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EXISTENCE OF SOLUTIONS FOR GRADIENT COUPLED DIRICHLET SYSTEMS

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A Italo, per i suoi 3/4 di secolo.

Abstract. In this paper, we prove existence of weak solutions in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ for the gradient coupled Dirichlet system

$$\begin{cases} u\in W_0^{1,2}(\Omega): -\operatorname{div}(M(x)\nabla u)+u+a(x)\,\nabla u\cdot\nabla\psi=f(x),\\ \psi\in W_0^{1,2}(\Omega): -\operatorname{div}(M(x)\nabla\psi)+\psi+a(x)\,\nabla u\cdot\nabla\psi=g(x).\end{cases}$$

We also prove that if $f(x)$, $g(x)\ge 0$ (of course $\not\equiv 0$ a.e.), then $u(x)$, $\psi(x)\ge 0$ and the sets $\{u=0\}$ and $\{\psi=0\}$ have zero Lebesgue measure.

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1. Introduction

In this paper we study existence and positivity of weak solutions of system (1.1) below.

The starting point of our research was the lecture "A short presentation of some results on the weak maximum principle" by Italo Capuzzo Dolcetta at Sapienza (november 2018), mainly concerning the paper [1], where maximum principle for systems of the type

$$\Delta u_i+\sum_{j=1}^{n}B_{ij}\nabla u_j+C_i\,u_i\ge 0,$$

is studied.

In this paper, we consider the following Dirichlet problem:

$$\begin{cases} u\in W_0^{1,2}(\Omega): -\operatorname{div}(M(x)\nabla u)+u+a(x)\,\nabla u\cdot\nabla\psi=f(x),\\ \psi\in W_0^{1,2}(\Omega): -\operatorname{div}(M(x)\nabla\psi)+\psi+a(x)\,\nabla u\cdot\nabla\psi=g(x).\end{cases}\tag{1.1}$$

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Here Ω is a bounded, open subset of $\mathbb{R}^N$, $N > 2$, $M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix-valued function such that

$$M(x)\,\xi\cdot\xi \geq \alpha\,|\xi|^2, \quad |M(x)| \leq \beta, \qquad \forall\,\xi \in \mathbb{R}^N, \tag{1.2}$$

$$f(x) \neq g(x) \in L^\infty(\Omega), \tag{1.3}$$

and

$$a(x) \in L^\infty(\Omega). \tag{1.4}$$

Note that each equation of system (1.1) is linear; however, the whole system is nonlinear due to the presence of the coupling first order term $a(x)\,\nabla u\cdot\nabla\psi$.

Under the above assumptions we prove, in Theorem 2.1, the existence of bounded weak solutions u and ψ. Moreover, a "weak" maximum principle is proved: if $\mathrm{m}(G)$ is the Lebesgue measure of a set G, and if $0 \not\equiv f(x) \geq 0$, then $u \geq 0$ and $\mathrm{m}(\{u = 0\}) = 0$; if $0 \not\equiv g(x) \geq 0$, then $\psi \geq 0$ and $\mathrm{m}(\{\psi = 0\}) = 0$. In Theorem 3.1 we prove, for unbounded data f and g, the existence of unbounded weak solutions u and ψ under a positivity assumption on $a(x)$ and $g(x)$ (or $f(x)$).

To the best of the authors' knowledge, both the existence of weak solutions in $W_0^{1,2}(\Omega)$ and the "weak" maximum principle are new. Some related results can be found in [2], where strict positivity of solutions is proved for a coupled cooperative system of the form

$$-\Delta u_i = \sum_{j=1}^{N} B_{ij}(x)\,\nabla u_j + f(x).$$

The plan of the paper is as follows: in Sections 2 and 3 we prove existence of solutions for system (1.1) in the case of bounded or unbounded data (see Thms. 2.1 and 3.1). In order to prove these results, we will use an existence theorem for a quasilinear elliptic equation having a term with quadratic growth with respect to the gradient, and a first order drift term (see Thm. 2.2). In Section 4 we prove an existence result for a system similar to (1.1), with lower order terms of the form $b(x)\,u$ and $b(x)\,\psi$, with $b(x)$ in $L^1(\Omega)$, under the assumption that $|f(x)| \leq Q\,b(x)$ and $|g(x)| \leq Q\,b(x)$ for some $Q > 0$. In the final Section 5, we prove an existence result for a system of m equations which generalizes (1.1).

2. Existence of bounded solutions

In this section, we are going to prove the following result.

Theorem 2.1. *Suppose that (1.2) holds, let $a(x)$ be such that (1.4) holds, and let $f(x)$ and $g(x)$ be such that (1.3) holds. Then there exist weak solutions u and ψ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ of system (1.1), that is*

$$\int_\Omega M(x)\nabla u\cdot\nabla\varphi + \int_\Omega u\,\varphi + \int_\Omega a(x)\,\nabla u\cdot\nabla\psi\,\varphi = \int_\Omega f(x)\,\varphi, \tag{2.1}$$

and

$$\int_\Omega M(x)\nabla\psi\cdot\nabla v + \int_\Omega \psi\,v + \int_\Omega a(x)\,\nabla u\cdot\nabla\psi\,v = \int_\Omega g(x)\,v, \tag{2.2}$$

for every v *and* φ *in* $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. *Furthermore, if* $0 \not\equiv f(x) \geq 0$*, then* $u \geq 0$ *and* $\mathrm{m}(\{u = 0\}) = 0$*, and if* $0 \not\equiv g(x) \geq 0$*, then* $\psi \geq 0$ *and* $\mathrm{m}(\{\psi = 0\}) = 0$.

The proof of Theorem 2.1 is based on the following existence result for a quasilinear elliptic equation having a term with quadratic growth with respect to the gradient, and a first order drift term depending on an $(L^2(\Omega))^N$ vector field $E(x)$.

