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CONTROLLED TRAVELING PROFILES FOR MODELS OF INVASIVE BIOLOGICAL SPECIES

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Abstract. We consider a family of controlled reaction-diffusion equations, describing the spatial spreading of an invasive biological species. For a given propagation speed $c \in \mathbb{R}$, we seek a control with minimum cost, which achieves a traveling profile with speed c. Since our goal is to slow down or even reverse the contamination, we always assume $c > c^*$, where c^* is the speed of an uncontrolled traveling profile. For various nonlinear models, the existence of an optimal control is proved, together with necessary conditions for optimality. In the last section, we study a case where the wave speed cannot be modified by any control with finite cost. The present analysis is motivated by the recent results in A. Bressan, *et al. Math. Models Methods Appl. Sci.* **32** (2022) 1109–1140. and A. Bressan, *et al. J. Differ. Equ.* **361** (2023) 97–137, showing how a control problem for a reaction-diffusion equation can be approximated by a simpler problem of optimal control of a moving set.

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1. Introduction

In the recent paper [1], an optimal control problem was studied, for traveling wave solutions to a reaction-diffusion equation of the form

$$u_t \;=\; \sigma \Delta u + f(u) - \alpha u. \tag{1.1}$$

Here $t \geq 0$ is time, $x \in \mathbb{R}^n$ is the spatial variable, while $u = u(t, x)$ denotes the density of an invasive biological species. By implementing a control $\alpha = \alpha(t, x) \geq 0$ (say, by spraying pesticides), the population can be partly removed. This will slow down, or even reverse, its spatial propagation. By a rescaling of variables, it is here assumed that $f(0) = f(1) = 0$, so that $u = 0$ and $u = 1$ are equilibrium states.

Traveling waves for (1.1) provide solutions to the 1-dimensional problem

$$u_t \;=\; \sigma u_{xx} + f(u) - \alpha u,$$

with $u = U(x - ct)$ and $\alpha = \alpha(x - ct)$. For a given speed $c \in \mathbb{R}$, a control $\alpha(\cdot)$ is sought, with minimum $\mathbf{L}^1$ norm, which produces a traveling wave with speed c. This leads to the problem

$$\text{minimize:} \quad \|\alpha\|_{\mathbf{L}^1(\mathbb{R})} \tag{1.2}$$

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among all integrable functions $\alpha \geq 0$ such that there exists a solution to the ODE

$$\sigma U'' + c\,U' + f(U) - \alpha U \;=\; 0, \tag{1.3}$$

with asymptotic conditions

$$U(-\infty) \;=\; 0, \qquad\qquad U(+\infty) \;=\; 1. \tag{1.4}$$

In the present paper, our first goal is to study controlled traveling waves for more general equations:

$$u_t \;=\; \sigma\Delta u + f(u,\alpha), \tag{1.5}$$

where the control $\alpha(\cdot)$ enters nonlinearly. In this nonlinear case, one cannot directly compare the cost of two traveling profiles by means of Stokes' theorem, as conveniently done in [1]. Because of this major difference, the existence of traveling profiles having minimum cost requires a different approach, and a more careful analysis.

In the second part of the paper, we focus the attention on two systems of PDEs, describing the interaction between disease-carrying insects and infected trees. A relevant example is provided by *Xylella fastidiosa*, which is a plant pathogenic bacterium that attacks olive trees. It is transmitted by a meadow spittlebug, the *Philaenus spumarius*, a sap-feeding insect. In [2] a detailed model for spatial propagation of a Xylella was introduced. This is described by a system of four equations for the densities of (i) healthy and infected insects, and (ii) healthy and infected trees. Here we consider two simplified models, that will allow a more detailed mathematical analysis.

Model 1. Assume that:

- The insect population spreads by diffusion and reproductive growth.
- By human action, some of the insects can be removed. This slows down, or even reverses, their spatial propagation.
- All insects carry the infection, and contaminate the trees.

Calling

- $u = u(t,x) \in [0,1]$ the density of insects,
- $\theta = \theta(t,x) \in [0,1]$ the fraction of trees that are infected,
- $\alpha = \alpha(t,x) \geq 0$ the control function,

the evolution of these variables can be described by

$$\begin{cases} u_t \;=\; \Delta u + f(u,\alpha), \\ \theta_t \;=\; \kappa_1 u(1-\theta). \end{cases} \tag{1.6}$$

Here the constant κ_1 is an infection rate. The function $f = f(u,\alpha)$, modeling the controlled population growth, can take different forms. For example:

(i) Insect reproduction + removal by pesticides or mosquito nets. As in [1], this leads to

$$f(u,\alpha) \;=\; f(u) - \alpha u. \tag{1.7}$$

(ii) Weed removal, reducing the carrying capacity of the ecosystem. A possible model is

$$f(u,\alpha) \;=\; u(u-u^*)\left[\frac{1}{1+\alpha} - u\right], \tag{1.8}$$

where $u^* \in [0, 1/2]$. Notice that in this case the maximum population supported by the environment shrinks to $(1+\alpha)^{-1} < 1$ as the control α increases. This is another way to reduce the density of insects.

Model 2. We here assume that

- Newly born insects are healthy. Only later in life they can be infected, by the presence of contaminated trees.
- Infected insects contaminate the trees, and contaminated trees infect the new insects.
- By human action, some of the insects can be removed.

In addition to the previous variables, calling

- $I = I(t,x) \in [0,1]$ the fraction of insects which are infected,
- $v = Iu$ the density of infected insects,

we thus consider the system of evolution equations

$$\left\{ \begin{array}{rcl} u_t &=& \Delta u + f(u) - \alpha u, \\ (Iu)_t &=& \Delta(Iu) + \kappa_2(1-I)u\theta - \alpha I u - \mathrm{d}\, Iu, \\ \theta_t &=& \kappa_1 I u (1-\theta). \end{array} \right. \tag{1.9}$$

The constants κ_1, κ_2 are infection rates, while d is a death rate.

For all three models (1.1), (1.6), (1.9), we are interested in (i) the existence of controlled traveling profiles having a given speed c, and (ii) control functions $\alpha(\cdot)$ which produce these traveling profiles and have minimum $\mathbf{L}^1$ norm.

The underlying motivation for studying optimally controlled traveling profiles was provided in [1]. In connection with the parabolic equation (1.5), given an initial density

$$u(0,x) \;=\; \bar{u}(x) \tag{1.10}$$

and a time interval $[0,T]$, a natural objective can be stated as

$$\text{minimize:} \quad \mathcal{J} \;\doteq\; \int_0^T \left(\int_{R^n} \big[u(t,x) + \alpha(t,x) \big] \mathrm{d}x \right) \mathrm{d}t. \tag{1.11}$$

The right hand side of (1.11) accounts for the population size, plus the cost of the control, integrated over time. Assuming that $u = 0$ and $u = 1$ are stable equilibrium states, in many cases the solution to (1.5) can be approximately described in terms of the set $\Omega(t)$ where $u(t,x) \approx 1$. Namely, if the diffusion coefficient $\sigma > 0$ is small, we expect that the difference $\big\| \chi_{\Omega(t)} - u(t,\cdot) \big\|_{\mathbf{L}^1}$ will also be small. The characteristic function $\chi_{\Omega(t)}$ of the set $\Omega(t)$ thus provides a good approximation to the density $u(t,\cdot)$ itself. Based on this observation, in [1] it was proposed to replace the problem (1.11) by an optimization problem for the moving set $\Omega(t)$. More precisely, let $c(t,x)$ be the speed at which the boundary $\partial\Omega(t)$ moves, in the direction of the interior normal, at a point $x \in \partial\Omega(t)$. The new optimization problem then takes the form

$$\text{minimize:} \quad \mathcal{J} \;=\; \int_0^T \left(\mathrm{meas}\big(\Omega(t)\big) + \int_{\partial\Omega(t)} E\big(c(t,x)\big)\, \mathrm{d}x \right) \mathrm{d}t. \tag{1.12}$$

The cost function $E(c)$, which is integrated over the boundary of the set $\Omega(t)$, measures the effort needed to push the boundary inward with speed c. The crucial link between the two problems (1.12) and (1.11) is obtained by defining

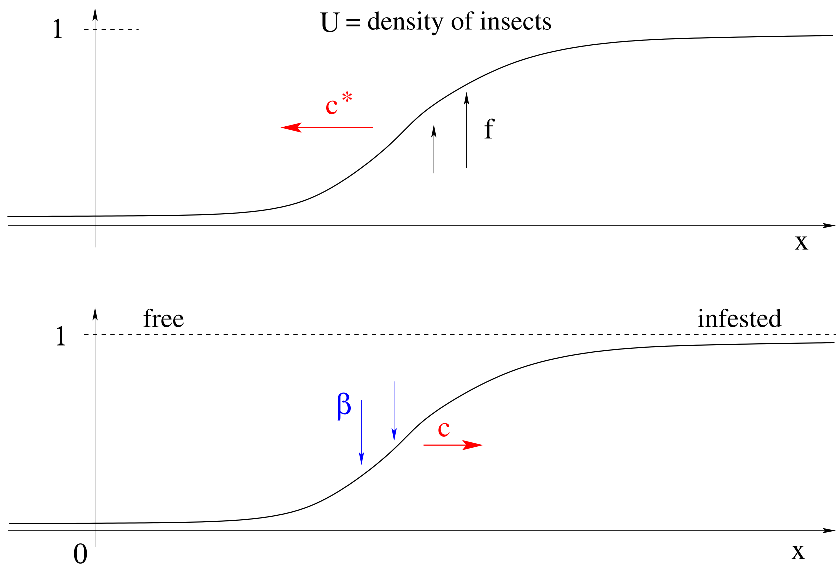

FIGURE 1. Traveling profiles for Model 1. Above: without any control, the insect population spreads toward the left, with a speed $c^* < 0$. Below: applying a control, part of the population is removed. This yields a new traveling wave profile, with speed $c > c^*$.

$E(c)$ = minimum cost of a control $\alpha(\cdot)$ which yields a traveling wave solution to (1.5) with speed c.

As shown in [1], a rigorous justification of this approximation procedure can be achieved *via* a sharp interface limit. The optimization problem (1.12) has been analyzed in greater detail in [3].

We now summarize the main results, proved in the remainder of this paper. In Section 2, we study the scalar equation (1.5). By a rescaling of variables, it is not restrictive to assume $\sigma = 1$. In the absence of control, by the standard theory in [4, 5] it is known that the equation admits a traveling wave solution with a suitable speed $c^* < 0$. Here we prove that, given any speed $c > c^*$, there exists a control function $\alpha(\cdot)$ with finite cost which yields a traveling profile with speed c. More precisely (see Fig. 1), setting

$$u = U(x - ct), \qquad \alpha = \alpha(x - ct),$$

we construct a solution to

$$U'' + cU' + f(U, \alpha) \ = 0, \tag{1.13}$$

with asymptotic conditions (1.4). In Section 3, we prove that a suitable control function $\alpha^*(\cdot)$ can be chosen, having minimum cost. Necessary conditions for optimality are then derived in Section 4. In turn, these can be used in a shooting method, to numerically compute optimal solutions. Plots of an optimal traveling profile, and of the minimum cost $E(c)$ as a function of the speed c, are shown in Figures 7 and 8, respectively.

In Section 5, we study Model 1. Here the main result shows that, for every wave speed $c \in [c^*, 0[$, the system (1.6) admits a controlled traveling wave with speed c. In other words, by removing part of the pest population, the speed at which the contamination advances can be slowed down to almost zero.

The last two sections are concerned with Model 2. Looking for traveling wave solutions of (1.9) of the form

$$u(t, x) \ = \ U(x - ct), \qquad I(t, x) \ = \ I(x - ct), \qquad \theta(t, x) \ = \ \Theta(x - ct),$$

we are led to the system of three ODEs:

$$\left\{\begin{array}{rcl} U'' + cU' + f(U) - \alpha U &=& 0, \\ (IU)'' + c(I'U + IU') + \kappa_2(1-I)U\Theta - \alpha IU - \mathrm{d}\, IU &=& 0, \\ c\Theta' + \kappa_1(1-\Theta)IU &=& 0. \end{array}\right. \tag{1.14}$$

Two scenarios can be considered. In Section 6, we study (1.14) with asymptotic conditions

$$\left\{\begin{array}{rcl} U(-\infty) &=& 0, \\ I(-\infty) &=& 0, \\ \Theta(-\infty) &=& 0, \end{array}\right. \qquad \left\{\begin{array}{rcl} U(+\infty) &=& 1, \\ I(+\infty) &=& I^*, \\ \Theta(+\infty) &=& 1, \end{array}\right. \tag{1.15}$$

where $I^* = \kappa_2/(\kappa_2 + d)$. In other words, the density of insects is vanishingly small as $x \to -\infty$, but large for $x \to +\infty$. All trees are healthy in the limit as $x \to -\infty$, while they are increasingly infected as $x \to +\infty$. In this case, controlling the contamination essentially amounts to slowing down the spreading of the insect population (see Fig. 1). Observing that the density of infected insects trivially satisfies $IU \leq U$, by a comparison argument we prove that, if the control $\alpha = \alpha(x - ct)$ yields a traveling profile with speed $c < 0$ for the first equation in (1.14), then the same control yields a traveling profile for the entire system (1.14), with the same speed.

