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THE EFFECT OF DIFFUSIONS AND SOURCES ON SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. This paper deals with properties of non-negative solutions of the boundary value problem in the presence of diffusion a and source f in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, where a and f are non-decreasing continuous functions on $[0, L_0)$ and f is positive. Part of the results are new even if we restrict ourselves to the Gelfand type case $L_0 = \infty$, $a(t) = t$ and f is a convex function. We study the behavior of related extremal parameters and solutions with respect to L_0 and also to a and f in the C^0 topology. The work is carried out in a unified framework for $0 < L_0 \leq \infty$ under some interactive conditions between a and f.

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1. Introduction

The present paper is concerned with properties of non-negative solutions for the boundary value problem

$$\begin{cases} -\Delta(a(u)) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, $n \geq 1$, λ is a positive parameter, a and f are non-decreasing continuous functions on a bounded or unbounded interval $[0, L_0)$ interacting in some senses and in addition f is a positive function.

The equation in ($\mathcal{P}_\lambda$) is the stationary counterpart of the generalized porous medium equation (GPME)

$$u_t - \Delta(a(u)) = \lambda f(u) \ \text{ in } \ \Omega \times (0, T), \quad u = u(x, t), \tag{1.1}$$

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in which a is defined on the interval $[0,\infty)$ ($L_0=\infty$), whose study has attracted great interest of researchers, see the book [30] for a comprehensive reference. The equation 1.1 arises in several important situations that include groundwater filtration [2], population dynamic models [14] and spatial diffusion of biological species [21]. Physically, it also appears in the study of how the speed of propagation of disturbances is affected by the function f, see for example [3, 13]. Other interesting applications of (GPME) are the study of black holes [17] and ancient solutions to Ricci flow [4].

For various models, the function a is called diffusion and the function f represents mass sources or sinks distributed in the medium. In most of them, it is natural to consider that the diffusion increases. More precisely, one usually assumes that a is continuous and increasing in $[0,\infty)$ with $a(0)=0$ and of C^1 class with $a'>0$ in $(0,\infty)$. Moreover, in some works (see [1, 2, 30]), one considers the additional condition:

$$\frac{ta'(t)}{a(t)} \geq \alpha > 1, \quad \forall\, t \geq t_0,$$

which yields the superlinear growth $a(t)\geq C_0 t^{\alpha}$ for t large enough. A typical example corresponds to the porous medium model $a(t)=t^m$, where $m>1$ is an integer number, which has been addressed by many authors, see [1, 13, 30] and references therein.

Besides the steady state equation of (GPME), two other models motivate widely our work. The first one is the famous electrostatic MEMS model

$$\begin{cases} -\Delta u &= \dfrac{\lambda}{(1-u)^k} \quad \text{in } \Omega, \\ \quad u &= 0 \qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $k>0$ is an integer number. There is a vast literature concerning some questions of interest related to the singular problem (1.2) where $L_0=1$. We refer to the book [18] for an overview and also to the papers [25, 32] and references therein. The second one is the celebrated Gelfand's problem introduced in 1963 by Gelfand [19], which studies the existence of positive classical solution of ($\mathcal{P}_\lambda$) when Ω is a ball, $a(t)=t$ and $f(t)=\mathrm{e}^t$. More precisely, the problem ($\mathcal{P}_\lambda$)

$$\begin{cases} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ \quad u &= 0 \qquad\;\; \text{on } \partial\Omega \end{cases} \tag{1.3}$$

has been addressed in the literature for smooth bounded domains Ω and non-decreasing convex C^1 functions f in $[0,\infty)$ satisfying the following conditions:

(a) $f(0)>0$;
(b) $\lim_{t\to\infty}\frac{f(t)}{t}=\infty$.

The above problem, known nowadays as Gelfand type problem, gave rise to a beautiful chapter in the history of elliptic PDEs. For more details, see the excellent monograph [16] and also the paper [10], where a long-time conjecture on extremal functions for (1.3) is proved, and references therein.

In summary, Keller and Cohen [24], Keller and Keener [23] and Crandall and Rabinowitz [15] established the existence of a positive parameter λ^* such that the problem (1.3) admits a minimal positive classical solution u_λ for any $0<\lambda<\lambda^*$. Besides, the family of solutions (u_λ) is always non-decreasing and continuous on $\lambda\in(0,\lambda^*)$ and, moreover, is differentiable and asymptotically stable provided that f is convex. Asymptotic stability here means that the first eigenvalue of the operator $-\Delta-\lambda f'(u_\lambda)$, corresponding to the linearized problem under Dirichlet boundary condition, is positive. Already for $\lambda>\lambda^*$, they also showed that there is no nonnegative classical solution. In 1996, by taking the pointwise limit of (u_λ) as $\lambda\uparrow\lambda^*$ and assuming that f is convex, Brezis, Cazenave, Martel and Ramiandrisoa [6] proved that a minimal nonnegative weak solution $u^*\in L^1(\Omega)$ exists for $\lambda=\lambda^*$, so called extremal solution of (1.3), and no exist such a solution for $\lambda>\lambda^*$.

A central question in the literature raised by Brezis and Vazquez [7] (and also by Brezis in [5]) is the well-known Extremal Conjecture, which consists in knowing whether the extremal solution u^* is bounded and so classical. There are a number of fundamental works concerning with it. In 1973, Joseph and Lundgren [22] showed that u^* is unbounded when $f(u) = \mathrm{e}^u$ and Ω is the unit ball in $\mathbb{R}^n$ with $n \geq 10$, because $\lambda^* = 2(n-2)$ and $u^*(x) = -2\log|x|$ and, in 1980, Mignot and Puel [27] also proved in this case that u^* is bounded in dimensions $n \leq 9$. Thenceforth, other relevant results were established during the last two decades: In 2000, Nedev [28] showed that u^* is bounded in dimensions $n \leq 3$ whenever f is convex; Cabré and Capella [9] proved in 2006 that u^* is bounded when Ω is the unit ball in $\mathbb{R}^n$ and $n \leq 9$ regarding general nonlinearities f; also for such functions Cabré [8] established in 2010 the boundedness of u^* for any convex domain Ω in $\mathbb{R}^n$ with $n \leq 4$; in 2013, Villegas [31] removed the convexity condition in dimension $n = 4$, while convexity of f was still required; also in 2013, Cabré and Ros-Oton [11] obtained boundedness up to $n = 7$ when f is convex and Ω is a convex domain of double revolution; Cabré, Sanchón and Spruck [12] in 2016 derived boundedness when $n = 5$ and f is convex and satisfies the condition

$$\limsup_{t\to\infty} \frac{f'(t)}{f(t)^{1+\varepsilon}} < +\infty$$

for every $\varepsilon > 0$; lastly, more recently, Cabré, Figalli, Ros-Oton and Serra [10] proved for any smooth bounded domains that u^* is bounded in dimensions $n \leq 9$ whenever f is convex, being $n = 9$ the optimal dimension in some cases.

Our main purpose here is to develop an unified approach for the study of stationary problem ($\mathcal{P}_\lambda$) including quite general diffusion and source. In particular, this work is devoted to the study on behavior of extremal parameters and solutions of the problem ($\mathcal{P}_\lambda$) with respect to L_0, a and f.

We start with the first assumptions on the functions a and f.

Let a and f be non-decreasing continuous functions in $[0, L_0)$, where $0 < L_0 \leq \infty$. Assume also that $f(0) > 0$, so f is positive.

In these circumstances, where ($\mathcal{P}_\lambda$) involves zero Dirichlet boundary condition, roughly speaking, no non-negative solution should exist for any $\lambda \neq 0$ when a is constant close to 0. Before making this statement accurate, we need to introduce the notions of solution that will be used in what follows.

We will deal with solutions in the strong and weak senses. In both notions, it is suitable to consider a and f from $[0, L_0]$ onto $\mathbb{R} \cup \{\infty\}$.

Let δ be the function distance to the boundary of Ω, namely $\delta(x) = dist(x, \partial\Omega)$.

Definition 1.1. We say that $u \in C(\overline{\Omega})$ is a positive strong solution (supersolution) of ($\mathcal{P}_\lambda$), if $0 < u \leq L_0$ in Ω, $a(u) \in L^1(\Omega)$, $f(u) \in L^1(\Omega, \delta)$ and

$$a(0)\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} \mathrm{d}\sigma - \int_\Omega a(u)\Delta\varphi \mathrm{d}x = (\geq)\ \lambda \int_\Omega f(u)\varphi \mathrm{d}x \tag{1.4}$$

for every (non-negative) $\varphi \in \mathcal{A}_0$, where $\mathcal{A}_0 := \{\varphi \in W^{2,n}(\Omega) \cap W_0^{1,n}(\Omega) : \Delta\varphi \in L^\infty(\Omega)\}$, ν is the outward unit normal to $\partial\Omega$ and $\mathrm{d}\sigma$ and dx are Euclidean area and volume forms associated to $\partial\Omega$ and Ω, respectively. If $u : \Omega \to \mathbb{R} \cup \{\infty\}$ is only measurable and non-negative, we say that it is a weak solution. Finally, by a strict strong supersolution, we mean a strong supersolution whose above inequality is strict for every nonzero non-negative $\varphi \in \mathcal{A}_0$.

From now on, consider the notations

$$a(L_0) := \lim_{t\to L_0^-} a(t) \quad \text{and} \quad f(L_0) := \lim_{t\to L_0^-} f(t).$$

Let $L_0 = \infty$ and $u : \Omega \to \mathbb{R} \cup \{\infty\}$ be a weak solution of $(\mathcal{P}_\lambda)$. Notice that the set $\mathcal{S} = \{x \in \Omega : u(x) = \infty\}$ has zero Lebesgue measure when $a(\infty) = \infty$ or $f(\infty) = \infty$, however, it does not seem clear what happens when a and f are bounded, except for special situations as shown below. This explains why we allow a non-negative weak solution to take infinite value.

