ESAIM: COCV 30 (2024) 90
https://doi.org/10.1051/cocv/2024082

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

NULL CONTROLLABILITY FOR ONE-DIMENSIONAL STOCHASTIC HEAT EQUATIONS WITH MIXED DIRICHLET-DYNAMIC BOUNDARY CONDITIONS

Mahmoud Baroun[1], Said Boulite[2,*], Abdellatif Elgrou[1] and Lahcen Maniar[1]

Abstract. In this paper, we study the null controllability of one-dimensional forward and backward linear stochastic heat equations with mixed Dirichlet-dynamic boundary conditions. Our equations incorporate noise not only within the domain but also at the boundary, represented by a two-dimensional Brownian motion. The primary tool will be global Carleman estimates, which yield the appropriate observability inequalities for the related adjoint systems. Hence, by classical duality arguments, we establish the corresponding null controllability results. Specifically, we first establish a Carleman estimate for a general adjoint backward stochastic heat equation using a weighted identity method. This approach combines two weighted identities: one for a stochastic parabolic operator and the other for a stochastic transport operator. Subsequently, we derive a Carleman estimate for a general adjoint forward stochastic heat equation by employing a duality method.

Mathematics Subject Classification. 93B05, 93B07, 93E20, 60H15.

Received April 8, 2023. Accepted November 21, 2024.

1. Introduction

Controllability theory is an active branch of applied mathematics, its main objective is to study the behavior of differential and evolution equations such as heat, wave, transport, Schrödinger equations, and so on. Roughly speaking, controllability theory can be divided into two parts, the first one is the controllability theory for deterministic equations which is still quite an active field containing a huge list of publications, see, *e.g.*, [1–10]. The second part is the controllability for stochastic equations, including stochastic partial differential equations which started attracting more and more attention only recently (see, for instance, [11–20] and the book [21]), and the field is still in its beginning development. By a classical duality argument, we know that the null controllability problem can be reduced to an observability inequality for the corresponding adjoint equation. To prove such observability inequality, one can use Carleman estimates introduced by Carleman in 1939 to prove the uniqueness of solutions to second-order elliptic partial differential equations in $\mathbb{R}^2$ (see [22]). Now, for both

Keywords and phrases: Null controllability, stochastic heat equations, Carleman estimates, observability inequality, dynamic boundary conditions.

[1] Cadi Ayyad University, Faculty of Sciences Semlalia, LMDP, UMMISCO (IRD-UPMC), B.P. 2390, Marrakesh, Morocco.
[2] Cadi Ayyad University, National School of Applied Sciences, LMDP, UMMISCO (IRD-UPMC), B.P. 575, Marrakesh, Morocco.

[*] Corresponding author: s.boulite@uca.ma

deterministic and stochastic partial differential equations, Carleman estimates become a powerful tool to study many problems such as controllability, observability, unique continuation, inverse problems, and so on.

In the existing literature, the null controllability problem of stochastic parabolic equations has been studied with the standard static boundary conditions, such as Dirichlet, Neumann, and Robin boundary conditions, see, *e.g.*, [11, 13–15, 17, 19, 21, 23]. Dynamic boundary conditions, characterized by a time derivative, are employed to describe physical models with dynamics either on the whole boundary or on part of the boundary. Consequently, dynamic boundary conditions differ qualitatively from the static boundary conditions. For more details and physical applications of such dynamical boundary conditions, see, for instance, [24–26]. In the literature, there are some results on the controllability of parabolic equations with dynamic boundary conditions. For deterministic equations, see, *e.g.*, [1, 8, 9]. For stochastic equations, we refer to [12, 27].

To state our problems, we introduce the following notations. Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ (with $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$) is a fixed complete filtered probability space on which a two-dimensional standard Brownian motion $W(\cdot) = \big(W^1(\cdot), W^2(\cdot)\big)^\top$ is defined such that $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$ and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $\mathcal{X}$ be a Banach space, and let $L^2_{\mathcal{F}_t}(\Omega;\mathcal{X})$ denote the Banach space of all $\mathcal{X}$-valued $\mathcal{F}_t$-measurable random variables X such that $\mathbb{E}|X|^2_{\mathcal{X}} < \infty$, with the canonical norm. $L^2_{\mathcal{F}}(0,T;\mathcal{X})$ is the Banach space consisting of all $\mathcal{X}$-valued $\mathbf{F}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;\mathcal{X})}) < \infty$, with the canonical norm. $L^\infty_{\mathcal{F}}(0,T;\mathcal{X})$ is the Banach space of all $\mathcal{X}$-valued $\mathbf{F}$-adapted essentially bounded processes, equipped with the usual norm denoted by $|\cdot|_\infty$. Subsequently, we further simply denote $L^2_{\mathcal{F}}(0,T)$ (resp., $L^\infty_{\mathcal{F}}(0,T)$) for $L^2_{\mathcal{F}}(0,T;\mathbb{R})$ (resp., $L^\infty_{\mathcal{F}}(0,T;\mathbb{R})$). The space $L^2_{\mathcal{F}}(\Omega;C([0,T];\mathcal{X}))$ is the Banach space consisting of all $\mathcal{X}$-valued $\mathbf{F}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X|^2_{C([0,T];\mathcal{X})}) < \infty$, with the canonical norm, where $C([0,T];\mathcal{X})$ denotes the Banach space of all $\mathcal{X}$-valued continuous functions.

The heat equation, also known as the diffusion equation, is one of the most fundamental partial differential equations. It describes the evolution over time of the density of various quantities such as heat, bacteria population, chemical concentration, *etc.*, which exhibit diffusion phenomena in typical applications. By incorporating a stochastic perturbation generated by the Brownian motion $W^1(\cdot)$ over the interval $(0,1) \subset \mathbb{R}$, we consider the stochastic heat equation. This equation accounts for all small independent random disturbances that may occur during the evolution of the density. In this paper, we study the evolution over the domain $(0,1)$, and in the absence of large-scale migration, the variation of the density is governed by the following linear stochastic heat equation

$$\mathrm{d}y(t,x) - y_{xx}(t,x)\,\mathrm{d}t = \beta_1(t,x)y(t,x)\,\mathrm{d}t + \beta_2(t,x)y(t,x)\,\mathrm{d}W^1(t), \tag{1.1}$$

where $y(t,x)$ represents the density at time $t \in (0,T)$ and position $x \in (0,1)$, for a sample point $\omega \in \Omega$. The coefficients β_1 and β_2 are suitable (random) coefficients dependent on the properties of the heat diffusion medium. The term $y_{xx}(t,x)$ characterizes the diffusion process from regions of high concentration to those of lower concentration. See [21], Chapter 5 for more details on this model. Now, if we consider the cooling of a cylindrical, thin, and uniform metal bar with faces positioned at $x = 0$ and $x = 1$, we assume that the lateral surface of the bar is thermally insulated. At the boundary $x = 0$, the bar is connected to an object maintaining the temperature at zero. We then focus on cooling the bar by acting on the boundary $x = 1$. This can be achieved by bringing this end of the bar into contact with a liquid, ensuring that the heat gained by the liquid at the end of the bar equals the heat lost by the bar. This heat transfer phenomenon between the bar and the liquid gives rise to the following mixed Dirichlet-dynamic boundary conditions

$$y(t,0) = 0, \qquad t \in (0,T), \tag{1.2}$$

$$\mathrm{d}y(t,1) + y_x(t,1)\,\mathrm{d}t = \eta_1(t)y(t,1)\,\mathrm{d}t + \eta_2(t)y(t,1)\,\mathrm{d}W^2(t), \qquad t \in (0,T), \tag{1.3}$$

where η_1 and η_2 represent suitable (random) coefficients, and the Brownian motion $W^2(\cdot)$ describes the noise on the boundary $x = 1$, which may differ from the noise in the domain. In the physical model, dynamic boundary conditions (sometimes referred to as Wentzell boundary conditions) are crucial, especially when there are dynamic interactions between the domain and the boundary. For some applications of this type of boundary conditions in the stochastic setting, we refer to [28–30].

Let $T>0$, $G=(0,1)$ and $G_0 \Subset G$ be any given non-empty open subset which is strictly contained in G (*i.e.*, $\overline{G_0}\subset G$). We indicate by $\mathbb{1}_{G_0}$ the characteristic function of G_0. Throughout this paper, C denotes a positive constant that may vary from one place to another. In what follows, we denote by

$$\mathbb{L}^2 := L^2(G)\times\mathbb{R},$$

which is a Hilbert space when equipped with the following standard inner product

$$\langle (y,\mathbf{a}),(z,\mathbf{b})\rangle_{\mathbb{L}^2} = \langle y,z\rangle_{L^2(G)} + \mathbf{ab}.$$

Subsequently, we also require the following Hilbert spaces

$$\mathbb{H}^k = \big\{(\phi,\phi(1))\in H^k(G)\times\mathbb{R}:\ \phi(0)=0\big\} \quad \text{for } \ k=1,2, \tag{1.4}$$

endowed with the usual inner product

$$\langle(\phi,\phi(1)),(\widetilde{\phi},\widetilde{\phi}(1))\rangle_{\mathbb{H}^k} = \langle \phi,\widetilde{\phi}\rangle_{H^k(G)} + \phi(1)\widetilde{\phi}(1),$$

where $H^k(G)$ are the usual Sobolev spaces.

By acting with controls u, v_1, and v_2, equations (1.1), (1.2) and (1.3) lead us to consider the following controlled forward stochastic heat equation

$$\begin{cases} \mathrm{d}y - y_{xx}\,\mathrm{d}t = (\beta_1 y + \mathbb{1}_{G_0}u)\,\mathrm{d}t + (\beta_2 y + v_1)\,\mathrm{d}W^1(t), & (t,x)\in(0,T)\times G,\\ y(t,0)=0, & t\in(0,T),\\ \mathrm{d}y(t,1) + y_x(t,1)\,\mathrm{d}t = \eta_1 y(t,1)\,\mathrm{d}t + (\eta_2 y(t,1)+v_2)\,\mathrm{d}W^2(t), & t\in(0,T),\\ (y(0,x),y(0,1)) = (y_0(x),y_0^1), & x\in G, \end{cases} \tag{1.5}$$

where $(y_0,y_0^1)\in L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)$ is the initial state, $\beta_1,\beta_2\in L^\infty_{\mathcal{F}}(0,T;L^\infty(G))$, $\eta_1,\eta_2\in L^\infty_{\mathcal{F}}(0,T)$ and the control variable consists of the triple

$$(u,v_1,v_2)\in L^2_{\mathcal{F}}(0,T;L^2(G_0))\times L^2_{\mathcal{F}}(0,T;L^2(G))\times L^2_{\mathcal{F}}(0,T).$$

In Section 2, we show that (1.5) is well-posed *i.e.*, there exists a unique mild (or weak) solution

$$(y,y(\cdot,1))\in L^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{L}^2))\bigcap L^2_{\mathcal{F}}(0,T;\mathbb{H}^1),$$

of equation (1.5) such that

$$\begin{aligned} &|(y,y(\cdot,1))|_{L^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{L}^2))} + |(y,y(\cdot,1))|_{L^2_{\mathcal{F}}(0,T;\mathbb{H}^1)}\\ &\leq C\left(|(y_0,y_0^1)|_{L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)} + |u|_{L^2_{\mathcal{F}}(0,T;L^2(G_0))} + |v_1|_{L^2_{\mathcal{F}}(0,T;L^2(G))} + |v_2|_{L^2_{\mathcal{F}}(0,T)}\right), \end{aligned}$$

where C is a positive constant depending on G, T, β_1, β_2, η_1, and η_2.

The first goal of this paper is to show that the system (1.5) is null controllable at time T *i.e.*, for any initial state $(y_0,y_0^1)\in L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)$, there exists controls $(u,v_1,v_2)\in L^2_{\mathcal{F}}(0,T;L^2(G_0))\times L^2_{\mathcal{F}}(0,T;L^2(G))\times L^2_{\mathcal{F}}(0,T)$

such that the corresponding solution $(y, y(\cdot,1))$ of (1.5) satisfies that

$$(y(T,\cdot), y(T,1)) = (0,0) \text{ in } G, \ \mathbb{P}\text{-a.s.}$$

The crucial step in establishing the null controllability of (1.5) is to derive an appropriate observability inequality for the corresponding (adjoint) backward stochastic heat equation

$$\begin{cases} \mathrm{d}z + z_{xx}\,\mathrm{d}t = (a_1 z + a_2 Z)\,\mathrm{d}t + Z\,\mathrm{d}W^1(t), & (t,x) \in (0,T)\times G, \\ z(t,0) = 0, & t \in (0,T), \\ \mathrm{d}z(t,1) - z_x(t,1)\,\mathrm{d}t = \big(b_1 z(t,1) + b_2 \widehat{Z}(t)\big)\,\mathrm{d}t + \widehat{Z}(t)\,\mathrm{d}W^2(t), & t \in (0,T), \\ (z(T,x), z(T,1)) = (z_T(x), z_T^1), & x \in G, \end{cases} \tag{1.6}$$

where $(z_T, z_T^1) \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$ is the terminal state, $a_1, a_2 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G))$, and $b_1, b_2 \in L^\infty_{\mathcal{F}}(0,T)$.

In Section 2, we prove that equation (1.6) admits a unique mild (or weak) solution

$$(z, z(\cdot,1); Z, \widehat{Z}) \in \Big(L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{L}^2)) \bigcap L^2_{\mathcal{F}}(0,T;\mathbb{H}^1)\Big) \times L^2_{\mathcal{F}}(0,T;\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2)),$$

so that

$$|(z, z(\cdot,1))|_{L^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{L}^2))} + |(z, z(\cdot,1))|_{L^2_{\mathcal{F}}(0,T;\mathbb{H}^1)} + |(Z,\widehat{Z})|_{L^2_{\mathcal{F}}(0,T;\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2))} \leq C\,|(z_T, z_T^1)|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)},$$

where C is a positive constant depending on G, T, a_1, a_2, b_1, and b_2.

