ii), $\widetilde{\varphi}_\eta(p)\ge\varphi(p)$. Recalling the definition (1.9), we have

$$c_\varphi^-\,|p|\le\widetilde{\varphi}_\eta(p)\le c_{\widetilde{\varphi}_\eta}^+\,|p|. \tag{5.5}$$

Let us also note a consequence of convexity of the function $p\mapsto(C|p|-k)_+$ for $C,k>0$, namely, the inequality

$$C\mathbf{1}_{C|p|>k}\,\frac{p}{|p|}\cdot(p-q)\ge(C|p|-k)_+-(C|q|-k)_+. \tag{5.6}$$

Using the fact that $(\widetilde{\varphi}_\eta(p)-k)_+\,\mathbf{1}_{c_{\widetilde{\varphi}_\eta}^+|p|>k}=(\widetilde{\varphi}_\eta(p)-k)_+$ and (5.6), we obtain the following estimate

$$\begin{aligned} G_k(u_x^{\varepsilon,\eta})\cdot(u_x^{\varepsilon,\eta}-h_x^\varepsilon) &= \frac{(\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta})-k)_+}{\widetilde{\varphi}_\eta^2(u_x^{\varepsilon,\eta})}\,f_\eta'(\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta}))\,u_x^{\varepsilon,\eta}\cdot(u_x^{\varepsilon,\eta}-h_x^\varepsilon) \\ &\overset{(5.6)}{\ge} \frac{(\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta})-k)_+}{\widetilde{\varphi}_\eta^2(u_x^{\varepsilon,\eta})}\,f_\eta'(\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta}))\,\frac{|u_x^{\varepsilon,\eta}|}{c_{\widetilde{\varphi}_\eta}^+}\left[\left(c_{\widetilde{\varphi}_\eta}^+\,|u_x^{\varepsilon,\eta}|-k\right)_+-\left(c_{\widetilde{\varphi}_\eta}^+\,|h_x^\varepsilon|-k\right)_+\right]. \end{aligned}$$

Since we do not know whether the r.h.s. of the inequality above is positive or negative, we need to handle the two terms which appear in it separately. In Lemma 2.12 (ii), it was shown that $f_\eta'\le f^\infty$. Bearing in mind this, as well as (5.5) and the fact that $(\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta})-k)_+\le\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta})$, we get

$$\frac{(\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta})-k)_+}{\widetilde{\varphi}_\eta^2(u_x^{\varepsilon,\eta})}f_\eta'(\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta}))\frac{|u_x^{\varepsilon,\eta}|}{c_{\widetilde{\varphi}_\eta}^+}\left(c_{\widetilde{\varphi}_\eta}^+\,|h_x^\varepsilon|-k\right)_+\le\frac{f^\infty}{c_\varphi^- c_{\widetilde{\varphi}_\eta}^+}\left(c_{\widetilde{\varphi}_\eta}^+\,|h_x^\varepsilon|-k\right)_+.$$

On the other hand, using monotonicity of f_η' and inequalities (5.5) again yields

$$\frac{(\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta})-k)_+}{\widetilde{\varphi}_\eta^2(u_x^{\varepsilon,\eta})}f_\eta'(\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta}))\frac{|u_x^{\varepsilon,\eta}|}{c_{\widetilde{\varphi}_\eta}^+}\left(c_{\widetilde{\varphi}_\eta}^+\,|u_x^{\varepsilon,\eta}|-k\right)_+\ge\frac{f_\eta'(k)}{(c_{\widetilde{\varphi}_\eta}^+)^3}A_k(|u_x^{\varepsilon,\eta}|).$$

All in all, we arrive at the following estimate

$$G_k(u_x^{\varepsilon,\eta})\cdot(u_x^{\varepsilon,\eta}-h_x^\varepsilon)\ge\frac{f_\eta'(k)}{(c_{\widetilde{\varphi}_\eta}^+)^3}A_k(|u_x^{\varepsilon,\eta}|)-\frac{f^\infty}{c_\varphi^- c_{\widetilde{\varphi}_\eta}^+}\left(c_{\widetilde{\varphi}_\eta}^+\,|h_x^\varepsilon|-k\right)_+. \tag{5.7}$$

At this point, we multiply the inequality (5.7) by ζ^2 and integrate it over I to obtain

$$\int_I \zeta^2 G_k(u_x^{\varepsilon,\eta})\cdot(u_x^{\varepsilon,\eta}-h_x^\varepsilon)\,\mathrm{d}\mathcal{L}^1\ge\int_I\zeta^2\frac{f_\eta'(k)}{(c_{\widetilde{\varphi}_\eta}^+)^3}A_k(|u_x^{\varepsilon,\eta}|)\,\mathrm{d}\mathcal{L}^1-\frac{f^\infty}{c_\varphi^- c_{\widetilde{\varphi}_\eta}^+}\int_I\zeta^2\left(c_{\widetilde{\varphi}_\eta}^+\,|h_x^\varepsilon|-k\right)_+\,\mathrm{d}\mathcal{L}^1. \tag{5.8}$$

In the second term on the r. h. s. of (5.8) we can pass to the limit with $\eta \to 0^+$ by the dominated convergence theorem. Indeed, h_x^ε is integrable and $c^+_{\widetilde{\varphi}_\eta} \to c^+_\varphi$ as shown in Proposition 2.15 (ii). Therefore,

$$\lim_{\eta\to 0} \frac{f^\infty}{c^-_\varphi c^+_{\widetilde{\varphi}_\eta}} \int_I \zeta^2 \left(c^+_{\widetilde{\varphi}_\eta} |h_x^\varepsilon| - k \right)_+ \mathrm{d}\mathcal{L}^1 = \frac{f^\infty}{c^-_\varphi c^+_\varphi} \int_I \zeta^2 (c^+_\varphi |h_x^\varepsilon| - k)_+ \,\mathrm{d}\mathcal{L}^1. \tag{5.9}$$

In the term involving $u_x^{\varepsilon,\eta}$ it will not be possible to take the limit but it will suffice to estimate this term from below. Set

$$r(k) := \liminf_{\eta\to 0} f'_\eta(k).$$

By Lemma 2.12 (ii), $r(k) \le f^\infty$. By convexity and the fact that f_η converge to f, as shown in Lemma 2.12 (iii), it is true that

$$r(k) \ge \liminf_{\eta\to 0} \lim_{h\to 0^+} \frac{1}{h}(f_\eta(k+h) - f_\eta(k)) \ge \liminf_{\eta\to 0} \frac{1}{k}(f_\eta(k) - f_\eta(0)) = \frac{1}{k}(f(k) - f(0)).$$

Consequently,

$$f^\infty \ge \lim_{k\to\infty} r(k) \ge \lim_{k\to\infty} \frac{1}{k}(f(k) - f(0)) = f^\infty,$$

which yields $\lim_{k\to\infty} r(k) = f^\infty$. Then, using once again the fact that $c^+_{\widetilde{\varphi}_\eta} \to c^+_\varphi$ and the definition of $r(k)$ we get

$$\liminf_{\eta\to 0} f'_\eta(k)(c^+_{\widetilde{\varphi}_\eta})^{-3} \int_I \zeta^2 A_k(|u_x^{\varepsilon,\eta}|)\,\mathrm{d}\mathcal{L}^1 \ge r(k)(c^+_\varphi)^{-3} \liminf_{\eta\to 0} \int_I \zeta^2 A_k(|u_x^{\varepsilon,\eta}|)\,\mathrm{d}\mathcal{L}^1. \tag{5.10}$$

Recall that Lemma 3.5 established that $u^{\varepsilon,\eta}$ converges weakly in $W^{1,2}$ to u^ε. Since $p \mapsto A_k(|p|)$ is convex, as discussed at the beginning of the proof, the functional $w \mapsto \int_I \zeta^2 A_k(|w_x|)\,\mathrm{d}\mathcal{L}^1$ is lower semicontinuous w. r. t. weak $W^{1,2}$-convergence by a standard result, see for example [1], Theorem 1.3. Consequently,

$$\liminf_{\eta\to 0} \int_I \zeta^2 A_k(|u_x^{\varepsilon,\eta}|)\,\mathrm{d}\mathcal{L}^1 \ge \int_I \zeta^2 A_k(|u_x^\varepsilon|)\,\mathrm{d}\mathcal{L}^1. \tag{5.11}$$

Thus, applying (5.9), (5.10) and (5.11) to (5.8), we arrive at the desired inequality (5.4).