Assume that $a(\infty), f(\infty) < \infty$ and a is increasing. In this case, $w = a(u) - a(0) : \Omega \to [0, a(\infty) - a(0)]$ is a weak solution of the problem

$$\left\{ \begin{array}{rcll} -\Delta w & = & \lambda(f \circ a^{-1})(w + a(0)) & \text{in } \ \Omega \\ w & = & 0 & \text{on } \ \partial\Omega \end{array} \right.$$

Observe that

$$\mathcal{S} = \{x \in \Omega : \ u(x) = \infty\} = \{x \in \Omega : \ w(x) = a(\infty) - a(0)\}$$

If we assume that $f \circ a^{-1}$ is real analytical, then the elliptic theory gives that w is real analytical too, so $|\mathcal{S}| = 0$. In particular, the conclusion holds for f constant.

Going back to the previous claim, suppose that a is constant in $[0, L_1]$ for some $0 < L_1 \leq L_0$. Then, the problem $(\mathcal{P}_\lambda)$ has no positive strong solution for any $\lambda \neq 0$. Indeed, for such a possible solution $u \in C(\overline{\Omega})$, taking a nontrivial non-negative test function $\varphi \in C_0^\infty(\Omega_1)$ in (1.4), where $\Omega_1 = \{x \in \Omega : 0 < u(x) < L_1\}$ is a nonempty open subset in Ω, we derive the contradiction

$$\begin{aligned} 0 \neq \lambda \int_\Omega f(u)\varphi \mathrm{d}x &= a(0) \int_{\partial\Omega} \frac{\partial \varphi}{\partial \nu} \mathrm{d}\sigma - \int_\Omega a(u)\Delta\varphi \mathrm{d}x \\ &= a(0) \int_{\partial\Omega} \frac{\partial \varphi}{\partial \nu} \mathrm{d}\sigma - a(0) \int_\Omega \Delta\varphi \mathrm{d}x = 0. \end{aligned}$$

For this reason, we are led to require the following local condition for a:

(H_0) a increases in $(0, \varepsilon)$ for some $\varepsilon > 0$ small.

By way of illustration, we exhibit two interesting examples that are covered by our assumptions.

Example 1.2. Gelfand type problem

$$\left\{ \begin{array}{rcll} -\Delta \left(\dfrac{u^m}{u+1} \right) & = & \lambda \mathrm{e}^u & \text{in } \ \Omega, \\ u & = & 0 & \text{on } \ \partial\Omega, \end{array} \right.$$

where $m > 0$ is an integer number. Here, $L_0 = \infty$, $a(\infty) < \infty$ for $m = 1$, $a(\infty) = \infty$ for $m > 1$ and $f(\infty) = \infty$.

Example 1.3. MEMS type problem

$$\left\{ \begin{array}{rcll} -\Delta(u - \ln(1 - u)) & = & \dfrac{\lambda}{(1 - u)^k} & \text{in } \ \Omega \\ u & = & 0 & \text{on } \ \partial\Omega, \end{array} \right.$$

where $k > 0$ is an integer number. Note that $L_0 = 1$, $a(1) = \infty$ and $f(1) = \infty$, whereas $a(1) < \infty$ in the classical MEMS problem.

The topics of interest involving the problem $(\mathcal{P}_\lambda)$, in their most comprehensive forms, that will guide us are:

- ▶ Extremal parameter and positive strong solutions;
- ▶ Extremal solution;

- ▶ Dependence of extremal parameter and strong and weak solutions with respect to L_0, a and f.

The results concerning the last topic involve more intricate arguments. Furthermore, due to their novelty and intrinsic importance, we would say that Theorems 1.6 and 1.7 and Corollary 1.8 are the most substantial of all the work.

Interactive assumptions between a and f are needed in order to approach each one of the above issues. For instance, the condition of superlinearity on f assumed in Gelfand type problems, that is $f(t)$ increases faster than $a(t) = t$ (linear diffusion) at the infinity, ensures the existence of an extremal parameter associated to $(\mathcal{P}_\lambda)$, while it does not exist for sublinear sources.

For existence of a finite extremal parameter connected to positive strong solutions, we assume the first interactive condition:

(H_1) $\liminf\limits_{t\to L_0^-} \dfrac{f(t)}{a(t)} > 0$, if $a(L_0) = \infty$.

Clearly, the above $\liminf$ is positive when $a(L_0) < \infty$.

Theorem 1.4. (*Separation*) *Assume* (H_0) *and* (H_1). *Then, there exists a number* $\lambda^*_{L_0} > 0$ *such that*

(i) For $0 < \lambda < \lambda^*_{L_0}$, *the problem* $(\mathcal{P}_\lambda)$ *has a minimal positive strong solution* $u_\lambda^{L_0}$ *and the map* $\lambda \mapsto u_\lambda^{L_0} \in C(\overline{\Omega})$ *is increasing and left continuous with respect to the uniform topology;*
(ii) For $\lambda > \lambda^*_{L_0}$, *the problem* $(\mathcal{P}_\lambda)$ *admits no positive strong solution.*

In most applications, the extremal parameter $\lambda^*_{L_0}$ and the minimal strong solutions $u_\lambda^{L_0}$ are not explicitly known, except in a few special cases. For example, assume that $f(t) = 1$, a is increasing in $(0, L_0)$ and $a(L_0) < \infty$. In these conditions, if we consider ξ_0 the classical solution of the problem

$$\begin{cases} -\Delta\xi &= 1 \quad \text{in } \Omega, \\ \xi &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

then $\lambda^*_{L_0} = (a(L_0) - a(0))/M_0$ and $u_\lambda^{L_0} = a^{-1}(a(0) + \lambda\xi_0) \in C(\overline{\Omega})$ for every $0 < \lambda < \lambda^*_{L_0}$, where $M_0 := \sup_\Omega \xi_0$. In particular, for $a(t) = 1{-}{-}\mathrm{e}^{-t}$, one has $\lambda^*_\infty = 1/M_0$ and $u_\lambda^\infty = -\ln(1 - \lambda\xi_0)$ and for $a(t) = \arctan t$, one has $\lambda^*_\infty = \pi/(2M_0)$ and $u_\lambda^\infty = \tan(\lambda\xi_0)$.

The next step is to know what happens when λ is equal to $\lambda^*_{L_0}$. In other words, whether or not $(\mathcal{P}_{\lambda^*_{L_0}})$ admits some kind of minimal solution, so-called extremal. We address this question under the stronger hypothesis than (H_1):

(H_2) $\lim\limits_{t\to L_0^-} \dfrac{f(t)}{a(t)} = \infty$, if $a(L_0) = \infty$.

As we should see, an extremal solution exists in the strong sense whenever $L_0 < \infty$ and $f(L_0) < \infty$. In the remaining cases, we guarantee its existence at least in the weak sense, provided that (H_2) occurs. On the other hand, it is worth emphasizing that an extremal solution does not always exist, as can easily to be verified in the simple example $a(t) = t$ and $f(t) = mt + 1$ for $t \geq 0$, where $m > 0$ is a constant.

Theorem 1.5. (*Extremal*) *Assume* (H_0) *and* (H_1). *Let* $\lambda^*_{L_0} > 0$ *be the extremal parameter provided in Theorem 1.4. Then, it holds that:*

(i) If $L_0 < \infty$ *and* $f(L_0) < \infty$, *then the problem* $(\mathcal{P}_{\lambda^*_{L_0}})$ *admits a unique minimal positive strong solution given in the uniform sense in* Ω *by*

$$u^{L_0}_{\lambda^*_{L_0}} = \lim_{\lambda\uparrow\lambda^*_{L_0}} u_\lambda^{L_0};$$

(ii) For other cases, if (H_2) is satisfied, then the problem $(\mathcal{P}_{\lambda^*_{L_0}})$ admits a unique minimal non-negative weak solution given by the above limit in the pointwise sense almost everywhere in Ω. Moreover, $a(u^{L_0}_\lambda)$ converges to $a(u^{L_0}_{\lambda^*_{L_0}})$ in $L^1(\Omega)$ and $f(u^{L_0}_\lambda)$ converges to $f(u^{L_0}_{\lambda^*_{L_0}})$ in $L^1(\Omega,\delta)$.

The next result address continuity and asymptotic behavior on L_0 of the extremal parameter and solutions provided in Theorems 1.4 and 1.5.

Theorem 1.6. (*Behavior and dependence on L_0*) *Assume (H_0) and (H_1). For each $0 < L < L_0$, by Theorems 1.4 and 1.5, consider the notations λ^*_L and u^L_λ for $0 < \lambda \le \lambda^*_L$ associated to the interval $[0, L]$. Then, it holds that:*

(i) λ^*_L is continuous and non-decreasing on $L \in (0, L_0)$ and, in addition,

$$\lim_{L\to L_0^-} \lambda^*_L = \lambda^*_{L_0};$$

(ii) The family of solutions u^L_λ steady for L near L_0 for any fixed $0 < \lambda < \lambda^*_{L_0}$. Indeed, $u^L_\lambda = u^{L_0}_\lambda$ in Ω for L near enough L_0;

(iii) If $L_0 < \infty$ and $f(L_0) < \infty$, then $u^L_{\lambda^*_L}$ converges uniformly to $u^{L_0}_{\lambda^*_{L_0}}$;

(iv) For other cases, if (H_2) is satisfied, then $u^L_{\lambda^*_L}$ converges pointwise to $u^{L_0}_{\lambda^*_{L_0}}$ almost everywhere in Ω, $a(u^L_{\lambda^*_L})$ converges to $a(u^{L_0}_{\lambda^*_{L_0}})$ in $L^1(\Omega)$ and $f(u^L_{\lambda^*_L})$ converges to $f(u^{L_0}_{\lambda^*_{L_0}})$ in $L^1(\Omega,\delta)$.

