ESAIM: COCV 30 (2024) 9
https://doi.org/10.1051/cocv/2023081

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

D-OBSERVABILITY FOR A COUPLED SYSTEM OF WAVE EQUATIONS ON RIGHT ANGLE TRIANGLES

Chengxia Zu*

Abstract. Based on the tiled method, we establish the equivalence between the D-observability for diagonalizable systems on the rectangular domain and the right angle triangle domain, and then get the sufficiency of Kalman's rank condition to the D-observability for diagonalizable systems on some right angle triangles.

Mathematics Subject Classification. 93B05, 93B07, 93C20, 35L53.

Received January 1, 2023. Accepted November 10, 2023.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipshitz boundary $\partial\Omega = \Gamma_1 \cup \Gamma_0$ such that $\bar{\Gamma}_1 \cap \bar{\Gamma}_0 = \emptyset$ and $\mathrm{mes}(\Gamma_1) > 0$. Let

$$\mathcal{H}_0 = \mathrm{L}^2(\Omega), \quad \mathcal{H}_1 = \mathrm{H}_0^1(\Omega), \quad \mathcal{L} = \mathrm{L}^2(0, +\infty; \mathrm{L}^2(\Gamma_1)).$$

The dual of $\mathcal{H}_1$ is denoted by $\mathcal{H}_{-1} = \mathrm{H}^{-1}(\Omega)$.

Let

$$U = (u^{(1)}, \cdots, u^{(N)})^T.$$

Consider the following coupled system of wave equations:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ U = DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \tag{1.1}$$

with the initial condition

$$t = 0: \quad U = \widehat{U}_0, \quad U' = \widehat{U}_1 \quad \text{in } \Omega, \tag{1.2}$$

Keywords and phrases: D-observability, Kalman's rank condition, diagonalizable systems of wave equations, tiling.

School of Mathematics, Hohai University, Nanjing, Jiangsu, PR China.

* Corresponding author: chengxiazu@hhu.edu.cn

where "$\prime$" stands for the time derivative; $\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator; the coupling matrix $A = (a_{ij})$ is of order N and the boundary control matrix $D = (d_{ij})$ is a full column-rank matrix of order $N \times M (M \leq N)$, both with constant elements; $H = (h^{(1)}, \cdots, h^{(M)})^T$ denotes the boundary control.

The following definition of approximate boundary null controllability of system (1.1) was introduced by Li and Rao in [3, 4].

Definition 1.1. System (1.1) is approximately null controllable at the time $T > 0$, if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$, there exists a sequence $\{H_n\}$ of boundary controls in $\mathcal{L}^M$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1.1)–(1.2) satisfies

$$(U_n(T), U_n'(T)) \to (0, 0) \ \text{ in } (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N \quad \text{ as } n \to +\infty. \tag{1.3}$$

Let $\Phi = (\phi^{(1)}, \cdots, \phi^{(N)})^T$. Consider the adjoint system

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in } (0, +\infty) \times \Omega, \\ \Phi = 0 & \text{on } (0, +\infty) \times \partial\Omega \end{cases} \tag{1.4}$$

with the initial data

$$t = 0: \ \Phi = \widehat{\Phi}_0, \ \Phi' = \widehat{\Phi}_1 \quad \text{ in } \Omega, \tag{1.5}$$

where the initial data $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$.

Definition 1.2. [3, 4] The adjoint system (1.4) is D-observable on the interval $[0, T]$, if the following partial Neumann observation

$$D^T \partial_\nu \Phi \equiv 0 \quad \text{on } [0, T] \times \Gamma_1, \tag{1.6}$$

∂_ν being the outward normal derivative, implies that $(\widehat{\Phi}_0, \widehat{\Phi}_1) = (0, 0)$, then $\Phi \equiv 0$.

Remark 1.3. To avoid the confusion, we will use "the D-observability of system (1.4) on $[0, T] \times \Gamma_1$ in the space $(\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$" instead of "the D-observability of system (1.4) on $[0, T]$" to indicate the observable boundary and the initial space.

Definition 1.4. For any given $\gamma \subset \partial\Omega$ and $V \times H \subseteq (\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$, the adjoint system (1.4) is D-observable on $[0, T] \times \gamma$ in the space $V \times H$, if the following partial Neumann observation

$$D^T \partial_\nu \Phi \equiv 0 \quad \text{on } [0, T] \times \gamma \tag{1.7}$$

implies that $(\widehat{\Phi}_0, \widehat{\Phi}_1) = (0, 0)$, and then $\Phi \equiv 0$, where Φ is the solution to problem (1.4)–(1.5) with initial data $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in V \times H$ and ∂_ν is the outward normal derivative on γ.

The relationship between the approximate boundary null controllability of system (1.1) and the D-observability of the adjoint system (1.4) was given by Li and Rao in [3, 4] as follows.

Theorem 1.5. *[3, 4] System (1.1) is approximately null controllable at the time $T > 0$ if and only if the adjoint system (1.4) is D-observable on the interval $[0, T]$.*

Theorem 1.5 shows the importance of D-observability for the adjoint system (1.4) in the study of approximate boundary null controllability for the coupled system (1.1) of wave equations. So far, the necessary condition of D-observability for the system (1.4) is clear.

Theorem 1.6. *[3, 4] Assume that the adjoint system (1.4) is D-observable. Then we have necessarily the following Kalman's rank condition:*

$$\text{rank } (D, AD, \cdots, A^{N-1}D) = N. \tag{1.8}$$

However, in the study of D-observability for the adjoint system (1.4), Kalman's rank condition (1.8) is not sufficient in general, otherwise, noting that Kalman's rank condition (1.8) is independent of the observation time $T > 0$, if system (1.4) is D-observable on the interval $[0, T]$, then for any given time $t > 0$, it should be D-observable on the interval $[0, t]$. Thus, the D-observability of the adjoint system (1.4) can be obtained almost immediately, which contradicts the finite speed of wave propagation. Hence, Kalman's rank condition (1.8) is not enough to ensure the D-observability of the adjoint system (1.4). We should require the observation time T large enough at least. But even if the above two requirements are satisfied, they are still not sufficient to guarantee the D-observability not only on the finite time interval $[0, T]$, but also on the infinite time interval $[T, +\infty)$ (see Thm. 3.2 in [3]). However, in some special cases, such as $N \times N$ cascade systems, 2×2 systems, diagonalizable systems (*i.e.*, the coupling matrix A is diagonalizable) in one-space-dimensional case, and in higher dimensional case on spherical domain and annular domain on the finite time interval $[0, T]$, Kalman's rank condition (1.8) is always sufficient to the D-observability of the adjoint system (1.4), provided that the observation time T is large enough (see [3, 8, 9]). However, for the general case, it is not clear whether the system (1.4) is D-observable.

Recently, some scholars are interested in the observability for the wave equation on triangle domain. Christianson and Stafford considered the boundary observability for a single wave equation with Dirichlet boundary condition on any given triangle domain and obtained the observability for any one side of the triangle with initial data in $(\mathrm{H}_0^1(\Omega) \cap \mathrm{H}^3(\Omega)) \times (\mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega))$ (see [1]). Lai considered the internal observability on special right angle triangle domain. He established the relation between the internal observability of the wave equation in a domain Ω and the internal observability in the tiled domain by Ω. Taking a right angle triangle domain $\mathcal{T}$ with vertices $(0,0), (0,1), (\frac{1}{\sqrt{3}}, 0)$ as an example, Lai transformed the internal observability for the wave equation on the right angle triangle domain $\mathcal{T}$ into that on the rectangular domain $\mathcal{R} = (0, \sqrt{3}) \times (0, 1)$(see [2]). Inspired by Lai, we will establish the relationship between the D-observability of system (1.4) on the rectangular domain $\mathcal{R}$ and the right angle triangle domain $\mathcal{T}$. Thus, the results of D-observability on the rectangular domain can be transferred to that on the unstudied right angle triangle domain.

We consider the D-observability of system (1.4), in which Ω is a right angle triangle domain with vertices $(0, 0), (a, 0), (0, b)$ denoted by $\mathcal{T}$ hereinafter and the coupling matrix A^T satisfying.

(H1) A is a diagonalizable matrix with s eigenvalues

$$-\lambda_1^2 < \delta_1 < \delta_2 < \cdots < \delta_s, \tag{1.9}$$

where λ_1^2 is the smallest eigenvalue of $-\Delta$ in Ω.

(H2) There exists a positive constant C such that $\|A - CI\|$ is small enough, or the elements of A are rational numbers and a, b are both algebraic numbers.

If the eigenfunction of the $-\Delta$ is known, the solution of system (1.4) can be obtained by separating variables (see [3]) and then it is easy to determine whether the system (1.4) is D-observable. However, it is so complicated to compute the eigenfunctions of the $-\Delta$ on $\mathcal{T}$ that this method is not suitable to apply directly. Lai gave the definition of tiling and obtained the eigenfunctions of $-\Delta$ on the right angle triangle domain $\mathcal{T}$ with vertices $(0, 0), (0, 1), (\frac{1}{\sqrt{3}}, 0)$ through that on the rectangular domain (see [2, 6, 7]). For $i = 1, 2$, we will establish the relation between the D-observability of system (1.4) on $[0, T] \times \Gamma_i$ and the D-observability of system (2.1) on $[0, T] \times \bar{\Gamma}_{i,j}$, where $j = 1, 2$, $\Gamma_1 = (0, a) \times \{0\}, \Gamma_2 = \{0\} \times (0, b)$ are the legs of $\mathcal{T}$, respectively and $\bar{\Gamma}_{1,1} = (0, a) \times \{0\}$, $\bar{\Gamma}_{1,2} = (0, a) \times \{b\}$, $\bar{\Gamma}_{2,1} = \{0\} \times (0, b)$ $\bar{\Gamma}_{2,2} = \{a\} \times (0, b)$ are the sides of rectangular domain $\mathcal{R} = (0, a) \times (0, b)$ (see Fig. 1). Then we will transform the D-observability of system (1.4) on the right angle triangle domain $\mathcal{T}$ into the D-observability of system (2.1) on the rectangular domain $\mathcal{R}$. However, for the

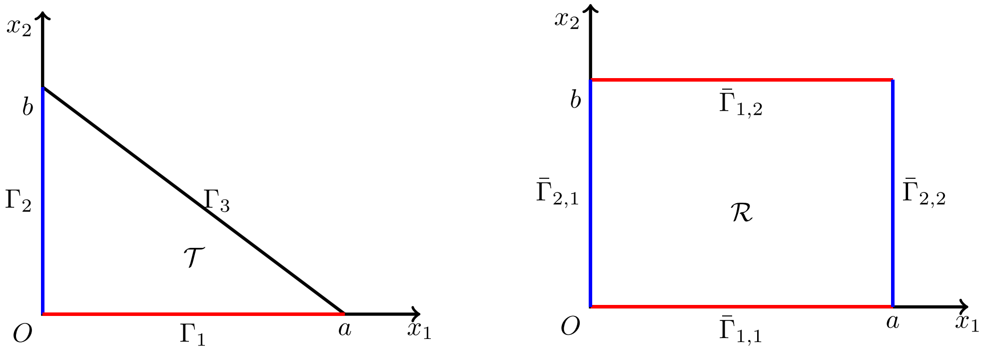

FIGURE 1. The boundary of the right angle triangle domain $\mathcal{T}$ and the rectangular domain $\mathcal{R}$.

hypotenuse $\Gamma_3 = \{(x_1, x_2)|x_2 = \frac{b}{a}(a - x_1), 0 < x_1 < a\}$ of $\mathcal{T}$, the observation on Γ_3 cannot reflect on the boundary of $\mathcal{R}$ in general. Therefore, we cannot obtain the D-observability of system (1.4) on $\mathcal{T}$ in general, except the special right angle triangle domain $\mathcal{T}$ with vertices $(0,0),(0,1),(1,0)$ and with vertices $(0,0),(1,0),(0,\sqrt{3})$.

