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EXISTENCE OF BOUNDED SOLUTIONS TO MULTIPLICATIVE POISSON EQUATIONS UNDER MIXING PROPERTY

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Abstract. We study the problem of Multiplicative Poisson Equation (MPE) bounded solution existence in a generic probabilistic discrete-time setup with preimposed mixing. In particular, we consolidate results based on the span-contraction framework and derive an explicit sharp bound that must be imposed on the cost function to guarantee the existence of a bounded MPE solution under mixing. Also, we study properties that the probability kernel must satisfy to ensure the existence of an MPE solution for any generic risk-reward function and characterise process behaviour in the complement of the invariant measure support. Finally, we present numerous examples and stochastic dominance based arguments that help to better understand the problems that arise when the mean is replaced by the entropy in the ergodic setup.

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1. Introduction

The purpose of this paper is to explore the conditions that lead to the existence of a bounded solution to the *Multiplicative Poisson Equation* (MPE) in a generic discrete-time system with preimposed mixing; see [1–4] for the general context. The problem considered in this paper is related to the long-run risk-sensitive averaged per unit of time performance assessment and its ergodic properties; see [5–7]. Namely, we consider an uncontrolled Markov process on a finite, countable, or general locally compact separable metric space with a bounded risk-reward setup. Given a risk-reward function g and a Markov process $(X_i)_{i\in\mathbb{N}}$ with starting point $X_0 = x$, we are interested in conditions which imply ergodic stability and the existence of the time-averaged value

$$\lim_{n\to\infty} \frac{\hat{\mu}_x\left(\sum_{i=0}^{n-1} g(X_i)\right)}{n}, \tag{1.1}$$

where $\hat{\mu}_x(\cdot) = \ln \mathbb{E}_x[\exp(\cdot)]$ is the normalised entropic utility under measure $\mathbb{P}_x$ linked to the starting point x, see Section 2 for the exact setup. The existence of a starting point independent limit in (1.1) is strictly related to a long-run equilibrium encoded in the MPE solution, that is, the existence of a function w and a constant λ,

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for which the equality

$$w(x) = g(x) - \lambda + \hat{\mu}_x(w(X_1)), \tag{1.2}$$

is satisfied for any starting point x; see Section 2.1 for details about the link between (1.1) and (1.2). Although the necessary and sufficient conditions that must be imposed on the probability kernel (or reward function) for the bounded MPE solution to exist are known in the finite setting (see [3]), the corresponding problem for countable or locally compact separable metric spaces remains open. The presence of a (non-linear) entropy operator in (1.1) makes the problem much harder when compared with the *Additive Poisson Equation* setting, where the standard (linear) expectation operator is used instead of $\hat{\mu}_x$, see Chapter 7 in [8] for details.

In a wider context, the objective value considered in (1.1) is linked to the *risk-sensitive criterion* which is a key object in the risk-sensitive stochastic control framework. The existence of a bounded MPE solution combined with the mixing property is often a prerequisite in controlled environments when the corresponding Bellman equations are analysed; see [2, 9–11] and references therein. In particular, solutions to MPEs are crucial in the study of unbounded-horizon risk-sensitive stopping problems, which in turn are the main tool to solve risk-sensitive impulse control problems, see [12, 13]. The usefulness of MPE analysis could be linked to the fact that a bounded MPE solution allows us to change probability measure and obtain a much simpler stopping impulse control problem, see *e.g.* [14]. We also refer to [7, 15–17] and the references therein, where MPE is studied in a wider context and is linked to spectral theory, multiplicative regularity, Donsker–Varadhan variational formulas, large deviations, Lyapunov conditions, and continuous-time SDEs.

Using the span-contraction framework, we analyse conditions that should be imposed on the underlying probability kernel or the reward function for the MPE bounded solution to always exist. In particular, we recall the local contraction property and derive an explicit sharp bound that needs to be imposed on the risk-reward function to guarantee the existence of the bounded solution under mixing, see Theorem 4.1 which is one of the key contributions of this paper. The results presented in this paper could be easily extended to the controlled framework and complement multiple conditions stated in the literature, *e.g.*, stating that for sufficiently small risk aversion the solution exists, or providing non-sharp bound conditions; see [7, 18, 19] and references therein. Also, we study the properties which the probability kernel must satisfy to ensure the existence of a bounded MPE solution for any generic risk-reward function and characterise process behaviour in the complement of the invariant measure support, see Proposition 5.2 and Proposition 6.2 for our key contributions. Our analysis allows us to better understand the classification made in [3] and its potential expansion to generic state spaces and nonnorm-like cost-reward functions, see [10, 20]. We also present numerous examples and develop some novel proof techniques that help to better understand the problems that arise when the linear mean operator is replaced with non-linear entropy in the ergodic setup, see *e.g.* Example 7.6 and Example 7.7.

This paper is organised as follows: In Section 2 we state the general setup and formulate all the assumptions that are the core of our study. Next, in Section 3 we recall the central result on which the span-contraction framework is built, that is, the local contraction property of the MPE operator. We also provide an extensive comment on the subtleties which might result in local contraction not becoming the global contraction. In Section 4 we present one of the main results of this paper linked to a bound that should be imposed on the risk-reward function for the bounded MPE solution to always exist under mixing. Then, in Section 5 we study the problem of the existence of a bounded MPE solution for the generic reward function, characterise how the process should behave in the complement of the invariant set, and show that the MPE solution is, in fact, typically expected to be unbounded. In Section 6 we study how various geometric process properties are linked to the existence of an MPE solution and provide a series of negative results that show many intricacies that are inherent in the risk-sensitive ergodic problem, even in the uncontrolled case. Finally, in Section 7 we provide multiple examples that complement the analysis made throughout the paper.

2. Preliminaries

Let $(E, \mathcal{E})$ be a locally compact separable metric space endowed with metric ρ and Borel σ-field $\mathcal{E}$ that corresponds to the *state space*. In particular, this covers the countable case, where, for simplicity, we often set

$E = \mathbb{N}$ or $E = \{1, 2, \ldots, N\}$, for $N \in \mathbb{N}$. We use $\mathcal{P}(E)$ to denote the space of probability measures on $(E, \mathcal{E})$ and $\mathcal{T}(E)$ to denote the space of probability transition kernels $\mathbb{P}$, such that $\mathbb{P}(x, \cdot) \in \mathcal{P}(E)$, for $x \in E$, and $\mathbb{P}$ satisfy the Feller property, that is, the mapping

$$x \mapsto \mathbb{P}f(x) := \int_E f(y)\mathbb{P}(x, \mathrm{d}y) \tag{2.1}$$

is continuous for every $f \in C(E)$, where $C(E)$ denotes the space of real valued continuous and bounded functions on E. For now, let us fix $\mathbb{P} \in \mathcal{T}(E)$. Then, for a fixed starting point $x \in E$ and $f \in C(E)$, the normalised entropic utility is given by

$$\mu_x(f) := \ln\left(\int_E \mathrm{e}^{f(y)}\mathbb{P}(x, \mathrm{d}y)\right). \tag{2.2}$$

For a fixed *reward function* $g \in C(E)$ we consider the *Multiplicative Poisson Equation* (MPE) given by

$$w(x) = g(x) - \lambda + \mu_x(w), \quad x \in E, \tag{2.3}$$

for $w \in C(E)$ and $\lambda \in \mathbb{R}$. If it exists, we call (λ, w) a solution to (2.3). The associated MPE operator is given by

$$Tf(x) := g(x) + \mu_x(f); \tag{2.4}$$

note that (2.3) could be simply restated as $Tw(x) = w(x) + \lambda$. For simplicity, we consider normalised MPE with positive risk aversion parameter equal to $\gamma = 1$, see [21] for details. That being said, all the results presented in this article can be transferred directly to a generic risk-sensitive MPE with the entropy given by $\mu_x^\gamma(f) = \frac{1}{\gamma}\mu_x(\gamma f)$, for $\gamma \neq 0$, by rescaling the values of g, λ, and w.

For completeness, we recall the basic notation linked to Markov processes that we use in this paper, see [22] for details. Given $\mathbb{P} \in \mathcal{T}(E)$, we use $(X_i)_{i\in\mathbb{N}}$ to denote the corresponding Markov process with the transition kernel $\mathbb{P}$. Typically, we fix $x \in E$, and consider the Markov process with a non-random starting point $X_0 = x$. In this context, we use $\mathbb{P}_x$ to denote the probability measure that constitutes the distribution of the process $(X_i)_{i\in\mathbb{N}}$, assuming $X_0 = x$. Also, for $x \in E$, we use $\mathbb{E}_x$ to denote the corresponding expectation operator, and

$$\hat{\mu}_x(\cdot) := \ln \mathbb{E}_x[\exp(\cdot)] \tag{2.5}$$

to denote the corresponding (normalised) entropy operator, both considered under $\mathbb{P}_x$. In some cases, we also consider non-random starting points and the corresponding conditional versions of the aforementioned objects. We hope that the notation introduced in (2.5) and (2.2) does not confuse the reader. Note that (2.5) could be recovered from (2.2) by considering a canonical Markov process with starting point $X_0 = x$, *i.e.* we get $\mu_x(f) = \hat{\mu}_x(f(X_1))$, for any $f \in C(E)$. In particular, note that (1.2) could be recovered from (2.3).

2.1. Long-run averaged entropy and MPE solution

Let us now show that given a (bounded) MPE solution (2.3) we can immediately recover the long-run average reward given by

$$J(x) := \liminf_{n\to\infty} \frac{\hat{\mu}_x\left(\sum_{i=0}^{n-1} g(X_i)\right)}{n}, \quad x \in E. \tag{2.6}$$

The value (2.6) is the robust time-averaged normalised long-run entropy for $(X_i)_{i\in\mathbb{N}}$ with starting point $X_0 = x \in E$, see [5] and [23] for economic interpretation of $J(\cdot)$ and connections to the risk-sensitive criterion performance measure. The next proposition recalls the fact that if there is an MPE solution (2.3), then λ is equal to (2.6), and the value does not depend on the starting point. For completeness, we provide the proof, as the substitution-based argumentation used in the proof will be utilised throughout the paper.

Proposition 2.1. *Let* $\mathbb{P} \in \mathcal{T}(E)$ *and* $g \in C(E)$ *be such that MPE solution* $(w, \lambda) \in C(E) \times \mathbb{R}$ *exists. Then*

$$\sup_{x\in E} \left| \frac{\hat{\mu}_x\left(\sum_{i=0}^{n-1} g(X_i)\right)}{n} - \lambda \right| \to 0, \quad n \to \infty. \tag{2.7}$$

Proof. From (2.3), using conditional translation invariance and strong time consistency of entropic utility measure (see [24]), for any $x \in E$, we get

$$\begin{aligned} w(x) &= \hat{\mu}_x(g(X_0) - \lambda + w(X_1)) \\ &= \hat{\mu}_x(g(X_0) - \lambda + \hat{\mu}_{X_1}(g(X_1) - \lambda + w(X_2))) \\ &= \hat{\mu}_x(\hat{\mu}_{X_1}(g(X_0) - \lambda + g(X_1) - \lambda + w(X_2))) \\ &= \hat{\mu}_x(g(X_0) - \lambda + g(X_1) - \lambda + w(X_2)); \end{aligned}$$

note that the substitution $w(X_1) = g(X_1) - \lambda + \hat{\mu}_{X_1}(w(X_2))$ in the second equality can be made due to (2.3). Performing similar calculations iteratively, for any $n \in \mathbb{N}$, we get

$$w(x) = \hat{\mu}_x \left(\sum_{i=0}^{n-1} (g(X_i) - \lambda) + w(X_n) \right), \quad x \in E.$$

By translation invariance of the entropic risk, this could be rewritten as

$$\frac{1}{n}\hat{\mu}_x \left(\sum_{i=0}^{n-1} g(X_i) + w(X_n) \right) - \lambda = \frac{1}{n} w(x), \quad x \in E.$$

Consequently, noting that $-\sup_{y\in E} |w(y)| \leq w(\cdot) \leq \sup_{y\in E} |w(y)|$, as well as using monotonicity and translation invariance of the entropic utility, for any $n \in \mathbb{N}$, we get

$$\sup_{x\in E} \left| \frac{1}{n}\hat{\mu}_x \left(\sum_{i=0}^{n-1} g(X_i) \right) - \lambda \right| \leq 2\frac{\sup_{x\in E} |w(x)|}{n}. \tag{2.8}$$

Taking the limit of (2.8) we complete the proof of (2.7). □

2.2. Ergodic conditions

The main goal of this paper is to investigate under what conditions the solution to (2.3) exists. Namely, we presume a mixing condition and assess what additional conditions need to be imposed on $g \in C(E)$ or $\mathbb{P} \in \mathcal{T}(E)$ to guarantee the existence of solution to (2.3). We say that the transition kernel $\mathbb{P} \in \mathcal{T}(E)$ satisfies the *mixing condition* with $\Lambda \in (0, 1)$ if

$$\sup_{x,x'\in E} |\mathbb{P}(x, B) - \mathbb{P}(x', B)| \leq \Lambda, \quad B \in \mathcal{E}. \tag{A.1}$$

For brevity, with slight abuse of notation, we often refer to Λ as the minimal constant for which (A.1) is satisfied. The assumption (A.1) could be relaxed considering multistep dynamics. Namely, we say that the transition kernel satisfies the *multistep mixing condition* with $\Lambda \in (0, 1)$ and $n \in \mathbb{N}$ if

$$\sup_{x,x'\in E} |\mathbb{P}_n(x, B) - \mathbb{P}_n(x', B)| \leq \Lambda, \quad B \in \mathcal{E}, \tag{A.1'}$$

where $\mathbb{P}_n(x,\cdot)$ denotes the n-step iterated transition kernel. In the literature, conditions (A.1) and (A.1') are sometimes replaced by a slightly stronger condition related to *global minorization*. For completeness, we also state these conditions and sometimes provide direct alternative proofs. We say that the transition kernel $\mathbb{P} \in \mathcal{T}(E)$ satisfies the *global minorisation* condition for $d > 0$ if there exists a probability measure $\eta \in \mathcal{P}(E)$, such that

$$\inf_{x\in E} \mathbb{P}(x,B) \geq \mathrm{d}\eta(B), \quad B \in \mathcal{E}. \tag{A.2}$$

Similarly, we say that the transition kernel satisfies the *multistep global minorization* condition for $d > 0$ and $n \in \mathbb{N}$, if there exists a probability measure η, such that

$$\inf_{x\in E} \mathbb{P}_n(x,B) \geq \mathrm{d}\eta(B), \quad B \in \mathcal{E}. \tag{A.2'}$$

For more information on conditions (A.1) and (A.2), we refer to [25]. Essentially, (A.1) refers to the total variation distance while (A.2) states the existence of the minorisation measure, which in turn is almost equivalent to the Doeblin condition; see Theorem 16.2.1 and Theorem 16.2.3 in [26] or Section 7.3 in [8].