Another physically important question is to know how $(\mathcal{P}_\lambda)$ behaves when the diffusion and the source vary. We present some partial answers in the next result.

Theorem 1.7. (*Behavior on a and f*) *Let a_k and f_k be sequences of non-decreasing continuous functions on $[0, L_0)$ that converge uniformly to a and f as $k \to \infty$, respectively. Assume also that a_k and a are increasing and f is positive. Denote by $\lambda^*_{L_0,k}$ the extremal parameter corresponding to a_k and f_k, by $u^{L_0}_{\lambda,k}$ the minimal positive strong solution associated to $0 < \lambda < \lambda^*_{L_0,k}$ and by $u^{L_0}_{\lambda^*_{L_0,k}}$ the extremal solution associated to $\lambda^*_{L_0,k}$. Then, it holds that:*

(i) The sequence $\lambda^*_{L_0,k}$ converges to $\lambda^*_{L_0}$;

(ii) For $0 < \lambda < \lambda^*_{L_0}$, the sequence $u^L_{\lambda,k}$ converges uniformly to some positive strong solution $u_{\lambda,0}$ of $(\mathcal{P}_\lambda)$, so $u_{\lambda,0} \ge u^{L_0}_\lambda$ in Ω;

(iii) If $L_0 < \infty$ and $f(L_0) < \infty$, then $u^{L_0}_{\lambda^*_{L_0,k}}$ converges uniformly to some positive strong solution u_0 of $(\mathcal{P}_{\lambda^*_{L_0}})$;

(iv) If $L_0 = \infty$ and $f(L_0) < \infty$, then $u^{L_0}_{\lambda^*_{L_0,k}}$ converges to u_0 almost everywhere in Ω and $a_k(u^{L_0}_{\lambda^*_{L_0,k}})$ converges uniformly to $a(u_0)$, where u_0 is a non-negative weak solution of $(\mathcal{P}_{\lambda^*_{L_0}})$;

(v) If $f(L_0) = \infty$ and (H_2) is satisfied, then $u^{L_0}_{\lambda^*_{L_0,k}}$ converges to u_0 almost everywhere in Ω and $a_k(u^{L_0}_{\lambda^*_{L_0,k}})$ converges to $a(u_0)$ in $L^p(\Omega)$ for every $p \ge 1$ and $p < n/(n-2)$ in case $n \ge 3$, where u_0 is a non-negative weak supersolution of $(\mathcal{P}_{\lambda^*_{L_0}})$.

Moreover, in (iii), (iv) *and* (v), *we have $u_0 \ge u^{L_0}_{\lambda^*_{L_0}}$ in Ω.*

The theorem is new even for the Gelfand and MEMS models where the diffusion $a(t) = t$ is fixed. In this situation, the statement can be rephrased in a slightly shorter form. Precisely, consider the problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & \lambda f(u) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{1.5}$$

for a positive non-decreasing continuous function $f : [0, L_0) \to \mathbb{R}$, where $L_0 > 0$ is finite or not.

Corollary 1.8. *Let f_k be a sequence of non-decreasing continuous functions on $[0, L_0)$ that converge uniformly to f as $k \to \infty$. Since f is positive, we fix the notations $\lambda^*_{L_0,k}$ and $u^{L_0}_{\lambda,k}$ for $0 < \lambda \leq \lambda^*_{L_0,k}$, as in Theorem 1.7. Then, it holds that:*

(i) The sequence $\lambda^*_{L_0,k}$ converges to $\lambda^*_{L_0}$;
(ii) For $0 < \lambda < \lambda^*_{L_0}$, the sequence $u^L_{\lambda,k}$ converges uniformly to some positive strong solution $u_{\lambda,0}$ of (1.5), so $u_{\lambda,0} \geq u^{L_0}_{\lambda}$ in Ω;
(iii) If $f(L_0) < \infty$, then $u^{L_0}_{\lambda^*_{L_0,k}}$ converges uniformly to some positive strong solution u_0 of (1.5);
(iv) If $f(L_0) = \infty$ and (H_2) is satisfied, then $u^{L_0}_{\lambda^*_{L_0,k}}$ converges to u_0 in $L^p(\Omega)$ for every $p \geq 1$ and $p < n/(n-2)$ in case $n \geq 3$, where u_0 is a non-negative weak supersolution of (1.5).

Moreover, in (iii) *and* (iv), *we have $u_0 \geq u^{L_0}_{\lambda^*_{L_0}}$ in Ω.*

The paper is organized as follows. Theorems 1.4 and 1.5 are proved in Section 2 and Theorems 1.6 and Theorem 1.7 are proved, respectively, in Sections 3 and 4.

2. Proof of Theorems 1.4 and 1.5

For a better organization, the proof of Theorems 1.4 and 1.5 will be done in several steps, from Lemma 2.1 to Lemma 2.9. We must detail the non-standard arguments, the rest will only be outlined briefly.

We start with the set

$$\Lambda_{L_0} = \{\lambda > 0 : (\mathcal{P}_\lambda) \text{ has a strong solution } u \in C_0(\overline{\Omega}) \text{ such that } 0 < u < L_0 \text{ in } \Omega\}.$$

Here, $C_0(\overline{\Omega})$ denotes the closed subspace of $C(\overline{\Omega})$ of functions vanishing on the boundary of Ω.

The following lemma implies that Λ_{L_0} is non-empty.

Lemma 2.1. *Assume (H_0). Then there exists $\lambda_0 > 0$ such that $(0, \lambda_0) \subset \Lambda_{L_0}$.*

Proof. Let $\varepsilon \in (0, L_0)$ as in (H_0) and v_ε be the $W^{2,n}$ solution of the problem

$$\begin{cases} -\Delta v = \lambda_0 f(\varepsilon) & \text{in } \Omega, \\ v = a(0) & \text{on } \partial\Omega \end{cases}$$

for a suitable $\lambda_0 > 0$ to be chosen below. It follows from the Aleksandrov-Bakelman-Pucci (ABP) estimate [20], Theorem 9.1 that there exists $C_0 = C_0(\Omega) > 0$ such that

$$\|v_\varepsilon\|_{L^\infty(\Omega)} \leq a(0) + C_0 \lambda_0 f(\varepsilon).$$

Setting

$$\lambda_0 = \frac{a(\varepsilon) - a(0)}{f(\varepsilon) C_0},$$

by the strong maximum principle, we get $a(0) < v_\varepsilon \leq a(\varepsilon)$. Since $a : [0, \varepsilon] \to [a(0), a(\varepsilon)]$ is invertible, the function $u_\varepsilon = a^{-1}(v_\varepsilon) \in C(\overline{\Omega})$ is positive and satisfies, in the strong sense,

$$\begin{cases} -\Delta(a(u_\varepsilon)) = \lambda_0 f(\varepsilon) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

As usual, for each fixed $0 < \lambda < \lambda_0$, one constructs inductively a sequence $(u_k)_{k\geq 0} \subset C(\overline{\Omega})$, by setting $u_0 = 0$ in Ω and

$$\begin{cases} -\Delta(a(u_{k+1})) &= \lambda f(u_k) \quad \text{in } \Omega, \\ u_{k+1} &= 0 \quad\quad\quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

satisfying $0 < u_k \leq u_{k+1} \leq u_\varepsilon \leq \varepsilon$ in Ω for $k \geq 1$. Here, the equations in (2.1) occur in the strong sense.

Define the function pointwise for $x \in \Omega$ by

$$u_\lambda(x) := \lim_{k\to\infty} u_k(x).$$

Thanks to the Lebesgue dominated convergence theorem, we obtain

$$a(0)\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\mathrm{d}\sigma - \int_\Omega a(u_\lambda)\Delta\varphi \mathrm{d}x = \lambda\int_\Omega f(u_\lambda)\varphi \mathrm{d}x$$

for every $\varphi \in \mathcal{A}_0$. Since $0 < u_\lambda \leq \varepsilon < L_0$ in Ω, we have $a(u_\lambda) \in L^1(\Omega)$ and $f(u_\lambda) \in L^1(\Omega, \delta(x))$, thus u_λ is a weak solution of $(\mathcal{P}_\lambda)$.

It only remains to verify that $u_\lambda \in C(\overline{\Omega})$. But this fact follows by showing easily that the $W^{2,n}$ solution v_λ of

$$\begin{cases} -\Delta v &= \lambda f(u_\lambda) \quad \text{in } \Omega, \\ v &= a(0) \quad\quad \text{on } \partial\Omega. \end{cases}$$

satisfies $v_\lambda = a(u_\lambda)$ and $0 < v_\lambda \leq a(\varepsilon)$ in Ω. This finishes the proof. □

The next lemma states that the set Λ_{L_0} is upper bounded.

