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THE DICHOTOMY PROPERTY IN STABILIZABILITY OF 2×2 LINEAR HYPERBOLIC SYSTEMS

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Abstract. This paper is devoted to discuss the stabilizability of a class of 2×2 non-homogeneous hyperbolic systems. Motivated by the example in the Section 5.6 of Bastin and Coron's book in 2016, we analyze the influence of the interval length L on stabilizability of the system. By spectral analysis, we prove that either the system is stabilizable for all $L > 0$ or it possesses the dichotomy property: there exists a critical length $L_c > 0$ such that the system is stabilizable for $L \in (0, L_c)$ but unstabilizable for $L \in [L_c, +\infty)$. In addition, for $L \in [L_c, +\infty)$, we obtain that the system can reach equilibrium state in finite time by backstepping control combined with observer. Finally, we also provide some numerical simulations to confirm our developed analytical criteria.

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1. Introduction and main result

Hyperbolic systems play a crucial role in representing physical phenomena and possess both theoretical and practical significance. Extensive research has focused on well-posedness and control problems, including the stability and stabilization of these systems. In particular, researchers have studied the exponential stability or stabilization of hyperbolic systems without source terms in both linear and nonlinear cases, under various boundary controls such as Proportional-Integral control and backstepping control [1]. Previous works by [2–7] have also investigated this topic. However, most physical equations, such as the Saint-Venant equations (see Chap. 5 of [8]), Euler equations (see [9] or [10]), and Telegrapher equations, cannot neglect the source term. Therefore, it is crucial to investigate the dynamics of hyperbolic systems with source terms.

Two main approaches have been used to achieve asymptotic stability of hyperbolic systems: analyzing the evolution of the solution along characteristic curves, as extensively studied in previous works such as [11–15]; and relying on a Lyapunov function approach, as thoroughly researched in [3, 4, 8, 16–21]. Both of these approaches are concerned with obtaining stability for the system.

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Another strategy that has been studied is the Backstepping method, which aims to design a control law that achieves stabilization. The Backstepping method has been applied in [1, 22] and typically requires full-state feedback control. However, it is possible to achieve boundary state feedback through backstepping control by designing an appropriate observer, as demonstrated in [23, 24].

In [25], p. 197, Bastin and Coron mention that for some systems of balance laws, there is an intrinsic limit of stabilization under local boundary control. It is proved that the following system

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + c y_2 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ \partial_t y_2 - \partial_x y_2 + c y_1 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ y_2(t,L) = y_1(t,L), & t \in (0,+\infty), \\ y_1(t,0) = k y_2(t,0), & t \in (0,+\infty). \end{cases} \tag{1.1}$$

cannot be stabilized for any $k \in \mathbb{R}$ if $L \in (\frac{\pi}{c}, +\infty)$. On the other hand, the system (1.1) is stabilizable if $L \in (0, \frac{\pi}{2c})$ from [26]. However, there remains a gap on L between the stabilizable and unstabilizable cases.

In [27], Gugat and Gerster analyze the limit of stabilizability for a network of hyperbolic systems. Remarkably, their results reveal that under certain conditions, the system may be inherently unstabilizable.

These results inspire us to investigate whether the stabilizability of the system (1.1) possesses the dichotomy property on L. Here, the dichotomy property on L can be described as follows: there exists a critical value $L_c > 0$ such that:

- While $L \geq L_c$, the system is not stabilizable, *i.e.* the system cannot be exponentially stable for any discussed control.
- While $0 < L < L_c$, the system is stabilizable, *i.e.* there exists certain control such that the system is exponentially stable.

In this paper we discuss the boundary feedback stabilization of the following 2×2 linear hyperbolic systems over a bounded interval $[0, L]$:

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + a y_2 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ \partial_t y_2 - \lambda \partial_x y_2 + b y_1 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ y_2(t,L) = y_1(t,L), & t \in (0,+\infty), \\ y_1(t,0) = u(t), & t \in (0,+\infty), \\ \big(y_1(0,x), y_2(0,x)\big) = \big(y_1^0(x), y_2^0(x)\big), & x \in (0,L). \end{cases} \tag{1.2}$$

where $\lambda > 0$ and $a, b \in \mathbb{R}$ are given constants, $y_1^0, y_2^0 \in L^2(0,L)$ are the initial data.

The boundary feedback law takes the proportional form

$$u(t) = k y_2(t,0), \tag{1.3}$$

where $k \in \mathbb{R}$ is the tuning parameter and $y_2(t,0)$ is the output measurement.

We are concerned about the exponential stability of the closed-loop system (1.2).

Definition 1.1. The linear hyperbolic system (1.2) (1.3) is said to be L^2 exponentially stable if there exists $C > 0$ and $\alpha > 0$ such that, for every $(y_1^0(x), y_2^0(x)) \in L^2(0,L) \times L^2(0,L)$ the solution to the system (1.2) (1.3) satisfies:

$$\Big\| \big(y_1(t,\cdot), y_2(t,\cdot)\big) \Big\|_{L^2(0,L)} \leq C \mathrm{e}^{-\alpha t} \big\| (y_1^0, y_2^0) \big\|_{L^2(0,L)}, \ t \geq 0. \tag{1.4}$$

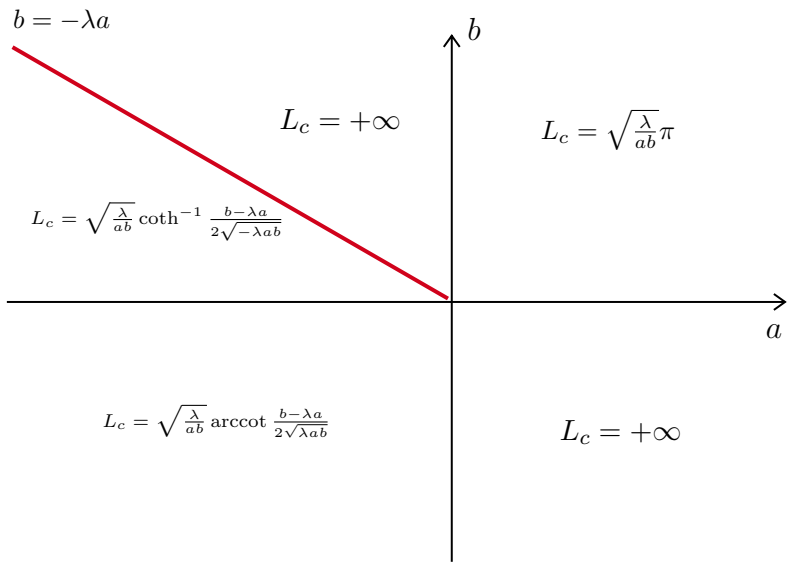


FIGURE 1. The expression of L_c.

Definition 1.2. The linear hyperbolic system (1.2) (1.3) is said to be stabilizable if there exists $k \in \mathbb{R}$ such that (1.2) (1.3) is L^2 exponentially stable.