We now turn to the passage with $\varepsilon \to 0$. Plugging $G(p) = (c^+_\varphi |p| - k)_+$ in Lemma 2.4 yields

$$\lim_{\varepsilon\to 0} \int_I \zeta^2 (c^+_\varphi |h_x^\varepsilon| - k)_+ \,\mathrm{d}\mathcal{L}^1 = \int_I \zeta^2 \,\mathrm{d}(c^+_\varphi |h_x| - k)_+.$$

As shown in Lemma 3.6, $u_x^\varepsilon \overset{*}{\rightharpoonup} u_x$. We can therefore apply Lemma 2.2 for the convex function of linear growth $p \mapsto A_k(|p|)$, the non-negative $\zeta^2 \in C^1(I)$ and the sequence $u_x^\varepsilon \overset{*}{\rightharpoonup} u_x$. We obtain

$$\liminf_{\varepsilon\to 0} \int_I \zeta^2 A_k(|u_x^\varepsilon|)\,\mathrm{d}\mathcal{L}^1 = \liminf_{\varepsilon\to 0} \int_I \zeta^2 \,\mathrm{d}A_k(|u_x^\varepsilon|) \ge \int_I \zeta^2 \,\mathrm{d}A_k(|u_x|).$$

Plugging the last two estimates into (5.4) finishes the proof. □

Since the next proposition is quite technical, let us comment on the essence of its proof. As mentioned in the Introduction, we aim to show that the r. h. s. of (5.3) converges to zero after limit passages with η, ε and k. We have chosen G_k to satisfy (1.15), *i. e.*,

$$\alpha := \left(G_k(u_x^{\varepsilon,\eta})\right)_x \cdot \left(DF_\eta(u_x^{\varepsilon,\eta})\right)_x \geq 0.$$

We use positivity of this term to compensate for the unknown sign of the term $\beta := 2\zeta\zeta_x G_k(u_x^{\varepsilon,\eta}) \cdot \left(DF_\eta(u_x^{\varepsilon,\eta})\right)_x$. That leads to the estimate (5) of $\alpha + \beta = \left(\zeta^2 G_k(u_x^{\varepsilon,\eta})\right)_x \cdot \left(DF_\eta(u_x^{\varepsilon,\eta})\right)_x$, from which the statement follows.

Proposition 5.2. *Let $\mathcal{R}(\varepsilon, k)$ denote the upper limit of the r. h. s. of (5.3) as $\eta \to 0$, i. e.,*

$$\mathcal{R}(\varepsilon, k) := \limsup_{\eta\to 0} -\int_I (\zeta^2 G_k(u_x^{\varepsilon,\eta}))_x \cdot \left(DF_\eta(u_x^{\varepsilon,\eta})\right)_x \, \mathrm{d}\mathcal{L}^1. \tag{5.12}$$

Then

$$\limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \mathcal{R}(\varepsilon, k) \leq 0.$$

Proof. The bulk of the proof consists of relatively complicated calculations. Throughout this part, the superscript ε, η on $u^{\varepsilon,\eta}$ will be omitted for readability. We compute

$$\begin{aligned}(\zeta^2 G_k(u_x))_x &= 2\zeta\zeta_x g_k(\widetilde{\varphi}_\eta(u_x))u_x + \zeta^2 g_k'(\widetilde{\varphi}_\eta(u_x))(\widetilde{\varphi}_\eta(u_x))_x u_x + \zeta^2 g_k(\widetilde{\varphi}_\eta(u_x))u_{xx},\\ (DF_\eta(u_x))_x &= (D(f_\eta \circ \widetilde{\varphi}_\eta)(u_x))_x + \frac{\eta}{2}\left(D\widetilde{\widetilde{\varphi}}_\eta(u_x)\right)_x.\end{aligned} \tag{5.13}$$

We deal with the terms including $f_\eta \circ \widetilde{\varphi}_\eta$ and $\widetilde{\widetilde{\varphi}}_\eta$ separately. First, we calculate

$$(D(f_\eta \circ \widetilde{\varphi}_\eta)(u_x))_x = f_\eta''(\widetilde{\varphi}_\eta(u_x))\,(\widetilde{\varphi}_\eta(u_x))_x\, D\widetilde{\varphi}_\eta(u_x) + f_\eta'(\widetilde{\varphi}_\eta(u_x))\, D^2\widetilde{\varphi}_\eta(u_x)\, u_{xx},$$

$$\begin{aligned}(\zeta^2 G_k(u_x))_x \cdot (D(f_\eta \circ \widetilde{\varphi}_\eta)(u_x))_x &= 2\zeta\zeta_x\, g_k(\widetilde{\varphi}_\eta(u_x))\, f_\eta''(\widetilde{\varphi}_\eta(u_x))\, \widetilde{\varphi}_\eta(u_x)\, (\widetilde{\varphi}_\eta(u_x))_x\\ &\quad + \zeta^2\, f_\eta''(\widetilde{\varphi}_\eta(u_x))\, (\widetilde{\varphi}_\eta(u_x)_x)^2 \left\{g_k'(\widetilde{\varphi}_\eta(u_x))\widetilde{\varphi}_\eta(u_x) + g_k(\widetilde{\varphi}_\eta(u_x))\right\}\\ &\quad + \zeta^2\, g_k(\widetilde{\varphi}_\eta(u_x))\, f_\eta'(\widetilde{\varphi}_\eta(u_x))\, u_{xx} \cdot D^2\widetilde{\varphi}_\eta(u_x)\, u_{xx}.\end{aligned} \tag{5.14}$$

Other terms vanish because $u_x \cdot D^2\widetilde{\varphi}_\eta(u_x) = 0$ by homogeneity of $\widetilde{\varphi}_\eta$, see Proposition 2.7. The last term on the r. h. s. of (5.14) is non-negative and we will ignore it. For $\sigma > 0$ there holds

$$g_k(\sigma) + g_k'(\sigma)\sigma = \sigma^{-2}\left(k f_\eta'(\sigma) + \sigma(\sigma - k) f_\eta''(\sigma)\right)\mathbf{1}_{\sigma > k} \geq 0.$$

Thus, the second term on the r. h. s. of (5.14) is also non-negative. Moreover, we can use it to estimate the first term by means of inequality $2ab \geq -a^2 - b^2$ with suitable choices of a, b, obtaining

$$(\zeta^2 G_k(u_x))_x \cdot \left(D(f_\eta \circ \widetilde{\varphi}_\eta)(u_x)\right)_x \geq -\zeta_x^2 f_\eta''(\widetilde{\varphi}_\eta(u_x)) \frac{g_k^2(\widetilde{\varphi}_\eta(u_x))\widetilde{\varphi}_\eta^2(u_x)}{g_k(\widetilde{\varphi}_\eta(u_x)) + g_k'(\widetilde{\varphi}_\eta(u_x))\widetilde{\varphi}_\eta(u_x)} = -\zeta_x^2 f_\eta''(\widetilde{\varphi}_\eta(u_x))\mathcal{K}(\widetilde{\varphi}_\eta(u_x)),$$

where we have denoted

$$\mathcal{K}(\sigma) := \frac{g_k^2(\sigma)\sigma^2}{g_k(\sigma) + g_k'(\sigma)\sigma} = \frac{(\sigma - k)_+^2 |f_\eta'(\sigma)|^2}{k f_\eta'(\sigma) + \sigma(\sigma - k) f_\eta''(\sigma)}. \tag{5.15}$$