We point out that for the right angle triangle domain $\mathcal{T}$ and the rectangular domain $\mathcal{R}$ in consideration, the tiling chosen can be different, which not only affects choosing the observation boundary of the right angle triangle domain $\mathcal{T}$, but also the estimation of the observation time T, even for the same observation boundary of $\mathcal{T}$.

Comparing with the results of Christianson and Stafford in [1], for the adjoint system (1.4) with coupling matrix A^T satisfying **(H1)**–**(H2)**, we reduce the regularity of initial data and the number of observations on boundary. The initial data is only required in the space $(\mathrm{H}_0^1(\mathcal{T}))^N \times (\mathrm{L}^2(\mathcal{T}))^N$ and the observable matrix D is only required to satisfy the Kalman's rank condition (1.8). However, due to the limitation of the existing results, it is impossible to obtain the relevant results to the D-observability of the system (1.4) on other triangular domains by tiling.

In this paper, we consider the D-observability of system (1.4) on the right angle triangle domain $\mathcal{T}$. To transform the D-observability on $\mathcal{T}$ into the D-observability on the rectangular domain $\mathcal{R}$, we first give the D-observability of system (2.1) in Section 2. Noting the fact that the right angle triangle domain $\mathcal{T}$ tiles the rectangular domain $\mathcal{R}$ in different way, we give the definition and corresponding properties of tiling and then establish the relation between the D-observability of system (1.4) on domain Ω and $\widetilde\Omega$ where Ω tiles $\widetilde\Omega$ in Sections 3 and 4. Applying this result, we show that for $i = 1, 2$, the Kalman's rank condition (1.8) is sufficient for the D-observability of system (1.4) with $\Omega = \mathcal{T}$ on $[0,T] \times \Gamma_i$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N \times (\mathrm{L}^2(\mathcal{T}))^N$, provided that T is large enough in Section 5. In Section 6, we further show that the Kalman's rank condition is sufficient for the D-observability of system (1.4) on $[0,T] \times \Gamma$ in the space $(\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$, provided that T is large enough, where Ω is a sector domain and Γ is the circular side of Ω.

2. D-OBSERVABILITY ON THE RECTANGULAR DOMAIN

Let $\mathcal{R} = (0,a) \times (0,b)$. Consider the following adjoint system:

$$\begin{cases} \bar\Phi'' - \Delta\bar\Phi + A^T\bar\Phi = 0 & \text{in } (0,+\infty) \times \mathcal{R}, \\ \bar\Phi = 0 & \text{on } (0,+\infty) \times \partial\mathcal{R} \end{cases} \tag{2.1}$$

with the initial condition

$$t = 0: \ \bar\Phi = \bar\Phi_0, \ \bar\Phi' = \bar\Phi_1 \quad \text{in } \mathcal{R}, \tag{2.2}$$

where $(\bar\Phi_0, \bar\Phi_1) \in (\mathrm{H}_0^1(\mathcal{R}))^N \times (\mathrm{L}^2(\mathcal{R}))^N$.

The following equivalent condition of Kalman's rank condition is useful to obtain the D-observability of system (2.1) (see [3, 4]).

Lemma 2.1. *Let $k \geq 0$ be an integer, A be a matrix of order N and D be a full column-rank matrix of order $N \times M (M \leq N)$. The Kalman's rank condition*

$$\mathit{rank}\ (D, AD, \cdots, A^{N-1}D) = N - k$$

holds if and only if the largest dimension of invariant subspace of A^T, contained in $\ker(D^T)$, is equal to k.

Recalling that $\mathcal{R}$ is a rectangular domain, the eigenfunctions of $-\Delta$ in $\mathrm{H}_0^1(\mathcal{R})$, corresponding to the eigenvalue

$$\bar{\lambda}_{m,n}^2 := \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \tag{2.3}$$

is given by

$$\bar{e}_{m,n}(x_1, x_2) = \frac{ab}{mn\pi^2} \sin(\frac{m\pi x_1}{a}) \sin(\frac{n\pi x_2}{b}), \quad m \in \mathbb{N},\ n \in \mathbb{N}. \tag{2.4}$$

The outward normal derivative of $\bar{e}_{m,n}(x_1, x_2)$ on the boundary $\partial\Omega$ is given by

$$\partial_\nu \bar{e}_{m,n}(x_1, x_2) = \begin{cases} -\dfrac{a}{m\pi} \sin(\dfrac{m\pi x_1}{a}) & \text{on } \bar{\Gamma}_{1,1}, \\ (-1)^n \dfrac{a}{m\pi} \sin(\dfrac{m\pi x_1}{a}) & \text{on } \bar{\Gamma}_{1,2}, \\ -\dfrac{b}{n\pi} \sin(\dfrac{n\pi x_2}{b}) & \text{on } \bar{\Gamma}_{2,1}, \\ (-1)^m \dfrac{b}{n\pi} \sin(\dfrac{n\pi x_2}{b}) & \text{on } \bar{\Gamma}_{2,2}. \end{cases} \tag{2.5}$$

Similar to the proof of Theorems 1.2 and 1.3 in [5], we have

Lemma 2.2. *Let $\mathcal{R} = (0, a) \times (0, b)$ be the rectangular domain and $\bar{e}_{m,n}(x_1, x_2)$ be the eigenfunction of $-\Delta$ corresponding to $\bar{\lambda}_{m,n}^2$. Assume that A satisfies **(H1)–(H2)**. Then*

$$\bar{\lambda}_{m,n}^2 + \delta_k = \bar{\lambda}_{m',n'}^2 + \delta_l, \quad (m, n, k) \neq (m', n', l)$$

implies $m \neq m'$ and $n \neq n'$.

Remark 2.3. In the case $\Omega = \mathcal{R} = (0, a) \times (0, b)$, the condition **(H1)** means the s eigenvalues of A satisfying

$$-\bar{\lambda}_{1,1}^2 < \delta_1 < \delta_2 < \cdots < \delta_s, \tag{2.6}$$

where $\bar{\lambda}_{1,1}^2$ is given by (2.3).

The following examples show the necessity of condition **(H2)**.

Example 2.4. Take the coupling matrix $A = \mathrm{diag}(0, 3\pi^2)$ in system (2.1), $a = b = 1$ and the observable matrix $D = (1, -1)^T$. Let

$$\Phi = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\sqrt{4+n^2}\pi t} \bar{e}_{2,n}(x_1, x_2) \\ \mathrm{e}^{\mathrm{i}\sqrt{4+n^2}\pi t} \bar{e}_{1,n}(x_1, x_2) \end{pmatrix}. \tag{2.7}$$

It is easy to check that Φ is a solution of system (2.1), and the observation of Φ on $[0,+\infty)\times\bar{\Gamma}_{2,1}$ is given by

$$\begin{aligned} D^T\partial_\nu\Phi|_{x_2=0} &= D^T\begin{pmatrix} \mathrm{e}^{\mathrm{i}\sqrt{4+n^2}\pi t}\dfrac{\partial\bar{e}_{2,n}(x_1,x_2)}{\partial x_1} \\ \mathrm{e}^{\mathrm{i}\sqrt{4+n^2}\pi t}\dfrac{\partial\bar{e}_{1,n}(x_1,x_2)}{\partial x_1}\end{pmatrix} \\ &= (1,-1)\begin{pmatrix} -\frac{1}{n\pi}\mathrm{e}^{\mathrm{i}\sqrt{4+n^2}\pi t}\sin n\pi x_2 \\ -\frac{1}{n\pi}\mathrm{e}^{\mathrm{i}\sqrt{4+n^2}\pi t}\sin n\pi x_2\end{pmatrix} \\ &= 0. \end{aligned} \tag{2.8}$$

Therefore, for any given $T>0$, system (2.1) is not D-observable on $[0,T]\times\bar{\Gamma}_{2,1}$ in the space $(\mathrm{H}_0^1(\widetilde{R}))^N\times(\mathrm{L}^2(\widetilde{R}))^N$.

Example 2.5. Take the coupling matrix $A=\operatorname{diag}(0,3)$ in system (2.1), $a=\pi$, $b=1$ and the observable matrix $D=(1,-1)^T$. Then Φ given by (2.7) is also the solution of system (2.1). By (2.8), system (2.1) is not D-observable on $[0,T]\times\bar{\Gamma}_{2,1}$ in the space $(\mathrm{H}_0^1(\widetilde{R}))^N\times(\mathrm{L}^2(\widetilde{R}))^N$.

Remark 2.6. (i) Example 2.4 shows that the condition "$\|A-CI\|$ is small enough" and "the elements of A are rational numbers" in **(H2)** is necessary for the D-observability of system (2.1).

(ii) Example 2.5 shows that the condition "a,b are both algebraic numbers" in **(H2)** is necessary for the D-observability of system (2.1).

Lemma 2.7. *[3, 4] Let $\mathbb{Z}^*$ denote the set of all non-zero integers and $\{\beta_n^{(l)}\}_{n\in\mathbb{Z}^*;1\le l\le s}$ be a strictly increasing real sequence:*

$$\cdots<\beta_{-1}^{(1)}<\beta_{-1}^{(2)}<\cdots<\beta_{-1}^{(s)}<\beta_1^{(1)}<\beta_1^{(2)}<\cdots<\beta_1^{(s)}<\beta_2^{(1)}<\cdots.$$

Assume that there exist positive constants c, τ and γ such that

$$\beta_{n+1}^{(l)}-\beta_n^{(l)}\ge s\gamma$$

and

$$\frac{c}{|n|^\tau}\le\beta_n^{(l+1)}-\beta_n^{(l)}\le\gamma,$$

for all $n\in\mathbb{Z}^*$ and $|n|$ large enough. Then the following condition

$$\sum_{n\in\mathbb{Z}^*}\sum_{l=1}^s a_n^{(l)}\mathrm{e}^{\mathrm{i}\beta_n^{(l)}t}\equiv 0 \qquad \text{on } [0,T]$$

with

$$\sum_{n\in\mathbb{Z}^*}\sum_{l=1}^s |a_k^{(l)}|^2<+\infty,$$

implies

$$a_n^{(l)}=0,\quad n\in\mathbb{Z}^*;1\le l\le s,$$

provided that $T > 2\pi D^+$*, where* D^+ *is the upper density of the sequence* $\{\beta_n^{(l)}\}_{n\in\mathbb{Z}^*;1\le l\le s}$*, defined by*

$$D^+ = \limsup_{R\to+\infty} \frac{N(R)}{2R},$$

where $N(R)$ *denotes the number of* $\{\beta_n^{(l)}\}_{n\in\mathbb{Z}^*;1\le l\le s}$ *contained in the interval* $[-R,R]$.