The observability problem in this case aims to establish the existence of a constant $C > 0$ depending only on G, G_0, T, $|a_1|_\infty$, $|a_2|_\infty$, $|b_1|_\infty$, and $|b_2|_\infty$ such that for all $(z_T, z_T^1) \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$, the corresponding solution $(z, z(\cdot,1); Z, \widehat{Z})$ of equation (1.6) fulfills that

$$\mathbb{E}|z(0,\cdot)|^2_{L^2(G)} + \mathbb{E}|z(0,1)|^2 \leq C\left[\mathbb{E}\int_0^T\int_{G_0} z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G Z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \widehat{Z}^2\,\mathrm{d}t\right]. \tag{1.7}$$

The second goal of this paper is to establish the null controllability of the following backward stochastic heat equation

$$\begin{cases} \mathrm{d}y + y_{xx}\,\mathrm{d}t = (\beta_1 y + \beta_2 Y + \mathbb{1}_{G_0} u)\,\mathrm{d}t + Y\,\mathrm{d}W^1(t), & (t,x) \in (0,T)\times G, \\ y(t,0) = 0, & t \in (0,T), \\ \mathrm{d}y(t,1) - y_x(t,1)\,\mathrm{d}t = \big(\eta_1 y(t,1) + \eta_2 \widehat{Y}(t)\big)\,\mathrm{d}t + \widehat{Y}(t)\,\mathrm{d}W^2(t), & t \in (0,T), \\ (y(T,x), y(T,1)) = (y_T(x), y_T^1), & x \in G, \end{cases} \tag{1.8}$$

where $(y_T, y_T^1) \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$, $\beta_1, \beta_2 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G))$, $\eta_1, \eta_2 \in L^\infty_{\mathcal{F}}(0,T)$, and $u \in L^2_{\mathcal{F}}(0,T;L^2(G_0))$ is the control function. That is, for all terminal state $(y_T, y_T^1) \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$, one can find a control $u \in L^2_{\mathcal{F}}(0,T;L^2(G_0))$ so that the corresponding solution $(y, y(\cdot,1); Y, \widehat{Y})$ of (1.8) satisfies that

$$(y(0,\cdot), y(0,1)) = (0,0) \text{ in } G, \ \mathbb{P}\text{-a.s.}$$

For this purpose, we consider the following (adjoint) forward stochastic heat equation

$$\begin{cases} \mathrm{d}z - z_{xx}\,\mathrm{d}t = a_1 z\,\mathrm{d}t + a_2 z\,\mathrm{d}W^1(t), & (t,x)\in(0,T)\times G,\\ z(t,0)=0, & t\in(0,T),\\ \mathrm{d}z(t,1) + z_x(t,1)\,\mathrm{d}t = b_1 z(t,1)\,\mathrm{d}t + b_2 z(t,1)\,\mathrm{d}W^2(t), & t\in(0,T),\\ (z(0,x),z(0,1)) = (z_0(x), z_0^1), & x\in G, \end{cases} \tag{1.9}$$

where $(z_0, z_0^1) \in L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)$, $a_1, a_2 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G))$, $b_1, b_2 \in L^\infty_{\mathcal{F}}(0,T)$, and show that there exists a constant $C>0$ depending only on G, G_0, T, $|a_1|_\infty$, $|a_2|_\infty$, $|b_1|_\infty$, and $|b_2|_\infty$ so that for all $(z_0, z_0^1) \in L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)$, the corresponding solution $(z, z(\cdot,1))$ of (1.9) fulfills the observability inequality

$$\mathbb{E}|z(T,\cdot)|^2_{L^2(G)} + \mathbb{E}|z(T,1)|^2 \leq C\,\mathbb{E}\int_0^T \int_{G_0} z^2\,\mathrm{d}x\mathrm{d}t. \tag{1.10}$$

To establish the observability inequality (1.7) (resp., (1.10)), we develop a new global Carleman estimate for solutions of equation (1.6) (resp., (1.9)). Our methodology for establishing these Carleman estimates follows the weighted identity method employed in [19, 21] for backward stochastic parabolic equations, and the duality method used in [14] for forward stochastic parabolic equations. However, our approach encounters some challenges due to several boundary terms arising from our boundary conditions (1.2)-(1.3), necessitating computations in a modified form. Additionally, the Carleman estimate for backward stochastic heat equations combines two weighted identities: one for the stochastic parabolic operator "$d + \partial_{xx}\mathrm{d}t$" and the other for the stochastic transport operator "$d - \partial_x \mathrm{d}t$".

Now, some remarks are in order.

Remark 1.1. Systems (1.5) and (1.8) can be regarded as a coupled system of dynamic equations, where the coupling is facilitated by the normal derivative term " $y_x(\cdot,1)$". Consequently, system (1.5) (resp., (1.8)) is directly controlled by the action of one localized control u in the drift part of the bulk equation and two additional controls v_1 and v_2 on the diffusion parts of the bulk and boundary equations (resp., by only one localized control u on the drift part of the bulk equation), while the control over the drift part of the boundary equation (resp., the whole boundary equation) is mediated through the coupling.

Remark 1.2. It is more natural to use only the control u in the drift part to establish the null controllability of (1.5). However, achieving this goal remains an open problem for forward stochastic heat equations due to the challenge of demonstrating the appropriate observability inequality for the corresponding adjoint backward equation. The primary difficulty stems from the presence of correction terms Z and $\widehat{Z}$, which are crucial for the well-posedness of such backward equations but pose challenges in Carleman estimates. In the literature, using the spectral method (Lebeau–Robbiano strategy), there are some interesting controllability results for this problem, particularly when the coefficients of the system are space-independent. For further details, we refer to [17, 31].

Remark 1.3. Applying the Carleman estimate in Theorem 3.3 (resp., Thm. 4.3), one can readily demonstrate the unique continuation property for equation (1.6) (resp., (1.9)). This property allows us to establish the approximate controllability of (1.5) (resp., (1.8)). For further details on the unique continuation for stochastic parabolic equations, see, for instance, [18, 20].

Remark 1.4. In our analysis, we aim to study the behavior of the solution $(y^\eta, y^\eta(\cdot,1))$ of equation (1.5) (and also the solution $(y^\eta, y^\eta(\cdot,1); Y^\eta, \widehat{Y}^\eta)$ of (1.8)) as η tends to infinity, with $\eta_2 \equiv 0$ and $\eta_1(t) \equiv \eta$ as a positive parameter. Our objective is to determine whether this solution converges to the solution of the corresponding Dirichlet problem on $(0,1)$, *i.e.*, with $y(t,0) = y(t,1) = 0$. Achieving this convergence result would enable us to deduce the well-known null controllability of stochastic heat equations with Dirichlet boundary conditions, as demonstrated in [14, 19]. However, this remains an open question in our research.

Remark 1.5. It would be interesting to study the controllability problems of equations (1.5) and (1.8) by incorporating dynamic boundary conditions alongside other boundary types, such as Neumann or Robin conditions. Additionally, investigating the controllability of their degenerate and singular forms could also be an interesting direction of research. For further results about the controllability of some degenerate and/or singular stochastic parabolic equations, we refer the readers to [15, 32, 33].

The rest of this paper is organized as follows. In the next section, we present the well-posedness results of our equations. Sections 3 and 4 are devoted to proving a Carleman estimate for the backward and forward stochastic heat equations, respectively, with Dirichlet-dynamic boundary conditions. Section 5 focuses on establishing our null controllability results for equations (1.5) and (1.8). Finally, the paper concludes with an appendix where we provide the proof of Proposition 4.2, which is essential for deriving our main Carleman estimate in Section 4.

2. Well-posedness results

In this section, we establish the well-posedness and regularity of solutions for our linear stochastic heat equations (1.5) and (1.6). The well-posedness of (1.8) and (1.9) can be deduced similarly. To achieve this, we will apply the semigroup approach. For more details on the well-posedness of stochastic evolution equations and backward stochastic evolution equations, we refer to [21, 28, 34, 35].

Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{L}^2 \to \mathbb{L}^2$ be the linear operator defined by

$$\mathcal{A}(\phi, \phi(1)) = (\phi_{xx}, -\phi_x(1)), \qquad \mathcal{D}(\mathcal{A}) = \mathbb{H}^2. \tag{2.1}$$

We have the following generating result.

Proposition 2.1. *The operator $\mathcal{A}$ is densely defined, dissipative, self-adjoint, and generates an analytic C_0-semigroup $(\mathrm{e}^{t\mathcal{A}})_{t\geq 0}$ on $\mathbb{L}^2$.*

Proof. Notice that $\{(y, y(1)) \in \mathbb{H}^1 : y \in C^\infty(\overline{G})\} \subset \mathbb{H}^2$ is dense in $\mathbb{L}^2$, so $\mathcal{A}$ is densely defined. For any $(\phi, \phi(1)) \in \mathbb{H}^2$, we have

$$\langle \mathcal{A}(\phi, \phi(1)), (\phi, \phi(1))\rangle_{\mathbb{L}^2} = -\int_G |\phi_x|^2 \,\mathrm{d}x \leq 0,$$

which implies that $\mathcal{A}$ is dissipative.

Following [8], let us introduce the bilinear form $\mathfrak{a} : \mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a}) \to \mathbb{R}$ defined by

$$\mathfrak{a}\left[(y, y(1)), (z, z(1))\right] = \int_G y_x z_x \,\mathrm{d}x + \int_G yz \,\mathrm{d}x + y(1)z(1),$$

with domain $\mathcal{D}(\mathfrak{a}) = \mathbb{H}^1$ on the Hilbert space $\mathbb{L}^2$. From [36], it is easy to show that $\mathfrak{a}$ is a densely defined, accretive, continuous, closed, and symmetric bilinear form. Thus, we can associate with the form $\mathfrak{a}$ an unbounded linear operator $\widetilde{\mathcal{A}}$ defined by

$$\begin{aligned} \mathcal{D}(\widetilde{\mathcal{A}}) = \Big\{&(y, y(1)) \in \mathbb{H}^1 : \text{ there exists } (\xi, \widehat{\xi}) \in \mathbb{L}^2 \text{ such that} \\ &\mathfrak{a}\left[(y, y(1)), (z, z(1))\right] = \langle (\xi, \widehat{\xi}), (z, z(1))\rangle_{\mathbb{L}^2} \text{ for all } \quad (z, z(1)) \in \mathbb{H}^1 \Big\}, \\ \widetilde{\mathcal{A}}(y, y(1)) &= (\xi, \widehat{\xi}) \quad \text{for all} \quad (y, y(1)) \in \mathcal{D}(\widetilde{\mathcal{A}}). \end{aligned}$$

It follows from Proposition 1.24 and Theorem 1.52 in [36] that the operator $-\widetilde{\mathcal{A}}$ is self-adjoint and generates an analytic C_0-semigroup on $\mathbb{L}^2$. Let us now show that $\widetilde{\mathcal{A}} = \mathrm{Id} - \mathcal{A}$.

Let $(y, y(1)) \in \mathbb{H}^2$ and $(z, z(1)) \in \mathbb{H}^1$. By integration by parts, we have

$$\begin{aligned}\mathfrak{a}\left[(y, y(1)),(z, z(1))\right] &= \int_G \left[-y_{xx} + y\right] z \,\mathrm{d}x + \left[y_x(1) + y(1)\right] z(1), \\ &= \langle (\mathrm{Id} - \mathcal{A})(y, y(1)), (z, z(1))\rangle_{\mathbb{L}^2}.\end{aligned}$$

Thus, $\mathbb{H}^2 \subset \mathcal{D}(\widetilde{\mathcal{A}})$ and $\widetilde{\mathcal{A}}(y, y(1)) = (\mathrm{Id} - \mathcal{A})(y, y(1))$. Therefore, $\widetilde{\mathcal{A}}$ is an extension of $\mathrm{Id} - \mathcal{A}$. Conversely, if $(y, y(1)) \in \mathcal{D}(\widetilde{\mathcal{A}})$, then there exists $(\xi, \widehat{\xi}) \in \mathbb{L}^2$ such that for any $(z, z(1)) \in \mathbb{H}^1$, we have

$$\mathfrak{a}\left[(y, y(1)),(z, z(1))\right] = \langle (\xi, \widehat{\xi}), (z, z(1))\rangle_{\mathbb{L}^2}. \tag{2.2}$$

In particular, if $z \in C_0^\infty(G)$, we deduce from (2.2) that

$$\int_G y_x z_x \,\mathrm{d}x + \int_G yz \,\mathrm{d}x = \int_G \xi z \,\mathrm{d}x.$$

This implies $-y_{xx} + y = \xi$ in $\mathcal{D}'(G)$, which gives $y_{xx} \in L^2(G)$. Since $y \in H^1(G)$, it follows that $y \in H^2(G)$. On the other hand, from (2.2), we have

$$\int_G \left[-y_{xx} + y\right] z \,\mathrm{d}x + \left[y_x(1) + y(1)\right] z(1) = \langle \widetilde{\mathcal{A}}(y, y(1)), (z, z(1))\rangle_{\mathbb{L}^2}.$$

Hence, $\mathcal{D}(\widetilde{\mathcal{A}}) \subset \mathbb{H}^2$ and $\widetilde{\mathcal{A}}(y, y(1)) = (\mathrm{Id} - \mathcal{A})(y, y(1))$. Therefore, we conclude that $\mathrm{Id} - \mathcal{A} = \widetilde{\mathcal{A}}$, and thus the operator $\mathcal{A}$ is self-adjoint and generates an analytic C_0-semigroup on $\mathbb{L}^2$. □

Let us first give the definitions of mild and weak solutions of equation (1.5).

Definition 2.2. a) A process $\mathcal{Y} = (y, y(\cdot, 1))$ is called a mild solution of (1.5) if for any $t \in [0, T]$, we have that

$$\mathcal{Y}(t) = \mathrm{e}^{t\mathcal{A}}\mathcal{Y}_0 + \int_0^t \mathrm{e}^{(t-s)\mathcal{A}} F_1(s, \mathcal{Y}(s)) \,\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)\mathcal{A}} F_2(s, \mathcal{Y}(s)) \,\mathrm{d}W(s), \quad \mathbb{P}\text{-a.s.},$$

where

$$\mathcal{Y}_0 = \begin{pmatrix} y_0 \\ y_0^1 \end{pmatrix}, \quad F_1(s, \mathcal{Y}(s)) = \begin{pmatrix} \beta_1 y + \mathbb{1}_{G_0} u \\ \eta_1 y(\cdot, 1) \end{pmatrix}, \quad \text{and} \quad F_2(s, \mathcal{Y}(s)) = \begin{pmatrix} \beta_2 y + v_1 & 0 \\ 0 & \eta_2 y(\cdot, 1) + v_2 \end{pmatrix}.$$

b) A process $(y, y(\cdot, 1))$ is said to be a weak solution of (1.5) if for any $t \in [0, T]$ and any $(\phi, \phi(1)) \in \mathbb{H}^1$, it holds that

$$\begin{aligned}&\int_G (y(t, x) - y_0(x))\phi(x) \,\mathrm{d}x + (y(t, 1) - y_0^1)\phi(1) \\ &= -\int_0^t \int_G y_x(s, x)\phi_x(x) \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_G \beta_1(s, x) y(s, x)\phi(x) \,\mathrm{d}x \,\mathrm{d}s \\ &\quad + \int_0^t \int_G \mathbb{1}_{G_0}(x) u(s, x)\phi(x) \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_G \beta_2(s, x) y(s, x)\phi(x) \,\mathrm{d}x \,\mathrm{d}W^1(s) \\ &\quad + \int_0^t \int_G v_1(s, x)\phi(x) \,\mathrm{d}x \,\mathrm{d}W^1(s) + \int_0^t \eta_1(s) y(s, 1)\phi(1) \,\mathrm{d}s \\ &\quad + \int_0^t \eta_2(s) y(s, 1)\phi(1) \,\mathrm{d}W^2(s) + \int_0^t v_2(s)\phi(1) \,\mathrm{d}W^2(s), \;\; \mathbb{P}\text{-a.s.}\end{aligned}$$

Similar to (1.5), we also provide the following definitions of mild and weak solutions for the backward equation (1.6).

Definition 2.3. a) We say that $(\mathcal{Z}, \widehat{\mathcal{Z}}) = (z, z(\cdot,1); Z, \widehat{Z})$ is a mild solution of (1.6) if for any $t \in [0,T]$

$$\mathcal{Z}(t) = \mathrm{e}^{(T-t)\mathcal{A}}\mathcal{Z}_T - \int_t^T \mathrm{e}^{(s-t)\mathcal{A}} F(s, \mathcal{Z}(s), \widehat{\mathcal{Z}}(s))\,\mathrm{d}s - \int_t^T \mathrm{e}^{(s-t)\mathcal{A}} \widehat{\mathcal{Z}}(s)\,\mathrm{d}W(s), \ \mathbb{P}\text{-a.s.},$$

where

$$\mathcal{Z}_T = \begin{pmatrix} z_T \\ z_T^1 \end{pmatrix}, \quad F(s, \mathcal{Z}(s), \widehat{\mathcal{Z}}(s)) = \begin{pmatrix} a_1 z + a_2 Z \\ b_1 z(\cdot,1) + b_2 \widehat{Z} \end{pmatrix}, \quad \text{and} \quad \widehat{\mathcal{Z}}(s) = \begin{pmatrix} Z & 0 \\ 0 & \widehat{Z} \end{pmatrix}.$$

b) A process $(z, z(\cdot,1); Z, \widehat{Z})$ is called a weak solution of (1.6) if for any $t \in [0,T]$ and all $(\phi, \phi(1)) \in \mathbb{H}^1$, we have that

$$\begin{aligned}
&\int_G (z_T(x) - z(t,x))\phi(x)\,\mathrm{d}x + (z_T^1 - z(t,1))\phi(1) \\
&= \int_t^T \int_G z_x(s,x)\phi_x(x)\,\mathrm{d}x\,\mathrm{d}s + \int_t^T \int_G a_1(s,x) z(s,x)\phi(x)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_t^T \int_G a_2(s,x) Z(s,x)\phi(x)\,\mathrm{d}x\,\mathrm{d}s + \int_t^T \int_G Z(s,x)\phi(x)\,\mathrm{d}x\,\mathrm{d}W^1(s) \\
&\quad + \int_t^T b_1(s) z(s,1)\phi(1)\,\mathrm{d}s + \int_t^T b_2(s)\widehat{Z}(s)\phi(1)\,\mathrm{d}s + \int_t^T \widehat{Z}(s)\phi(1)\,\mathrm{d}W^2(s), \ \mathbb{P}\text{-a.s.}
\end{aligned}$$

We have the following well-posedness result for equation (1.5).