Forgetting about the non-negative term kf'_η in the denominator and then using linear growth of f_η allows us to infer that

$$\zeta_x^2 f''_\eta(\sigma)\mathcal{K}(\sigma) \le \frac{\max \zeta_x^2}{\sigma}(\sigma - k)_+ |f'_\eta(\sigma)|^2 \le \frac{(f^\infty)^2 \max \zeta_x^2}{k}(\sigma - k)_+.$$

All in all, we get

$$\left(\zeta^2 G_k(u_x)\right)_x \cdot (D(f_\eta \circ \widetilde{\varphi}_\eta)(u_x))_x \ge -k^{-1}(f^\infty)^2 \max \zeta_x^2 \left(\widetilde{\varphi}_\eta(u_x) - k\right)_+. \tag{5.16}$$

Let us now turn to the neglected terms of order $\eta/2$. Recall that according to Proposition 2.16, $\widetilde{\widetilde{\varphi}}_\eta = \widetilde{\varphi}_\eta^2$ outside $B_\eta(0)$ and thus whenever $|u_x| > \eta$, it holds that

$$\frac{1}{2}\left(D\widetilde{\widetilde{\varphi}}_\eta(u_x)\right)_x = (\widetilde{\varphi}_\eta(u_x))_x\, D\widetilde{\varphi}_\eta(u_x) + \widetilde{\varphi}_\eta(u_x) D^2\widetilde{\varphi}_\eta(u_x)\, u_{xx}.$$

As long as $\eta < k$, we have, recalling (5.13)

$$\begin{aligned}\frac{1}{2}(\zeta^2 G_k(u_x))_x \cdot \left(D\widetilde{\widetilde{\varphi}}_\eta(u_x)\right)_x &= 2\zeta\zeta_x\, g_k(\widetilde{\varphi}_\eta(u_x))\, \widetilde{\varphi}_\eta(u_x)\, (\widetilde{\varphi}_\eta(u_x))_x \\ &+ \zeta^2 (\widetilde{\varphi}_\eta(u_x)_x)^2 \left\{g'_k(\widetilde{\varphi}_\eta(u_x))\, \widetilde{\varphi}_\eta(u_x) + g_k(\widetilde{\varphi}_\eta(u_x)\right\} + \zeta^2\, g_k(\widetilde{\varphi}_\eta(u_x))\, \widetilde{\varphi}_\eta(u_x)\, u_{xx} \cdot D^2\widetilde{\varphi}_\eta(u_x)\, u_{xx}.\end{aligned}$$

Similarly as before, the other terms vanish due to homogeneity of $\widetilde{\varphi}_\eta$. Observing that the last two terms on the r. h. s. are non-negative and estimating the first term as before, we obtain

$$\frac{1}{2}(\zeta^2 G_k(u_x))_x \cdot \left(D\widetilde{\widetilde{\varphi}}_\eta(u_x)\right)_x \ge -\zeta_x^2 \mathcal{K}(\widetilde{\varphi}_\eta(u_x)),$$

with $\mathcal{K}$ as in (5.15). Forgetting about the term with f''_η in the denominator of $\mathcal{K}$ and keeping in mind that $f'_\eta \le f^\infty$, as shown in Lemma 2.12 (ii), one gets

$$\zeta_x^2 \mathcal{K}(\sigma) \le \max \zeta_x^2\, f^\infty k^{-1} (\sigma - k)_+^2.$$

Consequently,

$$\frac{\eta}{2}\left(\zeta^2 G_k(u_x)\right)_x \cdot \left(D\widetilde{\widetilde{\varphi}}(u_x)\right)_x \ge -\eta \max \zeta_x^2\, f^\infty k^{-1}\, \left(\widetilde{\varphi}_\eta(u_x) - k\right)_+^2. \tag{5.17}$$

From now on, we reinstate the omitted superscript ε, η on $u^{\varepsilon,\eta}$. Adding up (5.17) and (5.16) and carefully multiplying by -1, we deduce

$$-\int_I (\zeta^2 G_k(u_x^{\varepsilon,\eta}))_x \cdot (DF_\eta(u_x^{\varepsilon,\eta}))_x \,\mathrm{d}\mathcal{L}^1 \le \frac{f^\infty \max \zeta_x^2}{k}\left(f^\infty \int_I (\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta}) - k)_+ \,\mathrm{d}\mathcal{L}^1 + \eta \int_I (\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta}) - k)_+^2 \,\mathrm{d}\mathcal{L}^1\right).$$

Since $c^+_{\widetilde{\varphi}_\eta} \to c^+_\varphi$ as shown in Proposition 2.15 (ii), we can assume that $c^+_{\widetilde{\varphi}_\eta} \le 2c^+_\varphi$ for sufficiently small η. Therefore, using Lemma 3.5,

$$\int_I (\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta}) - k)_+ \,\mathrm{d}\mathcal{L}^1 \le 2c^+_\varphi \int_I |u_x^{\varepsilon,\eta}| \,\mathrm{d}\mathcal{L}^1 \le c^+_\varphi C,$$

$$\int_I (\widetilde{\varphi}_\eta(u_x^{\varepsilon,\eta}) - k)_+^2 \,\mathrm{d}\mathcal{L}^1 \le 4(c_\varphi^+)^2 \int_I |u_x^{\varepsilon,\eta}|^2 \,\mathrm{d}\mathcal{L}^1 \le 4(c_\varphi^+)^2 \left(C_\varepsilon \eta + \int_I |h_x^\varepsilon|^2 \,\mathrm{d}\mathcal{L}^1 \right)$$

and so

$$\mathcal{R}(\varepsilon, k) = \limsup_{\eta\to 0} -\int_I \left(\zeta^2 G_k(u_x^{\varepsilon,\eta})\right)_x \cdot (DF_\eta(u_x^{\varepsilon,\eta}))_x \,\mathrm{d}\mathcal{L}^1 \le \frac{2c_\varphi^+ (f^\infty)^2 \max \zeta_x^2}{k} C,$$

whence the assertion clearly follows. □

We will now prove the main theorem of this section, *i. e.*, that under the structural assumption $F = f \circ \varphi$ for a convex, non-decreasing f of linear growth and a coercive anisotropy φ, it holds that $|u_x^s| \le (c_\varphi^+/c_\varphi^-)^2 |h_x^s|$.

Proof of Theorem 1.2. Recalling (5.3) and (5.12), Proposition 5.1 implies that the minimizer u of $\mathcal{E}$ satisfies

$$\frac{r(k)}{(c_\varphi^+)^3} \int_I \zeta^2 \,\mathrm{d}A_k(|u_x|) \le \frac{f^\infty}{c_\varphi^- c_\varphi^+} \int_I \zeta^2 \,\mathrm{d}(c_\varphi^+|h_x| - k)_+ + \lambda \limsup_{\varepsilon\to 0} \mathcal{R}(\varepsilon, k), \tag{5.18}$$

where $\lim_{k\to\infty} r(k) = f^\infty$. Let us now pass with $k \to \infty$. By definition of the measure $(c_\varphi^+|h_x| - k)_+$,

$$\int_I \zeta^2 \,\mathrm{d}(c_\varphi^+|h_x| - k)_+ = \int_I \zeta^2 (c_\varphi^+|h_x^{ac}| - k)_+ \,\mathrm{d}\mathcal{L}^1 + c_\varphi^+ \int_I \zeta^2 \,\mathrm{d}|h_x^s|.$$

As $\zeta^2|h_x^{ac}|$ is an integrable function, we can use the dominated convergence theorem to pass to the limit in the first integral on the r. h. s. and see that it vanishes. Therefore, by Proposition 5.2, the upper limit of the r. h. s. of (5.18) as $k \to \infty$ is bounded by $f^\infty (c_\varphi^-)^{-1} \int_I \zeta^2 \,\mathrm{d}|h_x^s|$.