Theorem 2.2. *Let* $h(x)$ *be a function in* $L^\infty(\Omega)$*, let* $E(x)$ *be a vector field in* $(L^2(\Omega))^N$*, and let* $a(x)$ *be a function such that* (1.4) *holds. Then there exists a weak solution* ψ *in* $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *of the Dirichlet problem*

$$\begin{cases} -\operatorname{div}(M(x)\nabla\psi) + \psi + a(x)\,|\nabla\psi|^2 = E(x)\cdot\nabla\psi + h(x), & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

that is

$$\int_\Omega M(x)\nabla\psi\cdot\nabla v + \int_\Omega \psi\, v + \int_\Omega a(x)\,|\nabla\psi|^2\, v = \int_\Omega E(x)\cdot\nabla\psi\, v + \int_\Omega h(x)\, v, \tag{2.4}$$

for every v *in* $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. *Furthermore,*

$$\|\psi\|_{L^\infty(\Omega)} \leq \|h\|_{L^\infty(\Omega)}, \tag{2.5}$$

and

$$0 \not\equiv h(x) \geq 0 \text{ implies } \psi \geq 0 \text{ and } \mathrm{m}(\{\psi = 0\}) = 0. \tag{2.6}$$

Remark 2.3. We observe that the above theorem can be seen as a unified presentation of the results of [3] and [4] (if $a(x) = 0$) and [5] (if $E(x) = 0$). We remark that for both these results, as well as for that of Theorem 2.2, the presence of the lower order term $+\psi$ is crucial.

Proof. We begin by using Schauder's theorem to prove existence of a solution for a problem which approximates (2.3). Let n in $\mathbb{N}$, let Φ in $W_0^{1,2}(\Omega)$, let $E_n(x) = \frac{E(x)}{1+\frac{1}{n}|E(x)|}$, and let Ψ in $W_0^{1,2}(\Omega)$ be the weak solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla\Psi) + \Psi = \Big(E_n(x) - \frac{a(x)\,\nabla\Phi}{1+\frac{1}{n}|\nabla\Phi|}\Big)\cdot\nabla\Psi + h(x) & \text{in } \Omega, \\ \Psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

that is

$$\int_\Omega M(x)\nabla\Psi\cdot\nabla v + \int_\Omega \Psi\, v = \int_\Omega \Big(E_n(x) - \frac{a(x)\,\nabla\Phi}{1+\frac{1}{n}|\nabla\Phi|}\Big)\cdot\nabla\Psi\, v + \int_\Omega h(x)\, v, \tag{2.8}$$

for every v in $W_0^{1,2}(\Omega)$. Such a solution exists thanks to the results of [3] (see also [4]) since the vector field

$$F(x) = E_n(x) - \frac{a(x)\,\nabla\Phi}{1+\frac{1}{n}|\nabla\Phi|},$$

belongs to $(L^2(\Omega))^N$. In the same paper it is proved that

$$\|\Psi\|_{L^\infty(\Omega)} \leq \|h\|_{L^\infty(\Omega)}. \tag{2.9}$$

Choosing Ψ as test function in (2.8), and using (1.2) and (2.9), one also has that

$$\|\Psi\|_{W_0^{1,2}(\Omega)} \leq C(\|E\|_{L^2(\Omega)}, \|h\|_{L^\infty(\Omega)}, \|a\|_{L^\infty(\Omega)}, \sqrt{n}) \stackrel{\text{def}}{=} R. \tag{2.10}$$

Thus, the ball $B_R(0)$ of $W_0^{1,2}(\Omega)$ is invariant for the map $S : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ defined by $S(\Phi) = \Psi$. We are going to prove that S is both continuous and compact, so that it will have a fixed point by Schauder's theorem.

Let $\{\Phi_m\}$ be a sequence bounded in $W_0^{1,2}(\Omega)$, and let $\Psi_m = S(\Phi_m)$ be the solution of (2.7) with $\Phi = \Phi_m$. Thanks to (2.9) and (2.10), the sequence $\{\Psi_m\}$ is bounded in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Note that Ψ_m is a weak solution of

$$\Psi_m \in W_0^{1,2}(\Omega) : -\operatorname{div}(M(x)\nabla\Psi_m) = F_m(x),$$

where

$$F_m(x) = \Big(E_n(x) - \frac{a(x)\,\nabla\Phi_m}{1 + \frac{1}{n}|\nabla\Phi_m|}\Big) \cdot \nabla\Psi_m + h(x) - \Psi_m.$$

Since $\{F_m\}$ is a bounded sequence in $L^2(\Omega)$, it is precompact in the dual of $W_0^{1,2}(\Omega)$. Thus Ψ_m strongly converges (up to subsequences) in $W_0^{1,2}(\Omega)$, which implies that S is compact.

If the sequence $\{\Phi_m\}$ is strongly convergent in $W_0^{1,2}(\Omega)$ to some function Φ_0, it is easy to prove that the strong limit Ψ_0 of the sequence $\{\Psi_m = S(\Phi_m)\}$ is such that $\Phi_0 = S(\Phi_0)$, so that S is also continuous.

Then, by Schauder's theorem, there exists a fixed point of S: a function ψ_n belonging to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that $S(\psi_n) = \psi_n$. Clearly, ψ_n is a weak solution of

$$\begin{cases} -\operatorname{div}(M(x)\nabla\psi_n) + \psi_n + \frac{a(x)\,|\nabla\psi_n|^2}{1+\frac{1}{n}|\nabla\psi_n|} = E_n(x)\cdot\nabla\psi_n + h(x) & \text{in } \Omega, \\ \psi_n = 0 & \text{in } \partial\Omega, \end{cases} \tag{2.11}$$

that is

$$\int_\Omega M(x)\nabla\psi_n \cdot \nabla v + \int_\Omega \psi_n\, v + \int_\Omega \frac{a(x)\,|\nabla\psi_n|^2}{1 + \frac{1}{n}|\nabla\psi_n|}\, v = \int_\Omega E_n(x)\cdot\nabla\psi_n\, v + \int_\Omega h(x)\, v, \tag{2.12}$$