Proposition 2.4. *1. Let $(y_0, y_0^1) \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{L}^2)$ and $(u, v_1, v_2) \in L^2_{\mathcal{F}}(0,T; L^2(G_0)) \times L^2_{\mathcal{F}}(0,T; L^2(G)) \times L^2_{\mathcal{F}}(0,T)$. Then, there exists a unique mild solution*

$$(y, y(\cdot,1)) \in L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{L}^2)) \bigcap L^2_{\mathcal{F}}(0,T; \mathbb{H}^1),$$

of equation (1.5) such that

$$\begin{aligned}
&|(y, y(\cdot,1))|_{L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{L}^2))} + |(y, y(\cdot,1))|_{L^2_{\mathcal{F}}(0,T;\mathbb{H}^1)} \\
&\leq C \left(|(y_0, y_0^1)|_{L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)} + |u|_{L^2_{\mathcal{F}}(0,T;L^2(G_0))} + |v_1|_{L^2_{\mathcal{F}}(0,T;L^2(G))} + |v_2|_{L^2_{\mathcal{F}}(0,T)} \right),
\end{aligned} \tag{2.3}$$

where C is a positive constant depending on G, T, β_1, β_2, η_1, and η_2.
2. A process $(y, y(\cdot,1))$ is a weak solution of (1.5) if and only if it is a mild solution to the same equation.

Proof. It is easy to see that the system (1.5) can be written as the following abstract Cauchy problem

$$\begin{cases} d\mathcal{Y} = \big(\mathcal{A}\mathcal{Y} + F_1(t, \mathcal{Y})\big)\,\mathrm{d}t + F_2(t, \mathcal{Y})\,\mathrm{d}W(t), \\ \mathcal{Y}(0) = \mathcal{Y}_0, \end{cases} \tag{2.4}$$

where $\mathcal{Y}_0 = (y_0, y_0^1)$ and $\mathcal{A}$ is the operator defined by (2.1).

The function $F_1 : [0,T] \times \Omega \times \mathbb{L}^2 \to \mathbb{L}^2$ is given by

$$F_1(t,(\mathbf{y},\mathbf{z})) = \begin{pmatrix} \beta_1 \mathbf{y} + \mathbb{1}_{G_0} u \\ \eta_1 \mathbf{z} \end{pmatrix},$$

and $F_2 : [0,T] \times \Omega \times \mathbb{L}^2 \to \mathcal{L}(\mathbb{R}^2; \mathbb{L}^2)$ is defined as

$$F_2(t,(\mathbf{y},\mathbf{z})) = \begin{pmatrix} \beta_2 \mathbf{y} + v_1 & 0 \\ 0 & \eta_2 \mathbf{z} + v_2 \end{pmatrix}.$$

Let $t \in [0,T]$ and $(\mathbf{y},\mathbf{z}), (\mathbf{y}',\mathbf{z}') \in \mathbb{L}^2$. It is easy to verify that

$$|F_1(t,(\mathbf{y},\mathbf{z})) - F_1(t,(\mathbf{y}',\mathbf{z}'))|_{\mathbb{L}^2} \leq \max(|\beta_1|_\infty, |\eta_1|_\infty) |(\mathbf{y},\mathbf{z}) - (\mathbf{y}',\mathbf{z}')|_{\mathbb{L}^2}, \quad \mathbb{P}\text{-a.s.}$$

and

$$|F_2(t,(\mathbf{y},\mathbf{z})) - F_2(t,(\mathbf{y}',\mathbf{z}'))|_{\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2)} \leq \max(|\beta_2|_\infty, |\eta_2|_\infty) |(\mathbf{y},\mathbf{z}) - (\mathbf{y}',\mathbf{z}')|_{\mathbb{L}^2}, \quad \mathbb{P}\text{-a.s.}$$

Thus, F_1 and F_2 are globally Lipschitz. Applying the abstract well-posedness result from [21], Theorem 3.24 to equation (2.4), we conclude that there exists a unique mild solution

$$\mathcal{Y} = (y,z) \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{L}^2)) \bigcap L^2_{\mathcal{F}}(0,T;\mathcal{D}((-\mathcal{A})^{1/2})),$$

such that

$$\begin{aligned} &|\mathcal{Y}|_{L^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{L}^2))} + |\mathcal{Y}|_{L^2_{\mathcal{F}}(0,T;\mathcal{D}((-\mathcal{A})^{1/2}))} \\ &\leq C \left(|\mathcal{Y}_0|_{L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)} + |u|_{L^2_{\mathcal{F}}(0,T;L^2(G_0))} + |v_1|_{L^2_{\mathcal{F}}(0,T;L^2(G))} + |v_2|_{L^2_{\mathcal{F}}(0,T)} \right), \end{aligned} \tag{2.5}$$

where C is a positive constant depending on the data of the problem. On the other hand, Theorem 4.36 of [37] yields that

$$\mathcal{D}((-\mathcal{A})^{1/2}) = (\mathbb{L}^2, \mathcal{D}(\mathcal{A}))_{1/2,2} = (\mathbb{L}^2, \mathbb{H}^2)_{1/2,2} = \mathbb{H}^1, \tag{2.6}$$

where $(\cdot,\cdot)_{1/2,2}$ denotes the real interpolation functor (see [37, 38] for more details). Therefore, the solution $(y,z) \in \mathbb{H}^1$ in space and from the definition of $\mathbb{H}^1$ in (1.4), we deduce that $z = y(\cdot,1)$. Moreover, the estimate (2.3) immediately follows from (2.5). This completes the proof of the well-posedness result for equation (1.5). The equivalence between mild and weak solutions for equation (1.5) is established for any abstract equation of the form (2.4). For this result, we refer to [21], Theorem 3.1. $\square$

The following result shows the well-posedness of equation (1.6).

Proposition 2.5. *1. Let $(z_T, z_T^1) \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$. Then, there exists a unique mild solution*

$$(z, z(\cdot,1); Z, \widehat{Z}) \in \left(L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{L}^2)) \bigcap L^2_{\mathcal{F}}(0,T;\mathbb{H}^1) \right) \times L^2_{\mathcal{F}}(0,T;\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2)),$$

of equation (1.6) such that

$$|(z,z(\cdot,1))|_{L^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{L}^2))} + |(z,z(\cdot,1))|_{L^2_{\mathcal{F}}(0,T;\mathbb{H}^1)} + |(Z,\widehat{Z})|_{L^2_{\mathcal{F}}(0,T;\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2))} \leq C\, |(z_T, z_T^1)|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)}, \tag{2.7}$$

where C is a positive constant depending on G, T, a_1, a_2, b_1, and b_2.

2. A process $(z,z(\cdot,1);Z,\widehat{Z})$ is a weak solution of (1.6) if and only if it is a mild solution to the same equation.

Proof. Notice that equation (1.6) can be rewritten as the following abstract backward Cauchy problem

$$\begin{cases} d\mathcal{Z} = \big(-\mathcal{A}\mathcal{Z} + F(t,\mathcal{Z},\widehat{\mathcal{Z}})\big)\,\mathrm{d}t + \widehat{\mathcal{Z}}\,\mathrm{d}W(t), \\ \mathcal{Z}(T) = \mathcal{Z}_T, \end{cases} \tag{2.8}$$

where $\mathcal{Z}_T = (z_T, z_T^1)$ and $\mathcal{A}$ is the same operator defined in (2.1).

The function $F : [0,T]\times\Omega\times\mathbb{L}^2\times\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2)\to\mathbb{L}^2$ is defined by

$$F(t,\mathbf{Z},\widehat{\mathbf{Z}}) = \begin{pmatrix} a_1\mathbf{z} + a_2 Z \\ b_1\mathbf{y} + b_2\widehat{Z} \end{pmatrix},$$

with

$$\mathbf{Z} = (\mathbf{z},\mathbf{y})^\top \in \mathbb{L}^2 \quad \text{and} \quad \widehat{\mathbf{Z}} = \begin{pmatrix} Z & 0 \\ 0 & \widehat{Z} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2;\mathbb{L}^2).$$

We first note that

$$F(t,\mathbf{Z},\widehat{\mathbf{Z}}) = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}\mathbf{Z} + \widehat{\mathbf{Z}}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

Thus, for $\mathbf{Z}_1,\mathbf{Z}_2\in\mathbb{L}^2$ and $\widehat{\mathbf{Z}}_1,\widehat{\mathbf{Z}}_2\in\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2)$, we can easily show that

$$|F(t,\mathbf{Z}_1,\widehat{\mathbf{Z}}_1) - F(t,\mathbf{Z}_2,\widehat{\mathbf{Z}}_2)|_{\mathbb{L}^2} \le C\left(|\mathbf{Z}_1-\mathbf{Z}_2|_{\mathbb{L}^2} + |\widehat{\mathbf{Z}}_1-\widehat{\mathbf{Z}}_2|_{\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2)}\right), \quad \mathbb{P}\text{-a.s.}, \tag{2.9}$$

where C is a positive constant depending on $|a_1|_\infty$, $|a_2|_\infty$, $|b_1|_\infty$, and $|b_2|_\infty$. Now, using the well-posedness result from [21], Theorem 4.11 for equation (2.8), we deduce that there exists a unique mild solution

$$(\mathcal{Z},\widehat{\mathcal{Z}}) \in \Big(L^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{L}^2))\bigcap L^2_{\mathcal{F}}(0,T;\mathcal{D}((-\mathcal{A})^{1/2}))\Big)\times L^2_{\mathcal{F}}(0,T;\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2)),$$

so that

$$|\mathcal{Z}|_{L^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{L}^2))} + |\mathcal{Z}|_{L^2_{\mathcal{F}}(0,T;\mathcal{D}((-\mathcal{A})^{1/2}))} + |\widehat{\mathcal{Z}}|_{L^2_{\mathcal{F}}(0,T;\mathcal{L}(\mathbb{R}^2;\mathbb{L}^2))} \le C\left|(z_T,z_T^1)\right|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)}, \tag{2.10}$$

where C is a positive constant depending on the data. Let us denote $(\mathcal{Z},\widehat{\mathcal{Z}})$ by $(z,y;Z,\widehat{Z})$. Recalling (2.6) and (1.4), we have $\mathcal{D}((-\mathcal{A})^{1/2}) = \mathbb{H}^1$, and it follows that $y = z(\cdot,1)$. The estimate (2.7) is a direct consequence of (2.10). This completes the proof of the well-posedness of equation (1.6). The second point of the proposition is deduced from [21], Theorem 4.8, which provides an abstract result of the equivalence between weak and mild solutions of backward stochastic evolution equations. □

Remark 2.6. In Section 3, the Carleman estimate requires the space regularity $\mathbb{H}^2$ for solutions instead of $\mathbb{H}^1$. However, the mild (or weak) solution of (1.6) does not have enough space regularity. Therefore, to address this issue, one can employ an approximating approach for (1.6) and then follow some results in [21], Theorem 4.12 for backward stochastic evolution equations. Here, without loss of generality and for the sake of presentation simplicity, we assume that the solutions have $\mathbb{H}^2$-space regularity.

3. Global Carleman estimate for backward stochastic heat equations

In this section, we derive a Carleman estimate for the general backward stochastic heat equation

$$\begin{cases} \mathrm{d}z + z_{xx}\,\mathrm{d}t = F_0\,\mathrm{d}t + Z\,\mathrm{d}W^1(t), & (t,x)\in(0,T)\times G,\\ z(t,0)=0, & t\in(0,T),\\ \mathrm{d}z(t,1) - z_x(t,1)\,\mathrm{d}t = F_1\,\mathrm{d}t + \widehat{Z}\,\mathrm{d}W^2(t), & t\in(0,T),\\ (z(T,x),z(T,1)) = (z_T(x), z_T^1), & x\in G, \end{cases} \tag{3.1}$$

where $(z_T, z_T^1)\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$, $F_0\in L^2_{\mathcal{F}}(0,T;L^2(G))$, and $F_1\in L^2_{\mathcal{F}}(0,T)$.

We first introduce the following known result. See [4] or [39], Lemma 2.68 for the proof.

Lemma 3.1. *For any nonempty open subset $G_1 \Subset G$, there exists a function $\psi\in C^4(\overline{G})$ such that*

$$\psi>0 \ \text{ in } G\,, \qquad \psi(0)=\psi(1)=0\,, \qquad |\psi_x(x)|>0 \ \text{ for all } \ x\in\overline{G}\setminus G_1. \tag{3.2}$$

Notice that from the above lemma, we deduce that

$$\psi_x(0)>0 \qquad \text{and} \qquad \psi_x(1)<0. \tag{3.3}$$

Remark 3.2. In the 1-D case, it is possible to provide an explicit formula for a function ψ that satisfies properties (3.2). For example:

- If $\frac{1}{2}\in G_1$, we can readily choose $\psi(x)=x(1-x)$.
- For the general case, let $a\in G_1$ then we construct the function ψ as follows

$$\psi_a(x) = \begin{cases} \dfrac{(1-a)^3}{a^3}\Big[a^3-(a-x)^3\Big], & x\in[0,a],\\ (1-a)^3-(x-a)^3, & x\in[a,1]. \end{cases}$$

For any parameters $\lambda>1$ and $\mu>1$, we choose the following weight functions

$$\alpha=\alpha(t,x)=\frac{\mathrm{e}^{\mu\psi(x)}-\mathrm{e}^{2\mu|\psi|_\infty}}{t(T-t)}\,, \qquad \ell=\lambda\alpha\,, \qquad \theta=\mathrm{e}^{\ell}, \qquad \varphi=\varphi(t,x)=\frac{\mathrm{e}^{\mu\psi(x)}}{t(T-t)}, \tag{3.4}$$

where ψ is the function defined in Lemma 3.1. It is easy to check that there exists a constant $C>0$ depending on G and T such that

$$\begin{aligned} &\varphi\geq C, \qquad |\varphi_t|\leq C\varphi^2, \qquad |\varphi_{tt}|\leq C\varphi^3,\\ &|\alpha_t|\leq C\mathrm{e}^{2\mu|\psi|_\infty}\varphi^2, \qquad |\alpha_{tt}|\leq C\mathrm{e}^{2\mu|\psi|_\infty}\varphi^3. \end{aligned} \tag{3.5}$$

In the rest of this section, let ψ be the function given in Lemma 3.1 with G_1 being any fixed nonempty open subset of G such that $G_1\Subset G_0$. Then, the desired global Carleman estimate for equation (3.1) is stated as follows.