As shown at the beginning of the proof of Proposition 5.1, A_k is a convex function of linear growth. Its recession function A_k^∞ equals

$$A_k^\infty = \lim_{s\to\infty} \frac{A_k(s)}{s} = \frac{1}{s^2}(c_\varphi^- s - k)_+ (c_\varphi^+ s - k)_+ = \lim_{s\to\infty} \left(c_\varphi^- - \frac{k}{s}\right)_+ \left(c_\varphi^+ - \frac{k}{s}\right)_+ = c_\varphi^- c_\varphi^+.$$

Therefore, by the definition of the measure $A_k(|u_x|)$,

$$\int_I \zeta^2 \,\mathrm{d}A_k(|u_x|) = \int_I \zeta^2 A_k(|u_x^{ac}|) \,\mathrm{d}\mathcal{L}^1 + c_\varphi^- c_\varphi^+ \int_I \zeta^2 \,\mathrm{d}|u_x^s|.$$

Thus, passing to the limit on the l. h. s. of (5.18) by dominated convergence, we obtain

$$\int_I \zeta^2 \,\mathrm{d}|u_x^s| \le \left(\frac{c_\varphi^+}{c_\varphi^-}\right)^2 \int_I \zeta^2 \,\mathrm{d}|h_x^s|. \tag{5.19}$$

To finish the proof, it suffices to apply Lemma 2.6 to $\mu = |u_x^s|$ and $\nu = (c_\varphi^+/c_\varphi^-)^2 |h_x^s|$. □

6. The regular case

This section is devoted to proving Theorem 1.3, *i. e.*, we assume that $F = f \circ \varphi$ is strictly convex and differentiable on $\mathbb{R}^n$ and show an improved estimate on $|u_x|$. Recall that, by Proposition 2.10, $f \in C^1([0,\infty))$ with $f'(0) = 0$ and that $\varphi \in C^1(\mathbb{R}^n \setminus \{0\})$.

We define $\mathcal{F}_M \colon M(I,\mathbb{R}^n) \to [0,\infty[$ by

$$\mathcal{F}_M(\mu) = F(\mu)(I).$$

Clearly, we have $\mathcal{F}(w) = \mathcal{F}_M(w_x)$ for $w \in BV(I,\mathbb{R}^n)$ where $\mathcal{F}(w) = F(w_x)(I)$, as defined in (1.5). Thanks to differentiability of F, we can differentiate $\mathcal{F}_M$ according to Proposition 6.1. This will enable us to learn more about regularity of composite function $DF(u_x^{ac})$ in Lemma 6.2.

Proposition 6.1. *([30], Theorem 2.4) Functional $\mathcal{F}_M$ is differentiable at $\alpha \in M(I,\mathbb{R}^n)$ in direction $\beta \in M(I,\mathbb{R}^n)$ if and only if $|\beta^s|$ is absolutely continuous w. r. t. $|\alpha^s|$. Then the derivative is of the form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathcal{F}_M(\alpha + t\beta) = \int_I DF(\alpha^{ac}) \cdot \beta^{ac} \,\mathrm{d}\mathcal{L}^1 + \int_I DF^\infty\left(\frac{\alpha^s}{|\alpha^s|}\right) \cdot \,\mathrm{d}\beta^s,$$

where $F^\infty(p) = \lim_{t\to\infty} \frac{1}{t} F(tp)$.

By $W_0^{1,2}(I,\mathbb{R}^n)$ we denote the space of $W^{1,2}(I,\mathbb{R}^n)$ functions that vanish at ∂I.

Lemma 6.2. *The following statements hold:*

(i) $DF(u_x^\varepsilon) \in W_0^{1,2}(I,\mathbb{R}^n)$ and $\lambda\,(DF(u_x^\varepsilon))_x = u^\varepsilon - h^\varepsilon$ a. e. in I;
(ii) $DF(u_x^{ac}) \in W_0^{1,2}(I,\mathbb{R}^n)$ and $\lambda\,(DF(u_x^{ac}))_x = u - h$ a. e. in I.

Proof. We will show only proof of (ii) as it is the more interesting one. Statement (i) is proved in the same way, except there is no need to invoke Proposition 6.1 to carry out the differentiation, as u_x^ε does not have a singular part.

By Proposition 6.1, we can write the derivative of $\mathcal{F}$ at point u in direction $\xi \in W^{1,1}(I,\mathbb{R}^n)$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathcal{F}(u + t\,\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathcal{F}_M(u_x + t\,\xi_x) = \int_I DF(u_x^{ac}) \cdot \xi_x \,\mathrm{d}\mathcal{L}^1,$$

since the integral with respect to the singular part vanishes. As u is the minimizer of $\mathcal{E}$, it holds that $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{E}(u + t\,\xi) = 0$, which translates to the Euler-Lagrange equation

$$-\lambda \int_I DF(u_x^{ac}) \cdot \xi_x \,\mathrm{d}\mathcal{L}^1 = \int_I \xi \cdot (u-h)\,\mathrm{d}\mathcal{L}^1 \quad \text{for } \xi \in W^{1,1}(I,\mathbb{R}^n). \tag{6.1}$$

This implies that the distributional derivative of $DF(u_x^{ac})$ coincides with $(u-h)/\lambda \in L^2(I,\mathbb{R}^n)$. Since DF is bounded, $DF(u_x^{ac}) \in L^2(I,\mathbb{R}^n)$. Consequently, $DF(u_x^{ac}) \in W^{1,2}(I,\mathbb{R}^n)$, which implies also that there exists a continuous representative of this function.

We can show that any continuous representative equals zero at the endpoints of the interval $I =]a,b[$ by the following standard argument. Let $\zeta^\ell(x) = (-\ell x + 1 + a\ell)\mathbf{1}_{[a,a+1/\ell]}$ be a piecewise linear function equal 1 for $x = a$ and 0 on $[a, a+1/\ell]$. From the Euler–Lagrange equation (6.1) we deduce

$$-\lambda \int_I DF(u_x^{ac})\zeta_x^\ell \,\mathrm{d}\mathcal{L}^1 = \int_I \zeta^\ell (u-h)\,\mathrm{d}\mathcal{L}^1.$$

As $\zeta_x^\ell = -\ell\mathbf{1}_{[a,a+1/\ell]}$, continuity of $DF(u_x^{ac})$ implies that the l. h. s. converges to $DF(u_x^{ac}(a))$ as $\ell \to \infty$. Moreover, dominated convergence theorem enables us to pass with the limit under the integral sign in order to get zero on the r. h. s., which implies that $DF(u_x^{ac}(a)) = 0$. To get $DF(u_x^{ac}(b)) = 0$, one needs to repeat this argument with $\zeta^\ell = (\ell x + 1 - \ell b) \cdot \mathbf{1}_{[b-1/\ell,b]}$. □

Limit passage with η

As in Section 5, we consider the approximate problem with F_η to be defined by (3.3), *i. e.*,

$$F_\eta = f_\eta \circ \widetilde{\varphi}_\eta + \frac{\eta}{2}\widetilde{\widetilde{\varphi}}_\eta.$$

We begin by showing that additional regularity of F allows us to deduce that up to a subsequence, $u_x^{\varepsilon,\eta}$ converge a. e. on I to u_x^ε. This will be crucial in obtaining better estimates than in Section 5.