for every v in $W_0^{1,2}(\Omega)$. Thanks to (2.9), we have that

$$\|\psi_n\|_{L^\infty(\Omega)} \leq \|h\|_{L^\infty(\Omega)}. \tag{2.13}$$

We now follow [6]: let $\lambda > 0$ to be chosen later, define $\varphi(s) = (\mathrm{e}^{\lambda\,|s|} - 1)\,\operatorname{sgn}(s)$, and choose $v = \varphi(\psi_n)$ as test function in (2.12); such a choice is admissible since every ψ_n belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. We obtain

$$\begin{aligned} \int_\Omega M(x)\nabla\psi_n \cdot \nabla\psi_n\, \varphi'(\psi_n) + \int_\Omega \psi_n\, \varphi(\psi_n) + \int_\Omega \frac{a(x)\,|\nabla\psi_n|^2}{1 + \frac{1}{n}|\nabla\psi_n|}\,\varphi(\psi_n) \\ = \int_\Omega E_n(x)\cdot\nabla\psi_n\,\varphi(\psi_n) + \int_\Omega h(x)\,\varphi(\psi_n). \end{aligned}$$

Recalling (1.2), using that $|\varphi(\psi_n)| \leq \mathrm{e}^{\lambda\,\|h\|_{L^\infty(\Omega)}} \stackrel{\text{def}}{=} C(\lambda, h)$ by (2.13), that $|E_n| \leq |E|$, and dropping a positive term, we obtain from the previous identity that

$$\begin{aligned} \int_\Omega |\nabla\psi_n|^2\,[\alpha\,\varphi'(\psi_n) - \|a\|_{L^\infty(\Omega)}\,|\varphi(\psi_n)|] \\ \leq C(\lambda, h)\int_\Omega |E(x)|\,|\nabla\psi_n| + C(\lambda, h)\int_\Omega |h(x)|. \end{aligned} \tag{2.14}$$

We now choose $\lambda = \|a\|_{L^\infty(\Omega)}/\alpha$, so that

$$\alpha\,\varphi'(s) - \|a\|_{L^\infty(\Omega)}\,|\varphi(s)| = (\alpha\,\lambda - \|a\|_{L^\infty(\Omega)})\,\mathrm{e}^{\lambda\,|s|} + \|a\|_{L^\infty(\Omega)} = \|a\|_{L^\infty(\Omega)}.$$

Thus, from (2.14) it follows that

$$\|a\|_{L^\infty(\Omega)} \int_\Omega |\nabla\psi_n|^2 \le C(\lambda,h) \int_\Omega |E(x)|\,|\nabla\psi_n| + C(\lambda,h)\int_\Omega |h(x)|,$$

from which it follows that

$$\|\psi_n\|_{W_0^{1,2}(\Omega)} \le C(\alpha,\Omega,\|E\|_{L^2(\Omega)},\|h\|_{L^\infty(\Omega)},\|a\|_{L^\infty(\Omega)}). \tag{2.15}$$

Thus, up to subsequences, the sequence $\{\psi_n\}$ converges, weakly in $W_0^{1,2}(\Omega)$, weakly* in $L^\infty(\Omega)$ and almost everywhere, to some function ψ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

We now remark that ψ_n is such that

$$-\operatorname{div}(M(x)\nabla\psi_n) = h_n(x), \tag{2.16}$$

with

$$h_n(x) = E_n(x)\cdot\nabla\psi_n + h(x) - \psi_n - \frac{a(x)\,|\nabla\psi_n|^2}{1+\frac{1}{n}|\nabla\psi_n|},$$

By Cauchy-Schwarz inequality, and by (2.13), (1.4) and the definition of $E_n(x)$, we have

$$\begin{aligned} |h_n(x)| &\le \tfrac{1}{2}|E_n(x)|^2 + \tfrac{1}{2}|\nabla\psi_n|^2 + \|h\|_{L^\infty(\Omega)} + \|\psi_n\|_{L^\infty(\Omega)} + \|a\|_{L^\infty(\Omega)}\,|\nabla\psi_n|^2 \\ &\le |E(x)|^2 + 2\|h\|_{L^\infty(\Omega)} + C(a)\,|\nabla\psi_n|^2 \\ &\le |E(x)|^2 + 2\|h\|_{L^\infty(\Omega)} + C(a)\,|\nabla(\psi_n-\psi)|^2 + C(a)\,|\nabla\psi|^2, \end{aligned} \tag{2.17}$$

where $C(a) > 0$ is a positive constant. Choosing $v = \varphi(\psi_n - \psi)$ as test function in (2.16), where $\varphi(s) = (\mathrm{e}^{\lambda\,|s|}-1)\,\mathrm{sgn}(s)$ as before, we have, by (2.17),

$$\begin{aligned} \int_\Omega M(x)\nabla\psi_n\cdot\nabla(\psi_n-\psi)\,\varphi' &= \int_\Omega h_n(x)\,\varphi \\ &\le \int_\Omega (|E(x)|^2 + 2\|h\|_{L^\infty(\Omega)})\,|\varphi| \\ &\quad + C(a)\int_\Omega |\nabla(\psi_n-\psi)|^2\,|\varphi| + C(a)\int_\Omega |\nabla\psi|^2\,|\varphi|, \end{aligned}$$

where we have omitted the dependence of φ and φ' from its argument $\psi_n - \psi$. Adding and subtracting the term

$$\int_\Omega M(x)\nabla\psi\cdot\nabla(\psi_n-\psi)\,\varphi',$$

and using (1.2), we arrive at

$$\begin{aligned} &\int_\Omega |\nabla(\psi_n-\psi)|^2\,[\alpha\,\varphi' - C(a)\,|\varphi|] \\ &\quad \le \int_\Omega (|E(x)|^2 + 2\|h\|_{L^\infty(\Omega)} + C(a)\,|\nabla\psi|^2)\,|\varphi| + \int_\Omega M(x)\nabla\psi\cdot\nabla(\psi_n-\psi)\,\varphi'. \end{aligned}$$