In this paper, we propose a method for finding the critical value L_c, given fixed parameters a, b, λ. Our approach is based on spectral analysis. For values of $|k| \geq 1$, we demonstrate that the closed-loop system (1.2) (1.3) is not exponentially stable by identifying unstable eigenvalues, specifically those located on the right half of the complex plane. To achieve this, we approximate the characteristic function at infinity and use Rouché's Theorem to obtain the roots. In the case of $|k| < 1$, we introduce the function $N_{a,b,\lambda}(k, L)$ to represent the degree of the characteristic function, see (3.16), on the right side of the complex plane. As stated in Lemma 3.2, $N_{a,b,\lambda}$ remains constant within each block that is separated by marginal curves determined by $\mathcal{A}_{a,b,\lambda}$. Furthermore, $N_{a,b,\lambda} \equiv 0$ within the block at the bottom. By applying Lemma 3.3, we show that $N_{a,b,\lambda}$ increases by 1 when (k, L) moves from one block to another block above it. Therefore, we conclude that the stability region is the block at the bottom. Finally, we determine the critical value L_c using the analytical results obtained from the marginal curves determined by $\mathcal{A}_{a,b,\lambda}$.

Theorem 1.3. *Let $\lambda > 0$ be fixed. Then, either the system (1.2) (1.3) is stabilizable for all $L > 0$ or it possesses the dichotomy property. More precisely, the expression of L_c in terms of $a, b \in \mathbb{R}$ is explicitly given as follows (see Fig. 1):*

$$L_c = \begin{cases} \sqrt{\frac{\lambda}{ab}}\pi, & \text{if } a > 0, b > 0. \\ \sqrt{\frac{\lambda}{ab}} \operatorname{arccot}\left(\frac{b-\lambda a}{2\sqrt{\lambda ab}}\right), & \text{if } a < 0, b < 0. \\ \sqrt{\frac{-\lambda}{ab}} \coth^{-1}(\frac{b-\lambda a}{2\sqrt{-\lambda ab}}), & \text{if } -\lambda a > b > 0. \\ -\frac{2}{a}, & \text{if } b = 0, a < 0. \\ +\infty, & \text{if else.} \end{cases} \tag{1.5}$$

Here $L_c \triangleq +\infty$ means that the system is stabilizable for all $L > 0$.

Remark 1.4. By setting $\lambda = 1$ and $a = b = c > 0$ in Theorem 1.3, we obtain $L_c = \frac{\pi}{c}$. This implies that the system (1.1) is stabilizable for $L \in (0, \frac{\pi}{c})$, but not stabilizable for $L \in [\frac{\pi}{c}, +\infty)$. The result presented in this paper bridges the gap between the stabilizable region $(0, \frac{\pi}{2c})$ (established in [26]) and the unstabilizable region $[\frac{\pi}{c}, +\infty)$ (demonstrated in [25]) for the system (1.1).

Remark 1.5. While L is sufficiently small, we can establish a Lyapunov function to demonstrate that the system (1.2) (1.3) is exponentially stable. For instance, if $ab > 0$, $|k| < \varepsilon < 1$, we define

$$V(y_1, y_2) \triangleq \int_0^L \left(\frac{y_1^2(t,x)}{\eta(x)} + \frac{\eta(x) y_2^2(t,x)}{\lambda} \right) \mathrm{d}x,$$

where $\eta(x)$ satisfies $\eta'(x) = (1+\varepsilon)(|a| + \frac{|b|}{\lambda}\eta^2)$, $\eta_\varepsilon(0) = \varepsilon$, *i.e.*

$$\eta(x) = \sqrt{\frac{\lambda a}{b}} \tan\big(\sqrt{\frac{ab}{\lambda}}(1+\varepsilon)x + \arctan\sqrt{\frac{b}{\lambda a}}\varepsilon\big).$$

Therefore, if $L < L_\varepsilon \triangleq \frac{1}{1+\varepsilon}\sqrt{\frac{\lambda}{ab}}\big(\arctan\sqrt{\frac{b}{\lambda a}} - \arctan\sqrt{\frac{b}{\lambda a}}\varepsilon\big)$, we verify that $V(y_1, y_2)$ is a Lyapunov function and the corresponding system is exponentially stable. Specially, for the case when $a = b \neq 0$, we can select a Lyapunov function for $L < \dfrac{\pi}{4|a|}$ when we choose $|k|, \varepsilon$ efficiently small. Using the method of spectral analysis (refer to Thm. 1.3), we show that the system (1.2) (1.3) is stabilizable for $L < \dfrac{\pi}{a}$ when $a = b > 0$, and $L < \dfrac{\pi}{2|a|}$ when $a = b < 0$. Therefore, the stabilizable region we obtain from Lyapunov approach is smaller than we obtain from the spectral analysis. This shows the superiority of spectral analysis over the use of Lyapunov functions for stability analysis in this particular case.

Remark 1.6. The general hyperbolic system with the rightward speed $\lambda_1 > 0$ and leftward speed $\lambda_2 > 0$

$$\begin{cases} \partial_t y_1 + \lambda_1 \partial_x y_1 + a y_2 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ \partial_t y_2 - \lambda_2 \partial_x y_2 + b y_1 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ y_2(t,L) = y_1(t,L), & t \in (0,+\infty), \\ y_1(t,0) = k y_2(t,0), & t \in (0,+\infty). \end{cases} \tag{1.6}$$

can be reduced, through a scaling of the space variable $x \to \lambda_1 x$, to a system with rightward speed 1 and leftward speed $\lambda_1 > 0$ in the form of (1.2). Thus, Theorem 1.3 can be extended to the general system (1.6).

According to Theorem 1.3, the proportional feedback control (1.3) cannot stabilize the system (1.2) for $L > L_c$. Therefore, an alternative control approach is worth exploring in this case. Building upon the works of Hu *et al.* [1, 22] and Holta *et al.* [23], we develop a Backstepping control combined with observer design that stabilizes the system even when $L \geq L_c$, and without the need to observe the full state. Notably, the proposed control law drives the system to its zero equilibrium in finite time. More details are presented in Section 5.

The main contribution of this paper can be summarized in three aspects: 1) we provide a complete characterization of the stabilizability of the hyperbolic system (1.2) under proportional feedback control (1.3) for all cases; 2) we show that the stabilizability of the system exhibits a dichotomy property on the interval L, indicating a clear boundary between the stabilizable and non-stabilizable regions; 3) we propose a new control method that combines backstepping control with observer design to stabilize the system when the proportional control fails.

The organization of this paper is as follows. In Section 2, we provide some preliminaries including Spectral Mapping Property and Implicit Function Theorem, which will be used in the following Sections. In Section 3, we provide the proof of Theorem 1.3. In Section 4, we provide some numerical simulations to confirm our developed

analytical criteria in Section 3. In Section 5, We give a sketch of the construction of the Backstepping control with observer design for the case of $L \geq L_c$.