Lemma 6.3. *There exists a sequence η_j with $\eta_j \to 0$ as $j \to \infty$ such that u_x^{ε,η_j} converges to u_x^ε a. e. on I.*

Proof. To begin with, we will show that the sequence $DF_\eta(u_x^{\varepsilon,\eta})$ converges to $DF(u_x^\varepsilon)$ in $C(\overline{I},\mathbb{R}^n)$. Due to the choice of $h^\varepsilon = h * \varrho_\varepsilon$ and properties established in Lemma 3.5, functions $u^{\varepsilon,\eta} - h^\varepsilon$ converge in L^2 to $u^\varepsilon - h^\varepsilon$. In light of Remark 3.1 and Lemma 6.2, $(DF_\eta(u_x^{\varepsilon,\eta}))_x$ converge to $(DF(u_x^\varepsilon))_x$ in L^2, and by Hölder's inequality also in L^1. For every $x_0 \in \overline{I}$, due to the Neumann boundary condition satisfied by $u^{\varepsilon,\eta}$ and u^ε,

$$DF_\eta(u_x^{\varepsilon,\eta})(x_0) = \int_a^{x_0} (DF_\eta(u_x^{\varepsilon,\eta}))_x \,\mathrm{d}\mathcal{L}^1 \quad \text{and} \quad DF(u_x^\varepsilon)(x_0) = \int_a^{x_0} (DF(u_x^\varepsilon))_x \,\mathrm{d}\mathcal{L}^1,$$

which allows to conclude that

$$\sup_{x_0\in\overline{I}} |DF_\eta(u_x^{\varepsilon,\eta})(x_0) - DF(u_x^\varepsilon)(x_0)| \le \int_a^b |(DF_\eta(u_x^{\varepsilon,\eta}))_x - (DF(u_x^\varepsilon))_x| \,\mathrm{d}\mathcal{L}^1 \xrightarrow{\eta\to 0} 0. \tag{6.2}$$

Secondly, we shall see that $DF_\eta(u_x^{\varepsilon,\eta}) - DF(u_x^{\varepsilon,\eta})$ converge in L^1 to zero. Recalling formula (3.3) for F_η,

$$\int_I |DF_\eta(u_x^{\varepsilon,\eta}) - DF(u_x^{\varepsilon,\eta})| \,\mathrm{d}\mathcal{L}^1 \le \int_I |D(f_\eta\circ\widetilde{\varphi}_\eta)(u_x^{\varepsilon,\eta}) - D(f\circ\varphi)(u_x^{\varepsilon,\eta})| \,\mathrm{d}\mathcal{L}^1 + \frac{\eta}{2}\int_I |D\widetilde{\widetilde{\varphi}}_\eta(u_x^{\varepsilon,\eta})| \,\mathrm{d}\mathcal{L}^1. \tag{6.3}$$

By Proposition 2.15 (iv), we can see that the first integral on the r. h. s. converges to zero with $\eta \to 0$. On the other hand, by Proposition 2.16, for small $\eta > 0$ we have

$$\int_I |D\widetilde{\widetilde{\varphi}}_\eta(u_x^{\varepsilon,\eta})| \,\mathrm{d}\mathcal{L}^1 \le C\int_I 1 + |u_x^{\varepsilon,\eta}| \,\mathrm{d}\mathcal{L}^1,$$

which is bounded independently of ε, η by Lemma 3.5.

Convergence in L^1 of $DF_\eta(u_x^{\varepsilon,\eta}) - DF(u_x^{\varepsilon,\eta})$ to zero with $\eta \to 0$ implies existence of a subsequence η_j for which

$$\lim_{j\to\infty} \left|DF_{\eta_j}(u_x^{\varepsilon,\eta_j}) - DF(u_x^{\varepsilon,\eta_j})\right| = 0 \text{ a. e. on } I,$$

see [24], Section 1.3, Theorem 5. Together with (6.2), this yields

$$\lim_{j\to\infty} |DF(u_x^{\varepsilon,\eta_j}) - DF(u_x^\varepsilon)| = 0 \text{ a. e. on } I.$$

Since F is strictly convex, DF is strictly monotone and hence injective. Moreover, DF is continuous and hence by Brouwer's invariance of domain theorem, DF is a homeomorphism onto its image, which means that there exists DF^{-1}, continuous inverse of DF. Therefore, at any point in which $DF(u_x^{\varepsilon,\eta_j})(x_0) \to DF(u_x^\varepsilon)(x_0)$ and the mappings $DF(u_x^{\varepsilon,\eta_j})$ and $DF(u_x^\varepsilon)$ are actual compositions, we have

$$u_x^{\varepsilon,\eta_j}(x_0) = DF^{-1}\left(DF(u_x^{\varepsilon,\eta_j}(x_0))\right) \to DF^{-1}\left(DF(u_x^\varepsilon(x_0))\right) = u_x^\varepsilon(x_0).$$

Since the set of such points is of full measure, we have established the desired a. e. convergence of u_x^{ε,η_j} to u_x^ε. □

In this section we use the same test function as in Section 5. In particular, recall that G_k was defined in (5.1) and that $\zeta \in C_c^1(U)$ where U is an open interval in I. Thus, we obtain the same equation (5.3). We deal with its r. h. s. exactly as in Proposition 5.2 but modify the reasoning on the l. h. s. of (5.3). Thanks to the a. e. convergent subsequence found in Lemma 6.3, we can perform a more subtle limit passage in the proposition below.

Proposition 6.4. *There exists a sequence η_j with $\eta_j \to 0$ as $j \to \infty$ for which the l. h. s. of (5.3) satisfies*

$$\lim_{j\to\infty} \int_I \zeta^2 G_k(u_x^{\varepsilon,\eta_j}) \cdot (u_x^{\varepsilon,\eta_j} - h_x^\varepsilon)\, \mathrm{d}\mathcal{L}^1 = \int_I \zeta^2 \frac{(\varphi(u_x^\varepsilon) - k)_+}{\varphi^2(u_x^\varepsilon)} f'(\varphi(u_x^\varepsilon))\, (u_x^\varepsilon - h_x^\varepsilon) \cdot u_x^\varepsilon \, \mathrm{d}\mathcal{L}^1.$$

Proof. Let us choose the sequence from Lemma 6.3, for which we know that the integrand is convergent a. e. Then, it suffices to check that it is uniformly integrable and apply the Vitali convergence theorem. Bearing in mind the formula for $G_k(\cdot)$, the absolute value of the integrand can be estimated from above by $C|u_x^{\varepsilon,\eta} - h_x^\varepsilon|$ for some constant C. Thus, uniform integrability follows from boundedness of the sequence $\{u_x^{\varepsilon,\eta}\}_\eta$ in L^2 shown in Lemma 3.5. □

Proposition 6.5. *The following estimate holds*

$$f'(k) \int_I \zeta^2 |u_x^\varepsilon|^2 \frac{(\varphi(u_x^\varepsilon) - k)_+}{\varphi^2(u_x^\varepsilon)}\, \mathrm{d}\mathcal{L}^1 \leq f^\infty \int_I \zeta^2 |h_x^\varepsilon||u_x^\varepsilon| \frac{(\varphi(u_x^\varepsilon) - k)_+}{\varphi^2(u_x^\varepsilon)}\, \mathrm{d}\mathcal{L}^1 + \lambda \mathcal{R}(\varepsilon, k),$$

where $\mathcal{R}(\varepsilon, k)$ was defined in Proposition 5.2.