We now remark that the right hand side tends to zero as n tends to infinity since $\varphi(\psi_n - \psi)$ converges to 0 weakly* in $L^\infty(\Omega)$, and since ψ_n weakly converges to ψ in $W_0^{1,2}(\Omega)$. Furthermore, if $\lambda = \frac{C(a)}{\alpha}$ one has that

$$\alpha\,\varphi'(s) - C(a)\,|\varphi(s)| = C(a), \qquad \forall\, s \in \mathbb{R}.$$

Thus we have

$$0 \leq \lim_{n\to+\infty} C(a) \int_\Omega |\nabla(\psi_n - \psi)|^2 \leq \lim_{n\to+\infty} \int_\Omega |\nabla(\psi_n - \psi)|^2\,[\alpha\,\varphi' - C(a)\,|\varphi|] = 0,$$

which implies that

$$\text{the sequence } \{\psi_n\} \text{ strongly converges to } \psi \text{ in } W_0^{1,2}(\Omega). \tag{2.18}$$

Let now v be a function in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$; choosing v as test function in (2.12) we can pass to the limit in every term, to have that

$$\int_\Omega M(x)\nabla\psi \cdot \nabla v + \int_\Omega \psi\, v + \int_\Omega a(x)\,|\nabla\psi|^2\, v = \int_\Omega E(x)\cdot\nabla\psi\, v + \int_\Omega h(x)\, v,$$

so that ψ is a weak solution of (2.3) in the sense of (2.4), and (2.5) holds as a consequence of (2.13).

The positivity of ψ if $h(x) \geq 0$, and the fact that $\mathrm{m}(\{\psi = 0\}) = 0$ if $0 \not\equiv h(x) \geq 0$ follows from the results of the paper [7], dedicated to Italo Capuzzo Dolcetta. □

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Let w be the unique weak solution in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of

$$\begin{cases} -\operatorname{div}(M(x)\nabla w) + w = g(x) - f(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.19}$$

which exists thanks to Lax-Milgram theorem, and is bounded by the results of Stampacchia (see [8]). Define $E(x) = a(x)\,\nabla w$, which belongs to $(L^2(\Omega))^N$. Therefore, since $g(x)$ belongs to $L^\infty(\Omega)$, by Theorem 2.2 there exists a weak solution ψ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of

$$\begin{cases} -\operatorname{div}(M(x)\nabla\psi) + \psi + a(x)\,|\nabla\psi|^2 = a(x)\,\nabla w\cdot\nabla\psi + g(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Rearranging terms, we have that

$$-\operatorname{div}(M(x)\nabla\psi) + \psi + a(x)\,\nabla(\psi - w)\cdot\nabla\psi = g(x).$$

Defining $u = \psi - w$ (so that u is a function in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$) we thus have that

$$-\operatorname{div}(M(x)\nabla\psi) + \psi + a(x)\,\nabla u\cdot\nabla\psi = g(x),$$

which is the second equation of the system. As for the first, we have

$$\begin{aligned} -\operatorname{div}(M(x)\nabla u) + u &= -\operatorname{div}(M(x)\nabla\psi) + \psi + \operatorname{div}(M(x)\nabla w) - w \\ &= -a(x)\nabla u\cdot\nabla\psi + g(x) + f(x) - g(x) \\ &= -a(x)\nabla u\cdot\nabla\psi + f(x), \end{aligned}$$

so that u is a weak solution in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of

$$-\operatorname{div}(M(x)\nabla u) + u + a(x)\,\nabla u \cdot \nabla\psi = f(x),$$

as desired.

As a consequence of the result of Theorem 2.2 (applied with $E(x) = a(x)\,\nabla w$ and $h(x) = g(x)$), if $0 \not\equiv g(x) \geq 0$, then $\psi \geq 0$ and $\mathrm{m}(\{\psi = 0\}) = 0$. Furthermore, since u is such that

$$-\operatorname{div}(M(x)\nabla u) + u + a(x)\,|\nabla u|^2 = f(x) - a(x)\,\nabla w \cdot \nabla u,$$

the same result of Theorem 2.2 (applied with $E(x) = -a(x)\nabla w$ and $h(x) = f(x)$) yields that if $0 \not\equiv f(x) \geq 0$, then $u \geq 0$ and $\mathrm{m}(\{u = 0\}) = 0$. Note that this strong positivity result for u (and/or for ψ) holds also if the subsets $\{f(x) = 0\}$ and $\{g(x) = 0\}$ have positive (but not full) measure in Ω. □

Remark 2.4. We observe that our proof also works if the system is of the following form:

$$\begin{cases} u \in W_0^{1,2}(\Omega) : -\operatorname{div}(M(x)\nabla u) + u + \mu\, a(x)\,\nabla u \cdot \nabla\psi = f(x), \\ \psi \in W_0^{1,2}(\Omega) : -\operatorname{div}(M(x)\nabla \psi) + \psi + a(x)\,\nabla u \cdot \nabla\psi = g(x). \end{cases} \tag{2.20}$$

with $\mu \neq 0$ belonging to $\mathbb{R}$. In this case, one has to define $w = \mu\,\psi - u$ and reason accordingly.

Remark 2.5. Observe that the solutions u and ψ belong to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, despite the fact that the term $a(x)\,\nabla u \cdot \nabla\psi$ only belongs to $L^1(\Omega)$.

Remark 2.6. If $g(x) \geq f(x)$, then the solution w of equation (2.19) is positive; recalling that $\psi = u + w$, we therefore have that $\psi \geq u$.