Notations. In this paper, we use standard notation and terminology. Specifically, $\mathbb{C}_+$, $\mathbb{C}_-$, and $\mathbb{C}_0$ denote the sets of complex numbers with positive real parts, negative real parts, and zero real parts, respectively. We use $\overline{\mathbb{C}_+}$ to denote the set $\mathbb{C}_+ \cup \mathbb{C}_0$. The sets of integers, positive integers, and non-negative integers are denoted by $\mathbb{Z}$, $\mathbb{N}^*$, and $\mathbb{N}$, respectively. The imaginary unit is denoted by i such that $\mathrm{i} = \sqrt{-1}$. For $\sigma \in \mathbb{C}$, we use $\mathrm{Re}\sigma$, $\mathrm{Im}\sigma$, $\arg\sigma$, and $|\sigma|$ to denote the real part, imaginary part, principal value of argument, and norm of σ, respectively. For an analytic function f, an open subset $\Omega \subseteq \mathbb{C}$ and $b \in \mathbb{C}$, $\deg(f, \Omega, b)$ denotes the number of roots in Ω of the equation $f(z) = b$, counted by multiplicity.

2. Preliminaries

Applying the results in Lichtner [28], we have the following lemmas:

Lemma 2.1. *Let $S(t)(t \geq 0)$ be the C^0 semigroup on $L^2(0, L)$ that corresponds to the solution map of (1.2) (1.3), and let $\mathscr{A}$ be the infinitesimal generator of the semigroup $S(t)(t \geq 0)$. Let us denote by $\sigma_p(\mathscr{A})$ and $\sigma(\mathscr{A})$ the point spectrum and the spectrum of $\mathscr{A}$, respectively. Then, $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A})$. Moreover, $\mathscr{A}$ has the Spectral Mapping Property (SMP), that is*

$$\text{(SMP)} \quad \sigma(\mathrm{e}^{\mathscr{A}t})\backslash\{0\} = \overline{\mathrm{e}^{\sigma(\mathscr{A})t}}\backslash\{0\}, \quad \text{for} \quad t \geq 0.$$

Hence (SMP) contains spectrum determined growth

$$\omega(\mathscr{A}) = s(\mathscr{A}),$$

with $\omega(\mathscr{A}) \triangleq \inf\{\omega \in \mathbb{R} \,|\, \exists\, M = M(\omega) : \|S(t)\| \leq M\mathrm{e}^{\omega t}, \forall\, t \geq 0\}$, $s(\mathscr{A}) \triangleq \sup\{\mathrm{Re}(\mu) \,|\, \mu \in \sigma(\mathscr{A})\}$

From Lemma 2.1, we obtain the following proposition:

Proposition 2.2. *The system (1.2) (1.3) is not exponentially stable if and only if $s(\mathscr{A}) \geq 0$.*

We will apply the analytic implicit function theorem in the proof of Lemma 3.3. The Implicit Function Theorem from [29] is stated following:

Lemma 2.3. *Let $\mathcal{B} \subset \mathbb{C}^n \times \mathbb{C}^m$ be an open set, $f = (f_1, ..., f_m) : \mathcal{B} \to \mathbb{C}^m$ a holomorphic mapping, and $(z_0, w_0) \in \mathcal{B}$ a point with $f(z_0, w_0) = 0$ and*

$$\det\left(\frac{\partial f_\mu}{\partial z_\mu}(z_0, w_0)\middle|\begin{matrix}\mu = 1, ..., m \\ \nu = n+1, ..., n+m\end{matrix}\right) \neq 0.$$

Then there is an open neighborhood $U = U' \times U'' \subset \mathcal{B}$ and a holomorphic map $g : U' \to U''$ such that

$$\{(z, w) \in U' \times U'' : f(z, w) = 0\} = \{(z, g(z)) : z \in U'\}.$$

3. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. We first establish the characteristic equation.

Let $\sigma \in \mathbb{C}$ be the eigenvalue of the system, we look for a nontrivial solution $(y_1, y_2)^{\mathrm{T}}$ of the system with the form:

$$\begin{cases} y_1(t,x) = \mathrm{e}^{\sigma t} f(x), & (t,x) \in [0,+\infty) \times (0,L), \\ y_2(t,x) = \mathrm{e}^{\sigma t} g(x), & (t,x) \in [0,+\infty) \times (0,L). \end{cases} \tag{3.1}$$

$f(x), g(x)$ are the corresponding eigenfunctions of the $y_1(t,x), y_2(t,x)$. Such a $(y_1, y_2)^{\mathrm{T}}$ is a solution of the system if and only if

$$\sigma f + \partial_x f + ag = 0, \tag{3.2}$$
$$\sigma g - \lambda \partial_x g + bf = 0, \tag{3.3}$$
$$f(L) = g(L), \tag{3.4}$$
$$f(0) = kg(0). \tag{3.5}$$

From equations (3.2) and (3.3), we have:

$$\lambda \partial_{xx} f + (\lambda - 1)\sigma \partial_x f + (ab - \sigma^2) f = 0.$$

We divide into two classes $\sigma^2 \neq \frac{4\lambda ab}{(\lambda+1)^2}$ and $\sigma^2 = \frac{4\lambda ab}{(\lambda+1)^2}$.
For the first case, $\sigma^2 \neq \frac{4\lambda ab}{(\lambda+1)^2}$, thus all eigenvalues are single. We obtain

$$f = \mathrm{e}^{\xi x}(A\mathrm{e}^{\eta x} + B\mathrm{e}^{-\eta x}), \tag{3.6}$$

$$g = -\frac{1}{a}\mathrm{e}^{\xi x}\big(A(\sigma + \xi + \eta)\mathrm{e}^{\eta x} + B(\sigma + \xi - \eta)\mathrm{e}^{-\eta x}\big), \tag{3.7}$$

where

$$\xi = \frac{-(\lambda - 1)}{2\lambda}\sigma, \ \eta^2 = \frac{(\lambda+1)^2\sigma^2 - 4\lambda ab}{4\lambda^2} \neq 0, \ A, B \in \mathbb{C}.$$

Equations (3.4) and (3.5) yield

$$A\big(a + k(\sigma + \xi + \eta)\big) + B\big(a + k(\sigma + \xi - \eta)\big) = 0, \tag{3.8}$$
$$A\big(a + \sigma + \xi + \eta\big)\mathrm{e}^{\eta L} + B\big(a + \sigma + \xi - \eta\big)\mathrm{e}^{-\eta L} = 0. \tag{3.9}$$

By computing the characteristic determinant of equation (3.8) and equation (3.9), we obtain that there exists $(A, B) \in \mathbb{C}^2 \backslash \{(0,0)\}$ such that equation (3.8) and equation (3.9) hold if and only if

$$(k-1)\cosh(\eta L) - \big((k+1)\frac{\lambda+1}{2\lambda}\sigma + (\frac{kb}{\lambda} + a)\big)\frac{\sinh(\eta L)}{\eta} = 0. \tag{3.10}$$