Proof. The proof boils down to observing what has already been proved about the equation (5.3), which we recall below:

$$\int_I \zeta^2 G_k(u_x^{\varepsilon,\eta}) \cdot (u_x^{\varepsilon,\eta} - h_x^\varepsilon)\, \mathrm{d}\mathcal{L}^1 = -\lambda \int_I (\zeta^2 G_k(u_x^{\varepsilon,\eta}))_x \cdot (DF_\eta(u_x^{\varepsilon,\eta}))_x \, \mathrm{d}\mathcal{L}^1.$$

After taking lim sup with $\eta \to 0$ on both sides of this equation, the term $\lambda\mathcal{R}(\varepsilon, k)$ appears on its r. h. s., as proved in Proposition 5.2. By Proposition 6.4,

$$\int_I \zeta^2 \frac{(\varphi(u_x^\varepsilon) - k)_+}{\varphi^2(u_x^\varepsilon)} f'(\varphi(u_x^\varepsilon))(u_x^\varepsilon - h_x^\varepsilon) \cdot u_x^\varepsilon \, \mathrm{d}\mathcal{L}^1 \leq \limsup_{\eta\to 0} \int_I \zeta^2 G_k(u_x^{\varepsilon,\eta}) \cdot (u_x^{\varepsilon,\eta} - h_x^\varepsilon)\, \mathrm{d}\mathcal{L}^1 = \lambda \mathcal{R}(\varepsilon, k),$$

which after rearranging terms and using Cauchy–Schwarz inequality yields

$$\int_I \zeta^2 |u_x^\varepsilon|^2 \frac{(\varphi(u_x^\varepsilon) - k)_+}{\varphi^2(u_x^\varepsilon)} f'(\varphi(u_x^\varepsilon))\, \mathrm{d}\mathcal{L}^1 \leq \int_I \zeta^2 |h_x^\varepsilon||u_x^\varepsilon| \frac{(\varphi(u_x^\varepsilon) - k)_+}{\varphi^2(u_x^\varepsilon)} f'(\varphi(u_x^\varepsilon))\, \mathrm{d}\mathcal{L}^1 + \lambda \mathcal{R}(\varepsilon, k).$$

Then, on the l. h. s. we use the facts that f' is non-decreasing and that the integration takes place on the set where $\varphi(u_x^\varepsilon) > k$, whereas on the r. h. s. the fact that $f'(t) \leq f^\infty$ for all t, as shown in Lemma 2.11 (iii). This concludes the proof. □

Limit passage with ε

In this section, we will make use of $\overline{\mathbb{R}^n}$, the spherical compactification of $\mathbb{R}^n$, introduced in Definition 2.17.

Lemma 6.6. *There exist functions $\overline{u}_x^\varepsilon, \overline{u}_x^{ac} \in C(\overline{I}, \overline{\mathbb{R}^n})$ s. t. $\overline{u}_x^\varepsilon \to \overline{u}_x^{ac}$ in $C(\overline{I}, \overline{\mathbb{R}^n})$ and $\overline{u}_x^\varepsilon(x_0) = u_x^\varepsilon(x_0)$, $\overline{u}_x^{ac}(x_0) = u_x^{ac}(x_0)$ for $\mathcal{L}^1$-a. e. $x_0 \in \overline{I}$.*

Proof. Lemma 6.2 implies (by reasoning similar to the beginning of the proof of Lemma 6.3) that $DF(u_x^\varepsilon)$ converge to $DF(u_x^{ac})$ in $C(\overline{I},\mathbb{R}^n)$ and therefore in $C(\overline{I},\overline{\mathbb{R}^n})$. The map $\overline{DF}$ is a homeomorphism onto its image, as established in Lemma 2.19, which prompts the following definition:

$$\overline{u}_x^\varepsilon := \overline{DF}^{-1} \circ DF(u_x^\varepsilon) \text{ and } \overline{u}_x^{ac} := \overline{DF}^{-1} \circ DF(u_x^{ac}).$$

Requested properties of these functions now follow from continuity of $\overline{DF}^{-1}$ and the fact that u_x^ε and u_x^{ac} are almost everywhere finite. ☐

Lemma 6.7 below in a sense shows the importance of the function $\overline{u}_x^{ac}$ introduced in Lemma 6.6. It says that the set of points in which u_x^{ac} does not blow up has $|u_x^s|$-measure zero. Therefore, to understand the behavior of $|u_x^s|$, we use $\overline{u}_x^{ac}$, the extension of u_x^{ac} into $\overline{\mathbb{R}^n}$, to gain insight into the set $\{x : |u_x^{ac}(x)| = \infty\}$.

Lemma 6.7. *For $|u_x^s|$-a. e. $x \in I$ it is true that $\overline{u}_x^{ac}(x) \in \mathbb{S}_\infty^{n-1}$.*

Proof. Choose x_0 such that $|\overline{u}_x^{ac}(x_0)| < \infty$. Since $\overline{u}_x^{ac}$ is continuous, there exists an open interval $J \subset I$, $x_0 \in J$, such that for any $y \in J$ it is true that $\overline{u}_x^{ac}(y) \in \mathbb{R}^n$. Then for any $\xi \in C_c(J)$,

$$\left|\int_J u_x^\varepsilon \xi \, \mathrm{d}\mathcal{L}^1 - \int_J u_x^{ac} \xi \, \mathrm{d}\mathcal{L}^1\right| = \left|\int_J \overline{u}_x^\varepsilon \xi \, \mathrm{d}\mathcal{L}^1 - \int_J \overline{u}_x^{ac} \xi \, \mathrm{d}\mathcal{L}^1\right| \le \sup_J |\overline{u}_x^\varepsilon - \overline{u}_x^{ac}| \int_J \xi \, \mathrm{d}\mathcal{L}^1 \xrightarrow{\varepsilon\to 0} 0,$$

due to the facts that $\overline{u}_x^\varepsilon$ coincides with u_x^ε $\mathcal{L}^1$-a. e., $\overline{u}_x^{ac}$ coincides with u_x^{ac} $\mathcal{L}^1$-a. e. and that $\overline{u}_x^\varepsilon$ converge in $C(\overline{I},\overline{\mathbb{R}^n})$ to $\overline{u}_x^{ac}$. Thus we have shown that $u_x^\varepsilon \overset{*}{\rightharpoonup} u_x^{ac}$ on J. On the other hand, Lemma 3.6 says that $u_x^\varepsilon \overset{*}{\rightharpoonup} u_x$ on J, which implies that $u_x^s \llcorner J = 0$ and, consequently, that

$$|u_x^s| \llcorner J = 0,$$

which in turn means that $x_0 \notin \operatorname{supp}|u_x^s|$ (the support of the measure $|u_x^s|$). We have therefore established that if $x_0 \in \operatorname{supp}|u_x^s|$, then $\overline{u}_x^{ac}(x_0) \in \mathbb{S}_\infty^{n-1}$. Recalling that the set $\operatorname{supp}|u_x^s|$ is of full $|u_x^s|$ measure we conclude the proof.

☐

We are now in the position to prove the main theorem of this section. Given $F = f \circ \varphi$ for a more regular f and φ than in Theorem 1.2, it is possible to show the estimate $|u_x^s| \le |h_x^s|$ without the multiplicative constant that appeared in Theorem 1.2.