Theorem 2.8. *Let* $\mathcal{R} = (0,a)\times(0,b)$ *be a rectangular domain and* A *satisfying* ***(H1)–(H2)****.*
(i)For $j = 1,2$*, Kalman's rank condition (1.8) is sufficient to the D-observability of system (2.1) on* $[0,T]\times\bar{\Gamma}_{1,j}$ *in the space* $(\mathrm{H}_0^1(\mathcal{R}))^N\times(\mathrm{L}^2(\mathcal{R}))^N$*, provided that* $T > 2bs$*.*
(ii)For $j = 1,2$*, Kalman's rank condition (1.8) is sufficient to the D-observability of system (2.1) on* $[0,T]\times\bar{\Gamma}_{2,j}$ *in the space* $(\mathrm{H}_0^1(\mathcal{R}))^N\times(\mathrm{L}^2(\mathcal{R}))^N$*, provided that* $T > 2as$*.*

Proof. Let $\omega_l^{(\mu)}(1\le\mu\le\mu_l)$ be the eigenvectors of A^T, corresponding to the eigenvalue δ_l, *i.e.*

$$A^T\omega_l^{(\mu)} = \delta_l\omega_l^{(\mu)}, \quad 1\le l\le s, \quad 1\le\mu\le\mu_l$$

with

$$\|\omega_l^{(\mu)}\|_{\mathbb{R}^N} = 1$$

and

$$\sum_{l=1}^{s}\mu_l = N.$$

(i) Choose $\bar{\Gamma}_{1,1}$ as the observable boundary. Define

$$\begin{cases}\bar{\beta}_{m,n}^{(l)} = \sqrt{\bar{\lambda}_{m,n}^2+\delta_l}, & l=1,2,\cdots,s,\ m\in\mathbb{N},\ n\in\mathbb{N},\\ \bar{\beta}_{m,-n}^{(s-l+1)} = -\bar{\beta}_{m,n}^{(l)}, & l=1,2,\cdots,s,\ m\in\mathbb{N},\ n\in\mathbb{N}\end{cases} \tag{2.9}$$

and

$$E_{m,n,l}^{(\mu)}(x_1,x_2) = \begin{pmatrix}\frac{1}{\mathrm{i}\bar{\beta}_{m,n}^{(l)}}\bar{e}_{m,n}(x_1,x_2)\omega_l^{(\mu)}\\ \bar{e}_{m,n}(x_1,x_2)\omega_l^{(\mu)}\end{pmatrix}, \qquad m\in\mathbb{Z}, n\in\mathbb{Z}^*,\ 1\le l\le s,\ 1\le\mu\le\mu_l,$$

where $\bar{e}_{m,-n}(x_1,x_2) = \bar{e}_{m,n}(x_1,x_2)$ for any given $n\in\mathbb{N}$. Thanks to (2.6), $\{\bar{\beta}_{m,n}^{(l)}\}$ is a real sequence. Since the eigenfunctions given by $\{\bar{e}_{m,n}\}_{m\in\mathbb{N};n\in\mathbb{N}}$ are orthogonal in $(\mathrm{L}^2(\mathcal{R}))^N$ as well as in $(\mathrm{H}_0^1(\mathcal{R}))^N$, the linear hull $\mathrm{Span}\{E_{m,n,l}^{(\mu)}\}_{1\le l\le s;1\le\mu\le\mu_l}$ with finite dimension N are mutually orthogonal for all $m\in\mathbb{N}; n\in\mathbb{N}$. On the other hand, $\{E_{m,n,l}^{(\mu)}\}_{m\in\mathbb{Z},n\in\mathbb{Z}^*,\ 1\le l\le s,\ 1\le\mu\le\mu_l}$ is complete in $(\mathrm{H}_0^1(\mathcal{R}))^N\times(\mathrm{L}^2(\mathcal{R}))^N$, so it forms a Hilbert basis of $(\mathrm{H}_0^1(\mathcal{R}))^N\times(\mathrm{L}^2(\mathcal{R}))^N$.

For any given initial data $(\bar{\Phi}_0,\bar{\Phi}_1)\in(\mathrm{H}_0^1(\mathcal{R}))^N\times(\mathrm{L}^2(\mathcal{R}))^N$, it can be decomposed by

$$\begin{pmatrix}\bar{\Phi}_0\\ \bar{\Phi}_1\end{pmatrix} = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^*}\sum_{1\le l\le s}\sum_{\mu=1}^{\mu_l} a_{m,n,l}^{(\mu)}E_{m,n,l}^{(\mu)},$$

with

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^*}\sum_{1\le l\le s}\sum_{\mu=1}^{\mu_l}|a^{(\mu)}_{m,n,l}|^2<+\infty.$$

Then the corresponding solution of problem (2.1)–(2.2) is given by

$$\bar\Phi=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{\mu=1}^{\mu_l}\frac{a^{(\mu)}_{m,n,l}}{\mathrm{i}\bar\beta^{(l)}_{m,n}}\omega^{(\mu)}_l\bar e_{m,n}(x_1,x_2)\mathrm{e}^{\mathrm{i}\bar\beta^{(l)}_{m,n}t}\quad \text{in } (0,T)\times\mathcal{R}.$$

By the observation (1.6) and (2.5), we have

$$0\equiv-\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{\mu=1}^{\mu_l}\frac{a^{(\mu)}_{m,n,l}}{\mathrm{i}\bar\beta^{(l)}_{m,n}}\frac{a}{m\pi}\sin(\frac{m\pi x_1}{a})D^T\omega^{(\mu)}_l\mathrm{e}^{\mathrm{i}\bar\beta^{(l)}_{m,n}t}\quad\text{on } \bar\Gamma_{1,1}. \tag{2.10}$$

Similar to the proof in [8], by Lemma 2.2 and the orthogonality of $\{\sin(\frac{m\pi x_1}{a})\}_{m\in\mathbb{N}}$, for any given $m\in\mathbb{N}$, we have

$$\sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{\mu=1}^{\mu_l}\frac{a^{(\mu)}_{m,n,l}}{\mathrm{i}\bar\beta^{(l)}_{m,n}}\mathrm{e}^{\mathrm{i}\bar\beta^{(l)}_{m,n}t}D^T\omega^{(\mu)}_l\equiv 0\quad\text{on } [0,T]. \tag{2.11}$$

For any given $m\in\mathbb{N}$, the sequence $\{\bar\beta^{(l)}_{m,n}\}_{n\in\mathbb{Z}^*;1\le l\le s}$ satisfies all the requirements of Lemma 2.7 by Lemma 2.2 and taking $\gamma_m=\dfrac{\pi}{bs},\tau_m=1$ and $D^+_m=\dfrac{bs}{\pi}$ in Lemma 2.7. Applying Lemma 2.7 to each line of (2.11), by Lemma 2.1, (1.8) and the linear independence of $\{\omega^{(l,\mu)}\}_{1\le\mu\le\mu_l}$, we get

$$a^{(\mu)}_{m,n,l}=0,\quad m\in\mathbb{N},n\in\mathbb{Z}^*,1\le l\le s,1\le\mu\le\mu_l,$$

and then $\bar\Phi\equiv0$, provided $T>2bs$. Namely, system (2.1) is D-observable on $[0,T]\times\bar\Gamma_{1,1}$ in the space $(\mathrm{H}^1_0(\mathcal{R}))^N\times(\mathrm{L}^2(\mathcal{R}))^N$, provided that $T>2bs$.

Choosing $\bar\Gamma_{1,2}$ as the observable boundary, by the observation (1.6) and (2.5), we have

$$0\equiv\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{\mu=1}^{\mu_l}\frac{a^{(\mu)}_{m,n,l}}{\mathrm{i}\bar\beta^{(l)}_{m,n}}\frac{(-1)^na}{m\pi}\sin(\frac{m\pi x_1}{a})D^T\omega^{(\mu)}_l\mathrm{e}^{\mathrm{i}\bar\beta^{(l)}_{m,n}t}\quad\text{on } \bar\Gamma_{1,2}. \tag{2.12}$$

The rest of proof is the same as in the case of $\bar\Gamma_{1,1}$. Namely, system (2.1) is D-observable on $[0,T]\times\bar\Gamma_{1,2}$ in the space $(\mathrm{H}^1_0(\mathcal{R}))^N\times(\mathrm{L}^2(\mathcal{R}))^N$, provided that $T>2bs$.

(ii) Choose $\bar\Gamma_{2,1}$ or $\bar\Gamma_{2,2}$ as the observable boundary. Define

$$\begin{cases}\tilde\beta^{(l)}_{m,n}=\sqrt{\bar\lambda^2_{m,n}+\delta_l}, & l=1,2,\cdots,s,\ m\in\mathbb{N},\ n\in\mathbb{N},\\ \tilde\beta^{(s-l+1)}_{-m,n}=-\tilde\beta^{(l)}_{m,n}, & l=1,2,\cdots,s,\ m\in\mathbb{N},\ n\in\mathbb{N}.\end{cases} \tag{2.13}$$

For any given $n\in\mathbb{N}$, the sequence $\{\bar\beta^{(l)}_{m,n}\}_{m\in\mathbb{Z}^*;1\le l\le s}$ satisfies all the requirements of Lemma 2.7 using Lemma 2.2 and taking $\gamma_n=\dfrac{\pi}{as},\tau_n=1$ and $D^+_n=\dfrac{as}{\pi}$ in Lemma 2.7. Similar to the proof of (i), for given $j=1,2$, system

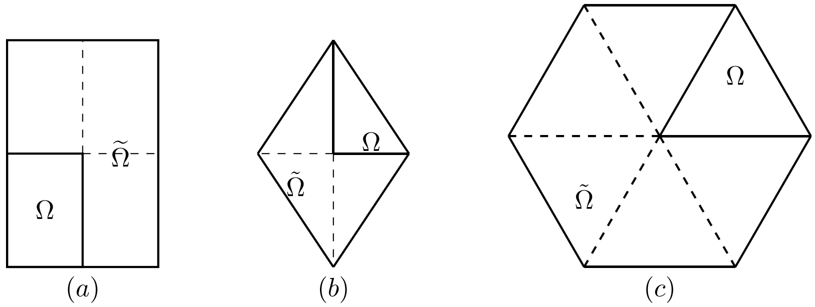

FIGURE 2. Some examples of tilings.

(2.1) is D-observable on $[0,T]\times\bar{\Gamma}_{2,j}$ in the space $(\mathrm{H}_0^1(\mathcal{R}))^N\times(\mathrm{L}^2(\mathcal{R}))^N$, provided that $T>2as$. The proof is complete. $\square$

3. Tiled domain and related properties

Based on the definition of tiled domain given by Lai (see [2]), we give

Definition 3.1. Let Ω and $\widetilde{\Omega}$ be two open bounded subsets of $\mathbb{R}^d$. The domain Ω tiles $\widetilde{\Omega}$, if there exists a set of rigid transformations (composite transformation of translations, flips and rotations) $\{K_h\}_{h=1}^k$ in $\mathbb{R}^d$ such that

$$\overline{\widetilde{\Omega}}=\bigcup_{h=1}^{k}K_h(\overline{\Omega}),\tag{3.1}$$

and

$$K_h(\Omega)\cap K_j(\Omega)=\emptyset,\quad \forall h,j=1,\cdots k,\ h\neq j,\tag{3.2}$$

where $\overline{\Omega}$ denotes the closure of Ω.

Remark 3.2. We point out that every rigid transformation K_h is invertible. In particular, since translations, flips and rotations are orthogonal transformation, the rigid transformation K_h is also an orthogonal transformation. Then the Jacobi matrix of K_h given by

$$J_h=(J_{ij}^h)_{d\times d}=\begin{pmatrix}\frac{\partial(K_h(x))_1}{\partial x_1} & \cdots & \frac{\partial(K_h(x))_1}{\partial x_d}\\ \vdots & \ddots & \vdots\\ \frac{\partial(K_h(x))_d}{\partial x_1} & \cdots & \frac{\partial(K_h(x))_d}{\partial x_d}\end{pmatrix},\tag{3.3}$$

is an orthogonal matrix.

We next give the prolongation of function $\phi:\Omega\to\mathbb{R}$ and the folding of $\widetilde{\phi}:\widetilde{\Omega}\to\mathbb{C}$ with spatial variables under the tiling $(\Omega,\{K_h\}_{h=1}^k)$.