Theorem 3.3. *There exists a constant $\mu_0>1$ depending only on G, G_0 and T such that for all $\mu\geq\mu_0$, one can find two constants $C=C(\mu)>0$ and $\lambda_0=\lambda_0(\mu)>1$ such that for all $\lambda\geq\lambda_0$ and for any $F_0\in L^2_{\mathcal{F}}(0,T;L^2(G))$,*

$F_1 \in L^2_{\mathcal{F}}(0,T)$, and $(z_T, z_T^1) \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$, the corresponding solution $(z, z(\cdot,1); Z, \widehat{Z})$ of (3.1) satisfies that

$$\begin{aligned}
&\lambda^3\mu^4\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \lambda^3\mu^3\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t + \lambda\mu^2\mathbb{E}\int_0^T\int_G \theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t\\
&\leq C\Big[\lambda^3\mu^4\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G \theta^2 F_0^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \theta^2(t,1)F_1^2(t)\,\mathrm{d}t\\
&\qquad + \lambda^2\mu^2\mathbb{E}\int_0^T\int_G \theta^2\varphi^2 Z^2\,\mathrm{d}x\mathrm{d}t + \lambda^2\mu\mathbb{E}\int_0^T \theta^2(t,1)\varphi^2(t,1)\widehat{Z}^2(t)\,\mathrm{d}t\Big].
\end{aligned}\tag{3.6}$$

To prove Theorem 3.3, we employ two weighted identities: one for the stochastic parabolic operator "$\mathrm{d}z + z_{xx}\mathrm{d}t$" and the other for the stochastic transport operator "$\mathrm{d}z - z_x\mathrm{d}t$". For simplicity, we denote by z_x (resp., z_{xx}) the first (resp., the second) partial derivative of the function z w.r.t the variable x and so on. In this context and subsequently, z_x^2 denotes $|z_x|^2$. Let us now consider an auxiliary function $\Psi \in C^{1,2}((0,T)\times G)$ and let

$$A = \ell_x^2 - \ell_{xx} - \Psi - \ell_t\,, \qquad B = 2[A\Psi + (A\ell_x)_x] - A_t + \Psi_{xx}, \tag{3.7}$$

where ℓ is the function defined in (3.4). We have the following weighted identities, which are derived from [21], Theorem 9.26 and [21], Proposition 8.9, respectively. These identities will play a key role in establishing the estimate (3.6).

Lemma 3.4. *1. Let z be an $H^2(G)$-valued Itô process. Set $\theta = \mathrm{e}^\ell$ and $v = \theta z$. Then, for any $(t,x) \in (0,T)\times \overline{G}$, we have that*

$$\begin{aligned}
&2\theta(v_{xx} + Av)(\mathrm{d}z + z_{xx}\mathrm{d}t) - 2(v_x dv)_x + 2\Big[\ell_x v_x^2 - \Psi v v_x + \Big(A\ell_x + \frac{\Psi_x}{2}\Big)v^2\Big]_x\mathrm{d}t\\
&= 2\ell_{xx}v_x^2\mathrm{d}t - 2\Psi v_x^2\mathrm{d}t + Bv^2\mathrm{d}t - d(v_x^2 - Av^2) + 2(v_{xx}+Av)^2\mathrm{d}t\\
&\quad + \theta^2|\mathrm{d}z_x + \ell_x\mathrm{d}z|^2 - \theta^2 A(\mathrm{d}z)^2, \qquad \mathbb{P}\text{-a.s.}
\end{aligned}\tag{3.8}$$

2. Let z be an $H^1(G)$-valued Itô process. Set $\theta = \mathrm{e}^\ell$ and $v = \theta z$. Then, for all $(t,x) \in (0,T)\times\overline{G}$, we have

$$\begin{aligned}
-\theta(\ell_t - \ell_x)v(\mathrm{d}z - z_x\mathrm{d}t) = &-\frac{1}{2}d[(\ell_t-\ell_x)v^2] + (\ell_t - \ell_x)vv_x\mathrm{d}t + \frac{1}{2}(\ell_{tx} - \ell_{xx})v^2\mathrm{d}t\\
&+ \frac{1}{2}[\ell_{tt} + \ell_{xx} - 2\ell_{tx}]v^2\mathrm{d}t + \frac{1}{2}(\ell_t - \ell_x)(\mathrm{d}v)^2\\
&+ (\ell_t-\ell_x)^2v^2\mathrm{d}t, \qquad \mathbb{P}\text{-a.s.}
\end{aligned}\tag{3.9}$$

Let us now give the proof of our Carleman estimate (3.6).

Proof of Theorem 3.3. The proof is divided into four steps.
Step 1. Choosing the auxiliary function $\Psi = -2\ell_{xx}$ in (3.7) and (3.8), integrating (3.8) on $(0,T)\times G$ and taking the expectation on both sides, we get that

$$\begin{aligned}
&6\mathbb{E}\int_0^T\int_G \ell_{xx}v_x^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G Bv^2\,\mathrm{d}x\mathrm{d}t\\
&\leq \mathbb{E}\int_0^T\int_G\theta^2F_0^2\,\mathrm{d}x\mathrm{d}t - 2\mathbb{E}\int_0^T\int_G (v_xdv)_x\,\mathrm{d}x + \mathbb{E}\int_0^T\int_G\theta^2AZ^2\,\mathrm{d}x\mathrm{d}t\\
&\quad + 2\mathbb{E}\int_0^T\int_G\Big[\ell_xv_x^2 + 2\ell_{xx}vv_x + (A\ell_x - \ell_{xxx})v^2\Big]_x\,\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.10}$$

Using the identity (3.9) for $x = 1$, integrating the resulting equality on $(0, T)$ and taking the expectation on both sides, it follows that

$$
\begin{aligned}
&2\mathbb{E}\int_0^T (\ell_t(t,1)-\ell_x(t,1))^2 v^2(t,1)\,\mathrm{d}t + \mathbb{E}\int_0^T (\ell_{tt}(t,1)-\ell_{tx}(t,1))v^2(t,1)\,\mathrm{d}t \\
&= -2\mathbb{E}\int_0^T (\ell_t(t,1)-\ell_x(t,1))v(t,1)v_x(t,1)\,\mathrm{d}t - 2\mathbb{E}\int_0^T \theta(t,1)(\ell_t(t,1)-\ell_x(t,1))v(t,1)F_1(t)\,\mathrm{d}t \\
&\quad + \mathbb{E}\int_0^T \theta^2(t,1)(\ell_t(t,1)-\ell_x(t,1))\widehat{Z}^2(t)\,\mathrm{d}t.
\end{aligned}
\tag{3.11}
$$

Applying Young's inequality for the second term on the right-hand side of (3.11), we find that

$$
\begin{aligned}
\mathbb{E}\int_0^T (\ell_{tt}(t,1)-\ell_{tx}(t,1))v^2(t,1)\,\mathrm{d}t \le &-2\mathbb{E}\int_0^T (\ell_t(t,1)-\ell_x(t,1))v(t,1)v_x(t,1)\,\mathrm{d}t \\
&+ \mathbb{E}\int_0^T \theta^2(t,1)F_1^2(t)\,\mathrm{d}t \\
&+ \mathbb{E}\int_0^T \theta^2(t,1)(\ell_t(t,1)-\ell_x(t,1))\widehat{Z}^2(t)\,\mathrm{d}t.
\end{aligned}
\tag{3.12}
$$

Adding (3.10) and (3.12), we obtain that

$$
\begin{aligned}
&6\mathbb{E}\int_0^T\int_G \ell_{xx}v_x^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G Bv^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T (\ell_{tt}(t,1)-\ell_{tx}(t,1))v^2(t,1)\,\mathrm{d}t \\
&\le \mathbb{E}\int_0^T\int_G \theta^2 F_0^2\,\mathrm{d}x\mathrm{d}t - 2\mathbb{E}\int_0^T\int_G (v_x dv)_x\,\mathrm{d}x + \mathbb{E}\int_0^T\int_G \theta^2 A Z^2\,\mathrm{d}x\mathrm{d}t \\
&\quad + 2\mathbb{E}\int_0^T\int_G \Big[\ell_x v_x^2 + 2\ell_{xx}vv_x + (A\ell_x - \ell_{xxx})v^2\Big]_x\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \theta^2(t,1)F_1^2(t)\,\mathrm{d}t \\
&\quad - 2\mathbb{E}\int_0^T (\ell_t(t,1)-\ell_x(t,1))v(t,1)v_x(t,1)\,\mathrm{d}t + \mathbb{E}\int_0^T \theta^2(t,1)(\ell_t(t,1)-\ell_x(t,1))\widehat{Z}^2(t)\,\mathrm{d}t.
\end{aligned}
\tag{3.13}
$$

Step 2. It is easy to check that

$$
\begin{aligned}
&\ell_t = \lambda\alpha_t, \qquad \ell_x = \lambda\mu\varphi\psi_x, \qquad \ell_{tt} = \lambda\alpha_{tt}, \\
&\ell_{tx} = \lambda\mu\varphi_t\psi_x, \qquad \ell_{xx} = \lambda\mu^2\varphi\psi_x^2 + \lambda\mu\varphi\psi_{xx}, \\
&\ell_{xxx} = \lambda\mu^3\varphi\psi_x\psi_x^2 + 3\lambda\mu^2\varphi\psi_x\psi_{xx} + \lambda\mu\varphi\psi_{xxx}.
\end{aligned}
\tag{3.14}
$$

In what follows, for a positive integer n, we denote by $O(\mu^n)$ a function of order μ^n for a sufficiently large μ (which is independent of λ). Similarly, we use the notation $O(\mathrm{e}^{\mu|\psi|_\infty})$, and so on. According to [21], Theorem 9.27, it is evident that when $|\psi_x(x)| > 0$, for a sufficiently large μ and for any $t \in [0, T]$, one has

$$
\begin{cases}
A = \lambda^2\mu^2\varphi^2\psi_x^2 + \lambda\mu^2\varphi\psi_x^2 + \lambda\mu\varphi\psi_{xx} + \lambda\varphi^2 O(\mathrm{e}^{2\mu|\psi|_\infty}), \\
B \ge 2\lambda^3\mu^4\varphi^3\psi_x^4 + \lambda^3\varphi^3 O(\mu^3) + \lambda^2\varphi^3 O(\mu^2\mathrm{e}^{2\mu|\psi|_\infty}) + \lambda\varphi^3 O(\mathrm{e}^{2\mu|\psi|_\infty}).
\end{cases}
\tag{3.15}
$$

Step 3. By some computations, we have that

$$\begin{aligned} -2\mathbb{E}\int_0^T\int_G (v_x dv)_x\,\mathrm{d}x =& -2\mathbb{E}\int_0^T \ell_x(t,1)\ell_t(t,1)\theta^2(t,1)z^2(t,1)\,\mathrm{d}t - 2\mathbb{E}\int_0^T \theta^2(t,1)z_x^2(t,1)\,\mathrm{d}t\\ &-2\mathbb{E}\int_0^T (\ell_t(t,1)+\ell_x(t,1))\theta^2(t,1)z(t,1)z_x(t,1)\,\mathrm{d}t\\ &-2\mathbb{E}\int_0^T \theta^2(t,1)z_x(t,1)F_1(t)\,\mathrm{d}t - 2\mathbb{E}\int_0^T \ell_x(t,1)\theta^2(t,1)z(t,1)F_1(t)\,\mathrm{d}t. \end{aligned} \tag{3.16}$$

From (3.14) and (3.5), there exists a large $\mu_0 > 1$ such that for all $\mu \geq \mu_0$, the equality (3.16) implies

$$\begin{aligned} &-2\mathbb{E}\int_0^T\int_G (v_x dv)_x\,\mathrm{d}x\\ &\leq C\lambda^2\mu\,\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T \varphi^3(t,1)\theta^2(t,1)z^2(t,1)\,\mathrm{d}t - 2\mathbb{E}\int_0^T \theta^2(t,1)z_x^2(t,1)\,\mathrm{d}t\\ &\quad + C\lambda\,\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T \varphi^2(t,1)\theta^2(t,1)|z(t,1)||z_x(t,1)|\,\mathrm{d}t + 2\mathbb{E}\int_0^T \theta^2(t,1)|z_x(t,1)||F_1(t)|\,\mathrm{d}t\\ &\quad + C\lambda\mu\mathbb{E}\int_0^T \varphi(t,1)\theta^2(t,1)|z(t,1)||F_1(t)|\,\mathrm{d}t. \end{aligned} \tag{3.17}$$

Using Young's inequality for the last three terms on the right-hand side of (3.17), we get that

$$\begin{aligned} &-2\mathbb{E}\int_0^T\int_G (v_x dv)_x\,\mathrm{d}x\\ &\leq C\lambda^2\mu\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T \varphi^3(t,1)\theta^2(t,1)z^2(t,1)\,\mathrm{d}t + C\mathbb{E}\int_0^T \theta^2(t,1)F_1^2(t)\,\mathrm{d}t\\ &\quad + C\lambda^2\,\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T \varphi^3(t,1)\theta^2(t,1)z^2(t,1)\,\mathrm{d}t + C\lambda^2\mu^2\mathbb{E}\int_0^T \varphi^3(t,1)\theta^2(t,1)z^2(t,1)\,\mathrm{d}t\\ &\quad + C\,\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T \varphi(t,1)\theta^2(t,1)z_x^2(t,1)\,\mathrm{d}t + C\mathbb{E}\int_0^T \varphi(t,1)\theta^2(t,1)z_x^2(t,1)\,\mathrm{d}t. \end{aligned} \tag{3.18}$$

Now, by choosing a large enough μ_0 in (3.18), we arrive at

$$\begin{aligned} -2\mathbb{E}\int_0^T\int_G (v_x dv)_x\,\mathrm{d}x \leq& \; C\lambda^2\mu\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t + C\mathbb{E}\int_0^T \theta^2(t,1)F_1^2(t)\,\mathrm{d}t\\ &+ C\,\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T \theta^2(t,1)\varphi(t,1)z_x^2(t,1)\,\mathrm{d}t. \end{aligned} \tag{3.19}$$

It is easy to see that

$$\begin{aligned} -2\mathbb{E}\int_0^T (\ell_t(t,1)-\ell_x(t,1))v(t,1)v_x(t,1)\,\mathrm{d}t =& -2\mathbb{E}\int_0^T \ell_x(t,1)(\ell_t(t,1)-\ell_x(t,1))\theta^2(t,1)z^2(t,1)\,\mathrm{d}t\\ &-2\mathbb{E}\int_0^T (\ell_t(t,1)-\ell_x(t,1))\theta^2(t,1)z(t,1)z_x(t,1)\,\mathrm{d}t. \end{aligned} \tag{3.20}$$

Using (3.14) and taking a large enough μ_0, the equality (3.20) provides that

$$\begin{aligned}-2\mathbb{E}\int_0^T(\ell_t(t,1)-\ell_x(t,1))v(t,1)v_x(t,1)\,\mathrm{d}t\leq\ & C\lambda^2\mu\,\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T\varphi^3(t,1)\theta^2(t,1)z^2(t,1)\,\mathrm{d}t\\ &+C\lambda\,\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T\varphi^2(t,1)\theta^2(t,1)|z(t,1)||z_x(t,1)|\,\mathrm{d}t.\end{aligned}\tag{3.21}$$

Applying Young's inequality for the last term on the right-hand side of (3.21), and taking a large enough μ_0, we get that

$$\begin{aligned}-2\mathbb{E}\int_0^T(\ell_t(t,1)-\ell_x(t,1))v(t,1)v_x(t,1)\,\mathrm{d}t\leq\ & C\lambda^2\mu\,\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\\ &+C\,\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T\theta^2(t,1)\varphi(t,1)z_x^2(t,1)\,\mathrm{d}t.\end{aligned}\tag{3.22}$$

From (3.14), we also have

$$\mathbb{E}\int_0^T(\ell_{tt}(t,1)-\ell_{tx}(t,1))v^2(t,1)\,\mathrm{d}t=\mathbb{E}\int_0^T\big[\lambda\alpha_{tt}(t,1)-\lambda\mu\varphi_t(t,1)\psi_x(1)\big]\theta^2(t,1)z^2(t,1)\,\mathrm{d}t.$$

Recalling (3.5) and taking a large μ_0, it follows that

$$\mathbb{E}\int_0^T(\ell_{tt}(t,1)-\ell_{tx}(t,1))v^2(t,1)\,\mathrm{d}t\geq-C\lambda\mathrm{e}^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t.\tag{3.23}$$

On the other hand, by (3.14) and (3.15), it is easy to see that

$$\begin{aligned}&6\mathbb{E}\int_0^T\int_G\ell_{xx}v_x^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G Bv^2\,\mathrm{d}x\mathrm{d}t\\ &\geq\ \mathbb{E}\int_0^T\int_G\Big[6(\lambda\mu^2\varphi\psi_x^2+\lambda\mu\varphi\psi_{xx})v_x^2+(2\lambda^3\mu^4\varphi^3\psi_x^4+\lambda^3\varphi^3O(\mu^3)\\ &\qquad+\lambda^2\varphi^3O(\mu^2\mathrm{e}^{2\mu|\psi|_\infty})+\lambda\varphi^3O(\mathrm{e}^{2\mu|\psi|_\infty}))v^2\Big]\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.24}$$

