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INSENSITIZING CONTROL PROBLEMS FOR THE STABILIZED KURAMOTO–SIVASHINSKY SYSTEM

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Abstract. In this work, we address the existence of insensitizing controls for a nonlinear coupled system of fourth- and second-order parabolic equations known as the stabilized Kuramoto–Sivashinsky model. The main idea is to look for controls such that some functional of the states (the so-called sentinel) is locally insensitive to the perturbations of the initial data. Since the underlying model is coupled, we shall consider a sentinel in which we may observe one or two components of the system in a localized observation set. By some classical arguments, the insensitizing problem can be reduced to a null-controllability one for a cascade system where the number of equations is doubled. Upon linearization, the null-controllability for this new system is studied by means of Carleman estimates but unlike other insensitizing problems for scalar models, the election of the Carleman tools and the overall control strategy depends on the initial choice of the sentinel due to the (lack of) couplings arising in the extended system. Finally, the local null-controllability of the extended (nonlinear) system (and thus the insensitizing property) is obtained by applying the inverse mapping theorem.

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1. Introduction and general setting

1.1. Problem statement and bibliographic comments

Let $T > 0$ and $\omega \subset (0, 1)$ be any nonempty open set. We set $Q_T := (0, T) \times (0, 1)$ and $\Sigma_T := (0, T) \times \{0, 1\}$. We also take another nonempty open set $\mathcal{O} \subset (0, 1)$ hereinafter referred as the observation set.

Let us consider the following control system with incomplete data

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + y y_x = z_x + h_1 \mathbb{1}_\omega + \xi_1 & \text{in } Q_T, \\ z_t - z_{xx} + \beta z_x = y_x + h_2 \mathbb{1}_\omega + \xi_2 & \text{in } Q_T, \\ y = y_x = z = 0 & \text{in } \Sigma_T, \\ y(0) = y_0 + \tau \bar{y}_0, \quad z(0) = z_0 + \tau \bar{z}_0 & \text{in } (0,1), \end{cases} \tag{1.1}$$

where $\gamma > 0$ and β is any real number.

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In (1.1), $y = y(t,x)$ and $z = z(t,x)$ are the state variables, $h_i = h_i(t,x)$, $i = 1,2$ are control functions acting on the control set ω, $\xi_i = \xi_i(t,x)$, $i = 1,2$, are given external source terms and the initial states $(y(0), z(0))$ are partially unknown in the following sense:

- $(y_0, z_0) \in [L^2(0,1)]^2$ are given,
- $(\bar{y}_0, \bar{z}_0) \in [L^2(0,1)]^2$ are unknown and satisfy $\|(\bar{y}_0, \bar{z}_0)\|_{[L^2(0,1)]^2} = 1$. They represent some *uncertainty* in the initial data.
- $\tau \in \mathbb{R}$ is unknown and small enough.

From the modelling point of view, when $h_i \equiv \xi_i \equiv 0$ $(i = 1,2)$ and $\tau = 0$, system (1.1) is the so-called stabilized Kuramoto–Sivashinsky system, which was proposed in [1] as a model of front propagation in reaction–diffusion phenomena and combines dissipative features with dispersive ones.

In this work, our main goal is to study the insensitizing control problem for (1.1). This problem, originally introduced by J.-L. Lions in [2], can be stated as follows: we observe the solution of system (1.1) through a functional J_τ (the so-called *sentinel*) defined on the set of solutions to (1.1), which is in this case given by

$$J_\tau(y,z) = \frac{\alpha}{2} \iint_{(0,T)\times\mathcal{O}} |y|^2 \,\mathrm{d}x\mathrm{d}t + \frac{1-\alpha}{2} \iint_{(0,T)\times\mathcal{O}} |z|^2 \,\mathrm{d}x\mathrm{d}t, \quad \alpha \in [0,1]. \tag{1.2}$$

Then, the insensitizing control problem is to find controls h_1, h_2 such that the uncertainty in the initial data does not affect the measurement J_τ, that is

$$\left.\frac{\partial J_\tau(y,z)}{\partial \tau}\right|_{\tau=0} = 0 \quad \forall (\bar{y}_0, \bar{z}_0) \in [L^2(0,1)]^2 \text{ with } \|(\bar{y}_0, \bar{z}_0)\|_{[L^2(0,1)]^2} = 1. \tag{1.3}$$

When (1.3) holds, the sentinel J_τ is said to be *locally insensitive* to the perturbations of the initial data. In other words, (1.3) indicates that the sentinel does not detect the variations of the initial data (y_0, z_0) by unknown small perturbation $\tau(\overline{y}_0, \overline{z}_0)$ in the observation domain $\mathcal{O}$.

The parameter α has been introduced in (1.2) to take into account the contribution of each state variable in the sentinel. Note that the insensitivity condition (1.3) should be satisfied for any perturbation of the initial data of both components, hence removing one observation in (1.2) (*i.e.* taking $\alpha = 0$ or $\alpha = 1$) reduces the information available and the problem becomes more interesting. Indeed, we will see later that additional difficulties arise in each of those cases and for that reason we shall mainly focus on them.

The first results concerning the existence of insensitizing controls were obtained for linear and semilinear heat equations in [3, 4]. After that, many works have been devoted to study the insensitizing problem from different perspectives: in [5–7], the authors study such problem for linear and semilinear heat equations with different types of nonlinearities and/or boundary conditions, while in [8] the problem of insensitizing a sentinel depending on the gradient of the solution of a linear parabolic equation is addressed. For insensitizing problems of equations in fluid mechanics, we refer to the works [9–12] and for a phase field system to [13]. Most recently, the insensitizing control problem has been addressed from a numerical point of view in [14], for fourth-order parabolic equations in [15] and with respect to shape variations in [16, 17]. We also mention a recent work [18] where the authors studied the insensitizing controls for the fourth-order dispersive nonlinear Schrödinger equation with cubic nonlinearity. Finally, it is worth mentioning that the insensitizing problem for a system of nonlinear KdV system has been analyzed in [19].

1.2. Main results

Our aim is to prove the existence of control functions (h_1, h_2) which insensitize the functional J_τ given by (1.2). In this spirit, our control result is the following.

Theorem 1.1. *Assume that $\mathcal{O}\cap\omega\neq\emptyset$ and $y_0\equiv z_0\equiv 0$. Then for any $\alpha\in[0,1]$, there exist constants $C>0$ and $\delta>0$ such that for any $(\xi_1,\xi_2)\in[L^2(Q_T)]^2$ verifying*

$$\|\mathrm{e}^{C/t}(\xi_1,\xi_2)\|_{[L^2(Q_T)]^2}\leq\delta, \tag{1.4}$$

one can prove the existence of some control $(h_1,h_2)\in[L^2((0,T)\times\omega)]^2$ which insensitizes the functional J_τ in the sense of (1.3).

To prove the above theorem, we shall equivalently prove the result given by Theorem 1.3 below. In fact, following well-known arguments (see *e.g.* [3], Prop. 1 or [20], Appendix), it can be proved that the insensitivity condition (1.3) is equivalent to a null-control problem for an extended system. More precisely, we have the following.

Proposition 1.2. *Consider the extended system*

$$\begin{cases} y_t+y_{xxxx}+\gamma y_{xx}+yy_x=z_x+h_1\mathbb{1}_\omega+\xi_1 & \text{in } Q_T,\\ z_t-z_{xx}+\beta z_x=y_x+h_2\mathbb{1}_\omega+\xi_2 & \text{in } Q_T,\\ y=y_x=z=0 & \text{in } \Sigma_T,\\ y(0)=y_0,\quad z(0)=z_0 & \text{in } (0,1),\end{cases} \tag{1.5}$$

$$\begin{cases} -p_t+p_{xxxx}+\gamma p_{xx}-yp_x=-q_x+\alpha y\mathbb{1}_{\mathcal{O}} & \text{in } Q_T,\\ -q_t-q_{xx}-\beta q_x=-p_x+(1-\alpha)z\mathbb{1}_{\mathcal{O}} & \text{in } Q_T,\\ p=p_x=q=0 & \text{in } \Sigma_T,\\ p(T)=0,\quad q(T)=0 & \text{in } (0,1).\end{cases} \tag{1.6}$$

Then, the controls (h_1,h_2) verify the insensitivity condition (1.3) *for the sentinel* (1.2) *if and only if the associated solution to* (1.5)–(1.6) *satisfies*

$$(p(0),q(0))=(0,0)\quad\text{in } (0,1). \tag{1.7}$$

In Appendix A, we present a sketch of the proof for Proposition 1.2.

In view of this result, we only focus on studying controllability properties for the extended system (1.5)–(1.6). Indeed, we prove the following theorem which is the main result of our paper.

Theorem 1.3 (Local null-controllability of the extended system)**.** *Assume that $\mathcal{O}\cap\omega\neq\emptyset$ and $y_0\equiv z_0\equiv 0$. Then for any $\alpha\in[0,1]$, there exist constants $C>0$ and $\delta>0$ such that for any $(\xi_1,\xi_2)\in[L^2(Q_T)]^2$ verifying*

$$\|\mathrm{e}^{C/t}(\xi_1,\xi_2)\|_{[L^2(Q_T)]^2}\leq\delta, \tag{1.8}$$

there exists controls $(h_1,h_2)\in[L^2((0,T)\times\omega)]^2$ such that the solution (y,z,p,q) to (1.5)–(1.6) *satisfies $p(0)=q(0)=0$ in $(0,1)$.*

We note that the control (h_1,h_2) acts indirectly on the state (p,q) by means of the coupling terms exerted on the observation set $\mathcal{O}$, that is, we have more equations than controls. As it has been pointed out in [21], this situation is more complicated than controlling scalar systems and, as we have said before, the parameter α introduces an additional difficulty. Note that when $\alpha=0$ or $\alpha=1$ one of the couplings in system (1.6) is removed and the action of the control (h_1,h_2) enters indirectly on the backward system only through one coupling term. As we will see later, this translates into using different Carleman tools for studying the observability of the corresponding adjoint system and establishing the controllability of (1.5)–(1.6).

Remark 1.4. Some remarks are in order.

- As in other insensitizing problems, the assumption on the zero initial condition is roughly related to the fact that system (1.5)–(1.6) is composed by forward and backward equations. As noticed in the work [20], even for the simple heat equation is not an easy task to characterize the space of initial data that can be insensitized; see also [4].
- In the present paper, the assumption $\mathcal{O} \cap \omega \neq \emptyset$ is essential to prove an observability inequality (see Prop. 4.3 for instance), which is the main ingredient in the proof of Theorem 1.3. Notwithstanding, in [22], the authors have proved that in the simpler case of heat equation this condition is not necessary if one considers an ϵ-insensitizing problem ($i.e.$, $\left|\frac{\partial J_\tau(y,z)}{\partial \tau}|_{\tau=0}\right| \leq \epsilon$) instead of the usual insensitizing problem as treated in our work. Hence, this ϵ-insensitizing problem remains as an open question for our coupled fourth- and second-order parabolic system.

To prove Theorem 1.3, we shall first prove a null-controllability result for the following linearized model (around zero) associated to (1.5)–(1.6),

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = z_x + h_1 \mathbb{1}_\omega + f_1 & \text{in } Q_T, \\ z_t - z_{xx} + \beta z_x = y_x + h_2 \mathbb{1}_\omega + f_2 & \text{in } Q_T, \\ y = y_x = z = 0 & \text{in } \Sigma_T, \\ y(0) = y_0, \quad z(0) = z_0 & \text{in } (0,1), \end{cases} \tag{1.9}$$

$$\begin{cases} -p_t + p_{xxxx} + \gamma p_{xx} = -q_x + \alpha y \mathbb{1}_{\mathcal{O}} + f_3 & \text{in } Q_T, \\ -q_t - q_{xx} - \beta q_x = -p_x + (1-\alpha) z \mathbb{1}_{\mathcal{O}} + f_4 & \text{in } Q_T, \\ p = p_x = q = 0 & \text{in } \Sigma_T, \\ p(T) = 0, \quad q(T) = 0 & \text{in } (0,1), \end{cases} \tag{1.10}$$

with source terms f_1, f_2, f_3, f_4 from the space $L^2(Q_T)$.

As usual, the controllability problem boils down to study the observability properties of the adjoint system associated to (1.9)–(1.10). In fact, the adjoint system is

$$\begin{cases} -u_t + u_{xxxx} + \gamma u_{xx} = F_1 - w_x + \alpha \zeta \mathbb{1}_{\mathcal{O}} & \text{in } Q_T, \\ -w_t - w_{xx} - \beta w_x = F_2 - u_x + (1-\alpha)\theta \mathbb{1}_{\mathcal{O}} & \text{in } Q_T, \\ \zeta_t + \zeta_{xxxx} + \gamma \zeta_{xx} = F_3 + \theta_x & \text{in } Q_T, \\ \theta_t - \theta_{xx} + \beta \theta_x = F_4 + \zeta_x & \text{in } Q_T, \\ u = u_x = w = \zeta = \zeta_x = \theta = 0 & \text{in } \Sigma_T, \\ u(T) = 0, \quad w(T) = 0, & \text{in } (0,1), \\ \zeta(0) = \zeta_0, \quad \theta(0) = \theta_0 & \text{in } (0,1), \end{cases} \tag{1.11}$$

with given data (ζ_0, θ_0) and source terms F_j for $j = 1, 2, 3, 4$ from some suitable Hilbert spaces.

The strategy amounts to apply Carleman estimates for each equation of system (1.11) and then use the first and second equations to estimate locally the terms related to ζ and θ. Note that, for $\alpha \in (0,1)$, we have a natural way to estimate such terms thanks to the hypothesis $\mathcal{O} \cap \omega \neq \emptyset$, but as soon as $\alpha = 0$ or $\alpha = 1$, we loose information on either ζ or θ and we have to use the first-order couplings from the third and fourth equations of (1.11) to do local energy estimates. To circumvent this, we shall use some Carleman tools from the works [23] and [24] allowing us to consider the derivatives w.r.t. x of the equations verified by ζ or θ, and then estimate locally the first-order derivatives of these variables.

Paper organization The rest of the paper is organized as follows.

– In Section 2, we shortly discuss the well-posedness of the extended system (1.5)–(1.6).

- In Section 3, we address several Carleman estimates for different α. More precisely, Sections 3.1, 3.2 and 3.3 contain the Carleman inequalities associated to the adjoint system (1.11) for $\alpha = 0$, $\alpha = 1$ and $\alpha \in (0,1)$ respectively.
- Thereafter, in Section 4, we discuss the main controllability results for diffrent $\alpha \in [0,1]$. We mainly focus on the two significant cases: $\alpha = 0$ and $\alpha = 1$.
  To be more precise, Sections 4.1.1–4.1.2 contain the observability inequality and null-controllability for the linearized model (1.9)–(1.10) when $\alpha = 0$. Then, in Section 4.1.3 we prove the local null-controllability of the extended system (1.5)–(1.6) for $\alpha = 0$.
  In the Sections 4.2.1–4.2.2, we discuss the null-controllability of the linearized model when $\alpha = 1$ and the proof of the local null-controllability for this case will be as similar as the case $\alpha = 0$ and thus we left this to the reader.
- Finally, Section 5 is devoted to present some concluding remarks.

Notations Throughout the paper, $C > 0$ denotes a generic constant that may vary line to line but is independent of the Carleman parameters s or λ.

By $\iint$, we denote the integral in Q_T and by $\iint_O$ we denote the integral in $(0,T) \times O$ for any non-empty open set $O \subset (0,1)$. For simplicity, the symbol "$\mathrm{d}x\mathrm{d}t$" will be omitted in all integrals.

2. Well-posedness

In this section, we briefly discuss the well-posedness of the 4×4 control system (1.5)–(1.6).

2.1. The linear adjoint system

Using the regularity results in Propositions C.1 and C.3 and utilizing the cascade structure of system (1.11) we can deduce the following result.

Proposition 2.1. *1. For given data $(\zeta_0, \theta_0) \in [L^2(0,1)]^2$ and source terms $F_j \in L^2(Q_T)$ for $j = 1,2,3,4$, there exists unique (u, w, ζ, θ) solution to (1.11) such that*

$$u, \zeta \in \mathcal{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^2(0,1)) \cap H^1(0,T; H^{-2}(0,1)), \tag{2.1}$$
$$w, \theta \in \mathcal{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1)) \cap H^1(0,T; H^{-1}(0,1)), \tag{2.2}$$

satisfying

$$\begin{aligned} &\|u\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)\cap H^1(H^{-2})} + \|w\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)\cap H^1(H^{-1})} + \|\zeta\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)\cap H^1(H^{-2})} \\ &\qquad + \|\theta\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)\cap H^1(H^{-1})} \leq C\Big(\|(\zeta_0,\theta_0)\|_{[L^2(0,1)]^2} + \sum_{j=1}^{4} \|F_j\|_{L^2(Q_T)}\Big). \end{aligned}$$

2. If we choose the data $(\zeta_0, \theta_0) \in H_0^2(0,1) \times H_0^1(0,1)$, then the solution to (1.11) satisfies the following regularity results:

$$u, \zeta \in \mathcal{C}^0([0,T]; H_0^2(0,1)) \cap L^2(0,T; H^4(0,1)) \cap H^1(0,T; L^2(0,1)), \tag{2.3}$$
$$w, \theta \in \mathcal{C}^0([0,T]; H_0^1(0,1)) \cap L^2(0,T; H^2(0,1)) \cap H^1(0,T; L^2(0,1)), \tag{2.4}$$

with in addition,

$$\|u\|_{\mathcal{C}^0(H_0^2)\cap L^2(H^4)\cap H^1(L^2)} + \|w\|_{\mathcal{C}^0(H_0^1)\cap L^2(H^2)\cap H^1(L^2)} + \|\zeta\|_{\mathcal{C}^0(H_0^2)\cap L^2(H^4)\cap H^1(L^2)}$$
$$+ \|\theta\|_{\mathcal{C}^0(H_0^1)\cap L^2(H^2)\cap H^1(L^2)} \leq C\Big(\|(\zeta_0,\theta_0)\|_{H_0^2(0,1)\times H_0^1(0,1)} + \sum_{j=1}^4 \|F_j\|_{L^2(Q_T)}\Big).$$

2.2. The linear forward system

In a same spirit, we have the following result for the direct linear system (1.9)–(1.10).

Proposition 2.2. *For given $(y_0, z_0) \in [L^2(0,1)]^2$ and $f_j \in L^2(Q_T)$ for $j = 1,2,3,4$ there exists unique (y,z,p,q) solution to (1.9)–(1.10) such that*

$$y, p \in \mathcal{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^2(0,1)), \tag{2.5}$$
$$z, q \in \mathcal{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1)), \tag{2.6}$$

satisfying

$$\|y\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)} + \|z\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)} + \|p\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)}$$
$$+ \|q\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)} \leq C\Big(\|(y_0,z_0)\|_{[L^2(0,1)]^2} + \sum_{j=1}^4 \|f_j\|_{L^2(Q_T)} + \|(h_1,h_2)\|_{[L^2((0,T)\times\omega)]^2}\Big). \tag{2.7}$$

2.3. The main nonlinear system

Theorem 2.3. *There is a positive real number δ_0 such that for any $(y_0, z_0) \in [L^2(0,1)]^2$, $(h_1, h_2) \in [L^2((0,T)\times\omega)]^2$ and $(\xi_1,\xi_2) \in [L^2(Q_T)]^2$ satisfying*

$$\|(y_0,z_0)\|_{[L^2(0,1)]^2} + \|(h_1,h_2)\|_{[L^2((0,T)\times\omega)]^2} + \|(\xi_1,\xi_2)\|_{[L^2(Q_T)]^2} \leq \delta_0, \tag{2.8}$$

the nonlinear problem (1.5)–(1.6) has unique solution (y,z,p,q) with

$$(y,z,p,q) \in \mathcal{C}^0([0,T];[L^2(0,1)]^4) \cap L^2(0,T; H_0^2(0,1)\times H_0^1(0,1)\times H_0^2(0,1)\times H_0^1(0,1)).$$

Proof. Let us define the map

$$\Lambda : [L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H_0^2(0,1))]^2 \to [L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H_0^2(0,1))]^2,$$
$$\Lambda(\widetilde{y},\widetilde{p}) = (y,p),$$

where (y,z,p,q) is the unique solution to (1.9)–(1.10) satisfying (2.5)–(2.6) with $f_1 = \xi_1 - \widetilde{y}\,\widetilde{y}_x$, $f_2 = \xi_2$, $f_3 = \widetilde{y}\,\widetilde{p}_x$ and $f_4 = 0$.

We compute that

$$\begin{aligned}\|\widetilde{y}\,\widetilde{y}_x\|_{L^2(Q_T)} = \Big(\int_0^T\int_0^1 |\widetilde{y}\,\widetilde{y}_x|^2\Big)^{\frac12} &\leq \Big(\int_0^T \|\widetilde{y}(t)\|^2_{L^2(0,1)}\|\widetilde{y}_x(t)\|^2_{L^\infty(0,1)}\Big)^{\frac12}\\ &\leq \|\widetilde{y}\|_{L^\infty(0,T;L^2(0,1))}\|\widetilde{y}_x\|_{L^2(0,T;H_0^1(0,1))}\\ &\leq \|\widetilde{y}\|_{L^\infty(0,T;L^2(0,1))}\|\widetilde{y}\|_{L^2(0,T;H_0^2(0,1))},\end{aligned}$$

and similarly,

$$\|\widetilde{y}\,\widetilde{p}_x\|_{L^2(Q_T)} \leq \|\widetilde{y}\|_{L^\infty(0,T;L^2(0,1))}\|\widetilde{p}\|_{L^2(0,T;H_0^2(0,1))}.$$

Therefore, by means of Proposition 2.2, we have

$$\begin{aligned}&\|y\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)}+\|z\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)}+\|p\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)}+\|q\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)}\\&\qquad\le C_0\Big(\|(y_0,z_0)\|_{[L^2(0,1)]^2}+\|(h_1,h_2)\|_{[L^2((0,T)\times\omega)]^2}\\&\qquad\quad+\|(\xi_1,\xi_2)\|_{[L^2(Q_T)]^2}+\|\widetilde{y}\|_{L^\infty(L^2)}\|\widetilde{y}\|_{L^2(H_0^2)}+\|\widetilde{y}\|_{L^\infty(L^2)}\|\widetilde{p}\|_{L^2(H_0^2)}\Big),\end{aligned}\tag{2.9}$$

for some constant $C_0>0$.

Now, we denote the set

$$\mathcal{B}_R=\Big\{(y,p)\in[L^\infty(0,T;L^2(0,1))\cap L^2(0,T;H_0^2(0,1))]^2 \;\mid\; \|y\|_{L^\infty(L^2)\cap L^2(H_0^2)}+\|p\|_{L^\infty(L^2)\cap L^2(H_0^2)}\le R\Big\}.$$

– Then starting with $(\widetilde{y},\widetilde{p})\in\mathcal{B}_R$, we have from (2.9)

$$\begin{aligned}&\|y\|_{L^\infty(L^2)\cap L^2(H_0^2)}+\|z\|_{L^\infty(L^2)\cap L^2(H_0^1)}+\|p\|_{L^\infty(L^2)\cap L^2(H_0^2)}+\|q\|_{L^\infty(L^2)\cap L^2(H_0^1)}\\&\le C_0\Big(\|(y_0,z_0)\|_{[L^2(0,1)]^2}+\|(h_1,h_2)\|_{[L^2((0,T)\times\omega)]^2}+\|(\xi_1,\xi_2)\|_{[L^2(Q_T)]^2}+R^2\Big).\end{aligned}\tag{2.10}$$

It follows that if $R<\frac{1}{C_0}$ and

$$\|(y_0,z_0)\|_{[L^2(0,1)]^2}+\|(h_1,h_2)\|_{[L^2((0,T)\times\omega)]^2}+\|(\xi_1,\xi_2)\|_{[L^2(Q_T)]^2}\le\frac{R-C_0R^2}{C_0},$$

one has $\Lambda(\mathcal{B}_R)\subset\mathcal{B}_R$ which implies $\mathcal{B}_R$ is stable under the map Λ. We take $\delta_0=\frac{R-C_0R^2}{C_0}$ in (2.8).