Definition 3.3. Let $(\Omega,\{K_h\}_{h=1}^k)$ be a tiling of $\widetilde{\Omega}$ and $\varrho=(\varrho_1,\cdots,\varrho_k)\in\{-1,1\}^k$ with $\varrho_h\stackrel{def}{:=}|J_h|(h=1,\cdots,k)$. The prolongation $\mathcal{P}_\varrho\phi:\widetilde{\Omega}\to\mathbb{C}$ of a function $\phi:\Omega\to\mathbb{C}$ with coefficient ϱ given by

$$\mathcal{P}_\varrho\phi(K_hx)=\varrho_h\phi(x),\quad \forall h=1,\cdots,k,\ \forall x\in\Omega.$$

The folding $\mathcal{F}_\varrho\bar{\phi}:\Omega\to\mathbb{C}$ of a function $\widetilde{\phi}:\widetilde{\Omega}\to\mathbb{C}$ with coefficient ϱ given by

$$\mathcal{F}_\varrho\widetilde{\phi}(x)=\frac{1}{k}\sum_{h=1}^{k}\varrho_h\widetilde{\phi}(K_hx),\quad \forall h=1,\cdots,k,\ \forall x\in\Omega. \tag{3.4}$$

Remark 3.4. (i) The definition of $\mathcal{F}_\varrho$ in this paper and that [2] are equal up to a multiplicative constant. We only change the constant coefficient of (3.4) to make calculations more concise.

(ii) For any given function $\phi:\Omega\to\mathbb{C}$, it is easy to check that

$$\mathcal{F}_\varrho(\mathcal{P}_\varrho\phi)(x)=\frac{1}{k}\sum_{h=1}^{k}\varrho_h\mathcal{P}_\varrho\phi(K_hx)=\frac{1}{k}\sum_{h=1}^{k}\varrho_h^2\phi(x)=\phi(x),\ \forall x\in\Omega. \tag{3.5}$$

Remark 3.5. The map $\mathcal{P}_\varrho$ is not a surjective from $(\mathrm{H}_0^1(\Omega))^N$ to $(\mathrm{H}_0^1(\widetilde{\Omega}))^N$ in general. For instance, choosing the tiling $\{K_h\}_{h=1}^2$ in Figure 1 and let $\widetilde{U}(x_1,x_2)\in(\mathrm{H}_0^1(\widetilde{\Omega}))^N$ satisfying $\widetilde{U}(x_1,\frac{b}{a}(a-x_1))\neq 0$, it is impossible to find $U(x_1,x_2)\in(\mathrm{H}_0^1(\Omega))^N$ such that $\mathcal{P}_\varrho U(x_1,x_2)=\widetilde{U}(x_1,x_2)$.

Denote $\{e_n^{(j)}(x)\}_{n\in\mathbb{N},1\le j\le j_n}$ the series of the unit eigenfunctions of $-\Delta$, satisfying

$$\begin{cases}-\Delta e_n^{(j)}(x)=\lambda_n^2e_n^{(j)}(x) & \text{in }\Omega,\\ e_n^{(j)}(x)=0 & \text{on }\partial\Omega,\end{cases} \tag{3.6}$$

where the eigenvalue λ_n with multiplicity j_n satisfies

$$0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots.$$

Let $(\Omega,\{K_h\}_{h=1}^k)$ be a tiling of $\widetilde{\Omega}$. We introduce the properties of the prolongation $\mathcal{P}_\varrho e_n^{(j)}(x)$ of $e_n^{(j)}(x)$ with coefficient ϱ.

Theorem 3.6. *Let $(\Omega,\{K_h\}_{h=1}^k)$ be a tiling of $\widetilde{\Omega}$. Then $\{\mathcal{P}_\varrho e_n^{(j)}(x)\}_{n\in\mathbb{N},1\le j\le j_n}\subset\mathrm{H}_0^1(\widetilde{\Omega})$, being the eigenfunctions of $-\Delta$ in the space $\mathrm{H}_0^1(\widetilde{\Omega})$, forms a complete basis of $\mathcal{P}_\varrho\mathrm{L}^2(\Omega)$.*

In particular, for any given $n\in\mathbb{N}$ and any given $1\le j\le j_n$, if $a_n^{(j)}$ is the expansion coefficient of $\phi\in\mathrm{L}^2(\Omega)$ corresponding to the eigenfunction $e_n^{(j)}$, then it is also the expansion coefficient of $\mathcal{P}_\varrho\phi$ corresponding to the eigenfunction $\mathcal{P}_\varrho e_n^{(j)}(x)$.

Proof. We first show that $\{\mathcal{P}_\varrho e_n^{(j)}(x)\}_{n\in\mathbb{N},1\le j\le j_n}\subset\mathrm{H}_0^1(\widetilde{\Omega})$ are the eigenfunctions of $-\Delta$ in the space $\mathrm{H}_0^1(\widetilde{\Omega})$.

Recalling that $e_n^{(j)}$ satisfies (3.6), we immediately have $\mathcal{P}_\varrho e_n^{(j)}(x)\in\mathrm{H}_0^1(\widetilde{\Omega})$. Since

$$\nabla_{K_h(x)}=\Big(\sum_{i=1}^{d}J_{1i}^{(h)}\partial_{x_i},\cdots,\sum_{i=1}^{d}J_{di}^{(h)}\partial_{x_i}\Big),$$

we have

$$
\begin{aligned}
\Delta_{K_h(x)} = \nabla_{K_h(x)} \cdot \nabla_{K_h(x)} &= \sum_{l=1}^{d} \Big(\sum_{i=1}^{d} J_{li}^{(h)2} \partial x_i x_i + 2 \sum_{i \neq j}^{d} J_{li}^{(h)} J_{lj}^{(h)} \partial x_i x_j \Big) \\
&= \sum_{i=1}^{d} \big(\sum_{l=1}^{d} J_{li}^{(h)2} \big) \partial x_i x_i + 2 \sum_{i \neq j}^{d} \big(\sum_{l=1}^{d} J_{li}^{(h)} J_{lj}^{(h)} \big) \partial x_i x_j \\
&= \sum_{i=1}^{d} (J_h^T J_h)_{ii} \partial x_i x_i + 2 \sum_{i \neq j}^{d} (J_h^T J_h)_{ij} \partial x_i x_j \\
&= \sum_{i=1}^{d} \partial x_i x_i = \Delta_x.
\end{aligned}
\tag{3.7}
$$

Therefore,

$$
\Delta_{K_h(x)} \mathcal{P}_\varrho e_n^{(j)}(K_h(x)) = \Delta_x \varrho_h e_n^{(j)}(x) = -\lambda_n^2 \varrho_h e_n^{(j)}(x) = -\lambda_n^2 \mathcal{P}_\varrho e_n^{(j)}(K_h(x)), \quad x \in \Omega,
$$

which implies

$$
\begin{cases}
-\Delta \mathcal{P}_\varrho e_n^{(j)}(x) = \lambda_n^2 \mathcal{P}_\varrho e_n^{(j)}(x) & \text{in } \widetilde{\Omega}, \\
\mathcal{P}_\varrho e_n^{(j)}(x) = 0 & \text{on } \partial \widetilde{\Omega}.
\end{cases}
$$

We next show that $\{\mathcal{P}_\varrho e_n^{(j)}(x)\}_{n \in \mathbb{N}, 1 \le j \le j_n}$ is complete in the space $\mathcal{P}_\varrho \mathrm{L}^2(\Omega)$.

For any given $\widetilde{u} \in \mathcal{P}_\varrho \mathrm{L}^2(\Omega)$, there exists a unique $u(x) \in \mathrm{L}^2(\Omega)$ such that $\widetilde{u} = \mathcal{P}_\varrho u(x)$. Therefore, we have

$$
\begin{aligned}
\int_{\widetilde{\Omega}} \mathcal{P}_\varrho u(x) \mathcal{P}_\varrho e_n^{(j)}(x) \, \mathrm{d}x &= \sum_{h=1}^{k} \int_{K_h(\Omega)} \mathcal{P}_\varrho u(x) \mathcal{P}_\varrho e_n^{(j)}(x) \, \mathrm{d}x \\
&= \sum_{h=1}^{k} \int_{\Omega} \varrho_h^2 u(x) e_n^{(j)}(x) |\varrho_h| \, \mathrm{d}x \\
&= k \int_{\Omega} u(x) e_n^{(j)}(x) \, \mathrm{d}x.
\end{aligned}
\tag{3.8}
$$

Then $\{\mathcal{P}_\varrho e_n^{(j)}(x)\}_{n \in \mathbb{N}, 1 \le j \le j_n} \subseteq \mathrm{H}_0^1(\widetilde{\Omega})$ is a complete basis of $\mathcal{P}_\varrho \mathrm{L}^2(\Omega)$. Noting that

$$
\int_{\widetilde{\Omega}} |\mathcal{P}_\varrho e_n^{(j)}(x)|^2 \, \mathrm{d}x = k \int_{\Omega} |e_n^{(j)}(x)|^2 \, \mathrm{d}x = k,
\tag{3.9}
$$

if $a_n^{(j)}$ is the expansion coefficient of $u(x) \in \mathrm{L}^2(\Omega)$ corresponding to the eigenfunction $e_n^{(j)}(x)$, *i.e.*

$$
\int_{\Omega} u(x) e_n^{(j)}(x) \, \mathrm{d}x = a_n^{(j)},
\tag{3.10}
$$

then by (3.8), (3.9) and (3.10), we have

$$
\int_{\widetilde{\Omega}} \mathcal{P}_\varrho u(x) \mathcal{P}_\varrho e_n^{(j)}(x) \, \mathrm{d}x = k a_n^{(j)} = a_n^{(j)} \int_{\widetilde{\Omega}} |\mathcal{P}_\varrho e_n^{(j)}(x)|^2 \, \mathrm{d}x.
$$

Namely, $a_n^{(j)}$ is the expansion coefficient of $\mathcal{P}_\varrho u(x)$ corresponding to the eigenfunction $\mathcal{P}_\varrho e_n^{(j)}(x)$. The proof is complete. □

Remark 3.7. Noting (3.7), the Laplace operator Δ is invariant under the rigid transformation K_h, which is the main reason why the D-observability of system (1.4) on Ω can be transformed into the D-observability of system (1.4) on the tiled domain $\widetilde{\Omega}$ through the tiled method.

Next we respectively give the prolongation and folding of $\Phi : \big(\mathbb{R}^+ \times \Omega\big)^N \to \mathbb{C}^N$ and $\widetilde{\Phi} : \big(\mathbb{R}^+ \times \widetilde{\Omega}\big)^N \to \mathbb{C}^N$ with time variables, respectively. The prolongation of $\Phi : \big(\mathbb{R}^+ \times \Omega\big)^N \to \mathbb{C}^N$ is given by

$$\mathcal{P}_\varrho \Phi(t, K_h x) = \varrho_h \Phi(t,x), \quad \forall h = 1, \cdots, k, \ \forall t \in \mathbb{R}^+, \ \forall x \in \Omega;$$

while, the folding of $\widetilde{\Phi} : \big(\mathbb{R}^+ \times \widetilde{\Omega}\big)^N \to \mathbb{C}^N$ is given by

$$\mathcal{F}_\varrho \widetilde{\Phi}(t,x) = \frac{1}{k}\sum_{h=1}^{k} \varrho_h \widetilde{\Phi}(t, K_h x), \quad \forall h = 1, \cdots, k, \ \forall t \in \mathbb{R}^+, \ \forall x \in \Omega.$$

Denote

$$\mathcal{P}_\varrho (\mathrm{L}^2(\Omega))^N := \{\mathcal{P}_\varrho U | U \in (\mathrm{L}^2(\Omega))^N\}$$

and

$$\mathcal{P}_\varrho (\mathrm{H}_0^1(\Omega))^N := \{\mathcal{P}_\varrho U | U \in (\mathrm{H}_0^1(\Omega))^N\}.$$

We will show that $\mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N \subseteq (\mathrm{L}^2(\widetilde{\Omega}))^N$ and $\mathcal{P}_\varrho(\mathrm{H}_0^1(\Omega))^N \subseteq (\mathrm{H}_0^1(\widetilde{\Omega}))^N$.