Finally, in Section 7 we consider again the system (1.14), but with asymptotic conditions

$$\left\{\begin{array}{rcl} U(-\infty) &=& 1, \\ I(-\infty) &=& 0, \\ \Theta(-\infty) &=& 0, \end{array}\right. \qquad \left\{\begin{array}{rcl} U(+\infty) &=& 1, \\ I(+\infty) &=& I^*, \\ \Theta(+\infty) &=& 1. \end{array}\right. \tag{1.16}$$

Notice that here the density of insects is large for $x \to +\infty$ as well as for $x \to -\infty$. Insects and trees are all healthy in the limit as $x \to -\infty$, while they are increasingly infected as $x \to +\infty$.

In the uncontrolled case where $\alpha = 0$, one would have a traveling wave profile where the insect population is everywhere constant: $U(x) = 1$. On the other hand, as shown at the top of Figure 2, the fraction of infected trees and insects keeps increasing. Indeed, the contamination advances toward the left, with speed $c^* < 0$.

An interesting question now arises. Assume that, by applying a control, we locally reduce the population density U. As shown at the bottom of Figure 2, this will create a buffer between a region (to the right) where most of the trees and insects are infected, and a region (to the left) where trees and insects are still largely healthy. Can this strategy effectively reduce the speed at which the contamination advances?

Our analysis shows that the answer is negative. Indeed, the speed of a traveling wave must satisfy a constraint stemming from the linearization of the system (1.14) at the asymptotic state $(U, I, \Theta) = (1, 0, 0)$. We now observe that any control $\alpha(\cdot)$ with finite cost must be integrable, hence vanishingly small as $x \to -\infty$. As a consequence, the presence of this additional control cannot remove the above constraint on the wave speed. A precise statement of the result is given in Theorem 7.1.

Traveling profiles for systems of parabolic equations is a classical subject, with an extensive literature. See for example [4–8, 30] and references therein. Control problems for nonlinear parabolic equations, such as optimal harvesting problems, were studied in [9–12]. In these cases the goal is not to eradicate the population, but to maximize the total harvest, subject to a control cost.

For more accurate models of the spreading and control of invasive populations we refer to [2, 13–16]. A major focus of the investigation in [13, 14] is the possibility of eradicating the population from the entire territory by means of a "regional control", acting only on a small subset of the entire domain.

The pioneering paper [17] was one of the first to study the effect of a human action in order to control an invading population. In recent years there has been a lot of interest in the control of traveling waves for models of

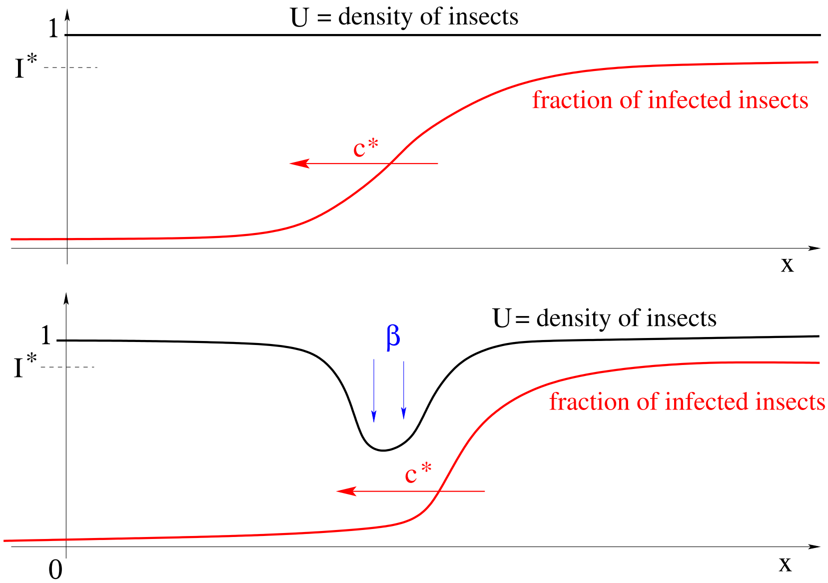

FIGURE 2. Traveling profiles for Model 2. Above: without any control, the insect population reaches everywhere its maximum value $U = 1$, while the fraction of infected insects keeps increasing, propagating to the left with speed $c^* < 0$. Below: applying a control, part of the population is removed, in a neighborhood of the interface between healthy and infected individuals. This yields a different traveling wave profile. However, our analysis shows that the propagation speed cannot be affected.

invasive species. The papers [18–20] analyze strategies based on the release of sterile males in the environment. On the other hand, [21, 22] also consider strategies where pesticides are sprayed, on a moving interval. In this setting, the existence of eradicating strategies, as well as the structure of optimal controls, are established.

Given an effort function $E(c)$, optimization problems for a moving set of the form (1.12) have been recently studied in [3], proving the existence of optimal strategies and establishing necessary conditions for optimality. Control problems for a moving set, describing the support of a population, have also been considered in [23–25, 29].

2. Controlling a traveling front

Given $c \in I\!R$, as in (1.2)–(1.4) we seek a control $\alpha(\cdot)$ with minimum $\mathbf{L}^1$ norm, that produces a traveling wave with speed c. Assuming for simplicity that $\sigma = 1$, and using the notation

$$\beta(u,\alpha) \;=\; f(u,0) - f(u,\alpha), \qquad\qquad f(u) \;=\; f(u,0), \tag{2.1}$$

we can write (1.1) in the form

$$u_t \;=\; \Delta u + f(u) - \beta. \tag{2.2}$$

In addition, for $\beta \geq 0$ we introduce the cost function L defined by

$$L(u,\beta) \;\doteq\; \inf\bigl\{\alpha \geq 0\,;\quad f(u) - f(u,\alpha) \;\geq\; \beta\bigr\}. \tag{2.3}$$

with the understanding that $L(u,\beta) = +\infty$ if there is no control value $\alpha \geq 0$ such that $f(u) - f(u,\alpha) \geq \beta$. Notice that, under the natural assumption that the map $\alpha \mapsto f(u,\alpha)$ is strictly decreasing, we have $L(u,\beta) = \alpha(u)$, where $\alpha(u)$ is implicitly defined by the first identity in (2.1).

Example 2.1. When $f(u,\alpha)$ is the nonlinear function in (1.8), one obtains

$$f(u) \;=\; u(u-u^*)(1-u), \qquad \beta \;=\; \left(1 - \frac{1}{1+\alpha}\right) u(u-u^*). \tag{2.4}$$

Notice that in this case the control $\alpha \geq 0$ will be effective only in the region where $u \in [u^*, 1]$, because for $u < u^*$ this control will not decrease the population growth. As Lagrangian function, according to (2.3) we take

$$L(u,\beta) \;=\; \begin{cases} 0 & \text{if } \ \beta = 0, \\ \dfrac{\beta}{(u-u^*)u - \beta} & \text{if } \ 0 \leq \beta < (u-u^*)u, \\ +\infty & \text{in all other cases.} \end{cases} \tag{2.5}$$

The optimization problem for traveling wave profiles can now be stated as follows.

(OTW) *Given functions $f(u)$ and $L(u,\beta)$, and a speed $c \in I\!R$, find a nondecreasing profile $U : I\!R \mapsto [0,1]$ and a control function $\beta : I\!R \mapsto I\!R_+$ which minimize the cost*

$$J(U,\beta) \;\doteq\; \int_{-\infty}^{+\infty} L\big(U(x), \beta(x)\big)\, \mathrm{d}x, \tag{2.6}$$

subject to

$$U'' + cU' + f(U) - \beta \;=\; 0, \qquad\qquad U(-\infty) = 0, \quad U(+\infty) = 1. \tag{2.7}$$

The optimization problem **(OTW)** will be studied under the following assumptions on the source function f and the cost function L.

(A1) *$f \in \mathcal{C}^2$, and moreover*

$$f(0) \;=\; f(1) \;=\; 0, \qquad f'(0) \;<\; 0, \qquad f'(1) \;<\; 0. \tag{2.8}$$

In addition, f vanishes at only one intermediate point $u^* \in]0,1[$, where $f'(u^*) > 0$.

(A2) *The cost function L is lower semicontinuous. For every $u \in]0,1[$ the map $\beta \mapsto L(u,\beta) \in I\!R_+ \cup \{+\infty\}$ is convex and has superlinear growth. More precisely, there exist constants $C_1 > 0$ and $p > 1$ such that*

$$L(u,0) \;=\; 0, \qquad L(u,\beta) \;\geq\; C_1\, \beta^p \qquad \textit{for all } \beta \geq 0 \textit{ and } u \in [0,1]. \tag{2.9}$$

Remark 2.2. In the main example studied in [1], the dynamics had the form $f(u,\alpha) = F(u) - \alpha$ or $f(u,\alpha) = F(u) - \alpha u$. In both of these cases, the corresponding function $L(u,\beta)$ has only linear growth w.r.t. β. For this reason, the optimal control can sometimes be measure-valued. In the present paper we use the assumption (2.9) to guarantee that the optimal control $\alpha(\cdot)$ is a function in $\mathbf{L}^p(I\!R)$. It is trivial to check that the function L at (2.5) satisfies (2.9). Indeed, it is always unbounded for $\beta \geq 1$.

As a preliminary, we review some basic facts on traveling waves for reaction-diffusion equations of the form

$$u_t \;=\; f(u) + u_{xx}. \tag{2.10}$$

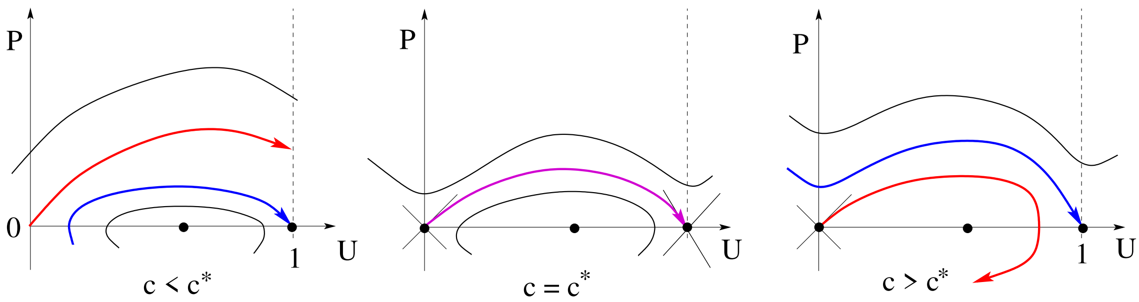

FIGURE 3. A traveling profile for (2.10) corresponds to a heteroclinic orbit for the system (2.13), connecting the points $(0,0)$ and $(1,0)$. Under the assumptions **(A2)**, such an orbit exists for one specific value $c = c^*$.

By definition, a traveling profile for (2.10) with speed c is a solution of the form

$$u(t,x) \;=\; U(x-ct). \tag{2.11}$$

This can be found by solving

$$U'' + cU' + f(U) \;=\; 0. \tag{2.12}$$

Assuming that $f(0) = f(1) = 0$, we seek a solution $U : I\!R \mapsto [0,1]$ of (2.12) with asymptotic conditions (1.4). Setting $P = U'$, we thus need to find a heteroclinic orbit of the system

$$\begin{cases} U' &= P, \\ P' &= -cP - f(U), \end{cases} \tag{2.13}$$

connecting the equilibrium points $(0,0)$ with $(0,1)$. A phase plane analysis of the system (2.13) yields

Theorem 2.3. *Consider the problem (2.12), (1.4), where f satisfies* **(A1)**. *Then, there exist a unique $c^* \in I\!R$ and a unique (up to a translation) traveling profile U with speed c^*.*

For a detailed proof, see Theorem 4.15 in [4]. It can be shown that the traveling profile U is monotone increasing. A phase portrait of the system (2.13) for various values of c is sketched in Figure 3.

For future reference, we observe that at the point $(u^*, 0)$, the Jacobian matrix (2.19) has complex eigenvalues with negative real part as long as

$$|c| \;\leq\; c^{**} \;\doteq\; 2\sqrt{f'(u^*)}. \tag{2.14}$$

However, for $c > c^{**}$, the eigenvalues are both real, with strictly negative real part.

The first result of this section is

Theorem 2.4. *Let f satisfy the assumptions* **(A1)** *and let c^* be as in Theorem 2.3. Then, for every $c > c^*$, there exist a bounded function $\beta :]0,1[\mapsto I\!R_+$ with compact support, such that the equation*

$$U'' + cU' + f(U) - \beta(U) \;=\; 0, \qquad U(-\infty) = 0, \quad U(+\infty) = 1. \tag{2.15}$$

admits a solution.

Proof. **1.** We will construct a solution of the first order system

$$\begin{cases} U' &= P, \\ P' &= -cP - f(U) + \beta(U), \end{cases} \tag{2.16}$$

with asymptotic conditions

$$(U,P)(-\infty) \;=\; (0,0), \qquad\qquad (U,P)(+\infty) \;=\; (1,0), \tag{2.17}$$

for some function $\beta(\cdot)$ of the form

$$\beta(U) \;=\; \begin{cases} \gamma & \text{if } \; u_0 < U < u^*, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Here u^* is the zero of f considered in **(A1)**, while $u_0 \in]0, u^*[$ and $\gamma > 0$ are suitable constants.