From Lemma 3.1, we know that $\min_{x\in\overline{G}\setminus G_1}|\psi_x|>0$. Then, there exists a large $\mu_0>1$ such that for all $\mu\geq\mu_0$, one can find $\lambda_0=\lambda_0(\mu)>0$ so that for any $\lambda\geq\lambda_0$, it holds that

$$\begin{aligned}&\mathbb{E}\int_0^T\int_{G\setminus G_1}\Big[6\big(\lambda\mu^2\varphi\min_{x\in\overline{G}\setminus G_1}\psi_x^2+\lambda\mu\varphi\psi_{xx}\big)v_x^2+\big(2\lambda^3\mu^4\varphi^3\min_{x\in\overline{G}\setminus G_1}\psi_x^4\\ &\qquad+\lambda^3\varphi^3O(\mu^3)+\lambda^2\varphi^3O(\mu^2\mathrm{e}^{2\mu|\psi|_\infty})+\lambda\varphi^3O(\mathrm{e}^{2\mu|\psi|_\infty})\big)v^2\Big]\,\mathrm{d}x\mathrm{d}t\\ &\geq\ c_0\lambda\mu^2\mathbb{E}\int_0^T\int_{G\setminus G_1}\varphi\big(v_x^2+\lambda^2\mu^2\varphi^2v^2\big)\,\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{3.25}$$

where $c_0 = \min\Big(\min_{x\in\overline{G}\setminus G_1} \psi_x^2\,,\ \min_{x\in\overline{G}\setminus G_1} \psi_x^4\Big) > 0$. By some direct computations, we obtain that

$$\frac{1}{C}\theta^2(z_x^2+\lambda^2\mu^2\varphi^2z^2) \le v_x^2+\lambda^2\mu^2\varphi^2v^2 \le C\theta^2(z_x^2+\lambda^2\mu^2\varphi^2z^2). \tag{3.26}$$

Now, from (3.24), (3.25) and (3.26), we conclude that

$$C\lambda\mu^2\mathbb{E}\int_0^T\int_{G\setminus G_1}\theta^2\varphi(z_x^2+\lambda^2\mu^2\varphi^2z^2)\,\mathrm{d}x\mathrm{d}t \le 6\mathbb{E}\int_0^T\int_G \ell_{xx}v_x^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G Bv^2\,\mathrm{d}x\mathrm{d}t. \tag{3.27}$$

It is easy to see that

$$\begin{aligned}
&2\mathbb{E}\int_0^T\int_G\Big[\ell_xv_x^2+2\ell_{xx}vv_x+(A\ell_x-\ell_{xxx})v^2\Big]_x\,\mathrm{d}x\mathrm{d}t\\
&=2\mathbb{E}\int_0^T\Big[\ell_x^3(t,1)+2\ell_x(t,1)\ell_{xx}(t,1)+A(t,1)\ell_x(t,1)-\ell_{xxx}(t,1)\Big]\theta^2(t,1)z^2(t,1)\,\mathrm{d}t\\
&\quad+2\mathbb{E}\int_0^T 2\Big[\ell_x^2(t,1)+\ell_{xx}(t,1)\Big]\theta^2(t,1)z(t,1)z_x(t,1)\,\mathrm{d}t+2\mathbb{E}\int_0^T\ell_x(t,1)\theta^2(t,1)z_x^2(t,1)\,\mathrm{d}t\\
&\quad-2\mathbb{E}\int_0^T\ell_x(t,0)\theta^2(t,0)z_x^2(t,0)\,\mathrm{d}t.
\end{aligned} \tag{3.28}$$

Note that the last term on the right-hand side of (3.28) is non-positive, thanks to (3.3). Then, by using (3.14) and (3.15), and taking a large enough λ_0, we arrive at

$$\begin{aligned}
&2\mathbb{E}\int_0^T\int_G\Big[\ell_xv_x^2+2\ell_{xx}vv_x+(A\ell_x-\ell_{xxx})v^2\Big]_x\,\mathrm{d}x\mathrm{d}t\\
&\le 3\lambda^3\mu^3\mathbb{E}\int_0^T\varphi^3(t,1)\psi_x^3(1)\theta^2(t,1)z^2(t,1)\,\mathrm{d}t+2\lambda\mu\mathbb{E}\int_0^T\varphi(t,1)\psi_x(1)\theta^2(t,1)z_x^2(t,1)\,\mathrm{d}t\\
&\quad+4\lambda^2\mu^2\mathbb{E}\int_0^T\varphi^2(t,1)\psi_x^2(1)\theta^2(t,1)|z(t,1)||z_x(t,1)|\,\mathrm{d}t+C\lambda\mu^2\mathbb{E}\int_0^T\varphi(t,1)\theta^2(t,1)|z(t,1)||z_x(t,1)|\,\mathrm{d}t.
\end{aligned} \tag{3.29}$$

Recalling (3.3) and using the inequality "$4ab\le\frac{5}{2}a^2+\frac{8}{5}b^2$", it is easy to check that

$$\begin{aligned}
4\lambda^2\mu^2\mathbb{E}\int_0^T\varphi^2(t,1)\psi_x^2(1)\theta^2(t,1)|z(t,1)||z_x(t,1)|\,\mathrm{d}t \le\ &-\frac{5}{2}\lambda^3\mu^3\mathbb{E}\int_0^T\varphi^3(t,1)\psi_x^3(1)\theta^2(t,1)z^2(t,1)\,\mathrm{d}t\\
&-\frac{8}{5}\lambda\mu\mathbb{E}\int_0^T\varphi(t,1)\psi_x(1)\theta^2(t,1)z_x^2(t,1)\,\mathrm{d}t.
\end{aligned} \tag{3.30}$$

By Young's inequality, we have that

$$\begin{aligned}
C\lambda\mu^2\mathbb{E}\int_0^T\varphi(t,1)\theta^2(t,1)|z(t,1)||z_x(t,1)|\,\mathrm{d}t \le\ &C\lambda^2\mu^3\mathbb{E}\int_0^T\varphi^3(t,1)\theta^2(t,1)z^2(t,1)\,\mathrm{d}t\\
&+C\mu\mathbb{E}\int_0^T\varphi(t,1)\theta^2(t,1)z_x^2(t,1)\,\mathrm{d}t.
\end{aligned} \tag{3.31}$$

Now, combining (3.29), (3.30) and (3.31) and taking a large λ_0, we obtain

$$\begin{aligned}&2\mathbb{E}\int_0^T\int_G\Big[\ell_x v_x^2+2\ell_{xx}vv_x+(A\ell_x-\ell_{xxx})v^2\Big]_x\,\mathrm{d}x\mathrm{d}t\\&\le -C\lambda^3\mu^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t-C\lambda\mu\mathbb{E}\int_0^T\theta^2(t,1)\varphi(t,1)z_x^2(t,1)\,\mathrm{d}t.\end{aligned}\tag{3.32}$$

From (3.15), by taking a large enough λ_0, it is easy to see that

$$\mathbb{E}\int_0^T\int_G\theta^2AZ^2\,\mathrm{d}x\mathrm{d}t\le C\lambda^2\mu^2\mathbb{E}\int_0^T\int_G\theta^2\varphi^2Z^2\,\mathrm{d}x\mathrm{d}t.\tag{3.33}$$

Using (3.14) and (3.5), we also have for a large λ_0 that

$$\mathbb{E}\int_0^T\theta^2(t,1)(\ell_t(t,1)-\ell_x(t,1))\widehat{Z}^2(t)\,\mathrm{d}t\le C\lambda^2\mu\mathbb{E}\int_0^T\theta^2(t,1)\varphi^2(t,1)\widehat{Z}^2(t)\,\mathrm{d}t.\tag{3.34}$$

Combining (3.13), (3.19), (3.22), (3.23), (3.27), (3.32), (3.33), (3.34) and taking a large enough λ_0, we conclude that

$$\begin{aligned}&\lambda^3\mu^4\mathbb{E}\int_0^T\int_G\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t+\lambda^3\mu^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t+\lambda\mu^2\mathbb{E}\int_0^T\int_G\theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t\\&\le C\Big[\lambda^3\mu^4\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t+\lambda\mu^2\mathbb{E}\int_0^T\int_{G_1}\theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G\theta^2F_0^2\,\mathrm{d}x\mathrm{d}t\\&\quad+\mathbb{E}\int_0^T\theta^2(t,1)F_1^2(t)\,\mathrm{d}t+\lambda^2\mu^2\mathbb{E}\int_0^T\int_G\theta^2\varphi^2Z^2\,\mathrm{d}x\mathrm{d}t+\lambda^2\mu\mathbb{E}\int_0^T\theta^2(t,1)\varphi^2(t,1)\widehat{Z}^2(t)\,\mathrm{d}t\Big].\end{aligned}\tag{3.35}$$

Step 4. We follow a classical strategy to eliminate the second undesired term on the right-hand side of (3.35). Let $\xi\in C_0^\infty(G_0;[0,1])$ be a cut-off function chosen such that $\xi\equiv 1$ in G_1. Applying Itô's formula, we have

$$d(\theta^2\varphi z^2)=(\theta^2\varphi)_tz^2\mathrm{d}t+2\theta^2\varphi z\mathrm{d}z+\theta^2\varphi(\mathrm{d}z)^2.\tag{3.36}$$

From (3.1), (3.4) and (3.36), we find that

$$\begin{aligned}0=&\,\mathbb{E}\int_0^T\int_G\theta^2\xi^2(2\lambda\varphi\alpha_t+\varphi_t)z^2\,\mathrm{d}x\mathrm{d}t-2\mathbb{E}\int_0^T\int_G\theta^2\xi^2\varphi zz_{xx}\,\mathrm{d}x\mathrm{d}t\\&+2\mathbb{E}\int_0^T\int_G\theta^2\xi^2\varphi zF_0\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G\theta^2\xi^2\varphi Z^2\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.37}$$

Then, employing integration by parts for the second term on the right-hand side of (3.37), we get

$$\begin{aligned}0=&\,\mathbb{E}\int_0^T\int_G\theta^2\Big[\xi^2(\varphi_t+2\lambda\varphi\alpha_t)z^2+2\mu\xi^2\varphi(1+2\lambda\varphi)\psi_xzz_x+4\xi\xi_x\varphi zz_x\\&+2\xi^2\varphi z_x^2+2\xi^2\varphi zF_0+\xi^2\varphi Z^2\Big]\,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

which provides that

$$
\begin{aligned}
2\mathbb{E}\int_0^T\int_G \theta^2\xi^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t = &-\mathbb{E}\int_0^T\int_G \theta^2\xi^2(\varphi_t+2\lambda\varphi\alpha_t)z^2\,\mathrm{d}x\mathrm{d}t - 4\mathbb{E}\int_0^T\int_G \theta^2\xi\xi_x\varphi z z_x\,\mathrm{d}x\mathrm{d}t\\
&-2\mu\mathbb{E}\int_0^T\int_G \theta^2\xi^2\varphi(1+2\lambda\varphi)\psi_x z z_x\,\mathrm{d}x\mathrm{d}t\\
&-2\mathbb{E}\int_0^T\int_G \theta^2\xi^2\varphi z F_0\,\mathrm{d}x\mathrm{d}t - \mathbb{E}\int_0^T\int_G \theta^2\xi^2\varphi Z^2\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.38}
$$

Using (3.5) and taking a large enough $\lambda_0 > 1$, we obtain

$$
-\mathbb{E}\int_0^T\int_G \theta^2\xi^2(\varphi_t+2\lambda\varphi\alpha_t)z^2\,\mathrm{d}x\mathrm{d}t \le C\lambda e^{2\mu|\psi|_\infty}\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t. \tag{3.39}
$$

On the other hand, it is easy to see that

$$
-2\mathbb{E}\int_0^T\int_G \theta^2\xi^2\varphi z F_0\,\mathrm{d}x\mathrm{d}t \le C\lambda^2\mu^2\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \frac{C}{\lambda^2\mu^2}\mathbb{E}\int_0^T\int_G \theta^2 F_0^2\,\mathrm{d}x\mathrm{d}t. \tag{3.40}
$$

Note also that for any $\varepsilon > 0$

$$
\begin{aligned}
&-2\mu\mathbb{E}\int_0^T\int_G \theta^2\xi^2\varphi(1+2\lambda\varphi)\psi_x z z_x\,\mathrm{d}x\mathrm{d}t - 4\mathbb{E}\int_0^T\int_G \theta^2\xi\xi_x\varphi z z_x\,\mathrm{d}x\mathrm{d}t\\
&\le \frac{C}{\varepsilon}\lambda^2\mu^2\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \varepsilon\mathbb{E}\int_0^T\int_{G_0}\theta^2\xi^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.41}
$$

Combining (3.38), (3.39), (3.40), (3.41) and taking a small enough ε and a large λ_0, it is straightforward to deduce that

$$
\begin{aligned}
\mathbb{E}\int_0^T\int_{G_1}\theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t \le C\Big[&\lambda^2\mu^2\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{\lambda^2\mu^2}\mathbb{E}\int_0^T\int_G \theta^2 F_0^2\,\mathrm{d}x\mathrm{d}t\\
&+\mathbb{E}\int_0^T\int_G \theta^2\varphi^2 Z^2\,\mathrm{d}x\mathrm{d}t\Big].
\end{aligned} \tag{3.42}
$$

Finally, by combining (3.35) and (3.42), we obtain our desired Carleman estimate (3.6). This completes the proof of Theorem 3.3. □

4. Global Carleman estimate for forward stochastic heat equations

This section is addressed to proving a Carleman estimate for the forward stochastic heat equation

$$
\begin{cases}
\mathrm{d}z - z_{xx}\,\mathrm{d}t = F_0\,\mathrm{d}t + F_1\,\mathrm{d}W^1(t), & (t,x)\in(0,T)\times G,\\
z(t,0)=0, & t\in(0,T),\\
\mathrm{d}z(t,1) + z_x(t,1)\,\mathrm{d}t = H_0\,\mathrm{d}t + H_1\,\mathrm{d}W^2(t), & t\in(0,T),\\
(z(0,x), z(0,1)) = (z_0(x), z_0^1), & x\in G,
\end{cases} \tag{4.1}
$$

where $(z_0, z_0^1)\in L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)$, $F_0, F_1 \in L^2_{\mathcal{F}}(0,T;L^2(G))$ and $H_0, H_1 \in L^2_{\mathcal{F}}(0,T)$.

Note that when $F_1 \equiv H_1 \equiv 0$, equation (4.1) becomes a random heat equation with mixed random Dirichlet-dynamic boundary conditions. Then, from the known Carleman estimate for deterministic parabolic equations in [9], Theorem A.2 and (3.4), we deduce the following Carleman estimate.

Lemma 4.1. *There exist constants $C > 0$, $\mu_0 > 1$, and $\lambda_0 > 1$ depending only on G, G_0 and T such that for all $\mu \geq \mu_0$ and $\lambda \geq \lambda_0$, $F_0 \in L^2_{\mathcal{F}}(0,T;L^2(G))$, $H_0 \in L^2_{\mathcal{F}}(0,T)$, and $(z_0, z_0^1) \in L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)$, the corresponding solution $(z, z(\cdot,1))$ of (4.1) (with $F_1 \equiv H_1 \equiv 0$) satisfies that*

$$\begin{aligned}
&\lambda^3\mu^4\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2 \,\mathrm{d}x\mathrm{d}t + \lambda^3\mu^3\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t + \lambda\mu\mathbb{E}\int_0^T\int_G \theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t \\
&\leq C\Big[\lambda^3\mu^4\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G \theta^2 F_0^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \theta^2(t,1)H_0^2(t)\,\mathrm{d}t\Big].
\end{aligned} \tag{4.2}$$

Throughout this section, let $\mu = \mu_0$ and $\lambda \geq \lambda_0$ where μ_0 and λ_0 are given in Lemma 4.1. We also choose the same weight functions α, φ and θ defined in (3.4). Here, we will employ a duality technique to prove the desired Carleman estimate for solutions of (4.1). To begin, we first consider the following controlled backward stochastic heat equation

$$\begin{cases}
\mathrm{d}r + r_{xx}\,\mathrm{d}t = (\lambda^3\theta^2\varphi^3 z + \mathbb{1}_{G_0}u)\,\mathrm{d}t + R_1\,\mathrm{d}W^1(t), & (t,x)\in(0,T)\times G,\\
r(t,0) = 0, & t\in(0,T),\\
\mathrm{d}r(t,1) - r_x(t,1)\,\mathrm{d}t = \lambda^3\theta^2(t,1)\varphi^3(t,1)z(t,1)\,\mathrm{d}t + R_2(t)\,\mathrm{d}W^2(t), & t\in(0,T),\\
(r(T,x), r(T,1)) = (0,0), & x\in G,
\end{cases} \tag{4.3}$$

where $(z, z(\cdot,1))$ is the solution of (4.1), $u \in L^2_{\mathcal{F}}(0,T;L^2(G_0))$ is the control variable and $(r, r(\cdot,1); R_1, R_2)$ is the state variable. Then, by applying the Carleman inequality (4.2), we will show the following controllability result for equation (4.3).