In fact, for the function $U \in (\mathrm{H}_0^1(\Omega))^N$, by the definition of $P_\varrho U$, we have $P_\varrho U|_{\partial\widetilde{\Omega}} = 0$. Moreover, by the definition of tiling, we have

$$\int_{\widetilde{\Omega}} |\mathcal{P}_\varrho U(x)|^2 \,\mathrm{d}x = \sum_{h=1}^{k} \int_{K_h(\Omega)} |\mathcal{P}_\varrho U(x)|^2 \,\mathrm{d}x = \sum_{h=1}^{k} \int_\Omega |\varrho_h U(x)|^2 \,\mathrm{d}x = k\|U(x)\|^2_{L^2(\Omega)}$$

and

$$\begin{aligned}
\int_{\widetilde{\Omega}} |\nabla \mathcal{P}_\varrho U(x)|^2 \,\mathrm{d}x &= \sum_{h=1}^{k} \int_{K_h(\Omega)} |\nabla \mathcal{P}_\varrho U(x)|^2 \,\mathrm{d}x \\
&= \sum_{h=1}^{k} \int_\Omega |\nabla_{K_h(x)} \varrho_h U(x)|^2 \,\mathrm{d}x \\
&= \sum_{h=1}^{k} \int_\Omega \sum_{k=1}^{d} \Big| \sum_{i=1}^{d} J_{ki}^{(h)} \partial_{x_i} U(x) \Big|^2 \,\mathrm{d}x \\
&= \sum_{h=1}^{k} \int_\Omega \sum_{i=1}^{d} \Big(\sum_{k=1}^{d} J_{ki}^{(h)2} \Big) \big|\partial_{x_i} U(x)\big|^2 \\
&\quad + \sum_{i\neq j}^{d} \Big(\sum_{k=1}^{d} J_{ki}^{(h)} J_{kj}^{(h)} \Big) \Big(\partial_{x_i} U(x) \overline{\partial_{x_j} U(x)}^T + \overline{\partial_{x_i} U(x)} \partial_{x_j} U(x)^T \Big) \,\mathrm{d}x.
\end{aligned} \tag{3.11}$$

Recalling that the J_h is a orthogonal matrix, we have

$$\sum_{k=1}^{d} J_{ki}^{(h)} J_{kj}^{(h)} = (J_h^T J_h)_{ij} = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. Then (3.11) becomes

$$\int_{\widetilde{\Omega}} |\nabla \mathcal{P}_\varrho U(x)|^2 \,\mathrm{d}x = \sum_{h=1}^{k} \int_\Omega \sum_{i=1}^{d} \left|\partial_{x_i} U(x)\right|^2 \mathrm{d}x = k \int_\Omega |\nabla U(x)|^2 \,\mathrm{d}x,$$

which implies that

$$\mathcal{P}_\varrho U(x) \in (\mathrm{H}_0^1(\widetilde{\Omega}))^N, \ \|\mathcal{P}_\varrho U(x)\|^2_{(L^2(\widetilde{\Omega}))^N} = k\|U(x)\|^2_{(L^2(\Omega))^N}$$

and

$$\|\mathcal{P}_\varrho U(x)\|^2_{(\mathrm{H}_0^1(\widetilde{\Omega}))^N} = k\|U(x)\|^2_{(\mathrm{H}_0^1(\Omega))^N},$$

namely, we have $\mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N \subseteq (\mathrm{L}^2(\widetilde{\Omega}))^N$ and $\mathcal{P}_\varrho(\mathrm{H}_0^1(\Omega))^N \subseteq (\mathrm{H}_0^1(\widetilde{\Omega}))^N$.

Remark 3.8. Lai gave the definition of "admissible tiling" in [2] to make sure that $\mathcal{F}_\varrho \phi \in \mathrm{H}_0^1(\Omega)$ for all $\phi \in \mathrm{H}_0^1(\widetilde{\Omega})$. Since $\mathcal{P}_\varrho(\mathrm{H}_0^1(\Omega)) \subseteq \mathrm{H}_0^1(\widetilde{\Omega})$ naturally, we consider the prolongation prolongation $\mathcal{P}_\varrho e_n^{(j)}(x)$ of $e_n^{(j)}(x)$ in this paper to avoid discussing "admissible tiling".

4. D-observability on tiled domains

Consider the prolongation system

$$\begin{cases} \widetilde{\Phi}'' - \Delta\widetilde{\Phi} + A^T\widetilde{\Phi} = 0 & \text{in } (0,+\infty)\times\widetilde{\Omega}, \\ \widetilde{\Phi} = 0 & \text{on } (0,+\infty)\times\partial\widetilde{\Omega} \end{cases} \tag{4.1}$$

with the initial condition:

$$t = 0: \ \widetilde{\Phi}(t,x) = \mathcal{P}_\varrho\widehat{\Phi}_0, \ \widetilde{\Phi}'(t,x) = \mathcal{P}_\varrho\widehat{\Phi}_1 \quad \text{in } \widetilde{\Omega}, \tag{4.2}$$

where the initial data $(\mathcal{P}_\varrho\widehat{\Phi}_0, \mathcal{P}_\varrho\widehat{\Phi}_1)$ belongs to the space $\mathcal{P}_\varrho(\mathrm{H}_0^1(\Omega))^N \times \mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N \subseteq (\mathrm{H}_0^1(\widetilde{\Omega}))^N \times (\mathrm{L}^2(\widetilde{\Omega}))^N$.

Let $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$. We first give the relation between the solutions of problem (1.4)–(1.5) and the prolongation problem (4.1)–(4.2).

Theorem 4.1. *Let $(\Omega, \{K_h\}_{h=1}^k)$ be a tiling of $\widetilde{\Omega}$. Assume that A satisfies **(H1)** and $\Phi(t,x)$ is the solution of problem (1.4)–(1.5), then $\mathcal{P}_\varrho\Phi(t,x)$ is the solution of the prolongation problem (4.1)–(4.2).*

Conversely, if $\widetilde{\Phi}$ is the solution of the prolongation problem (4.1)–(4.2), then $\mathcal{F}_\varrho\widetilde{\Phi}$ is the solution of the original problem (1.4)–(1.5). Moreover, for any given $h = 1, \cdots, k$, we have

$$\mathcal{F}_\varrho\widetilde{\Phi}(t,x) = \varrho_h\widetilde{\Phi}(t, K_h x), \quad \forall x \in \Omega. \tag{4.3}$$

Proof. Denote $\delta_l > -\lambda_1^2$ be the eigenvalue of A^T, satisfying

$$A^T \omega_l^{(\mu)} = \delta_l \omega_l^{(\mu)}, \quad 1 \le l \le s, \quad 1 \le \mu \le \mu_l,$$

where the multiplicity μ_l of eigenvalue δ_l satisfies

$$\sum_{l=1}^{s} \mu_l = N.$$

Define

$$\begin{cases} \beta_{n,l} = \sqrt{\lambda_n^2 + \delta_l}, & l = 1, 2, \cdots, s, \ n \in \mathbb{N}, \\ \beta_{-n,s-l+1} = -\beta_{n,l}, & l = 1, 2, \cdots, s, \ n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where $\lambda_n^2 > 0$ is given by (3.6). Thanks to condition **(H1)**, $\{\beta_{n,l}\}$ is a real sequence. Then we have

$$(-\Delta + A^T)\omega_l^{(\mu)} e_n^{(j)}(x) = \beta_{n,l}^2 \omega_l^{(\mu)} e_n^{(j)}(x).$$

Take

$$\|e_n^{(j)}(x)\|_{L^2(\Omega)} = 1 \qquad \text{and} \qquad \|\omega_l^{(\mu)}\|_{\mathbb{R}^N} = 1. \tag{4.5}$$

Define

$$E_{n,l}^{(j,\mu)}(x) = \begin{pmatrix} \frac{1}{\mathrm{i}\beta_{n,l}} e_n^{(j)}(x)\omega_l^{(\mu)} \\ e_n^{(j)}(x)\omega_l^{(\mu)} \end{pmatrix}, \qquad n \in \mathbb{Z}^*, \ 1 \le l \le s, \ 1 \le j \le j_n, \ 1 \le \mu \le \mu_l,$$

where $e_{-n}^{(j)}(x) = e_n^{(j)}(x)$ for any given $n \in \mathbb{N}$ and any given $1 \le j \le j_n$. Then by [4], $\{E_{n,l}^{(j,\mu)}\}_{n,l,j,\mu}$ forms a Hilbert basis in $(\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$.

For any given

$$\begin{pmatrix} \widehat{\Phi}_0 \\ \widehat{\Phi}_1 \end{pmatrix} = \sum_{n \in \mathbb{Z}^*} \sum_{1 \le l \le s} \sum_{j=1}^{j_n} \sum_{\mu=1}^{\mu_l} a_{n,l}^{(j,\mu)} E_{n,l}^{(j,\mu)},$$

with

$$\|\widehat{\Phi}_0\|^2_{(\mathrm{H}_0^1(\Omega))^N} + \|\widehat{\Phi}_1\|^2_{(\mathrm{L}^2(\Omega))^N} = \sum_{n \in \mathbb{Z}^*} \sum_{1 \le l \le s} \sum_{j=1}^{j_n} \sum_{\mu=1}^{\mu_l} |a_{n,l}^{(j,\mu)}|^2 < +\infty.$$

The corresponding solution of problem (1.4)–(1.5) is given by

$$\Phi = \sum_{n \in \mathbb{Z}^*} \sum_{l=1}^{s} \sum_{j=1}^{j_n} \sum_{\mu=1}^{\mu_l} \frac{a_{n,l}^{(j,\mu)}}{\mathrm{i}\beta_{n,l}} \omega_l^{(\mu)} e_n^{(j)}(x) \mathrm{e}^{\mathrm{i}\beta_{n,l} t} \quad \text{in } (0,T) \times \Omega.$$

By Theorem 3.6, the expansion coefficients of $(\mathcal{P}_\varrho\widehat{\Phi}_0, \mathcal{P}_\varrho\widehat{\Phi}_1)$ are given by $\{a_{n,l}^{(j,\mu)}\}$, then the solution of problem (4.1)–(4.2) is given by

$$\mathcal{P}_\varrho\Phi(t,x) = \sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_n}\sum_{\mu=1}^{\mu_l}\frac{a_{n,l}^{(j,\mu)}}{\mathrm{i}\beta_{n,l}}\omega_l^{(\mu)}\mathcal{P}_\varrho e_n^{(j)}(x)\mathrm{e}^{\mathrm{i}\beta_{n,l}t} \quad \text{in } (0,T)\times\widetilde{\Omega}.$$