For brevity, if not stated otherwise, given the transition kernel $\mathbb{P} \in \mathcal{T}(E)$, we always use ν to denote its invariant measure (assuming it exists), that is, a measure $\nu \in \mathcal{P}(E)$ such that

$$\nu(B) = \textstyle\int_E \mathbb{P}(x,B)\nu(\mathrm{d}x), \quad \text{for } B \in \mathcal{E}.$$

Note that ν exists under any of the ergodic conditions considered in this paper and can be approximated by iterating the underlying transition kernel; see Remark 2.2.

Remark 2.2 (Ergodicity and existence of the unique invariant measure)**.** Given the kernel $\mathbb{P} \in \mathcal{T}(E)$, any mixing condition considered in this paper, that is, (A.1), (A.1'), (A.2) or (A.2'), implies the existence of a unique invariant measure $\nu \in \mathcal{P}(E)$ for the kernel $\mathbb{P}$. We refer to Remark 7.3.13 in [8] and to [27] for a discussion of the generic relationship between the mixing conditions and properties of ν.

For completeness, we present simple relations between the stated conditions in the following proposition.

Proposition 2.3. *Let $\mathbb{P} \in \mathcal{T}(E)$. Then*

1. *If (A.2') holds for $d \in (0,1)$, then (A.1') holds for $\Lambda = 1-d$, for the same $n \in \mathbb{N}$.*
2. *If (A.1) holds for $\Lambda \in (0,1)$, then (A.1') holds for any $n \in \mathbb{N}$ with Λ^n.*
3. *If (A.1') holds and the unique invariant measure ν has an atom, then (A.2') holds for some $n \in \mathbb{N}$.*

Proof. (1.) For brevity we only show the proof for $n = 1$, when (A.1') and (A.2') reduce to (A.1) and (A.2), respectively; generalisation to $n \in \mathbb{N}$ is straightforward. Let us assume that $\mathbb{P} \in \mathcal{T}(E)$ satisfies (A.2) for $d \in (0,1)$ and $\eta \in \mathcal{P}(E)$. Then, for any $B \in \mathcal{E}$, we get

$$\begin{aligned}\sup_{x,x'\in E} |\mathbb{P}(x,B) - \mathbb{P}(x',B)| &= \sup_{x\in E} \mathbb{P}(x,B) - \inf_{x\in E} \mathbb{P}(x,B)\\ &= 1 - \left(\inf_{x\in E} \mathbb{P}(x,B^c) + \inf_{x\in E} \mathbb{P}(x,B)\right)\\ &\leq 1-d.\end{aligned}$$

(2.) Let us assume that $\mathbb{P} \in \mathcal{T}(E)$ satisfies (A.1) for $\Lambda \in (0,1)$. We know that (A.1') is satisfied for $n = 1$. Let us now show that if (A.1') is satisfied for $n \in \mathbb{N}$ and Λ^n then (A.1') is satisfied for $n+1$ and Λ^{n+1}. Fix $x, x' \in E$ and let $H \in \mathcal{E}$ denote the positive set of the Hahn–Jordan decomposition applied to the signed measure

$\nu_{x,x'} := (\mathbb{P}_n(x,\cdot) - \mathbb{P}_n(x',\cdot))$. Then, for any $B \in \mathcal{E}$, we get

$$\begin{aligned}
\mathbb{P}_{n+1}(x,B) - \mathbb{P}_{n+1}(x',B) &= \int_E \mathbb{P}(z,B)\nu_{x,x'}(\mathrm{d}z)\\
&\leq \int_H \sup_{z\in E} \mathbb{P}(z,B)\nu_{x,x'}(\mathrm{d}z) + \int_{H^c} \inf_{z\in E} \mathbb{P}(z,B)\nu_{x,x'}(\mathrm{d}z)\\
&= \left[\sup_{z\in E} \mathbb{P}(z,B) - \inf_{z\in E} \mathbb{P}(z,B)\right] \int_H \nu_{x,x'}(\mathrm{d}z)\\
&\leq \Lambda |\mathbb{P}_n(x,H) - \mathbb{P}_n(x',H)|\\
&\leq \Lambda^{n+1}.
\end{aligned}$$

Similarly, we get $\mathbb{P}_{n+1}(x,B) - \mathbb{P}_{n+1}(x',B) \geq -\Lambda^{n+1}$, which concludes the proof.
(3.) As in (1.), let us restrict ourselves to $n = 1$; generalisation to $n \in \mathbb{N}$ is straightforward. From [8], we know that for any $B \in \mathcal{E}$ we have

$$\sup_{x\in E} |\mathbb{P}_n(x,B) - \nu(B)| \to 0, \quad n \to \infty.$$

Let us assume there exists $y \in E$ such that $\nu(\{y\}) > 0$. Then, for sufficiently large $n \in \mathbb{N}$, we get $\sup_{x\in E} |\mathbb{P}_n(x,\{y\}) - \nu(\{y\}|) < \nu(\{y\})/2$ and consequently

$$\begin{aligned}
\inf_{x\in E} \mathbb{P}_n(x,\{y\}) &= \inf_{x\in E} (\mathbb{P}_n(x,\{y\}) - \nu(\{y\})) + \nu(\{y\})\\
&\geq \nu(\{y\}) - \sup_{x\in E} |\mathbb{P}_n(x,\{y\}) - \nu(\{y\})|\\
&\geq \nu(\{y\})/2,
\end{aligned}$$

which concludes the proof, as we can simply set $\eta(\{y\}) := 1$ and $d := \nu(\{y\})/2$. □

The second condition stated in 3. of Proposition 2.3 holds automatically if E is at most countable, so that in this case (A.1') and (A.2') are effectively equivalent. That being said, note that condition (A.2) is stronger than (A.1), see Example 7.1.

For simplicity, in this paper we state most of the results only for (A.1) having in mind that they are also true for (A.2) and could easily be generalised to a multistep framework considering (A.1') or (A.2').

3. Local contraction property under ergodic assumptions

In this section we state and prove an important result stating that for any $\mathbb{P} \in \mathcal{T}(E)$ satisfying assumption (A.1), the operator T defined in (2.4) is a local contraction in a suitable norm; this is an essential result for the span-contraction approach. For any $f \in C(E)$ we introduce the supremum norm $\|\cdot\|$ and the linked span seminorm $\|\cdot\|_{\mathrm{sp}}$ that are given by

$$\|f\| := \sup_{x\in E} |f(x)| \quad \text{and} \quad \|f\|_{\mathrm{sp}} := \frac{\sup_{x\in E} f(x) - \inf_{x\in E} f(x)}{2}.$$

Note that for any $f \in C(E)$ these two norms are linked by the relation

$$\inf_{c\in\mathbb{R}} \|f + c\| = \|f\|_{\mathrm{sp}}, \tag{3.1}$$

i.e. the span norm could be seen as the supremum norm for centered function f; see Proposition 2 in [19] for details. In particular, since the function w in (2.3) is defined up to an additive constant, it is often more convenient to use the span seminorm rather than the supremum norm. Furthermore, for any signed measure $\tilde{\mu} := \tilde{\mu}_1 - \tilde{\mu}_2$, where $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{P}(E)$, we define the total variation norm of $\tilde{\mu}$ by

$$\|\tilde{\mu}\|_{\mathrm{var}} := \frac{1}{2}\int_E |\tilde{\mu}|(\mathrm{d}y) = \sup_{B\in\mathcal{E}} |\tilde{\mu}_1(B) - \tilde{\mu}_2(B)|,$$

where $|\tilde{\mu}| := \mathbb{1}_A\tilde{\mu} - \mathbb{1}_{A^c}\tilde{\mu}$, and $A \in \mathcal{E}$ is the positive set for $\tilde{\mu}$ obtained by the Hahn–Jordan decomposition; see [28] for details. Also, we recall that for any bounded and measurable function $\varphi\colon E \to \mathbb{R}$ and $\tilde{\nu} \in \mathcal{P}(E)$, the corresponding entropic utility admits the dual (robust) representation, that is, we get

$$\ln\left(\int_E \mathrm{e}^{\varphi(x)}\tilde{\nu}(\mathrm{d}x)\right) = \sup_{\tilde{\mu}\in\mathcal{P}(E)}\left[\int_E \varphi(x)\tilde{\mu}(\mathrm{d}x) - \mathbb{H}[\tilde{\mu}\|\tilde{\nu}]\right], \tag{3.2}$$

where

$$\mathbb{H}[\tilde{\nu}\,\|\,\tilde{\mu}] := \begin{cases} \int_E \ln\left(\frac{\mathrm{d}\tilde{\nu}}{d\tilde{\mu}}(x)\right)\tilde{\nu}(\mathrm{d}x) & \text{if } \tilde{\nu} \ll \tilde{\mu}, \\ +\infty & \text{otherwise,}\end{cases} \tag{3.3}$$

is the relative entropy of $\tilde{\mu}$ with respect to $\tilde{\nu}$, see [29, 30] for details. We are now ready to present a theorem, which is a central tool of the risk-sensitive span-contraction framework. Although the statement and proof of the following theorem could be deduced from [25], for completeness, we decided to present a complete proof. The proof is adapted to the noncontrolled setting and modified in order to better expose the relationship stated in (3.1) and its link to the entropic utility, see also [19].

Theorem 3.1. *Assume $\mathbb{P} \in \mathcal{T}(E)$ satisfies (A.1) and let $g \in C(E)$. Then, the operator T defined in (2.4) is a local contraction in the span norm, i.e. there exists $\alpha\colon \mathbb{R}_+ \to (0,1)$ such that*

$$\|Tf_1 - Tf_2\|_{sp} \le \alpha(M)\,\|f_1 - f_2\|_{sp},$$

for any $f_1, f_2 \in \mathbb{C}(E)$ satisfying $\|f_1\|_{sp} \le M$ and $\|f_2\|_{sp} \le M$.

Proof. For any $x \in E$ and $f \in C(E)$ the *Esscher transform* measure $\tilde{\mu}_{(x,f)} \in \mathcal{P}(E)$ is defined as

$$\tilde{\mu}_{(x,f)}(B) := \frac{\int_B \mathrm{e}^{f(z)}\mathbb{P}(x,\mathrm{d}z)}{\int_E \mathrm{e}^{f(z)}\mathbb{P}(x,\mathrm{d}z)}, \quad B \in \mathcal{E}. \tag{3.4}$$

Fix $f_1, f_2 \in C(E)$ and $x_1, x_2 \in E$. Then, from the dual representation of the entropic risk measure (3.2) we get

$$\begin{aligned} Tf_1(x_1) &\ge g(x_1) + \int_E f_1(y)\tilde{\mu}_{(x_1,f_2)}(\mathrm{d}y) - \mathbb{H}\left[\tilde{\mu}_{(x_1,f_2)}\,\|\,\mathbb{P}(x_1,\cdot)\right], \\ Tf_2(x_2) &\ge g(x_2) + \int_E f_2(y)\tilde{\mu}_{(x_2,f_1)}(\mathrm{d}y) - \mathbb{H}\left[\tilde{\mu}_{(x_2,f_1)}\,\|\,\mathbb{P}(x_2,\cdot)\right]. \end{aligned} \tag{3.5}$$

Recalling (2.4) and taking into account that $\tilde{\mu}_{(x_1,f_2)}$ and $\tilde{\mu}_{(x_2,f_1)}$ are entropy maximising measures for (x_1, f_2) and (x_2, f_1), *i.e.* measures for which supremum in (3.2) is attained (see [29]), we get

$$\begin{aligned} Tf_1(x_2) &\le g(x_2) + \int_E f_1(y)\tilde{\mu}_{(x_2,f_1)}(\mathrm{d}y) - \mathbb{H}\left[\tilde{\mu}_{(x_2,f_1)}\,\|\,\mathbb{P}(x_2,\cdot)\right], \\ Tf_2(x_1) &\le g(x_1) + \int_E f_2(y)\tilde{\mu}_{(x_1,f_2)}(\mathrm{d}y) - \mathbb{H}\left[\tilde{\mu}_{(x_1,f_2)}\,\|\,\mathbb{P}(x_2,\cdot)\right]. \end{aligned} \tag{3.6}$$

Thus, combining (3.5) with (3.6), and setting $\tilde{\mu} := \tilde{\mu}_{(x_2,f_1)} - \tilde{\mu}_{(x_1,f_2)}$, we get

$$Tf_1(x_1) - Tf_2(x_1) - \big(Tf_1(x_2) - Tf_2(x_2)\big) \geq \int_E \big(f_1(y) - f_2(y)\big)\tilde{\mu}(\mathrm{d}y). \tag{3.7}$$

Let Γ denote the (positive) Hahn–Jordan decomposition set for the signed measure $\tilde{\mu}$. Then,

$$\begin{aligned}\int_E \big(f_1(y) - f_2(y)\big)\tilde{\mu}(\mathrm{d}y) &\leq \sup_{y\in E}\big(f_1(y) - f_2(y)\big)\tilde{\mu}(\Gamma) + \inf_{y\in E}\big(f_1(y) - f_2(y)\big)\tilde{\mu}(\Gamma^c)\\ &\leq \sup_{y\in E}\big(f_1(y) - f_2(y)\big)\tilde{\mu}(\Gamma) - \inf_{y\in E}\big(f_1(y) - f_2(y)\big)\tilde{\mu}(\Gamma)\\ &= 2\|f_1 - f_2\|_{\mathrm{sp}}\tilde{\mu}(\Gamma) = 2\|f_1 - f_2\|_{\mathrm{sp}}\|\tilde{\mu}\|_{\mathrm{var}}.\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8) and taking supremum over $x_1 \in E$ and $x_2 \in E$ we get

$$\|Tf_1 - Tf_2\|_{\mathrm{sp}} \leq \|f_1 - f_2\|_{\mathrm{sp}} \sup_{x,x'\in E} \|\tilde{\mu}_{(x,f_1)} - \tilde{\mu}_{(x',f_2)}\|_{\mathrm{var}}, \tag{3.9}$$