Lemma 2.2. *Assume* (H_0) *and* (H_1)*. Then,* Λ_{L_0} *is upper bounded.*

Proof. If $a(L_0) < \infty$, then there exists a constant $C_0 > 0$ such that $C_0 f(t) \geq a(t) - a(0)$ for all $0 \leq t < L_0$, since

$$a(t) - a(0) \leq a(L_0) - a(0) \leq \frac{|a(L_0) - a(0)| + 1}{f(0)} f(t), \quad \forall\, 0 \leq t < L_0.$$

For $a(L_0) = \infty$, the condition (H_1) implies that there are constants $\bar{C} > 0$ and $0 < \bar{L} < L_0$ such that $\bar{C} f(t) \geq a(t) - a(0)$ for every $\bar{L} \leq t < L_0$. On the other hand, clearly exists a constant $C_0 \geq \bar{C}$ such that $C_0 f(t) \geq a(t) - a(0)$ for $0 < t \leq \bar{L}$. Thus, in any case, we get

$$C_0 f(t) \geq a(t) - a(0), \quad \forall\, 0 \leq t < L_0.$$

By Lemma 2.1, we have $\Lambda_{L_0} \neq \emptyset$. Let $\lambda \in \Lambda_{L_0}$ and $u \in C(\overline{\Omega})$ be a strong solution of $(\mathcal{P}_\lambda)$ satisfying $0 < u < L_0$ in Ω. Take $\phi_1 \in \mathcal{A}_0$ a principal eigenfunction corresponding to the first Dirichlet eigenvalue λ_1 of $-\Delta$. From the above inequality, we derive

$$\begin{aligned} C_0\lambda_1 \int_\Omega f(u)\phi_1 \mathrm{d}x &\geq \lambda_1 \int_\Omega (a(u) - a(0))\,\phi_1 \mathrm{d}x \\ &= -\int_\Omega (a(u) - a(0))\,\Delta\phi_1 \mathrm{d}x \end{aligned}$$

$$= \lambda \int_{\Omega} f(u)\phi_1 \mathrm{d}x.$$

But this implies that $\Lambda_{L_0} \subset (0, C_0\lambda_1]$ and the proof is concluded. ◻

For any $0 < L_0 \leq \infty$, Lemmas 2.1 and 2.2 guarantee that the number

$$\lambda^*_{L_0} := \sup \Lambda_{L_0}$$

is finite and positive.

The next result will be useful in what follows.

Lemma 2.3. *Let $\lambda > 0$. Assume that a is increasing on $(0, L_0)$ and $(\mathcal{P}_\lambda)$ has a non-negative weak supersolution $\bar{u}$ such that $0 \leq \bar{u} \leq L_0$ in Ω. Then, $(\mathcal{P}_\lambda)$ admits a minimal non-negative weak solution $u_\lambda^{L_0}$ satisfying $0 < u_\lambda^{L_0} \leq \bar{u} \leq L_0$ in Ω.*

Proof. The argument uses the constructive idea in the proof of Lemma 2.1, except that the strong maximum principle is replaced by the weak maximum principle [6], Lemma 1. In this way, we obtain an increasing sequence $(u_k)_{k\geq 1} \subset C(\overline{\Omega})$ of functions satisfying $0 \leq u_k \leq \bar{u} \leq L_0$ in Ω for all k.

For $x \in \Omega$, define

$$u_\lambda^{L_0}(x) := \lim_{k\to\infty} u_k(x).$$

Clearly, $u_\lambda^{L_0} : \Omega \to \mathbb{R} \cup \{\infty\}$ is a measurable function such that $0 < u_\lambda^{L_0} \leq \bar{u} \leq L_0$ almost everywhere in Ω, $a(u_\lambda^{L_0}) \in L^1(\Omega)$ and $f(u_\lambda^{L_0}) \in L^1(\Omega, \delta)$. Moreover, Lebesgue dominated convergence theorem implies that $a(u_k)$ converges to $a(u_\lambda^{L_0})$ in $L^1(\Omega)$ and $f(u_k)$ converges to $f(u_\lambda^{L_0})$ in $L^1(\Omega, \delta)$. These limits along with (2.1) imply that $u_\lambda^{L_0}$ satisfies

$$a(0) \int_{\partial\Omega} \frac{\partial \varphi}{\partial \nu} \mathrm{d}\sigma - \int_{\Omega} a(u_\lambda^{L_0}) \Delta\varphi \mathrm{d}x = \lambda \int_{\Omega} f(u_\lambda^{L_0}) \varphi \mathrm{d}x$$

for every $\varphi \in \mathcal{A}_0$. In other words, $u_\lambda^{L_0}$ is a non-negative weak solution of $(\mathcal{P}_\lambda)$.

Finally, any non-negative weak solution $u_\lambda : \Omega \to \mathbb{R} \cup \{\infty\}$ is also a non-negative weak supersolution of $(\mathcal{P}_\lambda)$. Thus, using the above conclusion, we derive the inequality $u_\lambda^{L_0} \leq u_\lambda$ in Ω, thus $u_\lambda^{L_0}$ is minimal. This ends the proof. ◻

The next result is a stronger consequence of Lemma 2.3 which in particular implies existence of minimal positive strong solution for $(\mathcal{P}_\lambda)$ with $\lambda \in \Lambda_{L_0}$.

Corollary 2.4. *Let $\lambda > 0$. Assume that a is increasing on $(0, L_0)$ and $(\mathcal{P}_\lambda)$ has a (strict) strong supersolution $\bar{u}$ such that $0 < \bar{u} < L_0$ in Ω. Then, $(\mathcal{P}_\lambda)$ admits a minimal strong solution $u_\lambda^{L_0} \in C(\overline{\Omega})$ satisfying $0 < u_\lambda^{L_0} \leq \bar{u} < L_0$ in Ω ($0 < u_\lambda^{L_0} < \bar{u} < L_0$ in Ω).*

Proof. Let $u_\lambda^{L_0} : \Omega \to \mathbb{R} \cup \{\infty\}$ be as in Lemma 2.3. Clearly, $u_\lambda^{L_0} \in L^\infty(\Omega)$ and $f(u_\lambda^{L_0}) \in L^\infty(\Omega)$ since $0 < u_\lambda^{L_0} \leq \bar{u} < L_0$ in Ω. Thus, for the first part it suffices to show that $u_\lambda^{L_0} \in C(\overline{\Omega})$. Indeed, let $v_\lambda^{L_0}$ be the $W^{2,n}$ solution of

$$\begin{cases} -\Delta v &= \lambda f(u_\lambda^{L_0}) & \text{in } \Omega, \\ v &= a(0) & \text{on } \partial\Omega. \end{cases}$$

By the weak maximum principle [6], Lemma 1, we have $a(u_\lambda^{L_0}) = v_\lambda^{L_0} \in C(\overline{\Omega})$, thus $u_\lambda^{L_0} \in C(\overline{\Omega})$, since a is increasing and continuous on $[0, L_0)$.

Finally, we focus on strict inequality. Let $\bar{v} \in W^{2,n}(\Omega)$ be the solution of

$$\begin{cases} -\Delta v &= \lambda f(\bar{u}) \quad \text{in } \Omega, \\ v &= a(0) \quad \text{on } \partial\Omega. \end{cases}$$

Note that $a(\bar{u}), \bar{v} \in C(\overline{\Omega})$ and, since $\bar{u}$ is a strict strong supersolution, we get

$$\int_\Omega (a(\bar{u}) - \bar{v})\Delta\varphi \mathrm{d}x < 0$$

for every nonzero non-negative function $\varphi \in \mathcal{A}_0$. Hence, the strong maximum principle for superharmonic functions gives $\bar{v} < a(\bar{u})$ in Ω.

On the other hand, we also have $-\Delta\bar{v} = \lambda f(\bar{u}) \geq \lambda f(u_\lambda^{L_0}) = -\Delta v_\lambda^{L_0}$ in Ω and $\bar{v} = v_\lambda^{L_0}$ on $\partial\Omega$. Then, by the weak maximum principle, we obtain $v_\lambda^{L_0} = a(u_\lambda^{L_0}) \leq \bar{v}$ in Ω. Therefore, we have $a(u_\lambda^{L_0}) < a(\bar{u})$, so $u_\lambda^{L_0} < \bar{u}$ in Ω. □

Let $L_1 := \sup\{L > 0 : \ a \text{ is increasing on } (0, L)\}$. By (H_0), L_1 is well defined and $L_1 \leq L_0$. In addition, by definition, L_1 is the end point of the maximal growth interval of the diffusion a.

The next lemma is fundamental in what follows since it is assumed that a is only increasing close to 0.

Lemma 2.5. *Assume (H_0) and (H_1). If $L_1 < L_0$, then $u \leq L_1$ in Ω for every $u \in \Lambda_{L_0}$ and $\lambda^*_{L_1} = \lambda^*_{L_0}$.*

Proof. Let $\lambda \in \Lambda_{L_0}$ and u be a positive strong solution of $(\mathcal{P}_\lambda)$. We ensure that $u \leq L_1$ in Ω. Otherwise, since $L_1 < L_0$, we can choose $L_2 \in (L_1, L_0)$ so a is constant in (L_1, L_2). Consider the nonempty open set $\Omega_0 = \{x \in \Omega : L_1 < u(x) < L_2\} \subset \Omega$. For a nonzero non-negative function $\varphi \in C_0^\infty(\Omega_0)$ in (1.4), we derive the contradiction

$$\begin{aligned} 0 < \lambda \int_\Omega f(u)\varphi \mathrm{d}x &= a(0)\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\mathrm{d}\sigma - \int_\Omega a(u)\Delta\varphi \mathrm{d}x \\ &= a(0)\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\mathrm{d}\sigma - a(L_1)\int_\Omega \Delta\varphi \mathrm{d}x \\ &= a(0)\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\mathrm{d}\sigma - a(L_1)\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\mathrm{d}\sigma = 0. \end{aligned}$$

Clearly, we have $\lambda^*_{L_1} \leq \lambda^*_{L_0}$. For the reverse inequality, take $\overline{\lambda} \in \Lambda_{L_0}$. By definition, $(\mathcal{P}_{\overline{\lambda}})$ admits a positive strong solution $\overline{u} \in C(\overline{\Omega})$ such that $\overline{u} < L_0$. By the above argument, we have $\overline{u} \leq L_1$ in Ω and, by Corollary 2.4, we get $(0, \overline{\lambda}) \subset \Lambda_{L_1}$. Since $\overline{\lambda} \in \Lambda_{L_0}$ is arbitrary, we get $\lambda^*_{L_1} \geq \lambda^*_{L_0}$ and the desired equality follows. □

The next result asserts that the set Λ_{L_0} is an interval.