2. If $\beta \equiv 0$, computing the Jacobian matrix at a point (U,P) one finds

$$A(U,P) \;=\; \begin{pmatrix} 0 & 1 \\ -f'(U) & -c \end{pmatrix}. \tag{2.19}$$

Solving

$$\lambda^2 + c\lambda + f'(U) \;=\; 0,$$

one obtains

$$\lambda \;=\; \frac{-c \pm \sqrt{c^2 - 4f'(U)}}{2}. \tag{2.20}$$

We observe that the assumptions (2.8) imply that both $(0,0)$ and $(1,0)$ are saddle points. In particular, the ODE

$$\frac{\mathrm{d}}{\mathrm{d}U}P(U) \;=\; -c - \frac{f(U)}{P} \tag{2.21}$$

has a solution $U \mapsto P^\flat(U)$ through $(0,0)$ with slope

$$\frac{\mathrm{d}P^\flat}{\mathrm{d}U}(0) \;=\; \frac{-c + \sqrt{c^2 - 4f'(0)}}{2} \;>\; 0. \tag{2.22}$$

It also has a second solution $P^\sharp$ through the point $(1,0)$, with slope

$$\frac{\mathrm{d}P^\sharp}{\mathrm{d}U}(1) \;=\; \frac{-c - \sqrt{c^2 - 4f'(1)}}{2} \;<\; 0. \tag{2.23}$$

In the special case where $c = c^*$, these solutions exactly match, as in Figure 3, center. On the other hand, when $c > c^*$, as shown in Figure 4 these two solutions satisfy

$$P^\flat(U) \;<\; P^\sharp(U) \qquad\qquad \text{for all } U \in [0, u^*].$$

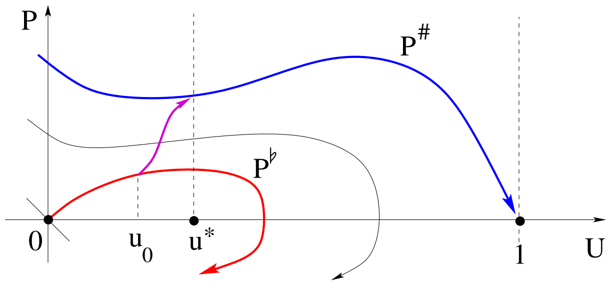

FIGURE 4. Trajectories of (2.16) in the case $c > c^*$, $\beta(U) \equiv 0$. Here $P^\flat$ and $P^\sharp$ are the trajectories through $(0,0)$ and through $(1,0)$, respectively.

Now consider the backward Cauchy problem

$$\frac{d}{dU}P(U) \;=\; -c - \frac{f(U)}{P} + \gamma, \qquad\qquad U \in [0, u^*], \tag{2.24}$$

with terminal data

$$P(u^*) \;=\; P^\sharp(u^*). \tag{2.25}$$

By choosing $\gamma > 0$ suitably large, the solution to (2.24)–(2.25) will satisfy

$$P(U) < P^\flat(U)$$

at some point $0 < U < u^*$. Calling $u_0 \in [0, u^*]$ the point where $P(u_0) = P^\flat(u_0)$, and defining $\beta(\cdot)$ as in (2.18), we achieve the desired conclusion. □

2.1. Existence of a control with finite cost.

According to Theorem 2.4, for every speed $c \geq c^*$ one can find a control $\beta = \beta(U)$ which yields a traveling wave with speed c. However, in some cases such as (2.5), one has

$$\begin{cases} L(U,\beta) \;<\; +\infty & \text{if } \beta < \widehat{\beta}(U), \\ L(U,\beta) \;=\; +\infty & \text{if } \beta \geq \widehat{\beta}(U), \end{cases} \tag{2.26}$$

for some function $\widehat{\beta}$. Therefore, some of the traveling waves considered in the above theorem may have infinite cost. The next theorem identifies some cases where a traveling wave exists, with finite cost,

Theorem 2.5. *Let the functions f, L satisfy the assumptions* **(A1)–(A2)**, *and let $\widehat{\beta}(\cdot)$ be as in (2.26). Consider any function $\widehat{f}$ that satisfies the same assumptions as in* **(A1)**, *together with*

$$f(u) - \widehat{\beta}(u) \;\leq\; \widehat{f}(u) \;\leq\; f(u) \qquad\qquad \textit{for all } u \in [0,1]. \tag{2.27}$$

Call $\widehat{c}$ the speed of a traveling wave for the corresponding equation

$$u_t \;=\; u_{xx} + \widehat{f}(u).$$

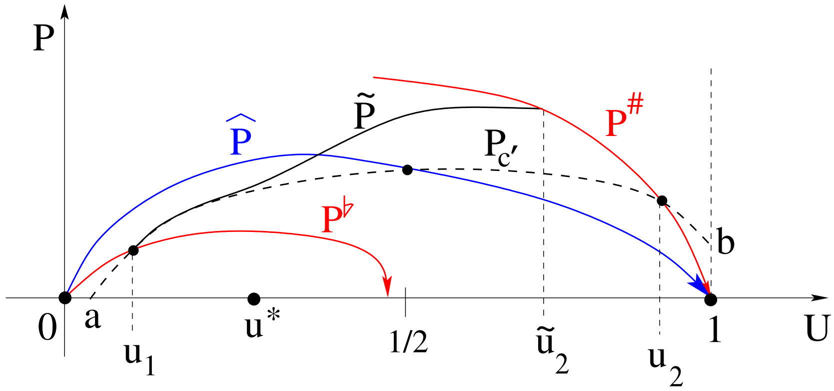

FIGURE 5. The trajectories considered in the proof of Theorem 2.5.

Then, for every speed $c \in [c^*, \widehat{c}[$ there exists a control $\beta = \beta(u)$ with finite cost, such that the equation (2.15) has a solution.

Proof. **1.** We can assume $\widehat{c} > c^*$, since otherwise there is nothing to prove. By assumption, the system

$$\begin{cases} U' &= P, \\ P' &= -\widehat{c}P - \widehat{f}(U) \end{cases} \tag{2.28}$$

has a heteroclinic orbit joining (0,0) with (1,0). With reference to Figure 5, we call $P = \widehat{P}(U)$ the corresponding solution to

$$\frac{\mathrm{d}P}{\mathrm{d}U} = -\widehat{c} - \frac{\widehat{f}(U)}{P}.$$

In addition, we denote by $P = P^\flat(U)$ and $P = P^\sharp(U)$ the solutions to

$$\frac{\mathrm{d}U}{\mathrm{d}U} = -c - \frac{f(U)}{P},$$

with boundary data

$$P^\flat(0) = 0 \qquad \text{and} \qquad P^\sharp(1) = 0,$$

respectively.

2. Next, consider a speed c' with

$$c^* < c < c' < \widehat{c}.$$

Call $P = P_{c'}(U)$ the solution to the system

$$P' = -c' - \frac{\widehat{f}(U)}{P}, \qquad P\left(\frac{1}{2}\right) = \widehat{P}\left(\frac{1}{2}\right). \tag{2.29}$$

Since $c' < \widehat{c}$, a comparison yields

$$\begin{aligned} U < \tfrac{1}{2} \quad &\Longrightarrow \quad P_{c'}(U) < \widehat{P}(U), \\ U > \tfrac{1}{2} \quad &\Longrightarrow \quad P_{c'}(U) > \widehat{P}(U). \end{aligned} \tag{2.30}$$

Moreover, as $c' \to \widehat{c}$ we have the convergence $P_{c'}(U) \to \widehat{P}(U)$ uniformly for U in any compact subinterval $[u_1, u_2] \subset]0,1[$.

We claim that, as U decreases, the solution $P_{c'}$ becomes zero at a point $a > 0$. If not, then $P_{c'}(U) > 0$ for all $U \in]0, 1/2[$, and

$$\lim_{U \to 0+} P_{c'}(U) \;=\; 0.$$

Repeating the analysis at (2.22) of the unstable manifold through the origin, we now obtain

$$\frac{\mathrm{d}P_{c'}}{\mathrm{d}U}(0) \;=\; \frac{-c' + \sqrt{(c')^2 - 4\widehat{f}'(0)}}{2}, \qquad\qquad \frac{\mathrm{d}\widehat{P}}{\mathrm{d}U}(0) \;=\; \frac{-\widehat{c} + \sqrt{\widehat{c}^{\,2} - 4\widehat{f}'(0)}}{2}.$$

This is a contradiction with (2.30), because $c' < \widehat{c}$ and an elementary computation with $\alpha = -4\widehat{f}'(0) > 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}c}\Big[- c + \sqrt{c^2 + \alpha}\Big] \;=\; -1 + \frac{2c}{2\sqrt{c^2+\alpha}} \;<\; 0.$$

Thanks to the convergence $P_{c'}(U) \to \widehat{P}(U)$ as $c' \to \widehat{c}$, uniformly on compact subintervals $[u_1, u_2] \subset]0,1[$, we can choose c' sufficiently close to $\widehat{c}$ so that the point a becomes arbitrarily close to zero. In particular, we can achieve $0 < a < u^*$.

A similar argument shows that $P_{c'}(1) = b > 0$.

3. Still referring to Figure 5, consider the intersection points $0 < u_1 < u_2 < 1$, defined by

$$P^\flat(u_1) \;=\; P_{c'}(u_1), \qquad\qquad P^\sharp(u_2) \;=\; P_{c'}(u_2).$$

Define the control

$$\widetilde{\beta}(U) \;\doteq\; \max\big\{\widehat{\beta}(U) - (c' - c)P_{c'}(U),\; 0\big\}. \tag{2.31}$$

Notice that this implies

$$-c' - \frac{f(U) - \widehat{\beta}(U)}{P_{c'}(U)} \;\leq\; -c - \frac{f(U) - \widetilde{\beta}(U)}{P_{c'}(U)} \qquad \text{for all } U \in [u_1, u_2].$$

Calling $P = \widetilde{P}(U)$ the solution to

$$\frac{\mathrm{d}U}{\mathrm{d}U} \;=\; -c - \frac{f(U) - \widetilde{\beta}(U)}{P}, \qquad\qquad \widetilde{P}(u_1) \;=\; P_{c'}(u_1),$$

a comparison argument yields

$$\widetilde{P}(U) \geq P_{c'}(U) \qquad \text{for all } U > u_1. \tag{2.32}$$

Therefore, the curve $P = \widetilde{P}(U)$ will intersect the trajectory $P^\sharp$ at some point $\widetilde{u}_2 \leq u_2$.

4. We claim that the concatenation of trajectories

$$P(U) \;=\; \begin{cases} P^\flat(U) & \text{if} \quad U\in[0,u_1],\\ \widetilde P(U) & \text{if} \quad U\in[u_1,\widetilde u_2],\\ P^\sharp(U) & \text{if} \quad U\in[\widetilde u_2,1],\end{cases} \tag{2.33}$$

provides a solution to (2.13) with finite cost.

Indeed, for $U\in[0,u_1]\cup[\widetilde u_2,1]$ the above solution corresponds to a control $\beta=0$, with zero cost.

Furthermore, for $U\in[u_1,\widetilde u_2]$, in view of (2.6) the cost is

$$\int_{u_1}^{\widetilde u_2} \frac{L\big(U,\widetilde\beta(U)\big)}{\widetilde P(U)}\,\mathrm{d}U. \tag{2.34}$$

By (2.31) we have

$$\widehat\beta(U)-\widetilde\beta(U)\;>\;\delta\;>\;0 \qquad\qquad \text{for all } U\in[u_1,\widetilde u_2].$$

Hence the numerator $L\big(U,\widetilde\beta(U)\big)$ remains uniformly bounded for $U\in[u_1,\widetilde u_2]$. Finally, the denominator $\widetilde P(U)$ is uniformly positive, because of (2.32). □

3. Existence of optimal traveling profiles

For any given speed $c>c^*$, we seek a control in feedback form $\beta=\beta(u)\geq 0$, with finite cost, that yields a traveling wave with speed c. In terms of the variables U-P considered at (2.16), this yields the problem of finding a control function $\beta=\beta(U)\geq 0$ that minimizes the cost functional

$$J(\beta)\;=\;\int_0^1 \frac{L\big(U,\beta(U)\big)}{P(U)}\,\mathrm{d}U, \tag{3.1}$$

subject to

$$\frac{\mathrm{d}P}{\mathrm{d}U}\;=\;-c+\frac{\beta(U)-f(U)}{P(U)}, \tag{3.2}$$

$$P(0)=P(1)=0, \qquad\qquad P(U)\geq 0 \quad \text{for all } U\in[0,1]. \tag{3.3}$$

In the following, we first consider a relaxed version of this optimization problem, using the variable change

$$Q\;=\;\frac{1}{2}P^2, \qquad\qquad P\;=\;\sqrt{2Q}. \tag{3.4}$$

(OP2) *Find a control function $\beta=\beta(U)\geq 0$ that minimizes the cost functional*

$$J(\beta)\;=\;\int_0^1 \frac{L\big(U,\beta(U)\big)}{\sqrt{2Q(U)}}\,\mathrm{d}U, \tag{3.5}$$

subject to

$$\frac{dQ}{dU} = -c\sqrt{2Q} + \beta - f(U). \tag{3.6}$$

$$Q(0) = Q(1) = 0, \qquad Q(U) > 0 \quad \text{for a.e. } U \in [0,1]. \tag{3.7}$$

Notice that, as long as $P, Q > 0$, the ODEs (3.2) and (3.6) are equivalent. The advantage of using Q as dependent variable is that now we only require the absolute continuity of Q, rather than P.