Proposition 4.2. *There exists a control $\widehat{u} \in L^2_{\mathcal{F}}(0,T;L^2(G_0))$ such that the associated solution $(\widehat{r}, \widehat{r}(\cdot,1); \widehat{R}_1, \widehat{R}_2)$ of (4.3) with $u = \widehat{u}$ satisfies that*

$$(\widehat{r}(0,\cdot), \widehat{r}(0,1)) = (0,0) \ \text{ in } G, \ \mathbb{P}\text{-a.s.}$$

Moreover, there exist constants $C > 0$ and $\lambda_0 > 1$ depending only on G, G_0, μ_0 and T such that for all $\lambda \geq \lambda_0$, we have that

$$\begin{aligned}
&\lambda^{-3}\mathbb{E}\int_0^T\int_G \theta^{-2}\varphi^{-3}\widehat{u}^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G \theta^{-2}\widehat{r}^2\,\mathrm{d}x\mathrm{d}t \\
&+ \mathbb{E}\int_0^T \theta^{-2}(t,1)\widehat{r}^2(t,1)\,\mathrm{d}t + \lambda^{-2}\mathbb{E}\int_0^T\int_G \theta^{-2}\varphi^{-2}\widehat{r}_x^2\,\mathrm{d}x\mathrm{d}t \\
&+ \lambda^{-2}\mathbb{E}\int_0^T\int_G \theta^{-2}\varphi^{-2}\widehat{R}_1^2\,\mathrm{d}x\mathrm{d}t + \lambda^{-2}\mathbb{E}\int_0^T \theta^{-2}(t,1)\varphi^{-2}(t,1)\widehat{R}_2^2(t)\,\mathrm{d}t \\
&\leq C\Big[\lambda^3\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \lambda^3\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\Big].
\end{aligned} \tag{4.4}$$

The proof of Proposition 4.2 is provided in the appendix at the end of this paper. Now, based on this result, we prove the following main Carleman estimate for equation (4.1).

Theorem 4.3. *There exist constants $C > 0$ and $\lambda_0 > 1$ depending only on G, G_0, μ_0 and T such that for all $\lambda \geq \lambda_0$, $F_0, F_1 \in L^2_{\mathcal{F}}(0,T;L^2(G))$, $H_0, H_1 \in L^2_{\mathcal{F}}(0,T)$, and $(z_0, z_0^1) \in L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)$, the associated solution $(z, z(\cdot,1))$ of (4.1) satisfies that*

$$\begin{aligned}
&\lambda^3\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2 \,\mathrm{d}x\mathrm{d}t + \lambda^3\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t + \lambda\mathbb{E}\int_0^T\int_G \theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t\\
&\leq C\Big[\lambda^3\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G\theta^2F_0^2\,\mathrm{d}x\mathrm{d}t + \lambda^2\mathbb{E}\int_0^T\int_G\theta^2\varphi^2F_1^2\,\mathrm{d}x\mathrm{d}t\\
&\qquad + \mathbb{E}\int_0^T\theta^2(t,1)H_0^2(t)\,\mathrm{d}t + \lambda^2\mathbb{E}\int_0^T\theta^2(t,1)\varphi^2(t,1)H_1^2(t)\,\mathrm{d}t\Big].
\end{aligned}\tag{4.5}$$

Proof. Let $(z, z(\cdot,1))$ be the solution of (4.1) and $(\widehat{r}, \widehat{r}(\cdot,1); \widehat{R}_1, \widehat{R}_2)$ be the solution of (4.3) with the control $u = \widehat{u}$ obtained in Proposition 4.2. Applying Itô's formula, we compute $d\langle(z, z(\cdot,1)),(\widehat{r},\widehat{r}(\cdot,1))\rangle_{\mathbb{L}^2}$, integrate the resulting equality over $(0,T)$, and then take the expectation on both sides. This yields

$$\begin{aligned}
&\lambda^3\mathbb{E}\int_0^T\int_G\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t + \lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\\
&= -\mathbb{E}\int_0^T\int_G\Big[\mathbb{1}_{G_0}z\widehat{u} + F_0\widehat{r} + F_1\widehat{R}_1\Big]\,\mathrm{d}x\mathrm{d}t - \mathbb{E}\int_0^T\Big[H_0(t)\widehat{r}(t,1) + H_1(t)\widehat{R}_2(t)\Big]\,\mathrm{d}t.
\end{aligned}\tag{4.6}$$

It follows that for all $\varepsilon > 0$, we have

$$\begin{aligned}
&\lambda^3\mathbb{E}\int_0^T\int_G\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t + \lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\\
&\leq \varepsilon\Big[\lambda^{-3}\mathbb{E}\int_0^T\int_G\theta^{-2}\varphi^{-3}\widehat{u}^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G\theta^{-2}\widehat{r}^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\theta^{-2}(t,1)|\widehat{r}(t,1)|^2\,\mathrm{d}t\\
&\qquad + \lambda^{-2}\mathbb{E}\int_0^T\int_G\theta^{-2}\varphi^{-2}\widehat{R}_1^2\,\mathrm{d}x\mathrm{d}t + \lambda^{-2}\mathbb{E}\int_0^T\theta^{-2}(t,1)\varphi^{-2}(t,1)\widehat{R}_2^2(t)\,\mathrm{d}t\Big]\\
&\quad + \frac{1}{4\varepsilon}\Big[\lambda^3\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G\theta^2F_0^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\theta^2(t,1)H_0^2(t)\,\mathrm{d}t\\
&\qquad + \lambda^2\mathbb{E}\int_0^T\int_G\theta^2\varphi^2F_1^2\,\mathrm{d}x\mathrm{d}t + \lambda^2\mathbb{E}\int_0^T\theta^2(t,1)\varphi^2(t,1)H_1^2(t)\,\mathrm{d}t\Big].
\end{aligned}\tag{4.7}$$

Combining (4.4) and (4.7), there exists $\lambda_0 > 1$ depending only on G, G_0, μ_0 and T so that

$$\begin{aligned}
&\lambda^3\mathbb{E}\int_0^T\int_G\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t + \lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\\
&\leq C\varepsilon\Big[\lambda^3\mathbb{E}\int_0^T\int_G\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t + \lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\Big]\\
&\quad + \frac{1}{4\varepsilon}\Big[\lambda^3\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G\theta^2F_0^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\theta^2(t,1)H_0^2(t)\,\mathrm{d}t\\
&\qquad + \lambda^2\mathbb{E}\int_0^T\int_G\theta^2\varphi^2F_1^2\,\mathrm{d}x\mathrm{d}t + \lambda^2\mathbb{E}\int_0^T\theta^2(t,1)\varphi^2(t,1)H_1^2(t)\,\mathrm{d}t\Big],
\end{aligned}\tag{4.8}$$

for all $\lambda \geq \lambda_0$. Now, choosing a small enough ε in (4.8), we end up with

$$\begin{aligned} &\lambda^3\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \lambda^3\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t \\ &\leq C\Big[\lambda^3\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G \theta^2 F_0^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \theta^2(t,1)H_0^2(t)\,\mathrm{d}t \\ &\qquad + \lambda^2\mathbb{E}\int_0^T\int_G \theta^2\varphi^2 F_1^2\,\mathrm{d}x\mathrm{d}t + \lambda^2\mathbb{E}\int_0^T \theta^2(t,1)\varphi^2(t,1)H_1^2(t)\,\mathrm{d}t\Big]. \end{aligned} \tag{4.9}$$

On the other hand, we apply Itô's formula to compute $d(\theta^2\varphi z^2)$ and $d(\theta^2(t,1)\varphi(t,1)z^2(t,1))$. We then proceed to compute

$$\mathbb{E}\int_0^T\int_G d(\theta^2\varphi z^2)\,\mathrm{d}x + \mathbb{E}\int_0^T d(\theta^2(t,1)\varphi(t,1)z^2(t,1)).$$

Thus, we get

$$\begin{aligned} 2\mathbb{E}\int_0^T\int_G \theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t =& \mathbb{E}\int_0^T\int_G (\theta^2\varphi)_t z^2\,\mathrm{d}x\mathrm{d}t - 2\mathbb{E}\int_0^T\int_G (\theta^2\varphi)_x z z_x\,\mathrm{d}x\mathrm{d}t \\ &+ 2\mathbb{E}\int_0^T\int_G \theta^2\varphi z F_0\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_G \theta^2\varphi F_1^2\,\mathrm{d}x\mathrm{d}t \\ &+ \mathbb{E}\int_0^T (\theta^2(t,1)\varphi(t,1))_t z^2(t,1)\,\mathrm{d}t + \mathbb{E}\int_0^T \theta^2(t,1)\varphi(t,1)H_1^2(t)\,\mathrm{d}t \\ &+ 2\mathbb{E}\int_0^T \theta^2(t,1)\varphi(t,1)z(t,1)H_0(t)\,\mathrm{d}t. \end{aligned} \tag{4.10}$$

It is easy to see that there is a constant $C = C(G,T) > 0$ and a large λ_0 such that

$$|(\theta^2\varphi)_t| \leq C\lambda\theta^2\varphi^3 \qquad \text{and} \qquad |(\theta^2\varphi)_x| \leq C\lambda\theta^2\varphi^2. \tag{4.11}$$

Using Young's inequality and (4.11), the equality (4.10) implies that

$$\begin{aligned} 2\mathbb{E}\int_0^T\int_G \theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t \leq& \mathbb{E}\int_0^T\int_G \theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t + C\Big[\lambda\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t \\ &+ \lambda^2\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \lambda\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t \\ &+ \lambda^{-1}\mathbb{E}\int_0^T\int_G \theta^2 F_0^2\,\mathrm{d}x\mathrm{d}t + \lambda^{-1}\mathbb{E}\int_0^T \theta^2(t,1)H_0^2(t)\,\mathrm{d}t \\ &+ \lambda\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t + \mathbb{E}\int_0^T\int_G \theta^2\varphi F_1^2\,\mathrm{d}x\mathrm{d}t \\ &+ \mathbb{E}\int_0^T \theta^2(t,1)\varphi(t,1)H_1^2(t)\,\mathrm{d}t\Big]. \end{aligned}$$

Therefore, taking a large enough λ_0, it follows that

$$\begin{aligned}\lambda\mathbb{E}\int_0^T\int_G \theta^2\varphi z_x^2\,\mathrm{d}x\mathrm{d}t \leq C\Big[&\lambda^3\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G \theta^2 F_0^2\,\mathrm{d}x\mathrm{d}t\\ &+\mathbb{E}\int_0^T \theta^2(t,1)H_0^2(t)\,\mathrm{d}t+\lambda\mathbb{E}\int_0^T\int_G \theta^2\varphi F_1^2\,\mathrm{d}x\mathrm{d}t\\ &+\lambda^2\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t+\lambda\mathbb{E}\int_0^T \theta^2(t,1)\varphi(t,1)H_1^2(t)\,\mathrm{d}t\Big].\end{aligned} \tag{4.12}$$

Combining (4.9) and (4.12) and taking a large λ_0, we finally deduce the desired Carleman estimate (4.5). This concludes the proof of Theorem 4.3. □

5. Null controllability of equations (1.5) and (1.8)

This section is devoted to establishing the null controllability results of equations (1.5) and (1.8). To achieve this, we first show the observability inequality (1.7), which is a consequence of the Carleman estimate (3.6) and an energy estimate for equation (1.6).

Proposition 5.1. *Solutions of equation (1.6) satisfy the observability inequality (1.7).*

Proof. Let us take

$$F_0 = a_1 z + a_2 Z \qquad \text{and} \qquad F_1 = b_1 z(\cdot,1) + b_2\widehat{Z},$$

in the Carleman estimate (3.6). Then, by choosing $\mu=\mu_0$ and a large enough λ, it is easy to see that there exists a constant $C>0$ depending only on G, G_0, μ_0, T, $|a_1|_\infty$, $|a_2|_\infty$, $|b_1|_\infty$, and $|b_2|_\infty$ such that

$$\begin{aligned}&\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\\ &\leq C\left[\mathbb{E}\int_0^T\int_{G_0} \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G \theta^2\varphi^2 Z^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T \theta^2(t,1)\varphi^2(t,1)\widehat{Z}^2(t)\,\mathrm{d}t\right].\end{aligned}$$

Then, it follows that

$$\begin{aligned}&\mathbb{E}\int_{T/4}^{3T/4}\int_G z^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_{T/4}^{3T/4} z^2(t,1)\,\mathrm{d}t\\ &\leq C\left[\mathbb{E}\int_0^T\int_{G_0} z^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G Z^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T \widehat{Z}^2(t)\,\mathrm{d}t\right].\end{aligned} \tag{5.1}$$

On the other hand, using the estimate (2.7), we can find a positive constant C depending only on G, T, $|a_1|_\infty$, $|a_2|_\infty$, $|b_1|_\infty$, and $|b_2|_\infty$, such that for all $t\in(0,T)$

$$\big|(z(0,\cdot),z(0,1))\big|^2_{L^2_{\mathcal{F}_0}(\Omega;\mathbb{L}^2)} \leq C\big|(z(t,\cdot),z(t,1))\big|^2_{L^2_{\mathcal{F}_t}(\Omega;\mathbb{L}^2)}.$$

Hence, the following energy estimate holds

$$\mathbb{E}|z(0,\cdot)|^2_{L^2(G)}+\mathbb{E}|z(0,1)|^2 \leq C\left[\mathbb{E}\int_G z^2(t,x)\,\mathrm{d}x+\mathbb{E}|z(t,1)|^2\right]. \tag{5.2}$$

We finally integrate (5.2) over the time interval $(T/4, 3T/4)$ and combine the resulting inequality with (5.1) to deduce the desired observability inequality (1.7). □

We are now in a position to prove our null controllability result of equation (1.5).