– Let us now prove that Λ is a contraction map. Choose two elements $(\widetilde{y},\widetilde{p})$, $(\widehat{y},\widehat{p})$ from the set $\mathcal{B}_R$ and denote the associated solutions by (y_1,z_1,p_1,q_1) and (y_2,z_2,p_2,q_2) respectively.

We further denote $(y,z,p,q)=(y_1-y_2,z_1-z_2,p_1-p_2,q_1-q_2)$ which satisfies the following set of equations

$$\begin{cases}y_t+y_{xxxx}+\gamma y_{xx}=z_x+\widehat{y}\,\widehat{y}_x-\widetilde{y}\,\widetilde{y}_x & \text{in } Q_T,\\ z_t-z_{xx}+\beta z_x=y_x & \text{in } Q_T,\\ y=y_x=z=0 & \text{in } \Sigma_T,\\ y(0)=0,\quad z(0)=0 & \text{in } (0,1),\end{cases}\tag{2.11}$$

$$\begin{cases}-p_t+p_{xxxx}+\gamma p_{xx}=-q_x+\alpha y\mathbb{1}_{\mathcal{O}}+\widetilde{y}\,\widetilde{p}_x-\widehat{y}\,\widehat{p}_x & \text{in } Q_T,\\ -q_t-q_{xx}-\beta q_x=-p_x+(1-\alpha)z\mathbb{1}_{\mathcal{O}} & \text{in } Q_T,\\ p=p_x=q=0 & \text{in } \Sigma_T,\\ p(T)=0,\quad q(T)=0 & \text{in } (0,1).\end{cases}\tag{2.12}$$

We first compute

$$\begin{aligned}\left(\int_0^T\int_0^1|\widetilde{y}\,\widetilde{p}_x-\widehat{y}\,\widehat{p}_x|^2\right)^{\frac12}&\le\left(2\int_0^T\int_0^1|\widetilde{y}(\widetilde{p}_x-\widehat{p}_x)|^2+|\widehat{p}_x(\widetilde{y}-\widehat{y})|^2\right)^{\frac12}\\&\le\sqrt{2}\|\widetilde{y}\|_{L^\infty(L^2)}\|\widetilde{p}-\widehat{p}\|_{L^2(H_0^2)}+\sqrt{2}\|\widehat{p}\|_{L^2(H_0^2)}\|\widetilde{y}-\widehat{y}\|_{L^\infty(L^2)}.\end{aligned}$$

Similarly, one has

$$\|\widetilde{y}\,\widetilde{y}_x - \widehat{y}\,\widehat{y}_x\|_{L^2(Q_T)} \leq \sqrt{2}\|\widetilde{y}\|_{L^\infty(L^2)}\|\widetilde{y}-\widehat{y}\|_{L^2(H_0^2)} + \sqrt{2}\|\widehat{y}\|_{L^2(H_0^2)}\|\widetilde{y}-\widehat{y}\|_{L^\infty(L^2)}.$$

Using the above two facts, the solution to (2.11)–(2.12) satisfies (by means of Prop. 2.2)

$$\begin{aligned}
&\|y\|_{L^\infty(L^2)\cap L^2(H_0^2)} + \|z\|_{L^\infty(L^2)\cap L^2(H_0^1)} + \|p\|_{L^\infty(L^2)\cap L^2(H_0^2)} + \|q\|_{L^\infty(L^2)\cap L^2(H_0^1)} \\
&\leq C_0\Big(\|\widetilde{y}\,\widetilde{y}_x - \widehat{y}\,\widehat{y}_x\|_{L^2(Q_T)} + \|\widetilde{y}\,\widetilde{p}_x - \widehat{y}\,\widehat{p}_x\|_{L^2(Q_T)}\Big) \\
&\leq \sqrt{2}C_0\|\widetilde{y}\|_{L^\infty(L^2)}\Big(\|\widetilde{y}-\widehat{y}\|_{L^2(H_0^2)} + \|\widetilde{p}-\widehat{p}\|_{L^2(H_0^2)}\Big) \\
&\qquad + \sqrt{2}C_0\|\widetilde{y}-\widehat{y}\|_{L^\infty(L^2)}\Big(\|\widehat{y}\|_{L^2(H_0^2)} + \|\widehat{p}\|_{L^2(H_0^2)}\Big) \\
&\leq \sqrt{2}C_0R\Big(\|\widetilde{y}-\widehat{y}\|_{L^\infty(L^2)\cap L^2(H_0^2)} + \|\widetilde{p}-\widehat{p}\|_{L^\infty(L^2)\cap L^2(H_0^2)}\Big).
\end{aligned} \tag{2.13}$$

Now, choose $R>0$ in such a way that $\sqrt{2}C_0R<1$, so that the map Λ is contracting. Thus, using the Banach fixed point theorem, there exists a unique fixed point of Λ in $\mathcal{B}_R$, which gives the unique solution (y,z,p,q) to (1.5)–(1.6).

The proof is complete. □

3. Carleman estimates for different α

This section is devoted to obtain Carleman inequalities for the extended adjoint system (1.11) when:

(i) $\alpha = 0$,
(ii) $\alpha = 1$,
(iii) $\alpha \in (0,1)$.

Each case will be treated differently. As mentioned in the Introduction (see the discussion below (1.11)), this is due to the kind of couplings that arise while setting different values of α.

Let us define some Carleman weights which have been introduced in the articles [8, 23].

Weight functions Recall that, by hypothesis, $\mathcal{O}\cap\omega \neq \emptyset$. Therefore, there is an open set $\omega_0 \subset\subset \mathcal{O}\cap\omega$. In what follows, we establish the Carleman estimates with the observation domain ω_0.

Consider a function $\nu \in \mathcal{C}^4([0,1])$ satisfying

$$\begin{cases} \nu(x) > 0 \quad \forall x \in (0,1), \quad \nu(0)=\nu(1)=0, \\ |\nu'(x)| \geq c > 0 \quad \forall x \in \overline{(0,1)\setminus\omega_0} \quad \text{for some } c>0. \end{cases} \tag{3.1}$$

In particular, we have $\nu'(0)>0$ and $\nu'(1)<0$.

Now, for some constants $\lambda>1$ and $k>m>0$, we define the weight functions

$$\varphi_m(t,x) = \frac{e^{\lambda(1+\frac{1}{m})k\|\nu\|_\infty} - e^{\lambda\big(k\|\nu\|_\infty + \nu(x)\big)}}{t^m(T-t)^m}, \quad \xi_m(t,x) = \frac{e^{\lambda\big(k\|\nu\|_\infty+\nu(x)\big)}}{t^m(T-t)^m}, \quad \forall (t,x)\in Q_T. \tag{3.2}$$

Here, observe that φ_m and ξ_m are positive functions in $[0,1]$ due to the choices of λ, k and m.

Some immediate results associated with the weights are following.

– For any $b > 0$, there exists some constant $C > 0$ such that

$$|(\mathrm{e}^{-2s\varphi_m}\xi_m^b)_x| \leq Cs\lambda\xi_m(\mathrm{e}^{-2s\varphi_m}\xi_m^b), \tag{3.3}$$

where $\lambda > 1$ and $k > m > 0$.

– Similarly, for any $b > 0$, there is some constant $C > 0$ such that

$$|(\mathrm{e}^{-2s\varphi_m}\xi_m^b)_t| \leq Cs\xi_m^{1+\frac{1}{m}}(\mathrm{e}^{-2s\varphi_m}\xi_m^b), \tag{3.4}$$

with $\lambda > 1$ and $k > m > 0$.

– For any $m > 1$, we observe that

$$\begin{aligned} t^{m-1}(T-t)^{m-1} &\leq CT^{2m-2}, \\ \textit{i.e.,}\ \ 1 \leq CT^{2m-2}\frac{t(T-t)}{t^m(T-t)^m} &\leq CT^{2m-2}\xi_m^{1-1/m} \\ \textit{i.e.,}\ \ \xi_m^{1/m} &\leq CT^{2m-2}\xi_m. \end{aligned} \tag{3.5}$$

Using this in (3.4), one has

$$|(\mathrm{e}^{-2s\varphi_m}\xi_m^b)_t| \leq CT^{2m-2}s\xi_m^2(\mathrm{e}^{-2s\varphi_m}\xi_m^b), \quad \text{for } m > 1. \tag{3.6}$$

Some standard Carleman estimates Let us write the following Carleman estimate for second order parabolic equations.

Theorem 3.1. *Let φ_m and ξ_m be given by (3.2) with $m \geq 1$ and $r \in \mathbb{R}$. Then, there exist positive constants λ_0, $s_0 = \sigma_0(T^{2m-1} + T^{2m})$ with some $\sigma_0 > 0$ and C, such that the solution $q \in L^2(0,T;H^2(0,1) \cap H_0^1(0,1)) \cap H^1(0,T;L^2(0,1))$ to the second-order parabolic equation*

$$\begin{cases} -q_t - q_{xx} = g & \textit{in } Q_T, \\ q = 0 & \textit{in } \Sigma_T, \\ q(T) = q_T & \textit{in } (0,1), \end{cases}$$

with given $q_T \in H_0^1(0,1)$ and $g \in L^2(Q_T)$, satisfies

$$\begin{aligned} I_H(q;r) := s^r\lambda^{r+1}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^r|q|^2 + s^{r-2}\lambda^{r-1}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^{r-2}|q_x|^2 \\ + s^{r-4}\lambda^{r-3}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^{r-4}\left(|q_t|^2 + |q_{xx}|^2\right) \\ \leq C\left(s^{r-3}\lambda^{r-3}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^{r-3}|g|^2 + s^r\lambda^{r+1}\iint_{\omega_0} \mathrm{e}^{-2s\varphi_m}\xi_m^r|q|^2\right), \end{aligned} \tag{3.7}$$

for every $\lambda \geq \lambda_0$, $s \geq s_0$.

For $r = 3$, the above result is well-known due to the pioneering work by Fursikov and Imanuvilov [25]. The general case $r \in \mathbb{R}$ can be obtained from the case $r = 3$ and adapting the procedure shown in [26], Lemma 2.3.

We also need the following Carleman estimate for fourth order parabolic equations.

Theorem 3.2. *Let φ_m and ξ_m be given by (3.2) with $m \geq 2/5$. Then, there exist positive constants λ_0, $s_0 = \sigma_0(T^{2m-2/5} + T^{2m})$ with some $\sigma_0 > 0$ and C, such that the solution $q \in L^2(0,T; H^4(0,1) \cap H_0^2(0,1)) \cap H^1(0,T; L^2(0,1))$ to the fourth-order parabolic equation*

$$\begin{cases} -q_t + q_{xxxx} + \gamma q_x = g & \text{in } Q_T, \\ q = q_x = 0 & \text{in } \Sigma_T, \\ q(T) = q_T & \text{in } (0,1), \end{cases}$$

with given $q_T \in H_0^2(0,1)$ and $g \in L^2(Q_T)$, satisfies

$$\begin{aligned} I_{KS}(q) :&= s^7\lambda^8 \iint e^{-2s\varphi_m}\xi_m^7|q|^2 + s^5\lambda^6 \iint e^{-2s\varphi_m}\xi_m^5|q_x|^2 + s^3\lambda^4 \iint e^{-2s\varphi_m}\xi_m^3|q_{xx}|^2 \\ &\quad + s\lambda^2 \iint e^{-2s\varphi_m}\xi_m|q_{xxx}|^2 + s^{-1} \iint e^{-2s\varphi_m}\xi_m^{-1}(|q_t|^2 + |q_{xxxx}|^2) \\ &\leq C\left(\iint e^{-2s\varphi_m}|g|^2 + s^7\lambda^8 \iint_{\omega_0} e^{-2s\varphi_m}\xi_m^7|q|^2\right), \end{aligned} \tag{3.8}$$

for every $\lambda \geq \lambda_0$, $s \geq s_0$.

The proof for the above result can be found in [27]. We also refer the work [23].

Remark 3.3. The above Carleman estimates (3.7) and (3.8) also hold if we consider forward-in-time parabolic equations instead of backward ones.

3.1. Carleman estimate for the case when $\alpha = 0$

The adjoint system (1.11) for the case $\alpha = 0$ reads as

$$\begin{cases} -u_t + u_{xxxx} + \gamma u_{xx} = F_1 - w_x & \text{in } Q_T, \\ -w_t - w_{xx} - \beta w_x = F_2 - u_x + \theta \mathbb{1}_{\mathcal{O}} & \text{in } Q_T, \\ \zeta_t + \zeta_{xxxx} + \gamma\zeta_{xx} = F_3 + \theta_x & \text{in } Q_T, \\ \theta_t - \theta_{xx} + \beta\theta_x = F_4 + \zeta_x & \text{in } Q_T, \\ u = u_x = w = \zeta = \zeta_x = \theta = 0 & \text{in } \Sigma_T, \\ u(T) = 0, \quad w(T) = 0, & \text{in } (0,1), \\ \zeta(0) = \zeta_0, \quad \theta(0) = \theta_0 & \text{in } (0,1). \end{cases} \tag{3.9}$$

Before proving the required Carleman inequality associated to system (3.9), we recall the weight functions φ_m and ξ_m as given by (3.2). We define

$$\begin{cases} \widehat{\varphi}_m(t) = \max\limits_{x\in[0,1]} \varphi_m(t,x) = \varphi_m(t,0) = \varphi_m(t,1), \\ \xi_m^*(t) = \min\limits_{x\in[0,1]} \xi_m(t,x) = \xi_m(t,0) = \xi_m(t,1). \end{cases} \tag{3.10}$$

We now prove the following Carleman inequality for the adjoint states (u, w, ζ, θ) for given data $(\zeta_0, \theta_0) \in H_0^2(0,1) \times H_0^1(0,1)$ and $F_j \in L^2(Q_T)$ for $j = 1,2,3,4$. Consequently, we have

$$\begin{cases} u, \zeta \in L^2(0,T; H^4(0,1) \cap H_0^2(0,1)) \cap H^1(0,T; L^2(0,1)), \\ w, \theta \in L^2(0,T; H^2(0,1) \cap H_0^1(0,1)) \cap H^1(0,T; L^2(0,1)). \end{cases} \tag{3.11}$$

Theorem 3.4 (Carleman inequality: the case $\alpha = 0$)**.** *Let (φ_m, ξ_m) and $(\widehat{\varphi}_m, \xi_m^*)$ be given by (3.2) and (3.10) respectively with $m \geq 2$. Then, there exist positive constants λ^*, $s^* := \sigma^*(T^m + T^{2m-2/5} + T^{2m-1} + T^{2m})$ with some $\sigma^* > 0$ and C such that the solution (u, w, ζ, θ), given by (3.11), to (3.9) satisfies:*

$$
\begin{aligned}
&I_{KS}(u) + I_H(w;7) + s^7\lambda^8 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta_x|^2 + s^7\lambda^8 \iint \mathrm{e}^{-2s\widehat{\varphi}_m}(\xi_m^*)^7|\zeta|^2 + I_H(\theta;9) \\
&\quad \leq C\bigg[\iint \mathrm{e}^{-2s\varphi_m}\left(|F_1|^2 + s^5\lambda^5\xi_m^5|F_3|^2 + s^7\lambda^8\xi_m^7|F_4|^2\right) + s^{37}\lambda^{22}\iint \mathrm{e}^{-10s\varphi_m+8s\widehat{\varphi}_m}\xi_m^{37}|F_2|^2 \\
&\quad +s^{39}\lambda^{24}\iint_{\omega_0} \mathrm{e}^{-10s\varphi_m+8s\widehat{\varphi}_m}\xi_m^{39}|u|^2 + s^{41}\lambda^{26}\iint_{\omega_0} \mathrm{e}^{-10s\varphi_m+8s\widehat{\varphi}_m}\xi_m^{41}|w|^2\bigg],
\end{aligned}
\tag{3.12}
$$

for all $\lambda \geq \lambda^*$ and $s \geq s^*$, where $I_H(\cdot;\cdot)$ and $I_{KS}(\cdot)$ are given by (3.7) and (3.8) respectively.

Let us briefly point out the idea behind the proof for the Carleman inequality (3.12).

(i) For the variable u satisfying fourth order parabolic equation, we use the Carleman estimate given by Theorem 3.2.

(ii) We observe that the Carleman estimate for ζ will always be associated with an observation integral of ζ and there is no chance to absorb it by any of the leading integrals. In fact, there is a coupling by ζ_x to the equation of θ and therefore, we are going to use a Carleman estimate for the variable ζ_x, see Lemma 3.8 below.
In this context, we recall the work [24], where such a Carleman estimate has been established.

(iii) For w and θ, we use the classical Carleman inequality by Fursikov and Imanuvilov (see Thm. 3.1) for the heat equation, maybe with different powers of Carleman parameters.

Remark 3.5. In the Carleman estimate (3.12), there appears the weight function $\mathrm{e}^{-10s\varphi_m+8s\widehat{\varphi}_m}$. But, by Lemma B.1, there exists some $c_0 > 0$ such that

$$
-10s\varphi_m + 8s\widehat{\varphi}_m \leq \frac{-c_0 s}{t^m(T-t)^m},
$$

which ensures obtaining a suitable observability inequality from the Carleman estimate (3.12), see Proposition 4.3.

Let us give the required Carleman estimates for the functions u, w, ζ_x and θ as mentioned above.

Lemma 3.6 (Carleman inequality for u, the case $\alpha = 0$)**.** *Let φ_m and ξ_m be as given by (3.2) with $m \geq 2/5$. Then, there exist positive constants $\bar{\lambda}_1$, $\bar{s}_1 := \bar{\sigma}_1(T^{2m} + T^{2m-2/5})$ with some $\sigma_1 > 0$ and C, such that the solution component u to (3.9), given by (3.11), satisfies:*

$$
I_{KS}(u) \leq C\left(\iint \mathrm{e}^{-2s\varphi_m}(|F_1|^2 + |w_x|^2) + s^7\lambda^8\iint_{\omega_0} \mathrm{e}^{-2s\varphi_m}\xi_m^7|u|^2\right), \tag{3.13}
$$

for all $\lambda \geq \bar{\lambda}_1$, $s \geq \bar{s}_1$, where $I_{KS}(\cdot)$ is defined by (3.8).

For the component w, we shall use the standard Carleman inequality for heat equations.

Lemma 3.7 (Carleman inequality for w, the case $\alpha = 0$)**.** *Let φ_m and ξ_m be defined as in (3.2) with $m \geq 1$. Then, there exist positive constants $\bar{\lambda}_2$, $\bar{s}_2 := \bar{\sigma}_2(T^{2m} + T^{2m-1})$ with some $\bar{\sigma}_2 > 0$ and C, such that w, the solution component to (3.9) given by (3.11), satisfies:*

$$
I_H(w;7) \leq C\left(s^4\lambda^4\iint \mathrm{e}^{-2s\varphi_m}\xi_m^4\left(|F_2|^2 + |u_x|^2 + |\theta|^2\right) + s^7\lambda^8\iint_{\omega_0} \mathrm{e}^{-2s\varphi_m}\xi_m^7|w|^2\right), \tag{3.14}
$$

for all $\lambda \geq \bar{\lambda}_2$ and $s \geq \bar{s}_2$, where $I_H(\cdot;\cdot)$ is defined by (3.7).

We now write the Carleman estimate for ζ_x.

Lemma 3.8 (Carlemn inequality for ζ_x, the case $\alpha = 0$)**.** *Let (φ_m, ξ_m) and $(\widehat{\varphi}_m, \xi_m^*)$ be given by (3.2) and (3.10) respectively with $m \geq 2$, $k > m$. Then, there exist positive constants $\bar{\lambda}_3$, $\bar{s}_3 := \bar{\sigma}_3(T^m + T^{2m})$ with some $\bar{\sigma}_3 > 0$ and C such that we have the following estimate for $\zeta_x \in L^2(0,T; H^3(0,1) \cap H_0^1(0,1))$,*

$$s^7\lambda^8 \iint \left(\mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta_x|^2 + \mathrm{e}^{-2s\widehat{\varphi}_m}(\xi_m^*)^7|\zeta|^2\right) + s^{5-\frac{2}{m}}\lambda^5\|\mathrm{e}^{-s\widehat{\varphi}_m}(\xi_m^*)^{\frac{5}{2}-\frac{1}{m}}\zeta\|^2_{L^2(0,T;H^4(0,1))}$$
$$\leq C\left[s^5\lambda^5 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^5\left(|\theta_x|^2 + |F_3|^2\right) + T^{10m}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^5|\theta_{xx}|^2 + s^7\lambda^8\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta_x|^2\right], \quad (3.15)$$

for all $\lambda \geq \bar{\lambda}_3$ and $s \geq \bar{s}_3$, where ζ is the solution component of (3.9) given by (3.11).

The proof of Lemma 3.8 can be verified using the techniques developed in [24], Section 3.2. For sake of completeness, we give a short sketch below.

Proof of Lemma 3.8. Recall the equation of ζ from the system (3.9) and consider the following equation for $\widetilde{\zeta} := \zeta_x$, given by

$$\begin{cases} \widetilde{\zeta}_t + \widetilde{\zeta}_{xxxx} + \gamma\widetilde{\zeta}_{xx} = \theta_{xx} + F_{3,x} & \text{in } Q_T, \\ \widetilde{\zeta} = 0, \quad \widetilde{\zeta}_x = \zeta_{xx} & \text{in } \Sigma_T, \\ \widetilde{\zeta}(0,\cdot) = 0 & \text{in } (0,1), \end{cases} \quad (3.16)$$

where it is clear that $\theta_{xx} \in L^2(Q_T)$ and we have $\widetilde{\zeta}_x(\cdot) \in L^2(0,T)$ in Σ_T.

Now, due to [24], Theorem 3.5, we have the following auxiliary estimate for ζ_x,

$$s^7\lambda^8\iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta_x|^2 \leq C\Big[\iint \mathrm{e}^{-2s\varphi_m}|\theta_{xx}|^2 + s^2\lambda^2\iint \mathrm{e}^{-2s\varphi_m}\xi_m^2|F_3|^2$$
$$+ s^5\lambda^5\int_0^T \mathrm{e}^{-2s\widehat{\varphi}_m}(\xi_m^*)^5\left(|\zeta_{xx}(t,0)|^2 + |\zeta_{xx}(t,1)|^2\right) + s^7\lambda^8\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta_x|^2\Big], \quad (3.17)$$

for every $\lambda \geq c_1$ and $s \geq c_2(T^m + T^{2m-2/5})$ for some $c_1, c_2 > 0$.

First observe that,

$$s^7\lambda^8\iint \mathrm{e}^{-2s\widehat{\varphi}_m}(\xi_m^*)^7|\zeta|^2 \leq Cs^7\lambda^8\iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta_x|^2, \quad (3.18)$$

since $\zeta(t,0) = \zeta(t,1) = 0$ $\forall t \in [0,T]$.