Lemma 2.6. *Assume (H_0) and (H_1). Then, it holds that $(0, \lambda^*_{L_0}) \subset \Lambda_{L_0}$. Moreover, for any $0 < \lambda < \lambda^*_{L_0}$, $(\mathcal{P}_\lambda)$ admits a minimal positive strong solution $u_\lambda^{L_0}$ which satisfies $u_\lambda^{L_0} < L_1$ in Ω.*

Proof. For a fixed number $0 < \lambda < \lambda^*_{L_0}$, by Lemma 2.5, there exists $\bar{\lambda} \in \Lambda_{L_1}$ such that $\bar{\lambda} > \lambda$. Thus, $(\mathcal{P}_\lambda)$ has a positive strong supersolution $\bar{u}$ satisfying $\bar{u} < L_1$. By Corollary 2.4, we deduce that $\lambda \in \Lambda_{L_1} \subset \Lambda_{L_0}$.

For the second part, Corollary 2.4 and Lemmas 2.5 and 2.6 guarantee the existence of a positive strong solution $u_\lambda^{L_0}$ of $(\mathcal{P}_\lambda)$ satisfying $u_\lambda^{L_0} < L_1$ in Ω.

We now prove the minimality of $u_\lambda^{L_0}$. Let $u : \Omega \to \mathbb{R} \cup \{\infty\}$ be a non-negative weak solution of $(\mathcal{P}_\lambda)$. In particular, $u \leq L_0$ in Ω.

We recall that $u_\lambda^{L_0}$ was introduced in the proof of Lemma 2.3 as the pointwise limit of an increasing sequence $(u_k)_{k\geq 1} \subset C(\overline{\Omega})$ of functions satisfying $0 \leq u_k \leq L_1$ in Ω for all k. Thus, it suffices to show that $u \geq u_k$ in Ω for all k.

Since $f(u) \geq f(0)$ in Ω, by weak maximum principle ([6], Lem. 1), it is clear that $a(u) \geq a(u_1)$ in Ω. Given $x \in \Omega$, if $u(x) < L_1$, then the previous inequality implies $u(x) \geq u_1(x)$, otherwise, $u(x) \geq L_1 > u_1(x)$. In any case, we deduce that $u \geq u_1$ in Ω. Finally, if $u \geq u_k$ in Ω for some k, the same argument produces $u \geq u_{k+1}$ in Ω. This ends the proof. $\square$

The next lemma ensures monotonicity and left continuous dependence on λ of these solutions.

Lemma 2.7. *Assume (H_0) and (H_1). Then, the map $\lambda \in (0, \lambda^*_{L_0}) \mapsto u_\lambda^{L_0} \in C(\overline{\Omega})$ is increasing and left continuous with respect to the uniform topology.*

Proof. We first show that the map $\lambda \in (0, \lambda^*_{L_0}) \mapsto u_\lambda^{L_0}$ is increasing. Take $0 < \underline{\lambda} < \overline{\lambda} < \lambda^*_{L_0}$. Notice that $u_{\overline{\lambda}}^{L_0}$ is a positive strict strong supersolution of

$$\begin{cases} -\Delta(a(u)) &= \underline{\lambda} f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Since $u_{\overline{\lambda}}^{L_0} < L_1$ in Ω, by Corollary 2.4, we get $u_{\underline{\lambda}}^{L_0} < u_{\overline{\lambda}}^{L_0}$ in Ω.

We now prove that the map $\lambda \in (0, \lambda^*_{L_0}) \mapsto u_\lambda^{L_0} \in C(\overline{\Omega})$ is left continuous at $\lambda_0 \in (0, \lambda^*_{L_0})$. Choose a fixed number $\overline{\lambda}$ so that $0 < \lambda_0 < \overline{\lambda} < \lambda^*_{L_0}$. By strict monotonicity, we have $0 < u_\lambda^{L_0} < u_{\overline{\lambda}}^{L_0}$ in Ω for every $0 < \lambda < \overline{\lambda}$. Then, we can define for $x \in \Omega$,

$$u_0(x) := \lim_{\lambda \to \lambda_0^-} u_\lambda^{L_0}(x).$$

Note that $0 < u_0 < u_{\overline{\lambda}}^{L_0} < L_1$ in Ω. We claim that u_0 is equal to $u_{\lambda_0}^{L_0}$. In fact, take λ close enough to λ_0 and let $v_\lambda^{L_0} \in W^{2,n}(\Omega)$ be the solution of

$$\begin{cases} -\Delta v &= \lambda f(u_\lambda^{L_0}) \quad \text{in } \Omega, \\ v &= a(0) \quad \text{on } \partial\Omega, \end{cases}$$

which satisfies $a(0) < v_\lambda^{L_0} = a(u_\lambda^{L_0}) < a(u_{\overline{\lambda}}^{L_0})$ in Ω. Invoking the Lebesgue dominated convergence theorem, one concludes that $f(u_\lambda^{L_0})$ converges to $f(u_0)$ in $L^n(\Omega)$ as $\lambda \to \lambda_0$. Thus, by standard elliptic estimates, it follows that $v_\lambda^{L_0} = a(u_\lambda^{L_0})$ converges to $v_0 = a(u_0)$ in $W^{2,n}(\Omega)$. Consequently, $a(u_\lambda^{L_0})$ converges to $a(u_0)$ in $C(\overline{\Omega})$. Since $0 < u_0 < L_1$ in Ω, then $u_0 \in C(\overline{\Omega})$ is a positive strong solution of $(\mathcal{P}_{\lambda_0})$ and by minimality, we have $u_{\lambda_0}^{L_0} \leq u_0$ in Ω.

On the other hand, by monotonicity, we deduce the reverse inequality

$$u_0(x) = \lim_{\lambda \to \lambda_0^-} u_\lambda^{L_0}(x) \leq u_{\lambda_0}^{L_0}(x), \quad \forall x \in \Omega.$$

Hence, $u_0 = u_{\lambda_0}^{L_0}$ in Ω and this completes the proof. $\square$

The next lemma is devoted to nonexistence of positive strong solution of $(\mathcal{P}_\lambda)$.

Lemma 2.8. *Assume (H_0) and (H_1). Then, the problem $(\mathcal{P}_\lambda)$ admits no positive strong solution for any $\lambda > \lambda^*_{L_0}$.*

Proof. Assume by contradiction that $(\mathcal{P}_{\overline{\lambda}})$ has a positive strong solution $\overline{u} \in C(\overline{\Omega})$ for some $\overline{\lambda} > \lambda^*_{L_0}$. By Lemma 2.5, we have $\overline{u} \leq L_1$ in case $L_1 < \infty$. Since $\overline{u}$ is a positive strong supersolution of $(\mathcal{P}_\lambda)$ for $0 < \lambda < \overline{\lambda}$, Corollary 2.4 yields $(0, \overline{\lambda}) \subset \Lambda_{L_0}$. But this inclusion contradicts the definition of $\lambda^*_{L_0}$. □

Finally, we study what happens when $\lambda = \lambda^*_{L_0}$, put another way, whether $(\mathcal{P}_\lambda)$ admits an extremal solution. The last result of this section establishes its existence taking values in the range $(0, L_0]$.

Lemma 2.9. *Assume (H_0) and (H_1). Then, it holds that:*

(i) If $L_0 < \infty$ and $f(L_0) < \infty$, then $(\mathcal{P}_{\lambda^*_{L_0}})$ admits a minimal positive strong solution given in the uniform sense in Ω by

$$u^{L_0}_{\lambda^*_{L_0}} = \lim_{\lambda \uparrow \lambda^*_{L_0}} u^{L_0}_\lambda;$$

(ii) For other cases, if (H_2) is satisfied, then $(\mathcal{P}_{\lambda^*_{L_0}})$ admits a minimal non-negative weak solution at $\lambda^*_{L_0}$ given by the above limit in the pointwise sense almost everywhere in Ω. Moreover, $a(u^{L_0}_\lambda)$ converges to $a(u^{L_0}_{\lambda^*_{L_0}})$ in $L^1(\Omega)$ and $f(u^{L_0}_\lambda)$ converges to $f(u^{L_0}_{\lambda^*_{L_0}})$ in $L^1(\Omega, \delta)$ as $\lambda \to \lambda^*_{L_0}$.

Proof. By the monotonicity of $u^{L_0}_\lambda$, define for $x \in \Omega$,

$$u_0(x) = \lim_{\lambda \uparrow \lambda^*_{L_0}} u^{L_0}_\lambda(x).$$

By Lemma 2.6, we derive $u^{L_0}_\lambda < L_1$ for every $0 < \lambda < \lambda^*_{L_0}$, so we have $0 < u_0 \leq L_1 \leq L_0$ in Ω. If $L_1 < L_0$, then $L_1 < \infty$ and $f(L_1) < \infty$ and so this case is covered by the assertion (i). Therefore, it suffices to prove the lemma when $L_1 = L_0$.