Now, we are going to show that for any $M > 0$ we get

$$\alpha(M) := \sup_{\substack{f_1\in C(E):\\ \|f_1\|_{\mathrm{sp}}\leq M}} \sup_{\substack{f_2\in C(E):\\ \|f_2\|_{\mathrm{sp}}\leq M}} \left(\sup_{x,x'\in E} \|\tilde{\mu}_{(x,f_1)} - \tilde{\mu}_{(x',f_2)}\|_{\mathrm{var}}\right) < 1. \tag{3.10}$$

Assuming that (3.10) is not true, there exists a sequence of objects $(f_{1n}, f_{2n}, B_n, x_n, x'_n)$, $n \in \mathbb{N}$, where $f_{1n}, f_{2n} \in C(E)$, $\|f_{1n}\|_{\mathrm{sp}} \leq M$, $\|f_{2n}\|_{\mathrm{sp}} \leq M$, $B_n \in \mathcal{E}$, and $x_n, x'_n \in E$ are such that

$$\left(\tilde{\mu}_{(x_n,f_{1n})} - \tilde{\mu}_{(x'_n,f_{2n})}\right)(B_n) \to 1 \quad \text{as} \quad n \to \infty.$$

In particular, this implies $\tilde{\mu}_{(x_n,f_{1n})}(B_n^c) \to 0$ and $\tilde{\mu}_{(x'_n,f_{2n})}(B_n) \to 0$, as $n \to \infty$. Now, from definition (3.4) it follows that

$$\mathrm{e}^{-\|f\|_{\mathrm{sp}}}\mathbb{P}(x, B) \leq \tilde{\mu}_{(x,f)}(B),$$

which in turn implies $\mathbb{P}(x_n, B_n^c) \to 0$ and $\mathbb{P}(x'_n, B_n) \to 0$, as $n \to \infty$. Consequently, we get

$$\lim_{n\to\infty} [\mathbb{P}(x_n, B_n) - \mathbb{P}(x'_n, B_n)] = 1,$$

which directly contradicts Assumption (A.1). □

Remark 3.2 (Local contraction for bounded measurable functions)**.** Note that, in general, we also have a local contraction property in the space $\mathbb{B}(E)$ of bounded measurable functions with the supremum norm. Furthermore, the constant $\alpha(M)$ does not depend on $g \in \mathbb{B}(E)$.

From Theorem 3.1 one can conclude that if the sequence of iterated MPE operators $(T^n(0))_{n\in\mathbb{N}}$ is bounded in the span norm, then we can apply standard Banach's fixed point arguments to obtain a solution to (2.3). Unfortunately, this is not always the case, that is, we could get

$$\sup_{n\in\mathbb{N}} \|T^n(0)\|_{\mathrm{sp}} = \infty,$$

and there might be no bounded solution to (2.3) under (A.1). We refer to Example 7.2 for a detailed analysis of a discrete (two state) example, where the interaction between the iterated operator span norm and the existence of MPE solution is illustrated.

Remark 3.3 (Difference between multiplicative and additive framework)**.** Under (A.1), the additive Poisson operator $\tilde{T}f(x) := g(x) + \int_E f(y)\mathbb{P}(x, \mathrm{d}y)$, defined for $f \in C(E)$, is contractive in the span norm so that there always exists a unique solution (up to an additive constant) to the corresponding Additive Poisson Equation; see Section 5.2.2 in [28] for details. This points to a fundamental difference between the (linear, additive) expectation-based framework and the (non-linear, multiplicative) entropy-based framework. This propagates directly to differences in the risk-neutral and risk-sensitive stochastic control problem; see [23, 31] for details. In Example 7.3, we illustrate this difference in a simple discrete (two state) space setting.

Remark 3.4 (Necessary and sufficient conditions for the existence of the MPE solution in a finite state space)**.** Assuming that the state space is finite, one could derive necessary and sufficient conditions for the existence of the solution, for any $g \in C(E)$, in both multiplicative and additive settings. On the finite state space, the MPE solution exists if and only if *Unichain condition* and *Strong Doeblin condition* are satisfied. Again, this is essentially different from the additive case, where the solution to Additive Poisson Equation exists if and only if *Unichain condition* is satisfied. We refer to [3] for more details.

Remark 3.5 (Split probability condition)**.** In [32], the split probability space technique is used to get an explicit formula for the solution of MPE. One can show that sufficient conditions for the existence of this formula are also necessary in the case of Example 7.2. We refer to [32] for more details.

4. Existence of a solution to MPE for the predetermined reward function

In this section, we fix $g \in C(E)$ and answer the question when (A.1) is sufficient to guarantee the existence of a solution to MPE for any $\mathbb{P} \in \mathcal{T}(E)$. As we show, the size of the span norm of g plays a key role, and one can find a sharp bound imposed on $\|g\|_{\mathrm{sp}}$. For brevity, we introduce the function

$$k(x) := -\tfrac{1}{2}\ln x, \quad x > 0,$$

and show the necessary and sufficient conditions for the existence of an MPE solution under (A.1) using some arguments adopted from [33]. We also refer to Example 7.2 for an illustration of this result in a simplified discrete (two state) setting.

Theorem 4.1. *Fix $g \in C(E)$ and $\Lambda \in (0,1)$. Then, there is a solution to MPE for any $\mathbb{P} \in \mathcal{T}(E)$ satisfying (A.1) for Λ if and only if $\|g\|_{sp} < k(\Lambda)$.*

Proof. ($\Rightarrow$) Let $g \in C(E)$ and $\Lambda \in (0,1)$ be such that $\|g\|_{\mathrm{sp}} < k(\Lambda)$ and let $\mathbb{P} \in \mathcal{T}(E)$ satisfy (A.1) for Λ. First, let us show that the iterated sequence of MPE operators is bounded in the span norm, *i.e.* there exists $M \in \mathbb{R}$ such that $\|T^n 0\|_{\mathrm{sp}} < M$ for $n \in \mathbb{N}$. From (A.1), for any nonnegative function $f \in C(E)$ and $x, x' \in E$ we get

$$\begin{aligned}\mathbb{P}f(x) - \mathbb{P}f(x') &\leq \int_E f(y)(\mathbb{P}(x, \mathrm{d}y) - \mathbb{P}(x', \mathrm{d}y)) \\ &\leq \int_H f(y)(\mathbb{P}(x, \mathrm{d}y) - \mathbb{P}(x', \mathrm{d}y)) \leq \sup_{y \in E} f(y)\Lambda, \end{aligned} \tag{4.1}$$

where H comes from the Hahn decomposition of $\mathbb{P}(x, \mathrm{d}y) - \mathbb{P}(x', \mathrm{d}y)$. Therefore, we have

$$
\begin{aligned}
Tf(x) - Tf(x') &\leq g(x) - g(x') + \ln \frac{\int_E \mathrm{e}^{f(y)}\mathbb{P}(x, \mathrm{d}y)}{\int_E \mathrm{e}^{f(y)}\mathbb{P}(x', \mathrm{d}y)} \\
&\leq 2\|g\|_{\mathrm{sp}} + \ln \frac{\int_E \mathrm{e}^{f(y)}\mathbb{P}(x', \mathrm{d}y) + \Lambda \mathrm{e}^{\sup_{y\in E} f(y)}}{\int_E \mathrm{e}^{f(y)}\mathbb{P}(x', \mathrm{d}y)} \\
&\leq 2\|g\|_{\mathrm{sp}} + \ln \left[1 + \Lambda \mathrm{e}^{2\|f\|_{\mathrm{sp}}}\right], \qquad (4.2)
\end{aligned}
$$

which implies $\|Tf\|_{\mathrm{sp}} \leq \|g\|_{\mathrm{sp}} + \frac{1}{2}\ln\left[1 + \Lambda \mathrm{e}^{2\|f\|_{\mathrm{sp}}}\right]$. Thus, by iteration, for $n \in \mathbb{N}$, we get

$$
\|T^n f\|_{\mathrm{sp}} \leq \|g\|_{\mathrm{sp}} + \frac{1}{2}\ln\left[1 + \sum_{i=1}^{n-1}\left(\Lambda \mathrm{e}^{2\|g\|_{\mathrm{sp}}}\right)^i + \Lambda^n \mathrm{e}^{(n-1)2\|g\|_{\mathrm{sp}} + 2\|f\|_{\mathrm{sp}}}\right]. \qquad (4.3)
$$

Since $\|g\|_{\mathrm{sp}} < k(\Lambda)$, we know that $\Lambda \mathrm{e}^{2\|g\|_{\mathrm{sp}}} < 1$. Therefore, using the sum of the infinite geometric sequence and setting $\epsilon_n := \Lambda^n \mathrm{e}^{(n-1)2\|g\|_{\mathrm{sp}} + 2\|f\|_{\mathrm{sp}}}$, we get

$$
\|T^n f\|_{\mathrm{sp}} \leq \|g\|_{\mathrm{sp}} + \frac{1}{2}\ln\left[1 + \frac{\Lambda \mathrm{e}^{2\|g\|_{\mathrm{sp}}}}{1 - \Lambda \mathrm{e}^{2\|g\|_{\mathrm{sp}}}} + \epsilon_n\right] = \|g\|_{\mathrm{sp}} + \frac{1}{2}\ln\left(\frac{1}{1 - \Lambda \mathrm{e}^{2\|g\|_{\mathrm{sp}}}} + \epsilon_n\right). \qquad (4.4)
$$

Now, since $\epsilon_n \searrow 0$, $n \to \infty$, and $1 - \Lambda \mathrm{e}^{2\|g\|_{\mathrm{sp}}} > 0$ we know that for sufficiently large $n \in \mathbb{N}$ we get a finite upper bound for $\|T^n f\|_{\mathrm{sp}}$ that does not depend on f. In particular, this holds for $f \equiv 0$, which concludes the proof that the sequence $\|T^n 0\|_{\mathrm{sp}}$, $n \in \mathbb{N}$, is bounded. Now, combining this result with Theorem 3.1 we can apply standard Banach fixed-point arguments to get a solution to MPE, *i.e.* (2.3); see [19]. This concludes this part of the proof.

($\Leftarrow$) We want to show that for any $g \in C(E)$ and $\Lambda \in (0, 1)$ such that $\|g\|_{\mathrm{sp}} \geq k(\Lambda)$ one can find kernel $\mathbb{P} \in \mathcal{T}(E)$ that satisfies (A.1) for Λ but for which a solution to MPE does not exist.

First, let us assume that $g \in C(E)$ is such that $\|g\|_{\mathrm{sp}} > k(\Lambda)$ or $\|g\|_{\mathrm{sp}} = k(\Lambda)$ and that the extremes are reached by g. For brevity, we only sketch the proof. In this case, one can find $x, x' \in E$, such that $g(x) - g(x') \geq k(\Lambda)$. Now, we can find a kernel $\mathbb{P} \in \mathcal{T}(E)$ with support on $\{x, x'\}$ such that it satisfies (A.1) for Λ, and for which

$$
\mathbb{P}(x, x) = 1, \quad \mathbb{P}(x, x') = 0, \quad \mathbb{P}(x', x) = 1 - \Lambda, \quad \mathbb{P}(x', x') = \Lambda.
$$

As a consequence, considering only the starting points x and x', we effectively end up with a simplified two state dynamics encoded in the transition matrix

$$
P = \begin{bmatrix} 1 & 0 \\ 1 - \Lambda & \Lambda \end{bmatrix},
$$

for states x and x'. One can easily show that in this case the solution to MPE exists if and only if $\|g\|_{\mathrm{sp}} < k(\Lambda)$. This contradicts the fact that $g(x) - g(x') \geq k(\Lambda)$. For more details, we refer to Example 7.2 where this is shown explicitly.

Second, let us assume that $\|g\|_{\mathrm{sp}} = k(\Lambda)$ and the upper extreme is not attained. Let $x_1 \in E$ be such that $x_1 = \inf_{x\in E} g(x)$ and let $(x_i)_{i=2}^{\infty}$ denote a sequence of points such that $(g(x_i))_{i=2}^{\infty}$ is increasing and $g(x_i) \to \sup_{x\in E} g(x)$ as $i \to \infty$. Let $\mathbb{P} \in \mathcal{T}(E)$ denote the kernel with support on $(x_i)_{i=1}^{\infty}$ such that for $i = 2, 3, \ldots$, we have

$$
\mathbb{P}(x_1, x_1) = 1, \quad \mathbb{P}(x_1, x_i) = 0, \quad \mathbb{P}(x_i, x_1) = 1 - \Lambda, \quad \mathbb{P}(x_i, x_i) = \Lambda.
$$

Then, directly from (2.3), for any $i \in \mathbb{N}$, we get

$$\begin{cases} g(x_1) = \lambda, \\ g(x_1) - g(x_i) = \ln\left[(1-\Lambda)\mathrm{e}^{w(x_1)-w(x_i)} + \Lambda\right]. \end{cases}$$

Now, noting that $g(x_1) - g(x_i) \to -2\|g\|_{\mathrm{sp}}$, as $i \to \infty$, and $-2\|g\|_{\mathrm{sp}} = -2k(\Lambda) = \ln\Lambda$, we get $w(x_i) \to \infty$ as $i \to \infty$, which contradicts the fact that $w \in C(E)$.

The remaining cases where both extremes are not attained or only the lower extreme is attained could be treated with similar logic; we omit the proof for brevity. □

For completeness, we state a similar result under assumption (A.2) and provide an alternative proof of the boundedness of the underlying operator, based directly on the analysis of the span norm size.