For the second case, $\sigma^2 = \frac{4\lambda ab}{(\lambda+1)^2}$. In this case, we have

$$f = \mathrm{e}^{\xi x}(Ax + B), \tag{3.11}$$

$$g = -\frac{1}{a}\mathrm{e}^{\xi x}\big(A(\xi + \sigma)x + (A + \xi B + \sigma B)\big). \tag{3.12}$$

Then equations (3.4) and (3.5) yield

$$A\big(aL + \sigma L + \xi L\big) + B\big(a + \sigma + \xi\big) = 0, \tag{3.13}$$

$$Ak + B\big(a + k\sigma + k\xi\big) = 0. \tag{3.14}$$

By computing the characteristic determinant of equations (3.13) and (3.14), we obtain that there exists $(A, B) \in \mathbb{C}^2 \backslash \{(0, 0)\}$ such that equation (3.13) equation (3.14) hold if and only if

$$(k-1) - (k\frac{\lambda+1}{2\lambda}\sigma + a)\big(1 + \frac{(\lambda+1)\sigma}{2\lambda a}\big)L = 0, \tag{3.15}$$

which can be included in equation (3.15) for $\eta = 0$ if we define $\frac{\sinh(\eta L)}{\eta} \triangleq L$ for $\eta = 0$.

Note that $\cosh(\eta L)$ and $\dfrac{\sinh(\eta L)}{\eta}$ are analytical functions with respect to η^2, which further implies that the left side of equation (3.10) is analytic with respect to σ. Denote by

$$F_{a,b,\lambda,k,L}(\sigma) \triangleq (k-1)\cosh(\eta L) - \big[(k+1)\frac{\lambda+1}{2\lambda}\sigma + (\frac{kb}{\lambda} + a)\big]\frac{\sinh(\eta L)}{\eta}, \tag{3.16}$$

with

$$\eta^2 = \frac{(\lambda+1)^2\sigma^2 - 4\lambda ab}{4\lambda^2},$$

and

$$\mathcal{U}_F \triangleq \Big\{\sigma \Big| F_{a,b,\lambda,k,L}(\sigma) = 0\Big\}.$$

The set of the roots of $F_{a,b,\lambda,k,L}(\sigma)$ satisfy

$$\mathcal{U}_F = \sigma(\mathscr{A}) = \sigma_p(\mathscr{A}).$$

By Lemma 2.1, we obtain that a sufficient and necessary condition for the stability of the system (1.2) (1.3):

$$\mathcal{U}_F \cap \mathbb{C}_+ \cap \mathbb{C}_0 = \emptyset. \tag{3.17}$$

Denote by

$$N_{a,b,\lambda}(k, L) \triangleq \deg(F_{a,b,\lambda,k,L}, \mathbb{C}_+, 0).$$

Our goal is to establish the regions for parameter a, b, λ, k, L such that $N_{a,b,\lambda}(k, L) = 0$. To achieve this, we fix a, b, λ and figure out the stability region for k, L on $\mathbb{R} \times [0, +\infty)$. We divide into two cases $|k| \geq 1$ and $|k| < 1$.

3.1. $|k| \geq 1$

Lemma 3.1. *For every $a, b, \lambda \in \mathbb{R}$ and $L > 0$, if $|k| \geq 1$, the system (1.2)(1.3) is non-exponentially stable.*

Proof of Lemma 3.1. Obviously, by Proposition 2.2 we just need to prove

$$\forall L \in \mathbb{R}, |k| \geq 1, \text{ the system (1.2) (1.3) has } s(\mathscr{A}) \geq 0.$$

We analyze the solution for the characteristic equation (3.10) in various cases:

Case $k=1$

While $k=1$, equation (3.10) yields the following equation with $\eta^2=\frac{(\lambda+1)^2\sigma^2-4\lambda ab}{4\lambda^2}$:

$$\big((\lambda+1)\sigma+(b+\lambda a)\big)(\mathrm{e}^{2\eta L}-1)=0. \tag{3.18}$$

The solutions of equation (3.18) include

$$\sigma_n=\frac{2\lambda}{(\lambda+1)}\sqrt{\frac{ab}{\lambda}-\frac{n^2\pi^2}{L^2}}\quad (n\in\mathbb{N}),$$

which are located on the imaginary axis for sufficiently large $n\in\mathbb{N}^*$.

Case $k=-1$

While $k=-1$, equation (3.10) yields the following equation with $\eta^2=\frac{(\lambda+1)^2\sigma^2-4\lambda ab}{4\lambda^2}$:

$$-2\eta(\mathrm{e}^{2\eta L}+1)=(a-\frac{b}{\lambda})(\mathrm{e}^{2\eta L}-1).$$

Considering the solutions on the imaginary axis $\eta=\alpha\mathrm{i}(\alpha\neq 0,\alpha\in\mathbb{R})$, we obtain

$$\cot\alpha L=\frac{a-\frac{b}{\lambda}}{2}\alpha,$$

which has infinitely many solutions. This implies that $F_{a,b,\lambda,k,L}(\sigma)$ has infinitely many roots on the imaginary axis.

Case $|k|>1$

Denote by $Q(\sigma)$ a single-valued branch that satisfy $4\lambda^2Q(\sigma)^2=(\lambda+1)^2\sigma^2-4\lambda ab$, and

$$\lim_{|\sigma|\to+\infty}\frac{Q(\sigma)}{\sigma}=\frac{\lambda+1}{2\lambda}.$$

By choosing $\eta=Q(\sigma)$ in equation (3.10) and multiplying $2\mathrm{e}^{\eta L}$ by both sides of it, we obtain

$$H(\sigma)\triangleq(k-1)(1+\mathrm{e}^{-2Q(\sigma)L})-\left[(k+1)\frac{\lambda+1}{2\lambda}\frac{\sigma}{Q(\sigma)}+\frac{kb+\lambda a}{Q(\sigma)}\right](1-\mathrm{e}^{-2Q(\sigma)L})=0.$$

This enlightens us to estimate $H(\sigma)$ as

$$G(\sigma)\triangleq(k-1)(1+\mathrm{e}^{-\frac{(\lambda+1)\sigma}{\lambda}L})-(k+1)(1-\mathrm{e}^{-\frac{(\lambda+1)\sigma}{\lambda}L}),$$

with $|\sigma|\to+\infty$, which vanishes at infinitely many points in $\mathbb{C}_+$ reading

$$\sigma_{k,n}=\frac{\lambda}{(\lambda+1)L}(\ln|k|+2\hat{n}\pi i),\ \hat{n}=\begin{cases}n, & k\in(1,+\infty),\\ n+\frac{1}{2}, & k\in(-\infty,-1).\end{cases}$$

Now we estimate $G(\sigma)-H(\sigma)$ as $|\sigma|\to+\infty,\sigma\in\mathbb{C}_+$.