The following analysis will show that

- For every $c > c^*$, the problem **(OP2)** has an optimal solution.
- When $c^* < c < c^{**}$, the above solution yields an optimal traveling profile for the original problem **(OTW)**.

Theorem 3.1. *Let f, L satisfy the assumptions* **(A1)** *and* **(A2)**. *For any wave speed $c > c^*$, if the equations (3.6)–(3.7) have a solution with finite cost $J(\beta) < +\infty$, then the problem* **(OP2)** *has an optimal solution.*

Proof. **1.** Following the direct method in the Calculus of Variations, we consider a minimizing sequence $(Q_n, \beta_n)_{n\geq 1}$. That is, a sequence of solutions to (3.6)–(3.7) such that

$$\lim_{n\to\infty} \int_0^1 \frac{L\big(U, \beta_n(U)\big)}{\sqrt{2Q_n(U)}}\, dU = m \doteq \inf_{\beta(\cdot)} \int_0^1 \frac{L\big(U, \beta(U)\big)}{\sqrt{2Q(U)}}\, dU. \tag{3.8}$$

Here the infimum m is taken over all solutions (Q, β) of (3.6)–(3.7).

2. We observe that all solutions to (3.6)–(3.7) are uniformly bounded. Indeed,

$$0 \leq \sqrt{2Q(U)} = P(U) \leq P^\sharp(U), \tag{3.9}$$

where $P^\sharp$ denotes the stable manifold of the system (2.13) through the point (1,0). Since $Q_n(U) > 0$ for a.e. U, another comparison argument yields

$$P^\flat(U) \leq P_n(U) = \sqrt{2Q_n(U)}, \tag{3.10}$$

where $P^\flat$ denotes the unstable manifold of the system (2.13) through the point (0,0).

Since (P_n, β_n) is a minimizing sequence, the coercivity assumption (2.9) implies a uniform bound of the form

$$\int_0^1 \beta_n^p(U)\, dU \leq C. \tag{3.11}$$

3. By assumption, the functions $Q_n(\cdot)$ are absolutely continuous and satisfy

$$\frac{d}{dU} Q_n(U) = -c\sqrt{2Q_n(U)} + \beta_n(U) - f(U). \tag{3.12}$$

In view of (3.9) functions $U \mapsto Q_n(U)$ are uniformly bounded. Moreover, by (3.11) all their derivatives have uniformly bounded $\mathbf{L}^p$ norms. By a standard Sobolev embedding theorem [26], they are uniformly Hölder continuous.

Thanks to Ascoli's compactness theorem, by possibly taking a subsequence, we can assume the uniform convergence $Q_n(U) \to Q(U)$, for some limit function $Q(\cdot)$. Taking square roots, this yields the convergence $P_n(U) \to P(U)$, uniformly for $U \in [0,1]$.

By taking a further subsequence, thanks to (3.11) we also obtain the weak convergence $\beta_n \rightharpoonup \beta$ in $\mathbf{L}^p\big([0,1]\big)$.

4. The limit function Q trivially satisfies the constraints $Q(0) = Q(1) = 0$. By assumption, for every $n \geq 1$ and $U \in [0,1]$ we have

$$Q_n(U) \;=\; \int_0^U \Big(-c\sqrt{2Q_n(w)} + \beta_n(w) - f(w)\Big)\, dw.$$

Taking the limit as $n \to \infty$, we conclude that $Q(\cdot)$ satisfies (3.6). On any open interval where $P(U) = \sqrt{Q(U)} > 0$, differentiating the square root we obtain that P satisfies (3.2). In particular, by (3.10) this is true on $]0, u^*[$.

5. It remains to show that $Q(U) > 0$ for a.e. $U \in [u^*, 1]$. Fix any compact interval $[u_1, u_2] \subset]u^*, 1[$. Since P_n is continuous and a.e. positive, we have the decomposition

$$[u_1, u_2] \;=\; \left(\bigcup_k I_k\right) \cup \mathcal{N},$$

where the $I_k =]a_k, b_k[$ are disjoint, maximal open intervals where P_n is strictly positive, while $\mathcal{N}$ is a compact set with zero measure.

Consider the constant

$$\kappa \;\doteq\; \min_{u\in[u_1,u_2]} f(U) \;>\; 0. \tag{3.13}$$

Then choose $\delta > 0$ such that

$$L(U,\beta) \;\geq\; \delta\beta \qquad\qquad \text{for all } \beta \geq \frac{\kappa}{2}. \tag{3.14}$$

At a.e. point $U \in I_k$, the function P_n satisfies

$$\frac{\mathrm{d}P_n}{\mathrm{d}U} \;\leq\; -c - \frac{\kappa}{2P_n(U)} + \frac{L(U,\beta_n)}{\delta\, P_n(U)}. \tag{3.15}$$

Indeed, one can check that (3.15) holds separately in the two cases $\beta \leq \kappa/2$ and $\beta \geq \kappa/2$.

For every compact subinterval $[\alpha, \beta] \subset]a_k, b_k[$, integrating (3.15) we obtain

$$P_n(\beta) - P_n(\alpha) \;\leq\; |c|\,(\beta - \alpha) - \int_\alpha^\beta \frac{\kappa}{2P_n(U)}\,\mathrm{d}U + \int_\alpha^\beta \frac{L(U,\beta_n)}{\delta\, P_n(U)}\,\mathrm{d}U.$$

Letting $\alpha \downarrow a_k$ and $\beta \uparrow b_k$, we conclude

$$\int_{a_k}^{b_k} \frac{\kappa}{2P_n(U)}\,\mathrm{d}U \;\leq\; \int_{a_k}^{b_k} \frac{L(U,\beta_n)}{\delta\, P_n(U)}\,\mathrm{d}U + P_n(a_k) + |c|\,(b_k - a_k). \tag{3.16}$$

Since the sequence (Q_n, β_n) is minimizing, we can assume that $J(\beta_n) < m + 1$ for every $n \geq 1$. Moreover, we observe that $P_n(a_k) = 0$ except for at most one index k, for which $a_k = u_1$. For this particular index, we use the bound

$$P_n(u_1) \;\leq\; P^\sharp(u_1) \;\leq\; \max_{u\in[0,1]} P^\sharp(u).$$

Summing over k, and observing that $\sum_k (b_k - a_k) = u_2 - u_1 < 1$, we obtain

$$\begin{aligned}\int_{u_1}^{u_2} \frac{1}{P_n(U)}\,\mathrm{d}U \quad &\leq\; \frac{2}{\kappa}\sum_k \left\{ \int_{a_k}^{b_k} \frac{L(U,\beta_n)}{\delta\, P_n(U)}\,\mathrm{d}U + |c|\,(b_k-a_k) + P_n(a_k)\right\}\\ &\leq\; \frac{2}{\kappa}\left(\frac{m+1}{\delta} + |c| + \max_{u\in[0,1]} P^\sharp(u)\right) \;\doteq\; \widehat{I}.\end{aligned} \tag{3.17}$$

6. By (3.17) and the uniform convergence $P_n \to P$, it now follows

$$\begin{aligned}\int_{u_1}^{u_2} \frac{1}{P(U)}\,\mathrm{d}U \quad &=\; \sup_{\varepsilon>0} \int_{u_1}^{u_2} \frac{1}{P(U)+\varepsilon}\,\mathrm{d}U\\ &\leq\; \limsup_{n\to\infty} \int_{u_1}^{u_2} \frac{1}{P_n(U)}\,\mathrm{d}U \;\leq\; \widehat{I}.\end{aligned} \tag{3.18}$$

Therefore $P(U) > 0$ for a.e. $U \in [u_1, u_2]$. Since $u^* < u_1 < u_2 < 1$ were arbitrary, we conclude that $P(U) > 0$ for a.e. $U \in [0,1]$. Hence $Q(U)$ is a.e. strictly positive as well.

7. Thanks to the previous step, we can write

$$\int_0^1 \frac{L(U,\beta)}{P(U)}\,\mathrm{d}U \;=\; \sup_{\varepsilon>0} \int_{\{P(U)>\varepsilon\}} \frac{L(U,\beta)}{P(U)}\,\mathrm{d}U. \tag{3.19}$$

On the other hand, by the uniform convergence $P_n \to P$, the weak convergence $\beta_n \rightharpoonup \beta$ and the convexity of L, for every fixed $\varepsilon > 0$ it follows

$$\int_{\{P(U)>\varepsilon\}} \frac{L(U,\beta)}{P(U)}\,\mathrm{d}U \;\leq\; \liminf_{n\to\infty} \int_{\{P(U)>\varepsilon\}} \frac{L(U,\beta_n)}{P_n(U)}\,\mathrm{d}U \;\leq\; m. \tag{3.20}$$

This yields the optimality of the limit solution (P,β). □

Corollary 3.2. *In the same setting as Theorem 3.1, if* **(OP2)** *has a solution* (β, Q) *for which the function* $\frac{1}{P(U)}$ *is integrable on a neighborhood of the point* u^**, then the optimization problem* **(OTW)** *has an optimal solution as well. This is always the case if* $c < c^{**} = 2\sqrt{f'(u^*)}$.

Proof. We observe that every increasing solution of (2.15) yields a solution to (2.16)–(2.17). Conversely, a solution to (2.16)–(2.17) corresponds to a traveling wave (up to a translation) provided that the change of variable

$$x(u) \;=\; \int_{u^*}^{u} \frac{1}{P(U)}\,\mathrm{d}U \tag{3.21}$$

is well defined for every $u \in \,]0,1[\,$.

By assumption, there exists $\delta > 0$ such that the above integral is bounded for all $u \in [u^* - \delta, u^* + \delta]$.

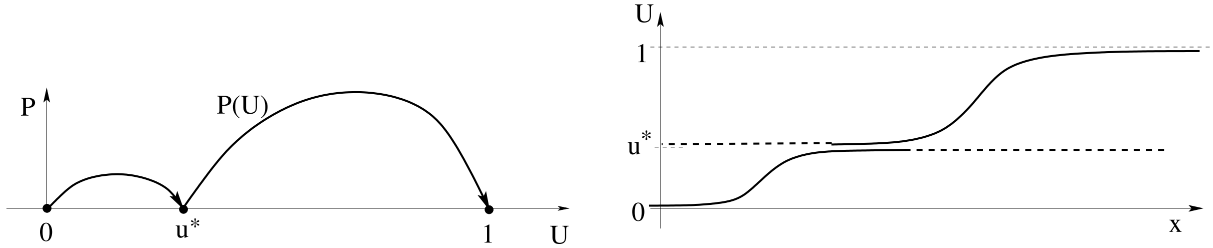

FIGURE 6. Left: an optimal solution to the problem **(OP2)** constructed in Theorem 3.1, in a case where $c >> c^{**}$. As explained in Remark 3.3, this does not lead to an optimal traveling profile for the original problem **(OTP)**.

Since the function $P(U)$ is bounded below by $P^\flat(U)$, the above integral is well defined also for $0 < u < u^* - \delta$. Moreover, using (3.18) with $u_1 = u^* + \delta$, we see that the integral (3.21) is bounded also for $u \in [u^* + \delta, 1[$.

Inverting the function $x(u)$ we thus obtain the profile of a traveling wave $u(x)$, which achieves the minimum cost.

The last assertion is clear, because if $c < c^{**}$ then $P(u^*) \geq P^\flat(u^*) > 0$. By continuity, the function $\frac{1}{P(U)}$ is bounded, hence integrable, in a neighborhood of u^*. □

Remark 3.3. In the case where $c \geq c^{**} = 2\sqrt{f'(u^*)}$, the above corollary does not apply. Even if the optimization problem **(OP2)** has an optimal solution, this solution may not lead to an optimal traveling profile for the original problem **(OTP)**. Indeed, as shown in Figure 6, setting $P(U) = \sqrt{2Q(U)}$ as in (3.4), one obtains a function such that

$$\int_{u^*-\varepsilon}^{u^*+\varepsilon} \frac{1}{P(U)}\, \mathrm{d}U \;=\; +\infty$$

for all $\varepsilon > 0$. This yields two separate traveling wave profiles: one joining the states $U(-\infty) = 0$ and $U(+\infty) = u^*$, and a second one joining the states $U(-\infty) = u^*$ and $U(+\infty) = 1$. These cannot be merged into a single optimal traveling profile.

4. Necessary conditions for optimality

Given a speed $c > c^*$, assume that (U, β) yield an optimally controlled traveling wave profile, as in Theorem 3.1. We seek necessary conditions to determine this profile.