Next, we define $\rho(t) = s^{5/2-1/m}\lambda^{5/2}\mathrm{e}^{-s\widehat{\varphi}_m(t)}(\xi_m^*(t))^{5/2-1/m}$ $\forall t \in [0,T]$, and consider the equation satisfied by $\zeta_\rho := \rho\zeta$,

$$\begin{cases} (\zeta_\rho)_t + (\zeta_\rho)_{xxxx} + \gamma(\zeta_\rho)_{xx} = \rho\theta_x + \rho F_3 - \rho_t\zeta & \text{in } Q_T, \\ \zeta_\rho = (\zeta_\rho)_x = 0 & \text{in } \Sigma_T, \\ \zeta_\rho(0,\cdot) = 0 & \text{in } (0,1). \end{cases}$$

Then, using the regularity result for KS equations given by Proposition C.3, we have

$$\|\zeta_\rho\|^2_{L^2(0,T;H^4(0,1))} \leq Cs^{5-2/m}\lambda^5\Big(\|\mathrm{e}^{-s\widehat{\varphi}_m}(\xi_m^*)^{5/2-1/m}\theta_x\|^2_{L^2(Q_T)}$$
$$+ \|\mathrm{e}^{-s\widehat{\varphi}_m}(\xi_m^*)^{5/2-1/m}F_3\|^2_{L^2(Q_T)} + s^7\lambda^5\|\mathrm{e}^{-s\widehat{\varphi}_m}(\xi_m^*)^{7/2}\zeta\|^2_{L^2(Q_T)}\Big), \quad (3.19)$$

for all $s \geq C(T^m + T^{2m})$.

Also, by a standard interpolation result and Young's inequality, one has

$$s^5\lambda^5\Big(\|e^{-s\widehat{\varphi}_m}(\xi_m^*)^{5/2}\zeta_{xx}(t,0)\|^2_{L^2(0,T)} + \|e^{-s\widehat{\varphi}_m}(\xi_m^*)^{5/2}\zeta_{xx}(t,1)\|^2_{L^2(0,T)}\Big) \\ \leq Cs^7\lambda^7\|e^{-s\widehat{\varphi}_m}(\xi_m^*)^{7/2}\zeta\|^2_{L^2(0,T;H^1(0,1))} + Cs^4\lambda^4\|e^{-s\widehat{\varphi}_m}(\xi_m^*)^{2}\zeta\|^2_{L^2(0,T;H^4(0,1))}. \quad (3.20)$$

So, thanks to the estimates (3.18), (3.19) and (3.20), we have from (3.17),

$$s^7\lambda^8\iint\Big(e^{-2s\varphi_m}\xi_m^7|\zeta_x|^2 + e^{-2s\widehat{\varphi}_m}(\xi_m^*)^7|\zeta|^2\Big) + s^{5-\frac{2}{m}}\lambda^5\|e^{-s\widehat{\varphi}_m}(\xi_m^*)^{\frac{5}{2}-\frac{1}{m}}\zeta\|^2_{L^2(0,T;H^4(0,1))} \\ \leq C\Big[\iint\Big(s^{5-\frac{2}{m}}\lambda^5 e^{-2s\widehat{\varphi}_m}(\xi_m^*)^{5-\frac{2}{m}}|\theta_x|^2 + e^{-2s\varphi_m}|\theta_{xx}|^2\Big) \\ + s^7\lambda^5\iint e^{-2s\widehat{\varphi}_m}(\xi_m^*)^7|\zeta|^2 + \iint\Big(s^{5-\frac{2}{m}}\lambda^5\xi^{5-\frac{2}{m}} + s^2\lambda^2\xi^2\Big)e^{-2s\varphi_m}|F_3|^2 \\ + s^7\lambda^7\iint e^{-2s\widehat{\varphi}_m}(\xi_m^*)^7|\zeta_x|^2 + s^4\lambda^4\|e^{-s\widehat{\varphi}}(\xi_m^*)^2\zeta\|^2_{L^2(0,T;H^4(0,1))} + s^7\lambda^8\iint_{\omega_0}e^{-2s\varphi_m}\xi_m^7|\zeta_x|^2\Big]. \quad (3.21)$$

It is clear that the terms associated to ζ and ζ_x in Q_T can be easily absorbed in terms of the l.h.s. of (3.21). Then, using the facts that $s^{-2/m} \leq C/T^4$ (since $s \geq CT^{2m}$), $\xi_m^{-2/m} \leq CT^4$, we can deduce the required Carleman inequality (3.15). □

Next, for the variable θ, we write the following Carleman inequality (usual for the heat equation).

Lemma 3.9 (Carleman inequality for θ, the case $\alpha = 0$)**.** *Let φ_m and ξ_m be defined as in (3.2) with $m \geq 1$. Then, there exist positive constants $\bar\lambda_4$, $\bar s_4 := \bar\sigma_4(T^{2m} + T^{2m-1})$ with some $\bar\sigma_4 > 0$ and C, such that we have the following estimate for θ (given by (3.9)–(3.11)):*

$$I_H(\theta;9) \leq C\left(s^6\lambda^6\iint e^{-2s\varphi_m}\xi_m^6(|\zeta_x|^2 + |F_4|^2) + s^9\lambda^{10}\iint_{\omega_0}e^{-2s\varphi_m}\xi_m^9|\theta|^2\right), \quad (3.22)$$

for all $\lambda \geq \bar\lambda_4$ and $s \geq \bar s_4$, where $I_H(\cdot,\cdot)$ is defined by (3.7).

Proof of Theorem 3.4. As a first step to obtain the Carleman estimate (3.12), let $m \geq 2$ be a fixed parameter and add all four Carleman estimates (3.13), (3.14), (3.15) and (3.22), which yields

$$I_{KS}(u) + I_H(w;7) + s^7\lambda^8\iint\Big(e^{-2s\varphi_m}\xi_m^7|\zeta_x|^2 + e^{-2s\widehat{\varphi}_m}(\xi_m^*)^7|\zeta|^2\Big) \\ + s^{5-\frac{2}{m}}\lambda^5\|e^{-s\widehat{\varphi}_m}(\xi_m^*)^{\frac{5}{2}-\frac{1}{m}}\zeta\|_{L^2(0,T;H^4(0,1))} + I_H(\theta;9) \\ \leq C\Big[\iint e^{-2s\varphi_m}\left(|w_x|^2 + s^4\lambda^4\xi_m^4|u_x|^2 + s^4\lambda^4\xi_m^4|\theta|^2 + s^5\lambda^5\xi_m^5|\theta_x|^2 + s^6\lambda^6\xi_m^6|\zeta_x|^2\right) \\ + \iint e^{-2s\varphi_m}\left(|F_1|^2 + s^4\lambda^4\xi_m^4|F_2|^2 + s^5\lambda^5\xi_m^5|F_3|^2 + s^6\lambda^6\xi_m^6|F_4|^2\right) \\ + T^{10m}\iint e^{-2s\varphi_m}\xi_m^5|\theta_{xx}|^2 + s^7\lambda^8\iint_{\omega_0}e^{-2s\varphi_m}\xi_m^7|u|^2 + s^7\lambda^8\iint_{\omega_0}e^{-2s\varphi_m}\xi_m^7|w|^2 \\ + s^7\lambda^8\iint_{\omega_0}e^{-2s\varphi_m}\xi_m^7|\zeta_x|^2 + s^9\lambda^{10}\iint_{\omega_0}e^{-2s\varphi_m}\xi_m^9|\theta|^2\Big], \quad (3.23)$$

for all $\lambda \geq \lambda^* := \max\{\bar\lambda_j : 1 \leq j \leq 4\}$ and $s \geq \bar c_1(T^m + T^{2m} + T^{2m-1} + T^{2m-2/5})$ for some $\bar c_1 \geq \max\{\bar\sigma_j : 1 \leq j \leq 4\}$.

Step 1: Absorbing the lower order integrals. In this step, we absorb the lower order integrals appearing in the r.h.s. of (3.23). Using that $1 \leq CT^{2m}\xi_m$, we can deduce the following

$$\begin{aligned}
&\iint e^{-2s\varphi_m}\left(|w_x|^2 + s^4\lambda^4\xi_m^4|u_x|^2 + s^4\lambda^4\xi_m^4|\theta|^2 + s^5\lambda^5\xi_m^5|\theta_x|^2 + s^6\lambda^6\xi_m^6|\zeta_x|^2\right)\\
&\qquad + T^{10m}\iint e^{-2s\varphi_m}\xi_m^5|\theta_{xx}|^2\\
&\leq C\iint e^{-2s\varphi_m}\left(T^{10m}\xi_m^5|w_x|^2 + s^4\lambda^4T^{2m}\xi_m^5|u_x|^2 + s^4\lambda^4T^{18m}\xi_m^9|\theta|^2\right)\\
&\quad + C\iint e^{-2s\varphi_m}\left(s^5\lambda^5T^{4m}\xi_m^7|\theta_x|^2 + s^6\lambda^6T^{2m}\xi_m^7|\zeta_x|^2 + T^{10m}\xi_m^5|\theta_{xx}|^2\right).
\end{aligned} \tag{3.24}$$

Thus by choosing $\lambda \geq \lambda^*$ and $s \geq \bar c_2T^{2m}$ for some $\bar c_2 > 0$, one can absorb the integrals appearing in the r.h.s. of (3.24) by the associated higher order integrals in the l.h.s. of (3.23). That is, putting together (3.23) and (3.24) gives

$$\begin{aligned}
&I_{KS}(u) + I_H(w;7) + s^7\lambda^8\iint\left(e^{-2s\varphi_m}\xi_m^7|\zeta_x|^2 + e^{-2s\widehat\varphi_m}(\xi_m^*)^7|\zeta|^2\right)\\
&\quad + s^{5-\frac{2}{m}}\lambda^5\|e^{-s\widehat\varphi_m}(\xi_m^*)^{\frac52-\frac1m}\zeta\|_{L^2(0,T;H^4(0,1))} + I_H(\theta;9)\\
&\quad\leq C\Bigg[s^7\lambda^8\iint_{\omega_0} e^{-2s\varphi_m}\xi_m^7|u|^2 + s^7\lambda^8\iint_{\omega_0} e^{-2s\varphi_m}\xi_m^7|w|^2\\
&\qquad + s^7\lambda^8\iint_{\omega_0} e^{-2s\varphi_m}\xi_m^7|\zeta_x|^2 + s^9\lambda^{10}\iint_{\omega_0} e^{-2s\varphi_m}\xi_m^9|\theta|^2\\
&\qquad + \iint e^{-2s\varphi_m}\left(|F_1|^2 + s^4\lambda^4\xi_m^4|F_2|^2 + s^5\lambda^5\xi_m^5|F_3|^2 + s^6\lambda^6\xi_m^6|F_4|^2\right)\Bigg],
\end{aligned} \tag{3.25}$$

for any $\lambda \geq \lambda^*$ and $s \geq \bar c_2T^{2m}$.

Step 2: Absorbing the observation integral associated to ζ_x. Using the equation $\zeta_x = \theta_t - \theta_{xx} + \beta\theta_x - F_4$, and then following the techniques developed in [24], Step 4–Proposition 3.6, we can eliminate the observation integral of ζ_x in the right-hand side of (3.25). The resulting estimate can be then written as

$$\begin{aligned}
&I_{KS}(u) + I_H(w;7) + s^7\lambda^8\iint\left(e^{-2s\varphi_m}\xi_m^7|\zeta_x|^2 + e^{-2s\widehat\varphi_m}(\xi_m^*)^7|\zeta|^2\right) + I_H(\theta;9)\\
&\quad\leq C\Bigg[s^7\lambda^8\iint_{\omega_0} e^{-2s\varphi_m}\xi_m^7|u|^2 + s^7\lambda^8\iint_{\omega_0} e^{-2s\varphi_m}\xi_m^7|w|^2 + s^{23}\lambda^{16}\iint_{\omega_0} e^{-6s\varphi_m+4s\widehat\varphi_m}\xi_m^{23}|\theta|^2\\
&\qquad + \iint e^{-2s\varphi_m}\left(|F_1|^2 + s^4\lambda^4\xi_m^4|F_2|^2 + s^5\lambda^5\xi_m^5|F_3|^2 + s^7\lambda^8\xi_m^7|F_4|^2\right)\Bigg],
\end{aligned} \tag{3.26}$$

for any $\lambda \geq \lambda^*$ and $s \geq \bar c_3(T^m + T^{2m} + T^{2m-1} + T^{2m-2/5})$, for some $\bar c_3 > 0$.

Step 3: Absorbing the observation integral associated to θ. We choose a smooth function $\phi \in \mathcal{C}_c^\infty(\omega_0)$ such that $0 \leq \phi \leq 1$ in ω_0 and $\phi = 1$ in $\widehat\omega \subset\subset \omega_0$ for some nonempty open set $\widehat\omega$ and without loss of generality, let us consider Carleman estimate (3.26) with observation domain $\widehat\omega$.

Then, we focus on the term

$$J := s^{23}\lambda^{16}\iint_{\widehat{\omega}} \mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}|\theta|^2 \leq s^{23}\lambda^{16}\iint_{\omega_0}\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}|\theta|^2. \tag{3.27}$$

From the second equation in (3.9), we have

$$\theta = -w_t - w_{xx} - \beta w_x + u_x - F_2 \quad \text{in } \mathcal{O} \text{ (consequently in } \omega_0).$$

Using this, we see

$$\begin{aligned}
&s^{23}\lambda^{16}\iint_{\omega_0}\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}|\theta|^2\\
=&s^{23}\lambda^{16}\iint_{\omega_0}\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}\theta(u_x - w_t - w_{xx} - \beta w_x - F_2)\\
=:&\sum_{1\leq i\leq 5} J_i.
\end{aligned}$$

– *Estimate for* J_1. We first look into the term J_1.

$$\begin{aligned}
J_1 &= s^{23}\lambda^{16}\iint_{\omega_0}\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}\theta u_x\\
&= -s^{23}\lambda^{16}\iint_{\omega_0}(\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23})_x\theta u - s^{23}\lambda^{16}\iint_{\omega_0}\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}\theta_x u.
\end{aligned}$$

Now, using (3.3), we observe that for any $n \in \mathbb{N}^*$ and $b > 0$, the n-th derivative with respect to x can be estimated as follows:

$$\left|\partial_x^n\left(\phi\mathrm{e}^{-2s\varphi_m}\xi_m^b\right)\right| \leq Cs^n\lambda^n\,\mathrm{e}^{-2s\varphi_m}\xi_m^{b+n}. \tag{3.28}$$

In particular, we can deduce that

$$|\partial_x^n(\phi\mathrm{e}^{-6s\varphi_m}\xi_m^{23})| \leq Cs^n\lambda^n\mathrm{e}^{-6s\varphi_m}\xi_m^{23+n} \quad \text{for } n \in \mathbb{N}^*. \tag{3.29}$$

Then, applying the Young's inequality with any $\epsilon > 0$, we obtain

$$\begin{aligned}
|J_1| &\leq Cs^{24}\lambda^{17}\iint_{\omega_0}\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{24}|\theta u| + s^{23}\lambda^{16}\iint_{\omega_0}\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}|\theta_x u|\\
&\leq \epsilon s^9\lambda^{10}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^9|\theta|^2 + \epsilon s^7\lambda^8\iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\theta_x|^2 + \frac{C}{\epsilon}s^{39}\lambda^{24}\iint_{\omega_0}\mathrm{e}^{-10s\varphi_m+8s\widehat{\varphi}_m}\xi_m^{39}|u|^2.
\end{aligned} \tag{3.30}$$

– *Estimate for* J_2. Next, we look into the term J_2,

$$\begin{aligned}
J_2 &= -s^{23}\lambda^{16}\iint_{\omega_0}\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}\theta w_t\\
&= s^{23}\lambda^{16}\iint_{\omega_0}\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}\theta_t w + s^{23}\lambda^{16}\iint_{\omega_0}\phi(\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23})_t\theta w,
\end{aligned}$$

since $\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}$ is 0 at $t=0$ and $t=T$, due to the result in Lemma B.1.

Now, recall estimate (3.5), so that one has

$$\left|(\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23})_t\right| \leq CT^{2m-2}s\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{25}.$$

Thanks to this and for any $\epsilon>0$, we have (again, by using the Young's inequality) that

$$|J_2| \leq \epsilon s^5\lambda^6 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^5|\theta_t|^2 + \epsilon s^9\lambda^{10}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^9|\theta|^2 + \frac{C}{\epsilon}s^{41}\lambda^{26}\iint \mathrm{e}^{-10s\varphi_m+8s\widehat{\varphi}_m}\xi_m^{41}|w|^2. \tag{3.31}$$

– *Estimate for* J_3. Let us now focus on J_3; upon consecutive integration by parts with respect to x, one can deduce that

$$\begin{aligned}|J_3| \leq s^{23}\lambda^{16}\iint_{\omega_0}\left|\left(\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}\right)_{xx}\theta w\right| + 2s^{23}\lambda^{16}\iint_{\omega_0}\left|\left(\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}\right)_x\theta_x w\right| \\ + s^{23}\lambda^{16}\iint_{\omega_0}\left|\phi\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}\theta_{xx}w\right| \\ \leq Cs^{25}\lambda^{18}\iint_{\omega_0}\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{25}|\theta w| + Cs^{24}\lambda^{17}\iint_{\omega_0}\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{24}|\theta_x w| \\ + Cs^{23}\lambda^{16}\iint_{\omega_0}\mathrm{e}^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}|\theta_{xx}w|,\end{aligned} \tag{3.32}$$

due to (3.29).

Now, for any $\epsilon>0$, applying Young's inequality, we have from (3.32),

$$\begin{aligned}|J_3| \leq \epsilon s^9\lambda^{10}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^9|\theta|^2 + \epsilon s^7\lambda^8\iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\theta_x|^2 + \epsilon s^5\lambda^6\iint \mathrm{e}^{-2s\varphi_m}\xi_m^5|\theta_{xx}|^2 \\ + \frac{C}{\epsilon}s^{41}\lambda^{26}\iint_{\omega_0}\phi\mathrm{e}^{-10s\varphi_m+8s\widehat{\varphi}_m}\xi_m^{41}|w|^2.\end{aligned} \tag{3.33}$$

– *Estimate for* J_4. Analogous to the estimate of J_1 in (3.30), the term J_4 satisfies

$$\begin{aligned}|J_4| \leq \epsilon s^9\lambda^{10}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^9|\theta|^2 + \epsilon s^7\lambda^8\iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\theta_x|^2 \\ + \frac{C}{\epsilon}s^{39}\lambda^{24}\iint_{\omega_0}\mathrm{e}^{-10s\varphi_m+8s\widehat{\varphi}_m}\xi_m^{39}|w|^2.\end{aligned} \tag{3.34}$$

– *Estimate for* J_5. It is esay to observe that

$$|J_5| \leq \epsilon s^9\lambda^{10}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^9|\theta|^2 + \frac{C}{\epsilon}s^{37}\lambda^{22}\iint \mathrm{e}^{-10s\varphi_m+8s\widehat{\varphi}_m}\xi_m^{37}|F_2|^2, \tag{3.35}$$

for any given $\epsilon>0$.

Finally, gathering the estimates of J_1, J_2, J_3, J_4 and J_5 given by (3.30), (3.31), (3.33), (3.34) and (3.35) respectively, we have

$$s^{23}\lambda^{16}\iint_{\omega_0}\phi e^{-6s\varphi_m+4s\widehat{\varphi}_m}\xi_m^{23}|\theta|^2 \leq C\epsilon s^9\lambda^{10}\iint e^{-2s\varphi_m}\xi_m^9|\theta|^2 + C\epsilon s^7\lambda^8\iint e^{-2s\varphi_m}\xi_m^7|\theta_x|^2 + C\epsilon s^5\lambda^6\iint e^{-2s\varphi_m}\xi_m^5\left(|\theta_t|^2+|\theta_{xx}|^2\right) + \frac{C}{\epsilon}\iint_{\omega_0} e^{-10s\varphi_m+8s\widehat{\varphi}_m}\left(s^{39}\lambda^{24}\xi_m^{39}|u|^2 + s^{41}\lambda^{26}\xi_m^{41}|w|^2\right) + \frac{C}{\epsilon}s^{37}\lambda^{22}\iint e^{-10s\varphi_m+8s\widehat{\varphi}_m}\xi_m^{37}|F_2|^2. \tag{3.36}$$

Fix $\epsilon > 0$ small enough so that, the integrals in Q_T can be absorbed in terms of the l.h.s. in (3.26), and this yields the required Carleman inequality (3.12) for every $\lambda \geq \lambda^*$ and $s \geq \sigma^*(T^m + T^{2m-1} + T^{2m-2/5} + T^{2m})$, for some $\lambda^* > 0$ and $\sigma^* > 0$ chosen largely enough.

Hence, the proof of Theorem 3.4 is finished. □

3.2. Carleman estimate for the case when $\alpha = 1$

The adjoint system (1.11) for the case $\alpha = 1$ reads as

$$\begin{cases} -u_t + u_{xxxx} + \gamma u_{xx} = F_1 - w_x + \zeta \mathbb{1}_{\mathcal{O}} & \text{in } Q_T, \\ -w_t - w_{xx} - \beta w_x = F_2 - u_x & \text{in } Q_T, \\ \zeta_t + \zeta_{xxxx} + \gamma\zeta_{xx} = F_3 + \theta_x & \text{in } Q_T, \\ \theta_t - \theta_{xx} + \beta\theta_x = F_4 + \zeta_x & \text{in } Q_T, \\ u = u_x = w = \zeta = \zeta_x = \theta = 0 & \text{in } \Sigma_T, \\ u(T) = 0, \quad w(T) = 0, & \text{in } (0,1), \\ \zeta(0) = \zeta_0, \quad \theta(0) = \theta_0 & \text{in } (0,1). \end{cases} \tag{3.37}$$

Now, we prove the Carleman inequality for the adjoint states (u, w, ζ, θ) to (3.37) for given data $(\zeta_0, \theta_0) \in H_0^2(0,1) \times (H^2(0,1) \cap H_0^1(0,1))$, $F_1, F_2, F_3 \in L^2(Q_T)$ and $F_4 \in L^2(0,T; H^1(0,1))$. With these in hand, the solution to (3.37) satisfies

$$\begin{cases} u, \zeta \in L^2(0,T; H^4(0,1) \cap H_0^2(0,1)) \cap H^1(0,T; L^2(0,1)), \\ w \in L^2(0,T; H^2(0,1) \cap H_0^1(0,1)) \cap H^1(0,T; L^2(0,1)), \\ \theta \in L^2(0,T; H^3(0,1) \cap H_0^1(0,1)) \cap H^1(0,T; H^1(0,1)). \end{cases} \tag{3.38}$$

Theorem 3.10 (Carleman inequality: the case $\alpha = 1$)**.** *Let φ_m and ξ_m be given by (3.2) with $m > 3$. Then, there exist positive constants λ_0, $s_0 := \sigma_0(T^m + T^{2m} + T^{2m-1} + T^{2m-2/5} + T^{4m/3} + T^{3m/2})$ with some $\sigma_0 > 0$ and C, such that the solution (u, w, ζ, θ), given by (3.38), to (3.37) satisfies:*

$$I_{KS}(u) + I_H(w;3) + I_{KS}(\zeta) + s^3\lambda^4\iint e^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + s\lambda^2\iint e^{-2s\varphi_m}\xi_m|\theta_{xx}|^2 \leq C\iint e^{-2s\varphi_m}\left(s^{71}\lambda^{72}\xi_m^{71}|F_1|^2 + |F_2|^2 + s^3\lambda^4\xi_m^3|F_3|^2\right) + Cs\lambda^2\iint e^{-2s\varphi_m}\xi_m^2\left(|F_4|^2 + |F_{4,x}|^2\right) + Cs^{79}\lambda^{80}\iint_{\omega_0} e^{-2s\varphi_m}\xi_m^{79}|u|^2 + Cs^{73}\lambda^{74}\iint_{\omega_0} e^{-2s\varphi_m}\xi_m^{73}|w|^2, \tag{3.39}$$

for all $\lambda \geq \lambda_0$ and $s \geq s_0$, where $I_H(\cdot;\cdot)$ and $I_{KS}(\cdot)$ are introduced in (3.7) and (3.8) respectively.