The proof of (i) is carried out in the same spirit of the proof of Lemma 2.6. Assume that $L_0 < \infty$ and $f(L_0) < \infty$. This implies that $a(L_0) < \infty$ because otherwise (H_1) would not be valid. Proceeding as in that proof, for each $0 < \lambda < \lambda^*_{L_0}$, let $v^{L_0}_\lambda \in W^{2,n}(\Omega)$ be the solution of

$$\begin{cases} -\Delta v &= \lambda f(u^{L_0}_\lambda) \quad \text{in } \Omega, \\ \quad v &= a(0) \qquad\quad \text{on } \partial\Omega. \end{cases}$$

Note that $0 < v^{L_0}_\lambda = a(u^{L_0}_\lambda) < a(L_0)$ in Ω. By Lebesgue dominated convergence theorem, $f(u^{L_0}_\lambda)$ converges to $f(u_0)$ in $L^n(\Omega)$ as $\lambda \to \lambda^*_{L_0}$. Thus, we deduce that $v^{L_0}_\lambda = a(u^{L_0}_\lambda)$ converges to $v_0 := a(u_0)$ in $W^{2,n}(\Omega)$ and hence $a(u^{L_0}_\lambda)$ converges to $a(u_0)$ in $C(\overline{\Omega})$ as $\lambda \to \lambda^*_{L_0}$. Since a is increasing in $[0, L_0]$, then $u^{L_0}_\lambda$ also converges uniformly to $u_0 \in C(\overline{\Omega})$ and, in addition, u_0 is a positive strong solution of the problem $(\mathcal{P}_{\lambda^*_{L_0}})$ and satisfies $u_0 \leq L_0$.

Now, by Corollary 2.4, $(\mathcal{P}_{\lambda^*_{L_0}})$ admits a minimal positive strong solution $u^{L_0}_{\lambda^*_{L_0}}$ and so $u^{L_0}_{\lambda^*_{L_0}} \leq u_0$ in Ω. On the other hand, by that corollary, we also have $u^{L_0}_\lambda \leq u^{L_0}_{\lambda^*_{L_0}}$ in Ω for all $0 < \lambda < \lambda^*_{L_0}$, so $u_0 \leq u^{L_0}_{\lambda^*_{L_0}}$ in Ω and the proof of (i) is completed.

For the proof of (ii), assume first that $f(L_0) = \infty$. We have two possibilities: either $a(L_0) < \infty$ or $a(L_0) = \infty$ and (H_2) occurs. In any case, there exists a constant $C > 0$ such that $f(t) \geq 2\frac{\lambda_1}{\lambda^*_L} a(t) - C$ for all $0 \leq t < L_0$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta$. Let ϕ_1 be a principal eigenfunction associated to λ_1. Taking $\phi_1 \in \mathcal{A}_0$ as a test function in (1.4), we get

$$\lambda \int_\Omega f(u^{L_0}_\lambda)\phi_1 \mathrm{d}x = \lambda_1 \int_\Omega a(u^{L_0}_\lambda)\phi_1 \mathrm{d}x \leq \frac{\lambda^*_L}{2} \int_\Omega (f(u^{L_0}_\lambda) + C)\phi_1 \mathrm{d}x.$$

Letting $\lambda \uparrow \lambda^*_{L_0}$, by usual arguments, it follows that $a(u_\lambda^{L_0})$ converges to $a(u_0)$ in $L^1(\Omega)$ and $f(u_\lambda^{L_0})$ converges to $f(u_0)$ in $L^1(\Omega,\delta)$.

If $f(L_0) < \infty$ then, by (H1), we have $a(L_0) < \infty$. Consequently, $a(u_\lambda^{L_0})$ and $f(u_\lambda^{L_0})$ are uniformly bounded on λ, respectively, in $L^1(\Omega)$ and $L^1(\Omega,\delta)$. In particular, the above conclusions of convergence also occur in this case.

In any situation, taking the limit in the integral equality (1.4) satisfied by $u_\lambda^{L_0}$, we conclude that u_0 is a non-negative weak solution of $(\mathcal{P}_{\lambda^*_{L_0}})$. By Lemma 2.3, we have $u^{L_0}_{\lambda^*_{L_0}} \le u_0$ in Ω. But the same lemma also yields $u_\lambda^{L_0} \le u^{L_0}_{\lambda^*_{L_0}}$ in Ω for all $0 < \lambda < \lambda^*_{L_0}$, since $u^{L_0}_{\lambda^*_{L_0}}$ is a non-negative weak supersolution of $(\mathcal{P}_\lambda)$. Hence, $u_0 \le u^{L_0}_{\lambda^*_{L_0}}$ in Ω and the proof is ended. □

3. Proof of Theorem 1.6

This section is dedicated to the complete proof of Theorem 1.6.

Proof. Let $(0, L_1)$ be the maximal growth interval of a. If $L_1 < L_0$, by Lemma 2.5, we have $\lambda^*_{L_1} = \lambda^*_L = \lambda^*_{L_0}$ for $L_1 < L < L_0$. This clearly leads to $u_\lambda^L = u_\lambda^{L_0}$ in Ω for $0 < \lambda \le \lambda^*_{L_0}$ for $L_1 < L < L_0$. In particular, all assertions concerning asymptotic behavior near L_0 and continuity of λ^*_L at $L \in (L_1, L_0)$ follow trivially in this case. Thus, it suffices to assume that $L_1 = L_0$.

For the proof of (i), we first notice that $\Lambda_{\tilde{L}_0} \subset \Lambda_{\tilde{L}} \subset \Lambda_{L_0}$ for $0 < \tilde{L}_0 \le \tilde{L} < L_0$. Therefore, the extremal parameter λ^*_L is non-decreasing on L and upper bounded by $\lambda^*_{L_0}$, then for any fixed $0 < \tilde{L}_0 < L_0$, the limits

$$\underline{\lambda} := \lim_{L \to \tilde{L}_0^-} \lambda^*_L \quad \text{and} \quad \overline{\lambda} := \lim_{L \to \tilde{L}_0^+} \lambda^*_L$$

exist and satisfy $\underline{\lambda} \le \lambda^*_{\tilde{L}_0} \le \overline{\lambda}$. We wish to show that these two inequalities become equality.

Assume by contradiction that the first of them is strict and take λ_0 so that $\underline{\lambda} < \lambda_0 < \lambda^*_{\tilde{L}_0}$. By (i) of Theorem 1.4, $(\mathcal{P}_{\lambda_0})$ has a minimal positive strong solution $u_{\lambda_0} \in C(\overline{\Omega})$ with $u_{\lambda_0} < \tilde{L}_0$ in Ω. Fix $0 < \overline{L} < \tilde{L}_0$ such that $0 < u_{\lambda_0} < \overline{L}$ in Ω. Then, we have $\lambda_0 \in \Lambda_{\overline{L}}$, leading readily to the contradiction $\lambda_0 \le \lambda^*_{\overline{L}} \le \underline{\lambda}$. Note that the first limit shows that

$$\lambda^*_{\tilde{L}_0} = \lim_{L \to \tilde{L}_0^-} \lambda^*_L.$$

Proceeding also by contradiction, assume that the second inequality is strict. Take λ_0 so that $\lambda^*_{\tilde{L}_0} < \lambda_0 < \overline{\lambda}$. Set $L_\varepsilon := (1+\varepsilon)\tilde{L}_0$ for $\varepsilon > 0$. By Lemma 2.6, $(\mathcal{P}_{\overline{\lambda}})$ admits a strong solution $u_\varepsilon \in C(\overline{\Omega})$ such that $0 < u_\varepsilon < L_\varepsilon$ in Ω. Arguing as in the proof of (i) of Theorem 1.5, it follows that u_ε converges uniformly to a strong solution $u \in C(\overline{\Omega})$ of $(\mathcal{P}_{\overline{\lambda}})$ which clearly satisfies $0 < u \le \tilde{L}_0$ in Ω. Therefore, by Corollary 2.4, we conclude that $(\mathcal{P}_{\lambda_0})$ has a strong solution $u_0 \in C(\overline{\Omega})$ such that $0 < u_0 < \tilde{L}_0$, so $\lambda_0 \le \lambda^*_{\tilde{L}_0}$. But this contradicts the assumption by absurd $\lambda^*_{\tilde{L}_0} < \lambda_0$.

As a key ingredient for the proof of (ii), (iii) and (iv), we first show that $u_\lambda^L = u_\lambda^{L_0}$ in Ω for any $0 < L < L_0$ and $0 < \lambda < \lambda^*_L$. By Corollary 2.4 and Lemma 2.6, $(\mathcal{P}_\lambda)$ admits a minimal strong solution $u_\lambda^L \in C(\overline{\Omega})$ verifying $0 < u_\lambda^L < L$ in Ω.

On the one hand, using a minimality argument and the inequality $\lambda^*_L \le \lambda^*_{L_0}$, we deduce that $u_\lambda^{L_0} \le u_\lambda^L$ for $0 < L < L_0$ and $0 < \lambda < \lambda^*_L$. On the other hand, this implies that $0 < u_\lambda^{L_0} < L$ in Ω and thus, again by minimality, we get $u_\lambda^L \le u_\lambda^{L_0}$, so $u_\lambda^L = u_\lambda^{L_0}$ in Ω, as claimed.

The proof of (ii) is easily deduced from the assertion (i). Precisely, given a fixed number $0 < \lambda < \lambda^*_{L_0}$, (i) guarantees that $0 < \lambda < \lambda^*_L$ for L near enough L_0 and thus the conclusion follows from the above step.

The same tool is used in the proof of the parts (iii) and (iv). Precisely, letting $\lambda \uparrow \lambda_L^*$ in the equality $u_\lambda^L = u_\lambda^{L_0}$ which holds in Ω, we obtain $u_{\lambda_L^*}^L = u_{\lambda_L^*}^{L_0}$ in Ω. Finally, taking the limit as $L \to L_0^-$ on the right-hand side of this equality, using the part (i) of this lemma and (i) and (ii) of Lemma 2.9, we end the proof of (iii) and (iv). □

4. Proof of Theorem 1.7

The proof will be based on a central lemma. Before stating it, let h be a non-decreasing continuous function in $[0, M_0)$ with $0 < M_0 \leq \infty$ satisfying $h(0) > 0$. Under the additional condition $\liminf_{t\to M_0} \frac{h(t)}{t} > 0$ when $M_0 = \infty$, we know that there exists $\lambda^* > 0$ such that the problem

$$\begin{cases} -\Delta w = \lambda h(w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

admits a minimal positive solution $w_\lambda \in W^{2,n}(\Omega)$ for any $0 < \lambda < \lambda^*$. If moreover $\lim_{t\to M_0} \frac{h(t)}{t} = \infty$ when $M_0 = \infty$, then a minimal non-negative weak solution $w^* \in L^1(\Omega)$ exists for $\lambda = \lambda^*$. If both M_0 and $h(M_0)$ are finite, we recall that $w^* \in C(\overline{\Omega})$, by Theorem 1.4.