To apply the Pontryagin Maximum Principle [27, 28], we first compute

$$\frac{\partial}{\partial P}\left(\frac{L(U,\beta)}{P}\right) \;=\; -\frac{L(U,\beta)}{P^2}, \qquad\qquad \frac{\partial}{\partial P}\left(-c + \frac{\beta - f(U)}{P}\right) \;=\; -\frac{\beta - f(U)}{P^2}.$$

The PMP now yields the existence of a nontrivial adjoint variable $Y(\cdot)$ satisfying the linear equation

$$\frac{\mathrm{d}Y}{\mathrm{d}U} \;=\; \frac{\beta(U) - f(U)}{P^2(U)}\, Y + \frac{L\big(U, \beta(U)\big)}{P^2(U)}, \tag{4.1}$$

such that, at a.e. $U \in [0,1]$, the following optimality condition holds:

$$\beta(U) \;=\; \operatorname*{argmin}_{\beta\geq 0}\left\{\left(-c+\frac{\beta-f(U)}{P(U)}\right)Y(U)+\frac{L\big(U,\beta\big)}{P(U)}\right\}. \tag{4.2}$$

Equivalently,

$$\beta(U) \;=\; \operatorname*{argmin}_{\beta\geq 0}\Big\{\beta Y(U)+L(U,\beta)\Big\}. \tag{4.3}$$

Assuming that $L(U,\cdot)$ is continuously differentiable in the region where $\beta>0$, by (4.3) it follows

$$Y(U)+L_\beta\big(U,\beta(U)\big) \;=\; 0. \tag{4.4}$$

Example 4.1. For sake of illustration, we consider here the optimization problem for a traveling profile, choosing $f(u)$ and $L(u,\beta)$ as in (2.4)–(2.5), with $u^*=1/3$. If no control is present, a numerical simulation shows that the speed of the traveling profile is $c^*\approx -0.2356$.

For a given speed $c\in\,]c^*,0[\,$, we seek a control $\beta(\cdot)$ with minimum cost, that produces a traveling profile with speed c. By (2.5) it trivially follows that $\beta(u)=0$ for $u\in[0,u^*]$. Calling

$$u_1 \doteq \inf\,\big\{u\in[0,1]\,;\quad \beta(u)>0\big\},$$

by the optimality condition (4.3) and the continuity of Y it follows

$$L_\beta(u_1,0+) \;\doteq\; \lim_{\beta\to 0+}\frac{L(u_1,\beta)}{\beta} \;=\; -Y(u_1).$$

Recalling (2.5), we obtain the boundary conditions

$$\left\{\begin{array}{rcl} Y(u_1) &=& \dfrac{-1}{(u_1-u^*)u_1},\\[2mm] P(u_1) &=& P^\flat(u_1).\end{array}\right. \tag{4.5}$$

In addition, define

$$u_2 \doteq \sup\,\big\{u\in[u^*,1]\,;\quad \beta(u)>0\big\}.$$

We then need to consider two cases.

CASE 1: $u_2<1$. In this case, at u_2 the additional conditions hold:

$$\left\{\begin{array}{rcl} Y(u_2) &=& \dfrac{-1}{(u_2-u^*)u_2},\\[2mm] P(u_2) &=& P^\sharp(u_2).\end{array}\right. \tag{4.6}$$

CASE 2: $u_2=1$. In this case, the solution satisfies $0<P(u)<P^\sharp(u)$ for all $u<1$.

In the following simulations, by a shooting method, we determine $u_1\in\,]u^*,1]$ in (4.5) such that the additional identity (4.6) is satisfied.

In the case where the wave speed is $c=-0.1$, a numerical simulation of the optimal traveling profile is shown in Figure 7. The minimum cost, for increasing values of the wave speed $c\geq c^*$, is shown in Figure 8.

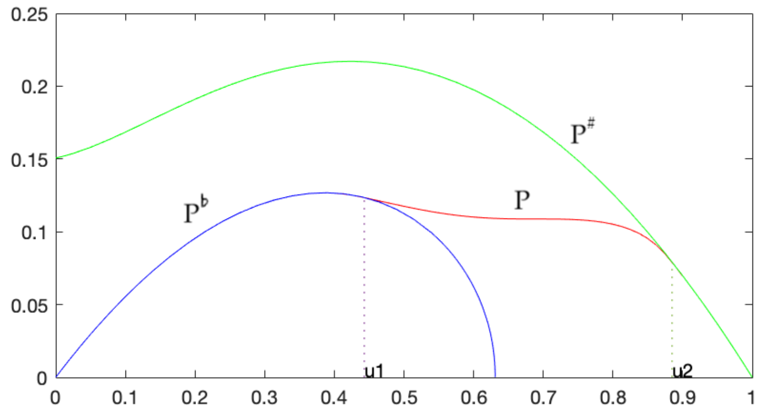

FIGURE 7. The optimal traveling profile in Example 4.1, in the U, P coordinates, computed for the speed $c = -0.1$. Here the control is active for $U \in [u_1, u_2]$.

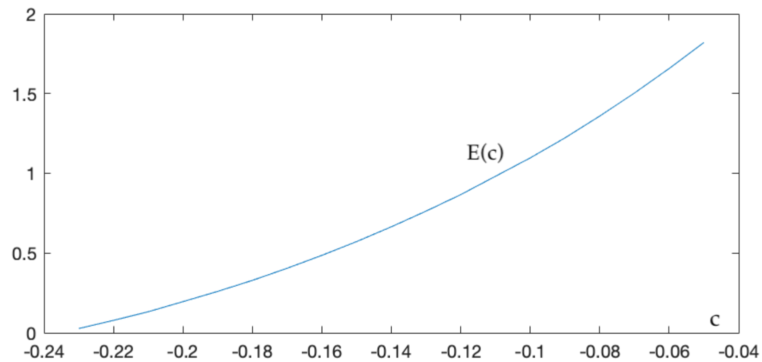

FIGURE 8. The minimum cost $E(c)$ of a control which achieves a traveling profile with speed c, depending on the wave speed $c \in [c^*, 0]$.

5. Controlled traveling profiles for Model 1

In this section, we consider controlled traveling profiles for the system (1.6), say with

$$u(t,x) = U(x-ct), \qquad \theta(t,x) = \Theta(x-ct), \qquad \alpha = \alpha(x-ct).$$

Since (1.6) is in triangular form, for any speed $c > c^*$ the existence of an optimal traveling profile U for the first equation has already been proved in Theorem 3.1. The next result shows that, if $c < 0$, then the second equation in (1.6) also admits a traveling profile with speed c.

We recall that the functions $x \mapsto \big(U(x), \Theta(x)\big)$ should satisfy

$$U'' + cU' + f(U, \alpha(U)) = 0, \qquad U(-\infty) = 0, \quad U(+\infty) = 1, \tag{5.1}$$

$$c\Theta' + \kappa_1 U(1-\Theta) = 0, \qquad \Theta(-\infty) = 0, \qquad \Theta(+\infty) = 1. \tag{5.2}$$

A solution Θ of (5.2) will be constructed assuming the integrability condition

$$\int_{-\infty}^{0} U(x)\,\mathrm{d}x \;<\; +\infty. \tag{5.3}$$

Theorem 5.1. *Let $U : I\!\!R \mapsto [0,1]$ be an increasing solution to (5.1), such that (5.3) holds. Then a solution to (5.2) exists if and only if $c < 0$.*

Proof. To construct the function Θ in (5.2), we begin by solving

$$\frac{-\Theta'}{1-\Theta} \;=\; \frac{\kappa_1}{c} U, \qquad\qquad \Theta(-\infty) = 0.$$

An integration yields

$$\ln\bigl(1-\Theta(x)\bigr) \;=\; \frac{\kappa_1}{c}\int_{-\infty}^{x} U(y)\,dy,$$

$$\Theta(x) \;=\; 1 - \exp\left\{\frac{\kappa_1}{c}\int_{-\infty}^{x} U(y)\,dy\right\}.$$

Since $\kappa_1 > 0$, if $c < 0$ then

$$\lim_{x\to-\infty} \Theta(x) \;=\; 0, \qquad\qquad \lim_{x\to+\infty} \Theta(x) \;=\; 1.$$

On the other hand, if $c > 0$ then

$$\lim_{x\to+\infty} \ln\bigl(1-\Theta(x)\bigr) \;=\; \lim_{x\to+\infty} \frac{\kappa_1}{c}\int_{-\infty}^{x} U(y)\,dy \;=\; +\infty.$$

This contradicts the condition $\Theta(x) \in [0,1]$. Hence, no such traveling profile exists. □

A key assumption of the previous theorem was the boundedness of the integral in (5.3). We now show that this is always satisfied in the setting considered in Theorems 2.4 and 3.1.

Lemma 5.2. *Assume that $f : [0,1] \mapsto I\!\!R$ satisfies the assumptions* **(A1)***. Then for any $c > c^*$ and any solution U of (2.15) with $\beta(U) \geq 0$, the integrability condition (5.3) holds.*

Proof. As before, let $P^\flat(U)$ be the solution to the ODE (2.21) through the origin. By a comparison argument, under the assumptions **(A1)** any traveling wave solution must satisfy

$$P(U) \;\geq\; P^\flat(U) \;\geq\; C\,U \qquad\qquad \text{for all } U \in [0, u^* - \delta],$$

for suitable constants $C, \delta > 0$. Therefore

$$U'(x) \;\geq\; C\,U(x) \qquad \text{whenever} \qquad U(x) \in [0, u^* - \delta]. \tag{5.4}$$

Calling $x^* \in I\!\!R$ the point where $U(x^*) = u^*$, from the differential inequality (5.4) we deduce

$$U(x) \;\leq\; \mathrm{e}^{-C(x^*-x)} u^* \qquad\qquad \text{for all } x \in\,]-\infty, x^*]. \tag{5.5}$$

This implies that, as $x \to -\infty$, the function $U(x)$ converges to zero exponentially fast. Hence the integrability condition (5.3) holds. □

6. Traveling profiles for Model 2

In this section, we begin a study of the system (1.9), assuming that the function f satisfies the assumptions in **(A1)** together with

$$f(u) \;\geq\; -\mathrm{d}u \qquad\qquad u \in [0,1]. \tag{6.1}$$

Introducing the variable $v = Iu$ = density of infected insects, we thus consider the system

$$\begin{cases} u_t &=& u_{xx} + f(u) - \alpha u,\\ v_t &=& v_{xx} + \kappa_2(u-v)\theta - \alpha v - \mathrm{d}\, v,\\ \theta_t &=& \kappa_1(1-\theta)\, v. \end{cases} \tag{6.2}$$

For future use, we recall a basic definition [4, 5].

Definition 6.1. A $\mathcal{C}^1$ function $F : I\!R^m \mapsto I\!R^m$, say $F(w) = \big(F_1(w), \ldots, F_m(w)\big)$ is **quasi-monotone** on a convex domain $\mathcal{D} \subseteq I\!R^m$ if

$$\frac{\partial F_i}{\partial w_j}(w) \;\geq\; 0 \qquad\qquad \text{for all } i \neq j, \quad w = (w_1, \ldots, w_m) \in \mathcal{D}.$$

Motivated by (6.2), we observe that the map $F : I\!R^3 \mapsto I\!R^3$ defined by

$$F(u,v,\theta) \;=\; \Big(f(u) - \alpha(x)u,\;\kappa_2(u-v)\theta - \alpha(x)v - \mathrm{d}\,v,\;\kappa_1(1-\theta)v\Big), \tag{6.3}$$

is quasi-monotone on the domain

$$\mathcal{D} \;\doteq\; \Big\{(u,v,\theta)\,; \qquad 0 \leq v \leq u \leq 1, \quad \theta \in [0,1]\Big\}. \tag{6.4}$$

By a comparison argument we obtain

Lemma 6.2. *Let f satisfy the assumptions* **(A1)** *together with the inequality (6.1). Then the domain $\mathcal{D}$ is positively invariant for the system (6.2). Namely, for any control function $\alpha = \alpha(t,x) \geq 0$, let (u,v,θ) be a solution to (6.2) such that, at time $t = 0$, $(u,v,\theta)(0,x) \in \mathcal{D}$ for all $x \in I\!R$. Then $(u,v,\theta)(t,x) \in \mathcal{D}$ for all $x \in I\!R$ and $t \geq 0$.*

Proof. We first observe that the triples

$$(u^-, v^-, \theta^-)(t,x) \;=\; (0,0,0), \qquad\qquad (u^+, v^+, \theta^+)(t,x) \;=\; (1,1,1),$$

provide a lower and an upper solution to the system (6.2), respectively. This implies that the three functions u, v, θ all take values within the interval $[0,1]$.

Next, let (u,v,θ) be any solution. Then the function $w = u - v$ satisfies

$$w_t \;=\; u_t - v_t \;=\; \Delta w + \big[f(u) + dv\big] - \kappa_2\theta w - \alpha w \;\geq\; \Delta w - \big[\kappa_2\theta + \alpha + \mathrm{d}\big]w. \tag{6.5}$$

Indeed, by (6.1) it follows

$$f(u) + \mathrm{d}\, v \;=\; f(u) + \mathrm{d}u - [\mathrm{d}u - \mathrm{d}v] \;\geq\; -\mathrm{d}(u-v).$$

From (6.5) we conclude that, if $w(0,x) \geq 0$ for all $x \in I\!R$, then also $w(t,x) \geq 0$ for all $t \geq 0, x \in I\!R$. □

In this section, we focus the analysis on

CASE 1: *The density of insects is large for $x \to +\infty$, but vanishingly small as $x \to -\infty$. All trees and insects are healthy in the limit as $x \to -\infty$, while they are increasingly infected as $x \to +\infty$.*

We seek traveling wave solutions of (6.2), having the form

$$u(t,x) \;=\; U(x-ct), \qquad v(t,x) \;=\; V(x-ct), \qquad \theta(t,x) \;=\; \Theta(x-ct), \qquad \alpha = \alpha(x-ct). \tag{6.6}$$

This leads to the system

$$\begin{cases} U'' + cU' + f(U) - \alpha(x)\, U &=\; 0, \\ V'' + cV' + \kappa_2(U-V)\Theta - \mathrm{d}\, V - \alpha(x)\, V &=\; 0, \\ c\,\Theta' + \kappa_1 V(1-\Theta) &=\; 0, \end{cases} \tag{6.7}$$

with asymptotic conditions

$$\begin{cases} U(-\infty) &=\; 0, \\ V(-\infty) &=\; 0, \\ \Theta(-\infty) &=\; 0. \end{cases} \qquad\qquad \begin{cases} U(+\infty) &=\; 1, \\ V(+\infty) &=\; V^*, \\ \Theta(+\infty) &=\; 1. \end{cases} \tag{6.8}$$

Here $V^* = \kappa_2/(\kappa_2 + d)$.