Conversely, assume that

$$\widetilde{\Phi}(t,x) = \sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_n}\sum_{\mu=1}^{\mu_l}\frac{\tilde{a}_{n,l}^{(j,\mu)}}{\mathrm{i}\beta_{n,l}}\omega_l^{(\mu)}\mathcal{P}_\varrho e_n^{(j)}(x)\mathrm{e}^{\mathrm{i}\beta_{n,l}t} \quad \text{in } (0,T)\times\widetilde{\Omega}$$

is the solution of problem (4.1)–(4.2). Let $a_{n,l}^{(j,\mu)} = \tilde{a}_{n,l}^{(j,\mu)}$. Then

$$\begin{aligned}\Phi(t,x) &:= \mathcal{F}_\varrho\widetilde{\Phi}(t,x)\\ &= \mathcal{F}_\varrho\sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_n}\sum_{\mu=1}^{\mu_l}\frac{\tilde{a}_{n,l}^{(j,\mu)}}{\mathrm{i}\beta_{n,l}}\omega_l^{(\mu)}\mathcal{P}_\varrho e_n^{(j)}(x)\mathrm{e}^{\mathrm{i}\beta_{n,l}t}\\ &= \sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_n}\sum_{\mu=1}^{\mu_l}\frac{a_{n,l}^{(j,\mu)}}{\mathrm{i}\beta_{n,l}}\omega_l^{(\mu)}e_n^{(j)}(x)\mathrm{e}^{\mathrm{i}\beta_{n,l}t} \quad \text{in } (0,T)\times\Omega\end{aligned}$$

is the solution of problem (1.4)–(1.5). Finally, noting that $|\varrho_h|\equiv 1$, for any given $h=1,\cdots,k$, we have

$$\begin{aligned}\mathcal{F}_\varrho\widetilde{\Phi}(t,x) &= \sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_n}\sum_{\mu=1}^{\mu_l}\frac{\tilde{a}_{n,l}^{(j,\mu)}}{\mathrm{i}\beta_{n,l}}\omega_l^{(\mu)}\mathrm{e}^{\mathrm{i}\beta_{n,l}t}e_n^{(j)}(x)\\ &= \varrho_h\sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_n}\sum_{\mu=1}^{\mu_l}\frac{\tilde{a}_{n,l}^{(j,\mu)}}{\mathrm{i}\beta_{n,l}}\omega_l^{(\mu)}\mathrm{e}^{\mathrm{i}\beta_{n,l}t}\varrho_h e_n^{(j)}(x)\\ &= \varrho_h\sum_{n\in\mathbb{Z}^*}\sum_{l=1}^{s}\sum_{j=1}^{j_n}\sum_{\mu=1}^{\mu_l}\frac{\tilde{a}_{n,l}^{(j,\mu)}}{\mathrm{i}\beta_{n,l}}\omega_l^{(\mu)}\mathrm{e}^{\mathrm{i}\beta_{n,l}t}\mathcal{P}_\varrho e_n^{(j)}(K_h x)\\ &= \varrho_h\widetilde{\Phi}(t,K_h x) \quad x\in\Omega.\end{aligned}$$

Then we get (4.3). The proof is complete. □

The aim of this section is to establish the relation between the D-observability of system (1.4) and system (4.1). By Theorem 4.1, Φ is the solution of problem (1.4)–(1.5) if and only if $\mathcal{P}_\varrho\Phi$ is the solution of the prolongation problem (4.1)–(4.2). In particular, $\Phi\equiv 0$ if and only if $\mathcal{P}_\varrho\Phi\equiv 0$.

Let $(\Omega,\{K_h\}_{h=1}^k)$ be a tiling of $\widetilde{\Omega}$. We next show that the D-observability of system (1.4) on $[0,T]\times\Gamma$ in the space $(\mathrm{H}_0^1(\Omega))^N\times(\mathrm{L}^2(\Omega))^N$ is equivalent to the D-observability of system (4.1) on $[0,T]\times\widetilde{\Gamma}$ in the space $\mathcal{P}_\varrho(\mathrm{H}_0^1(\Omega))^N\times\mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N$, where $\widetilde{\Gamma}\subset\partial\widetilde{\Omega}$ and Γ defined by

$$\Gamma = \bigcup_{h=1}^{k} K_h^{-1}(\widetilde{\Gamma})\cap\partial\Omega. \tag{4.6}$$

Remark 4.2. Noting that $\mathcal{P}_\varrho(\mathrm{H}_0^1(\Omega))^N \subseteq (\mathrm{H}_0^1(\widetilde{\Omega}))^N$ and $\mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N \subseteq (\mathrm{L}^2(\widetilde{\Omega}))^N$, then the D-observability of system (4.1) on $[0,T]\times\widetilde{\Gamma}$ in the space $(\mathrm{H}_0^1(\widetilde{\Omega}))^N \times (\mathrm{L}^2(\widetilde{\Omega}))^N$ implies the D-observability of system (4.1) on $[0,T]\times\widetilde{\Gamma}$ in the space $\mathcal{P}_\varrho(\mathrm{H}_0^1(\Omega))^N \times \mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N$.

Theorem 4.3. *Let $(\Omega, \{K_h\}_{h=1}^k)$ be a tiling of $\widetilde{\Omega}$. Assume that $\widetilde{\Gamma}\subset\partial\widetilde{\Omega}$ and Γ is given by (4.6). Then the system (1.4) is D-observable on $[0,T]\times\Gamma$ in the space $(\mathrm{H}_0^1(\Omega))^N\times(\mathrm{L}^2(\Omega))^N$ if and only if the system (4.1) is D-observable on $[0,T]\times\widetilde{\Gamma}$ in the space $\mathcal{P}_\varrho(\mathrm{H}_0^1(\Omega))^N\times\mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N$.*

Proof. Assume that Φ is the solution of problem (1.4)–(1.5), then the prolongation $\mathcal{P}_\varrho\Phi$ of Φ is the solution of problem (4.1)–(4.2). Since Ω tiles $\widetilde{\Omega}$, defining $\Gamma_h := K_h^{-1}(\widetilde{\Gamma})\cap\partial\Omega$, we have

$$\Gamma=\bigcup_{h=1}^k \Gamma_h, \qquad \text{and} \qquad \widetilde{\Gamma}=\bigcup_{h=1}^k K_h(\Gamma_h).$$

Denote ∂_{ν_h}, $\partial_{\widetilde{\nu}}$ and $\partial_{\widetilde{\nu}_h}$ the outward normal derivative on Γ_h, $\widetilde{\Gamma}$ and $K_h(\Gamma_h)$, respectively. By $|\varrho_h|\equiv 1$ and Remark 3.2, we have

$$\begin{aligned}
\int_0^T\int_{\widetilde{\Gamma}}|D^T\partial_{\widetilde{\nu}}\mathcal{P}_\varrho\Phi(t,x)|^2\,\mathrm{d}\widetilde{\Gamma}\,\mathrm{d}t \le& \sum_{h=1}^k\int_0^T\int_{K_h(\Gamma_h)}|D^T\partial_{\widetilde{\nu}_h}\mathcal{P}_\varrho\Phi(t,x)|^2\,\mathrm{d}\Gamma'\,\mathrm{d}t\\
=&\sum_{h=1}^k\int_0^T\int_{\Gamma_h}|D^T\partial_{K_h(\nu_h)}\mathcal{P}_\varrho\Phi(t,K_hx)|^2\,\mathrm{d}\Gamma\,\mathrm{d}t\\
=&\sum_{h=1}^k\varrho_h^2\int_0^T\int_{\Gamma_h}|D^T\partial_{K_h(\nu_h)}\frac{1}{\varrho_h}\mathcal{P}_\varrho\Phi(t,K_hx)|^2\,\mathrm{d}\Gamma\,\mathrm{d}t\\
=&\sum_{h=1}^k\int_0^T\int_{\Gamma_h}|D^T\partial_{\nu_h}\Phi(t,x)|^2\,\mathrm{d}\Gamma\,\mathrm{d}t\\
\le& k\int_0^T\int_{\Gamma}|D^T\partial_\nu\Phi(t,x)|^2 d\Gamma\,\mathrm{d}t
\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}
\int_0^T\int_{\Gamma}|D^T\partial_\nu\Phi(t,x)|^2 d\Gamma\,\mathrm{d}t \le& \sum_{h=1}^k\int_0^T\int_{\Gamma_h}|D^T\partial_{\nu_h}\Phi(t,x)|^2\,\mathrm{d}\Gamma\,\mathrm{d}t\\
=&\sum_{h=1}^k\int_0^T\int_{\Gamma_h}|D^T\partial_{K_h(\nu_h)}\frac{1}{\varrho_h}\mathcal{P}_\varrho\Phi(t,K_hx)|^2\,\mathrm{d}\Gamma\,\mathrm{d}t\\
=&\sum_{h=1}^k\int_0^T\int_{K_h(\Gamma_h)}|D^T\partial_{\widetilde{\nu}_h}\mathcal{P}_\varrho\Phi(t,x)|^2\,\mathrm{d}\Gamma'\,\mathrm{d}t\\
\le& k\int_0^T\int_{\widetilde{\Gamma}}|D^T\partial_{\widetilde{\nu}}\mathcal{P}_\varrho\Phi(t,x)|^2\,\mathrm{d}\widetilde{\Gamma}\,\mathrm{d}t.
\end{aligned}\tag{4.8}$$

Therefore,

$$D^T\partial_\nu\Phi\equiv 0 \quad \text{on } [0,T]\times\Gamma \tag{4.9}$$

if and only if

$$D^T \partial_{\widetilde{\nu}} \mathcal{P}_\varrho \Phi \equiv 0 \quad \text{on } [0,T] \times \widetilde{\Gamma}. \tag{4.10}$$

Finally, by Theorem 3.6, we have

$$\|\mathcal{P}_\varrho \widehat{\Phi}_0\|^2_{(\mathrm{H}^1_0(\widetilde{\Omega}))^N} = k\|\widehat{\Phi}_0\|^2_{(\mathrm{H}^1_0(\Omega))^N} \quad \text{and} \quad \|\mathcal{P}_\varrho \widehat{\Phi}_1\|^2_{(\mathrm{L}^2(\widetilde{\Omega}))^N} = k\|\widehat{\Phi}_1\|^2_{(\mathrm{L}^2(\Omega))^N}.$$

Then $\mathcal{P}_\varrho$ is a bijection from $(\mathrm{H}^1_0(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$ to $\mathcal{P}_\varrho(\mathrm{H}^1_0(\Omega))^N \times \mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N$. Hence, the partial Neumann observation (1.6) of system (1.4) in the space $(\mathrm{H}^1_0(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$ is equivalent to the partial Neumann observation of the prolongation system (4.1) in the space $\mathcal{P}_\varrho(\mathrm{H}^1_0(\Omega))^N \times \mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N$:

$$D^T \partial_{\widetilde{\nu}} \widetilde{\Phi} \equiv 0 \quad \text{on } [0,T] \times \widetilde{\Gamma}. \tag{4.11}$$

It follows that the D-observability of system (1.4) on $[0,T] \times \Gamma$ in the space $(\mathrm{H}^1_0(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$ is equivalent to the D-observability of system (4.1) on $[0,T] \times \widetilde{\Gamma}$ in the space $\mathcal{P}_\varrho(\mathrm{H}^1_0(\Omega))^N \times \mathcal{P}_\varrho(\mathrm{L}^2(\Omega))^N$. The proof is complete. $\square$

Remark 4.4. We point out that Theorem 4.3 and Theorem 4.6 below are valid for arbitrary coupling matrix A.

The following example shows that the D-observability of system (1.4) on $[0,T] \times \Gamma$ in the space $(\mathrm{H}^1_0(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$ does not imply the D-observability of the prolongation system (4.1) on $[0,T] \times \widetilde{\Gamma}$ in the space $(\mathrm{H}^1_0(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$.

Example 4.5. Taking $\Omega = (0,\pi) \times (0,\pi)$ and $\widetilde{\Omega} = (0,\pi) \times (0,2\pi)$, $(\Omega, \{K_h\}_{h=1}^2)$ tiles $\widetilde{\Omega}$(see Fig. 3), where the transformation K_h from Ω to $K_h(\Omega)$ is defined by

$$K_1 := id; \quad K_2: \ (x_1,x_2) \to (x_1, 2\pi - x_2).$$

Let $m,n \in \mathbb{N}$. The eigenfunction of $-\Delta$ on the domain Ω is given by

$$e_{m,n} = \sin(mx_1)\sin(nx_2),$$

corresponding to the eigenvalue $\lambda_{m,n} = \sqrt{m^2+n^2}$; while the eigenfunction of $-\Delta$ on the domain $\widetilde{\Omega}$ is given by

$$\tilde{e}_{m,n} = \sin(mx_1)\sin(\frac{nx_2}{2}),$$

corresponding to the eigenvalue $\tilde{\lambda}_{m,n} = \sqrt{m^2 + \frac{n^2}{4}}$.