If, instead $g(x) \equiv f(x)$, then $w \equiv 0$, so that $u = \psi$; in this case, system (1.1) becomes the equation (repeated twice)

$$-\operatorname{div}(M(x)\nabla\psi) + \psi + a(x)\,|\nabla\psi|^2 = g(x),$$

for which existence of solutions follows from Theorem 2.2, applied with $E(x) \equiv 0$.

3. Existence of unbounded solutions

We will now deal with the case of unbounded solutions under different assumptions on $a(x)$ and on the data $f(x)$ and $g(x)$ of the problem. More precisely, we will assume that

$$a \in L^\infty(\Omega), \qquad a(x) \geq \alpha_0 > 0, \text{ almost everywhere in } \Omega, \tag{3.1}$$

and

$$f(x) \in L^{2_*}(\Omega), \qquad g(x) \in L^{2_*}(\Omega), \qquad g(x) \geq 0, \text{ almost everywhere in } \Omega, \tag{3.2}$$

where

$$2_* = (2^*)' = \frac{2N}{N+2}.$$

During the proof, we will make use of the following real valued functions, depending on a parameter $k > 0$:

$$T_k(s) = \min(s,k), \quad G_k(s) = s - T_k(s) = (s-k)^+, \qquad \forall\, s \geq 0.$$

Our result is the following.

Theorem 3.1. *Let $a(x)$ be such that (3.1) holds, and let $f(x)$ and $g(x)$ be such that (3.2) holds. Then there exist weak solutions u and ψ in $W_0^{1,2}(\Omega)$ of system (1.1), that is*

$$\int_\Omega M(x)\nabla u\cdot\nabla\varphi+\int_\Omega u\,\varphi+\int_\Omega a(x)\,\nabla u\cdot\nabla\psi\,\varphi=\int_\Omega f(x)\,\varphi, \tag{3.3}$$

and

$$\int_\Omega M(x)\nabla\psi\cdot\nabla v+\int_\Omega \psi\,v+\int_\Omega a(x)\,\nabla u\cdot\nabla\psi\,v=\int_\Omega g(x)\,v, \tag{3.4}$$

for every v and φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Remark 3.2. Differently from the results of the previous section, in order to prove Theorem 3.1 we need a strict positivity assumption on the function $a(x)$; the reason for this assumption is that we will use the quadratic lower order term in (3.5) below in order to prove *a priori* estimates on the sequence $\{\psi_n\}$: see (3.6).

Proof. Let n in $\mathbb{N}$ and define $g_n(x)=T_n(g(x))$. If $E(x)$ belongs to $(L^2(\Omega))^N$, thanks to Theorem 2.2, there exists a weak solution ψ_n in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla\psi_n)+\psi_n+a(x)\,|\nabla\psi_n|^2=E(x)\cdot\nabla\psi_n+g_n(x) & \text{in } \Omega,\\ \psi_n=0 & \text{su } \partial\Omega.\end{cases} \tag{3.5}$$

Furthermore, since $g_n(x)\ge 0$, we have that $\psi_n\ge 0$. Choose now $v=T_1(\psi_n)$ as test function in (3.5). We obtain

$$\begin{aligned}\int_\Omega M(x)\nabla\psi_n\cdot\nabla T_1(\psi_n)+\int_\Omega \psi_n\,T_1(\psi_n)+\int_\Omega a(x)\,|\nabla\psi_n|^2\,T_1(\psi_n)\\ =\int_\Omega E(x)\cdot\nabla\psi_n\,T_1(\psi_n)+\int_\Omega g_n(x)\,T_1(\psi_n).\end{aligned}$$

Dropping one positive term, using (1.2) and (3.1), and the fact that $0\le T_1(\psi_n)\le 1$, we have

$$\alpha\int_{\{\psi_n\le 1\}}|\nabla\psi_n|^2+\alpha_0\int_{\{\psi_n\ge 1\}}|\nabla\psi_n|^2\le\int_\Omega|E(x)|\,|\nabla\psi_n|+\int_\Omega|g_n(x)|,$$

that is

$$\min(\alpha,\alpha_0)\int_\Omega|\nabla\psi_n|^2\le\int_\Omega|E(x)|\,|\nabla\psi_n|+\|g\|_{L^1(\Omega)}. \tag{3.6}$$

From this inequality, it follows that there exists a constant $C>0$ such that

$$\|\psi_n\|^2_{W_0^{1,2}(\Omega)}\le C\,\|E\|^2_{L^2(\Omega)}+C\,\|g\|_{L^1(\Omega)}.$$

Hence, the sequence $\{\psi_n\}$ is bounded in $W_0^{1,2}(\Omega)$, and so it weakly converges, up to subsequences, to some function ψ in $W_0^{1,2}(\Omega)$. Once this result has been proved, we have, as in the proof of Theorem 2.2, that

$$-\operatorname{div}(M(x)\nabla\psi_n)=h_n(x),$$

where

$$h_n(x)=E(x)\cdot\nabla\psi_n+g_n(x)-\psi_n-a(x)\,|\nabla\psi_n|^2,$$

is a sequence bounded in $L^1(\Omega)$. From the results of [9] it then follows that

$$\text{the sequence } \{\nabla\psi_n\} \text{ converges almost everywhere to } \nabla\psi. \tag{3.7}$$

We are now going to prove that ψ is a weak solution of the equation

$$\begin{cases} -\operatorname{div}(M(x)\nabla\psi) + \psi + a(x)\,|\nabla\psi|^2 = E(x)\cdot\nabla\psi + g(x) & \text{in } \Omega, \\ \psi = 0 & \text{su } \partial\Omega. \end{cases} \tag{3.8}$$

In order to do that, we are going to prove (following [6]) that ψ is both a weak subsolution and a supersolution of (3.8).

Step 1: ψ is a subsolution of (3.8).