To prove the above Carleman inequality with the observations only on u and w, we do the following steps.

(i) First, observe that the usual Carleman estimate for θ will always give an observation integral of θ and there is no chance to absorb it by any leading integrals. In fact, there is a coupling by θ_x to the equation of ζ and therefore, it is relevant to seek for a Carleman estimate associated with the variable θ_x.
In this context, we recall the work [23], where they proved such a Carleman estimate to demonstrate a joint Carleman inequality for the adjoint to the KS system coupled with a heat equation and we shall use it in our present article.
(ii) For the variables u and ζ, we use a Carleman estimate from the work [27] or [23] as mentioned earlier, see Theorem 3.2 in the present paper.
(iii) Finally for w, we make use the standard Carleman inequality for the heat equation, see Theorem 3.1.

Below, we state the individual Carleman estimates for each of u, w, ζ and θ_x.

Lemma 3.11 (Carleman inequality for u, the case $\alpha = 1$)**.** *Let φ_m and ξ_m be as given by (3.2) with $m \geq 2/5$. Then, there exist positive constants λ_1, $s_1 := \sigma_1(T^{2m} + T^{2m-2/5})$ with some $\sigma_1 > 0$ and C, such that the solution component u to (3.37) given by (3.38) satisfies:*

$$I_{KS}(u) \leq C \left(\iint \mathrm{e}^{-2s\varphi_m} \left(|F_1|^2 + |w_x|^2 + |\zeta|^2 \right) + s^7 \lambda^8 \iint_{\omega_0} \mathrm{e}^{-2s\varphi_m} \xi_m^7 |u|^2 \right), \tag{3.40}$$

for all $\lambda \geq \lambda_1$, $s \geq s_1$, where $I_{KS}(\cdot)$ is defined by (3.8).

The next lemma is concerned with a Carleman estimate for w.

Lemma 3.12 (Carleman inequality for w, the case $\alpha = 1$)**.** *Let φ_m and ξ_m be defined as in (3.2) with $m \geq 1$. Then, there exist positive constants λ_2, $s_2 := \sigma_2(T^{2m} + T^{2m-1})$ with some $\sigma_2 > 0$ and C, such that the solution component w to (3.37) given by (3.38) satisfies:*

$$I_H(w; 3) \leq C \left(\iint \mathrm{e}^{-2s\varphi_m} (|F_2|^2 + |u_x|^2) + s^3 \lambda^4 \iint_{\omega_0} \mathrm{e}^{-2s\varphi_m} \xi_m^3 |w|^2 \right), \tag{3.41}$$

for all $\lambda \geq \lambda_2$ and $s \geq s_2$, where $I_H(\cdot;\cdot)$ is defined by (3.7).

We also write a Carleman estimate for ζ which is similar with the one for u.

Lemma 3.13 (Carleman inequality for ζ, the case $\alpha = 1$)**.** *Let φ_m and ξ_m be defined as in (3.2) with $m \geq 2/5$. Then, there exist positive constants λ_3, $s_3 := \sigma_3(T^{2m} + T^{2m-2/5})$ with some $\sigma_3 > 0$ and C, such that we have the following estimate for ζ given by (3.38) (solution component of (3.37)):*

$$I_{KS}(\zeta) \leq C \left(\iint \mathrm{e}^{-2s\varphi_m} (|F_3|^2 + |\theta_x|^2) + s^7 \lambda^8 \iint_{\omega_0} \mathrm{e}^{-2s\varphi_m} \xi_m^7 |\zeta|^2 \right), \tag{3.42}$$

for all $\lambda \geq \lambda_3$ and $s \geq s_3$, where $I_{KS}(\cdot)$ has been defined in (3.8).

Lastly, by following [23], Theorem 3.1, we have a Carleman estimate for θ_x as given below, which can be proved using a result given by [12], Lemma 6.

Lemma 3.14 (Carleman inequality for θ_x, the case $\alpha = 1$)**.** *Let φ_m and ξ_m be defined by (3.2) with $m > 3$, $k > m$. Then, there exist positive constants λ_4, $s_4 := \sigma_4(T^{2m} + T^{2m-1})$ with some $\sigma_4 > 0$ and C, such that we have the following estimate for $\theta_x \in L^2(0,T; H^2(0,1) \cap H_0^1(0,1)) \cap H^1(0,T; L^2(0,1))$,*

$$s^3\lambda^4 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + s\lambda^2 \iint \mathrm{e}^{-2s\varphi_m}\xi_m|\theta_{xx}|^2 \\ \leq Cs\lambda^2 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^2 \left(|F_4|^2 + |F_{4,x}|^2 + |\zeta_x|^2 + |\zeta_{xx}|^2\right) + Cs^3\lambda^4 \iint_{\omega_0} \mathrm{e}^{-2s\varphi_m}\xi_m^3|\theta_x|^2, \quad (3.43)$$

for all $\lambda \geq \lambda_4$, $s \geq s_4$, where θ is the solution component to (3.37) given by (3.38).

Now, we are in the situation to prove our main Carleman inequality, that is Theorem 3.10.

Proof of Theorem 3.10. We divide it into several steps.

Step 1: Absorbing the lower order integrals. Observe the following result:

$$1 \leq T^{2pm}\xi_m^p \quad \forall p \in \mathbb{N}^*.$$

– Using this, the lower order integrals in the r.h.s. of the Carleman inequality (3.40) can be estimated as

$$\iint \mathrm{e}^{-2s\varphi_m}\left(|w_x|^2 + |\zeta|^2\right) \leq T^{2m} \iint \mathrm{e}^{-2s\varphi_m}\xi_m|w_x|^2 + T^{14m}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta|^2, \quad (3.44)$$

whereas, the source integral in the r.h.s. of (3.41) satisfies

$$\iint \mathrm{e}^{-2s\varphi_m}|u_x|^2 \leq T^{10m}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^5|u_x|^2. \quad (3.45)$$

Thus, by choosing any $\lambda \geq \lambda_0 := \max\{\lambda_j : 1 \leq j \leq 4\}$ fixed and $s \geq \sigma_5 T^{2m}$ for some $\sigma_5 > 0$, one can absorb all the integrals appearing in the r.h.s. of (3.44) and (3.45) by the associated leading integrals in the l.h.s. of (3.40) (3.41) and (3.42).

– Next, the source integral in the r.h.s. of (3.42) enjoys

$$\iint \mathrm{e}^{-2s\varphi_m}|\theta_x|^2 \leq T^{6m}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\theta_x|^2, \quad (3.46)$$

and the r.h.s. of the Carleman inequality (3.43) (for θ_x) can be estimated as

$$s\lambda^2 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^2\left(|\zeta_x|^2 + |\zeta_{xx}|^2\right) \\ \leq s\lambda^2\left(T^{6m}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_x|^2 + T^{2m}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2\right). \quad (3.47)$$

Then, by choosing any $\lambda \geq \lambda_0$ and $s \geq \sigma_6(T^m + T^{3m/2})$ for some $\sigma_6 > 0$, the quantity in (3.46) can be absorbed by the 1st leading integral in the l.h.s. of (3.43) and the quantities appearing in the r.h.s. of (3.47), by the associated leading integrals in the l.h.s. of (3.41) and (3.42).

So, after adding the inequalities: (3.40), (3.41), (3.42) and (3.43), and using the above absorption techniques, we obtain the following auxiliary estimate:

$$I_{KS}(u) + I_H(w,3) + I_{KS}(\zeta) + s^3\lambda^4\iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + s\lambda^2\iint \mathrm{e}^{-2s\varphi_m}\xi_m|\theta_{xx}|^2 \\ \leq Cs^7\lambda^8\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^7|u|^2 + Cs^3\lambda^4\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^3|w|^2 + Cs^7\lambda^8\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta|^2$$

$$+ Cs^3\lambda^4 \iint_{\omega_0} e^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + C\iint e^{-2s\varphi_m}(|F_1|^2+|F_2|^2+|F_3|^2) + Cs\lambda^2\iint e^{-2s\varphi_m}\xi_m^2(|F_4|^2+|F_{4,x}|^2), \quad (3.48)$$

for all $\lambda \geq \lambda_0$ and $s \geq \sigma_0(T^m + T^{2m} + T^{2m-1} + T^{2m-2/5} + T^{3m/2} + T^{4m/3})$ where $\sigma_0 := \max\{\sigma_j\,;\, 1 \leq j \leq 6\}$.

Now, our duty is to absorb the observation integrals associated with θ_x and ζ by some leading integrals in the l.h.s. of (3.48).

Step 2: Absorbing the observation integral associated to θ_x. In the sequel, we choose a nonempty set $\widehat{\omega}_2 \subset\subset \widehat{\omega}_1 \subset\subset \omega_0$ and a function

$$\phi \in \mathcal{C}_c^\infty(\widehat{\omega}_1) \ \text{ with } \ 0 \leq \phi \leq 1 \text{ in } \widehat{\omega}_1, \quad \phi = 1 \text{ in } \widehat{\omega}_2,$$

and we consider the auxiliary Carleman estimate (3.48) with the observation domain $\widehat{\omega}_2$ instead of ω_0.

Our goal is to eliminate the observation integral of θ_x. Using the equation of ζ from (3.37), we have

$$\begin{aligned} & s^3\lambda^4\iint_{\widehat{\omega}_2} e^{-2s\varphi_m}\xi_m^3|\theta_x|^2 \qquad (3.49)\\ &\leq s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3\theta_x(\zeta_t+\zeta_{xxxx}+\gamma\zeta_{xx}-F_3)\\ &= s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3\theta_x(\zeta_t+\zeta_{xx}) + s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3\theta_x(\zeta_{xxxx}+(\gamma-1)\zeta_{xx})\\ &\quad - s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3\theta_x F_3\\ &:= I_1 + I_2 + I_3. \end{aligned}$$

– *Estimate for I_1.* We have

$$\begin{aligned} I_1 &= s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3\theta_x(\zeta_t+\zeta_{xx})\\ &= -s^3\lambda^4\iint_{\widehat{\omega}_1}\phi(e^{-2s\varphi_m}\xi_m^3)_t\theta_x\zeta - s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3(\theta_t)_x\zeta - s^3\lambda^4\iint_{\widehat{\omega}_1}(\phi e^{-2s\varphi_m}\xi_m^3)_x\theta_x\zeta_x\\ &\quad + s^3\lambda^4\iint_{\widehat{\omega}_1}(\phi e^{-2s\varphi_m}\xi_m^3)_x\theta_{xx}\zeta + s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3(\theta_{xx})_x\zeta. \qquad (3.50) \end{aligned}$$

In the above equation, there is no boundary terms at $t = 0, T$ (while integrating by parts in time), since $e^{-2s\varphi_m(t,x)} \to 0$ as $t \to 0^+$ or T^-. We also perfomed two consecutive integration by parts in space as follows

$$\begin{aligned} s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3\theta_x\zeta_{xx} &= -s^3\lambda^4\iint_{\widehat{\omega}_1}(\phi e^{-2s\varphi_m}\xi_m^3)_x\theta_x\zeta_x - s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3\theta_{xx}\zeta_x\\ &= -s^3\lambda^4\iint_{\widehat{\omega}_1}(\phi e^{-2s\varphi_m}\xi_m^3)_x\theta_x\zeta_x + s^3\lambda^4\iint_{\widehat{\omega}_1}(\phi e^{-2s\varphi_m}\xi_m^3)_x\theta_{xx}\zeta + s^3\lambda^4\iint_{\widehat{\omega}_1}\phi e^{-2s\varphi_m}\xi_m^3(\theta_{xx})_x\zeta. \end{aligned}$$

Now, going back to (3.50), and using the equation $\theta_t - \theta_{xx} = -\beta\theta_x + \zeta_x$, we have

$$I_1 = -s^3\lambda^4 \iint_{\widehat{\omega}_1} \phi e^{-2s\varphi_m}\xi_m^3(-\beta\theta_x + \zeta_x)_x\zeta + X_1, \tag{3.51}$$

where

$$X_1 \quad := \quad -s^3\lambda^4 \iint_{\widehat{\omega}_1} \phi(e^{-2s\varphi_m}\xi_m^3)_t\theta_x\zeta \; - \; s^3\lambda^4 \iint_{\widehat{\omega}_1} (\phi e^{-2s\varphi_m}\xi_m^3)_x\theta_x\zeta_x \; + \; s^3\lambda^4 \iint_{\widehat{\omega}_1} (\phi e^{-2s\varphi_m}\xi_m^3)_x\theta_{xx}\zeta.$$

Now, recall (3.6) (for $m > 1$) to write

$$|(e^{-2s\varphi_m}\xi_m^3)_t| \leq CT^{2m-2}s\xi_m^2(e^{-2s\varphi_m}\xi_m^3).$$

We use this fact in the first integral of X_1, and then thanks to the Cauchy–Schwarz inequality, we have for any $\epsilon > 0$, that

$$\begin{aligned}|X_1| \leq \epsilon s^3\lambda^4 \iint e^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + \epsilon s\lambda^2 \iint e^{-2s\varphi_m}\xi_m|\theta_{xx}|^2 \\ + \frac{C}{\epsilon}s^5\lambda^6 \iint_{\widehat{\omega}_1} e^{-2s\varphi_m}\xi_m^5|\zeta_x|^2 + \frac{C}{\epsilon}s^5\lambda^6 \iint_{\widehat{\omega}_1} e^{-2s\varphi_m}\xi_m^5|\zeta|^2.\end{aligned} \tag{3.52}$$

Let us estimate the other integrals of I_1 in (3.51). We have for any $\epsilon > 0$ (again by applying Cauchy–Schwarz inequality)

$$\begin{aligned}s^3\lambda^4 \left| \iint_{\widehat{\omega}_1} \phi e^{-2s\varphi_m}\xi_m^3\Big(-\beta\theta_x + \zeta_x\Big)_x\zeta \right| \leq \epsilon s\lambda^2 \iint e^{-2s\varphi_m}\xi_m|\theta_{xx}|^2 \\ + \epsilon s^3\lambda^4 \iint e^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2 + \frac{C}{\epsilon}s^5\lambda^6 \iint_{\widehat{\omega}_1} e^{-2s\varphi_m}\xi_m^5|\zeta|^2.\end{aligned} \tag{3.53}$$

Therefore, the estimates (3.52) and (3.53) yield

$$\begin{aligned}|I_1| \leq \epsilon s^3\lambda^4 \iint e^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + 2\epsilon s\lambda^2 \iint e^{-2s\varphi_m}\xi_m|\theta_{xx}|^2 + \epsilon s^3\lambda^4 \iint e^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2 \\ + \frac{C}{\epsilon}s^5\lambda^6 \iint_{\widehat{\omega}_1} e^{-2s\varphi_m}\xi_m^5|\zeta_x|^2 + \frac{C}{\epsilon}s^5\lambda^6 \iint_{\widehat{\omega}_1} e^{-2s\varphi_m}\xi_m^5|\zeta|^2.\end{aligned} \tag{3.54}$$

– *Estimate for I_2.* Let us recall the quantity I_2 from (3.49) and we have

$$\begin{aligned}I_2 &= s^3\lambda^4 \iint_{\widehat{\omega}_1} \phi e^{-2s\varphi_m}\xi_m^3\theta_x(\zeta_{xxxx} + (\gamma-1)\zeta_{xx}) \\ &= -s^3\lambda^4 \iint_{\widehat{\omega}_1} (\phi e^{-2s\varphi_m}\xi_m^3)_x\theta_x\zeta_{xxx} - s^3\lambda^4 \iint_{\widehat{\omega}_1} \phi e^{-2s\varphi_m}\xi_m^3\big(\theta_{xx}\zeta_{xxx} - (\gamma-1)\theta_x\zeta_{xx}\big),\end{aligned}$$

where we perform an integration by parts on the term involving fourth order derivative in ζ. It follows that

$$|I_2| \leq \epsilon s^3\lambda^4 \iint_{\widehat{\omega}_1} \mathrm{e}^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + \epsilon s\lambda^2 \iint_{\widehat{\omega}_1} \mathrm{e}^{-2s\varphi_m}\xi_m|\theta_{xx}|^2 + \frac{C}{\epsilon}s^3\lambda^4 \iint_{\widehat{\omega}_1} \mathrm{e}^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2 + \frac{C}{\epsilon}s^5\lambda^6 \iint_{\widehat{\omega}_1} \mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_{xxx}|^2. \quad (3.55)$$

– *Estimate for I_3.* Finally, we have

$$|I_3| \leq \epsilon s^3\lambda^4 \iint_{\widehat{\omega}_1} \mathrm{e}^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + \frac{C}{\epsilon}s^3\lambda^4 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|F_3|^2, \quad (3.56)$$

for any given $\epsilon > 0$.

Now, using the estimates of I_1, I_2 and I_3 given by (3.54), (3.55) and (3.56) respectively, we have from (3.49) that

$$\begin{aligned} s^3\lambda^4 \iint_{\widehat{\omega}_1} \mathrm{e}^{-2s\varphi_m}\xi_m^3|\theta_x|^2 \leq{}& C\epsilon s^3\lambda^4 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + C\epsilon s\lambda^2 \iint \mathrm{e}^{-2s\varphi_m}\xi_m|\theta_{xx}|^2 \\ &+ C\epsilon s^3\lambda^4 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2 + \frac{C}{\epsilon}s^5\lambda^6 \iint_{\widehat{\omega}_1} \mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta|^2 \\ &+ \frac{C}{\epsilon}\iint_{\widehat{\omega}_1}\left(s^5\lambda^6\mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_x|^2 + s^3\lambda^4\mathrm{e}^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2 + s^5\lambda^6\mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_{xxx}|^2\right) \\ &+ \frac{C}{\epsilon}s^3\lambda^4 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|F_3|^2. \end{aligned} \quad (3.57)$$

Fix $\epsilon > 0$ small enough so that we can absorb the first three integrals in the r.h.s. of (3.57) by the associated leading terms in the l.h.s. of (3.48).

Next, one can deduce the following result: for any $\epsilon > 0$, there exists $C > 0$ such that,

$$\begin{aligned} &\iint_{\widehat{\omega}_1}\left(s^5\lambda^6\mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_x|^2 + s^3\lambda^4\mathrm{e}^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2 + s^5\lambda^6\mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_{xxx}|^2\right) \\ &\quad\leq \epsilon\iint \mathrm{e}^{-2s\varphi_m}\left((s\xi_m)^{-1}|\zeta_{xxxx}|^2 + s\lambda^2\xi_m|\zeta_{xxx}|^2 + s^3\lambda^4\xi_m^3|\zeta_{xx}|^2\right) + \frac{C}{\epsilon}s^{39}\lambda^{40}\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^{39}|\zeta|^2, \end{aligned} \quad (3.58)$$

assuming that there is a set $\widehat{\omega}_0$ such that $\widehat{\omega}_1 \subset\subset \widehat{\omega}_0 \subset\subset \omega_0$.

The above proof can be done by performing several integration by parts in space and applying the Cauchy–Schwarz inequalty accordingly. We omit the details here.

Then, for $\epsilon > 0$ small enough, we can absorb the first three integrals in the r.h.s. of (3.58) by the corresponding leading integrals in the l.h.s. of (3.48) and as a consequence, we have

$$\begin{aligned} &I_{KS}(u) + I_H(w,3) + I_{KS}(\zeta) + s^3\lambda^4\iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\theta_x|^2 + s\lambda^2\iint \mathrm{e}^{-2s\varphi_m}\xi_m|\theta_{xx}|^2 \\ &\quad\leq Cs^7\lambda^8\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^7|u|^2 + Cs^3\lambda^4\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^3|w|^2 + Cs^{39}\lambda^{40}\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^{39}|\zeta|^2 \\ &\qquad + C\iint \mathrm{e}^{-2s\varphi_m}(|F_1|^2 + |F_2|^2) + Cs^3\lambda^4\iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|F_3|^2 \\ &\qquad + Cs\lambda^2\iint \mathrm{e}^{-2s\varphi_m}\xi_m^2(|F_4|^2 + |F_{4,x}|^2). \end{aligned} \quad (3.59)$$

Step 3: Absorbing the observation integral associated to ζ. In the previous step, one can assume a couple of nonempty sets $\widehat{\omega}_2 \subset\subset \widehat{\omega}_1 \subset\subset \omega_1 \subset\subset \omega_0$ and prove the auxiliary inequality (3.59) with the observation domain ω_1 instead of ω_0.