Let now h_k, $k \geq 1$, be a sequence of non-decreasing continuous functions in $[0, M_k)$ with $0 < M_k \leq \infty$. Assume that h_k converges uniformly to h as $k \to \infty$ in the sense that for any $\varepsilon > 0$ there exists $k_0 \geq 1$ such that $|h_k(t) - h(t)| < \varepsilon$ for every $t \in [0, M_k) \cap [0, M_0)$ and $k \geq k_0$.

If $M_0 = \infty$ and $\liminf_{t\to M_0} \frac{h(t)}{t} > 0$, then h_k also satisfies $\liminf_{t\to M_k} \frac{h_k(t)}{t} > 0$ for k large enough whenever $M_k = \infty$. Indeed, by the assumption assumed on h, we know that there exists $c_0 > 0$ such that $h(t) \geq 2c_0 t$ for every $t \geq 0$. On the other hand, for some $k_0 \geq 1$, we have $|h_k(t) - h(t)| \leq c_0$ for every $t \geq 0$ and $k \geq k_0$. Combining these inequalities, we obtain $h_k(t) \geq c_0 t$ for every $t \geq 1$ and $k \geq k_0$.

If $M_0 = \infty$ and $\lim_{t\to M_0} \frac{h(t)}{t} = \infty$, then $\lim_{t\to M_k} \frac{h_k(t)}{t} = \infty$ for k large enough whenever $M_k = \infty$. In fact, given any $D > 0$ there exists $M > 0$ such that $h(t) \geq (D+1)t \geq Dt + M$ for every $t \geq M$. Then, the uniform convergence of h_k yields $h_k(t) \geq Dt$ for every $t \geq M$ and k large enough.

These conclusions along with Theorem 1.4 and 1.5 provide, for any k large enough, a number $\lambda_k^* > 0$ such that the problem

$$\begin{cases} -\Delta w = \lambda h_k(w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

possesses a minimal positive solution $w_{\lambda,k} \in W^{2,n}(\Omega)$ for any $0 < \lambda < \lambda_k^*$ and a minimal non-negative weak solution $w_k^* \in L^1(\Omega)$ for $\lambda = \lambda_k^*$. Clearly, $h(M_0)$ (or $h_k(M_k)$) is a positive number or infinite.

With the above notations, we are ready to state the main tool of this section.

Lemma 4.1. *Let h be a non-decreasing continuous function in $[0, M_0)$ for some $0 < M_0 \leq \infty$ satisfying $h(0) > 0$ and $\liminf_{t\to M_0} \frac{h(t)}{t} > 0$ if $M_0 = \infty$. Let also h_k be a sequence of non-decreasing continuous functions in $[0, M_k)$, where $0 < M_k \leq \infty$, converging uniformly to h. Assume that M_k and $h_k(M_k)$ converge respectively to M_0 and $h(M_0)$ as $k \to \infty$, provided that M_0 and $h(M_0)$ are finite (the latter limit always occurs when $M_k \leq M_0$ for k large). Then:*

(i) The sequence λ_k^* converges to λ^*;
(ii) For $0 < \lambda < \lambda^*$, the sequence $w_{\lambda,k}$ converges uniformly to some positive solution $w_{\lambda,0}$ of (4.1), so $w_{\lambda,0} \geq w_\lambda$ in Ω;
(iii) If $M_0 < \infty$ and $h(M_0) < \infty$, then w_k^* converges uniformly to some positive solution w_0 of (4.1) with $\lambda = \lambda^*$. In particular, we have $w_0 \geq w^*$ in Ω;

(iv) If $M_0 = \infty$ and

$$\liminf_{t\to M_0} \frac{h(t)}{t} = \infty,$$

then w_k^* converges in $L^p(\Omega)$ to some non-negative weak supersolution w_0 of (4.1) with $\lambda = \lambda^*$ for every $p \geq 1$ and $p < n/(n-2)$ in case $n \geq 3$. In particular, we have $w_0 \geq w^*$ in Ω

Proof. For the proof of (i), assume first that $M_0 < \infty$ and $h(M_0) < \infty$. Let $M_1 := M_0 + 1$. Since $M_k \to M_0$ and $h_k(M_k) \to h(M_0)$, we can take non-decreasing continuous extensions to the interval $[0, M_1]$ so that $h_k \to h$ uniformly in $[0, M_1]$. Examples of such extensions are:

$$h(t) = \begin{cases} h(t) & \text{if } \ 0 \leq t < M_0, \\ t + h(M_0) - M_0 & \text{if } \ M_0 \leq t \leq M_0 + 1 \end{cases}$$
$$h_k(t) = \begin{cases} h_k(t) & \text{if } \ 0 \leq t < M_k, \\ h_k(M_k) + \frac{h(M_0)-h_k(M_k)+1}{M_0-M_k+1}(t - M_k) & \text{if } \ M_k \leq t \leq M_0 + 1 \end{cases}$$

Consider the sets associated to (4.1) and (4.2), respectively,

$$\Lambda_{M_0} = \{\lambda > 0 : \ (4.1) \text{ has a solution } w \in W^{2,n}(\Omega) \text{ such that } 0 < w < M_0 \text{ in } \Omega\}$$

and

$$\Lambda^k_{M_k} = \{\lambda > 0 : \ (4.2) \text{ has a solution } w \in W^{2,n}(\Omega) \text{ such that } 0 < w < M_k \text{ in } \Omega\}.$$

Given a number $\varepsilon > 0$, there is an integer $k_0 \geq 1$ such that, for any $k \geq k_0$,

$$|h_k(t) - h(t)| \leq \varepsilon h(0) \leq \varepsilon h(t)$$

for every $t \in [0, M_1]$, in other words,

$$(1 - \varepsilon)h(t) \leq h_k(t) \leq (1 + \varepsilon)h(t)$$

for every $t \in [0, M_1]$.

For $\lambda \in \Lambda_{M_0}$, let $w \in W^{2,n}(\Omega)$ be a solution of (4.1) verifying $0 < w < M_0$ in Ω. Then,

$$\begin{cases} -\Delta w &= \lambda h(w) \geq \frac{\lambda}{1+\varepsilon} h_k(w) & \text{in } \ \Omega, \\ \quad w &= 0 & \text{on } \ \partial\Omega, \end{cases}$$

so $\frac{\lambda}{1+\varepsilon} \in \Lambda^k_{M_k}$ for k large enough. But this clearly implies that, for any $\lambda \in \Lambda_{M_0}$, $\frac{\lambda}{1+\varepsilon} \leq \lambda_k^*$ for k large enough. Thus,

$$\frac{1}{1+\varepsilon}\lambda^* \leq \liminf_{k\to\infty} \lambda_k^*.$$

Let now a fixed $0 < \delta \le 1$. For $\lambda \in \Lambda^k_{M_k}$, we know that (4.2) has a solution $w_k \in W^{2,n}(\Omega)$ such that $0 < w_k < M_k$ in Ω. Then, for any k large, we have

$$\begin{cases} -\Delta w_k = \lambda h_k(w_k) \ge (1-\varepsilon)\lambda h(w_k) & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega, \end{cases}$$

so $(1-\varepsilon)\lambda \in \Lambda_{M_0+\delta}$ and thus $(1-\varepsilon)\lambda \le \lambda^*_\delta$ for every $\lambda \in \Lambda^k_{M_k}$ and k large enough (not depending on λ), where $\lambda^*_\delta = \sup \Lambda_{M_0+\delta}$. Therefore, for any k large,

$$\lambda^*_k \le \frac{1}{1-\varepsilon}\lambda^*_\delta,$$

so

$$\limsup_{k\to\infty} \lambda^*_k \le \frac{1}{1-\varepsilon}\lambda^*_\delta$$

for every $0 < \delta \le 1$. Letting $\delta \to 0^+$ and using the part (i) of Theorem 1.6, it follows that λ^*_δ converges to λ^*, and so λ^*_k to λ^*, as wished.

We separate the remaining cases into two types:

If $M_0 < \infty$ and $h(M_0) = \infty$, then the uniform convergence of h_k to h guarantees that $M_k \le M_0$ for k large enough and so h_k converges uniformly to h in $[0, M_k)$.

If $M_0 = \infty$, then $M_k \to \infty$ and h_k again converges uniformly to h in $[0, M_k)$.
In both situations, we work with the sets Λ_{M_0} and $\Lambda^k_{M_k}$ and carry out the proof in the same spirit of the previous case, without needed of extending the functions h and h_k. Thus, the proof of (i) is finished.