Proof of Theorem 1.3. Let us recall the inequality proved in Proposition 6.5

$$L_{\varepsilon,k} := f'(k) \int_I \zeta^2 |u_x^\varepsilon|^2 \frac{(\varphi(u_x^\varepsilon) - k)_+}{\varphi^2(u_x^\varepsilon)} \, \mathrm{d}\mathcal{L}^1 \le f^\infty \int_I \zeta^2 |h_x^\varepsilon||u_x^\varepsilon| \frac{(\varphi(u_x^\varepsilon) - k)_+}{\varphi^2(u_x^\varepsilon)} \, \mathrm{d}\mathcal{L}^1 + \lambda \mathcal{R}(\varepsilon, k) := R_{\varepsilon,k},$$

where $\zeta \in C_c^1(I)$ and $\limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \mathcal{R}(\varepsilon, k) \le 0$ as stated in Proposition 5.2. We will firstly pass with ε to zero. For $k > 0$, we define a continuous function $b_k \colon \overline{\mathbb{R}^n} \to \mathbb{R}$ by

$$b_k(p) := \begin{cases} 0 & \text{if } p = 0, \\ \frac{|p|(\varphi(p)-k)_+}{\varphi^2(p)} & \text{if } p \in \mathbb{R}^n \setminus \{0\}, \\ \frac{1}{\varphi(\omega)} & \text{if } p = \infty\omega \in \mathbb{S}_\infty^{n-1}. \end{cases}$$

Using Lemma 6.6 and the definition of b_k, we can rewrite

$$L_{\varepsilon,k} = f'(k) \int_I \zeta^2 \left(b_k(\overline{u}_x^\varepsilon) - b_k(\overline{u}_x^{ac})\right) |u_x^\varepsilon| \, \mathrm{d}\mathcal{L}^1 + f'(k) \int_I \zeta^2 b_k(\overline{u}_x^{ac}) |u_x^\varepsilon| \, \mathrm{d}\mathcal{L}^1.$$

Again in view of Lemma 6.6, we see that $b_k(\overline{u}_x^\varepsilon)$ converges to $b_k(\overline{u}_x^{ac})$ in $C(\overline{I},\mathbb{R})$, which implies that the first term converges to zero (recall that $\{u_x^\varepsilon\}_\varepsilon$ is bounded in L^1, see Lem. 3.6). The second term is lower semicontinuous w. r. t. weak* convergence of u_x^ε to u_x (see Lem. 2.2) and so

$$\liminf_{\varepsilon\to 0} L_{\varepsilon,k} \geq f'(k)\int_I \zeta^2 b_k(\overline{u}_x^{ac})\,\mathrm{d}|u_x|. \tag{6.4}$$

By Lemma 2.4 applied to $G = |\cdot|$, $|h_x^\varepsilon| \overset{*}{\rightharpoonup} |h_x|$. Therefore, similarly as in the previous paragraph, we show that

$$\limsup_{\varepsilon\to 0} R_{\varepsilon,k} \leq f^\infty \int_I \zeta^2 b_k(\overline{u}_x^{ac})\,\mathrm{d}|h_x| + \limsup_{\varepsilon\to 0} \lambda\mathcal{R}(\varepsilon,k). \tag{6.5}$$

As a result, collecting inequalities obtained above, we get

$$\begin{aligned} f'(k)\int_I \zeta^2 b_k(\overline{u}_x^{ac})\,\mathrm{d}|u_x^s| &\leq f'(k)\int_I \zeta^2 b_k(\overline{u}_x^{ac})\,\mathrm{d}|u_x| \overset{(6.4)}{\leq} \liminf_{\varepsilon\to 0} L_{\varepsilon,k} \leq \limsup_{\varepsilon\to 0} R_{\varepsilon,k} \\ &\overset{(6.5)}{\leq} f^\infty \int_I \zeta^2 b_k(\overline{u}_x^{ac})|h_x^{ac}|\,\mathrm{d}\mathcal{L}^1 + f^\infty\int_I \zeta^2 b_k(\overline{u}_x^{ac})\,\mathrm{d}|h_x^s| + \limsup_{\varepsilon\to 0}\lambda\mathcal{R}(\varepsilon,k). \end{aligned} \tag{6.6}$$

We will now pass with k to infinity. At any point $x\in I$ for which $\overline{u}_x^{ac}(x)\in\mathbb{S}_\infty^{n-1}$, set $\omega(x)\in\mathbb{S}_1^{n-1}$ to be such that $\overline{u}_x^{ac}(x) = \infty\,\omega(x)$. Clearly, for any $x\in I$

$$b_k(\overline{u}_x^{ac}(x)) \xrightarrow{k\to\infty} b(x) := \frac{1}{\varphi(\omega(x))}\mathbf{1}_{\{x\colon \overline{u}_x^{ac}(x)\in\mathbb{S}_\infty^{n-1}\}}(x).$$

Observe that by Lemma 6.7, b is positive $|u_x^s|$-a. e. Since $b_k(p)\leq 1/c_\varphi^-$ for $p\in\mathbb{R}^n$, we can use dominated convergence theorem to pass with k to infinity in (6.6) and thus obtain

$$f^\infty\int_I \zeta^2 b\,\mathrm{d}|u_x^s| \leq \int_I \zeta^2 b\,|h_x^{ac}|\,\mathrm{d}\mathcal{L}^1 + f^\infty\int_I\zeta^2 b\,\mathrm{d}|h_x^s| = f^\infty\int_I \zeta^2 b\,\mathrm{d}|h_x^s|.$$

The second equality follows from the fact that $|\overline{u}_x^{ac}|$ is finite $\mathcal{L}^1$-a. e. Dividing by f^∞ yields

$$\int_I \zeta^2 b\,\mathrm{d}|u_x^s| \leq \int_I \zeta^2 b\,\mathrm{d}|h_x^s|. \tag{6.7}$$

Denoting $\mathrm{d}\mu = b\,\mathrm{d}|u_x^s|$, $\mathrm{d}\nu = b\,\mathrm{d}|h_x^s|$ we deduce $\mu\leq\nu$ as Borel measures by Lemma 2.6, and therefore also $\widetilde{\mu}\leq\widetilde{\nu}$, where

$$\mathrm{d}\widetilde{\mu} := \varphi(\omega)\,\mathrm{d}\mu = \mathbf{1}_{\{x\colon\overline{u}_x^{ac}(x)\in\mathbb{S}_\infty^{n-1}\}}\,\mathrm{d}|u_x^s|, \qquad \mathrm{d}\widetilde{\nu} := \varphi(\omega)\,\mathrm{d}\nu = \mathbf{1}_{\{x\colon\overline{u}_x^{ac}(x)\in\mathbb{S}_\infty^{n-1}\}}\,\mathrm{d}|h_x^s|.$$

By Lemma 6.7, $\widetilde{\mu}$ actually coincides with $|u_x^s|$. On the other hand, clearly $\widetilde{\nu}\leq|h_x^s|$, which concludes the proof. □

7. Extensions

Gradient flows

We recall that the convex, lower semicontinuous function $\mathcal{F}$ on the Hilbert space $L^2(I,\mathbb{R}^n)$, given by (1.5), defines a unique gradient flow [31]. In other words, for any $v_0\in L^2(I,\mathbb{R}^n)$ there exists a unique function

$v \in C([0,\infty[, L^2(I,\mathbb{R}^n)) \cap W^{1,2}_{loc}(0,T; L^2(I,\mathbb{R}^n))$ such that

$$v_t(t) \in -\partial\mathcal{F}(v(t)) \quad \text{for a.e. } t>0, \qquad v(0)=v_0. \tag{7.1}$$