Theorem 4.2. *Fix $g \in C(E)$ and $d \in (0,1)$. Then, there is a solution to MPE for any $\mathbb{P} \in \mathcal{T}(E)$ satisfying (A.2) for d if and only if $\|g\|_{sp} < k(1-d)$.*

Proof. The proof follows directly from Theorem 4.1 combined with Proposition 2.3 since (A.2) implies (A.1). However, we decided to present an alternative proof of the fact that under (A.2) for d, the iterated operator T is bounded for any $g \in C(E)$ such that $\|g\|_{\mathrm{sp}} < k(1-d)$. The proof is based on a span norm centering and distance analysis that is aligned with the method introduced in the proof of Theorem 3.1. Let $g \in C(E)$ be such that $\|g\|_{\mathrm{sp}} < k(1-d)$ and let $\mathbb{P} \in \mathcal{T}(E)$ satisfy (A.2) for d. For $n \in \mathbb{N}$, let $g_n := T^n 0$ and $c_n := \inf_{c\in\mathbb{R}} \|g_n + c\|$. Then, we get

$$\begin{aligned} \|g_{n+1}\|_{\mathrm{sp}} &= \sup_{x,y\in E} \frac{g(x) + \mu_x(g_n) - g(y) - \mu_y(g_n)}{2} \\ &\leq \|g\|_{\mathrm{sp}} + \sup_{x,y\in E} \frac{\mu_x(g_n + c_n) - \mu_y(g_n + c_n)}{2}. \end{aligned} \tag{4.5}$$

From $\|g\|_{\mathrm{sp}} < k(d)$ we know that there exists $\epsilon \in \mathbb{R}$ such that $d > \frac{\epsilon}{2} > 0$ and

$$\|g\|_{\mathrm{sp}} < -\tfrac{1}{2}\ln(1-d+2\epsilon). \tag{4.6}$$

Let $K := \ln(d-\epsilon) - \ln\epsilon$ and $A_n := \{x \in E \colon g_n(x) + c_n \leq 0\}$. For any fixed $n \in \mathbb{N}$ we consider three disjoint cases: (1) $\|g_n\|_{\mathrm{sp}} \leq K$; (2) $\|g_n\|_{\mathrm{sp}} > K$ and $\inf_{x\in E} \mathbb{P}(x, A_n^c) \geq \epsilon$; (3) $\|g_n\|_{\mathrm{sp}} > K$ and $\inf_{x\in E} \mathbb{P}(x, A_n^c) < \epsilon$.

In the first case, noting that $\|g_n + c_n\| = \|g_n\|_{\mathrm{sp}}$, and using (4.5), we get

$$\|g_{n+1}\|_{\mathrm{sp}} \leq K + \|g\|_{\mathrm{sp}}. \tag{4.7}$$

In the second case, using (4.5) and the fact that $1_{A_n}(g_n(x) + c_n) \geq -1_{A_n}\|g_n\|_{\mathrm{sp}}$ and noting that for any $x, y \in E$ we have

$$\begin{aligned} \mu_y(g_n + c_n) &\geq \mu_y(-1_{A_n}\|g_n\|_{\mathrm{sp}}) \\ &= \ln\left[\mathrm{e}^{-\|g_n\|_{\mathrm{sp}}}\mathbb{P}(y, A_n) + \mathrm{e}^0\mathbb{P}(y, A_n^c)\right] \\ &\geq \inf_{y\in E} \ln\mathbb{P}(y, A_n^c) \\ &\geq \ln\epsilon \end{aligned}$$

and $\mu_x(g_n + c_n) \leq \|g_n\|_{\mathrm{sp}}$, we get

$$\|g_{n+1}\|_{\mathrm{sp}} \leq \tfrac{1}{2}\|g_n\|_{\mathrm{sp}} + \|g\|_{\mathrm{sp}} - \tfrac{1}{2}\ln\epsilon. \tag{4.8}$$

In the third case, we get $\sup_{x\in E}\mathbb{P}(x,A_n)>1-\epsilon$. Thus, recalling assumption (A.1), we get $\epsilon\geq\inf_{x\in E}\mathbb{P}(x,A_n^c)\geq d\nu(A_n^c)\geq d-\mathbb{P}(x,A_n)$ so that $\inf_{x\in E}\mathbb{P}(x,A_n)\geq d-\epsilon$. Consequently, for any $x,y\in E$ we have

$$\begin{aligned}\mu_x(g_n+c_n) &\leq \mu_x(1_{A_n^c}\|g_n\|_{\mathrm{sp}})\\ &\leq \|g_n\|_{\mathrm{sp}}+\mu_x(-1_{A_n}K)\\ &= \|g_n\|_{\mathrm{sp}}+\ln\left[\mathrm{e}^{-K}\mathbb{P}(x,A_n)+\mathrm{e}^{0}\mathbb{P}(x,A_n^c)\right]\\ &\leq \|g_n\|_{\mathrm{sp}}+\ln\left[1-\left(1-\frac{\epsilon}{d-\epsilon}\right)\inf_{x\in E}\mathbb{P}(x,A_n)\right]\\ &\leq \|g_n\|_{\mathrm{sp}}+\ln\left[1-\left(\frac{d-2\epsilon}{d-\epsilon}\right)(d-\epsilon)\right]\\ &= \|g_n\|_{\mathrm{sp}}+\ln\left[1-d+2\epsilon\right]\end{aligned}$$

and $\mu_y(g_n+c_n)\geq-\|g_n\|_{\mathrm{sp}}$. Thus, recalling (4.5) and (4.6) we get

$$\|g_{n+1}\|_{\mathrm{sp}}\leq\|g_n\|_{\mathrm{sp}}+\|g\|_{\mathrm{sp}}+\tfrac{1}{2}\ln\left[1-d+2\epsilon\right]\leq\|g_n\|_{\mathrm{sp}}. \tag{4.9}$$

Combining all three cases, *i.e.* (4.7), (4.8), and (4.9), for any $n\in\mathbb{N}$ we get

$$\|g_{n+1}\|_{\mathrm{sp}}\leq\max\left\{K+\|g\|_{\mathrm{sp}},\ \tfrac{1}{2}\|g_n\|_{\mathrm{sp}}+\|g\|_{\mathrm{sp}}-\tfrac{1}{2}\ln\epsilon,\ \|g_n\|_{\mathrm{sp}}\right\},$$

from which the proof follows using standard geometric series arguments. □

Theorem 4.1 sheds some light on the interaction between the risk neutral and risk sensitive frameworks represented by the additive and multiplicative Poisson equations, respectively. The risk-neutral Additive Poisson Equation (APE) given by

$$w_0(x)=g(x)-\lambda_0+\int_E w_0(y)\mathbb{P}(x,\mathrm{d}y),\quad x\in E,$$

could be seen as a limit of risk-sensitive MPEs with risk-aversion $\gamma\neq0$ given by

$$w_\gamma(x)=\gamma g(x)-\lambda_\gamma+\ln\int_E \mathrm{e}^{w_\gamma(y)}\mathbb{P}(x,\mathrm{d}y),\quad x\in E,$$

for $\gamma\to0$, see [23, 34] for the economic context and differences between the risk-neutral (expectation based) and risk-sensitive (entropy based) frameworks. Note that the risk-averse parameter $\gamma\in\mathbb{R}\setminus\{0\}$ could be seen as a scaling factor that is used to substitute g with γg in the standardised MPE problem. Consequently, regardless of the initial choice of $g\in C(E)$, we can find γ small enough, that is, such that the bound presented in Theorem 4.1 (or Thm. 4.2) is satisfied by the rescaled function γg. This indicates that a solution to APE should always exist under (A.1), which is indeed the case, see [7]. Also, note that some results presented in the literature state the existence of a solution to MPE (or its controlled version) for sufficiently small γ which is consistent with the results presented herein, see, *e.g.*, Theorem 1 in [18] or Theorem 5.4 in [1].

5. Existence of bounded MPE solution for generic reward function

In this section, we investigate what additional assumptions, in addition to (A.1), could be imposed on a transition kernel $\mathbb{P}\in\mathcal{T}(E)$, so that the bounded solution to MPE exists for an arbitrary $g\in C(E)$. It should be noted that while the assumption (A.1) is natural for compact spaces, it could not be satisfied for general locally compact spaces. In fact, for a noncompact state space, one can show that the MPE solution is typically unbounded under very generic conditions related to the C_0-*Feller property*, saying that the transition operator $\mathbb{P}$ transforms the space $C_0(E)$ of continuous and bounded functions vanishing at infinity into itself.

The typical condition imposed within the mixing framework relates to *strong mixing*. We say that the transition kernel satisfies *multi-step strong mixing condition* for $n \in \mathbb{N}$ and $L > 0$ if

$$\sup_{x,x' \in E} \frac{\mathbb{P}_n(x,B)}{\mathbb{P}_n(x',B)} \leq L, \quad B \in \mathcal{E}, \tag{A.3}$$

where the convention $\frac{0}{0} = 0$ and $\frac{1}{0} = \infty$ is used. Note that under (A.3) we have that the iterated measures $\mathbb{P}_n(x,\cdot)$, for $x \in E$, are equivalent. Furthermore, (A.3) implies (A.1') for the same $n \in \mathbb{N}$, which effectively leads to the local contraction property of the operator T^n; see Theorem 3.1. For example, the condition (A.3) is satisfied by regular reflected diffusions in bounded domains, see [35], Remark 2.1 and references therein. Let us now recall a result which states that under (A.3) the iterated operator T is bounded in the span norm, so that we get the existence of a solution to MPE; the proof is recalled for completeness.

Proposition 5.1. *Let $\mathbb{P} \in \mathcal{T}(E)$ satisfy (A.3). Then, there is a solution to the MPE for any $g \in C(E)$.*

Proof. Due to Theorem 3.1 it is sufficient to show that the iterated sequence of MPE operators is bounded, *i.e.* there exists $M \in \mathbb{R}$ such that $\|T^n 0\|_{\text{sp}} < M$ for $n \in \mathbb{N}$. For any $f \in C(E)$ and $x, x' \in E$ we get

$$\begin{aligned} T^n f(x) - T^n f(x') &\leq 2n\|g\|_{\text{sp}} + \ln \frac{\int_E \mathrm{e}^{f(y)} \mathbb{P}_n(x, \mathrm{d}y)}{\int_E \mathrm{e}^{f(y)} \mathbb{P}_n(x', \mathrm{d}y)} \\ &\leq 2n\|g\|_{\text{sp}} + \ln \left[L + \frac{\int_E \mathrm{e}^{f(y)} \left[\mathbb{P}_n(x,\mathrm{d}y) - L\mathbb{P}_n(x',\mathrm{d}y)\right]}{\int_E \mathrm{e}^{f(y)} \mathbb{P}_n(x',\mathrm{d}y)} \right] \\ &\leq 2n\|g\|_{\text{sp}} + \ln L. \end{aligned} \tag{5.1}$$

Thus, for any $f \in C(E)$ we get $\|T^n f\|_{\text{sp}} \leq n\|g\|_{\text{sp}} + \frac{1}{2}\ln L$. This implies that the iterated sequence $(T^{kn}0)_{k\in\mathbb{N}}$, is jointly bounded in the span norm, and so is $(T^n 0)_{n\in\mathbb{N}}$. This concludes the proof. □

Although (A.3) guarantees the existence of an MPE solution, it could be seen as restrictive and designed for compact state spaces. In Example 7.4, we show that (A.3) is not a necessary condition even in a simplified discrete time setup; a similar example could be constructed for a dynamics with full invariant measure support.

Next, we investigate the problem of the existence of an MPE solution on the complement of the invariant measure support, which should shed some light on the required process dynamics. Please recall that for $\mathbb{P} \in \mathcal{T}(E)$ satisfying (A.1) there exists a unique invariant measure $\nu \in \mathcal{P}(E)$, see Remark 2.2. Also, to ease the notation, for any $x \in E$, we define the hitting time as

$$\tau_B := \inf\{n \in \mathbb{N} : X_n \in B\}, \quad B \in \mathcal{E}, \tag{5.2}$$

where $(X_i)_{i\in\mathbb{N}}$ is the underlying Markov process. In a nutshell, we show that if the underlying Markov process escapes quickly from the complement of the invariant measure support and a local solution in the invariant measure support exists, then a global MPE solution also exist; the escape time must be faster than geometric. This intuition is formalised in the following result.

Proposition 5.2. *Let $\mathbb{P} \in \mathcal{T}(E)$ satisfy (A.1) and let ν correspond to its invariant measure. Let E_ν be an invariant set, and let a bounded MPE solution on E_ν exist for any $g \in C(E_\nu)$. Assume that*

$$\forall \alpha \in (0,1)\, \exists\, n \in \mathbb{N} \colon \sup_{x \in E_\nu^c} \mathbb{P}_x[\tau_{E_\nu} > n] \leq \alpha^n. \tag{5.3}$$

Then, a bounded MPE solution on E exists for any $g \in C(E)$.

Proof. Let $g \in C(E)$. Without loss of generality, we can assume that $g \not\equiv 0$ and $\inf_{x\in E} g(x) \geq 0$. Let $\alpha := \exp(-2\|g-\lambda\|-1)$. From (5.3) we know that there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in E_\nu^c} \mathbb{P}_x[\tau_{E_\nu} > n_0] \leq \alpha^{n_0}.$$

Let $(w,\lambda) \in C(E_\nu)\times\mathbb{R}$ denote the MPE solution for g on the invariant set E_ν such that $\inf_{x\in E_\nu} w(x) = 0$. Note that $\lambda \geq 0$ since g is nonnegative. For $x \in E$, let

$$w_0(x) := \begin{cases} w(x) & \text{if } x \in E_\nu \\ 0 & \text{if } x \notin E_\nu. \end{cases}$$

and let $T_\lambda(\cdot) = T(\cdot)-\lambda$. First, recalling that E_ν is an invariant set, for any $x \in E_\nu$ we get $T_\lambda w_0(x) = w(x)$ and consequently $T^n w_0(x) = w(x)$, for $n \in \mathbb{N}$. Thus, for any $k \in \mathbb{N}$ and $x \in E$, recalling that $\inf_{x\in E_\nu} w(x) = 0$, we get

$$\begin{aligned} T_\lambda^{(k+1)n_0} w_0(x) &\geq \inf_{x\in E} \hat{\mu}_x\left(\sum_{i=0}^{n_0-1}(g(X_i)-\lambda) + T_\lambda^{kn_0} w_0(X_n)\right) \\ &\geq -n_0\|g-\lambda\| + \inf_{x\in E}\left(\hat{\mu}_x\left(\mathbb{1}_{\{\tau_{E_\nu}\leq n_0\}} w(X_n) + \mathbb{1}_{\{\tau_{E_\nu}>n_0\}} T_\lambda^{kn_0} w_0(X_n)\right)\right) \\ &\geq -n_0\|g-\lambda\| + \inf_{x\in E}\left(\hat{\mu}_x\left(\mathbb{1}_{\{\tau_{E_\nu}>n_0\}} T_\lambda^{kn_0} w_0(X_n)\right)\right) \\ &\geq -n_0\|g-\lambda\| + \inf_{x\in E} \ln\left(\mathbb{P}_x[\tau_{E_\nu}\leq n_0]\mathrm{e}^0 + \mathbb{P}_x[\tau_{E_\nu}>n_0]\mathrm{e}^{-\|T_\lambda^{kn_0} w_0\|}\right) \\ &\geq -n_0\|g-\lambda\| + \ln(1-\alpha^{n_0}). \end{aligned} \tag{5.4}$$