For the proof of (ii), we take a fixed $0 < \lambda < \lambda^*$. We claim that $w_{\lambda,k}$ converges uniformly to a positive solution $w_{\lambda,0}$ of (4.1). Choose $\varepsilon > 0$ small so that $\lambda_\varepsilon := (1+\varepsilon)\lambda < \lambda^*$. By definition, there exists a function $w^\varepsilon \in W^{2,n}(\Omega)$ such that $0 < w^\varepsilon < M_0$ in Ω and

$$\begin{cases} -\Delta w^\varepsilon = (1+\varepsilon)\lambda h(w^\varepsilon) \ge \lambda h_k(w^\varepsilon) & \text{in } \Omega, \\ w^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

so by minimality $w_{\lambda,k} \le w^\varepsilon$ in Ω for k large enough. This boundedness along with the usual elliptic theory imply that $w_{\lambda,k}$ converges to some function $w_{\lambda,0}$ in $W^{2,n}(\Omega)$, up to a subsequence. In particular, $w_{\lambda,0}$ satisfies $0 < w_{\lambda,0} < M_0$ in Ω and

$$\begin{cases} -\Delta w_{\lambda,0} = \lambda h(w_{\lambda,0}) & \text{in } \Omega, \\ w_{\lambda,0} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by minimality, $w_\lambda \le w_{\lambda,0}$ in Ω and this completes the proof of (ii).

For the proof of (iii), assume that $M_0 < \infty$ and $h(M_0) < \infty$. By (i) of Theorem 1.5, we know that w^* and w^*_k belong to $W^{2,n}(\Omega)$ and satisfy $0 < w^* \le M_0$ and $0 < w^*_k \le M_k$ in Ω. Arguing as in the beginning of the proof of (i), we have non-decreasing extensions of h and h_k to $[0, M_1]$, with $M_1 > M_0$, such that h_k converges to h uniformly in $[0, M_1]$. Since $h_k(w_k)$ is bounded in $L^\infty(\Omega)$ uniformly on k, elliptic estimates then yield $w^*_k \to w_0$ in $W^{2,n}(\Omega)$, module a subsequence, so w_0 satisfies $0 < w_0 \le M_0$ in Ω and

$$\begin{cases} -\Delta w_0 = \lambda^* h(w_0) & \text{in } \Omega, \\ w_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

By minimality, we clearly have $w^* \le w_0$ in Ω.

Finally, we prove the assertion (iv). Assume $M_0 = \infty$ and

$$\liminf_{t \to M_0} \frac{h(t)}{t} = \infty.$$

We recall from the proof of (ii) that $M_k \leq M_0$ and so h_k converges uniformly to h in $[0, M_k)$. In addition, there exists a constant $C > 0$ such that $h(t) \geq 2\frac{\lambda_1}{\lambda^*} t - C$ for all $0 \leq t < M_0$, where λ_1 denotes the first eigenvalue of the laplacian on $H_0^1(\Omega)$. Then, using (i) and the uniform convergence of h_k, we derive $h_k(t) \geq 2\frac{\lambda_1}{\lambda_k^*} t - C$ for all $0 \leq t < M_k$.

Proceeding now as in the proof of the statement (ii) of Theorem 1.5, we readily deduce that w_k^* is bounded in $L^1(\Omega)$ and $h_k(w_k^*)$ is bounded in $L^1(\Omega, \delta)$. By a regularity result of Stampacchia (see [26], Thm. 5.1 or [29], Thm. 9.1), we can assume that w_k^* converges to some function w_0 in $L^1(\Omega)$ and almost everywhere in Ω. We assert that $h_k(w_k^*)$ converges to $h(w_0)$ almost everywhere in Ω. The conclusion is direct for points $x \in \Omega$ such that $w_0(x) < M_0$. If $w_0(x) = M_0$, we then analyze two possibilities:

If $w_k^*(x) = M_k$ and $h_k(M_k) = \infty$, then $h_k(w_k^*(x)) = \infty = h(w_0(x))$.

If $w_k^*(x) \leq M_k$ and $h_k(M_k) < \infty$, then $h_k(w_k^*(x)) \geq (1-\varepsilon) h(w_k^*(x)) \to \infty = h(M_0) = h(w_0(x))$.

Since $h_k(w_k^*)$ converges to $h(w_0)$ almost everywhere in Ω and $h_k(w_k^*)$ is uniformly bounded in $L^1(\Omega, \delta)$, by Fatou's lemma, we deduce that $h(w_0) \in L^1(\Omega, \delta)$. Consequently, letting $k \to \infty$ in the integral equation satisfied by w_k^* with non-negative test functions and again applying Fatou's lemma, we derive that w_0 is a non-negative weak supersolution of (4.1), and so the proof is completed.

□

Finally, we prove Theorem 1.7 with the aid of Lemma 4.1.

Proof of Theorem 1.7. Let a, a_k and f, f_k be increasing continuous functions in $[0, L_0)$ for some $0 < L_0 \leq \infty$. Note that $a_k(0) < a_k(L_0) \leq \infty$ and $a(0) < a(L_0) \leq \infty$. Set

$$M_k := a_k(L_0) \;\text{ and }\; M_0 := a(L_0).$$

Clearly, $0 < M_k, M_0 \leq \infty$ and, in addition, if $M_0 < \infty$, then $M_k < \infty$ for k large and $M_k \to M_0$. Define the functions h_k and h by

$$\begin{aligned} h_k(t) &= (f_k \circ a_k^{-1})(t + a_k(0)), \;\forall 0 \leq t < M_k; \\ h(t) &= (f \circ a^{-1})(t + a(0)), \;\forall 0 \leq t < M_0. \end{aligned}$$

Note that $h_k(M_k) = f_k(L_0)$ and $h(M_0) = f(L_0)$. Moreover, h_k and h are non-decreasing continuous functions verifying $h_k(0) > 0$ and $h(0) > 0$. Since $a_k \to a$ and $f_k \to f$ uniformly in $[0, L_0)$, it also follows that h_k converges uniformly to h in the sense described in the beginning of this section.

Furthermore, $M_k = a_k(L_0) \to a(L_0) = M_0$ and $h_k(M_k) = f_k(L_0) \to f(L_0) = h(M_0)$ in case $M_0 < \infty$ and $h(M_0) < \infty$. Finally, we have the relation

$$\liminf_{t \to M_0^-} \frac{h(t)}{t} = \liminf_{t \to L_0^-} \frac{f(t)}{a(t)}.$$

Therefore, from the assumptions satisfied by a_k, f_k, a and f in the statement of Theorem 1.7, we conclude that h_k and h satisfy all conditions stated in Lemma 4.1.

On the other hand, non-negative weak (strong) solutions u (satisfying $u < L_0$ in Ω) of $(\mathcal{P}_\lambda)$ are connected to weak (strong) solutions w (satisfying $w < M_0$ in Ω) of (4.1) by mean of the relation $w = a(u) - a(0)$. Consequently, $u_\lambda^{L_0}$ is a minimal positive strong solution and $u_{\lambda_{L_0}^*}^{L_0}$ is an extremal solution of $(\mathcal{P}_\lambda)$ if, and only if, w_λ and w^* are solutions of same type of (4.1). In particular, the extremal parameters of both problems are equal and the proof of (i) of Theorem 1.7 follows readily from (i) of Lemma 4.1.

For the proof of (ii), we recall from (ii) of Lemma 4.1 that $w_{\lambda,k}$ converges to $w_{\lambda,0}$ uniformly in Ω, so $a_k(u_{\lambda,k}^{L_0})$ also converges uniformly to $a(u_{\lambda,0})$. We also know that $w_{\lambda,0} < M_0$ in Ω, so $w_{\lambda,k} < M_0$ in Ω. This implies that $a_k(u_{\lambda,k}^{L_0})$ and $a(u_{\lambda,0})$ are smaller than both $a_k(L_0)$ and $a(L_0)$ for k large. Therefore, it follows from the uniform convergence that $u_{\lambda,k}^{L_0}$ converges uniformly to $u_{\lambda,0}$, as wished.

For the proof of (iii), we invoke similar ideas as above. Assume that $L_0 < \infty$ and $f(L_0) < \infty$, so $a(L_0) < \infty$ by (H1), and then $M_0 < \infty$ and $h(M_0) < \infty$. Consider extensions of a and f, keeping the same properties, to some interval $[0, \overline{L}]$, where $\overline{L} > L_0$ is a fixed number. This leads us to an extension of h to some range $[0, \overline{M}]$ with $M_0 < \overline{M} < a(\overline{L}) - a(0)$. Invoking now (iii) of Lemma 4.1, we have the uniform convergence $a_k(u^{L_0}_{\lambda^*_{L_0,k}}) \to a(u_0)$ as $k \to \infty$, where $u_0 := a^{-1}(w_0) \in C(\overline{\Omega})$. Since both $a_k(u^{L_0}_{\lambda^*_{L_0,k}})$ and $a(u_0)$ are smaller that $a_k(\overline{L})$ and $a(\overline{L})$ for k large, it follows that $u^{L_0}_{\lambda^*_{L_0,k}}$ converges uniformly to u_0. Passing the limit in the definition of strong solution to $u^{L_0}_{\lambda^*_{L_0,k}}$ and using (i), one concludes readily that u_0 is a positive strong solution of $(\mathcal{P}_{\lambda^*_{L_0}})$, and thus $u_0 \geq u^{L_0}_{\lambda^*_{L_0}}$ in Ω. The proof of (iv) is carried out in a similar way, however the conclusion is that u_0 is a weak solution once $L_0 = \infty$.

Finally, for the proof of (v), assume $f(L_0) = \infty$ and (H_2). The assertion (iv) of Lemma 4.1 then ensures that $a_k(u^{L_0}_{\lambda^*_{L_0,k}})$ converges to $a(u_0)$ in $L^1(\Omega)$ and $f(u_0) \in L^1(\Omega, \delta)$, where $u_0 := a^{-1}(w_0) : \Omega \to \mathbb{R} \cup \{\infty\}$. Moreover, u_0 is a non-negative weak supersolution of $(\mathcal{P}_{\lambda^*_{L_0}})$, so $u_0 \geq u^{L_0}_{\lambda^*_{L_0}}$ in Ω. This completes the proof. $\square$

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