Theorem 5.2. *For any given $T>0$, $G_0 \Subset G$ a nonempty open subset of G, and for all initial state $(y_0, y_0^1) \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{L}^2)$, there exists a triple of controls*

$$(u, v_1, v_2) \in L^2_{\mathcal{F}}(0,T; L^2(G_0)) \times L^2_{\mathcal{F}}(0,T; L^2(G)) \times L^2_{\mathcal{F}}(0,T),$$

such that the corresponding solution $(y, y(\cdot,1))$ of (1.5) satisfies that

$$(y(T,\cdot), y(T,1)) = (0,0) \text{ in } G, \ \mathbb{P}\text{-a.s.}$$

Proof. Let $(y_0, y_0^1) \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{L}^2)$ be the initial state of (1.5) and let us consider a linear subspace of the space $L^2_{\mathcal{F}}(0,T; L^2(G_0)) \times L^2_{\mathcal{F}}(0,T; L^2(G)) \times L^2_{\mathcal{F}}(0,T)$ as follows

$$\begin{aligned}\mathcal{H} := \Big\{ \big(\mathbb{1}_{G_0} z, Z, \widehat{Z}\big) \ \big| \ \big(z, z(\cdot,1); Z, \widehat{Z}\big) &\text{ be the solution of (1.6) with } a_1 = -\beta_1, \ \ a_2 = -\beta_2, \\ b_1 = -\eta_1, \ \ b_2 = -\eta_2 \ \text{ and a terminal state } &(z_T, z_T^1) \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{L}^2) \Big\}.\end{aligned}$$

Define the following linear functional F on $\mathcal{H}$ by

$$F\big(\mathbb{1}_{G_0} z, Z, \widehat{Z}\big) = -\mathbb{E}\int_G y_0(x) z(0,x)\,\mathrm{d}x - \mathbb{E}\left[y_0^1 z(0,1)\right]. \tag{5.3}$$

Using the observability inequality (1.7), it is easy to see that F is a bounded linear functional on $\mathcal{H}$. Then, by Hahn–Banach theorem, F can be extended to be a bounded linear functional on the whole space $L^2_{\mathcal{F}}(0,T; L^2(G_0)) \times L^2_{\mathcal{F}}(0,T; L^2(G)) \times L^2_{\mathcal{F}}(0,T)$. For simplicity, we still use F to denote such an extension. Now, by Riesz representation theorem, there exists $(u, v_1, v_2) \in L^2_{\mathcal{F}}(0,T; L^2(G_0)) \times L^2_{\mathcal{F}}(0,T; L^2(G)) \times L^2_{\mathcal{F}}(0,T)$ such that

$$F(\mathbb{1}_{G_0} z, Z, \widehat{Z}) = \mathbb{E}\int_0^T \int_{G_0} z u \,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \int_G Z v_1 \,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \widehat{Z} v_2 \,\mathrm{d}t. \tag{5.4}$$

We claim that the above obtained u, v_1, and v_2 are our desired controls. Indeed, noting that from (5.3) and (5.4), we have that

$$-\mathbb{E}\int_G y_0(x) z(0,x)\,\mathrm{d}x - \mathbb{E}\big[y_0^1 z(0,1)\big] = \mathbb{E}\int_0^T \int_{G_0} z u \,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \int_G Z v_1 \,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \widehat{Z} v_2 \,\mathrm{d}t. \tag{5.5}$$

On the other hand, using Itô's formula for the solutions of (1.5) and the adjoint equation (1.6) with coefficients $(a_1, a_2, b_1, b_2) \equiv (-\beta_1, -\beta_2, -\eta_1, -\eta_2)$, we obtain the following duality relation

$$\begin{aligned}&\mathbb{E}\int_G y(T,x) z_T(x)\,\mathrm{d}x - \mathbb{E}\int_G y_0(x) z(0,x)\,\mathrm{d}x + \mathbb{E}\big[y(T,1) z_T^1\big] - \mathbb{E}\big[y_0^1 z(0,1)\big] \\ &= \mathbb{E}\int_0^T \int_{G_0} z u \,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \int_G Z v_1 \,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \widehat{Z} v_2 \,\mathrm{d}t.\end{aligned} \tag{5.6}$$

Combining (5.6) and (5.5), we conclude that

$$\mathbb{E}\int_G y(T,x)z_T(x)\,\mathrm{d}x + \mathbb{E}\big[y(T,1)z_T^1\big] = 0. \tag{5.7}$$

Since (z_T, z_T^1) can be chosen arbitrarily in $L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$, then (5.7) implies that

$$(y(T,\cdot), y(T,1)) = (0,0) \ \text{ in } G, \ \mathbb{P}\text{-a.s.}$$

This completes the proof of Theorem 5.2. □

For the null controllability of equation (1.8), we first establish the key observability inequality (1.10).

Proposition 5.3. *Solutions of equation* (1.9) *satisfy the observability inequality* (1.10).

Proof. In the Carleman estimate (4.5), we choose

$$F_0 = a_1 z, \quad F_1 = a_2 z, \quad H_0 = b_1 z(\cdot,1), \quad \text{and} \quad H_1 = b_2 z(\cdot,1).$$

Now, by taking a large enough λ, we get that

$$\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t \leq C\,\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t,$$

where C is a positive constant depending only on G, G_0, μ_0, T, $|a_1|_\infty$, $|a_2|_\infty$, $|b_1|_\infty$, and $|b_2|_\infty$. Then, it follows that

$$\mathbb{E}\int_{T/4}^{3T/4}\int_G z^2\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_{T/4}^{3T/4} z^2(t,1)\,\mathrm{d}t \leq C\,\mathbb{E}\int_0^T\int_{G_0} z^2\,\mathrm{d}x\mathrm{d}t. \tag{5.8}$$

From the estimate (2.3), we have that for any $t\in(0,T)$

$$|(z(T,\cdot), z(T,1))|^2_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)} \leq C|(z(t,\cdot), z(t,1))|^2_{L^2_{\mathcal{F}_t}(\Omega;\mathbb{L}^2)},$$

for a constant $C>0$ which depends only on G, T, $|a_1|_\infty$, $|a_2|_\infty$, $|b_1|_\infty$, and $|b_2|_\infty$. Therefore, we deduce the following energy estimate

$$\mathbb{E}|z(T,\cdot)|^2_{L^2(G)} + \mathbb{E}|z(T,1)|^2 \leq C\left[\mathbb{E}\int_G z^2(t,x)\,\mathrm{d}x + \mathbb{E}|z(t,1)|^2\right]. \tag{5.9}$$

Integrating (5.9) on $(T/4, 3T/4)$ and combining the obtained estimate with (5.8), we deduce the desired observability inequality (1.10). □

Proceeding as in the proof of Theorem 5.2 and using the observability inequality (1.10), we establish our null controllability result of equation (1.8).

Theorem 5.4. *For any given $T>0$, $G_0 \Subset G$ a nonempty open subset of G, and for all terminal state $(y_T, y_T^1)\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$, there exists a control $u\in L^2_{\mathcal{F}}(0,T;L^2(G_0))$ such that the corresponding solution $(y, y(\cdot,1); Y, \widehat{Y})$ of* (1.8) *satisfies that*

$$(y(0,\cdot), y(0,1)) = (0,0) \ \text{ in } G, \ \mathbb{P}\text{-a.s.}$$

Appendix A. Proof of Proposition 4.2

Here, we apply the penalized Hilbert Uniqueness method, which involves constructing the control $\widehat{u}$ as the weak limit of a sequence of controls from an optimal control problem associated with equation (4.3). For insights and steps regarding this method, we refer to [5, 14]. Within our framework, we consider an optimal control problem that incorporates our Dirichlet-dynamic boundary conditions.

Proof of Proposition 4.2. The proof is divided into three steps.

Step 1. For any $\varepsilon > 0$, we define the functions

$$\alpha_\varepsilon \equiv \alpha_\varepsilon(t,x) = \frac{\mathrm{e}^{\mu_0\psi(x)} - \mathrm{e}^{2\mu_0|\psi|_\infty}}{(t+\varepsilon)(T-t+\varepsilon)}, \qquad \theta_\varepsilon = \mathrm{e}^{\lambda\alpha_\varepsilon}.$$

Let us now consider the minimization problem

$$\inf\{J_\varepsilon(u)\,,\ \ u\in\mathcal{U}\}, \tag{A.1}$$

where

$$\begin{aligned} J_\varepsilon(u) = &\frac{1}{2}\mathbb{E}\int_0^T\int_{G_0}\lambda^{-3}\theta^{-2}\varphi^{-3}u^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}r^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\mathbb{E}\int_0^T\theta_\varepsilon^{-2}(t,1)r^2(t,1)\,\mathrm{d}t \\ &+ \frac{1}{2\varepsilon}\mathbb{E}\int_G r^2(0,x)\,\mathrm{d}x + \frac{1}{2\varepsilon}\mathbb{E}r^2(0,1), \end{aligned}$$

and

$$\mathcal{U} = \left\{u\in L^2_{\mathcal{F}}(0,T;L^2(G_0))\,,\quad \mathbb{E}\int_0^T\int_{G_0}\theta^{-2}\varphi^{-3}u^2\,\mathrm{d}x\mathrm{d}t<\infty\right\}.$$

It is easy to see that the functional J_ε is well-defined, continuous, strictly convex, and coercive. Then, the minimization problem (A.1) admits a unique optimal solution u_ε. Therefore, by the Euler–Lagrange equation (*i.e.*, Fréchet derivative $J'_\varepsilon(u_\varepsilon)=0$) and the optimality system (see, *e.g.*, [5, 7, 14]), u_ε can be characterized by

$$u_\varepsilon = \mathbb{1}_{G_0}\lambda^3\theta^2\varphi^3 q^\varepsilon \quad \text{in } (0,T)\times G,\ \ \mathbb{P}\text{-a.s.}, \tag{A.2}$$

where $(q^\varepsilon, q^\varepsilon(\cdot,1))$ is the solution of the following random heat equation

$$\begin{cases} dq^\varepsilon - q^\varepsilon_{xx}\,\mathrm{d}t = \theta_\varepsilon^{-2}r^\varepsilon\,\mathrm{d}t, & (t,x)\in(0,T)\times G,\\ q^\varepsilon(t,0)=0, & t\in(0,T),\\ dq^\varepsilon(t,1) + q^\varepsilon_x(t,1)\,\mathrm{d}t = \theta_\varepsilon^{-2}(t,1)r^\varepsilon(t,1)\,\mathrm{d}t, & t\in(0,T),\\ (q^\varepsilon(0,x), q^\varepsilon(0,1)) = \left(\frac{1}{\varepsilon}r^\varepsilon(0,x), \frac{1}{\varepsilon}r^\varepsilon(0,1)\right), & x\in G, \end{cases} \tag{A.3}$$

where $(r^\varepsilon, r^\varepsilon(\cdot,1); R_1^\varepsilon, R_2^\varepsilon)$ is the solution of (4.3) with the control $u=u_\varepsilon$.

Step 2. By Itô's formula, computing $d\langle (r^\varepsilon, r^\varepsilon(\cdot,1)),(q^\varepsilon, q^\varepsilon(\cdot,1))\rangle_{\mathbb{L}^2}$, integrating the equality on $(0,T)$, taking the expectation on both sides and recalling (A.2), we obtain that

$$\begin{aligned}
&\lambda^3\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3|q^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t\\
&+\mathbb{E}\int_0^T\theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G|r^\varepsilon(0,x)|^2\,\mathrm{d}x+\frac{1}{\varepsilon}\mathbb{E}|r^\varepsilon(0,1)|^2\\
&=\ -\lambda^3\mathbb{E}\int_0^T\int_G\theta^2\varphi^3zq^\varepsilon\,\mathrm{d}x\mathrm{d}t-\lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z(t,1)q^\varepsilon(t,1)\,\mathrm{d}t.
\end{aligned}$$

By Young's inequality, it follows that for any $\rho>0$

$$\begin{aligned}
&\lambda^3\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3|q^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t\\
&+\mathbb{E}\int_0^T\theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G|r^\varepsilon(0,x)|^2\,\mathrm{d}x+\frac{1}{\varepsilon}\mathbb{E}|r^\varepsilon(0,1)|^2\\
&\leq\ \rho\Big[\lambda^3\mathbb{E}\int_0^T\int_G\theta^2\varphi^3|q^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)|q^\varepsilon(t,1)|^2\,\mathrm{d}t\Big]\\
&\quad+\frac{1}{4\rho}\Big[\lambda^3\mathbb{E}\int_0^T\int_G\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t+\lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\Big].
\end{aligned}\tag{A.4}$$

Noting that $\theta^2\theta_\varepsilon^{-2}\leq 1$, then by using Lemma 4.1 for solutions of (A.3), there exists $\lambda_0>1$ such that

$$\begin{aligned}
&\lambda^3\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3|q^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t\\
&+\mathbb{E}\int_0^T\theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G|r^\varepsilon(0,x)|^2\,\mathrm{d}x+\frac{1}{\varepsilon}\mathbb{E}|r^\varepsilon(0,1)|^2\\
&\leq\ C\rho\Big[\lambda^3\mathbb{E}\int_0^T\int_{G_0}\theta^2\varphi^3|q^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t\Big]\\
&\quad+\frac{1}{4\rho}\Big[\lambda^3\mathbb{E}\int_0^T\int_G\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t+\lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\Big],
\end{aligned}\tag{A.5}$$

for all $\lambda\geq\lambda_0$. Now, by taking a small enough ρ in (A.5) and recalling (A.2), we conclude that

$$\begin{aligned}
&\lambda^{-3}\mathbb{E}\int_0^T\int_G\theta^{-2}\varphi^{-3}u_\varepsilon^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t\\
&+\mathbb{E}\int_0^T\theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G|r^\varepsilon(0,x)|^2\,\mathrm{d}x+\frac{1}{\varepsilon}\mathbb{E}|r^\varepsilon(0,1)|^2\\
&\leq\ C\Big[\lambda^3\mathbb{E}\int_0^T\int_G\theta^2\varphi^3z^2\,\mathrm{d}x\mathrm{d}t+\lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\Big].
\end{aligned}\tag{A.6}$$

On the other hand, by Itô's formula, we compute

$$\mathbb{E}\int_0^T\int_G d(\theta_\varepsilon^{-2}\varphi^{-2}|r^\varepsilon|^2)\,\mathrm{d}x+\mathbb{E}\int_0^T d(\theta_\varepsilon^{-2}(t,1)\varphi^{-2}(t,1)|r^\varepsilon(t,1)|^2),$$

then we get that

$$\begin{aligned}
&2\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|r_x^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|R_1^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T \theta_\varepsilon^{-2}(t,1)\varphi^{-2}(t,1)|R_2^\varepsilon(t)|^2\,\mathrm{d}t\\
&=-\mathbb{E}\int_0^T\int_G(\theta_\varepsilon^{-2}\varphi^{-2})_t|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t-\mathbb{E}\int_0^T(\theta_\varepsilon^{-2}(t,1)\varphi^{-2}(t,1))_t|r^\varepsilon(t,1)|^2\,\mathrm{d}t\\
&\quad-2\mathbb{E}\int_0^T\int_{G_0}\theta_\varepsilon^{-2}\varphi^{-2}r^\varepsilon u_\varepsilon\,\mathrm{d}x\mathrm{d}t-2\mathbb{E}\int_0^T\int_G(\theta_\varepsilon^{-2}\varphi^{-2})_x r^\varepsilon r_x^\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&\quad-2\lambda^3\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}\theta^2\varphi z r^\varepsilon\,\mathrm{d}x\mathrm{d}t-2\lambda^3\mathbb{E}\int_0^T\theta_\varepsilon^{-2}(t,1)\theta^2(t,1)\varphi(t,1)z(t,1)r^\varepsilon(t,1)\,\mathrm{d}t.
\end{aligned}\tag{A.7}$$

It is not difficult to see that there exists a constant $C=C(G,T)>0$ and a large λ_0 such that

$$\left|(\theta_\varepsilon^{-2}\varphi^{-2})_t\right|\le C\lambda\theta_\varepsilon^{-2}\qquad\text{and}\qquad\left|(\theta_\varepsilon^{-2}\varphi^{-2})_x\right|\le C\lambda\theta_\varepsilon^{-2}\varphi^{-1}.\tag{A.8}$$

Notice that $\theta_\varepsilon^{-1}\le\theta^{-1}$ and combining (A.8) and (A.7), it follows that

$$\begin{aligned}
&2\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|r_x^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|R_1^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T \theta_\varepsilon^{-2}(t,1)\varphi^{-2}(t,1)|R_2^\varepsilon(t)|^2\,\mathrm{d}t\\
&\le C\lambda\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+C\lambda\mathbb{E}\int_0^T\theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t\\
&\quad+2\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-1}\theta^{-1}\varphi^{-2}|r^\varepsilon||u_\varepsilon|\,\mathrm{d}x\mathrm{d}t+C\lambda\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}\varphi^{-1}|r^\varepsilon||r_x^\varepsilon|\,\mathrm{d}x\mathrm{d}t\\
&\quad+2\lambda^3\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-1}\theta\varphi|z||r^\varepsilon|\,\mathrm{d}x\mathrm{d}t+2\lambda^3\mathbb{E}\int_0^T\theta_\varepsilon^{-1}(t,1)\theta(t,1)\varphi(t,1)|z(t,1)||r^\varepsilon(t,1)|\,\mathrm{d}t.
\end{aligned}\tag{A.9}$$