Assuming that the function f satisfies **(A1)**, there exists a unique speed $c^* < 0$ such that the uncontrolled scalar equation

$$u_t \;=\; u_{xx} + f(u)$$

admits a traveling wave solution with speed c^*. Moreover, by the analysis in [1] combined with Corollary 3.2 it follows that, for every $c \in]c^*, 0]$, there exists a non-negative control function $\alpha(\cdot)$ with minimum $\mathbf{L}^1$ norm, such that the first equation in (6.2) admits a traveling profile with speed c.

Relying on quasi-monotonicity, a traveling profile for the entire system (6.2) will be obtained by the method of upper and lower solutions.

Definition 6.3. Let $\mathcal{D}$ be the domain at (6.4). Given an integrable function $\alpha \in \mathbf{L}^1(I\!R)$ and a constant $c < 0$, we say that the triple of functions $(U, V, \Theta) : I\!R \mapsto \mathcal{D}$ is an upper solution (respectively, a lower solution) of the system (6.7) if

(i) The functions U, V, Θ are absolutely continuous on bounded intervals.
(ii) The derivatives U', V' have locally bounded variation.
(iii) The left hand sides of (6.7) are negative (respectively, positive) measures.

Remark 6.4. Assuming that U', V' have locally bounded variation already implies that the second derivatives U'', V'' are measures. Adopting a common technique, we shall construct an upper solution where the derivative V' (6.7) is absolutely continuous separately for $x < \bar{x}$ and for $x > \bar{x}$, and satisfies the downward jump condition

$$V'(\bar{x}-) \;\geq\; V'(\bar{x}+).$$

Starting with a solution to the first equation, constructing an upper solution to the whole system (6.7) is an easy matter.

Lemma 6.5. *Let $u = U(x)$ be a stationary solution for the first equation in (6.7), for some control $\alpha \in \mathbf{L}^1(\mathbb{R})$ and some speed $c < 0$. Define*

$$\overline{\Theta}(x) \;=\; 1 - \exp\left\{\frac{\kappa_1}{c}\int_{-\infty}^{x} U(y)\,dy\right\}.$$

Then the triple of functions

$$(u^+, v^+, \theta^+)(t,x) \;\doteq\; \Big(U(x),\, \min\big\{U(x), V^*\big\},\, \overline{\Theta}(x)\Big) \tag{6.9}$$

provides an upper solution to the system (6.7).

Proof. **1.** As a preliminary, we check that the functions u^+, v^+, θ^+ satisfy the regularity conditions (i)–(ii). Since the control $\alpha(\cdot)$ is assumed to be non-negative and integrable, as observed in step **2** of proof of Theorem 3.1 the first derivative U' is uniformly bounded. Hence the function $u^+(x) = U(x)$ is Lipschitz continuous. In turn, $v^+(x) = \min\{U(x), V^*\}$ is Lipschitz continuous as well. The absolute continuity of θ^+ follows immediately from the definition of $\overline{\Theta}$.

By the first equation in (6.7) it follows that the second derivative U'' is locally integrable. Hence $(u^+)' = U'$ has locally bounded variation. Observing that, for any open interval $\,]a,b[\,$ one has

$$\text{Tot.Var.}\Big\{\frac{\mathrm{d}}{\mathrm{d}x}\min\big\{U(x), V^*\big\}\,;\;\,]a,b[\,\Big\} \;\le\; \text{Tot.Var.}\Big\{\frac{\mathrm{d}}{\mathrm{d}x}U(x)\,;\;\,]a,b[\,\Big\},$$

we conclude that $(v^+)'$ has locally bounded variation as well.

2. The heart of the matter is to show that, by inserting the functions u^+, v^+, θ^+ in (6.7), the left hand sides are all ≤ 0. A direct computation yields

$$c(\theta^+)' + \kappa_1(1-\theta^+)v^+ \;=\; \kappa_1(1-\overline{\Theta})(v^+ - U) \;\le\; 0.$$

Moreover, at points where $v^+(x) = U(x)$, by (6.1), one has

$$(v^+)'' + c(v^+)' + \kappa_2(U - v^+)\theta^+ - \alpha(x)v^+ - dv \;=\; U'' + cU' - \alpha(x)U - \mathrm{d}U \;=\; -f(U) - dU \;\le\; 0.$$

Finally, at points where $U(x) \ge V^*$ and hence $v^+(x) = V^*$ one has

$$\begin{aligned}(v^+)'' + c(v^+)' + \kappa_2(U - v^+)\theta^+ - \alpha(x)v^+ - dv^+ \;&=\; \kappa_2(U - V^*)\overline{\Theta} - \alpha(x)V^* - dV^*\\ &\le\; \kappa_2(1-V^*) - dV^* \;=\; 0,\end{aligned}$$

completing the proof. □

In the remainder of this section, we will show that the same control $\alpha(\cdot)$ yields a traveling profile for the system (6.2), *i.e.* a stationary solution to (6.7) with asymptotic conditions (6.8) as $x \to \pm\infty$. In view of Lemma 6.5, relying on the monotonicity property of the system (6.7), to prove the result it remains to construct a lower solution (u^-, v^-, θ^-), with the same asymptotic conditions (6.8).

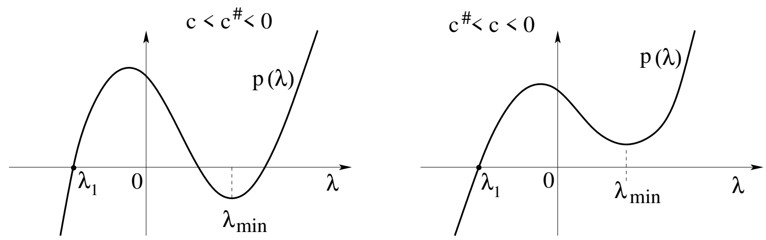

FIGURE 9. Left: the characteristic polynomial (6.13) in the case $c < c^\sharp$. Right: the case $c^\sharp < c < 0$.

Introducing the variable $W = V'$, the last two equations in (6.7) are equivalent to the first order system

$$\begin{cases} V' &= W, \\ W' &= -cW - \kappa_2(U-V)\Theta + \alpha V + \mathrm{d}\,V, \\ \Theta' &= -\dfrac{\kappa_1}{c}\,V(1-\Theta), \end{cases} \tag{6.10}$$

Linearizing (6.10) at the point $(U, V, W, \Theta) = (1, 0, 0, 0)$ we obtain

$$\begin{pmatrix} V' \\ W' \\ \Theta' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ d & -c & -\kappa_2 \\ -\kappa_1/c & 0 & 0 \end{pmatrix} \begin{pmatrix} V \\ W \\ \Theta \end{pmatrix} + G(V,\Theta) + H(V,\Theta,x), \tag{6.11}$$

where

$$G(V,\Theta) = \begin{pmatrix} 0 \\ \kappa_2 V\Theta \\ \kappa_1 V\Theta/c \end{pmatrix}, \qquad H(V,\Theta,x) = \begin{pmatrix} 0 \\ \kappa_2\Theta\big(1-U(x)\big) + V\alpha(x) \\ 0 \end{pmatrix}. \tag{6.12}$$

The eigenvalues of the 3×3 matrix in (6.11) are the roots of the characteristic polynomial

$$p(\lambda) = \det \begin{pmatrix} \lambda & -1 & 0 \\ -d & c+\lambda & \kappa_2 \\ \kappa_1/c & 0 & \lambda \end{pmatrix} = \lambda^3 + c\lambda^2 - \mathrm{d}\lambda - \frac{\kappa_1\kappa_2}{c}. \tag{6.13}$$

Since

$$p(0) = -\frac{\kappa_1\kappa_2}{c} > 0,$$

as shown in Figure 9 the polynomial $p(\lambda)$ will have two positive real roots if and only if $p(\lambda_{\min}) \le 0$, where

$$\lambda_{\min} = \frac{-c + \sqrt{c^2 + 3d}}{3}$$

is the positive zero of $p'(\lambda)$. That is

$$p(\lambda_{\min}) = -\frac{\kappa_1\kappa_2}{c} + \frac{cd}{3} + \frac{2}{27}c^3 - \frac{2}{27}(c^2+3d)^{\frac{3}{2}} \le 0. \tag{6.14}$$

Differentiating the left hand side of (6.14) w.r.t. c, we obtain

$$\frac{\kappa_1\kappa_2}{c^2} + \frac{\mathrm{d}}{3} + \frac{2}{9}c^2 - \frac{2}{9}(c^2+3d)^{\frac{1}{2}}c > 0 \qquad \text{for all} \quad c < 0.$$

Therefore, if $c^\sharp < 0$ is a value for which (6.14) is satisfied as an equality, than any value $c \le c^\sharp$ will satisfy the inequality (6.14).

To explicitly determine the value $c = c^\sharp$ for which the expression in (6.14) vanishes, we move the last term to the right side, square both sides and simplify the equation to get

$$\frac{c^2d^2}{27} + \frac{4d^3}{27} + \frac{4}{27}\kappa_1\,\kappa_2\,c^2 + \frac{2\,\kappa_1\,\kappa_2\,d}{3} - \frac{\kappa_1^2\kappa_2^2}{c^2} = 0.$$

We solve the above equation for c^2, and take the negative square root. This yields

$$c^\sharp \doteq -\left(\frac{-2d^3 - 9\kappa_1\kappa_2 d + 2\big(d^2+3\kappa_1\kappa_2\big)^{3/2}}{d^2+4\kappa_1\kappa_2}\right)^{1/2}. \tag{6.15}$$

From the above analysis it follows

Lemma 6.6. *For $c^\sharp < c < 0$, the 3×3 Jacobian matrix at (6.11) has one negative eigenvalue and two complex conjugate eigenvalues, with positive real part.*

Calling

$$\lambda_1 < 0, \qquad \lambda_2 = a+ib, \qquad \lambda_3 = a-ib, \tag{6.16}$$

the three eigenvalues, with $a, b > 0$, we obtain three corresponding eigenvectors:

$$\mathbf{v}_i = \begin{pmatrix} 1 \\ \lambda_i \\ -\kappa_1/c\lambda_i \end{pmatrix}, \qquad i = 1,2,3. \tag{6.17}$$

Notice that $\mathbf{v}_1$ has real entries, while $\mathbf{v}_2, \mathbf{v}_3$ are complex valued. Taking the real and imaginary parts, we obtain the two vectors

$$\mathbf{w}_2 = \begin{pmatrix} 1 \\ a \\ -\dfrac{\kappa_1 a}{c(a^2+b^2)} \end{pmatrix}, \qquad \mathbf{w}_3 = \begin{pmatrix} 0 \\ b \\ \dfrac{\kappa_1 b}{c(a^2+b^2)} \end{pmatrix}, \tag{6.18}$$

which satisfy

$$\Sigma \doteq \mathrm{span}\{\mathbf{w}_2, \mathbf{w}_3\} = \mathrm{span}\{\mathbf{v}_2, \mathbf{v}_3\}. \tag{6.19}$$

In particular, a direct computation shows that the linear system

$$\begin{pmatrix} V' \\ W' \\ \Theta' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ d & -c & -\kappa_2 \\ -\kappa_1/c & 0 & 0 \end{pmatrix}\begin{pmatrix} V \\ W \\ \Theta \end{pmatrix} \tag{6.20}$$

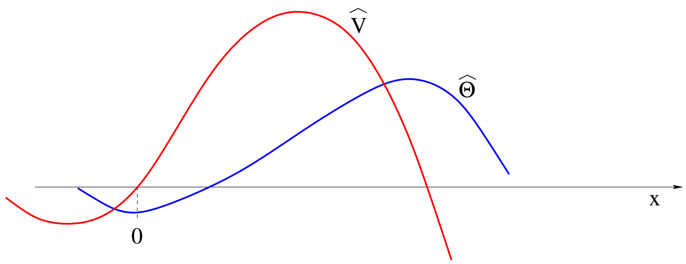

FIGURE 10. A particular solution (6.22) to the linear system (6.20).

admits the solution

$$\begin{pmatrix} V \\ W \\ \Theta \end{pmatrix}(x) \;=\; Ae^{ax}\begin{pmatrix} \sin bx \\ b\cos bx + a\sin bx \\ \frac{\kappa_1}{c(a^2+b^2)}\big(b\cos bx - a\sin bx\big)\end{pmatrix}. \tag{6.21}$$

We can now prove the main result of this section, on the existence of controlled traveling waves for Model 2.