Noting that $\lambda_{m,n} = \tilde{\lambda}_{m,2n}$ and

$$\begin{aligned} P_\varrho e_{m,n}(x_1,x_2) = & \begin{cases} \sin(mx_1)\sin(nx_2), & (x_1,x_2) \in (0,\pi)\times(0,\pi), \\ \sin(mx_1)\sin(n(2\pi - x_2)), & (x_1,x_2) \in (0,\pi)\times(\pi,2\pi), \end{cases} \\ = & \sin(mx_1)\sin(nx_2), \qquad (x_1,x_2) \in (0,\pi)\times(0,2\pi), \end{aligned}$$

we have $P_\varrho e_{m,n} = \tilde{e}_{m,2n}$.

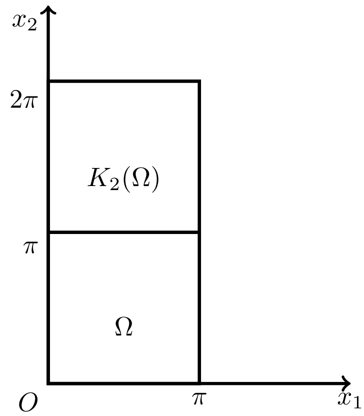

Figure 3. The rectangular domain Ω tiles the rectangular domain $\widetilde{\Omega}$.

Take the coupling matrix $A = \mathrm{diag}(\frac{5}{4}, 2)$ in system (1.4) and the prolongation system (4.1), the observation boundary $\Gamma = \widetilde{\Gamma} = (0, \pi) \times \{0\}$ and the observable matrix $D = (1, -1)^T$. Let

$$\Phi = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\sqrt{13}}{2}t}\tilde{e}_{1,2}(x_1, x_2) \\ 2\mathrm{e}^{\mathrm{i}\frac{\sqrt{13}}{2}t}\tilde{e}_{1,1}(x_1, x_2) \end{pmatrix}.$$

It is easy to check that Φ is a solution of the prolongation system (4.1), and the observation of Φ on $[0, +\infty) \times (0, \pi) \times \{0\}$ is given by

$$\begin{aligned} D^T \partial_\nu \Phi|_{x_2=0} &= D^T \begin{pmatrix} -\mathrm{e}^{\mathrm{i}\frac{\sqrt{13}}{2}t}\dfrac{\partial \tilde{e}_{1,2}}{\partial x_2}|_{x_2=0} \\ -2\mathrm{e}^{\mathrm{i}\frac{\sqrt{13}}{2}t}\dfrac{\partial \tilde{e}_{1,1}}{\partial x_2}|_{x_2=0} \end{pmatrix} \\ &= (1, -1) \begin{pmatrix} -\mathrm{e}^{\mathrm{i}\frac{\sqrt{13}}{2}t}\sin x_1 \\ -\mathrm{e}^{\mathrm{i}\frac{\sqrt{13}}{2}t}\sin x_1 \end{pmatrix} \\ &= 0. \end{aligned}$$

Therefore, for any given $T > 0$, the prolongation system (4.1) is not D-observable on $[0, T] \times \widetilde{\Gamma}$ in the space $(\mathrm{H}_0^1(\widetilde{\Omega}))^N \times (\mathrm{L}^2(\widetilde{\Omega}))^N$.

We next show that system (1.4) is D-observable on $[0, T] \times \Gamma$ in the space $(\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$, provided that $T > 4\pi$.

Define

$$\begin{cases} \beta_{m,n}^{(l)} = \sqrt{\lambda_{m,n}^2 + \delta_l}, & l = 1, 2,\ m \in \mathbb{N},\ n \in \mathbb{N}, \\ \beta_{m,-n}^{(3-l)} = -\beta_{m,n}^{(l)}, & l = 1, 2,\ m \in \mathbb{N},\ n \in \mathbb{N}, \end{cases}$$

where $\delta_1 = \dfrac{5}{4}, \delta_2 = 2$. It is easy to check that

$$\beta_{m,n}^{(l)} = \beta_{m,n'}^{(l')} \iff n = n', l = l',$$

for any given $m \in \mathbb{N}$.

Similar to the proof of Theorem 4.1, for any given initial data $(\Phi(0), \Phi'(0)) \in (\mathrm{H}_0^1(\Omega))^2 \times (\mathrm{L}^2(\Omega))^2$, there exist $\{a_{m,n,l}\} \in \mathrm{L}^2$ such that

$$\begin{pmatrix} \Phi(0) \\ \Phi'(0) \end{pmatrix} = \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{Z}^*} \sum_{l=1}^{2} a_{m,n,l} \begin{pmatrix} \dfrac{1}{\mathrm{i}\beta_{m,n}^{(l)}} e_{m,n}(x)\omega_l \\ e_{m,n}(x)\omega_l \end{pmatrix},$$

where $\omega_1 = (1,0)^T$ and $\omega_2 = (0,1)^T$ are the eigenvectors of A corresponding to the eigenvalues $\delta_1 = \dfrac{5}{4}$ and $\delta_2 = 2$, respectively. Then the solution of prolongation system (4.1) is given by

$$\Phi(t,x) = \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{Z}^*} \sum_{l=1}^{2} \frac{a_{m,n,l}}{\mathrm{i}\beta_{m,n}^{(l)}} e_{m,n}(x)\omega_l \mathrm{e}^{\mathrm{i}\beta_{m,n}^{(l)} t}.$$

Consider the observation of Φ on $[0,T] \times (0,\pi) \times \{0\}$

$$0 \equiv \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{Z}^*} \sum_{l=1}^{2} \frac{n a_{m,n,l}}{\mathrm{i}\beta_{m,n}^{(l)}} \mathrm{e}^{\mathrm{i}\beta_{m,n}^{(l)} t} \sin(mx_1) D^T\omega_l \quad \text{ on } [0,T].$$

By the orthogonality of functions $\{\sin(mx_1)\}_{m\in\mathbb{N}}$ in the space $\mathrm{L}^2(0,\pi)$, we have

$$\sum_{n\in\mathbb{Z}^*} \sum_{l=1}^{2} \frac{n a_{m,n,l}}{\mathrm{i}\beta_{m,n}^{(l)}} \mathrm{e}^{\mathrm{i}\beta_{m,n}^{(l)} t} D^T\omega_l = 0 \quad \text{ on } [0,T]. \tag{4.12}$$

For any given $m \in \mathbb{N}$, taking the positive constants $\gamma_n = \dfrac{1}{2}, \tau_n = 1$ and $D_n^+ = 2$ respectively, the sequence $\{\beta_{m,n}^{(l)}\}_{n\in\mathbb{Z}^*;1\le l\le 2}$ satisfies all the requirements of Lemma 2.7. Appling Lemma 2.7 to each line of (4.12), we have

$$a_{m,n,l} = 0, \quad m \in \mathbb{N}, n \in \mathbb{Z}^*, \ l = 1, 2,$$

provided that $T > 4\pi$. Thus, system (1.4) is D-observable on $[0,T] \times \Gamma$ in the space $(\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$.

By Remark 4.2 and Theorem 4.3, we immediately have

Theorem 4.6. *Let $(\Omega, \{K_h\}_{h=1}^k)$ be a tiling of $\widetilde{\Omega}$. Assume that $\widetilde{\Gamma} \subset \partial\widetilde{\Omega}$ and Γ is defined by (4.6). If the prolongation system (4.1) is D-observable on $[0,T] \times \widetilde{\Gamma}$ in the space $(\mathrm{H}_0^1(\widetilde{\Omega}))^N \times (\mathrm{L}^2(\widetilde{\Omega}))^N$, then system (1.4) is D-observable on $[0,T] \times \Gamma$ in the space $(\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$.*

5. D-OBSERVABILITY ON THE RIGHT ANGLE TRIANGLE DOMAIN

Let $\mathcal{T}$ be a right angle triangle domain with vertices $(0,0)$, $(a,0)$, $(0,b)$. Consider the adjoint system (1.4) with the initial condition (1.5), where $\Omega = \mathcal{T}$ and the initial data $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (\mathrm{H}_0^1(\mathcal{T}))^N \times (\mathrm{L}^2(\mathcal{T}))^N$.

Denote $\Gamma_1 = (0,a) \times \{0\}$ and $\Gamma_2 = \{0\} \times (0,b)$ be the two legs of the right angle triangle domain $\mathcal{T}$, respectively and $\Gamma_3 = \{(x_1,x_2) | x_2 = \frac{b}{a}(a - x_1), 0 < x_1 < a\}$ be the hypotenuse of $\mathcal{T}$.

For $i = 1, 2$, we first consider the D-observability of system (1.4) on $[0,T] \times \Gamma_i$.

The right angle triangle domain $\mathcal{T}$ tiles the rectangular domain $\mathcal{R}$ (see Fig. 4).

By Theorem 1.6, Theorem 2.8 and Theorem 4.6, we have

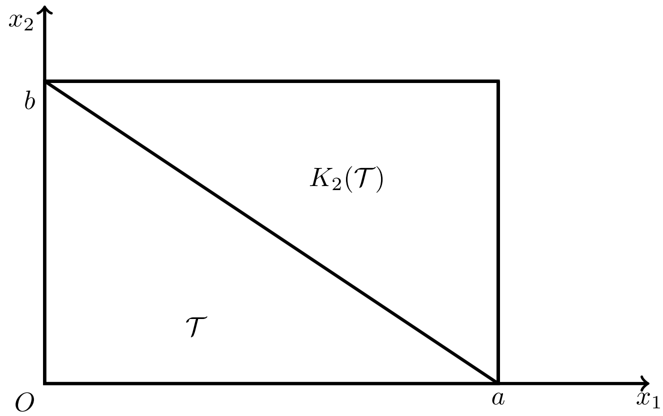

FIGURE 4. The right angle triangle domain $\mathcal{T}$ tiles the rectangular domain $\mathcal{R}$.

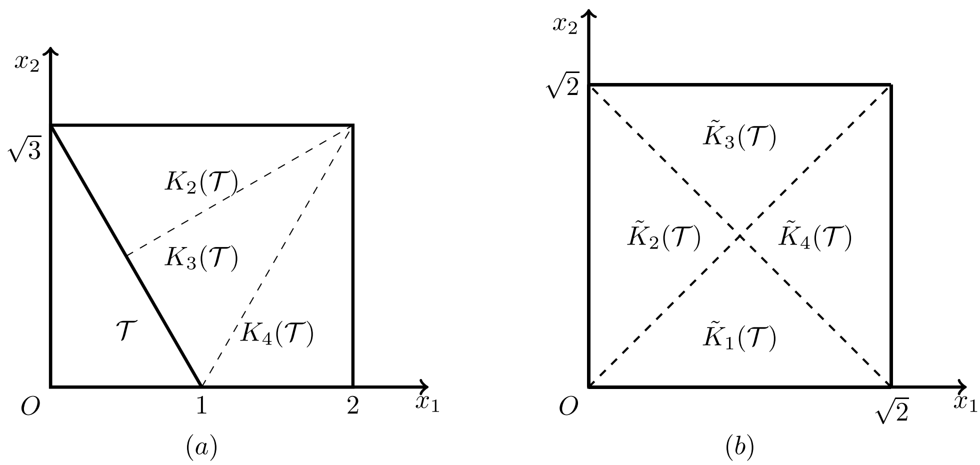

FIGURE 5. The special right angle triangle domain $\mathcal{T}$ tiles the rectangular domain $\mathcal{R}$.