Then, choose a function $\phi_1 \in \mathcal{C}_c^\infty(\omega_0)$ with $0 \leq \phi_1 \leq 1$ in ω_0 and $\phi_1 = 1$ in ω_1. Also, one has from the 4×4 adjoint system (3.37) that

$$\zeta = -u_t + u_{xxxx} + \gamma u_{xx} + w_x - F_1 \ \text{ in } \omega_0, \text{ since } \omega_0 \subset\subset \mathcal{O}.$$

Therefore, we observe that

$$\begin{aligned} s^{39}\lambda^{40} \iint_{\omega_1} \mathrm{e}^{-2s\varphi_m}\xi_m^{39}|\zeta|^2 &\leq s^{39}\lambda^{40}\iint_{\omega_0} \phi_1 \mathrm{e}^{-2s\varphi_m}\xi_m^{39}|\zeta|^2 \\ &= s^{39}\lambda^{40}\iint_{\omega_0} \phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\zeta\big(-u_t + u_{xxxx} + \gamma u_{xx} + w_x - F_1\big) := I_4 + I_5 + I_6 + I_7 + I_8. \end{aligned} \tag{3.60}$$

– *Estimate for I_4.* First, we compute

$$\begin{aligned} I_4 := -s^{39}\lambda^{40}\iint_{\omega_0} \phi_1 \mathrm{e}^{-2s\varphi_m}\xi_m^{39}\zeta u_t = s^{39}\lambda^{40}\iint_{\omega_0}\phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\zeta_t u \\ + s^{39}\lambda^{40}\iint_{\omega_0}\phi_1\big(\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\big)_t\zeta u. \end{aligned} \tag{3.61}$$

Let us recall (3.6) (for $m > 1$), to write

$$|(\mathrm{e}^{-2s\varphi_m}\xi_m^{39})_t| \leq CT^{2m-2}s\mathrm{e}^{-2s\varphi_m}\xi_m^{41}.$$

Using the above fact and Cauchy–Schwarz inequality, we have for any $\epsilon > 0$ that

$$|I_4| \leq \epsilon s^{-1}\iint \mathrm{e}^{-2s\varphi_m}\xi_m^{-1}|\zeta_t|^2 + \epsilon s^7\lambda^8\iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta|^2 + \frac{C}{\epsilon}s^{79}\lambda^{80}\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^{79}|u|^2. \tag{3.62}$$

– *Estimate for I_5.* Next, by performing a successive number of integration by parts on I_5 w.r.t. x, we get

$$\begin{aligned} |I_5| := \left|s^{39}\lambda^{40}\iint_{\omega_0}\phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\zeta u_{xxxx}\right| \leq C\Big(s^{39}\lambda^{40}\iint_{\omega_0}\big(\phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\big)_{xxxx}\zeta u \\ + s^{39}\lambda^{40}\iint_{\omega_0}\big(\phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\big)_{xxx}\zeta_x u + s^{39}\lambda^{40}\iint_{\omega_0}\big(\phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\big)_{xx}\zeta_{xx}u \\ + s^{39}\lambda^{40}\iint_{\omega_0}\big(\phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\big)_x\zeta_{xxx}u + s^{39}\lambda^{40}\iint_{\omega_0}\phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\zeta_{xxxx}u\Big). \end{aligned} \tag{3.63}$$

Let us recall the result (3.28), so that we have the following:

$$\big|\partial_x^n\big(\phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\big)\big| \leq Cs^n\lambda^n\,\mathrm{e}^{-2s\varphi_m}\xi_m^{39+n} \quad \text{for } n \in \mathbb{N}^*,$$

and the estimate (3.63) follows

$$\begin{aligned} |I_5| \leq C\Big(s^{43}\lambda^{44}\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^{43}\zeta u + s^{42}\lambda^{43}\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^{42}\zeta_x u + s^{41}\lambda^{42}\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\zeta_{xx}u \\ + s^{40}\lambda^{41}\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^{40}\zeta_{xxx}u + s^{39}\lambda^{40}\iint_{\omega_0}\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\zeta_{xxxx}u\Big). \end{aligned} \tag{3.64}$$

Then, for any $\epsilon > 0$, we obtain by using Cauchy–Schwarz inequality that

$$|I_5| \leq \epsilon\bigg(s^7\lambda^8 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta|^2 + s^5\lambda^6 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_x|^2 + s^3\lambda^4 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2 + s\lambda^2 \iint \mathrm{e}^{-2s\varphi_m}\xi_m|\zeta_{xxx}|^2 + s^{-1} \iint \mathrm{e}^{-2s\varphi_m}\xi_m^{-1}(|\zeta_t|^2 + |\zeta_{xxxx}|^2)\bigg) + \frac{C}{\epsilon}s^{79}\lambda^{80} \iint_{\omega_0} \mathrm{e}^{-2s\varphi_m}\xi_m^{79}|u|^2. \quad (3.65)$$

– *Estimate for I_6.* In a similar manner, one can obtain an estimate for I_6, given by

$$|I_6| \leq \epsilon\bigg(s^7\lambda^8 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta|^2 + s^5\lambda^6 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_x|^2 + s^3\lambda^4 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2\bigg) + \frac{C}{\epsilon}s^{75}\lambda^{76} \iint_{\omega_0} \mathrm{e}^{-2s\varphi_m}\xi_m^{75}|u|^2. \quad (3.66)$$

– *Estimate for I_7.* Let us focus on the term I_7, we have

$$\begin{aligned}|I_7| &\leq \left|-s^{39}\lambda^{40} \iint_{\omega_0} (\phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39})_x\zeta w - s^{39}\lambda^{40} \iint_{\omega_0} \phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\zeta_x w\right| \quad (3.67)\\ &\leq \epsilon\left(s^7\lambda^8 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta|^2 + s^5\lambda^6 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_x|^2\right) + \frac{C}{\epsilon}s^{73}\lambda^{74} \iint_{\omega_0} \mathrm{e}^{-2s\varphi_m}\xi_m^{73}|w|^2.\end{aligned}$$

– *Estimate for I_8.* Finally, we see

$$\begin{aligned}|I_8| &= \left|s^{39}\lambda^{40} \iint_{\omega_0} \phi_1\mathrm{e}^{-2s\varphi_m}\xi_m^{39}\zeta F_1\right| \quad (3.68)\\ &\leq \epsilon s^7\lambda^8 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta|^2 + \frac{C}{\epsilon}s^{71}\lambda^{72} \iint \mathrm{e}^{-2s\varphi_m}\xi_m^{71}|F_1|^2,\end{aligned}$$

for any $\epsilon > 0$.

Thus, using the estimates (3.62), (3.65), (3.66), (3.67) and (3.68) in (3.60), we have

$$s^{39}\lambda^{40} \iint_{\omega_1} \mathrm{e}^{-2s\varphi_m}\xi_m^{39}|\zeta|^2 \leq C\epsilon\bigg(s^7\lambda^8 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^7|\zeta|^2 + s^5\lambda^6 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^5|\zeta_x|^2 + s^3\lambda^4 \iint \mathrm{e}^{-2s\varphi_m}\xi_m^3|\zeta_{xx}|^2 + s\lambda^2 \iint \mathrm{e}^{-2s\varphi_m}\xi_m|\zeta_{xxx}|^2 + s^{-1} \iint \mathrm{e}^{-2s\varphi_m}\xi_m^{-1}(|\zeta_t|^2 + |\zeta_{xxxx}|^2)\bigg) + \frac{C}{\epsilon}s^{79}\lambda^{80} \iint_{\omega_0} \mathrm{e}^{-2s\varphi_m}\xi_m^{79}|u|^2 + \frac{C}{\epsilon}s^{73}\lambda^{74} \iint_{\omega_0} \mathrm{e}^{-2s\varphi_m}\xi_m^{73}|w|^2 + \frac{C}{\epsilon}s^{71}\lambda^{72} \iint \mathrm{e}^{-2s\varphi_m}\xi_m^{71}|F_1|^2. \quad (3.69)$$

Now, fix $\epsilon > 0$ in (3.69) small enough so that all the integrals in Q_T can be absorbed by the quantity $I_{KS}(\zeta)$ in (3.59), and this yields the required Carleman inequality (3.39). □

3.3. Carleman estimate for the case when $\alpha \in (0,1)$

We just state the Carleman estimate for the case $\alpha \in (0,1)$. In this case, the proof is simpler than the previous two cases since we just need to use the standard Carleman estimates of the fourth and second order parabolic equations.

Recall the adjoint system (1.11) for any $\alpha \in (0,1)$. Then, one can obtain the following Carleman inequality.

Theorem 3.15 (Carleman inequality: the case $\alpha \in (0,1)$)**.** *Let the weight functions (φ_m, ξ_m) be given by (3.2) with $m \geq 1$. Then, there exist positive constants $\widehat{\lambda}$, $\widehat{s} := \widehat{\sigma}(T^m + T^{2m-2/5} + T^{2m-1} + T^{2m})$ with some $\widehat{\sigma} > 0$ and C such that we have the following estimate satisfied by the solution to (1.11) with given data $(\zeta_0, \theta_0) \in H^2_0(0,1) \times H^1_0(0,1)$ and $F_j \in L^2(Q_T)$ for $j = 1,2,3,4$,*

$$\begin{aligned} &I_{KS}(u) + I_H(w;3) + I_{KS}(\zeta) + I_H(\theta;3) \\ &\qquad \leq C \iint e^{-2s\varphi_m} \left(s^7\lambda^8\xi_m^7|F_1|^2 + s^3\lambda^4\xi_m^3|F_2|^2 + |F_3|^2 + |F_4|^2 \right) \\ &\qquad\qquad + Cs^{15}\lambda^{16} \iint_{\omega_0} e^{-2s\varphi_m}\xi_m^{15}|u|^2 + Cs^9\lambda^{10} \iint_{\omega_0} e^{-2s\varphi_m}\xi_m^9|w|^2, \end{aligned} \tag{3.70}$$

for all $\lambda \geq \widehat{\lambda}$ and $s \geq \widehat{s}$, where $I_H(\cdot;\cdot)$ and $I_{KS}(\cdot)$ are given by (3.7) and (3.8) respectively.

As stated before, the technique for proving Theorem 3.15 is the following.

(i) For the variables u and ζ, satisfying fourth order parabolic equations, we use the Carleman estimate given by Theorem 3.2.
(ii) For w and θ, we use the classical Carleman inequalities for the heat equations, see Theorem 3.1.

A sketch of the proof for Theorem 3.15 is available in the arXiv v1 of this work, see [28], Section 2.1.

4. Null-controllability of the studied systems for different α

In this section, we establish the local null-controllability of our extended system (1.5)–(1.6) for different values of $\alpha \in [0,1]$. But as mentioned earlier, the most interesting cases are $\alpha = 0$ and $\alpha = 1$, so we mainly discuss the controllability for these two cases.

The main ingredient to prove Theorem 1.3 is to obtain a suitable observability inequality for system (1.11) which will ensure the null-controllability for the extended linearized systems (1.9)–(1.10). Due to the presence of the parameter α in the system, we have used different strategies leading to different Carleman estimates, see Theorems 3.4, 3.10 and 3.15.

In our present work, we mainly give a detailed proof of the observability inequality derived from Theorem 3.4. The case $\alpha = 1$ will be then shortly described in Section 4.2; more precisely, there we point out the main changes in the proof of observability inequality and the null-controllablity in comparison with the case $\alpha = 0$. After the study of linear cases, we use the so-called *inverse mapping theorem* to handle the local null-controllability of the nonlinear systems (1.5)–(1.6).

Remark 4.1. The required proofs for the more standard case $\alpha \in (0,1)$ can be done in a similar fashion following Section 4.1 below and making minor and straightforward adjustments. For the sake of brevity, we omit any further comments about this case.

4.1. The case when $\alpha = 0$

4.1.1. Observability inequality ($\alpha = 0$)

To do this, we shall first prove a refined Carleman inequality with weight functions that do not vanish at $t = T$. More precisely, let us consider

$$\ell(t) = \begin{cases} t(T-t), & 0 \le t \le T/2, \\ T^2/4, & T/2 \le t \le T, \end{cases} \tag{4.1}$$

and the following associated weight functions

$$\mathfrak{S}_m(t,x) = \frac{\mathrm{e}^{\lambda(1+\frac{1}{m})k\|\nu\|_\infty} - \mathrm{e}^{\lambda\left(k\|\nu\|_\infty + \nu(x)\right)}}{\ell(t)^m}, \quad \mathfrak{Z}_m(t,x) = \frac{\mathrm{e}^{\lambda\left(k\|\nu\|_\infty+\nu(x)\right)}}{\ell(t)^m}, \quad \forall (t,x) \in Q_T, \tag{4.2}$$

for any constants $\lambda > 1$ and $k > m > 0$. Additionally, we define

$$\widehat{\mathfrak{S}}_m(t) = \max_{x\in[0,1]} \mathfrak{S}_m(t,x), \quad \widehat{\mathfrak{Z}}_m(t) = \max_{x\in[0,1]} \mathfrak{Z}_m(t,x), \tag{4.3}$$

$$\mathfrak{S}^*_m(t) = \min_{x\in[0,1]} \mathfrak{S}_m(t,x), \quad \mathfrak{Z}^*_m(t) = \min_{x\in[0,1]} \mathfrak{Z}_m(t,x). \tag{4.4}$$

We have the following.

Proposition 4.2 (A refined Carleman estimate: the case $\alpha = 0$)**.** *Let m, k, s and λ be fixed constants according to Theorem 3.4. Then, there exists a positive constant C depending at most on $\omega, \mathcal{O}$, T, m, k, s, and λ such that*

$$\begin{aligned} &\iint \left(\mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^7|u|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^7|w|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^7|\zeta|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^9|\theta|^2\right) \\ &\quad + \|\zeta(T)\|^2_{L^2(0,1)} + \|\theta(T)\|^2_{L^2(0,1)} \le C\left[\iint \mathrm{e}^{-2s\mathfrak{S}^*_m}|F_1|^2 + \iint \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{37}_m|F_2|^2 \right. \\ &\quad \left. + \iint \mathrm{e}^{-2s\mathfrak{S}^*_m}\left(\widehat{\mathfrak{Z}}^5_m|F_3|^2 + \widehat{\mathfrak{Z}}^7_m|F_4|^2\right) + \iint_{\omega_0} \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{39}_m|u|^2 + \iint_{\omega_0} \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{41}_m|w|^2\right], \end{aligned} \tag{4.5}$$

where (u, w, ζ, θ) is the solution associated to (3.9) for any given $(\zeta_0, \theta_0) \in [L^2(0,1)]^2$ and $F_j \in L^2(Q_T)$ for $j = 1, 2, 3, 4$.

Proof. Let us first fix the Carleman parameters $s = s^*$ and $\lambda = \lambda^*$ in the estimate (3.12) given by Theorem 3.4.

By construction $\varphi_m = \mathfrak{S}_m$ and $\xi_m = \mathfrak{Z}_m$ in $(0, T/2) \times (0,1)$, hence

$$\begin{aligned} &\int_0^{\frac{T}{2}}\int_0^1 \left(\mathrm{e}^{-2s\mathfrak{S}_m}\mathfrak{Z}^7_m|u|^2 + \mathrm{e}^{-2s\mathfrak{S}_m}\mathfrak{Z}^7_m|w|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*)^7_m|\zeta|^2 + \mathrm{e}^{-2s\mathfrak{S}_m}\mathfrak{Z}^9_m|\theta|^2\right) \\ &= \int_0^{\frac{T}{2}}\int_0^1 \left(\mathrm{e}^{-2s\varphi_m}\xi^7_m|u|^2 + \mathrm{e}^{-2s\varphi_m}\xi^7_m|w|^2 + \mathrm{e}^{-2s\widehat{\varphi}_m}(\xi^*)^7_m|\zeta|^2 + \mathrm{e}^{-2s\varphi_m}\xi^9_m|\theta|^2\right). \end{aligned}$$

Therefore, using the Carleman estimate (3.12) and the definitions (4.3)–(4.4), we readily get

$$\int_0^{\frac{T}{2}}\int_0^1 \left(\mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*)_m^7|u|^2+\mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*)_m^7|w|^2+\mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*)_m^7|\zeta|^2+\mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*)_m^9|\theta|^2\right)$$
$$\leq C\Big[\iint \mathrm{e}^{-2s\mathfrak{S}_m^*}|F_1|^2+\iint \mathrm{e}^{-10s\mathfrak{S}_m^*+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}_m^{37}|F_2|^2+\iint \mathrm{e}^{-2s\mathfrak{S}_m^*}\left(\widehat{\mathfrak{Z}}_m^5|F_3|^2+\widehat{\mathfrak{Z}}_m^7|F_4|^2\right)$$
$$+\iint_{\omega_0}\mathrm{e}^{-10s\mathfrak{S}_m^*+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}_m^{39}|u|^2+\iint_{\omega_0}\mathrm{e}^{-10s\mathfrak{S}_m^*+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}_m^{41}|w|^2\Big]. \quad (4.6)$$

For the domain $(T/2,T)\times(0,1)$, we argue as follows. Let us introduce a function $\eta\in\mathcal{C}^1([0,T])$ such that

$$\eta=0 \text{ in } [0,T/4], \quad \eta=1 \text{ in } [T/2,T], \quad |\eta'|\leq C/T. \quad (4.7)$$

Using Proposition C.1, we apply the energy estimate to the equation verified by $(\eta\zeta,\eta\theta)$ with $((\eta\zeta)(0),(\eta\theta)(0))=(0,0)$, so that one can deduce that

$$\begin{aligned}&\|\eta\zeta\|^2_{L^\infty(T/4,T;L^2(0,1))}+\|\eta\theta\|^2_{L^\infty(T/4,T;L^2(0,1))}\\ &\leq C\Big(\|\eta F_3\|^2_{L^2((T/4,T)\times(0,1))}+\|\eta F_4\|^2_{L^2((T/4,T)\times(0,1))}\\ &\qquad+\frac{1}{T^2}\|\zeta\|^2_{L^2((T/4,T/2)\times(0,1))}+\frac{1}{T^2}\|\theta\|^2_{L^2((T/4,T/2)\times(0,1))}\Big).\end{aligned} \quad (4.8)$$

Analogously, for the equation verified by $(\eta u,\eta w)$, we have

$$\begin{aligned}&\|\eta u\|^2_{L^\infty(T/4,T;L^2(0,1))}+\|\eta w\|^2_{L^\infty(T/4,T;L^2(0,1))}\\ &\leq C\Big(\|\eta F_1\|^2_{L^2((T/4,T)\times(0,1))}+\|\eta F_2\|^2_{L^2((T/4,T)\times(0,1))}\\ &\qquad+\|\eta\theta\|^2_{L^2((T/4,T)\times\mathcal{O})}+\frac{1}{T^2}\|u\|^2_{L^2((T/4,T/2)\times(0,1))}+\frac{1}{T^2}\|w\|^2_{L^2((T/4,T/2)\times(0,1))}\Big).\end{aligned} \quad (4.9)$$

Using the estimate of $\eta\theta$ from (4.8) in the right hand side of (4.9) and then combining both (4.8)–(4.9) we obtain

$$\begin{aligned}&\|\eta u\|^2_{L^\infty(T/4,T;L^2(0,1))}+\|\eta w\|^2_{L^\infty(T/4,T;L^2(0,1))}+\|\eta\zeta\|^2_{L^\infty(T/4,T;L^2(0,1))}+\|\eta\theta\|^2_{L^\infty(T/4,T;L^2(0,1))}\\ &\leq C\Big[\sum_{j=1}^4\|\eta F_j\|^2_{L^2((T/4,T)\times(0,1))}+\int_{\frac{T}{4}}^{\frac{T}{2}}\int_0^1\left(|u|^2+|w|^2+|\zeta|^2+|\theta|^2\right)\Big].\end{aligned} \quad (4.10)$$

Now, we need to find proper estimates of the states u,w,ζ,θ in the r.h.s. of (4.10). Since the weight functions $\mathfrak{S}_m$ and $\mathfrak{Z}_m$ are bounded by below in $[T/4,T]$, we estimate

$$\begin{aligned}&\int_{\frac{T}{4}}^{\frac{T}{2}}\int_0^1\left(|u|^2+|w|^2+|\zeta|^2+|\theta|^2\right)\\ &\leq\int_{\frac{T}{4}}^{\frac{T}{2}}\int_0^1\left(\mathrm{e}^{-2s\mathfrak{S}_m}\mathfrak{Z}_m^7|u|^2+\mathrm{e}^{-2s\mathfrak{S}_m}\mathfrak{Z}_m^7|w|^2+\mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*)_m^7|\zeta|^2+\mathrm{e}^{-2s\mathfrak{S}_m}\mathfrak{Z}_m^9|\theta|^2\right)\\ &\leq C\Big[\iint \mathrm{e}^{-2s\mathfrak{S}_m}|F_1|^2+\iint \mathrm{e}^{-10s\mathfrak{S}_m+8s\widehat{\mathfrak{S}}_m}\mathfrak{Z}_m^{37}|F_2|^2+\iint \mathrm{e}^{-2s\mathfrak{S}_m}\left(\mathfrak{Z}_m^5|F_3|^2+\mathfrak{Z}_m^7|F_4|^2\right)\\ &\quad+\iint_{\omega_0}\mathrm{e}^{-10s\mathfrak{S}_m+8s\widehat{\mathfrak{S}}_m}\mathfrak{Z}_m^{39}|u|^2+\iint_{\omega_0}\mathrm{e}^{-10s\mathfrak{S}_m+8s\widehat{\mathfrak{S}}_m}\mathfrak{Z}_m^{41}|w|^2\Big],\end{aligned} \quad (4.11)$$

due to the Carleman estimate (3.12).

We also can incorporate the weight functions in the left-hand side of (4.10) in the interval $[T/2, T]$, which yields together with (4.11) and from the definitions (4.3)–(4.4), that

$$\begin{aligned}
&\int_{\frac{T}{2}}^{T}\int_0^1 \Big(\mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{z}^*)_m^7|u|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{z}^*)_m^7|w|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{z}^*)_m^7|\zeta|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{z}^*)_m^9|\theta|^2\Big)\\
&\quad + \|\eta u\|^2_{L^\infty(T/4,T;L^2(0,1))} + \|\eta w\|^2_{L^\infty(T/4,T;L^2(0,1))} + \|\eta \zeta\|^2_{L^\infty(T/4,T;L^2(0,1))} + \|\eta \theta\|^2_{L^\infty(T/4,T;L^2(0,1))}\\
&\quad\quad \le C\Big[\iint \mathrm{e}^{-2s\mathfrak{S}^*_m}|F_1|^2 + \iint \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{z}}_m^{37}|F_2|^2 + \iint \mathrm{e}^{-2s\mathfrak{S}^*_m}\big(\widehat{\mathfrak{z}}_m^5|F_3|^2 + \widehat{\mathfrak{z}}_m^7|F_4|^2\big)\\
&\quad\quad\quad + \iint_{\omega_0} \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{z}}_m^{39}|u|^2 + \iint_{\omega_0} \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{z}}_m^{41}|w|^2\Big] \qquad (4.12)
\end{aligned}$$

for some $C > 0$.

Combining (4.6) and (4.12), we have the desired result (4.5). □

As a consequence of Proposition 4.2, we have the following result.