On the other hand, for any $k \in \mathbb{N}$ and $x \in E$, we get

$$\begin{aligned} T_\lambda^{(k+1)n_0} w_0(x) &\leq n_0\|g-\lambda\| + \sup_{x\in E}\left(\hat{\mu}_x\left(\mathbb{1}_{\{\tau_{E_\nu}\leq n_0\}} w(X_n) + \mathbb{1}_{\{\tau_{E_\nu}>n_0\}}\|T_\lambda^{kn_0} w_0\|\right)\right) \\ &\leq n_0\|g-\lambda\| + \ln(\mathrm{e}^{\|w\|} + \alpha^{n_0}\mathrm{e}^{\|T_\lambda^{kn_0} w_0\|}). \end{aligned} \tag{5.5}$$

Now, let $a := \mathrm{e}^{-n_0(\|g-\lambda\|+1)}$, and $C := \mathrm{e}^{n_0\|g-\lambda\|}(\mathrm{e}^{\|w\|} + (1-\alpha^{n_0})^{-1})$. Then, combining (5.4) with (5.5), noting that $a < 1$ as well as $C < \infty$, and taking the exponent, for any $k \in \mathbb{N}$, we get

$$\begin{aligned} \mathrm{e}^{\|T_\lambda^{(k+1)n_0} w_0\|} &\leq \mathrm{e}^{n_0\|g-\lambda\|}\left(\mathrm{e}^{\|w\|} + \alpha^{n_0}\mathrm{e}^{\|T_\lambda^{kn_0} w_0\|}\right) + \mathrm{e}^{n_0\|g-\lambda\|-\ln(1-\alpha^{n_0})} \\ &= \mathrm{e}^{n_0\|g-\lambda\|+n_0(-2\|g-\lambda\|-1)}\mathrm{e}^{\|T_\lambda^{kn_0} w_0\|} + C \\ &\leq a\mathrm{e}^{\|T_\lambda^{kn_0} w_0\|} + C \\ &\leq \sum_{i=0}^{k} a^i C + a^k \mathrm{e}^{\|T_\lambda^{n_0} w_0\|} \\ &\leq \frac{C}{1-a} + \mathrm{e}^{\|T_\lambda^{n_0} w_0\|}. \end{aligned} \tag{5.6}$$

Noting that $\|T_\lambda^{n_0} w_0\|$ is bounded (as T_λ is a C-Feller operator), we conclude that

$$\sup_{k\in\mathbb{N}} \|T_\lambda^{kn_0} w_0\| < \infty,$$

which implies $\sup_{n\in\mathbb{N}} \|T_\lambda^n w_0\| < \infty$ and consequently $\sup_{n\in\mathbb{N}} \|T^n w_0\|_{\mathrm{sp}} < \infty$. Using Theorem 3.1 and the Banach fixed-point theorem, we find that there exist $\tilde{w} \in C(E)$ and $\tilde{\lambda} \in \mathbb{R}$ that satisfies MPE for g on E, which concludes the proof. Note that from $\sup_{n\in\mathbb{N}} \|T^n w_0\|_{\mathrm{sp}} < \infty$ we also get $\sup_{n\in\mathbb{N}} \|T^n 0\|_{\mathrm{sp}} < \infty$, as iterations of the operator converge in the span norm. Consequently, $\tilde{w} \in C(E)$ could be recovered from iterations of $\tilde{g} \in C(E)$, where $\tilde{g} \equiv 0$. □

Later, in Proposition 6.2, we show that condition (5.3) is not restrictive – it is also necessary under some additional technical continuity assumptions. Furthermore, Proposition 5.2 allows us to immediately recover the necessary and sufficient conditions for the existence of a (bounded) MPE solution for a finite state space and recover the key result from [3]. Namely, the probability kernel must be such that it has a unique invariant measure, and all states that lie outside of its support must be transient with return probability equal to zero, as otherwise condition (5.3) would not hold. In particular, assuming that the number of states is equal to $N \in \mathbb{N}$, condition (5.3) immediately implies that, after at most N steps, the process must be in the invariant measure support. We refer to [3] where various finite state space characterisations are provided and discussed in detail.

Remark 5.3 (Difference between finite and denumerable case)**.** Finite state space necessary and sufficient conditions for the existence of the MPE solution stated in [3] do not transfer directly to denumerable spaces. Namely, we can have a situation where we do not get into the support of the invariant measure in finite number of steps (with probability one), but the MPE solution always exists. In contrast to the finite setting, condition (5.3) does not imply finite-step entry into the invariant measure support in the denumerable case. See Example 7.4 for details.

6. Geometric bounds and negative existence results

In this section, we investigate how the process dynamics is linked to the existence of an MPE solution for any $g \in C(E)$. Although most of the results presented in this section do not require assumption (A.1), we will generally assume that $\mathbb{P} \in \mathcal{T}(E)$ is such that it has a unique invariant measure $\nu \in \mathcal{P}(E)$; recall that the existence of the invariant measure is effectively implied by (A.1), see Remark 2.2.

We start with a simple lemma which states that geometric stay in a subset of states for which the reward value is high sets a lower bound on the optimal value.

Lemma 6.1. *Let $g \in C(E)$ be nonnegative and such that it has the MPE solution (w, λ). Moreover, assume that there are $k > 0$, $K \subset E$, and $\alpha \in (0, 1)$ such that $g(x) \geq k$, $x \in K$, and $\sup_{x\in E} \mathbb{P}[\tau_{K^c} > n] > \alpha^n$, $n \in \mathbb{N}$. Then, we have $\lambda \geq k + \ln \alpha$.*

Proof. Using the MPE iteration, as in the proof of Proposition 2.1, for any $n \in \mathbb{N}$, we get

$$n\lambda \geq \sup_{x\in E} \hat{\mu}_x \left(\sum_{i=0}^{n-1} g(X_i) \right) - 2\|w\| \geq \ln \left[\alpha^n \mathrm{e}^{nk} \right] - 2\|w\| = n(k + \ln \alpha) - 2\|w\|.$$

Dividing both sides by n and taking the limit as $n \to \infty$, we get $\lambda \geq k + \ln \alpha$. □

In the following result, we show that condition (5.3) considered in Proposition 5.3 could be seen as necessary, *i.e.* if the process remains in a compact set that is outside of the support of the invariant measure with geometric probability, then one can construct $g \in C(E)$ for which no bounded MPE solution exists.

Proposition 6.2. *Let $\mathbb{P} \in \mathcal{T}(E)$ have a unique invariant measure $\nu \in \mathcal{P}(E)$. Assume there exists a compact set $K \in \mathcal{E}$ for which*

1. *There is* $\epsilon > 0$*, such that* $\nu(K_\epsilon) = 0$*, where* $K_\epsilon := \{x \in E \colon \rho(x, K) \leq \epsilon\}$*;*
2. *There is* $h \in (0, 1)$*, such that* $\sup_{x\in K} \mathbb{P}_x [\tau_{K^c} > n] > h^n$*, for* $n \in \mathbb{N}$*.*

Then, there exists $g \in C(E)$ *for which a bounded solution to MPE does not exist.*

Proof. Assume there is such $K \in \mathcal{E}$ and let $\epsilon, h \in (0, 1)$ denote the corresponding constants. Let $g \in C(E)$ be given by

$$g(x) = \begin{cases} \ln(2/h) & \text{if } \rho(x, K) = 0, \\ \ln(2/h) \cdot (1 - \rho(x, K)/\epsilon) & \text{if } \rho(x, K) \in (0, \epsilon) \\ 0 & \text{if } \rho(x, K) \geq \epsilon. \end{cases} \tag{6.1}$$

Assume that there is a bounded MPE solution for g; let us denote the solution by $w \in C(E)$ and $\lambda \in \mathbb{R}$. Using iteration we get

$$n\lambda = \hat{\mu}_x \left(\sum_{i=0}^{n-1} g(X_i) + w(X_n) - w(x) \right), \quad x \in E, n \in \mathbb{N}. \tag{6.2}$$

Recalling that $\nu(K_\epsilon) = 0$, we know that there exists $Z \in \mathcal{E}$ such that $\nu(Z) = 1$, and for any $i \in \mathbb{N}$ and $x \in Z$, we have $\mathbb{P}_x[X_i \in K_\epsilon^c] = 1$. Consequently, using (6.2) and noting that $\sup_{z\in K_\epsilon^c} g(z) = 0$, we have

$$\begin{aligned} n\lambda &\leq \inf_{x\in E} \hat{\mu}_x \left(\sum_{i=0}^{n-1} g(X_i) \right) + 2\|w\|_{\text{sp}} \\ &\leq \int_Z \hat{\mu}_x \left(\sum_{i=0}^{n-1} g(X_i) \right) \nu(\mathrm{d}x) + 2\|w\|_{\text{sp}} \\ &\leq \int_Z \hat{\mu}_x \left(\sum_{i=0}^{n-1} \mathbb{1}_{K_\epsilon^c}(X_i) \sup_{z\in K_\epsilon^c} g(z) \right) \nu(\mathrm{d}x) + 2\|w\|_{\text{sp}} \\ &\leq 2\|w\|_{\text{sp}}. \end{aligned}$$

Thus, letting $n \to \infty$, we conclude that $\lambda \leq 0$. On the other hand, using (6.2) and property $\sup_{x\in K} \mathbb{P}_x [\tau_{K^c} > n] > h^n$, for $n \in \mathbb{N}$, we get

$$\begin{aligned} n\lambda &\geq \sup_{x\in E} \hat{\mu}_x \left(\sum_{i=0}^{n-1} g(X_i) \right) - 2\|w\|_{\text{sp}} \\ &\geq \sup_{x\in K} \hat{\mu}_x \left(\ln \tfrac{2}{h} \cdot \sum_{i=0}^{n-1} \mathbb{1}_K(X_i) \right) - 2\|w\|_{\text{sp}} \\ &\geq \ln \left[\mathrm{e}^{\ln 2/h\cdot(n-1)} \cdot \sup_{x\in K} \mathbb{P}_x[\tau_{K^c} > n - 1] \right] - 2\|w\|_{\text{sp}} \\ &\geq \ln \frac{2^{n-1}h^{n-1}}{h^{n-1}} - 2\|w\|_{\text{sp}} \\ &= (n - 1) \ln 2 - 2\|w\|_{\text{sp}}. \end{aligned}$$

Thus, letting $n \to \infty$ we get $\lambda \geq \ln 2$, which leads to a contradiction. □

In the remaining part of this section, let us show a series of more technical results which illustrate why the MPE solution is typically unbounded under very generic conditions linked to the C_0-Feller property and how it interacts with mixing, that is, assumption (A.1'). To do this, we need some additional notation that will allow us to study dynamic propagation on an arbitrary compact ball. For any $x \in E$ and $\epsilon \in \mathbb{R}_+$ let

$$B(x,\epsilon) := \{y \in E : \rho(x,y) \leq \epsilon\}.$$

With slight abuse of notation, given a fixed $\bar{x} \in E$ and $\eta > 1$, till the end of this section, we use the notation,

$$\begin{aligned} B &:= B(\bar{x},\eta), \\ g_m(x) &:= \left[m \cdot \rho\left(x, B(\bar{x}, \eta - \tfrac{1}{m})\right)\right] \wedge 1, \quad m \in \mathbb{N}, \end{aligned}$$

and use $\lambda_m \in \mathbb{R}$ and $w_m \colon E \to \mathbb{R}$ to denote a generic (*i.e.* not necessarily bounded) solution to MPE for g_m, assuming it exists. Note that $g_m \in C(E)$ and for any $x \in E$ and $m \in \mathbb{N}$ we know that $1 \geq g_m(x) \geq \mathbb{1}_{\bar{B}(\bar{x},\eta)}(x)$, $g_m(\cdot)$ is decreasing, and $g_m(x) \searrow \mathbb{1}_{\bar{B}^c(\bar{x},\eta)}(x)$, as $m \to \infty$. Also, recall that τ_B denotes the first hitting time to $B \in \mathcal{E}$, see (5.2).

Proposition 6.3. *Fix $\bar{x} \in E$, $\eta > 1$, and $m \in \mathbb{N}$. Assume $g_m \in C(E)$ admits MPE solution $(w_m, \lambda_m) \in C(E) \times \mathbb{R}$. Then, we have*

$$\sup_{x\in E} \hat{\mu}_x(\tau_B(1-\lambda_m)) \leq 2\|w_m\| + 1, \tag{6.3}$$

$$\sup_{x\in E} \mathbb{P}_x\left[\tau_B > n\right] \leq e^{2\|w_m\|+1} e^{-(1-\lambda_m)n}. \tag{6.4}$$

Proof. First, we show (6.3). Iterating MPE, for any $n \in \mathbb{N}$ and $x \in E$, we get

$$w_m(x) = \hat{\mu}_x\left(\sum_{i=0}^{\tau_B\wedge n-1}(g_m(X_i) - \lambda_m) + w_m(X_{\tau_B\wedge n})\right). \tag{6.5}$$

Consequently, letting $n \to \infty$, using Fatou lemma, and monotonicity of entropic utility, for any $x \in E$ we have

$$w_m(x) + \mathbb{1}_B(x) \geq \hat{\mu}_x\left(\tau_B(1-\lambda_m) + w_m(X_{\tau_B})\right). \tag{6.6}$$

which implies (6.3). The equality (6.4) follows directly from (6.3), which concludes the proof. □

In the next corollary, we show that Proposition 6.3 implies that MPE solutions are typically unbounded under some additional assumptions.

Corollary 6.4. *Fix $\bar{x} \in E$, $\eta > 1$, and $m \in \mathbb{N}$. Assume $g_m \in C(E)$ admits MPE solution (w_m, λ_m), and we have*

$$\lambda_m < 1, \tag{6.7}$$

$$\sup_{n\in\mathbb{N}} \inf_{x\in E} \mathbb{P}_x\left[\exists_{s\leq n} \quad x_s \in B\right] = 0. \tag{6.8}$$

Then, the function w_m must be unbounded.