$$\begin{aligned}&G(\sigma)-H(\sigma)\\=&2k\big(\mathrm{e}^{-\frac{(\lambda+1)\sigma}{\lambda}L}-\mathrm{e}^{-2Q(\sigma)L}\big)+\frac{kb+\lambda a}{Q(\sigma)}(1-\mathrm{e}^{-2Q(\sigma)L})+(k+1)\bigg(\frac{\lambda+1}{2\lambda}\frac{\sigma}{Q(\sigma)}-1\bigg)(1-\mathrm{e}^{-2Q(\sigma)L}).\end{aligned}$$

Notice that:

$$\lim_{|\sigma|\to+\infty,\sigma\in\mathbb{C}_+}\Big|2Q(\sigma)-\frac{(\lambda+1)\sigma}{\lambda}\Big|L=\Big|\frac{4ab}{2\lambda Q(\sigma)+(\lambda+1)\sigma}\Big|L=0.$$

We obtain

$$\limsup_{|\sigma|\to+\infty,\sigma\in\mathbb{C}_+}\big|\mathrm{e}^{-\frac{(\lambda+1)\sigma}{\lambda}L}-\mathrm{e}^{-2Q(\sigma)L}\big|=\big|\mathrm{e}^{-\frac{(\lambda+1)\sigma}{\lambda}L}\big|\cdot\big|1-\mathrm{e}^{\left(-2Q(\sigma)+\frac{(\lambda+1)\sigma}{\lambda}\right)L}\big|=0.$$

Furthermore,

$$|\mathrm{e}^{-2Q(\sigma)L}|=\mathrm{e}^{-2\mathrm{Re}(Q(\sigma))L}=(\mathrm{e}^{-2\mathrm{Re}(\sigma)L})^{\frac{\mathrm{Re}(Q(\sigma))}{\mathrm{Re}(\sigma)}},$$

which implies that

$$\limsup_{|\sigma|\to+\infty,\sigma\in\mathbb{C}_+}|\mathrm{e}^{-2Q(\sigma)L}|\le 1.$$

Then we have

$$\lim_{|\sigma|\to+\infty,\sigma\in\mathbb{C}_+}|G(\sigma)-H(\sigma)|=0. \tag{3.19}$$

We apply Rouché's Theorem for analytic functions to show that $H(\sigma)$ has infinitely many roots in $\mathbb{C}_+$.
At $\sigma_{k,n}$, let $\varepsilon>0$ small enough such that

$$\Omega^{\varepsilon}_{k,n}\triangleq\big\{\sigma\in\mathbb{C}\big||G(\sigma)|<\varepsilon,|\sigma-\sigma_{k,n}|<1\big\}\subset\mathbb{C}_+.$$

Notice that $\Omega^{\varepsilon}_{k,n}$ is a bounded open set of $\mathbb{C}_+$ and $\deg(G,\Omega^{\varepsilon}_{k,n},0)=1$. For the definition of deg, please refer to **Notations**.
If $\varepsilon>0$ is small enough, we have

$$\partial\Omega^{\varepsilon}_{k,n}\subset\big\{\sigma\in\mathbb{C}\big||G(\sigma)|=\varepsilon\big\},\quad \Omega^{\varepsilon}_{k,n}\subset\mathbb{C}_+,\quad \forall n\in\mathbb{Z}.$$

For all $\sigma\in\partial\Omega^{\varepsilon}_{k,n}$, from equation (3.19), there exists $N>0$, such that for all $n>N$, we have

$$|G(\sigma)-H(\sigma)|<\varepsilon=|G(\sigma)|,\quad \forall\sigma\in\partial\Omega^{\varepsilon}_{k,n}.$$

Applying Rouché's Theorem, we obtain

$$\deg(H,\Omega^{\varepsilon}_{k,n},0)=\deg(G,\Omega^{\varepsilon}_{k,n},0)=1,\quad \forall n>N.$$

Thus, there exists N such that while $n>N$, $F(\sigma)$ vanishes at $\hat{\sigma}_{k,n}\in\Omega^{\varepsilon}_{k,n}\subset\mathbb{C}_+$. For $|k|>1$, the spectrum of the solution $s(\mathscr{A})$ satisfies $s(\mathscr{A})>0$.

These analysis for three cases complete the proof of Lemma 3.1. □

3.2. $|k| < 1$

It follows from Lemma 3.1 that when $|k| \geq 1$, the system (1.2)(1.3) is non-exponentially stable. In the following, we investigate the situation when $|k| < 1$ by spectral analysis. Denote by

$$\mathcal{A}_{a,b,\lambda} \triangleq \big\{(k, L)||k| < 1, L \geq 0, \text{ there exists } \beta \in \mathbb{R} \text{ such that } F_{a,b,\lambda,k,L}(\mathrm{i}\beta) = 0\big\}.$$

Lemma 3.2. *For any continuous path* $(k(t), L(t))_{t\in[0,1]} \subset (-1, 1) \times [0, +\infty)$*, if*

$$\big\{(k(t), L(t))|t \in [0, 1]\big\} \cap \mathcal{A}_{a,b,\lambda} = \emptyset,$$

then $N_{a,b,\lambda}(k(t), L(t))$ *is a constant on* $[0, 1]$.

Proof of Lemma 3.2. Since $k(t)$ is continuous with respect to t, there exists $\delta > 0$ such that $|k(t)| < 1 - \delta$. First, we prove that for $L \geq 0$ and $|k| < 1 - \delta$, there exists $R > 0$, such that

$$(\mathcal{U}_F \cap \mathbb{C}_+) \subset \{\sigma||\sigma|<R\}. \tag{3.20}$$

Recall the roots of $F_{a,b,\lambda,k,L}(\sigma)$ satisfy

$$(k-1)(1+\mathrm{e}^{-2Q(\sigma)L}) - \left[(k+1)\frac{\lambda+1}{2\lambda}\frac{\sigma}{Q(\sigma)} + \frac{(\frac{kb}{\lambda}+a)}{Q(\sigma)}\right](1-\mathrm{e}^{-2Q(\sigma)L}) = 0. \tag{3.21}$$

Here $Q(\sigma)$ is the single-valued branch that satisfied $4\lambda^2 Q(\sigma)^2 = (\lambda+1)^2\sigma^2 - 4\lambda ab$, and

$$\lim_{|\sigma|\to+\infty} \frac{\sigma}{Q(\sigma)} = \frac{2\lambda}{\lambda+1}, \quad \lim_{|\sigma|\to+\infty} |Q(\sigma)| = +\infty. \tag{3.22}$$

Equation (3.21) is equivalent to

$$(k-1)(1+\mathrm{e}^{-2Q(\sigma)L}) - (k+1)\frac{\lambda+1}{2\lambda}\frac{\sigma}{Q(\sigma)}(1-\mathrm{e}^{-2Q(\sigma)L}) = \frac{(\frac{kb}{\lambda}+a)}{Q(\sigma)}(1-\mathrm{e}^{-2Q(\sigma)L}). \tag{3.23}$$