Theorem 6.7. *Let f satisfy the assumptions* **(A1)** *together with (6.1). Let $c^\sharp < c < 0$ and let $U : I\!\!R \mapsto [0,1]$ be an increasing solution to the first equation in (6.7), with asymptotic conditions as in (6.8), for some nonnegative control function $\alpha \in \mathbf{L}^1(I\!\!R)$ with bounded support. Then there exist solutions V, Θ of the remaining two equations in (6.7), with asymptotic conditions (6.8).*

Proof. By Lemma 6.5 we already have an upper solution of (6.7) satisfying the asymptotic conditions (6.8). It remains to construct a lower solution.

1. Let $\varphi_0 \in]0, \pi/2[$ be the angle such that

$$\cos\varphi_0 \;=\; \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\varphi_0 \;=\; \frac{b}{\sqrt{a^2+b^2}}.$$

Then by (6.21) the functions

$$\widehat{V}(x) \;=\; e^{ax}\sin bx, \qquad\qquad \widehat{\Theta}(x) \;=\; -\frac{\kappa_1}{c\sqrt{a^2+b^2}}e^{ax}\sin\big[bx-\varphi_0\big] \tag{6.22}$$

provide one particular solution to the linear system (6.20), as shown in Figure 10.

2. Given $x_0 \in I\!\!R$, for any $\varepsilon > 0$, call $(V_\varepsilon, W_\varepsilon, \Theta_\varepsilon)$ the solution to the system (6.7) with initial data

$$V_\varepsilon(x_0) \;=\; 0, \qquad W_\varepsilon(x_0) \;=\; V_\varepsilon'(x_0) \;=\; \varepsilon\widehat{V}'(0), \qquad \Theta_\varepsilon(x_0) \;=\; \varepsilon\widehat{\Theta}(0), \tag{6.23}$$

in the special case where $\alpha(x) = 0$ and $U(x) = 1$ for all x. By standard ODE theory, as $\varepsilon \to 0$ we have the convergence

$$e^{-1}V_\varepsilon(x+x_0) \;\to\; \widehat{V}(x), \qquad e^{-1}V_\varepsilon'(x+x_0) \;\to\; \widehat{V}'(x), \qquad e^{-1}\Theta_\varepsilon(x+x_0) \;\to\; \widehat{\Theta}(x), \tag{6.24}$$

uniformly for x in bounded intervals.

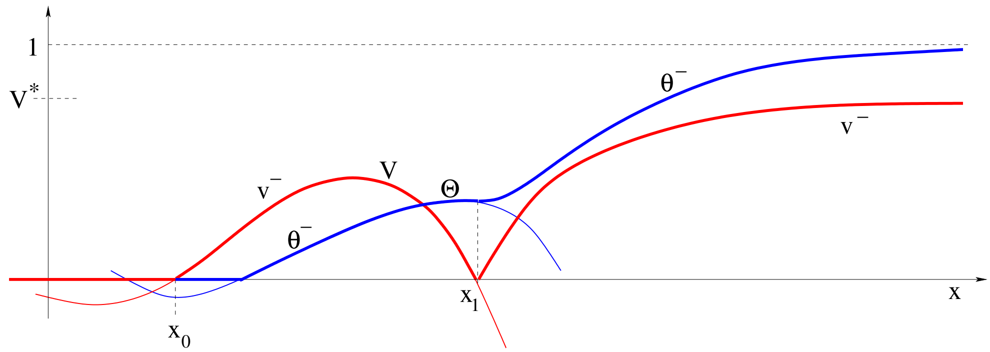

FIGURE 11. The lower solution constructed at (6.27) and at (6.30)–(6.32), separately on the half lines where $x \leq x_1$ and $x \geq x_1$.

Since we are assuming that the control $\alpha(\cdot)$ has bounded support, for any $\epsilon_0 > 0$ we can choose $x_0 > 0$ large enough so that

$$\alpha(x) = 0, \qquad 1-\epsilon_0 \leq U(x) \leq 1, \qquad \text{for all } x \geq x_0. \tag{6.25}$$

By choosing $\varepsilon, \epsilon_0 > 0$ small enough, we obtain an exact solution of (6.7) on an interval $[x_0, x_1]$, with $x_1 - x_0 \leq 4\pi/b$ and α, U as in (6.25), such that

$$\begin{cases} V_\varepsilon(x_0) = 0, \\ V_\varepsilon(x_1) = 0, \end{cases} \qquad \begin{cases} \Theta_\varepsilon(x_0) < 0, \\ \Theta_\varepsilon(x_1) > 0, \end{cases} \qquad \begin{cases} V_\varepsilon(x) > 0 \quad \text{for} \quad x_0 < x < x_1, \\ V'_\varepsilon(x_1) < 0. \end{cases} \tag{6.26}$$

Restricted to the half line $]-\infty, x_1]$, our lower solution (see Fig. 11) is then defined as

$$v^-(x) = \begin{cases} 0 & \text{if } x < x_0, \\ V_\varepsilon(x) & \text{if } x \in [x_0, x_1], \end{cases} \qquad \theta^-(x) = \begin{cases} 0 & \text{if } x < x_0 \\ \max\{\Theta_\varepsilon(x), 0\} & \text{if } x \in [x_0, x_1]. \end{cases} \tag{6.27}$$

Notice that, by choosing $\varepsilon > 0$ small enough, we obtain $\Theta_\varepsilon(x_1) < 1$. This inequality will be used in Step **4.**

3. Next, we extend this lower solution to the remaining half line $[x_1, +\infty[$.
As a first step, we define the constant function

$$\widetilde{\theta}(x) \doteq \Theta_\varepsilon(x_1) > 0,$$

and let $\widetilde{v}$ be the solution to

$$v'' = -cv' - \kappa_2(1-\epsilon_0 - v)\widetilde{\theta} + dv, \tag{6.28}$$

on the domain $x \in [x_1, +\infty[$, with boundary conditions

$$v(x_1) = 0, \qquad v(+\infty) = V^\dagger \doteq \frac{\kappa_2(1-\epsilon_0)\Theta_\varepsilon(x_1)}{\kappa_2\Theta_\varepsilon(x_1) + d}. \tag{6.29}$$

An explicit computation yields

$$\widetilde{v}(x) \;=\; V^\dagger\big(1-e^{\lambda_0(x-x_1)}\big),\qquad\qquad \lambda_0 \;=\; \frac{-c-\sqrt{c^2+4(\kappa_2\Theta_\varepsilon(x_1)+d)}}{2}.$$

Notice that, for $x>x_1$, the couple $(\widetilde{v},\widetilde{\theta})$ provides a lower solution to the last two equations in (6.7). However, this lower solution does not yet satisfy the asymptotic conditions in (6.8). One more step is thus needed.

4. For $x>x_1$ we let θ^- be the solution to

$$\theta' \;=\; \frac{-\kappa_1}{c}\widetilde{v}(x)(1-\theta),\qquad\qquad \theta(x_1)=\Theta_\varepsilon(x_1), \tag{6.30}$$

where $\widetilde{v}$ is the function constructed in the previous step. More explicitly, this means

$$\theta^-(x) \;=\; 1-\big(1-\Theta_\varepsilon(x_1)\big)\exp\left\{\int_{x_1}^x \frac{\kappa_1}{c}\widetilde{v}(z)\,dz\right\}.$$

Observe that, since $c<0$ and $\widetilde{v}(x)\to V^\dagger>0$ as $x\to+\infty$, the above solution θ^- is monotone increasing and satisfies $\theta^-(x)\geq\widetilde{\Theta}_\varepsilon(x_1)$ as $x\in[x_1,+\infty)$ and $\theta^-(x)\to 1$ as $x\to+\infty$.

We then define v^- to be the solution of

$$v'' \;=\; -cv'-\kappa_2(U(x)-v)\theta^-(x)+dv, \tag{6.31}$$

on the domain $x\in[x_1,+\infty[$, with boundary conditions

$$v(x_1)\;=\;0,\qquad\qquad v(+\infty)\;=\;V^*\;\doteq\;\frac{\kappa_2}{\kappa_2+d}. \tag{6.32}$$

Observing that

$$U(x)\;\geq\;1-\epsilon_0,\qquad\qquad \theta^-(x)\;\geq\;\widetilde{\theta}\qquad\qquad \text{for all } x\geq x_1,$$

by a comparison argument we conclude

$$v^-(x)\;\geq\;\widetilde{v}(x)\qquad\qquad \text{for all } x\geq x_1.$$

It is now clear that the couple (v^-,θ^-) provides a lower solution, restricted to the half line $[x_1,+\infty[$. Since $v^-(x)\geq 0$ for all $x\in I\!R$ while $v^-(x_1)=0$, it follows that at the junction point x_1 the left and right derivatives of v^- satisfy

$$(v^-)'(x_1-)\;\leq\;0\;\leq\;(v^-)'(x_1+). \tag{6.33}$$

Hence (v^-,θ^-) is a lower solution defined on the whole real line, which satisfies all the asymptotic conditions in (6.8).

5. Having constructed an upper and a lower solution of (6.7) with

$$v^-(x)\leq v^+(x),\qquad\qquad \theta^-(x)\leq\theta^+(x)\qquad \text{for all } x\in I\!R, \tag{6.34}$$

the existence of an exact solution follows by a standard monotonicity argument. Namely, since the (componentwise) supremum of two lower solutions is also a lower solution, we can define

$$\big(V(x),\Theta(x)\big) \;=\; \sup_{(v^-,\theta^-)\in\mathcal{S}} \big(v^-(x),\theta^-(x)\big),$$

where the supremum is taken over the set $\mathcal{S}$ of all lower solutions which satisfy (6.34). More precisely:

$$v^-(x) \;\leq\; \min\big\{U(x),V^*\big\}, \qquad\qquad \theta^-(x) \;\leq\; \overline{\Theta}(x).$$

By construction, our lower an upper solutions all satisfy the same asymptotic conditions at (6.8). Hence the same holds for the exact solution. □

7. Nonexistence of controlled traveling profiles with slow speed

In this section, we continue the analysis of the system (6.2), focusing on

CASE 2: *The density of insects is large for $x\to+\infty$ as well as for $x\to-\infty$. Insects and trees are all healthy in the limit as $x\to-\infty$, while they are increasingly infected as $x\to+\infty$.*

We consider the possibility of using a control $\alpha(\cdot)$ to reduce the density of insects in the intermediate region between the healthy and contaminated zone. In principle, this should provide a "buffer zone", separating the healthy population from the sick one, thus slowing down the spread of the contamination. Our analysis, however, will show that this strategy is not effective. Namely, it cannot yield any traveling wave profile with slower propagation speed.

To state a precise result in this direction, we first study the asymptotic behavior of a traveling wave as $x\to-\infty$. To fix ideas, let a speed $c<0$ be given. We seek a control $\alpha\in\mathbf{L}^1(I\!\!R)$ and a solution of (6.2) in the form of a traveling wave (6.6). This leads again to the system (6.7). However, the asymptotic conditions (6.8) are now replaced by

$$\left\{\begin{array}{rcl} U(-\infty) &=& 1,\\ V(-\infty) &=& 0,\\ \Theta(-\infty) &=& 0,\end{array}\right. \qquad\qquad \left\{\begin{array}{rcl} U(+\infty) &=& 1,\\ V(+\infty) &=& V^*,\\ \Theta(+\infty) &=& 1.\end{array}\right. \tag{7.1}$$

Theorem 7.1. *Let $c^\sharp$ be the constant in (6.15), and consider any speed c with $c^\sharp<c<0$. Then the system (6.7) does not admit any solution $x\mapsto\big(U(x),V(x),\Theta(x)\big)\in\mathcal{D}$ with asymptotic conditions (7.1), for any control $\alpha\in\mathbf{L}^1(I\!\!R)$.*

Proof. The proof will be achieved by showing that, even by adding a control $\alpha\in\mathbf{L}^1(I\!\!R)$ in the equations (6.10), one cannot achieve solutions such that $V(x),\Theta(x)$ converge to zero as $x\to-\infty$, and satisfy the constraint $V(x),\Theta(x)\in[0,1]$ for all $x\in I\!\!R$. The argument will be given in several steps.

1. Assume that, on the contrary, a traveling wave solution (U,V,Θ), exists, with the prescribed asymptotic behavior as $x\to-\infty$. A contradiction will be obtained by showing that the control $\alpha(\cdot)$ cannot be integrable.

As a preliminary, we observe that the assumption $\alpha\in\mathbf{L}^1(I\!\!R)$ trivially implies that $\beta=\alpha U\in\mathbf{L}^1(I\!\!R)$. In turn, the traveling wave profile $U(\cdot)$, *i.e.* the solution to

$$U''+cU'+f(U)-\alpha(x)U \;=\; 0, \qquad\qquad U(-\infty)\;=\;U(+\infty)\;=\;1, \tag{7.2}$$

satisfies

$$\int_{-\infty}^{0} \big(1 - U(x)\big)\, \mathrm{d}x \;<\; +\infty. \tag{7.3}$$

On the space $I\!R^3$, it will be convenient to use a new system of coordinates $y = (y_1, y_2, y_3)$ corresponding to the basis $\{\mathbf{v}_1, \mathbf{w}_2, \mathbf{w}_3\}$ defined in (6.17), (6.18). Let $x \mapsto Y(x) = \big(Y_1(x), Y_2(x), Y_3(x)\big)$ be the coordinates of the traveling profile (V, W, Θ) w.r.t. this new basis. By construction, the system (6.10) can be written as

$$\begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \end{pmatrix} \;=\; \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \widetilde{G}(Y) + \widetilde{H}(Y, x), \tag{7.4}$$

where, in view of (6.12) and (7.3), the nonlinear perturbations $\widetilde{G}, \widetilde{H}$ satisfy the bounds

$$\big|\widetilde{G}(Y)\big| \;\le\; C_0\, |Y|^2, \qquad\qquad \big|\widetilde{H}(Y, x)\big| \;\le\; C_0\, |Y|\, \beta(x), \tag{7.5}$$

for some constant C_0 and some integrable function $\beta \in \mathbf{L}^1\big(]-\infty, 0]\big)$.