Proposition 4.3 (Observability inequality: the case $\alpha = 0$)**.** *Let m, k, s and λ be fixed constants according to Theorem 3.4. Then, there exists a positive constant C depending at most on $\omega, \mathcal{O}$, T, m, k, s, and λ such that for any given $(\zeta_0, \theta_0) \in [L^2(0,1)]^2$ and $F_j \in L^2(Q_T)$, $j = 1,2,3,4$, the solution (u, w, ζ, θ) to (3.9) satisfies*

$$\begin{aligned}
&\|\mathrm{e}^{-s\widehat{\mathfrak{S}}_m}(\mathfrak{z}^*)^{5/2-1/m}u\|^2_{L^2(Q_T)} + \|\mathrm{e}^{-s\widehat{\mathfrak{S}}_m}(\mathfrak{z}^*)^{5/2-1/m}w\|^2_{L^2(Q_T)}\\
&\quad + \|\mathrm{e}^{-s\widehat{\mathfrak{S}}_m}(\mathfrak{z}^*)^{5/2-1/m}\zeta\|^2_{L^2(Q_T)} + \|\mathrm{e}^{-s\widehat{\mathfrak{S}}_m}(\mathfrak{z}^*)^{5/2-1/m}\theta\|^2_{L^2(Q_T)}\\
&\quad + \int_0^1 \big(|\zeta(T,x)|^2 + |\theta(T,x)|^2\big) \le C\Big[\iint \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{z}}_m^{37}\big(|F_1|^2 + |F_2|^2 + |F_3|^2 + |F_4|^2\big)\\
&\quad\quad + \iint_{\omega_0} \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{z}}_m^{39}|u|^2 + \iint_{\omega_0} \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{z}}_m^{41}|w|^2\Big]. \qquad (4.13)
\end{aligned}$$

Proof. Let us define $\rho^*(t) = \mathrm{e}^{-s\widehat{\mathfrak{S}}_m}(\mathfrak{z}^*)^{5/2-1/m}$ so that $\rho^*(0) = 0$. Then, the system for $(\zeta^*, \theta^*) = (\rho^*\zeta, \rho^*\theta)$ looks like

$$\begin{cases}
\zeta^*_t + \zeta^*_{xxxx} + \gamma\zeta^*_{xx} = \theta^*_x + \rho^* F_3 + \rho^*_t\zeta & \text{in } Q_T,\\
\theta^*_t - \theta^*_{xx} + \beta\theta^*_x = \zeta^*_x + \rho^* F_4 + \rho^*_t\theta & \text{in } Q_T,\\
\zeta^* = \zeta^*_x = \theta^* = 0 & \text{in } (0,T),\\
\zeta^*(0) = \theta^*(0) = 0 & \text{in } (0,1),
\end{cases}$$

and it satisfies the following estimate

$$\begin{aligned}
\|\zeta^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|\theta^*\|^2_{L^\infty(0,T;L^2(0,1))} \le & C\Big(\|\rho^* F_3\|^2_{L^2(Q_T)} + \|\rho^* F_4\|^2_{L^2(Q_T)} \qquad (4.14)\\
&+ \|\rho^*_t\zeta\|^2_{L^2(Q_T)} + \|\rho^*_t\theta\|^2_{L^2(Q_T)}\Big).
\end{aligned}$$

Similarly, the system for $(u^*, w^*) = (\rho^* u, \rho^* w)$ is

$$\begin{cases} -u^*_t + u^*_{xxxx} + \gamma u^*_{xx} = -w^*_x + \rho^* F_1 - \rho^*_t u & \text{in } Q_T, \\ -w^*_t - w^*_{xx} - \beta w^*_x = -u^*_x + \rho^* F_2 + \rho^* \theta \mathbb{1}_{\mathcal{O}} - \rho^*_t w & \text{in } Q_T, \\ u^* = u^*_x = w = 0 & \text{in } (0,T), \\ u^*(T) = w^*(T) = 0 & \text{in } (0,1), \end{cases}$$

which satisfies

$$\|u^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|w^*\|^2_{L^\infty(0,T;L^2(0,1))} \leq C\Big(\|\rho^* F_1\|^2_{L^2(Q_T)} + \|\rho^* F_2\|^2_{L^2(Q_T)} + \|\rho^*_t u\|^2_{L^2(Q_T)} + \|\rho^*_t w\|^2_{L^2(Q_T)} + \|\rho^* \theta\|^2_{L^2(Q_T)} \Big). \tag{4.15}$$

Now, it can be checked that $|(\widehat{\mathfrak{S}}_m)_t| \leq C(\mathfrak{Z}^*_m)^{1+1/m}$ and thus

$$|\rho^*_t| \leq C e^{-s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}^*_m)^{7/2}, \quad |\rho^*| \leq C e^{-s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}_m^{7/2}.$$

Using this and together with (4.14)–(4.15), we get

$$\|u^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|w^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|\zeta^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|\theta^*\|^2_{L^\infty(0,T;L^2(0,1))}$$
$$\leq \iint \left(e^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}^*_m)^7 |u|^2 + e^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}^*_m)^7 |w|^2 + e^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}^*_m)^7 |\zeta|^2 + e^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}^*_m)^7 |\theta|^2 \right) + \iint e^{-2s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}_m^7 \left(|F_1|^2 + |F_2|^2 + |F_3|^2 + |F_4|^2 \right). \tag{4.16}$$

Then, by applying the modified Carleman estimate (4.5) and using the fact that $-10s\mathfrak{S}^*_m + 8s\widehat{\mathfrak{S}}_m \geq -2s\mathfrak{S}^*_m$ (thanks to the definitions of $\widehat{\mathfrak{S}}_m$ and $\mathfrak{S}^*_m$ given by (4.3)–(4.4)), we get the required observability inequality (4.13). ☐

4.1.2. Null-controllability of the linearized system ($\alpha = 0$)

We denote the following operators

$$\mathcal{L}_1 := \partial_t + \partial_{xxxx} + \gamma \partial_{xx}, \tag{4.17}$$
$$\mathcal{L}_2 := \partial_t - \partial_{xx} + \beta \partial_x, \tag{4.18}$$

and their respecting adjoint operators by

$$\mathcal{L}^*_1 = -\partial_t + \partial_{xxxx} + \gamma \partial_{xx}, \tag{4.19}$$
$$\mathcal{L}^*_2 = -\partial_t - \partial_{xx} - \beta \partial_x. \tag{4.20}$$

Also, we denote the following Banach space

$$\mathcal{E} := \Big\{ (y, z, p, q, h_1, h_2) \mid e^{5s\mathfrak{S}^*_m - 4s\widehat{\mathfrak{S}}_m} \widehat{\mathfrak{Z}}_m^{-37/2} (y, z, p, q) \in [L^2(Q_T)]^4, \tag{4.21}$$
$$e^{5s\mathfrak{S}^*_m - 4s\widehat{\mathfrak{S}}_m} \big(\widehat{\mathfrak{Z}}_m^{-39/2} h_1 \mathbb{1}_\omega, \widehat{\mathfrak{Z}}_m^{-41/2} h_2 \mathbb{1}_\omega \big) \in [L^2((0,T) \times \omega)]^2,$$
$$e^{5s\mathfrak{S}^*_m - 4s\widehat{\mathfrak{S}}_m} \widehat{\mathfrak{Z}}_m^{-41/2} (y, z, p, q) \in \mathcal{C}^0([0,T]; [L^2(0,1)]^4)$$

$$\begin{aligned}
&\cap L^2(0,T;H^2_0(0,1)\times H^1_0(0,1)\times H^2_0(0,1)\times H^1_0(0,1)),\\
&e^{s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^{-5/2+1/m}(\mathcal{L}_1 y - z_x - h_1\mathbb{1}_\omega)\in L^2(Q_T),\\
&e^{s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^{-5/2+1/m}(\mathcal{L}_2 z - y_x - h_2\mathbb{1}_\omega)\in L^2(Q_T),\\
&e^{s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^{-5/2+1/m}(\mathcal{L}^*_1 p + q_x)\in L^2(Q_T),\\
&e^{s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^{-5/2+1/m}(\mathcal{L}^*_2 q + p_x - z\mathbb{1}_{\mathcal{O}})\in L^2(Q_T),\\
&\quad p(T,\cdot)=q(T,\cdot)=0 \text{ in } (0,1)\Big\}.
\end{aligned}$$

Proposition 4.4 (Null-controllability: the case $\alpha = 0$)**.** *Let m, k, s and λ be fixed constants according to Theorem 3.4. Let f_1, f_2, f_3, f_4 satisfy*

$$e^{s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^{-5/2+1/m}(f_1,f_2,f_3,f_4)\in [L^2(Q_T)]^4. \tag{4.22}$$

Then, there exists controls (h_1,h_2) and a solution (y,z,p,q) to (1.9)–(1.10) (when $\alpha=0$) such that we have $p(0)=q(0)=0$ in $(0,1)$.

Proof. We consider the following space

$$\mathcal{Q}_0 := \Big\{(u,w,\zeta,\theta)\in\mathcal{C}^\infty(\overline{Q_T}) \mid u=u_x=w=\zeta=\zeta_x=\theta=0 \text{ in } \Sigma_T\Big\},$$

and define the bi-linear operator $\mathcal{K}:\mathcal{Q}_0\times\mathcal{Q}_0\to\mathbb{R}$ given by

$$\begin{aligned}
&\mathcal{K}((u,w,\zeta,\theta),(\underline{u},\underline{w},\underline{\zeta},\underline{\theta})) \qquad (4.23)\\
&:= \iint e^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{37}_m\Big[(\mathcal{L}^*_1 u + w_x)(\mathcal{L}^*_1\underline{u}+\underline{w}_x) + (\mathcal{L}^*_2 w + u_x + \theta\mathbb{1}_{\mathcal{O}})(\mathcal{L}^*_2\underline{w}+\underline{u}_x+\underline{\theta}\mathbb{1}_{\mathcal{O}})\\
&\qquad + (\mathcal{L}_1\zeta-\theta_x)(\mathcal{L}_1\underline{\zeta}-\underline{\theta}_x) + (\mathcal{L}_2\theta-\zeta_x)(\mathcal{L}_2\underline{\theta}-\underline{\zeta}_x)\Big]\\
&\quad + \int_0^T\int_\omega e^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\big(\widehat{\mathfrak{Z}}^{39}_m u\underline{u} + \widehat{\mathfrak{Z}}^{41}_m w\underline{w}\big),
\end{aligned}$$

as well as the linear operator $l:\mathcal{Q}_0\to\mathbb{R}$ given by

$$l((u,w,\zeta,\theta)) := \langle f_1,u\rangle_{L^2(Q_T)} + \langle f_2,w\rangle_{L^2(Q_T)} + \langle f_3,\zeta\rangle_{L^2(Q_T)} + \langle f_4,\theta\rangle_{L^2(Q_T)}.$$

It is clear that the product (4.23) is defines an inner product since the observability inequality (4.13) holds. We denote by $\mathcal{Q}$ the closure of $\mathcal{Q}_0$ w.r.t. the norm $\mathcal{K}(\cdot,\cdot)^{1/2}$, which is indeed a Hilbert space endowed with the inner product (4.23). The linear functional l is also bounded due to (4.13) and the hypothesis (4.22). Therefore, the Lax-Milgram's theorem ensures the existence of unique $(\widehat{u},\widehat{w},\widehat{\zeta},\widehat{\theta})\in\mathcal{Q}\times\mathcal{Q}$ satisfying

$$\mathcal{K}((\widehat{u},\widehat{w},\widehat{\zeta},\widehat{\theta}),(\underline{u},\underline{w},\underline{\zeta},\underline{\theta})) = l((\underline{u},\underline{w},\underline{\zeta},\underline{\theta})) \quad \forall(\underline{u},\underline{w},\underline{\zeta},\underline{\theta})\in\mathcal{Q}.$$

Now, we set

$$\widehat{y} = e^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{37}_m(\mathcal{L}^*_1\widehat{u}+\widehat{w}_x), \quad \widehat{z} = e^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{37}_m(\mathcal{L}^*_2\widehat{w}+\widehat{u}_x+\widehat{\theta}\mathbb{1}_{\mathcal{O}}), \tag{4.24}$$

$$\widehat{p} = e^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{37}_m(\mathcal{L}_1\widehat{\zeta}-\widehat{\theta}_x), \quad \widehat{q} = e^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{37}_m(\mathcal{L}_2\widehat{\theta}-\widehat{\zeta}_x), \tag{4.25}$$

and

$$\widehat{h}_1 = \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{39}_m\widehat{u}\mathbb{1}_\omega, \quad \widehat{h}_2 = \mathrm{e}^{-10s\mathfrak{S}^*_m+8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{41}_m\widehat{w}\mathbb{1}_\omega.$$

Then, due to the observability inequality (4.13), we have

$$\iint \mathrm{e}^{10s\mathfrak{S}^*_m-8s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-37}_m\left(|\widehat{y}|^2+|\widehat{z}|^2+|\widehat{p}|^2+|\widehat{q}|^2\right) + \int_0^T\int_\omega \mathrm{e}^{10s\mathfrak{S}^*_m-8s\widehat{\mathfrak{S}}_m}\left(\widehat{\mathfrak{Z}}^{-39}_m|\widehat{h}_1|^2+\widehat{\mathfrak{Z}}^{-41}_m|\widehat{h}_2|^2\right) < +\infty, \quad (4.26)$$

and this $(\widehat{y},\widehat{z},\widehat{p},\widehat{q})$ is unique solution to the linearized system (1.9)–(1.10) (with $\alpha = 0$) in the sense of transposition with the control functions $\widehat{h}_1$ and $\widehat{h}_2$. Moreover, from (4.25) it is clear that

$$\widehat{p}(0) = 0, \quad \widehat{q}(0) = 0 \quad \text{in } (0,1).$$

We further set

$$(y^*,z^*,p^*,q^*) = \mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-41/2}_m(\widehat{y},\widehat{z},\widehat{p},\widehat{q}),$$

which satisfies

$$\begin{cases} \mathcal{L}_1 y^* - z^*_x = h^*_1\mathbb{1}_\omega + f^*_1 + \left(\mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-41/2}_m\right)_t\widehat{y} & \text{in } Q_T,\\ \mathcal{L}_2 z^* - y^*_x = h^*_2\mathbb{1}_\omega + f^*_2 + \left(\mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-41/2}_m\right)_t\widehat{z} & \text{in } Q_T,\\ \mathcal{L}^*_1 p^* + q^*_x = f^*_3 - \left(\mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-41/2}_m\right)_t\widehat{p} & \text{in } Q_T,\\ \mathcal{L}^*_2 q^* + p^*_x = f^*_4 + z^*\mathbb{1}_{\mathcal{O}} - \left(\mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-41/2}_m\right)_t\widehat{q} & \text{in } Q_T,\\ y^* = y^*_x = z^* = p^* = p^*_x = q^* = 0 & \text{in } \Sigma_T,\\ y^*(0) = z^*(0) = p^*(T) = q^*(T) = 0 & \text{in } (0,1), \end{cases} \quad (4.27)$$

where $h^*_i = \mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-41/2}_m\widehat{h}_i$ for $i = 1,2$ and $f^*_j = \mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-41/2}_m f_j$ for $j = 1,2,3,4$.

We first observe that

$$\mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-41/2}_m \le CT^{36m+1}\mathrm{e}^{s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}^*_m)^{-5/2+1/m},$$

since $\mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m} \le \mathrm{e}^{s\mathfrak{S}^*_m}$ and $\mathfrak{Z}^{-1}_m \le CT^{2m}$.

On the other hand, note that $|(\widehat{\mathfrak{S}}_m)_t| \le C(\widehat{\mathfrak{Z}}_m)^{1+1/m}$ (also, $|(\mathfrak{S}^*{}_m)_t| \le C(\widehat{\mathfrak{Z}}_m)^{1+1/m}$), which yields

$$\left|\left(\mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-41/2}_m\right)_t\right| \le C\mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-39/2+1/m}_m \le CT^{2m-1}\mathrm{e}^{5s\mathfrak{S}^*_m-4s\widehat{\mathfrak{S}}_m}\widehat{\mathfrak{Z}}^{-37/2}_m,$$

Using the above facts, the conditions on f_j given by (4.22) and the bound (4.26), we first conclude that the right hand sides of the first two pdes in (4.27) belong to $L^2(Q_T)$. Therefore, by means of Proposition C.1, we have

$$(y^*,z^*) \in \mathcal{C}^0([0,T];[L^2(0,1)]^2) \cap L^2(0,T;H^2_0(0,1)\times H^1_0(0,1)).$$

As a result, the right hand sides of the third and fourth equations of (4.27) also belong to $L^2(Q_T)$, which ensures

$$(p^*, q^*) \in \mathcal{C}^0([0,T];[L^2(0,1)]^2) \cap L^2(0,T;H_0^2(0,1)\times H_0^1(0,1)).$$

Moreover, one has

$$\|y^*\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)} + \|z^*\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)} + \|p^*\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)} + \|q^*\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)} \leq C, \tag{4.28}$$

for some $C > 0$, thanks to the assumption (4.22) and the fact (4.26).

Thus, the functions $(\widehat{y}, \widehat{z}, \widehat{p}, \widehat{q}, \widehat{h}_1, \widehat{h}_2) \in \mathcal{E}$ defined in (4.21). The proof is complete. □

4.1.3. Local null-controllability of the nonlinear system ($\alpha = 0$)

In this section, we shall prove the main theorem of our paper for in the case when $\alpha = 0$, that is Theorem 1.1. But as we mentioned in the beginning, this is equivalent to prove the local null-controllability of the extended system (1.5)–(1.6), which is precisely Theorem 1.3.

To prove the local controllability result, we use the following well-known theorem.

Theorem 4.5 (Inverse mapping theorem)**.** *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two Banach spaces and $\mathcal{Y} : \mathcal{B}_1 \to \mathcal{B}_2$ satisfying $\mathcal{Y} \in \mathcal{C}^1(\mathcal{B}_1;\mathcal{B}_2)$. Assume that $b_1 \in \mathcal{B}_1$ and $\mathcal{Y}(b_1) = b_2 \in \mathcal{B}_2$ and $\mathcal{Y}'(b_1) : \mathcal{B}_1 \to \mathcal{B}_2$ is surjective. Then, there exists $\delta > 0$ such that for every $\widetilde{b} \in \mathcal{B}_2$ satisfying $\|\widetilde{b} - b_2\|_{\mathcal{B}_2} < \delta$, there exists a solution of the equation*

$$\mathcal{Y}(b) = \widetilde{b}, \quad b \in \mathcal{B}_1.$$

Proof of Theorem 1.3. We now prove the main result of this work. The idea is to apply the above theorem with

$$\mathcal{B}_1 = \mathcal{E},$$
$$\mathcal{B}_2 = \mathcal{F} \times \mathcal{F} \times \mathcal{F} \times \mathcal{F},$$

where

$$\mathcal{F} := \left\{ \xi \mid e^{s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^{-5/2+1/m} \xi \in L^2(Q_T) \right\}, \tag{4.29}$$

and consider $\mathcal{Y} : \mathcal{B}_1 \to \mathcal{B}_2$ such that

$$\begin{aligned} &\mathcal{Y}(y,z,p,q,h_1,h_2) \\ =&(\mathcal{L}_1 y - z_x - h_1 \mathbb{1}_\omega + y y_x, \mathcal{L}_2 z - y_x - h_2 \mathbb{1}_\omega, \mathcal{L}_1^* p + q_x - y p_x, \mathcal{L}_2^* q + p_x - z \mathbb{1}_{\mathcal{O}}). \end{aligned} \tag{4.30}$$

- Let us first check that $\mathcal{Y} \in \mathcal{C}^1(\mathcal{B}_1, \mathcal{B}_2)$. In fact, all the terms of $\mathcal{Y}$ are well-defined and linear except the terms $y y_x$ and $-y p_x$. Thus, it is enough to show that the map

  $$\big((y,z,p,q,h_1,h_2), (\widetilde{y}, \widetilde{z}, \widetilde{p}, \widetilde{q}, \widetilde{h}_1, \widetilde{h}_2)\big) \longmapsto \big(y\widetilde{y}_x, -y\widetilde{p}_x\big)$$

  from $\mathcal{B}_1 \times \mathcal{B}_1 \to \mathcal{F} \times \mathcal{F}$ is continuous. Before that, we observe that

  $$e^{s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^{-5/2+1/m} \leq C e^{2(5s\mathfrak{S}_m^* - 4s\widehat{\mathfrak{S}}_m)} \widehat{\mathfrak{Z}}_m^{-41}, \tag{4.31}$$

which can be seen as follows: there exists some $d_0 > 0$ such that

$$\begin{aligned}2(5s\mathfrak{S}_m^* - 4s\widehat{\mathfrak{S}}_m) - s\widehat{\mathfrak{S}}_m &= \frac{s}{\ell(t)^m}\left[\mathrm{e}^{\lambda(1+\frac{1}{m})k\|\nu\|_\infty} - 10\mathrm{e}^{\lambda(k\|\nu\|_\infty+\|\nu\|_\infty)} + 9\mathrm{e}^{\lambda k\|\nu\|_\infty}\right]\\ &= \frac{s\mathrm{e}^{\lambda k\|\nu\|_\infty}}{\ell(t)^m}\left[\mathrm{e}^{\lambda\frac{k}{m}\|\nu\|_\infty} - 10\mathrm{e}^{\lambda\|\nu\|_\infty} + 9\right]\\ &\geq \frac{d_0 s\mathrm{e}^{\lambda k\|\nu\|_\infty}}{\ell(t)^m},\end{aligned}$$

for chosen $k > m$ large enough. In other words,

$$s\widehat{\mathfrak{S}}_m \leq 2(5s\mathfrak{S}_m^* - 4s\widehat{\mathfrak{S}}_m) - \frac{d_0 s\mathrm{e}^{\lambda k\|\nu\|_\infty}}{\ell(t)^m},$$

which yields

$$\mathrm{e}^{s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^{-5/2+1/m} \leq \mathrm{e}^{2(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41} \times \mathrm{e}^{-\frac{sd_0\mathrm{e}^{\lambda k\|\nu\|_\infty}}{\ell(t)^m}}\widehat{\mathfrak{Z}}_m^{41}(\mathfrak{Z}_m^*)^{-5/2+1/m},$$

and that (4.31) follows.
Now, using (4.31), we have

$$\begin{aligned}&\left\|\mathrm{e}^{s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^{-5/2+1/m}\big(y\widetilde{y}_x, -y\widetilde{p}_x\big)\right\|_{[L^2(Q_T)]^2}\\ \leq& C\left\|\mathrm{e}^{2(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41}y\widetilde{y}_x\right\|_{L^2(Q_T)} + C\left\|\mathrm{e}^{2(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41}y\widetilde{p}_x\right\|_{L^2(Q_T)}\\ \leq& C\left\|\mathrm{e}^{(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41/2}y\right\|_{L^\infty(0,T;L^2(0,1))}\left\|\mathrm{e}^{(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41/2}\widetilde{y}\right\|_{L^2(0,T;H^2(0,1))}\\ &+ C\left\|\mathrm{e}^{(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41/2}y\right\|_{L^\infty(0,T;L^2(0,1))}\left\|\mathrm{e}^{(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41/2}\widetilde{p}\right\|_{L^2(0,T;H^2(0,1))}\\ \leq& C\big\|(y,z,p,q,h_1,h_2)\big\|_{\mathcal{B}_1}\big\|(\widetilde{y},\widetilde{z},\widetilde{p},\widetilde{q},\widetilde{h}_1,\widetilde{h}_2)\big\|_{\mathcal{B}_1},\end{aligned} \tag{4.32}$$

Therefore, the claim $\mathcal{Y} \in \mathcal{C}^1(\mathcal{B}_1, \mathcal{B}_2)$ is established.

Remark 4.6. The estimate (4.32) can be obtained as follows:

$$\begin{aligned}&\int_0^T\int_0^1 \mathrm{e}^{4(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41\times 2}|y\widetilde{y}_x|^2\\ \leq& \int_0^T \mathrm{e}^{4(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41\times 2}\|y\|^2_{L^2(0,1)}\|\widetilde{y}_x\|^2_{L^\infty(0,1)}\\ \leq& \int_0^T \mathrm{e}^{4(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41\times 2}\|y\|^2_{L^2(0,1)}\|\widetilde{y}_x\|^2_{H_0^1(0,1)}\\ \leq& \left\|\mathrm{e}^{(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41/2}y\right\|^2_{L^\infty(0,T;L^2(0,1))}\left\|\mathrm{e}^{(5s\mathfrak{S}_m^*-4s\widehat{\mathfrak{S}}_m)}\widehat{\mathfrak{Z}}_m^{-41/2}\widetilde{y}\|\right\|^2_{L^2(0,T;H_0^2(0,1))}.\end{aligned}$$

A similar estimate can be found for the term associated to $y\widetilde{p}_x$.

- Next, we show that $\mathcal{Y}'(0,0,0,0,0)$ is surjective. In fact, we have $\mathcal{Y}(0,0,0,0,0) = (0,0,0,0)$ and $\mathcal{Y}'(0,0,0,0,0) : \mathcal{B}_1 \to \mathcal{B}_2$ given by

$$\begin{aligned}&\mathcal{Y}'(0,0,0,0,0)(y,z,p,q,h_1,h_2)\\ =&(\mathcal{L}_1 y - z_x - h_1 \mathbb{1}_\omega, \mathcal{L}_2 z - y_x - h_2 \mathbb{1}_\omega, \mathcal{L}_1^* p + q_x, \mathcal{L}_2^* q + p_x - z\mathbb{1}_{\mathcal{O}})\end{aligned}$$

is surjective, thanks to the null-controllability result Proposition 4.4.
Finally, we consider $b_1 = (0,0,0,0,0)$, $b_2 = (0,0,0,0)$ and $\widetilde{b} = (\xi_1, \xi_2, 0, 0) \in \mathcal{B}_2$, where (ξ_1,ξ_2) is the given external source term in (1.1) or in (1.5)–(1.6). Then, according to Theorem 4.5, there exists a $\delta > 0$ verifying

$$\|(\xi_1,\xi_2,0,0)\|_{\mathcal{B}_2} < \delta,$$

we have the existence of solution-control pair $(y,z,p,q,h_1,h_2) \in \mathcal{B}_1 = \mathcal{E}$ to the system (1.5)–(1.6). In particular, $p(0) = q(0) = 0$ in $(0,1)$. This completes the proof of Theorem 1.3 which implies the proof for Theorem 1.1.