Note that

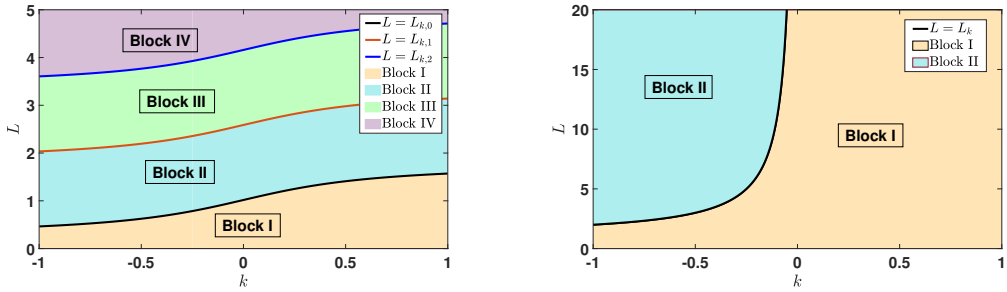
$$|\mathrm{e}^{-2Q(\sigma)L}| = \mathrm{e}^{-2\mathrm{Re}Q(\sigma)L} = (\mathrm{e}^{-2\mathrm{Re}\sigma L})^{\frac{\mathrm{Re}Q(\sigma)}{\mathrm{Re}\sigma}},$$

which implies that

$$\limsup_{|\sigma|\to+\infty,\sigma\in\cup\overline{\mathbb{C}_+}} |\mathrm{e}^{-2Q(\sigma)L}| \leq 1.$$

As $|\sigma| \to +\infty, \sigma \in \overline{\mathbb{C}_+}$, the right side of equation (3.23) can be estimated as

$$\lim_{|\sigma|\to+\infty,\sigma\in\cup\overline{\mathbb{C}_+}} \frac{(\frac{kb}{\lambda}+a)}{Q(\sigma)}(1-\mathrm{e}^{-2Q(\sigma)L}) = 0. \tag{3.24}$$



(a) (b)

FIGURE 2. $k - L$ plane is separated by marginal curves determined by $\mathcal{A}_{a,b,\lambda}$ for different cases.

The left side of equation (3.23) can be estimated as

$$\begin{aligned}
&\liminf_{|\sigma|\to+\infty,\sigma\in\cup\overline{\mathbb{C}_+}}\left|(k-1)(1+\mathrm{e}^{-2Q(\sigma)L})-(k+1)\frac{\lambda+1}{2\lambda}\frac{\sigma}{Q(\sigma)}(1-\mathrm{e}^{-2Q(\sigma)L})\right| \\
\geq &\liminf_{|\sigma|\to+\infty,\sigma\in\cup\overline{\mathbb{C}_+}}\left|(k-1)-(k+1)\frac{\lambda+1}{2\lambda}\frac{\sigma}{Q(\sigma)}\right|-\left|(k-1)+(k+1)\frac{\lambda+1}{2\lambda}\frac{\sigma}{Q(\sigma)}\right|\cdot\left|\mathrm{e}^{-2Q(\sigma)L}\right| \\
\geq &2-2|k|\geq\delta.
\end{aligned} \tag{3.25}$$

The limit in equations (3.24) and (3.25) are taken uniformly with respect to $L \geq 0, |k| < 1 - \delta$. Therefore, there exists a sufficiently large $R > 0$ such that equation (3.21) has no roots located in $\sigma \in \overline{\mathbb{C}_+} \cap \{\sigma||\sigma| \geq R\}$. Then we establish equation (3.20). Equation (3.20) yields

$$N_{a,b,\lambda}(k(t), L(t)) = \deg(F_{a,b,\lambda,k(t),L(t)}, \mathbb{C}_+, 0) = \deg(F_{a,b,\lambda,k(t),L(t)}, \mathbb{C}_+ \cap \{\sigma||\sigma| < R\}, 0).$$

We denote the contour $C \triangleq \{R\mathrm{e}^{\mathrm{i}\theta}|\theta : -\frac{\pi}{2} \to \frac{\pi}{2}\} \cup \{\mathrm{i}\beta|\beta : R \to -R\}$. If $\{k(t), L(t)\} \cap \mathcal{A}_{a,b,\lambda} = \emptyset$, using argument principle, we obtain

$$N_{a,b,\lambda}(k(t), L(t)) = \frac{1}{2\pi\mathrm{i}}\int_C \frac{F'_{a,b,\lambda,k(t),L(t)}(\sigma)}{F_{a,b,\lambda,k(t),L(t)}(\sigma)}\mathrm{d}\sigma. \tag{3.26}$$

Since $F_{a,b,\lambda,k(t),L(t)}(\sigma) \neq 0$ on C, the right side of equation (3.26) is continuous with respect to t. Furthermore, since $N_{a,b,\lambda}(k(t), L(t))$ is an integer, we know that $N_{a,b,\lambda}(k(t), L(t))$ is a constant. □

As shown in Figure 2, $\mathcal{A}_{a,b,\lambda}$, which consists of several curves, divides the $k - L$ plane into several blocks.

The calculation of $\mathcal{A}_{a,b,\lambda}$ is provided in Appendix A. It is worth noting that $N_{a,b,\lambda}$ is a constant within each block as per Lemma 3.2. Moreover, for $L = 0$, we have $F_{a,b,\lambda,k,L}(\sigma) = k - 1 \neq 0$ for $k \in (-1, 1)$, which implies that $N_{a,b,\lambda}(k, 0) = 0$. As a result, $N_{a,b,\lambda}(\text{Block I}) = 0$, and the corresponding system (1.2) (1.3) with (k, L) in Block I is exponentially stable.

We can further demonstrate that if a point (k, L) moves from one block to a block above it, then $N_{a,b,\lambda}$ increases by 1. Therefore, for any point (k, L) in a block other than Block I, the system (1.2) (1.3) possesses at least one eigenvalue in $\overline{\mathbb{C}}_+$ and cannot be exponentially stable. □

Lemma 3.3. *For $(k_0, L_0) \in \mathcal{A}_{a,b,\lambda}$, there exists $\varepsilon > 0$, such that*

$$N_{a,b,\lambda}(k_0, L_0 + \varepsilon) = N_{a,b,\lambda}(k_0, L_0 - \varepsilon) + 1.$$

Proof of Lemma 3.3. From Appendix A, if $(k_0, L_0) \in \mathcal{A}_{a,b,\lambda}$, we have:

$$F_{a,b,\lambda,k_0,L_0}(0) = 0,$$

with

$$F_{a,b,\lambda,k,L}(\sigma) = (k-1)\cosh(\eta L) - \big[(k+1)\frac{\lambda+1}{2\lambda}\sigma + (\frac{kb}{\lambda}+a)\big]\frac{\sinh(\eta L)}{\eta}.$$