Theorem 5.1. *Let $\mathcal{T}$ be a right angle triangle domain with vertices $(0,0)$, $(a,0)$, $(0,b)$. Assume that A satisfies **(H1)–(H2)**.*

(i) Assume that $T > 2bs$, then system (1.4) is D-observable on $[0,T]\times\Gamma_1$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N \times (\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8) holds.

(ii) Assume that $T > 2as$, then system (1.4) is D-observable on $[0,T]\times\Gamma_2$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N \times (\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8) holds.

We next consider the D-observability of system (1.4) on $[0,T]\times\Gamma_3$.

In general, it is hard to transform the hypotenuse Γ_3 of $\mathcal{T}$ into the partial boundary of $\mathcal{R}$ through rigid transformation. Therefore, we can not establish the D-observability of the adjoint system (1.4) on $[0,T]\times\Gamma_3$. However, for special right angle triangle domain $\mathcal{T}$ with vertices $(0,0)$, $(1,0)$, $(0,\sqrt{3})$ or with vertices $(0,0)$, $(1,0)$, $(0,1)$, the adjoint system (1.4) is D-observable on $[0,T]\times\Gamma_3$.

Let $\mathcal{T}$ be a right angle triangle domain with vertices $(0,0)$, $(0,\sqrt{3})$, $(1,0)$. Then the right angle triangle domain $\mathcal{T}$ tiles $\mathcal{R}=(0,2)\times(0,\sqrt{3})$ (see Fig. 5(a)). By Theorem 1.6, Theorem 2.8 and Theorem 4.6, we have

Theorem 5.2. *Let $\mathcal{T}$ be a right angle triangle domain with vertices $(0,0)$, $(0,\sqrt{3})$, $(1,0)$. Assume that A satisfies the condition **(H1)**. Assume furthermore that there exists a positive constant C such that $\|A-CI\|$ is small enough, or the elements of A are rational numbers.*

(i) Assume that $T>2\sqrt{3}s$, then for $i=1,3$, system (1.4) is D-observable on $[0,T]\times\Gamma_i$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N\times(\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8) holds, where $\Gamma_1=\{(x_1,x_2)|x_2=0,0<x_1<1\}$ is a leg of $\mathcal{T}$ and $\Gamma_3=\{(x_1,x_2)|x_2=\sqrt{3}-\sqrt{3}x_1,x_1\in[0,1]\}$ is the hypotenuse of $\mathcal{T}$.

(ii) Assume that $T>4s$, then system (1.4) is D-observable on $[0,T]\times\Gamma_2$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N\times(\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8) holds, where $\Gamma_2=\{(x_1,x_2)|x_1=0,0<x_2<\sqrt{3}\}$ is another leg of $\mathcal{T}$.

Remark 5.3. Assume that $T>2s$. Assume furthermore that $\mathcal{T}$ is a right angle triangle domain with vertices $(0,0)$, $(0,\sqrt{3})$, $(1,0)$. Under the assumptions of Theorem 5.1, in the tiling given by Figure 4, system (1.4) is D-observable on $[0,T]\times\Gamma_2$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N\times(\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8), where $\Gamma_2=\{(x_1,x_2)|x_1=0,0<x_2<1\}$ is a leg of $\mathcal{T}$. Therefore, the bound of observation time T maybe different in different tilings.

Let $\mathcal{T}$ be a right angle triangle domain with vertices $(0,0)$, $(0,1)$, $(1,0)$. Then the right angle triangle domain $\mathcal{T}$ tiles $\mathcal{R}=(0,\sqrt{2})\times(0,\sqrt{2})$ (see Fig. 5(b)). By Theorem 1.6, Theorem 2.8 and Theorem 4.6, we have

Theorem 5.4. *Let $\mathcal{T}$ be a right angle triangle domain with vertices $(0,0)$, $(0,1)$, $(1,0)$. Assume that A satisfies the condition **(H1)**. Assume furthermore that there exists a positive constant C such that $\|A-CI\|$ is small enough, or the elements of A are rational numbers. Assume that $T>2\sqrt{2}s$, then system (1.4) is D-observable on $[0,T]\times\Gamma_3$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N\times(\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8) holds, where $\Gamma_3=\{(x_1,x_2)|x_2=1-x_1,x_1\in(0,1)\}$ is the hypotenuse of $\mathcal{T}$.*

Remark 5.5. Assume that $\mathcal{T}$ is a right angle triangle domain with vertices $(0,0)$, $(0,1)$, $(1,0)$. Theorem 5.4 is obtained in the tiling given by Figure 5(b). In particular, we can not get the D-observability of system (1.4) on $[0,T]\times\Gamma_3$ in the tiling given by Figure 4, while for $i=1,2$, we can not get the D-observability of system (1.4) on $[0,T]\times\Gamma_i$ in the tiling given by Figure 5(b).

Remark 5.6. For the general right angle triangle domain $\mathcal{T}$, it is hard to transform the hypotenuse Γ_3 of $\mathcal{T}$ into the partial boundary of $\mathcal{R}$ through rigid transformation. As a consequence, it is not clear whether the system (1.4) is D-observable on $[0,T]\times\Gamma_3$ for the general right angle triangle domain. Figure 5 just shows two possibilities that the hypotenuse Γ_3 of $\mathcal{T}$ can transformed into the partial boundary of $\mathcal{R}$.

Choosing different tilings, we will get different estimation of the observation time T. Under the above mentioned tilings, choosing the smallest bound of observation time T, we have.

Corollary 5.7. *Under the assumptions of Theorem 5.2, we have*

(i) Assume that $T>2\sqrt{3}s$, then for $i=1,3$, system (1.4) is D-observable on $[0,T]\times\Gamma_i$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N\times(\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8) holds, where $\Gamma_1=\{(x_1,x_2)|x_2=0,0<x_1<1\}$ is a leg of $\mathcal{T}$ and $\Gamma_3=\{(x_1,x_2)|x_2=\sqrt{3}-\sqrt{3}x_1,x_1\in[0,1]\}$ is the hypotenuse of $\mathcal{T}$.

(ii) Assume that $T>2s$, then system (1.4) is D-observable on $[0,T]\times\Gamma_2$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N\times(\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8) holds, where $\Gamma_2=\{(x_1,x_2)|x_1=0,0<x_2<\sqrt{3}\}$ is another leg of $\mathcal{T}$.

Corollary 5.8. *Under the assumptions of Theorem 5.4, we have*

(i) Assume that $T>2s$, then for $i=1,2$, system (1.4) is D-observable on $[0,T]\times\Gamma_i$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N\times(\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8) holds, where $\Gamma_1=\{(x_1,x_2)|x_2=0,0<x_1<\frac{\sqrt{2}}{2}\}$ and $\Gamma_2=\{(x_1,x_2)|x_1=0,0<x_2<\frac{\sqrt{2}}{2}\}$ are the legs of $\mathcal{T}$.

(ii) Assume that $T>2\sqrt{2}s$, then system (1.4) is D-observable on $[0,T]\times\Gamma_3$ in the space $(\mathrm{H}_0^1(\mathcal{T}))^N\times(\mathrm{L}^2(\mathcal{T}))^N$ if and only if the Kalman's rank condition (1.8) holds, where $\Gamma_3=\{(x_1,x_2)|x_2=1-x_1,x_1\in[0,1]\}$ is the hypotenuse of $\mathcal{T}$.

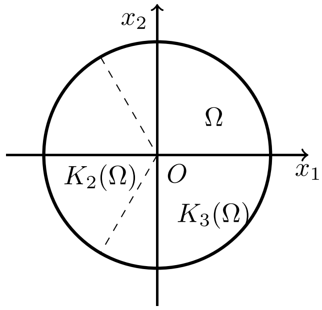

FIGURE 6. The sector domain Ω tiles the circle domain $\widetilde{\Omega}$.

Remark 5.9. The estimation of the observation time T in Corollary 5.7 and Corollary 5.8 is not optimal. We may get a smaller observation time T if using another tiling.

Remark 5.10. For the general triangular domain or even other domains, this method does not make the study of D-observability of system (1.4) any easier. In addition, due to the limitation of the existing results, it is hard to obtain the relevant results to the D-observability of the system (1.4) by tiled method.

6. D-OBSERVABILITY ON THE SECTOR DOMAIN

Let Ω be a sector domain with radius r and central angle $\dfrac{2\pi}{n}$, where $r > 0$ and $n \in \mathbb{N}$. Let $\widetilde{\Omega} = \{x||x| < r\}$. Then Ω tiles $\widetilde{\Omega}$ (see Fig. 6). By the result in [8], we have

Lemma 6.1. *Let $\widetilde{\Omega} = \{x : |x| < r\}$ be a bounded domain in $\mathbb{R}^d$, where $r \in \mathbb{R}^+$ and $d \geq 2$. Assume that A is a diagonalizable matrix with s eigenvalues given by*

$$-\lambda_1^2 < \delta_1 < \delta_2 < \cdots < \delta_s, \tag{6.1}$$

where λ_1^2 is the smallest eigenvalue of $-\Delta$ on $\widetilde{\Omega}$. Assume furthermore that $\|A\|$ is small enough. Then Kalman's rank condition (1.8) is sufficient to the D-observability of system (1.4) on $[0, T] \times \partial\widetilde{\Omega}$ in $(\mathrm{H}_0^1(\widetilde{\Omega}))^N \times (\mathrm{L}^2(\widetilde{\Omega}))^N$, provided that $T > 2rs$.

By Lemma 6.1, Theorem 1.6 and Theorem 4.6, we have

Theorem 6.2. *Let Ω be a sector domain with radius r and central angle $\dfrac{2\pi}{n}$, where $r > 0$ and $n \in \mathbb{N}$. Assume that A is a diagonalizable matrix with s eigenvalues given by (6.1) and $\|A\|$ is small enough. Then system (1.4) is D-observable on $[0, T] \times \Gamma$ in the space $(\mathrm{H}_0^1(\Omega))^N \times (\mathrm{L}^2(\Omega))^N$ if and only if the Kalman's rank condition (1.8) holds, provided that $T > 2rs$, where $\Gamma = \{(x_1, x_2) \in \partial\Omega | x_1^2 + x_2^2 = r^2\}$ is the circular side of Ω.*

REFERENCES

[1] C. Hans and S. Evan, Asymptotic boundary observability for the wave equation on one side of a planar triangle. *Ann. Henri Poincaré* **20** (2019) 2987–3006.

[2] A.C. Lai, Internal observability of the wave equation in tiled domains. *Rend. Mat. Appl.* **40** (2019) 1–16.

[3] T. Li and B. Rao, Criteria of Kalman's type to the approximate controllability and the approximate synchronization for a coupled system of wave equations with Dirichlet boundary controls. *SIAM J. Control Optim.* **54** (2016) 49–72.

[4] T. Li and B. Rao, *Boundary synchronization for hyperbolic systems.* Vol. 94 *Progress in Nonlinear Differential Equations and their Applications.* Subseries in Control. Birkhäuser/Springer (2019).

[5] T. Li and B. Rao, Asymptotic synchronization of a coupled system of wave equations on a rectangular domain with reflecting sides. *Asymptot. Anal.* **128** (2022) 85–102.

[6] M. Práger, Eigenvalues and eigenfunctions of the Laplace operator on an equilateral triangle. *Appl. Math.* **43** (1998) 311–320.

[7] M. Práger, Eigenvalues and eigenfunctions of the Laplace operator on an equilateral triangle for the discrete case. *Appl. Math.* **46** (2001) 231–239.

[8] C. Zu, T. Li and B. Rao, Sufficiency of Kalman's rank condition for the approximate boundary controllability at finite time on spherical domain. *Math. Methods Appl. Sci.* **44** (2021) 13509–13525.

[9] C. Zu, Sufficiency of Kalman's rank condition for the approximate boundary controllability at finite time on an annular domain. *Chin. Ann. Math. Ser. B* **43** (2022) 209–222.