Proof. Assume that w_m is bounded and that (6.7) holds. Let (large) $n \in \mathbb{N}$ be such that

$$e^{2\|w_m\|+1} e^{-(1-\lambda_m)n} < 1.$$

Then, directly from (6.4), we get $\sup_{x\in E}\mathbb{P}_x\left[\tau_B>n\right]<1$, and consequently

$$\inf_{x\in E}\mathbb{P}_x\left[\exists_{s\leq n}\;\; x_s\in B\right]=1-\sup_{x\in E}\mathbb{P}_x\left[\tau_B>n\right]>0,$$

which contradicts (6.8). □

Property (6.8) from Corollary 6.4 could be seen as a weaker version of the C_0-Feller property. In addition, property (6.7) does not appear to be strong, as it is true under mild assumptions related to the invariant measure convergence, as illustrated in the following result.

Proposition 6.5. *Fix $\bar{x}\in E$, $\eta>1$, and $m\in\mathbb{N}$. Assume that $g_m\in C(E)$ admits the generic MPE solution (w_m,λ_m) and $\mathbb{P}$ has a unique invariant measure ν. Moreover, assume that $\nu(g_m):=\int_E g_m d\nu<1$, and there exists $x\in E$ such that for any $\epsilon>0$ there is $p>0$ such that for a sufficiently large $n\in\mathbb{N}$ we have*

$$\mathbb{P}_x\left[\left|\tfrac{1}{n}\textstyle\sum_{i=0}^{n-1}g_m(X_i)-\nu(g_m)\right|\geq\epsilon\right]\leq \mathrm{e}^{-np}. \tag{6.9}$$

Then, we have $\lambda_m<1$.

Proof. Let $x\in E$ be such that (6.9) is satisfied and let $\epsilon>0$ be such that $\epsilon<1-\nu(g_m)$. Then, there exists $p>0$ such that for sufficiently big $n\in\mathbb{N}$, we have

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{\sum_{i=0}^{n-1}g_m(X_i)}\right]&\leq \mathrm{e}^{n}\mathbb{P}_x\left[\left|\frac{1}{n}\sum_{i=0}^{n-1}g_m(X_i)-\nu(g_m)\right|\geq\epsilon\right]+\mathrm{e}^{n(\nu(g_m)+\epsilon)}\\&\leq \mathrm{e}^{n(1-p)}+\mathrm{e}^{n(\nu(g_m)+\epsilon)}.\end{aligned}\tag{6.10}$$

Without loss of generality, we can assume that $p<1-\nu(g_m)-\epsilon$; note that if (6.10) holds for some $p>0$, then it also holds for any smaller positive p. Consequently, we get

$$\hat{\mu}_x\left(\sum_{i=0}^{n-1}g_m(X_i)\right)=\ln\mathbb{E}_x\left[\mathrm{e}^{\sum_{i=0}^{n-1}g_m(X_i)}\right]\leq\ln(2\mathrm{e}^{n(1-p)})=\ln 2+n(1-p) \tag{6.11}$$

which implies $\lambda_m\leq 1-p$. □

Remark 6.6 (Estimates of the empirical measures for Feller–Markov processes)**.** The assumption (6.9) follows from estimates for empirical measures of Feller–Markov processes. In the form used here, sufficient conditions can be found in Theorem 3 (and its proof) in [36]; the theory comes back to famous papers [37] and [38].

In the next result, we show that the mixing condition combined with (6.8) leads to a compact-type dynamics for which we do not get the condition (6.7).

Proposition 6.7. *Fix $\bar{x}\in E$, $\eta>1$, and $m\in\mathbb{N}$. Assume that $g_m\in C(E)$ admits the MPE solution (w_m,λ_m) and $\mathbb{P}$ satisfies (A.1') and (6.8). Then $\lambda_m=1$.*

Proof. Fix $\bar{x}\in E$, $\eta>1$, and $m\in\mathbb{N}$ such that g_m admits the MPE solution (w_m,λ_m). From (A.1'), it follows that the value $\nu(B)$, where ν is the unique invariant measure, is approximated from below by an increasing sequence $m_n(B):=\inf_{y\in E}\mathbb{P}_n(y,B)$. For details, see the proof of case (b) in Section 5, Chapter 5 of [27].

Now, since for any $x \in E$ we have $m_n(B) \leq \mathbb{P}_n(x, B)$ and $\mathbb{P}_n(x, B) \to 0$ as $n \to \infty$ due to (6.8), we get $\nu(B) = 0$. Consequently, using Jensen inequality, we get

$$\begin{aligned} \lambda_m &= \lim_{n\to\infty} \tfrac{1}{n} \ln \mathbb{E}_x \left[\mathrm{e}^{\sum_{i=0}^{n-1} g_m(X_i)} \right] \\ &\geq \liminf_{n\to\infty} \tfrac{1}{n} \ln \mathbb{E}_x \left[\mathrm{e}^{\sum_{i=0}^{n-1} \mathbb{1}_{B^c}(X_i)} \right] \\ &\geq \liminf_{n\to\infty} \tfrac{1}{n} \mathbb{E}_x \left[\textstyle\sum_{i=0}^{n-1} \mathbb{1}_{B^c}(X_i) \right] \\ &= \nu(B^c) = 1, \end{aligned}$$

which concludes the proof as $1 \geq \lambda_m$ due to the fact that $1 \geq g_m(\cdot)$. □

Proposition 6.8. *Fix $\bar{x} \in E$, $\eta > 1$, and $m \in \mathbb{N}$. Assume $g_m \in C(E)$ admits the MPE solution $(w_m, \lambda_m) \in C(E) \times \mathbb{R}$ and $\mathbb{P}$ has a unique invariant measure ν. If $\lambda_m = 1$, then $\nu(B(\bar{x}, \eta - \frac{1}{m})) = 0$.*

Proof. Iterating MPE with $\lambda_m = 1$, in the same way as in the proof of Proposition 2.1, for any $x \in E$ and $n \in \mathbb{N}$, we have

$$w(x) = \hat{\mu}_x \left(\sum_{i=0}^{n-1} (g_m(X_i) - 1) + w(X_n) \right). \tag{6.12}$$

Consequently, since $g_m(\cdot) - 1 \leq \mathbb{1}_{B(\bar{x}, \eta - \frac{1}{m})^c}(\cdot) - 1 = -\mathbb{1}_{B(\bar{x}, \eta - \frac{1}{m})}(\cdot)$, for any $n \in \mathbb{N}$, we have

$$w(x) - \|w\| \leq \hat{\mu}_x \left(-N(B(\bar{x}, \eta - \tfrac{1}{m}), n) \right), \tag{6.13}$$

where $N(A, n)$ denotes the number of visits in set $A \in \mathcal{E}$ in the time interval $[0, n]$. Letting $n \to \infty$, we get $\hat{\mu}_x \left(-N(B(\bar{x}, \eta - \frac{1}{m})) \right) > -\infty$, where $N(A)$ denotes the number of visits in A over the entire time interval. In particular, this implies that for any $x \in E$ we get

$$\mathbb{P}_x \left[N(B(\bar{x}, \eta - \tfrac{1}{m})) < \infty \right] > 0.$$

This property cannot be satisfied if $\nu(B(\bar{x}, \eta - \frac{1}{m})) > 0$, which concludes the proof. □

Corollary 6.9. *Under assumptions of Proposition 6.8, if $\nu(B(\bar{x}, \eta - \frac{1}{m})) > 0$, then $\lambda_m < 1$, and for sufficiently large $n \in \mathbb{N}$ we must have*

$$\sup_{x \in E} \mathbb{P}_x [\tau_B > n] < 1 \quad \text{or} \quad \inf_{x \in E} \mathbb{P}_x [\tau_N \leq n] > 0.$$

Proof. Property $\nu(B(\bar{x}, \eta - \frac{1}{m})) > 0$ combined with Proposition 6.8 implies $\lambda_m < 1$ and it remains to use (6.4). □

Let us now provide a concluding summary of the technical results presented in this section that apply to the general locally compact separable metric space E: The study of the existence of local bounded MPE solutions for reward functions that approximate (scaled) set indicator functions proved to be a useful analytical tool. In particular, we conclude that if the underlying Feller–Markov process has a unique invariant measure, then it is quite natural for the MPE solution to be unbounded, as we expect $\lambda_m = 1$ for some nontransient set and sufficiently large $m \in \mathbb{N}$. Indeed, if the MPE solution is bounded and we have $\lambda_m = 1$, then $\nu(B(\bar{x}, \eta - \frac{1}{m})) = 0$ which means that the corresponding ball must be transient – this points to a relatively compact-like process dynamics.

7. Illustrative examples and counterexamples

In this section, we provide a series of examples that complement the results stated in this paper. In particular, we show a series of counterexamples on discrete state spaces that show situations under which the MPE solution does not always exist. The nonexistence of an MPE solution is justified by using various approaches, often linked to stochastic dominance. We hope that the analyses introduced in this section could be used in the future to help formulate natural necessary and sufficient conditions for the existence of an MPE solution in general spaces.

7.1. Illustration of results stated in the paper

We start with a very simple example, which complements Proposition 2.3, *i.e.* shows that the implication stated in point 1. of Proposition 2.3 cannot be reversed.

Example 7.1 (One-step Mixing does not imply one-step uniform ergodicity)**.** Let $E = \{1,2,3\}$ and consider the transition matrix

$$P := \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

It can be easily checked that the transition kernel linked to matrix P satisfies (A.1) with $\Lambda = \frac{1}{2}$ but does not satisfy (A.2) since zero-diagonal entries imply that the minorization measure $\eta \in \mathcal{P}(E)$ from (A.2) must be zero for all states.

In the next example, we show that the bound stated in Theorem 4.1 is sharp even in a simplified discrete (two state) dynamics, present how the span norm size is linked to the existence of MPE solution, and illustrate why the local contraction property does not necessarily imply the global contraction property.

Example 7.2. (No generic solution to MPE under ergodicity assumption) Let $E := \{x_1, x_2\}$, $\Lambda \in (0,1)$, and let the transition matrix be given by

$$P = \begin{bmatrix} 1 & 0 \\ 1-\Lambda & \Lambda \end{bmatrix}.$$

Note that both (A.1) and (A.2) are satisfied.

First, let us show that the solution to MPE for $g \in C(E)$ exists if and only if $g(x_1) > g(x_2)$ or $\|g\|_{\mathrm{sp}} < -\ln\sqrt{\Lambda}$. Under the assumed dynamics, MPE could be restated as

$$\begin{cases} w(x_1) + \lambda = g(x_1) + w(x_1), \\ w(x_2) + \lambda = g(x_2) + \ln\left[(1-\Lambda)\mathrm{e}^{w(x_1)} + \Lambda \mathrm{e}^{w(x_2)}\right]. \end{cases} \tag{7.1}$$

Easy algebraic check shows that (7.1) is equivalent to

$$\begin{cases} g(x_1) = \lambda, \\ g(x_1) - g(x_2) = \ln\left[(1-\Lambda)\mathrm{e}^{w(x_1)-w(x_2)} + \Lambda\right]. \end{cases} \tag{7.2}$$

From (7.2) we see that if $g \in C(E)$ is such that $g(x_1) - g(x_2) \leq \ln\Lambda$ then there is no solution to MPE due to the monotonicity of the logarithmic function. In other words, the solution to MPE exists if and only if $g(x_1) > g(x_2)$ or

$$\|g\|_{\mathrm{sp}} < -\ln\sqrt{\Lambda}. \tag{7.3}$$

Second, to illustrate how this affects the values of $\|T^n 0\|_{\mathrm{sp}}$ let us set $\Lambda = \frac{1}{2}$, $g(x_1) = 0$ and $g(x_2) = \ln k$, for some $k > 0$. Then, we get

$$\|T^n 0\|_{\mathrm{sp}} = \frac{|T^n 0(x_1) - T^n 0(x_2)|}{2} = \left| 0 + \frac{1}{2} \ln \left[\frac{1}{2} + \frac{1}{2} \sum_{i=1}^{n} \left(\frac{k}{2} \right)^i \right] \right|.$$

Clearly, if $k \geq 2$ then $\|T^n(0)\|_{\mathrm{sp}} \to \infty$, as $n \to \infty$. On the other hand, if $k < 2$, then we get $\sup_{n\in\mathbb{N}} \|T^n(0)\|_{\mathrm{sp}} < \infty$ and the solution exists.

Third, for completeness, let us also show how this interacts with the local contraction property. For any $f_1, f_2 \in C(E)$, such that $\|f_1 - f_2\|_{\mathrm{sp}} \neq 0$, we get

$$\begin{aligned}
\|Tf_1 - Tf_2\|_{\mathrm{sp}} &= \frac{|(f_1(x_1) - f_2(x_1)) - \left(\ln\left[\frac{1}{2}\mathrm{e}^{f_1(x_1)} + \frac{1}{2}\mathrm{e}^{f_1(x_2)}\right] - \ln\left[\frac{1}{2}\mathrm{e}^{f_2(x_1)} + \frac{1}{2}\mathrm{e}^{f_2(x_2)}\right]\right)|}{2} \\
&= \frac{\left|(f_1 - f_2)(x_1) - (f_1 - f_2)(x_2) + \ln\left[\frac{1+\mathrm{e}^{f_2(x_1)-f_2(x_2)}}{1+\mathrm{e}^{f_1(x_1)-f_1(x_2)}}\right]\right|}{2} \\
&= \left| 1 + \frac{\ln\left(\frac{1+\mathrm{e}^{M_2}}{1+\mathrm{e}^{M_1}}\right)}{M_1 - M_2} \right| \cdot \|f_1 - f_2\|_{\mathrm{sp}},
\end{aligned}$$

where $M_1 := f_1(x_1) - f_1(x_2)$, $M_2 := f_2(x_1) - f_2(x_2)$, and $\|f_1 - f_2\|_{\mathrm{sp}} = \frac{1}{2}|M_1 - M_2|$. Consequently, it is easy to show that for $f_1, f_2 \in C(E)$ satisfying $\|f_1\|_{\mathrm{sp}} \leq M$ and $\|f_2\|_{\mathrm{sp}} \leq M$, we get the local contraction property, for example, for the (local) shrinkage constant

$$\alpha(M) = 1/(1 + \mathrm{e}^{-M}).$$

Indeed, without loss of generality, let us assume that $M_1 - M_2 > 0$. Then, it is sufficient to show that

$$-(\alpha(M) + 1)(M_1 - M_2) < \ln\left(\tfrac{1+\mathrm{e}^{M_2}}{1+\mathrm{e}^{M_1}}\right) < (\alpha(M) - 1)(M_1 - M_2). \tag{7.4}$$

Let $h(x) := \ln(1 + \mathrm{e}^x) - (\alpha(M) + 1) \cdot x$. Then, the left inequality in (7.4) could be rewritten as $h(M_1) < h(M_2)$. Now, since for $x \leq M$ we have

$$h'(x) = \frac{\mathrm{e}^x}{1 + \mathrm{e}^x} - (\alpha(M) + 1) \leq \frac{\mathrm{e}^M}{1 + \mathrm{e}^M} - (\alpha(M) + 1) < -\alpha(M) < 0,$$

we get left inequality in (7.4). Next, set $k(x) := \ln(1 + \mathrm{e}^x) + (\alpha(M) - 1) \cdot x$. Then, the right inequality in (7.4) could be rewritten as $k(M_1) > k(M_2)$. Noting that for $x \geq -M$ we have

$$k'(x) = \frac{\mathrm{e}^x}{1 + \mathrm{e}^x} + \alpha(M) - 1 \geq \frac{\mathrm{e}^{-M}}{1 + \mathrm{e}^{-M}} + \alpha(M) - 1 = 0,$$

we conclude the proof of (7.4).