Denote $\sigma_0 = 0$ and the corresponding η is denoted by η_0, from equation (3.10),

$$\eta_0 \coth(\eta_0 L_0) = \frac{(\frac{kb}{\lambda}+a)}{(k-1)}. \tag{3.27}$$

Denote

$$H_{a,b,\lambda,k}(\sigma, L) \triangleq F_{a,b,\lambda,k,L}(\sigma) = (k-1)\cosh(\eta L) - \big[(k+1)\frac{\lambda+1}{2\lambda}\sigma + (\frac{kb}{\lambda}+a)\big]\frac{\sinh(\eta L)}{\eta},$$

with

$$\eta^2 = \frac{(\lambda+1)^2\sigma^2 - 4\lambda ab}{4\lambda^2}, \tag{3.28}$$

and

$$\widetilde{H}_{a,b,\lambda,k}(\eta, \sigma, L) \triangleq (k-1)\cosh(\eta L) - \big[(k+1)\frac{\lambda+1}{2\lambda}\sigma + (\frac{kb}{\lambda}+a)\big]\frac{\sinh(\eta L)}{\eta}.$$

- If $ab \neq 0$, $\eta_0 \neq 0$, at point (σ_0, L_0), we have

$$\frac{\partial \eta}{\partial \sigma}\bigg|_{(\sigma,L)=(0,L_0)} = \frac{(\lambda+1)^2}{4\lambda^2}\frac{\sigma}{\eta} = 0,$$

  where $\eta = \eta(\sigma)$ is a single-valued branch in the neighbourhood of $\sigma = 0$ that satisfy (3.28). The partial differential derivative of $H_{a,b,\lambda,k}(\sigma, L)$ with respect to σ is

$$\begin{aligned}
\frac{\partial H_{a,b,\lambda,k}}{\partial \sigma}\bigg|_{(\sigma,L)=(0,L_0)} &= \frac{\partial \widetilde{H}_{a,b,\lambda,k}}{\partial \eta}\cdot\frac{\partial \eta}{\partial \sigma} + \frac{\partial \widetilde{H}_{a,b,\lambda,k}}{\partial \sigma}\bigg|_{(\sigma,L)=(0,L_0)} \\
&= \frac{\partial \widetilde{H}_{a,b,\lambda,k}}{\partial \sigma}\bigg|_{(\sigma,L)=(0,L_0)} \\
&= -(k+1)\frac{\lambda+1}{2\lambda}\frac{\sinh(\eta_0 L_0)}{\eta_0} \neq 0.
\end{aligned}$$

Notice η_0 is a pure imaginary number ($ab > 0$) or a real number ($ab < 0$), from equation (3.27), the partial differential derivative of $H_{a,b,\lambda,k}(\sigma, L)$ with respect to L is:

$$\begin{aligned}\frac{\partial H_{a,b,\lambda,k}}{\partial L}\bigg|_{(\sigma,L)=(0,L_0)} &= \left[(k-1)\sinh(\eta_0 L_0) - (\frac{kb}{\lambda}+a)\frac{\cosh(\eta_0 L_0)}{\eta_0}\right]\eta_0 \\ &= \left[(k-1)\sinh(\eta_0 L_0) - (k-1)\frac{\cosh^2(\eta_0 L_0)}{\sinh(\eta_0 L_0)}\right]\eta_0 \\ &= -\frac{(k-1)\eta_0}{\sinh(\eta_0 L_0)}.\end{aligned}$$

Using the Implicit Function Theorem 2.3, there exists an implicit function $\sigma_{k_0}(L)$ defined on $(L_0-\varepsilon, L_0+\varepsilon)$ for small sufficiently $\varepsilon > 0$ such that

$$H_{a,b,\lambda,k_0}(\sigma_{k_0}(L), L) = 0, \sigma_{k_0}(L_0) = 0,$$

and $\sigma'_{k_0}(L) = -\left(\frac{\partial H_{a,b,\lambda,k}}{\partial L}\right)/\left(\frac{\partial H_{a,b,\lambda,k}}{\partial \sigma}\right)$. More precisely,

$$\sigma'_{k_0}(L_0) = \frac{-2\lambda(k-1)}{(\lambda+1)(k+1)} \cdot \frac{\eta_0^2}{\sinh^2(\eta_0 L_0)} > 0.$$

The last inequality is obtained by $\frac{\eta_0}{\sinh(\eta_0 L_0)} \in \mathbb{R}\backslash\{0\}$. This demonstrates that as L varies within the range from $L_0 - \varepsilon$ to $L_0 + \varepsilon$, the solution $\sigma(L)$ of the equation $F_{a,b,\lambda,k_0,L}(\sigma) = 0$, undergoes a continuous transition from $\mathbb{C}_-$ to $\mathbb{C}_+$, crossing the imaginary axis at $\sigma = 0$, when $L = L_0$. This further indicates that $N_{a,b,\lambda}(k_0, L_0+\varepsilon) = N_{a,b,\lambda}(k_0, L_0-\varepsilon) + 1$. Then we finish the proof with $ab \neq 0$.

- If $ab = 0$, we have the following characteristic equation

$$(k-1)\cosh(\eta L) - \big((k+1)\frac{\lambda+1}{2\lambda}\sigma + (\frac{kb}{\lambda}+a)\big)\frac{\sinh(\eta L)}{\eta} = 0, \tag{3.29}$$

with $\eta = \frac{(\lambda+1)\sigma}{2\lambda}$.
At point (σ_0, L_0), we obtain

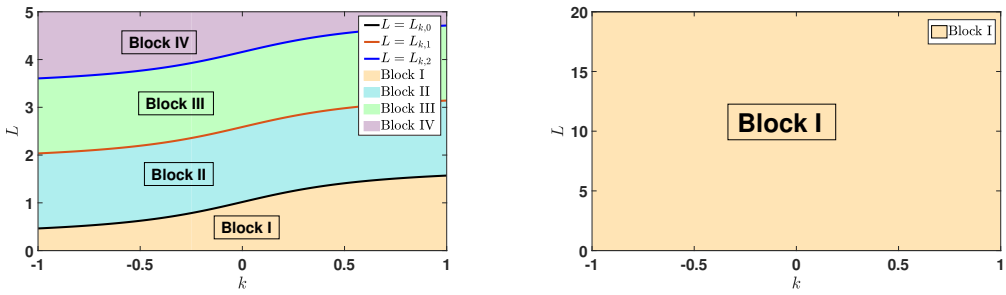
$$(k-1) - (\frac{kb}{\lambda}+a)L_0 = 0.$$

Moreover,

$$\frac{\partial \eta}{\partial \sigma}|_{(\sigma,L)=(0,L_0)} = \frac{\lambda+1}{2\lambda}.$$

Since $\cosh(\eta L)$ and $\frac{\sinh(\eta L)}{\eta}$ are analytic functions with respect to η^2, we obtain

$$\frac{\partial \widetilde{H}_{a,b,\lambda,k}}{\partial \eta}|_{(\eta,\sigma,L)=(0,0,L_0)} = 0.$$



(a) (b)

FIGURE 3. $k - L$ plane is separated by marginal curves determined by $\mathcal{A}_{a,b,\lambda}$ for different cases.