2. By (7.4)–(7.5) there exists a constant C_1 such that

$$\left| \frac{\mathrm{d}}{\mathrm{d}x} \big|Y(x)\big| \right| \;\le\; C_1 \Big(1 + \big|\beta(x)\big|\Big) \big|Y(x)\big|.$$

Since $\beta \in \mathbf{L}^1$, we conclude that the vector $Y(x)$ cannot vanish at any point $-\infty < x \le 0$.

3. Introducing the radius $r(x) = \big|Y(x)\big|$, we now consider the normalized vector ξ, such that

$$\xi(x) \;=\; (\xi_1, \xi_2, \xi_3)(x) \;=\; \frac{Y(x)}{\big|Y(x)\big|}, \qquad\qquad Y(x) \;=\; r(x)\xi(x).$$

By (7.4), denoted the 3×3 matrix in (7.4) as A, this vector ξ satisfies

$$\xi'(x) \;=\; A\xi + \tilde{g}(r, \xi) + \tilde{h}(r, \xi, x) - \Big\langle A\xi + \tilde{g}(r, \xi) + \tilde{h}(r, \xi, x)\ ,\ \xi \Big\rangle \xi, \tag{7.6}$$

where

$$\tilde{g}(r, \xi) \;=\; r^{-1} \widetilde{G}(r\xi), \qquad\qquad \tilde{h}(r, \xi, x) \;=\; r^{-1} \widetilde{H}(r\xi, x). \tag{7.7}$$

Since $r(x) \to 0$ as $x \to -\infty$, by (7.5) and (7.7) we have

$$\big|\tilde{g}(r, \xi)\big| \le C_0 \big|r(x)\big|, \qquad \big|h(r, \xi, x)\big| \;\le\; C_0 \big|\beta(x)\big| \qquad\qquad \lim_{x \to -\infty} \Big|\tilde{g}\big(r(x), \xi\big)\Big| \;=\; 0,$$

uniformly for all $|\xi| = 1$.

4. Based on the previous step, we observe that, as $x \to -\infty$, the evolution of the normalized vector $\xi = (\xi_1, \xi_2, \xi_3)$ satisfies an equation of the form

$$\xi' \;=\; A\xi - \big\langle A\xi,\ \xi \big\rangle \xi + g(x) + h(x), \tag{7.8}$$

where $h \in \mathbf{L}^1$ while $\lim_{x\to-\infty} g(x) = 0$. We claim that, as $x \to -\infty$, two cases are possible

Case 1: $\xi_1(x) \to \pm 1$.
Case 2: $\xi_1(x) \to 0$.

Indeed, by (7.6), the first component of the vector ξ satisfies the ODE

$$\xi_1'(x) \;=\; (\lambda_1 - a)\xi_1(1-\xi_1^2) + g_1(x) + h_1(x), \tag{7.9}$$

where $h_1 \in \mathbf{L}^1$ while $\lim_{x\to-\infty} g_1(x) = 0$.
For any $\delta \in]0, 1/2]$, consider the set

$$\begin{aligned} I_\delta \;=\; \Big\{\bar x \le 0\,;\quad \big|\xi_1(\bar x)\big| \ge \delta,\quad \int_{-\infty}^{\bar x} \big|h_1(y)\big|\,dy < \frac{\delta}{2},& \\ \big|g_1(x)\big| < (a-\lambda_1)\frac{\delta(1-\delta^2)}{2} \qquad \text{for all } x \le \bar x\Big\}.& \end{aligned} \tag{7.10}$$

Assume that one of these sets I_δ is nonempty, say $\bar x \in I_\delta$. We claim that

$$\big|\xi_1(x)\big| \;>\; \frac{\delta}{2} \qquad \text{for all } x \le \bar x. \tag{7.11}$$

Indeed, consider the function

$$\phi(x) \;\doteq\; \big|\xi_1(x)\big| - \int_{-\infty}^{x} \big|h_1(y)\big|\,dy. \tag{7.12}$$

Recalling that $\lambda_1 < 0 < a$, for $x \le \bar x$ by (7.10) we have the implication

$$|\xi_1| \in \left[\frac{\delta}{2}, \delta\right] \quad \Longrightarrow \quad \phi'(x) \;\le\; (\lambda_1 - a)|\xi_1|(1-\xi_1^2) + \big|g_1(x)\big| \;<\; 0.$$

If there exist $x_1 < x_2 \le \bar x$ such that

$$\frac{\delta}{2} \;=\; \big|\xi_1(x_1)\big| \;<\; \big|\xi_1(x_2)\big| \;=\; \delta, \tag{7.13}$$

then

$$\frac{\delta}{2} \;=\; \big|\xi_1(x_2)\big| - \big|\xi_1(x_1)\big| \;=\; \phi(x_2) - \phi(x_1) + \int_{x_1}^{x_2} \big|h_1(y)\big|\,dy \;<\; \int_{-\infty}^{\bar x} \big|h_1(y)\big|\,dy \;<\; \frac{\delta}{2},$$

reaching a contradiction.
Using (7.11), we now show that

$$\lim_{x\to-\infty} \big|\xi_1(x)\big| \;=\; \lim_{x\to-\infty} \phi(x) \;=\; 1. \tag{7.14}$$

Indeed, the first identity is an immediate consequence of (7.12). To prove the second equality, let any $\varepsilon \in]0, \delta/4]$ be given. Choose $x^* < \bar{x}$ such that

$$\int_{-\infty}^{x^*} \big|h_1(y)\big|\,dy < \varepsilon, \qquad \big|g_1(x)\big| < \varepsilon \qquad \text{for all } x \le x^*.$$

Observing that

$$\frac{\delta}{4} \;\le\; \big|\xi_1(x)\big| - \varepsilon \;\le\; \phi(x) \;\le\; \big|\xi_1(x)\big| \qquad \text{for all } x \le x^*,$$

from (7.9) we obtain

$$\phi'(x) \;\le\; (\lambda_1 - a)\frac{\delta}{4}\big(1 - \phi^2(x)\big) + \varepsilon \;<\; 0,$$

where the last inequality holds as long as

$$1 - \phi^2(x) \;\ge\; \frac{4\varepsilon}{(a-\lambda_1)\delta}.$$

We thus conclude

$$\liminf_{x\to-\infty}\big(1 - \phi^2(x)\big) \;\le\; \frac{4\varepsilon}{(a-\lambda_1)\delta}.$$

Since $\varepsilon > 0$ can be arbitrarily small, arguing by contradiction one obtains the second identity in (7.14).

The previous analysis has shown that, if one of the sets I_δ is nonempty, then (7.14) holds, hence Case 1 occurs.

The remaining possibility is that all sets I_δ are empty. In this case, for every $\delta > 0$ we can find $x^* < 0$ such that

$$\int_{-\infty}^{x^*} \big|h_1(y)\big|\,dy < \frac{\delta}{2}, \qquad \big|g_1(x)\big| < (a-\lambda_1)\frac{\delta(1-\delta^2)}{2} \qquad \text{for all } x \le x^*.$$

This implies $\big|\xi_1(\bar{x})\big| < \delta$ for all $\bar{x} \le x^*$, otherwise $\bar{x} \in I_\delta$ against the assumption. We thus conclude that Case 2 holds true.

5. We show that Case 1 leads to a contradiction. Indeed, given $\varepsilon > 0$, by choosing $x_0 << 0$ we achieve

$$|Y(x)| \le \varepsilon, \qquad \int_{-\infty}^{x} \big|\beta(x)\big|\,\mathrm{d}x \;\le\; \varepsilon, \qquad |Y(x)| \;\le\; 2|Y_1(x)|, \qquad \text{for all } x < x_0. \tag{7.15}$$

In this case, for any $x_1 < x < x_0$ we have

$$\big|Y_1(x)\big| \;\le\; \mathrm{e}^{\lambda_1(x-x_1)}\big|Y_1(x_1)\big| + C\int_{x_1}^{x} \mathrm{e}^{\lambda_1(x-y)}\Big(\big|Y_1(y)\big| + \big|\beta(y)\big|\Big)\big|Y_1(y)\big|\,dy.$$

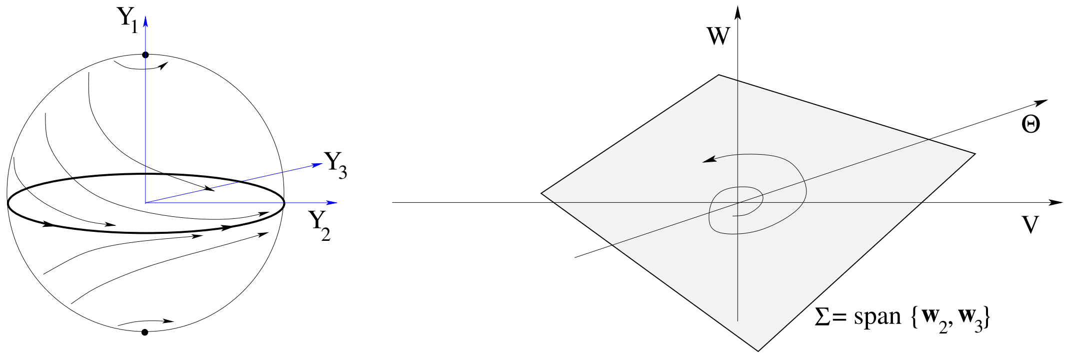

FIGURE 12. Left: the dynamics of the unit vector $\xi = Y/|Y|$, on the surface of the unit ball in $I\!R^3$. Right: on the plane $\Sigma = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, for certain values of the angular component ϑ, the point P with polar coordinates (r, ϑ) lies outside the admissible set where $V \geq 0$ and $\Theta \geq 0$.

Letting $x_1 \to -\infty$ we obtain

$$\begin{aligned} \big|Y_1(x)\big| \;&\leq\; C\int_{-\infty}^{x} \mathrm{e}^{\lambda_1(x-y)}\Big(\big|Y_1(y)\big| + \big|\beta(y)\big|\Big)\big|Y_1(y)\big|\,dy \\ &\leq\; C\varepsilon\int_{-\infty}^{x} \mathrm{e}^{\lambda_1(x-y)}\Big(\varepsilon + \big|\beta(y)\big|\Big)\,dy \\ &\leq\; C\frac{\mathrm{e}^2}{|\lambda_1|}\mathrm{e}^{\lambda_1 x} + C\varepsilon\int_{-\infty}^{x} \lambda_1\mathrm{e}^{\lambda_1(x-y)}\left(\int_y^x |\beta(z)|\,dz\right)dy \\ &\leq\; C\varepsilon\Big(C_1\varepsilon + C_2\int_{-\infty}^{x}|\beta(z)|\,dz\Big) \;\leq\; \frac{\varepsilon}{4}, \end{aligned} \tag{7.16}$$

provided that $\varepsilon > 0$ is chosen sufficiently small. By the third inequality on (7.15) we conclude $\big|Y(x)\big| \leq \varepsilon/2$ for all $x < x_0$.

Iterating this argument, we obtain $\big|Y(x)\big| \leq 2^{-k}\varepsilon$ for every $k \geq 1$, hence $Y(x) = 0$ for all $x \in]-\infty, x_0]$, reaching a contradiction.

6. We now show that Case 2 also leads to a contradiction. By step **3**, the last two components satisfy an ODE of the form

$$\begin{pmatrix}\xi_2' \\ \xi_3'\end{pmatrix} \;=\; \begin{pmatrix}0 & -b \\ b & 0\end{pmatrix}\begin{pmatrix}\xi_2 \\ \xi_3\end{pmatrix} + \phi_1(x) + \phi_2(x), \tag{7.17}$$

with

$$\lim_{x\to-\infty}\phi_1(x) \;=\; 0, \qquad\qquad \int_{-\infty}^{0}\phi_2(x)\,\mathrm{d}x \;<\; +\infty.$$

On the plane Σ at (6.19), it will be convenient to use polar coordinates (r, ϑ). More precisely, by (7.17) the evolution of the angle variable has the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\vartheta(x) \;=\; b + \widetilde{\phi}_1(x) + \widetilde{\phi}_2(x). \tag{7.18}$$

where

$$\widetilde{\phi}_1(x) \to 0, \qquad\qquad \int_{-\infty}^{x_0} \bigl|\widetilde{\phi}_2(x)\bigr|\,\mathrm{d}x \;<\; +\infty.$$

As shown in Figure 12, this implies that the trajectory makes infinitely many loops around the origin, close to the plane Σ. But this is impossible, because in this case, for some values of x, one of the components $V(x)$, $\Theta(x)$ must be negative. This concludes the proof of the theorem. □

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