In the next example, we briefly expand the analysis performed in Example 7.2 to the linear setting and show that mixing is sufficient to guarantee the existence of a solution to the additive Poisson equation.

Example 7.3. (On interaction between MPE and APE under ergodicity assumption) Let us assume the dynamics introduced in Example 7.2 and apply it to the additive framework. Namely, we want to show that the mixing property is sufficient to guarantee the existence of a solution $(w,\lambda)\in C(E)\times\mathbb{R}$ to equation

$$w(x)=g(x)-\lambda+\int_E w(y)\mathbb{P}(x,\mathrm{d}y),\quad x\in E. \tag{7.5}$$

Similarly as in (7.1), the equation (7.5) could be stated as

$$\begin{cases} w(x_1)+\lambda=g(x_1)+w(x_1),\\ w(x_2)+\lambda=g(x_2)+\left[(1-\Lambda)w(x_1)+\Lambda w(x_2)\right]. \end{cases}$$

It is easy to check that this equation has a solution for any $g\in C(E)$ and no additional restriction is required. Indeed, the exemplary solution is given by $\lambda:=g(x_1)$, $w(x_1):=(1-d)^{-1}[g(x_1)-g(x_2)]$, and $w(x_2):=0$. This shows that the additive framework requires weaker conditions than the multiplicative framework.

In the next example. we show that strong mixing, that is, condition (A.3), is not a necessary condition.

Example 7.4 (Existence of MPE solution without multi-step strong mixing)**.** Let $E=\mathbb{N}$ and consider a Markov process with transition matrix

$$P:=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots\\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \dots\\ \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & \dots\\ \frac{7}{8} & 0 & 0 & 0 & \frac{1}{8} & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Let us now show that assumption (A.3) is not satisfied but MPE solution exists for any $g\in C(E)$. While $\mathbb{P}\in\mathcal{P}(E)$ corresponding to the transition matrix P satisfies (A.1) with $\Lambda=\frac{1}{2}$, the condition (A.3) is not satisfied, since for any $n\in\mathbb{N}$ we get $\mathbb{P}_n(1,\{n+1\})=0$ and $\mathbb{P}_n(2,\{n+1\})>0$. Fix $g\in C(E)$. For simplicity, assume that $g(1)=0$. It is sufficient to show that $\sup_{n\in\mathbb{N}}\|T^n 0\|_{\mathrm{sp}}<\infty$. Since for any $n\in\mathbb{N}$ and $k\in\mathbb{N}$ we have $T^n0(k)\geq-\|g\|+\ln\frac{1}{2}$, it is sufficient to show that (T^n0) is uniformly bounded from above. For $n\in\mathbb{N}$, we have $T^n0(1)=0$ and consequently, for any $k=2,3,\ldots$, we get

$$\begin{aligned} T^n0(k)&=g(k)+\ln\left[(1-2^{-k})+2^{-k}\mathrm{e}^{T^{n-1}0(k+1)}\right]\\ &\leq\|g\|+\ln\left[1+2^{-1}\mathrm{e}^{T^{n-1}0(k+1)}\right]\\ &\leq\|g\|+\ln\left[1+2^{-1}\mathrm{e}^{\|g\|+\ln[1+2^{-2}\mathrm{e}^{T^{n-2}0(k+2)}]}\right]\\ &\leq\|g\|+\ln\left[1+2^{-1}\mathrm{e}^{\|g\|}+2^{-3}\mathrm{e}^{\|g\|}\mathrm{e}^{T^{n-2}0(k+2)}\right]\\ &\leq\|g\|+\ln\left[1+2^{-1}\mathrm{e}^{\|g\|}+2^{-3}\mathrm{e}^{2\|g\|}+2^{-6}\mathrm{e}^{2\|g\|}\mathrm{e}^{T^{n-3}0(k+3)}\right]\\ &\leq\|g\|+\ln\left[\sum_{i=0}^{\infty}2^{-i(i+1)/2}\mathrm{e}^{i\|g\|}\right]\\ &\leq\|g\|+\ln\left[\sum_{i=0}^{\infty}\mathrm{e}^{i[\|g\|-(i+1)/2\cdot\ln 2]}\right]. \end{aligned}$$

Noting that the sequence $(\|g\|-(i+1)/2\cdot\ln 2)_{i\in\mathbb{N}}$ does not depend on $n\in\mathbb{N}$, is decreasing, and for sufficiently large $i\in\mathbb{N}$ we have $(\|g\|-(i+1)/2\cdot\ln 2)<0$, we conclude that $\sup_{n\in\mathbb{N}}\|T^n\|<\infty$, which implies

$\sup_{n\in\mathbb{N}} \|T^n\|_{\mathrm{sp}} < \infty$ due to (3.1). This implies the existence of an MPE solution for g due to Theorem 3.1 and the Banach fixed-point theorem.

7.2. Counterexamples

In this section, we present a series of counterexamples on discrete state spaces, that is, we consider various transition matrices for which mixing is satisfied but the MPE solution does not exist for any $g \in C(E)$.

First, we show that even a relatively fast (geometric) process entry into the invariant measure support is not sufficient to guarantee the existence of an MPE solution under (A.1). In particular, even if the invariant measure has full support, the solution to MPE might still not exist.

Example 7.5 (No bounded MPE solution under full support invariant measure)**.** Let $E = \mathbb{N}$ and consider a Markov process with transition matrix

$$P := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \dots \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \end{bmatrix}.$$

Let us show that there exists $g \in C(E)$ for which bounded MPE solution does not exist; note that (A.1) is satisfied with $\Lambda = \frac{1}{2}$. Let $\epsilon > 0$ and let $g \in C(E)$ be given by

$$g(i) := \begin{cases} 0, & \text{if } i \in \bigcup_{k\in\mathbb{N}} [2^k, 2^k + 2^{k-1}) \\ 2(\ln 2 + \epsilon), & \text{if } i \in \bigcup_{k\in\mathbb{N}} [2^k + 2^{k-1}, 2^{k+1}) \end{cases}.$$

Let us assume that bounded MPE solution, say $(w, \lambda) \in C(E) \times \mathbb{R}$, exists; without loss of generality we assume that $w(1) = 0$; recall that the solution is defined up to an additive constant, so that we can apply any additive shift to w. Let $(X_i)_{i\in\mathbb{N}}$ denote the underlying Markov process starting at $X_0 = 1$; $(X_i)_{i\in\mathbb{N}}$ takes values in $\mathbb{N}$ and its dynamics is defined *via* the transition matrix P. Noting that for any $n \in \mathbb{N}$ we get

$$\sum_{i=0}^{n} g(X_i) \leq \frac{n \cdot 2(\ln 2 + \epsilon)}{2},$$

we conclude that $\lambda \leq \ln 2 + \epsilon$. Moreover, recalling MPE, we know that for any $i \in \mathbb{N}$ we get

$$\begin{aligned} w(i) - w(i+1) &= g(i) - \lambda + \ln\left[\tfrac{1}{2}\mathrm{e}^{0} + \tfrac{1}{2}\mathrm{e}^{w(i+1)}\right] - w(i+1) \\ &\geq g(i) - (\ln 2 + \epsilon) - \ln 2 + \ln\left[1 + \mathrm{e}^{w(i+1)}\right] - w(i+1) \\ &\geq g(i) - (\epsilon + 2\ln 2). \end{aligned}$$

In particular, for any $k \in \mathbb{N}$, summing up, we get

$$\begin{aligned} w(2^k + 2^{k-1}) - w(2^{k+1}) &\geq \left[\sum_{i=2^k+2^{k-1}}^{2^{k+1}-1} g(i)\right] - (2^{k+1} - 2^k - 2^{k-1})(\epsilon + 2\ln 2) \\ &= (2^{k+1} - 2^k - 2^{k-1})(2(\ln 2 + \epsilon) - \epsilon - 2\ln 2) \\ &\geq 2^{k-1}\epsilon. \end{aligned}$$

This contradicts the assumption that $w \in C(E)$. In fact, one could also show nonexistence by iterating MPE. Indeed, following similar reasoning as in the proof of Proposition 2.1, for any $n \in \mathbb{N}$, we get

$$\begin{aligned}
\lambda &\leq \tfrac{1}{n}\hat{\mu}_1\left(\sum_{i=0}^{n-1} g_k(X_i)\right) + \tfrac{1}{n}\|w\|_{\mathrm{sp}} \\
&\leq \epsilon + \ln 2 + \tfrac{1}{n}\|w\|_{\mathrm{sp}}, \\
\lambda &\geq \tfrac{1}{n}\hat{\mu}_{2^n+2^{n-1}}\left(\sum_{i=0}^{n-1} g_k(X_i)\right) - \tfrac{1}{n}\|w\|_{\mathrm{sp}} \\
&\geq \tfrac{1}{n}\ln\left[\tfrac{1}{2^{n-1}}\mathrm{e}^{n2(\ln 2+\epsilon)}\right] - \tfrac{1}{n}\|w\|_{\mathrm{sp}} \\
&\geq \tfrac{1}{n}(-(n-1)\ln 2 + n2(\ln 2+\epsilon)) - \tfrac{1}{n}\|w\|_{\mathrm{sp}} \\
&= 2\epsilon + \tfrac{n+1}{n}\ln 2 - \tfrac{1}{n}\|w\|_{\mathrm{sp}}.
\end{aligned}$$

Taking the limit, this leads to a contradiction, as we get $2\epsilon \leq \lambda - \ln 2 \leq \epsilon$.

Now, let us build on Example 7.5 and present a modification of this example in which we permit a one-step state recurrence. In the next example, we show how to prove the nonexistence of an MPE bounded solution by introducing two reward functions and use stochastic dominance to analyse the relation between them. For brevity, given random variables X and Y, we use $X \preceq_{\tilde{\mathbb{P}}} Y$ to denote the first-order (weak) stochastic dominance of Y over X in $\tilde{\mathbb{P}}$, that is, the condition $\tilde{\mathbb{P}}[X \geq t] \leq \tilde{\mathbb{P}}[Y \geq t]$, for $t \in \mathbb{R}$.

Example 7.6 (No bounded MPE solution under full support invariant measure and one step recurrent states)**.** Let $E = \mathbb{N}_+$ and consider a Markov process with transition matrix

$$P := \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \dots \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \dots \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Then, there exists $g \in C(E)$ for which bounded MPE solution does not exist; note that (A.1) is satisfied with $\Lambda = \frac{1}{2}$. Let us assume that the MPE solution for P exists for any $g \in C(E)$. For a fixed $k \in \mathbb{R}_+$, we define two functions $g_1, g_2 \in C(E)$ by setting

$$g_1(i) := \begin{cases} 0, & \text{if } i \in 2\mathbb{N}+1 \\ k, & \text{if } i \in 2\mathbb{N}, \end{cases}, \quad g_2(i) := \begin{cases} 0, & \text{if } i \in \bigcup_{n\in\mathbb{N}}[2^n, 2^n+2^{n-1}) \\ k, & \text{if } i \in \bigcup_{n\in\mathbb{N}}[2^n+2^{n-1}, 2^{n+1}) \end{cases}.$$

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $w_1, w_2 \in C(E)$ be solutions to MPE for g_1 and g_2, respectively. Without loss of generality, assume that $w_1(1) = w_2(1) = 0$. Let $S_n^1 := \sum_{i=1}^{n-1} g_1(X_i)$ and $S_n^2 := \sum_{i=1}^{n-1} g_2(X_i)$, where $(X_i)_{i\in\mathbb{N}}$ denotes the underlying Markov process with a fixed starting point $X_0 := x_0 \in E$. Assuming $x_0 = 1$, we have $S_n^1 \preceq_{\mathbb{P}_{x_0}} S_n^2$, for $n \in \mathbb{N}$. Now, since

$$\lambda_1 = \lim_{n\to\infty} \tfrac{1}{n}\mu_{x_0}\left(S_n^1\right) \quad \text{and} \quad \lambda_2 = \lim_{n\to\infty} \tfrac{1}{n}\mu_{x_0}\left(S_n^2\right),$$

we conclude that

$$\lambda_1 \geq \lambda_2. \tag{7.6}$$

Furthermore, by analysing the structure of the matrix P, it is easy to check that for $i \in \mathbb{N} \setminus \{1\}$ we get $w_1(i) = w_1(i+2)$. This allows us to explicitly calculate the value of λ_1. Indeed, for any even $i > 1$, from MPE, we get

$$\begin{aligned} w_1(i) &= g_1(i) - \lambda_1 + \ln\left(\frac{1}{2}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^{w_1(i)} + \frac{1}{4}\mathrm{e}^{w_1(i+1)}\right) \\ &= 0 - \lambda_1 + \ln\left(\frac{1}{2}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^{w_1(i)} + \frac{1}{4}\mathrm{e}^{w_1(i+1)}\right), \\ w_1(i+1) &= g_1(i+1) - \lambda_1 + \ln\left(\frac{1}{2}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^{w_1(i+1)} + \frac{1}{4}\mathrm{e}^{w_1(i+2)}\right) \\ &= k - \lambda_1 + \ln\left(\frac{1}{2}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^{w_1(i+1)} + \frac{1}{4}\mathrm{e}^{w_1(i)}\right), \end{aligned}$$

which implies $w_1(i) - w_1(i+1) = -k$. Consequently, for $i = 2$, we get

$$w_1(2) = -\lambda_1 + \ln\left(\frac{1}{2}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^{w_1(2)} + \frac{1}{4}\mathrm{e}^{w_1(2)+k}\right),$$

which implies $\lambda_1 = \ln\left(\frac{1}{2}\mathrm{e}^{-w_1(2)} + \frac{1}{4}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^k\right)$. On the other hand, using MPE for $i = 1$, we also know that $\lambda_1 = \ln\left(\frac{3}{4}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^{w_1(2)}\right)$, which implies

$$\tfrac{3}{4}\mathrm{e}^0 + \tfrac{1}{4}\mathrm{e}^{w_1(2)} = \tfrac{1}{2}\mathrm{e}^{-w_1(2)} + \tfrac{1}{4}\mathrm{e}^0 + \tfrac{1}{4}\mathrm{e}^k.$$