In the case $F = |\cdot|$ this is a vectorial version of the so-called *total variation flow*, formally given by the equation

$$v_t = \left(\frac{v_x}{|v_x|}\right)_x \text{ in }]0,\infty[\times I, \qquad \frac{v_x}{|v_x|} = 0 \text{ on }]0,\infty[\times \partial I,$$

cf. [32]. Let us mark here the dependence of $\mathcal{E}$ on h and λ by denoting $\mathcal{E}^\lambda_h \equiv \mathcal{E}$. The map associating the minimizer of $\mathcal{E}^\lambda_h$ to a given h coincides with the resolvent operator for the subdifferential $-\partial\mathcal{F}$. This is the basis of the *minimizing movements scheme*: for a given $N \in \mathbb{N}$ we iteratively define

$$v_0^N = v_0, \qquad v_j^N = \arg\min \mathcal{E}^{1/N}_{v^N_{j-1}} \quad \text{for } j = 1,2,\ldots$$

Then, for a. e. $t > 0$ we have

$$v_j^N \to v(t) \quad \text{in } L^2(I,\mathbb{R}^n) \quad \text{if } N\to\infty,\ j/N \to t. \tag{7.2}$$

Let $F = \varphi$ be a coercive anisotropy. Then, by Theorem 1.1, for all $N, j \in \mathbb{N}$, it is true that $v_j^N \in BV(U,\mathbb{R}^n)$ and

$$|(v_j^N)_x|(V) \le |(v_{j-1}^N)_x|(V) \le \ldots \le |(v_0)_x|(V) \quad \text{for any open } V \subset U.$$

Using (7.2) and Theorem 2.1, we deduce $|v_x(t)|(V) \le |(v_0)_x|(V)$ for any open $V \subset U$ and a. e. $t > 0$. Thus, we obtain

Corollary 7.1. *Suppose that $F = \varphi$ is a coercive anisotropy. If $v_0 \in BV(U,\mathbb{R}^n)$ with open $U \subset I$, then*

$$|v_x(t)| \le |v_{0,x}| \qquad \text{as Borel measures on } U. \tag{7.3}$$

This generalizes an analogous result for the scalar 1D total variation flow from [8, 9]. We note also a recent paper [33], where similar estimate was obtained for more general parabolic equations in the scalar 1D case $m = n = 1$. Moreover, the authors provide conditions for instantaneous regularization $BV \to W^{1,1}_{loc}$ and $L^1 \to W^{1,1}_{loc}$.

We are unable to transfer Theorems 1.2 and 1.3 to the gradient flow setting since $|u_x^s|$ does not have good semicontinuity properties. On the other hand, intermediate estimates (5.18) and (6.6) do not behave well under iteration.

General fidelity

One can consider a generalized version of $\mathcal{E}$ given by

$$\mathcal{E}(w) = \mathcal{F}(w) + \mathcal{H}(w-h) \quad \text{with } \mathcal{H}(w-h) = \int_\Omega \Phi(w-h)$$

where $\Phi\colon \mathbb{R}^n \to [0,\infty[$ is convex. The function $\mathcal{H}$ is called *fidelity*. A typical example is the L^p fidelity, $p \ge 1$, given by

$$\mathcal{H}(w-h) = \frac{1}{p}\int_\Omega |w-h|^p.$$

Until now, we restricted ourselves to the L^2 case, however other choices also appear in applications, in particular the L^1 fidelity. In order to discuss general fidelities rigorously, we would have to adapt various auxiliary facts and lemmata that we use. This is possible, but we will not do this here. One reason is to avoid increased obfuscation and/or length of the paper. Instead, we will include a brief discussion of the general fidelity setting on heuristic level.

Our results can be generalized to other fidelities in at least two ways. First, let us consider a general convex function Φ of class C^2 satisfying

$$\ell\,\mathrm{Id} \le D^2\Phi(z) \le L\,\mathrm{Id} \quad \text{for } |z| < R, \quad 0 < \ell < L \tag{7.4}$$

and suppose that $f \in L^\infty(I, \mathbb{R}^n)$. Under our usual hypotheses of convexity and linear growth on F, one can show that there exists a unique minimizer $u \in BV(I, \mathbb{R}^n) \subset L^\infty(I, \mathbb{R}^n)$ of $\mathcal{E}$. Let us assume that

$$|u - f| < R \quad \text{on } I. \tag{7.5}$$

Formally, u satisfies the Euler-Lagrange equation

$$D\Phi(u-h) = \lambda(DF(u_x))_x \text{ in } I, \quad DF(u_x) = 0 \text{ on } \partial I. \tag{7.6}$$

As before, we test (7.6) with test functions of form $(\zeta^2 g(u_x)u_x)_x$. We deal with the r. h. s. like in previous sections. On the l. h. s. we integrate by parts and estimate the result using (7.4) and (7.5), obtaining (we denote $A := D^2\Phi(u-h)$)

$$\begin{aligned}\int_I \zeta^2 g(u_x)u_x \cdot A\,(h_x - u_x) &= \int_I \zeta^2 g(u_x)u_x \cdot \sqrt{A}\sqrt{A}\,h_x - \int_I \zeta^2 g(u_x)u_x \cdot A\,u_x \\ &\le \int_I \zeta^2 g(u_x)|\sqrt{A}\,u_x|\,|\sqrt{A}\,h_x| - \int_I \zeta^2 g(u_x)u_x \cdot A\,u_x \le L\int_I \zeta^2 g(u_x)|u_x|\,|h_x| - \ell\int_I \zeta^2 g(u_x)|u_x|^2.\end{aligned}$$

Using these estimates, we can proceed with the proofs of Theorems 1.1, 1.2 and 1.3, obtaining estimates analogous to (1.10), (1.11) and (1.12) with additional multiplicative constant L/ℓ on the r. h. s. In the "regular" case of Theorem 1.3, one should be able to avoid this, but the proof would become even more technical, since the product $(\zeta^2 g(u_x)u_x)_x \cdot (D\Phi(u-h))_x = (\zeta^2 g(u_x)u_x)_x \cdot D^2\Phi(u-h)(u_x - h_x)$ depends explicitly on u and h which may be discontinuous.

If Φ is of form $\Phi(z) = \widetilde{\Phi}(|z|)$, there is a simple trick that lets us avoid the additional constant, at the same time allowing us to admit singular cases such as L^p fidelities with $1 < p < 2$. In this case, the Euler-Lagrange equation (7.6) takes form

$$|u-h|^{p-2}(u-h) = \lambda(DF(u_x))_x \text{ in } I, \quad DF(u_x) = 0 \text{ on } \partial I.$$

Before testing it with $(\zeta^2 g(u_x)u_x)_x$, we multiply both sides by $|u-h|^{2-p}$, obtaining eventually

$$\int_I \zeta^2 g(u_x)u_x \cdot (h_x - u_x) = \lambda \int_I |u-h|^{2-p}(\zeta^2 g(u_x)u_x)_x \cdot (DF(u_x))_x.$$

The l. h. s. is exactly as in previous sections. We can also estimate the quantity $(\zeta^2 g(u_x)u_x)_x \cdot DF(u_x)$ on the r. h. s. pointwise as before. Since the additional factor $|u-h|^{2-p}$ under the integral is bounded, it does not cause problems. Thus, we can obtain the same estimates as in Theorems 1.1, 1.2 and 1.3.

The same conclusion can be reached in the case of L^1 fidelity, provided that we know that the minimizer is unique, which can be ensured by requiring that $\mathcal{F}$ is strictly convex. (We note that technically in the L^1 setting the fidelity term does not give sufficient compactness for the limit passages in the course of the proof, however enough compactness is provided by $\mathcal{F}$.) Without the strict convexity assumption, we can only obtain existential statements of type *there exists a minimizer* u *of* $\mathcal{E}$ *satisfying* (1.10) (respectively (1.11), (1.12)). It would seem to require a substantially different proof to demonstrate that the estimates hold for *all* minimizers. In fact, it is not clear to us whether such a proposition is true.

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