Let $\phi \geq 0$ be a function in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Choosing $v = \phi$ as test function in (3.5), we have that

$$\int_\Omega M(x)\nabla\psi_n\cdot\nabla\phi + \int_\Omega \psi_n\,\phi + \int_\Omega a(x)\,|\nabla\psi_n|^2\,\phi = \int_\Omega E(x)\cdot\nabla\psi_n\,\phi + \int_\Omega g_n(x)\,\phi.$$

Passing to the limit using the weak convergence of ψ_n to ψ in $W_0^{1,2}(\Omega)$, as well as the almost everywhere convergence of $\nabla\psi_n$ to $\nabla\psi$, and using Fatou lemma, we arrive at

$$\int_\Omega M(x)\nabla\psi\cdot\nabla\phi + \int_\Omega \psi\,\phi + \int_\Omega a(x)\,|\nabla\psi|^2\,\phi \leq \int_\Omega E(x)\cdot\nabla\psi\,\phi + \int_\Omega g(x)\,\psi, \tag{3.9}$$

for every $\phi \geq 0$ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Step 2: ψ is a supersolution of (3.8).

Let $\rho \geq 0$ be a function in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, and choose $v = \rho\,\mathrm{e}^{-\lambda\,\psi_n}$ as test function in (3.5), with $\lambda = \|a\|_{L^\infty(\Omega)}/\alpha$. We have

$$\begin{aligned}\int_\Omega M(x)\nabla\psi_n\cdot\nabla\rho\,\mathrm{e}^{-\lambda\,\psi_n} - \lambda\int_\Omega M(x)\nabla\psi_n\cdot\nabla\psi_n\,\rho\,\mathrm{e}^{-\lambda\,\psi_n} + \int_\Omega \psi_n\,\rho\,\mathrm{e}^{-\lambda\,\psi_n} \\ + \int_\Omega a(x)\,|\nabla\psi_n|^2\,\rho\,\mathrm{e}^{-\lambda\,\psi_n} = \int_\Omega E(x)\cdot\nabla\psi_n\,\rho\,\mathrm{e}^{-\lambda\,\psi_n} + \int_\Omega g_n(x)\,\rho\,\mathrm{e}^{-\lambda\,\psi_n}.\end{aligned}$$

Rearranging terms, we have that

$$\begin{aligned}&\int_\Omega M(x)\nabla\psi_n\cdot\nabla\rho\,\mathrm{e}^{-\lambda\,\psi_n} + \int_\Omega \psi_n\,\rho\,\mathrm{e}^{-\lambda\,\psi_n} \\ &\quad= \int_\Omega [\lambda\,M(x)\nabla\psi_n\cdot\nabla\psi_n - a(x)|\nabla\psi_n|^2]\,\rho\,\mathrm{e}^{-\lambda\,\psi_n} + \int_\Omega E(x)\cdot\nabla\psi_n\,\rho\,\mathrm{e}^{-\lambda\,\psi_n} \\ &\qquad+ \int_\Omega g_n(x)\,\rho\,\mathrm{e}^{-\lambda\,\psi_n}.\end{aligned}$$

We now remark that, by (1.2),

$$\lambda\,M(x)\nabla\psi_n\cdot\nabla\psi_n - a(x)|\nabla\psi_n|^2 \geq [\alpha\,\lambda - \|a\|_{L^\infty(\Omega)}]\,|\nabla\psi_n|^2 = 0,$$

thanks to the assumption made on λ. Therefore, we can apply again Fatou lemma to obtain, using the other convergence properties of ψ_n and g_n, that

$$\begin{aligned}\int_\Omega M(x)\nabla\psi\cdot\nabla\rho\,\mathrm{e}^{-\lambda\,\psi} + \int_\Omega \psi\,\rho\,\mathrm{e}^{-\lambda\,\psi} \geq \int_\Omega [\lambda\,M(x)\nabla\psi\cdot\nabla\psi - a(x)|\nabla\psi|^2]\,\rho\,\mathrm{e}^{-\lambda\,\psi} \\ + \int_\Omega E(x)\cdot\nabla\psi\,\rho\,\mathrm{e}^{-\lambda\,\psi} + \int_\Omega g(x)\,\rho\,\mathrm{e}^{-\lambda\,\psi}.\end{aligned}$$

We now fix $k > 0$ and choose $\rho = \phi\, \mathrm{e}^{\lambda\, T_k(\psi)}$, with $\phi \geq 0$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ to obtain that

$$\begin{aligned}&\int_\Omega M(x)\nabla\psi\cdot\nabla\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)} + \lambda\int_\Omega M(x)\nabla\psi\cdot\nabla T_k(\psi)\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)} + \int_\Omega \psi\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)}\\ &\quad\geq \int_\Omega [\lambda\,M(x)\nabla\psi\cdot\nabla\psi - a(x)|\nabla\psi|^2]\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)} + \int_\Omega E(x)\cdot\nabla\psi\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)}\\ &\qquad+\int_\Omega g(x)\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)}.\end{aligned}$$

Rearranging terms once again, we have that

$$\begin{aligned}&\int_\Omega M(x)\nabla\psi\cdot\nabla\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)} + \int_\Omega \psi\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)} + \int_\Omega a(x)\,|\nabla\psi|^2\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)}\\ &\quad\geq \lambda\int_\Omega M(x)\nabla G_k(\psi)\cdot\nabla G_k(\psi)\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)} + \int_\Omega E(x)\cdot\nabla\psi\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)}\\ &\qquad+\int_\Omega g(x)\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)}.\end{aligned}\tag{3.10}$$

We now remark that since ψ belongs to $W_0^{1,2}(\Omega)$ we have that

$$\lim_{k\to+\infty}\int_\Omega M(x)\nabla G_k(\psi)\cdot\nabla G_k(\psi)\,\phi\,\mathrm{e}^{-\lambda\,G_k(\psi)} = 0.$$

Therefore, passing to the limit in (3.10) as k tends to infinity, we obtain

$$\int_\Omega M(x)\nabla\psi\cdot\nabla\phi + \int_\Omega \psi\,\phi + \int_\Omega a(x)\,|\nabla\psi|^2\,\phi \geq \int_\Omega E(x)\cdot\nabla\psi\,\phi + \int_\Omega g(x)\,\phi,\tag{3.11}$$

for every $\phi \geq 0$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Putting together (3.9) and (3.11), we have that

$$\int_\Omega M(x)\nabla\psi\cdot\nabla\phi + \int_\Omega \psi\,\phi + \int_\Omega a(x)\,|\nabla\psi|^2\,\phi = \int_\Omega E(x)\cdot\nabla\psi\,\phi + \int_\Omega g(x)\,\phi,$$

for every $\phi \geq 0$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Writing a generic function ϕ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ as $\phi = \phi^+ - \phi^-$, with $\phi^\pm \geq 0$, we then recover (3.4): ψ is a weak solution of the second equation of system (1.1).