□

Remark 4.7. Observe that, we have chosen $L^2(Q_T)$ right hand sides in the equations (1.9)–(1.10) and initial data $(y_0,z_0) \in [L^2(0,1)]^2$ to handle the controllability for the nonlinear system (1.5)–(1.6). In several previous works regarding the controllability study of stabilized KS systems, for instance [23, 24], the authors considered $L^1(0,T;W^{-1,1}(0,1))$– and $L^2(0,T;H^{-1}(0,1))$–source terms (for the KS and heat equations resp.) and $H^{-2}(0,1)\times H^{-1}(0,1)$ initial data. In those cases, the only nonlinear term is yy_x and this can be formally handled integrating by parts so

$$yy_x \in L^1(0,T;W^{-1,1}(0,1)) \Longleftrightarrow |y|^2 \in L^1(0,T;L^1(0,1)).$$

However, it seems that this strategy is not useful to deal with the extra nonlinear term "yp_x" appearing in our 4×4 coupled system. Obtaining (4.32) starting with data weaker than L^2 is an open problem.

4.2. The case when $\alpha = 1$

4.2.1. Observability inequality ($\alpha = 1$)

We recall the Carleman estimate (3.39) in Theorem 3.10 for $\alpha = 1$ and state a modified version of it.

Proposition 4.8 (A refined Carleman estimate: the case $\alpha = 1$)**.** *Let m, k, s and λ be fixed constants according to Theorem 3.10. Then, there exists a positive constant C depending at most on $\omega, \mathcal{O}$, T, m, k, s, and λ such that*

$$\begin{aligned}\iint \Big(&e^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^7|u|^2 + e^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^3|w|^2 + e^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^7|\zeta|^2 + e^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^3|\theta_x|^2\Big)\\ &+ \|\zeta(T)\|^2_{L^2(0,1)} + \|\theta(T)\|^2_{L^2(0,1)} \leq C\bigg[\iint e^{-2s\mathfrak{S}_m^*}\Big(\widehat{\mathfrak{Z}}_m^{71}|F_1|^2 + |F_2|^2 + \widehat{\mathfrak{Z}}_m^3|F_3|^2\Big)\\ &+ \iint e^{-2s\mathfrak{S}_m^*}\widehat{\mathfrak{Z}}_m^2\left(|F_4|^2 + |F_{4,x}|^2\right) + \iint_{\omega_0} e^{-2s\mathfrak{S}_m^*}\widehat{\mathfrak{Z}}_m^{79}|u|^2 + \iint_{\omega_0} e^{-2s\mathfrak{S}_m^*}\widehat{\mathfrak{Z}}_m^{73}|w|^2\bigg], \quad (4.33)\end{aligned}$$

where (u,w,ζ,θ) is the solution associated to (3.37), for given data $(\zeta_0,\theta_0) \in [L^2(0,1)]^2$ and $F_j \in L^2(Q_T)$, $j = 1,2,3,4$ such that $\iint e^{-2s\mathfrak{S}_m^*}\big(\widehat{\mathfrak{Z}}_m^{71}|F_1|^2 + |F_2|^2 + \widehat{\mathfrak{Z}}_m^3|F_3|^2 + \widehat{\mathfrak{Z}}_m^2|F_4|^2 + \widehat{\mathfrak{Z}}_m^2|F_{4,x}|^2\big) < +\infty$.

To prove the above proposition, we shall use the similar technique as applied in the proof of Proposition 4.2 (the case $\alpha = 0$). In this case, we write the main steps of the proof.

Proof. In the beginning, we fix the Carleman parameters $s = s_0$ and $\lambda = \lambda_0$ in the estimate (3.39) given by Theorem 3.10, and recall that $\varphi_m = \mathfrak{S}_m$ and $\xi_m = \mathfrak{Z}_m$ in $(0, T/2) \times (0, 1)$, hence

$$\begin{aligned}&\int_0^{\frac{T}{2}} \int_0^1 \left(\mathrm{e}^{-2s\mathfrak{S}_m} \mathfrak{Z}_m^7 |u|^2 + \mathrm{e}^{-2s\mathfrak{S}_m} \mathfrak{Z}_m^3 |w|^2 + \mathrm{e}^{-2s\mathfrak{S}_m} \mathfrak{Z}_m^7 |\zeta|^2 + \mathrm{e}^{-2s\mathfrak{S}_m} \mathfrak{Z}_m^3 |\theta_x|^2\right) \\ &= \int_0^{\frac{T}{2}} \int_0^1 \left(\mathrm{e}^{-2s\varphi_m} \xi_m^7 |u|^2 + \mathrm{e}^{-2s\varphi_m} \xi_m^3 |w|^2 + \mathrm{e}^{-2s\varphi_m} \xi_m^7 |\zeta|^2 + \mathrm{e}^{-2s\varphi_m} \xi_m^3 |\theta_x|^2\right).\end{aligned}$$

Therefore, using the Carleman estimate (3.39), and the definitions (4.3)–(4.4), we readily get

$$\begin{aligned}\int_0^{\frac{T}{2}} \int_0^1 &\left(\mathrm{e}^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^7 |u|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^3 |w|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^7 |\zeta|^2 + \mathrm{e}^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^3 |\theta_x|^2\right) \\ &\leq C \iint \mathrm{e}^{-2s\mathfrak{S}_m^*} \left(\widehat{\mathfrak{Z}}_m^{71} |F_1|^2 + |F_2|^2 + \widehat{\mathfrak{Z}}_m^3 |F_3|^2 + \widehat{\mathfrak{Z}}_m^2 |F_4|^2 + \widehat{\mathfrak{Z}}_m^2 |F_{4,x}|^2\right) \\ &\qquad + \iint_{\omega_0} \mathrm{e}^{-2s\mathfrak{S}_m^*} \widehat{\mathfrak{Z}}_m^{79} |u|^2 + \iint_{\omega_0} \mathrm{e}^{-2s\mathfrak{S}_m^*} \widehat{\mathfrak{Z}}_m^{73} |w|^2 \Bigg]. \end{aligned} \tag{4.34}$$

We now recall the cut-off function η given by

$$\eta = 0 \text{ in } [0, T/4], \quad \eta = 1 \text{ in } [T/2, T], \quad |\eta'| \leq C/T.$$

Then, as we obtained the energy estimate (4.10) (for the case $\alpha = 0$), we have the following estimate for the equation satisfied by $(\eta u, \eta w, \eta\zeta, \eta\theta)$ with $((\eta\zeta)(0), (\eta\theta)(0)) = (0, 0)$,

$$\begin{aligned}&\|\eta u\|^2_{L^\infty(T/4,T;L^2(0,1))} + \|\eta w\|^2_{L^\infty(T/4,T;L^2(0,1))} + \|\eta\zeta\|^2_{L^\infty(T/4,T;L^2(0,1))} \\ &\quad + \|\eta\theta\|^2_{L^\infty(T/4,T;L^2(0,1))} + \|\eta\theta\|^2_{L^2(T/4,T;H_0^1(0,1))} \\ &\leq C\Bigg(\sum_{j=1}^4 \|\eta F_j\|^2_{L^2((T/4,T)\times(0,1))} + \|u\|^2_{L^2((T/4,T/2)\times(0,1))} + \|w\|^2_{L^2((T/4,T/2)\times(0,1))} \\ &\qquad + \|\zeta\|^2_{L^2((T/4,T/2)\times(0,1))} + \|\theta\|^2_{L^2((T/4,T/2)\times(0,1))}\Bigg) \\ &\leq C\Bigg(\sum_{j=1}^4 \|\eta F_j\|^2_{L^2((T/4,T)\times(0,1))} + \int_{\frac{T}{4}}^{\frac{T}{2}} \left(|u|^2 + |w|^2 + |\zeta|^2 + |\theta_x|^2\right),\end{aligned} \tag{4.35}$$

using the Poincaré inequality for θ. Now, since the weight functions $\mathfrak{S}_m$ and $\mathfrak{Z}_m$ are bounded by below in $[T/4, T]$ (see (4.2)), we have

$$\begin{aligned}&\int_{\frac{T}{4}}^{\frac{T}{2}} \int_0^1 \left(|u|^2 + |w|^2 + |\zeta|^2 + |\theta_x|^2\right) \\ &\leq \int_{\frac{T}{4}}^{\frac{T}{2}} \int_0^1 \left(\mathrm{e}^{-2s\mathfrak{S}_m} \mathfrak{Z}_m^7 |u|^2 + \mathrm{e}^{-2s\mathfrak{S}_m} \mathfrak{Z}_m^3 |w|^2 + \mathrm{e}^{-2s\mathfrak{S}_m} \mathfrak{Z}_m^7 |\zeta|^2 + \mathrm{e}^{-2s\mathfrak{S}_m} \mathfrak{Z}_m^3 |\theta_x|^2\right) \\ &\leq C \iint \mathrm{e}^{-2s\mathfrak{S}_m^*} \left(\widehat{\mathfrak{Z}}_m^{71} |F_1|^2 + |F_2|^2 + \widehat{\mathfrak{Z}}_m^3 |F_3|^2 + \widehat{\mathfrak{Z}}_m^2 |F_4|^2 + \widehat{\mathfrak{Z}}_m^2 |F_{4,x}|^2\right) \\ &+ \iint_{\omega_0} \mathrm{e}^{-2s\mathfrak{S}_m^*} \widehat{\mathfrak{Z}}_m^{79} |u|^2 + \iint_{\omega_0} \mathrm{e}^{-2s\mathfrak{S}_m^*} \widehat{\mathfrak{Z}}_m^{73} |w|^2 \Bigg],\end{aligned} \tag{4.36}$$

due to the Carleman estimate (3.39) and the definitions (4.3)–(4.4).

We also can incorporate the weight functions in the left hand side of (4.35), which yields together with (4.36) and from the definitions (4.3)–(4.4), that

$$\begin{aligned}
&\int_{\frac{T}{2}}^{T}\int_0^1 \left(e^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^7|u|^2 + e^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^3|w|^2 + e^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^7|\zeta|^2 + e^{-2s\widehat{\mathfrak{S}}_m}(\mathfrak{Z}_m^*)^3|\theta_x|^2\right)\\
&\quad + \|\eta u\|^2_{L^\infty(T/4,T;L^2(0,1))} + \|\eta w\|^2_{L^\infty(T/4,T;L^2(0,1))} + \|\eta\zeta\|^2_{L^\infty(T/4,T;L^2(0,1))} + \|\eta\theta\|^2_{L^\infty(T/4,T;L^2(0,1))}\\
&\qquad \le C\bigg[\iint e^{-2s\mathfrak{S}_m^*}\left(\widehat{\mathfrak{Z}}_m^{71}|F_1|^2 + |F_2|^2 + \widehat{\mathfrak{Z}}_m^{3}|F_3|^2 + \widehat{\mathfrak{Z}}_m^{2}|F_4|^2 + \widehat{\mathfrak{Z}}_m^{2}|F_{4,x}|^2\right)\\
&\qquad\qquad + \iint_{\omega_0} e^{-2s\mathfrak{S}_m^*}\widehat{\mathfrak{Z}}_m^{79}|u|^2 + \iint_{\omega_0} e^{-2s\mathfrak{S}_m^*}\widehat{\mathfrak{Z}}_m^{73}|w|^2\bigg], \qquad (4.37)
\end{aligned}$$

for some $C > 0$.

Combining (4.34) and (4.37), we have the desired result (4.33). □

Let us find the observability inequality in this case.

Proposition 4.9 (Observability inequality: the case $\alpha = 1$). *Let m, k, s and λ be fixed constants according to Theorem 3.10. Then, there exists a positive constant C depending at most on $\omega, \mathcal{O}$, T, m, k, s, and λ such that for any $(\zeta_0, \theta_0) \in [L^2(0,1)]^2$, the solution (u, w, ζ, θ) to (3.37) satisfies*

$$\begin{aligned}
&\|e^{-s\widehat{\mathfrak{S}}_m}u\|^2_{L^2(Q_T)} + \|e^{-s\widehat{\mathfrak{S}}_m}w\|^2_{L^2(Q_T)} + \|e^{-s\widehat{\mathfrak{S}}_m}\zeta\|^2_{L^2(Q_T)}\\
&\qquad + \|e^{-s\widehat{\mathfrak{S}}_m}\theta\|^2_{L^2(Q_T)} + \int_0^1\left(|\zeta(T,x)|^2 + |\theta(T,x)|^2\right)\\
&\quad \le C\bigg[\iint e^{-2s\mathfrak{S}_m^*}\widehat{\mathfrak{Z}}_m^{71}\left(|F_1|^2 + |F_2|^2 + |F_3|^2 + |F_4|^2 + |F_{4,x}|^2\right)\\
&\qquad\qquad + \iint_{\omega_0} e^{-2s\mathfrak{S}_m^*}\widehat{\mathfrak{Z}}_m^{79}|u|^2 + \iint_{\omega_0} e^{-2s\mathfrak{S}_m^*}\widehat{\mathfrak{Z}}_m^{73}|w|^2\bigg], \qquad (4.38)
\end{aligned}$$

where the source terms $F_j \in L^2(Q_T)$, $j = 1, 2, 3, 4$ in (3.37) are verifying $\iint e^{-2s\mathfrak{S}_m^*}\widehat{\mathfrak{Z}}_m^{71}\Big(|F_1|^2 + |F_2|^2 + |F_3|^2 + |F_4|^2 + |F_{4,x}|^2\Big) < +\infty$.

Proof. Let us define $\rho^*(t) = e^{-s\widehat{\mathfrak{S}}_m(t)}$ so that $\rho^*(0) = 0$. Then, the equation of $(\zeta^*, \theta^*) = (\rho^*\zeta, \rho^*\theta)$ satisfies (recall first the adjoint system (3.37))

$$\begin{cases}
\zeta_t^* + \zeta_{xxxx}^* + \gamma\zeta_{xx}^* = \theta_x^* + \rho^*F_3 + \rho_t^*\zeta & \text{in } Q_T,\\
\theta_t^* - \theta_{xx}^* + \beta\theta_x^* = \zeta_x^* + \rho^*F_4 + \rho_t^*\theta & \text{in } Q_T,\\
\zeta^* = \zeta_x^* = \theta^* = 0 & \text{in } (0,T),\\
\zeta^*(0) = \theta^*(0) = 0 & \text{in } (0,1),
\end{cases}$$

and it satisfies the following estimate

$$\begin{aligned}
\|\zeta^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|\theta^*\|^2_{L^\infty(0,T;L^2(0,1))} \le{}& C\Big(\|\rho^*F_3\|^2_{L^2((0,T)\times(0,1))} + \|\rho^*F_4\|^2_{L^2((0,T)\times(0,1))}\\
&+ \|\rho_t^*\zeta\|^2_{L^2((0,T)\times(0,1))} + \|\rho_t^*\theta\|^2_{L^2((0,T)\times(0,1))}\Big). \qquad (4.39)
\end{aligned}$$

Similarly, the equation of $(u^*, w^*) = (\rho^* u, \rho^* w)$ is

$$\begin{cases} -u_t^* + u_{xxxx}^* + \gamma u_{xx}^* = -w_x^* + \zeta^* \mathbb{1}_{\mathcal{O}} + \rho^* F_1 - \rho_t^* u & \text{in } Q_T, \\ -w_t^* - w_{xx}^* - \beta w_x^* = -u_x^* + \rho^* F_2 - \rho_t^* w & \text{in } Q_T, \\ u^* = u_x^* = w = 0 & \text{in } (0,T), \\ u^*(T) = w^*(T) = 0 & \text{in } (0,1), \end{cases}$$

which satisfies

$$\begin{aligned} \|u^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|w^*\|^2_{L^\infty(0,T;L^2(0,1))} \leq C\Big(& \|\rho^* F_1\|^2_{L^2((0,T)\times(0,1))} + \|\rho^* F_2\|^2_{L^2((0,T)\times(0,1))} \\ & + \|\rho_t^* u\|^2_{L^2((0,T)\times(0,1))} + \|\rho_t^* w\|^2_{L^2((0,T)\times(0,1))} + \|\rho^* \zeta\|^2_{L^2((0,T)\times(0,1))}\Big). \end{aligned} \tag{4.40}$$

Now, we check that $|(\widehat{\mathfrak{S}}_m)_t| \leq C(\mathfrak{Z}_m^*)^{1+1/m}$ so that $|\rho_t^*| \leq C e^{-s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^{3/2}$ (since $m > 3$ in this case). Using this, together with (4.39)–(4.40) (also by applying the Poincaré inequality on θ in the r.h.s. of (4.39)), we get

$$\begin{aligned} & \|u^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|w^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|\zeta^*\|^2_{L^\infty(0,T;L^2(0,1))} + \|\theta^*\|^2_{L^\infty(0,T;L^2(0,1))} \\ & \quad \leq \iint \left(e^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^3 |u|^2 + e^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^3 |w|^2 + e^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^3 |\zeta|^2 + e^{-2s\widehat{\mathfrak{S}}_m} (\mathfrak{Z}_m^*)^3 |\theta_x|^2 \right) \\ & \qquad + \iint e^{-2s\mathfrak{S}_m^*} \left(|F_1|^2 + |F_2|^2 + |F_3|^2 + |F_4|^2 + |F_{4,x}|^2 \right). \end{aligned} \tag{4.41}$$

Then, by applying the modified Carleman estimate (4.37), we get the required observability inequality (4.38). ☐

4.2.2. Null-controllability of the linearized system ($\alpha = 1$)

We recall the operators $\mathcal{L}_1$, $\mathcal{L}_2$ from (4.17)–(4.18) and their adjoints $\mathcal{L}_1^*$, $\mathcal{L}_2^*$ from (4.19)–(4.20). Denote the following Banach space

$$\begin{aligned} \mathcal{E}_1 := \Big\{ (y, z, p, q, h_1, h_2) \mid\ & e^{s\mathfrak{S}_m^*} \widehat{\mathfrak{Z}}_m^{-71/2} (y, z, p, q) \in [L^2(Q_T)]^3 \times L^2(0,T; H^{-1}(0,1)), \\ & e^{s\mathfrak{S}_m^*} \left(\widehat{\mathfrak{Z}}_m^{-79/2} h_1 \mathbb{1}_\omega, \widehat{\mathfrak{Z}}_m^{-73/2} h_2 \mathbb{1}_\omega \right) \in [L^2((0,T)\times\omega)]^2, \\ & e^{s\mathfrak{S}_m^*} \widehat{\mathfrak{Z}}_m^{-79/2} (y, z, p, q) \in \mathcal{C}^0([0,T]; [L^2(0,1)]^4) \\ & \quad \cap L^2(0,T; H_0^2(0,1) \times H_0^1(0,1) \times H_0^2(0,1) \times H_0^1(0,1)), \\ & e^{s\widehat{\mathfrak{S}}_m} \left(\mathcal{L}_1 y - z_x - h_1 \mathbb{1}_\omega \right) \in L^2(Q_T), \\ & e^{s\widehat{\mathfrak{S}}_m} \left(\mathcal{L}_2 z - y_x - h_2 \mathbb{1}_\omega \right) \in L^2(Q_T), \\ & e^{s\widehat{\mathfrak{S}}_m} \left(\mathcal{L}_1^* p + q_x - y \mathbb{1}_{\mathcal{O}} \right) \in L^2(Q_T), \\ & e^{s\widehat{\mathfrak{S}}_m} \left(\mathcal{L}_2^* q + p_x \right) \in L^2(Q_T), \\ & \quad p(T,\cdot) = q(T,\cdot) = 0 \text{ in } (0,1) \Big\}. \end{aligned} \tag{4.42}$$

Proposition 4.10 (Null-controllability: the case $\alpha = 1$)**.** *Let m, k, s and λ be fixed constants according to Theorem 3.10. Let f_1, f_2, f_3, f_4 be the functions satisfying*

$$e^{s\widehat{\mathfrak{S}}_m} (f_1, f_2, f_3, f_4) \in [L^2(Q_T)]^4. \tag{4.43}$$

Then, there exists controls (h_1, h_2) *and a solution* (y, z, p, q) *to* (1.9)–(1.10) *(when* $\alpha = 1$*) such that we have* $p(0) = q(0) = 0$ *in* $(0, 1)$.