Applying Young's inequality for the fourth term on the right-hand side of (A.9), one has that

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|r_x^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|R_1^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T \theta_\varepsilon^{-2}(t,1)\varphi^{-2}(t,1)|R_2^\varepsilon(t)|^2\,\mathrm{d}t\\
&\le C\lambda\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+C\lambda\mathbb{E}\int_0^T\theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t\\
&\quad+2\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-1}\theta^{-1}\varphi^{-2}|r^\varepsilon||u_\varepsilon|\,\mathrm{d}x\mathrm{d}t+C\lambda^2\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad+2\lambda^3\mathbb{E}\int_0^T\int_G\theta_\varepsilon^{-1}\theta\varphi|z||r^\varepsilon|\,\mathrm{d}x\mathrm{d}t+2\lambda^3\mathbb{E}\int_0^T\theta_\varepsilon^{-1}(t,1)\theta(t,1)\varphi(t,1)|z(t,1)||r^\varepsilon(t,1)|\,\mathrm{d}t.
\end{aligned}\tag{A.10}$$

Multiplying (A.10) by λ^{-2} and using Young's inequality, we arrive at

$$\begin{aligned}&\lambda^{-2}\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|r_x^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\lambda^{-2}\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|R_1^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\lambda^{-2}\mathbb{E}\int_0^T \theta_\varepsilon^{-2}(t,1)\varphi^{-2}(t,1)|R_2^\varepsilon(t)|^2\,\mathrm{d}t\\&\leq C\Big[\lambda^{-1}\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\lambda^{-1}\mathbb{E}\int_0^T \theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t+\lambda^{-3}\mathbb{E}\int_0^T\int_G \theta^{-2}\varphi^{-3}u_\varepsilon^2\,\mathrm{d}x\mathrm{d}t\\&\quad+\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\lambda^3\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2 dx dt+\lambda^3\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\Big].\end{aligned}$$

We now take a large enough λ_0, it follows that

$$\begin{aligned}&\lambda^{-2}\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|r_x^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\lambda^{-2}\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|R_1^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\lambda^{-2}\mathbb{E}\int_0^T \theta_\varepsilon^{-2}(t,1)\varphi^{-2}(t,1)|R_2^\varepsilon(t)|^2\,\mathrm{d}t\\&\leq C\Big[\lambda^{-3}\mathbb{E}\int_0^T\int_G \theta^{-2}\varphi^{-3}u_\varepsilon^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T \theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t\\&\quad+\lambda^3\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t+\lambda^3\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\Big].\end{aligned}\tag{A.11}$$

Combining (A.6) and (A.11), we end up with

$$\begin{aligned}&\lambda^{-3}\mathbb{E}\int_0^T\int_G \theta^{-2}\varphi^{-3}u_\varepsilon^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}|r^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T \theta_\varepsilon^{-2}(t,1)|r^\varepsilon(t,1)|^2\,\mathrm{d}t\\&+\lambda^{-2}\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|r_x^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\lambda^{-2}\mathbb{E}\int_0^T\int_G \theta_\varepsilon^{-2}\varphi^{-2}|R_1^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t\\&+\lambda^{-2}\mathbb{E}\int_0^T \theta_\varepsilon^{-2}(t,1)\varphi^{-2}(t,1)|R_2^\varepsilon(t)|^2\,\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G |r^\varepsilon(0,x)|^2\,\mathrm{d}x+\frac{1}{\varepsilon}\mathbb{E}|r^\varepsilon(0,1)|^2\\&\leq C\Big[\lambda^3\mathbb{E}\int_0^T\int_G \theta^2\varphi^3 z^2\,\mathrm{d}x\mathrm{d}t+\lambda^3\mathbb{E}\int_0^T \theta^2(t,1)\varphi^3(t,1)z^2(t,1)\,\mathrm{d}t\Big].\end{aligned}\tag{A.12}$$

Step 3. From (A.12), we conclude that there exist

$$\big(\widehat{u},\widehat{r},\widehat{r}(\cdot,1),\widehat{R}_1,\widehat{R}_2\big)\in L^2_{\mathcal{F}}(0,T;L^2(G_0))\times L^2_{\mathcal{F}}(0,T;H^1(G))\times L^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;L^2(G))\times L^2_{\mathcal{F}}(0,T),$$

such that as $\varepsilon\to 0$,

$$\begin{aligned}&u_\varepsilon\longrightarrow\widehat{u}\quad\text{weakly in }\;L^2((0,T)\times\Omega;L^2(G_0));\\&r^\varepsilon\longrightarrow\widehat{r}\quad\text{weakly in }\;L^2((0,T)\times\Omega;H^1(G));\\&r^\varepsilon(\cdot,1)\longrightarrow\widehat{r}(\cdot,1)\quad\text{weakly in }\;L^2((0,T)\times\Omega);\\&R_1^\varepsilon\longrightarrow\widehat{R}_1\quad\text{weakly in }\;L^2((0,T)\times\Omega;L^2(G));\\&R_2^\varepsilon\longrightarrow\widehat{R}_2\quad\text{weakly in }\;L^2((0,T)\times\Omega).\end{aligned}\tag{A.13}$$

Let us now show that $(\widehat{r},\widehat{r}(\cdot,1);\widehat{R}_1,\widehat{R}_2)$ is the solution of (4.3) associated to the control $\widehat{u}$. To prove this fact, let us assume that $(\widetilde{r},\widetilde{r}(\cdot,1);\widetilde{R}_1,\widetilde{R}_2)$ is the unique solution of (4.3) with $u=\widehat{u}$. Set $f_1,f_2\in L^2_{\mathcal{F}}(0,T;L^2(G))$,

$g_1, g_2 \in L^2_{\mathcal{F}}(0,T)$ and let $(\zeta, \zeta(\cdot,1))$ be the solution of the forward stochastic heat equation

$$\begin{cases} d\zeta - \zeta_{xx}\,\mathrm{d}t = f_1\,\mathrm{d}t + f_2\,\mathrm{d}W^1(t), & (t,x)\in(0,T)\times G,\\ \zeta(t,0)=0, & t\in(0,T),\\ d\zeta(t,1)+\zeta_x(t,1)\mathrm{d}t = g_1\,\mathrm{d}t + g_2\,\mathrm{d}W^2(t), & t\in(0,T),\\ (\zeta(0,x),\zeta(0,1))=(0,0), & x\in G. \end{cases} \tag{A.14}$$

Computing $d\langle(\zeta,\zeta(\cdot,1)),(\widetilde{r},\widetilde{r}(\cdot,1))\rangle_{\mathbb{L}^2}$, integrating the equality on $(0,T)$ and taking the expectation on both sides, we arrive at

$$\begin{aligned} &\mathbb{E}\int_0^T\int_G(\lambda^3\theta^2\varphi^3 z + \mathbb{1}_{G_0}\widehat{u})\zeta\,\mathrm{d}x\mathrm{d}t + \lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)\zeta(t,1)z(t,1)\,\mathrm{d}t\\ &+\mathbb{E}\int_0^T\int_G(\widetilde{r}f_1+\widetilde{R}_1 f_2)\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T(\widetilde{r}(t,1)g_1(t)+\widetilde{R}_2(t)g_2(t))\,\mathrm{d}t = 0. \end{aligned} \tag{A.15}$$

Similarly to (A.15), computing $d\langle(\zeta,\zeta(\cdot,1)),(r^\varepsilon,r^\varepsilon(\cdot,1))\rangle_{\mathbb{L}^2}$, we obtain that

$$\begin{aligned} &\mathbb{E}\int_0^T\int_G(\lambda^3\theta^2\varphi^3 z + \mathbb{1}_{G_0}u_\varepsilon)\zeta\,\mathrm{d}x\mathrm{d}t + \lambda^3\mathbb{E}\int_0^T\theta^2(t,1)\varphi^3(t,1)\zeta(t,1)z(t,1)\,\mathrm{d}t\\ &+\mathbb{E}\int_0^T\int_G(r^\varepsilon f_1+R_1^\varepsilon f_2)\,\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T(r^\varepsilon(t,1)g_1(t)+R_2^\varepsilon(t)g_2(t))\,\mathrm{d}t = 0. \end{aligned} \tag{A.16}$$

Letting $\varepsilon\to 0$ in (A.16), and combining the obtained equality with (A.15), we find that

$$\begin{aligned} &\mathbb{E}\int_0^T\int_G\Big[(\widetilde{r}-\widehat{r})f_1+(\widetilde{R}_1-\widehat{R}_1)f_2\Big]\,\mathrm{d}x\mathrm{d}t\\ &+\mathbb{E}\int_0^T\Big[(\widetilde{r}(t,1)-\widehat{r}(t,1))g_1(t)+(\widetilde{R}_2(t)-\widehat{R}_2(t))g_2(t)\Big]\,\mathrm{d}t = 0, \end{aligned}$$

which provides that $(\widetilde{r},\widetilde{R}_1)=(\widehat{r},\widehat{R}_1)$ in $(0,T)\times G$, $\mathbb{P}$-a.s., and $(\widetilde{r}(\cdot,1),\widetilde{R}_2)=(\widehat{r}(\cdot,1),\widehat{R}_2)$ in $(0,T)$, $\mathbb{P}$-a.s. Finally, we conclude that $(\widehat{r},\widehat{r}(\cdot,1);\widehat{R}_1,\widehat{R}_2)$ is the unique solution of equation (4.3) associated to the control $\widehat{u}$. Now, by combining the uniform estimate (A.12) and the weak convergence result (A.13), we deduce the null controllability result of (4.3) with the desired estimate (4.4). This completes the proof of Proposition 4.2. □

Acknowledgements

The authors gratefully thank the editor and the anonymous referees for their valuable comments and suggestions for improving this paper.

References

[1] S.E. Chorfi, G. Guermai, L. Maniar and W. Zouhair, Logarithmic convexity and impulsive controllability for the one-dimensional heat equation with dynamic boundary conditions. *IMA J. Math. Control Inform.* **29** (2022) 861–891.

[2] E. Fernández-Cara, M. González-Burgos, S. Guerrero and J. Puel, Null controllability of the heat equation with boundary Fourier conditions: the linear case. *ESAIM: Control Optim. Calc. Var.* **12** (2006) 442–465.

[3] E. Fernández-Cara and S. Guerrero, Global Carleman inequalities for parabolic systems and applications to controllability. *SIAM J. Control Optim.* **45** (2006) 1395–1446.

[4] A.V. Fursikov and O. Yu. Imanuvilov, Controllability of evolution equations, in *Lecture Notes Series*, vol. 34 Seoul National University, Seoul, Korea (1996).

[5] O. Yu Imanuvilov and M. Yamamoto, Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations. *Publ. Res. Inst. Math. Sci.* **39** (2003) 227–274.

[6] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur. *Commun. Part. Differ. Equ.* **20** (1995) 335–356.

[7] J.-L. Lions, Some Aspects of the Optimal Control of Distributed Parameter Systems. SIAM (1972).

[8] L. Maniar, M. Meyries and R. Schnaubelt, Null controllability for parabolic equations with dynamic boundary conditions. *Evol. Equ. Control Theory* **6** (2017) 381–407.

[9] R.A. Morales Ponce, Contribution to inverse problems and controllability issues of hyperbolic and parabolic partial differential equations. PhD thesis, Universidad de Chile (2019).

[10] E. Zuazua, Controllability and observability of partial differential equations: some results and open problems, in Handbook of Differential Equations. Elsevier Science *Evol. Differ. Equ.* **3** (2007) 527–621.

[11] V. Barbu, A. Răşcanu and G. Tessitore, Carleman estimates and controllability of linear stochastic heat equations. *Appl. Math. Optim.* **47** (2003) 97–120.

[12] M. Baroun, S. Boulite, A. Elgrou and L. Maniar, Null controllability for stochastic parabolic equations with dynamic boundary conditions. *J. Dyn. Control Syst.* **29** (2023) 1727–1756.

[13] M. Chen, Null controllability with constraints on the state for the stochastic heat equation. *J. Dyn. Control Syst.* **24** (2018) 39–50.

[14] X. Liu, Global Carleman estimate for stochastic parabolic equations, and its application. *ESAIM: Control Optim. Calc. Var.* **20** (2014) 823–839.

[15] X. Liu and Y. Yu, Carleman estimates of some stochastic degenerate parabolic equations and application. *SIAM J. Control Optim.* **57** (2019) 3527–3552.

[16] Q. Lü, Exact controllability for stochastic transport equations. *SIAM J. Control Optim.* **52** (2014) 397–419.

[17] Q. Lü, Some results on the controllability of forward stochastic heat equations with control on the drift. *J. Funct. Anal.* **260** (2011) 832–851.

[18] Q. Lü and Y. Zhongqi, Unique continuation for stochastic heat equations. *ESAIM: Control Optim. Calc. Var.* **21** (2015) 378–398.

[19] S. Tang and X. Zhang, Null controllability for forward and backward stochastic parabolic equations. *SIAM J. Control Optim.* **48** (2009) 2191–2216.

[20] X. Zhang, Unique continuation for stochastic parabolic equations. *Differ. Integral Equ.* **21** (2008) 81–93.

[21] Q. Lü and X. Zhang, Mathematical Control Theory for Stochastic Partial Differential Equations. Springer (2021).

[22] T. Carleman, Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat. Astr. Fys.* **17** (1939) 1–9.

[23] Y. Yan, Carleman estimates for stochastic parabolic equations with Neumann boundary conditions and applications. *J. Math. Anal. Applic.* **457** (2018) 248–272.

[24] G. Coclite, A. Favini, C. Gal, G. Goldstein, J. Goldstein, E. Obrecht and S. Romanelli, The role of Wentzell boundary conditions in linear and nonlinear analysis. *Tübinger Berichte* **3** (2009) 279–292.

[25] C. Gal, The role of surface diffusion in dynamic boundary conditions: where do we stand? *Milan J. Math.* **83** (2015) 237–278.

[26] G. Goldstein, Derivation and physical interpretation of general boundary conditions. *Adv. Differ. Equ.* **11** (2006) 457–480.

[27] M. Baroun, S. Boulite, A. Elgrou and L. Maniar, Null controllability for backward stochastic parabolic convection-diffusion equations with dynamic boundary conditions. *Math. Control Related Fields* (2024).

[28] I. Chueshov and B. Schmalfuss, Parabolic stochastic partial differential equations with dynamical boundary conditions. *Differ. Integral Equ.* **17** (2004) 751–780.

[29] P. Müller, Stochastic forcing of oceanic motions, in Stochastic Models in Geosystems, vol. 85. Springer New York, New York, NY (1997) 219–237.

[30] D. Yang and J. Duan, An impact of stochastic dynamic boundary conditions on the evolution of the Cahn–Hilliard system. *Stochast. Anal. Applic.* **25** (2007) 613–639.

[31] D. Yang and J. Zhong, Observability inequality of backward stochastic heat equations for measurable sets and its applications. *SIAM J. Control Optim.* **54** (2016) 1157–1175.

[32] L. Yan and B. Wu, Null controllability for a class of stochastic singular parabolic equations with the convection term, *Discrete Continuous Dyn. Syst. B* **27** (2022) 3213–3240.

[33] Y. Yongyi and Q. Zhao, Controllability and observability for some forward stochastic complex degenerate/singular Ginzburg Landau equations. *ESAIM: Control Optim. Calc. Var.* **29** (2023) 34.

[34] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions. Cambridge University Press (2014).

[35] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.* **14** (1990) 55–61.

[36] E.M. Ouhabaz, Analysis of Heat Equations on Domains. LMS Monograph Series 31., Princeton University Press (2004).

[37] A. Lunardi, Interpolation Theory, 2nd edn. Edizioni della Normale, Pisa (2009).

[38] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, J. A. Barth, Heidelberg (1995).

[39] J.-M. Coron, *Control and Nonlinearity.* American Mathematical Society (2007).