Once we have proved that ψ is a solution, the solution u can be obtained as $u = \psi - w$, where, as in the proof of Theorem 2.1, w is the unique solution in $W_0^{1,2}(\Omega)$ of

$$\begin{cases} -\operatorname{div}(M(x)\nabla w) + w = g(x) - f(x) & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega,\end{cases}$$

which exists thanks to Lax-Milgram theorem. □

Remark 3.3. Note that the assumption $f(x)$ in $L^{2_*}(\Omega)$ and $g(x)$ in $L^{2_*}(\Omega)$ has only been used to obtain that the function w belongs to $W_0^{1,2}(\Omega)$, so that the vector field $E(x) = a(x)\,\nabla w$ belongs to $(L^2(\Omega))^N$, while the existence of the solution ψ to the problem

$$-\operatorname{div}(M(x)\nabla\psi) + \psi + a(x)\,|\nabla\psi|^2 = E(x)\cdot\nabla\psi + g(x),$$

can be obtained under the weaker assumption that $g(x)$ belongs to $L^1(\Omega)$. Therefore, if $f(x)$ and $g(x)$ are functions in $L^1(\Omega)$ such that $g(x) - f(x)$ belongs to $L^{2_*}(\Omega)$, one can prove existence of solutions u and ψ of system (1.1).

Note that if $f(x) \geq 0$ (instead of $g(x) \geq 0$), one can reverse the role of u and ψ and prove again the existence of solutions for system (1.1).

4. Existence under a Q-condition

In this section we will study existence of solutions for the following system, which is a modified version of system (1.1):

$$\begin{cases} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : -\operatorname{div}(M(x)\nabla u) + b(x)\, u + a(x)\nabla\psi \cdot \nabla u = f(x), \\ \psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : -\operatorname{div}(M(x)\nabla \psi) + b(x)\, \psi + a(x)\nabla\psi \cdot \nabla u = g(x), \end{cases} \tag{4.1}$$

where

$$0 \leq b(x) \in L^1(\Omega), \tag{4.2}$$

and there exists $Q > 0$ such that

$$|f(x)| \leq Q\, b(x), \qquad |g(x)| \leq Q\, b(x). \tag{4.3}$$

For a single equation (for example, the first one), in the case $a(x) \equiv 0$ in [10] is proved that the assumption $|f(x)| \leq Q\, b(x)$, even if $b(x)$ only belongs to $L^1(\Omega)$, yields that there exists a bounded weak solution (regularizing effect of (4.3)). In the following theorem, we prove the same regularizing effect for system (4.1).

Theorem 4.1. *Under assumptions (4.2), (4.3), there exist bounded weak solutions u and ψ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of system (4.1).*

Proof. The idea of the proof of existence of solutions for system (4.1) is the same as the proof of Theorem 2.1; first of all, if $E(x)$ belongs to $(L^2(\Omega))^N$, and $E_n(x) = \frac{E(x)}{1+\frac{1}{n}|E(x)|}$, we prove existence of a solution ψ_n in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the equation

$$\begin{cases} -\operatorname{div}(M(x)\nabla\psi_n) + b_n(x)\,\psi_n = \Big(E_n(x) - \frac{a(x)\,\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|}\Big) \cdot \nabla\psi_n + g_n(x) & \text{in } \Omega, \\ \psi_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $b_n(x) = T_n(b(x))$ and $g_n(x) = T_{Q\,n}(g(x))$. The existence of ψ_n can be proved exactly as in the proof of Theorem 2.2, using Schauder's theorem, since the term $b_n(x)$ does not yield any difficulty being bounded. Once the existence of ψ_n belonging to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ has been proved, let $z_n \geq 0$ in $W_0^{1,2}(\Omega)$ be the solution of

$$\begin{cases} -\operatorname{div}(M(x)\nabla z_n) + b_n(x) z_n = -\operatorname{div}\Big(z_n\Big(E_n(x) - \frac{a(x)\,\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|}\Big)\Big) + \chi_E(x) & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\chi_E(x)$ is the characteristic function of a measurable subset E of Ω. Such a solutions exists by the results of [11], and is such that

$$0 \leq \int_\Omega b_n(x)\, z_n \leq \int_\Omega \chi_E(x) = \mathrm{m}(E). \tag{4.4}$$

We now choose ψ_n as test function in the equation solved by z_n, and z_n as test function in the equation solved by ψ_n to have that

$$\int_\Omega M(x)\nabla\psi_n\cdot\nabla z_n+\int_\Omega b_n(x)\,\psi_n\,z_n=\int_\Omega\Big(E_n(x)-\frac{a(x)\,\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|}\Big)\cdot\nabla\psi_n\,z_n+\int_\Omega g_n(x)\,z_n,$$

and

$$\int_\Omega M(x)\nabla\psi_n\cdot\nabla z_n+\int_\Omega b_n(x)\,\psi_n\,z_n=\int_\Omega\Big(E_n(x)-\frac{a(x)\,\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|}\Big)\cdot\nabla\psi_n\,z_n+\int_\Omega \chi_E\,\psi_n.$$