Proof. We consider the following space

$$\mathcal{Q}_1 := \Big\{(u, w, \zeta, \theta) \in \mathcal{C}^\infty(\overline{Q_T}) \mid u = u_x = w = \zeta = \zeta_x = \theta = 0 \text{ in } \Sigma_T\Big\},$$

and define the bi-linear operator $\mathcal{K}_1 : \mathcal{Q}_1 \times \mathcal{Q}_1 \to \mathbb{R}$ given by

$$\begin{aligned}
&\mathcal{K}_1((u, w, \zeta, \theta), (\underline{u}, \underline{w}, \underline{\zeta}, \underline{\theta})) \\
&:= \iint e^{-2s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{71}_m \Big[(\mathcal{L}^*_1 u + w_x - \zeta \mathbb{1}_{\mathcal{O}})(\mathcal{L}^*_1 \underline{u} + \underline{w}_x - \zeta \mathbb{1}_{\mathcal{O}}) + (\mathcal{L}^*_2 w + u_x)(\mathcal{L}^*_2 \underline{w} + \underline{u}_x) \\
&\qquad + (\mathcal{L}_1 \zeta - \theta_x)(\mathcal{L}_1 \underline{\zeta} - \underline{\theta}_x) + (\mathcal{L}_2 \theta - \zeta_x)(\mathcal{L}_2 \underline{\theta} - \underline{\zeta}_x) + \big(\mathcal{L}_2 \theta - \zeta_x\big)_x \big(\mathcal{L}_2 \underline{\theta} - \underline{\zeta}_x\big)_x\Big] \\
&\quad + \int_0^T \int_\omega e^{-2s\mathfrak{S}^*_m} \big(\widehat{\mathfrak{Z}}^{79}_m u\underline{u} + \widehat{\mathfrak{Z}}^{73}_m w\underline{w}\big),
\end{aligned} \tag{4.44}$$

as well as the linear operator $l : \mathcal{Q}_1 \to \mathbb{R}$ given by

$$l_1((u, w, \zeta, \theta)) := \langle f_1, u\rangle_{L^2(Q_T)} + \langle f_2, w\rangle_{L^2(Q_T)} + \langle f_3, \zeta\rangle_{L^2(Q_T)} + \langle f_4, \theta\rangle_{L^2(Q_T)}.$$

It is clear that the product (4.44) is defines an inner product since the observability inequality (4.38) holds. We denote by $\mathcal{Q}_\flat$ the closure of $\mathcal{Q}_1$ w.r.t. the norm $\mathcal{K}_1(\cdot, \cdot)^{1/2}$, which is indeed a Hilbert space endowed with the inner product (4.44). The linear functional l_1 is also bounded due to (4.38) and the hypothesis (4.43), therefore the Lax-Milgram's theorem ensures the existence of unique $(\widehat{u}, \widehat{w}, \widehat{\zeta}, \widehat{\theta}) \in \mathcal{Q}_\flat \times \mathcal{Q}_\flat$ satisfying

$$\mathcal{K}_1((\widehat{u}, \widehat{w}, \widehat{\zeta}, \widehat{\theta}), (\underline{u}, \underline{w}, \underline{\zeta}, \underline{\theta})) = l_1((\underline{u}, \underline{w}, \underline{\zeta}, \underline{\theta})) \quad \forall (\underline{u}, \underline{w}, \underline{\zeta}, \underline{\theta}) \in \mathcal{Q}_\flat. \tag{4.45}$$

Now, we set

$$\widehat{y} = e^{-2s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{71}_m (\mathcal{L}^*_1 \widehat{u} + \widehat{w}_x - \widehat{\zeta} \mathbb{1}_{\mathcal{O}}), \quad \widehat{z} = e^{-2s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{71}_m (\mathcal{L}^*_2 \widehat{w} + \widehat{u}_x), \tag{4.46}$$

$$\widehat{p} = e^{-2s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{71}_m (\mathcal{L}_1 \widehat{\zeta} - \widehat{\theta}_x), \quad \widehat{q} = e^{-2s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{71}_m \Big[(\mathcal{L}_2 \widehat{\theta} - \widehat{\zeta}_x) - (\mathcal{L}_2 \widehat{\theta} - \widehat{\zeta}_x)_{xx}\Big], \tag{4.47}$$

and

$$\widehat{h}_1 = e^{-2s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{79}_m \widehat{u} \mathbb{1}_\omega, \quad \widehat{h}_2 = e^{-2s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{73}_m \widehat{w} \mathbb{1}_\omega.$$

Then, using the observability inequality (4.38) we have (from the equation (4.45))

$$\begin{aligned}
&\|e^{s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{-71/2}_m \widehat{y}\|_{L^2(Q_T)} + \|e^{s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{-71/2}_m \widehat{z}\|_{L^2(Q_T)} \\
&+ \|e^{s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{-71/2}_m \widehat{p}\|_{L^2(Q_T)} + \|e^{s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{-71/2}_m \widehat{q}\|_{L^2(0,T;H^{-1}(0,1))} \\
&+ \|e^{s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{-79/2}_m \widehat{h}_1\|_{L^2((0,T)\times\omega)} + \|e^{s\mathfrak{S}^*_m} \widehat{\mathfrak{Z}}^{-73/2}_m \widehat{h}_2\|_{L^2((0,T)\times\omega)} < +\infty,
\end{aligned} \tag{4.48}$$

which can be seen by performing a series of computations. We just show it only for the quantity $\|e^{s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-71/2}\widehat{q}\|_{L^2(0,T;H^{-1}(0,1))}$. Indeed, we find that

$$
\begin{aligned}
&\int_0^T e^{2s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-71}\|\widehat{q}\|^2_{H^{-1}(0,1)}\\
&=\int_0^T e^{2s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-71}\sup_{\|\vartheta\|_{H^1_0}=1}|\langle\widehat{q},\vartheta\rangle|^2_{H^{-1},H^1_0}\\
&=\int_0^T e^{2s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-71}\sup_{\|\vartheta\|_{H^1_0}=1}\left|\left\langle e^{-2s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{71}\left[(\mathcal{L}_2\widehat{\theta}-\widehat{\zeta}_x)-(\mathcal{L}_2\widehat{\theta}-\widehat{\zeta}_x)_{xx}\right],\vartheta\right\rangle\right|^2_{H^{-1},H^1_0}\\
&\leq 2\iint e^{-2s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{71}\left|(\mathcal{L}_2\widehat{\theta}-\widehat{\zeta}_x)\right|^2+2\iint e^{-2s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{71}\left|(\mathcal{L}_2\widehat{\theta}-\widehat{\zeta}_x)_x\right|^2\\
&\leq \mathcal{K}_1\big((\widehat{u},\widehat{w},\widehat{\zeta},\widehat{\theta}),(\widehat{u},\widehat{w},\widehat{\zeta},\widehat{\theta})\big)<+\infty.
\end{aligned}
\tag{4.49}
$$

The above $(\widehat{y},\widehat{z},\widehat{p},\widehat{q})$ is unique solution to the linearized system (1.9)–(1.10) (with $\alpha=1$) in the sense of transposition with the control functions $\widehat{h}_1$ and $\widehat{h}_2$. Moreover, from (4.47) it is clear that

$$\widehat{p}(0)=0,\quad \widehat{q}(0)=0\quad \text{in } (0,1).$$

At this stage, we set

$$(y^*,z^*,p^*,q^*)=e^{s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-79/2}(\widehat{y},\widehat{z},\widehat{p},\widehat{q}),$$

which satisfies the following set of equations

$$
\begin{cases}
\mathcal{L}_1 y^*-z^*_x=h^*_1\mathbb{1}_\omega+f^*_1+\big(e^{s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-79/2}\big)_t\widehat{y} & \text{in } Q_T,\\
\mathcal{L}_2 z^*-y^*_x=h^*_2\mathbb{1}_\omega+f^*_2+\big(e^{s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-79/2}\big)_t\widehat{z} & \text{in } Q_T,\\
\mathcal{L}^*_1 p^*+q^*_x=f^*_3+y^*\mathbb{1}_{\mathcal{O}}-\big(e^{s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-79/2}\big)_t\widehat{p} & \text{in } Q_T,\\
\mathcal{L}^*_2 q^*+p^*_x=f^*_4-\big(e^{s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-79/2}\big)_t\widehat{q} & \text{in } Q_T,\\
y^*=y^*_x=z^*=p^*=p^*_x=q^*=0 & \text{in } \Sigma_T,\\
y^*(0)=z^*(0)=p^*(T)=q^*(T)=0 & \text{in } (0,1),
\end{cases}
\tag{4.50}
$$

where $h^*_i=e^{s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-79/2}\widehat{h}_i$ for $i=1,2$ and $f^*_j=e^{s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-79/2}f_j$ for $j=1,2,3,4$.

Now, thanks to the choice of f_j in (4.43) for $j=1,2,3,4$, and by virtue of the bounds (4.48), we first observe that the right hand sides of the first two pdes in (4.50) belong to $L^2(Q_T)$. Therefore, by means of Proposition C.1, we have

$$(y^*,z^*)\in\mathcal{C}^0([0,T];[L^2(0,1)]^2)\cap L^2(0,T;H^2_0(0,1)\times H^1_0(0,1)).$$

Consequently, the right hand side of the third equation of (4.50) now belong to $L^2(Q_T)$ and thanks to (4.49), one has $f^*_4-\big(e^{s\mathfrak{S}^*_m}\widehat{\mathfrak{Z}}_m^{-79/2}\big)_t\widehat{q}\in L^2(0,T;H^{-1}(0,1))$. This gives (using Prop. C.1)

$$(p^*,q^*)\in\mathcal{C}^0([0,T];[L^2(0,1)]^2)\cap L^2(0,T;H^2_0(0,1)\times H^1_0(0,1)).$$

Moreover, the following estimate holds true

$$\|y^*\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)} + \|z^*\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)} + \|p^*\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^2)} + \|q^*\|_{\mathcal{C}^0(L^2)\cap L^2(H_0^1)} \leq C, \tag{4.51}$$

for some $C > 0$, thanks to the assumption (4.43) and the fact (4.48).

Thus, the functions $(\widehat{y}, \widehat{z}, \widehat{p}, \widehat{q}, \widehat{h}_1, \widehat{h}_2) \in \mathcal{E}_1$ defined in (4.42). The proof is finished. □

4.2.3. Local null-controllability of the nonlinear system ($\alpha = 1$)

The proof of the main result, that is Theorem 1.1 for the case $\alpha = 1$ is exactly similar as we describe for the case $\alpha = 0$. We refer Section 4.1.3 for the details.

5. Concluding remarks and comments

In this work, we have proved the existence of insensitizing controls for a coupled system of fourth- and second-order parabolic PDEs. As usual, this problem is reformulated as a null-control problem for an extended system in cascade form (see (1.5)–(1.6)) but due to the presence of the parameter α in the sentinel (1.2), this system may change its structure and different Carleman tools should be employed for studying the observability of the corresponding adjoint equations. Let us present a concluding remark concerning the problems addressed in this work.

Less control than equations. An important question related to the controllability of coupled systems is: what is (are) the minimum number of control(s) required to accomplish a given task. In the works [23] and [24], it has been shown that for proving the null-controllability of the system

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + y y_x = z_x & \text{in } Q_T, \\ z_t - z_{xx} + \beta z_x = y_x & \text{in } Q_T, \\ y = y_x = z = 0 & \text{in } \Sigma_T, \\ y(0) = y_0, \quad z(0) = z_0 & \text{in } (0,1), \end{cases} \tag{5.1}$$

it is needed only one control localized in either equation.

In our insensitizing problem we have used two controls, one for each component, but determining if we can reduce its number it is not so clear. As far as we know, there are very few papers devoted to the insensitizing control problems for coupled systems, see [9, 10, 13]. Similar to our case, in those works, the original problem is transformed into a control problem for an extended system of four equations (two forward and two backward in time). In particular, in [13], the authors have used only one control to prove their result by using a sentinel depending (explicitly) only on one of the components of the system. This is comparable to choosing $\alpha = 0$ in our case (1.2). At a first glance, it seems that we can follow the ideas similar to [13] to eliminate the extra control, but in our case, the first order couplings make things difficult. Indeed, recalling our adjoint system (3.9) (*i.e.* the case $\alpha = 0$), we see that the only way to remove an observation related to w is to differentiate the second equation of (3.9) (changing $\mathbb{1}_{\mathcal{O}}$ by some suitable smooth approximation) and use the coupling in the first one. By doing so we can obtain a Carleman estimate with localized terms depending on u, ζ and θ. Then, the only way to estimate θ locally is by using the second equation of (3.9) which reintroduces a local term of w (see Eq. (3.36)), and we failed! Therefore, dealing with cascade systems (forward–backward) of the original coupled systems can be tricky and its controllability and observability properties deserve further attention. This fact has been also pointed out in [29] in the context of hierarchic control problems.

Appendix A. Sketch of the proof of Proposition 1.2

Sketch of the proof. Let (y_τ, z_τ) be the solution to (1.1) for given $\tau \in \mathbb{R}$. It can be checked that $(y_\tau, z_\tau) \to (y, z)$ strongly in $L^2(Q_T)$ as $\tau \to 0$ where we have denoted (y, z) the solution to (1.1) with $\tau = 0$. In turn, this yields that $(u, v) = \lim_{\tau\to 0}(\frac{y_\tau - y}{\tau}, \frac{z_\tau - z}{\tau})$ solves the system

$$\begin{cases} u_t + u_{xxxx} + \gamma u_{xx} + (yu)_x = v_x & \text{in } Q_T, \\ v_t - v_{xx} + \beta v_x = u_x & \text{in } Q_T, \\ u = u_x = v = 0 & \text{in } \Sigma_T, \\ u(0) = \bar{y}_0, \quad v = \bar{z}_0 & \text{in } (0,1). \end{cases} \tag{A.1}$$

So, recalling (1.2), we have

$$\left.\frac{\partial J_\tau(y_\tau, z_\tau)}{\partial \tau}\right|_{\tau=0} = \alpha \iint_{Q_T} yu\mathbb{1}_{\mathcal{O}} + (1-\alpha)\iint_{Q_T} zv\mathbb{1}_{\mathcal{O}}.$$

Now, consider the adjoint system to (A.1) with source terms $\alpha y\mathbb{1}_{\mathcal{O}}$ and $(1-\alpha)z\mathbb{1}_{\mathcal{O}}$, that is

$$\begin{cases} -p_t + p_{xxxx} + \gamma p_{xx} - yp_x = -q_x + \alpha y\mathbb{1}_{\mathcal{O}} & \text{in } Q_T, \\ -q_t - q_{xx} - \beta q_x = -p_x + (1-\alpha)z\mathbb{1}_{\mathcal{O}} & \text{in } Q_T, \\ p = p_x = q = 0 & \text{in } \Sigma_T, \\ p(T) = 0, \quad q(T) = 0 & \text{in } (0,1). \end{cases} \tag{A.2}$$

By a standard duality argument, integrating by parts and using the set of equations (A.1) and (A.2), we get

$$\left.\frac{\partial J_\tau(y_\tau, z_\tau)}{\partial \tau}\right|_{\tau=0} = \int_0^1 p(0)\bar{y}_0 + \int_0^1 q(0)\bar{z}_0.$$

Thus, the insensitizing condition (1.3) is equivalent to the condition

$$\int_0^1 p(0)\bar{y}_0 + \int_0^1 q(0)\bar{z}_0 = 0 \quad \forall (\bar{y}_0, \bar{z}_0) \in [L^2(0,1)]^2 \text{ with } \|(\bar{y}_0, \bar{z}_0)\|_{[L^2(0,1)]^2} = 1.$$

In other words, the insensitizing condition is equivalent to the null-controllability criterion (1.7) for the system (1.5)–(1.6). ☐

Appendix B. An auxiliary result

Lemma B.1. *Recall the weight functions φ_m and $\widehat{\varphi}_m$ defined by (3.2) and (3.10) respectively for $\lambda > 1$ and $k > m > 0$. Then, for any $s > 0$ and $p \in \mathbb{N}^*$, there exists $c_0 > 0$, such that we have*

$$-ps\varphi_m + (p-1)s\widehat{\varphi}_m \leq \frac{-c_0 s}{t^m(T-t)^m}. \tag{B.1}$$

Proof. The proof can be deduced from the explicit expressions of the weight functions. We see

$$\begin{aligned} &- ps\varphi_m + (p-1)s\widehat{\varphi}_m \\ =& - ps\frac{e^{\lambda(1+\frac{1}{m})k\|\nu\|_\infty} - e^{\lambda\left(k\|\nu\|_\infty + \nu(x)\right)}}{t^m(T-t)^m} + (p-1)s\frac{e^{\lambda(1+\frac{1}{m})k\|\nu\|_\infty} - e^{\lambda k\|\nu\|_\infty}}{t^m(T-t)^m} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{s e^{\lambda(1+\frac{1}{m})k\|\nu\|_\infty}}{t^m(T-t)^m} + \frac{s e^{\lambda k\|\nu\|_\infty}}{t^m(T-t)^m}\left(p\left(e^{\lambda\nu(x)}-1\right)+1\right)\\
&= -\frac{s e^{\lambda k\|\nu\|_\infty}}{t^m(T-t)^m}\left(e^{\frac{\lambda}{m}k\|\nu\|_\infty} - p e^{\lambda\nu(x)} + p - 1\right).
\end{aligned}
$$

Thus, for fixed $m>0$ and any $\lambda>0$, one may choose $k>m$ large enough such that there exists some $c_0>0$ verifying that the quantity $\left(e^{\frac{\lambda}{m}k\|\nu\|_\infty} - p e^{\lambda\nu(x)} + p - 1\right) \geq c_0 > 0$ for all $x\in[0,1]$. Hence, the result (B.1) follows. □

Appendix C. Well-posedness of the linear stabilized KS system

Let us consider the following coupled system

$$
\begin{cases}
y_t + y_{xxxx} + \gamma y_{xx} = z_x + f_1 & \text{in } Q_T,\\
z_t - z_{xx} + \beta z_x = y_x + f_2 & \text{in } Q_T,\\
y = y_x = z = 0 & \text{in } \Sigma_T,\\
y(0) = y_0 \quad z(0) = z_0 & \text{in } (0,1),
\end{cases}
\tag{C.1}
$$

where $\gamma>0$ and β is any real number.

Below, we write the standard well-posedness and some regularity result concerning the coupled system (C.1).

Proposition C.1 (Well-posedness & energy estimate)**.** *For any given $(y_0,z_0)\in[L^2(0,1)]^2$ and $(f_1,f_2)\in L^2(0,T;H^{-2}(0,1)\times H^{-1}(0,1))$, there exists unique weak solution*

$$
(y,z)\in[\mathcal{C}^0([0,T];L^2(0,1))]^2\cap L^2(0,T;H^2_0(0,1)\times H^1_0(0,1))\cap H^1(0,T;H^{-2}(0,1)\times H^{-1}(0,1))
$$

to (C.1), such that it satisfies

$$
\begin{aligned}
&\|(y,z)\|_{[\mathcal{C}^0([0,T];L^2(0,1))]^2} + \|(y,z)\|_{L^2(0,T;H^2_0(0,1)\times H^1_0(0,1))} + \|(y_t,z_t)\|_{L^2(0,T;H^{-2}(0,1)\times H^{-1}(0,1))}\\
&\quad\leq C\left(\|(y_0,z_0)\|_{[L^2(0,1)]^2} + \|(f_1,f_2)\|_{L^2(0,T;H^{-2}(0,1)\times H^{-1}(0,1))}\right),
\end{aligned}
\tag{C.2}
$$

for some $C>0$.

Proof. Considering regular enough data, we multiply the first and second equations of (C.1) by y and z respectively, and integrating on $(0,1)$, we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1\left(|y|^2+|z|^2\right) + \int_0^1\left(|y_{xx}|^2+|z_x|^2\right) &\leq \epsilon\int_0^1\left(|y_{xx}|^2+|z_x|^2\right) + \frac{C}{\epsilon}\int_0^1\left(|y|^2+|z|^2\right)\\
&\quad+\epsilon\left(\|y\|^2_{H^2_0(0,1)}+\|z\|^2_{H^1_0(0,1)}\right) + \frac{C}{\epsilon}\left(\|f_1\|^2_{H^{-2}(0,1)}+\|f_2\|^2_{H^{-1}(0,1)}\right).
\end{aligned}
$$

Using Poincaré inequality and by choosing $\epsilon>0$ small enough, we further obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left(|y|^2+|z|^2\right) + \|y\|^2_{H^2_0(0,1)} + \|z\|^2_{H^1_0(0,1)}\\
&\leq C\int_0^1\left(|y|^2+|z|^2\right) + C\left(\|f_1\|^2_{H^{-2}(0,1)}+\|f_2\|^2_{H^{-1}(0,1)}\right).
\end{aligned}
\tag{C.3}
$$

Using Grönwall's lemma, we then deduce that

$$\|(y,z)\|_{[L^\infty(0,T;L^2(0,1))]^2} \leq C\left(\|(y_0,z_0)\|_{[L^2(0,1)]^2} + \|(f_1,f_2)\|_{L^2(0,T;H^{-2}(0,1)\times H^{-1}(0,1))}\right). \tag{C.4}$$

Next, by integrating (C.3) w.r.t. $t \in (0,T)$ and using (C.3), we get

$$\|(y,z)\|_{L^2(0,T;H_0^2(0,1)\times H_0^1(0,1))} \leq C\left(\|(y_0,z_0)\|_{[L^2(0,1)]^2} + \|(f_1,f_2)\|_{L^2(0,T;H^{-2}(0,1)\times H^{-1}(0,1))}\right). \tag{C.5}$$

Once we get the information $y \in L^2(0,T;H_0^2(0,1))$ and $z \in L^2(0,T;H_0^1(0,1))$, from the equations (C.1), we have $y_t \in L^2(0,T;H^{-2}(0,1))$ and $z_t \in L^2(0,T;H^{-1}(0,1))$, together with the estimate:

$$\|(y_t,z_t)\|_{L^2(0,T;H^{-2}(0,1)\times H^{-1}(0,1))} \leq C\left(\|(y_0,z_0)\|_{[L^2(0,1)]^2} + \|(f_1,f_2)\|_{L^2(0,T;H^{-2}(0,1)\times H^{-1}(0,1))}\right). \tag{C.6}$$

Then, from the classical properties of these spaces (see for instance [30], Thm. 4, Chap. 5.9), we have $(y,z) \in [\mathcal{C}^0([0,T];L^2(0,1))]^2$. Accordingly, the estimate (C.2) holds by means of (C.4), (C.5) and (C.6).

Finally, by using the density argument, we can prove the estimate (C.2) for any data as given in the hypothesis of this proposition. □

Remark C.2. Considering $(f_1,f_2) \in [L^2(Q_T)]^2$ in (C.1), the solution (y,z) to (C.1) satisfies (performing the similar analysis as described in the proof of Proposition C.1)

$$\begin{aligned}
&\|(y,z)\|_{[\mathcal{C}^0([0,T];L^2(0,1))]^2} + \|(y,z)\|_{L^2(0,T;H_0^2(0,1)\times H_0^1(0,1))} + \|(y_t,z_t)\|_{L^2(0,T;H^{-2}(0,1)\times H^{-1}(0,1))} \\
&\quad \leq C\left(\|(y_0,z_0)\|_{[L^2(0,1)]^2} + \|(f_1,f_2)\|_{[L^2(Q_T)]^2}\right).
\end{aligned}$$

Proposition C.3 (Regularity)**.** *For any given $(y_0,z_0) \in H_0^2(0,1)\times H_0^1(0,1)$ and $(f_1,f_2) \in [L^2(Q_T)]^2$, the solution*

$$(y,z) \in \mathcal{C}^0([0,T];H_0^2(0,1)\times H_0^1(0,1)) \cap L^2(0,T;H^4(0,1)\times H^2(0,1)) \cap [H^1(0,T;L^2(0,1))]^2$$

to (C.1) satisfies

$$\begin{aligned}
&\|(y,z)\|_{\mathcal{C}^0([0,T];H_0^2(0,1)\times H_0^1(0,1))} + \|(y,z)\|_{L^2(0,T;H^4(0,1)\times H^2(0,1))} + \|(y_t,z_t)\|_{[L^2(Q_T)]^2} \\
&\quad \leq C\Big(\|(y_0,z_0)\|_{H_0^2(0,1)\times H_0^1(0,1)} + \|(f_1,f_2)\|_{[L^2(Q_T)]^2}\Big),
\end{aligned} \tag{C.7}$$

for some $C>0$.

A sketch of the proof for above regularity result can be found in [23], Proposition 2.1. For sake of completeness, we give the proof below.

Proof. Considering regular enough data, we multiply the equations of (y,z) by (y_{xxxx}, z_{xx}) and integrating we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(|y_{xx}|^2 + |z_x|^2\right) + \int_0^1 \left(|y_{xxxx}|^2 + |z_{xx}|^2\right) &\leq \epsilon\int_0^1 \left(|y_{xxxx}|^2 + |z_{xx}|^2\right) \\
&+ \frac{C}{\epsilon}\int_0^1 \left(|z_x|^2 + |y_{xx}|^2\right) + \frac{C}{\epsilon}\int_0^1 \left(|y_x|^2 + |f_1|^2 + |f_2|^2\right).
\end{aligned}$$

Choosing $\epsilon > 0$ small enough, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(|y_{xx}|^2 + |z_x|^2\right) + \int_0^1 \left(|y_{xxxx}|^2 + |z_{xx}|^2\right) \\ &\leq C\int_0^1 \left(|z_x|^2 + |y_{xx}|^2\right) + C\int_0^1 \left(|y_x|^2 + |f_1|^2 + |f_2|^2\right).\end{aligned} \tag{C.8}$$

Applying Grönwall's lemma to the above inequality and by using the fact that $y_x \in L^2(Q_T)$ (at least) from the well-posedness result in Proposition C.1, we get

$$\|(y,z)\|_{L^\infty(0,T;H_0^2(0,1)\times H_0^1(0,1))} \leq C\left(\|(y_0,z_0)\|_{H_0^2(0,1)\times H_0^1(0,1)} + \|(f_1,f_2)\|_{[L^2(Q_T)]^2}\right). \tag{C.9}$$

Once we have (C.9), by integrating the inequality (C.8) in $(0,T)$, we get

$$\|(y,z)\|_{L^2(0,T;H^4(0,1)\times H^2(0,1))} \leq C\left(\|(y_0,z_0)\|_{H_0^2(0,1)\times H_0^1(0,1)} + \|(f_1,f_2)\|_{[L^2(Q_T)]^2}\right). \tag{C.10}$$

Then, from the equations of y and z, we have $y_t \in L^2(Q_T)$ and $z \in L^2(Q_T)$ with the desired estimate as in (C.7) (due to (C.9) and (C.10)). Moreover, we have $y \in \mathcal{C}^0([0,T];H_0^2(0,1))$ and $z \in \mathcal{C}^0([0,T];H_0^1(0,1))$ (using [30], Thm. 4, Chap. 5.9).

The proof is finished. □

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