Thus, simple algebraic calculations yield

$$\mathrm{e}^{w_1(2)} = \tfrac{1}{2}\left(\sqrt{12 + \mathrm{e}^{2k} - 4\mathrm{e}^k} + \mathrm{e}^k - 2\right)$$

and consequently we get

$$\lambda_1 = \ln\left(\frac{1}{2} + \frac{1}{8}\left(\sqrt{12 + \mathrm{e}^{2k} - 4\mathrm{e}^k} + \mathrm{e}^k\right)\right). \tag{7.7}$$

Now, let us show that solution for g_2 does not exists. From (7.6) and (7.7) we know that for sufficiently large $k > 0$ (used in the definition of g_1 and g_2), and the corresponding MPE solutions, we get

$$\lambda_2 + \ln 2 - k \leq \lambda_1 + \ln 2 - k < \ln\left(\mathrm{e}^{-k} + \frac{1}{4}\left(\mathrm{e}^{-k}\sqrt{12 + \mathrm{e}^{2k}} + 1\right)\right) < 0. \tag{7.8}$$

Recalling that $w_2(1) = 0$ and using the MPE for $i \geq 1$, we get

$$\begin{aligned} w_2(i) &= g_2(i) - \lambda_2 + \ln\left(\frac{1}{2}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^{w_2(i)} + \frac{1}{4}\mathrm{e}^{w_2(i+1)}\right) \\ &\geq g_2(i) - \lambda_2 + \ln\left(\frac{1}{4}\mathrm{e}^{w_2(i)} + \frac{1}{4}\mathrm{e}^{w_2(i+1)}\right) \\ &\geq g_2(i) - \lambda_2 - \ln 2 + \ln\left(\frac{1}{2}\mathrm{e}^{w_2(i)} + \frac{1}{2}\mathrm{e}^{w_2(i+1)}\right) \\ &\geq g_2(i) - \lambda_2 - \ln 2 + \frac{1}{2}w_2(i) + \frac{1}{2}w_2(i+1) \end{aligned}$$

which implies

$$w_2(i) - w_2(i+1) \geq 2(g_2(i) - \lambda_2 - \ln 2) \quad \text{for } i \geq 1. \tag{7.9}$$

In particular, recalling the definition of g_2 and using (7.9), for any $n \in \mathbb{N}$, we get

$$w_2(2^{n+1}) - w_2(2^n + 2^{n-1}) \geq 2^{n-1} \cdot 2(k - \lambda_2 - \ln 2).$$

Now, recalling the fact that $k - \lambda_2 - \ln 2 > 0$ due to (7.8) and setting $n \to \infty$, we get $\|w_2\|_{\text{sp}} = \infty$, which contradicts the assumption that the MPE solution for g_2 is bounded.

In the previous two counterexamples, we assumed that there are states which cannot be connected (with positive probability) in a finite number of steps and that there is a single path that goes to infinity and we can stay on it with geometric probability. In the next example, we show that those conditions could be relaxed, *i.e.* we can allow two-step connection (with positive probability) and relax the assumption about the existence of an infinite path, *i.e.* force the process to go back to a prefixed state in a finite number of steps, from any state.

Example 7.7 (No bounded MPE solution under full support invariant measure with disjoint geometric dynamics)**.** Let $E = \mathbb{N}_+$ and consider a Markov process with transition matrix

$$P := \begin{bmatrix} 1/2+p & \frac{e^{-2}}{2} & \frac{e^{-2}}{2} & \frac{e^{-4}}{3} & \frac{e^{-4}}{3} & \frac{e^{-4}}{3} & \frac{e^{-8}}{4} & \frac{e^{-8}}{4} & \frac{e^{-8}}{4} & \frac{e^{-8}}{4} & \frac{e^{-16}}{5} & \frac{e^{-16}}{5} & \dots \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & \dots \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

where $p := \frac{1}{2} - \sum_{i=1}^{\infty} e^{-2^i}$. Let us show that there exists $g \in C(E)$ for which a bounded MPE solution does not exist. Let us divide the state space into disjoint sets $A_0 = \{1\}$, $A_1 := \{2,3\}$, $A_2 := \{4,5,6\}$, $A_3 := \{7,8,9,10\}$, *etc.* Using convention $A_i := \{a_1^i, \ldots, a_{i+1}^i\}$, $i \in \mathbb{N}$, the transition kernel is given by the following transition probabilities

$$\mathbb{P}(a_j^i, 1) = \begin{cases} \frac{1}{2}, & i \in \mathbb{N}_+,\, j = 1, \ldots, i \\ 1, & i \in \mathbb{N}_+,\, j = i+1 \end{cases}, \quad \mathbb{P}(a_j^i, a_j^i + 1) = \begin{cases} \frac{1}{2} & i \in \mathbb{N}_+,\, j = 1, \ldots, i \\ 0 & i \in \mathbb{N}_+,\, j = i+1 \end{cases}.$$

$$\mathbb{P}(1, a_j^i) = \begin{cases} \frac{1}{2} + p & i = 0, j = 1 \\ \frac{1}{i+1} e^{-2^i} & i \neq 0,\, j = 1, 2, \ldots, i+1 \end{cases};$$

note that the stated values fully allocate non-zero probability transitions. Next, let $g \in C(E)$ be given by

$$g(a_j^i) := k(1 - \mathbb{1}_{\{j=i+1\}}), \quad i \in \mathbb{N},\, j = 1, \ldots, i+1,$$

where $k \in \mathbb{R}_+$ is a fixed constant. Note that g could be related to a vector in $\mathbb{N}$ that is simply given by

$$(0, k, 0, k, k, 0, k, k, k, 0, k, k, k, k, 0, \ldots).$$

For simplicity, we also use p_i to denote the probability of entering the set A_i from state 1, *i.e.* we set

$$p_i := \mathbb{P}(1, A_i) = \sum_{j=1}^{i+1} \mathrm{e}^{-2^i}/(i+1) = \mathrm{e}^{-2^i}.$$

Let us assume that there is a bounded MPE solution for g and reach a contradicition; as usual, we use $\lambda \in \mathbb{R}$ and $w \in C(E)$ to refer to the MPE solution. First, let us show that

$$\lambda \geq k + \ln \tfrac{1}{2}. \tag{7.10}$$

For any $n \in \mathbb{N}$, noting that g is nonnegative, we have

$$n\lambda \geq \hat{\mu}_{a_1^n} \left(\sum_{i=0}^{n-1} g(X_i) \right) - \|w\| \geq \ln \left[\frac{1}{2^n} \mathrm{e}^{nk} \right] - \|w\| = nk + n \ln \tfrac{1}{2} - \|w\|.$$

Dividing both sides by n and letting $n \to \infty$ we get (7.10). Second, let us contradict the existence of a bounded MPE solution. It is sufficient to show that for there exists $k \in \mathbb{R}_+$ such that

$$\lambda < k + \ln \tfrac{1}{2}. \tag{7.11}$$

For simplicity, fix $k := 2$. To show (7.11), let us first define a sequence of discrete i.i.d. random variables $(Z_i)_{i \in \mathbb{N}}$, with a probability mass function given by

$$\mathbb{P}_1[Z_1 = 0] = 1 - \textstyle\sum_{i=1}^{\infty} p_i, \quad \mathbb{P}_1[Z_1 = ik] = p_i, \quad i \in \mathbb{N}.$$

Assuming that the underlying Markov process $(X_i)_{i \in \mathbb{N}}$ has a starting point $X_0 := 1$, one could show that for any $n \in \mathbb{N}$, we get

$$\textstyle\sum_{i=0}^{n} g(X_i) \preceq_{\mathbb{P}_1} \sum_{i=1}^{n} Z_i, \tag{7.12}$$

For completeness, we outline the proof of (7.12). Fix $n \in \mathbb{N}$ and, for $j = 1, \ldots, n$, let τ_j denote the jth time the process $(X_i)_{i \in \mathbb{N}}$ re-enters its initial state $X_0 = 1$. Then, recalling that $g(X_0) = 0$ and g is non-negative we have

$$\textstyle\sum_{i=0}^{n} g(X_i) \leq \sum_{i=0}^{\tau_n} g(X_i) \leq \sum_{i=1}^{n} k(\tau_i - \tau_{i-1} - 1),$$

with $\tau_0 := 0$. Consequently, since the sequence $(\tau_i - \tau_{i-1} - 1)_{i=1}^{n}$ is i.i.d. due to the strong Markov property, it is sufficient to show that $\tau_1 - 1 \preceq Z_1/k$. This is indeed the case since for any $t \in \mathbb{N}$ we get

$$\mathbb{P}_1[\tau_1 - 1 \geq t] \leq \sum_{j=t}^{\infty} \mathbb{P}(1, A_j) = \sum_{j=t}^{\infty} p_j = \sum_{j=t}^{\infty} \mathbb{P}_1[Z_1 = jk] = \mathbb{P}_1\left[Z_1/k \geq t\right].$$

Let us provide an intuition behind the construction of the dominating process: the dynamics of $(V_j)_{j=0}^{n}$, where $V_j := \sum_{i=0}^{j} g(X_i)$, as a function of the process $(X_i)_{i \in \mathbb{N}}$, could be described as follows: (1) the process starts in $X_0 = 1$ with value $V_0 = 0$; (2) in the first time step, the process remains in state 1 (no increase) or enters the subset A_i; (3) if the process is in A_i then the process re-enters the state 1 in the (random) number of steps bounded from above by i and the total aggregated reward before re-entering state 1 is bounded by ik; (4) when the process re-enters state 1, the dynamics is reset. Now, the sequence of sums of (Z_i) could be linked to an immediate gratification sequence, that is, we assume that if the process enters A_i (from state 1) then we are paid an immediate full gratification ik in one step and reset the state instantly. This is better compared to the previous case since we assume that we are paid immediately the maximal possible reward in a single step, and we profit both from getting maximal reward with probability one and from increasing the number of times the process 'resets' itself (to 1) after getting the reward.

Next, to show why this helps to prove (7.11) let us note that the entropy of Z_1 is bounded from above by $2+\ln\frac{1}{2}-\epsilon$, for a fixed small $\epsilon>0$, since

$$\hat{\mu}_1(Z_1)\leq \ln\left[1\mathrm{e}^0+\sum_{i=1}^{\infty}p_i\mathrm{e}^{ik}\right]=\ln\left[1+\sum_{i=1}^{\infty}\mathrm{e}^{ik-2^i}\right]\leq \ln(3.15)<2+\ln\tfrac{1}{2}-\epsilon. \tag{7.13}$$

Now, recalling that entropic utility is monotone, law invariant, and additive with respect to independent random variables (see [24]), using (7.12) and (7.13), we get

$$\frac{1}{n}\hat{\mu}_1\left(\sum_{i=0}^{n-1}g(X_i)\right)\leq \frac{1}{n}\hat{\mu}_1\left(\sum_{i=1}^{n}Z_i\right)=\frac{n}{n}\hat{\mu}_1(Z_1)<2+\ln\tfrac{1}{2}-\epsilon,\quad n\in\mathbb{N}.$$

Thus, letting $n\to\infty$, we get

$$\lambda\leq \lim_{n\to\infty}\left[\frac{1}{n}\hat{\mu}_1\left(\sum_{i=0}^{n-1}g(X_i)\right)+\frac{\|w\|}{n}\right]<2+\ln\tfrac{1}{2},$$

which proves (7.11) and leads to a contradiction with the existence of a bounded MPE solution.

Finally, let us illustrate that local geometric dynamics, that is, a positive lower bound imposed on diagonal values in the transition matrix, is sufficient to guarantee the nonexistence of an MPE solution for some $g\in C(E)$.

Example 7.8 (No bounded MPE solution under full support invariant measure with local geometric dynamics)**.** Let $E=\mathbb{N}_+$ and consider a Markov process with transition matrix

$$P:=\begin{bmatrix}\frac{1}{2} & a_1 & a_2 & a_3 & a_4 & \dots\\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots\\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \dots\\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots\end{bmatrix},$$

where $(a_i)_{i=1}^{\infty}$ is such that $a_i>0$, $i\in\mathbb{N}$, and $\sum_{i=1}^{\infty}a_i=\frac{1}{2}$. Let us provide a comment on why there exists $g\in C(E)$ for which a bounded MPE solution does not exist. The proof follows directly from the results presented in Section 9 in [10] and is omitted for brevity. To see the link, observe that there exists a decreasing subsequence $(a_{i_k})_{k\in\mathbb{N}}$ for which we have $a_{i_k}<\zeta(3)/(1+k)^3$, where $\zeta(3):=\sum_{i=1}^{\infty}1/i^3$ is the Apéry's constant and set

$$g(x)=\begin{cases}2\left(\ln 2+\ln\left(1-\frac{1}{k+2}\right)\right) & \text{if } x=a_{i_k} \text{ and } k\geq N,\\ 0 & \text{otherwise,}\end{cases}$$

for some large constant $N \in \mathbb{N}$; see Proposition 9.1 and Proposition 9.2 in [10] for details. Then, one can show that constant λ in the MPE must depend on the state, *i.e.*, for state 1 it must be strictly smaller than for a sufficiently large state, which would lead to contradiction, see Proposition 9.3 and Proposition 9.4 in [10] for details.

From the examples introduced in this section, one can deduce that the Markov process should propagate in a rather compact and uniform manner. The core idea behind all the denumerable examples introduced in this section was to construct a process for which: (1) when starting from the state 1, it is hard to reach the states associated with large integers; (2) when starting far away from 1, it is possible to stay far away from 1 with some geometric probability. Hopefully, this intuition could be expanded to provide in the future full MPE solution existence characterisation results, with conditions possibly linked to geometric dynamics as the one presented in Proposition 5.2. For now, the full characterisation problem remains open.

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