Therefore, we have

$$\int_\Omega \chi_E(x)\,\psi_n=\int_\Omega g_n(x)\,z_n,$$

which implies, using (4.3), (4.4), and the fact that $|g_n(x)|\le Q\,b_n(x)$, that

$$\Big|\int_\Omega \chi_E(x)\,\psi_n\Big|=\Big|\int_\Omega g_n(x)\,z_n\Big|\le\int_\Omega|g_n(x)|\,z_n\le Q\int_\Omega b_n(x)\,z_n\le Q\ \mathrm{m}(E).$$

Thus

$$\Big|\frac{1}{\mathrm{m}(E)}\int_E\psi_n\Big|\le Q,$$

which implies that

$$\|\psi_n\|_{L^\infty(\Omega)}\le Q. \tag{4.5}$$

Once this estimate has been proved, one can continue as in the proof of Theorem 2.2 to prove that it exists a solution ψ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ of

$$\begin{cases} -\operatorname{div}(M(x)\nabla\psi)+b(x)\,\psi+a(x)\,|\nabla\psi|^2=E(x)\cdot\nabla\psi+g(x) & \text{in } \Omega,\\ \psi=0 & \text{on } \partial\Omega.\end{cases} \tag{4.6}$$

To conclude the proof, since $|f(x)-g(x)|\le 2Q\,b(x)$, there exists (by the results of [10]) a weak solution w in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ of

$$\begin{cases} -\operatorname{div}(M(x)\nabla w)+b(x)\,w=g(x)-f(x) & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega,\end{cases} \tag{4.7}$$

with

$$\|w\|_{L^\infty(\Omega)}\le 2Q.$$

Defining $u=\psi+w$ one can conclude the proof as in the proof of Theorem 2.1. □

5. Systems with m equations

The same techniques used to prove an existence result for system (1.1) can be used to prove existence of solutions for systems of m equations of the following form:

$$-\operatorname{div}(M(x)\nabla u_i) + u_i + a(x)\,\nabla u_i \cdot \Big(\sum_{j<i} \nabla u_j - \sum_{j>i} \nabla u_j\Big) = f_i(x), \quad i = 1, \dots m, \tag{5.1}$$

with the obvious convention that the sum on $j < 1$ is zero, as well as the sum on $j > m$. Note that in the equation for u_i the term with $\nabla u_i \cdot \nabla u_i$ is missing and that, if $m = 2$, system (5.1) becomes system (2.20) for $\mu = -1$.

We remark that

$$\sum_{j<i} \nabla u_j - \sum_{j>i} \nabla u_j = 2\sum_{j<i} \nabla u_j + \nabla u_i - \sum_{j=1}^{m} \nabla u_j.$$

Therefore, if we define

$$w = \sum_{i=1}^{m} u_i,$$

then

$$\sum_{j<i} \nabla u_j - \sum_{j>i} \nabla u_j = 2\sum_{j<i} \nabla u_j + \nabla u_i - \nabla w,$$

so that the equation for u_i can be rewritten as

$$-\operatorname{div}(M(x)\,\nabla u_i) + u_i + a(x)\,|\nabla u_i|^2 = a(x)\Big(\nabla w - 2\sum_{j<i} \nabla u_j\Big) \cdot \nabla u_i + f_i(x). \tag{5.2}$$

We now sum all the equations with i ranging from 1 to m; we have, since the operator is linear,

$$-\operatorname{div}(M(x)\,\nabla w) + w + a(x)\sum_{i=1}^{m} |\nabla u_i|^2 = a(x)|\nabla w|^2 - 2a(x)\sum_{i=1}^{m}\sum_{j<i} \nabla u_i \cdot \nabla u_j + \sum_{i=1}^{n} f_i(x).$$

We now remark that

$$|\nabla w|^2 - 2\sum_{i=1}^{m}\sum_{j<i} \nabla u_i \cdot \nabla u_j = \sum_{i=1}^{m} |\nabla u_i|^2,$$

so that w is the unique weak solution in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of

$$-\operatorname{div}(M(x)\nabla w) + w = \sum_{i=1}^{m} f_i(x). \tag{5.3}$$

whose existence follows from Lax-Milgram theorem, and whose boundedness follows from the results of Stampacchia (see [8])

We now consider the first equation of the system, which is, from (5.2)

$$-\operatorname{div}(M(x)\,\nabla u_1) + u_1 + a(x)\,|\nabla u_1|^2 = a(x)\,\nabla w \cdot \nabla u_1 + f_1(x). \tag{5.4}$$

This is an equation of the form

$$-\operatorname{div}(M(x)\,\nabla \psi) + \psi + a(x)\,|\nabla \psi|^2 = E(x) \cdot \nabla \psi + h(x), \tag{5.5}$$

with $h(x) = f_1(x)$ in $L^\infty(\Omega)$ and $E(x) = a(x)\,\nabla w$ in $(L^2(\Omega))^N$. Thus, by Theorem 2.2, there exists a solution u_1 in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of equation (5.4).

We now consider the second equation of the system; again from (5.2), it is

$$-\operatorname{div}(M(x)\,\nabla u_2) + u_2 + a(x)\,|\nabla u_2|^2 = a(x)\,(\nabla w - 2\,\nabla u_1) \cdot \nabla u_2 + f_2(x), \tag{5.6}$$

which is again an equation of the form (5.5), with $h(x) = f_2(x)$ in $L^\infty(\Omega)$ and $E(x) = a(x)\,(\nabla w - 2\,\nabla u_1)$ in $(L^2(\Omega))^N$, a known datum (since we know both w and u_1). Thus, by Theorem 2.2, there exists a solution u_2 in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of equation (5.6).

In general, once $u_1, \ldots, u_{i-1}$ are known (together with w), one can find u_i by solving (5.2) using Theorem 2.2. Once we have proved the existence of $u_1, \ldots, u_{m-1}$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, we define $u_m = w - (u_1 + \ldots + u_{m-1})$ to conclude the proof of the existence.

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