Thus,

$$\frac{\partial H_{a,b,\lambda,k}}{\partial \sigma}|_{(\sigma,L)=(0,L_0)} = \frac{\partial \widetilde{H}_{a,b,\lambda,k}}{\partial \eta} \cdot \frac{\partial \eta}{\partial \sigma} + \frac{\partial \widetilde{H}_{a,b,\lambda,k}}{\partial \sigma}$$
$$= -(k+1)\frac{\lambda+1}{2\lambda}\frac{\sinh(\eta_0 L_0)}{\eta_0} = -(k+1)\frac{\lambda+1}{2\lambda}L_0 \neq 0.$$

The partial differential derivative of $H_{a,b,\lambda,k}(\sigma, L)$ with respect to L is

$$\frac{\partial H_{a,b,\lambda,k}}{\partial L}|_{(\sigma,L)=(0,L_0)} = -(\frac{kb}{\lambda} + a).$$

Using the Implicit Function Theorem 2.3, there exists an implicit function $\sigma(L)$ defined on $(L_0 - \varepsilon, L_0 + \varepsilon)$ for small sufficiently $\varepsilon > 0$ such that

$$H_{a,b,\lambda,k}(\sigma(L), L) = 0, \sigma(L_0) = 0,$$

and $\sigma'(L) = -\left(\frac{\partial H_{a,b,\lambda,k}}{\partial L}\right) / \left(\frac{\partial H_{a,b,\lambda,k}}{\partial \sigma}\right)$. More precisely,

$$\sigma'(L_0) = -\frac{2\lambda(k-1)}{(\lambda+1)(k+1)L_0^2} > 0.$$

Similarly, we finish the proof of $ab = 0$.

□

Proof of Theorem 1.3

Lemma 3.3 gives us a way to determine $N_{a,b,\lambda}$ for each block. For example, for the case $ab > 0$, the $k - L$ plane is separated into infinitely blocks by marginal curves determined by $\mathcal{A}_{a,b,\lambda}$ (see Fig. 3a). Thus, we know that $N_{a,b,\lambda}(\text{Block I}) = 0$, $N_{a,b,\lambda}(\text{Block II}) = 1$, $N_{a,b,\lambda}(\text{Block III}) = 2$, $N_{a,b,\lambda}(\text{Block IV}) = 3$ and so on. For the case $a > 0, b = 0$, there are no marginal curves (see Fig. 3b). Thus, we know that for all the point $(k, L) \in (-1, 1) \times [0, +\infty)$, we have $N_{a,b,\lambda}(k, L) = 0$.

Thus we know that the stability region is the block at the bottom which contains $\{(k, 0)|k \in (-1, 1)\}$. Denote $D(a, b, \lambda)$ is the block that contains $\{(k, 0)|k \in (-1, 1)\}$, we obtain

$$L_c(a, b, \lambda) = \max\{L|(k, L) \in D(a, b, \lambda)\}.$$

Moreover, if $\mathcal{A}_{a,b,\lambda} = \emptyset$, for any $L > 0$, the corresponding system (1.2) (1.3) with (k, L) is exponentially stable with $L_c \triangleq +\infty$ that is defined in Theorem 1.3. More precisely, from (A.6), (A.8), (A.10) in Appendix A, we obtain

$$L_c = \begin{cases} \sqrt{\frac{\lambda}{ab}}\pi, & \text{if } a > 0, b > 0. \\ \sqrt{\frac{\lambda}{ab}} \operatorname{arccot}\left(\frac{b-\lambda a}{2\sqrt{\lambda ab}}\right), & \text{if } a < 0, b < 0. \\ \sqrt{\frac{-\lambda}{ab}} \coth^{-1}(\frac{b-\lambda a}{2\sqrt{-\lambda ab}}), & \text{if } -\lambda a > b > 0. \\ -\frac{2}{a}, & \text{if } b = 0, a < 0. \\ +\infty, & \text{if else.} \end{cases} \tag{3.30}$$

□

Remark 3.4. As indicated in Remark 1.5, when $L > 0$ is sufficiently small, the choice of $k = 0$ consistently yields the optimal conditions for the existence of a Lyapunov function. However, for specific values of $L > 0$, small value of $|k|$ does not represent the most advantageous option. An instance of this is demonstrated in Figure 4b (purple dashed line), where the stability region for k is $(-1, k_*)$ with $k_* \approx -0.23$. This intriguing and counter-intuitive phenomenon demonstrates the superiority of spectral analysis over the Lyapunov function method.

4. Numerical simulations

In this section we present some numerical simulations generated with MATLAB of upwind scheme with implicit methods for the system (1.2) (1.3). We adopt the finite difference method in both the time and the space domain, which can be written as follows. The grid size $N = 100$ and the time step $\Delta t = 1.1\lambda\Delta x = 1.1\lambda L/N$ are used.

$$\begin{cases} \frac{u_j^{n+1}-u_j^n}{\Delta t} + \frac{u_j^{n+1}-u_{j-1}^{n+1}}{\Delta x} + av_j^{n+1} = 0, & j = 1, ..., N; n = 0, ..., \frac{T}{\Delta t}, \\ \frac{v_j^{n+1}-v_j^n}{\Delta t} - \lambda\frac{v_{j+1}^{n+1}-v_j^{n+1}}{\Delta x} + bu_j^{n+1} = 0, & j = 0, ..., N-1; n = 0, ..., \frac{T}{\Delta t}, \\ u_0^n = kv_0^n, & n = 0, ..., \frac{T}{\Delta t}, \\ u_N^n = v_N^n, & n = 0, ..., \frac{T}{\Delta t}. \end{cases}$$

Here u_j^n and v_j^n provide an approximation of $y_1(x_j, t_n)$ and $y_2(x_j, t_n)$, respectively. The initial conditions are chosen as

$$\begin{cases} y_1(0, x) = x + \sin^2 x, \\ y_2(0, x) = \frac{L+\sin^2 L}{L+L^2}(x^2 + x). \end{cases}$$

Energy is measured in the L_2-norm for

$$E_t \triangleq \int_0^L [y_1(t, x)^2 + y_2(t, x)^2]\mathrm{d}x.$$

We choose four triple $(a, b, \lambda) = (1, 1, 1), (-1, -2, 1), (0, 1, 1), (-1, 0, 1)$. For each triple value of (a, b, λ), we numerically implement system (1.2) for the parameter $(k, L) \in (-1, 1) \times (0, 3)$. As shown in Figure 4, the analytical criteria for stability region established in Section 3 can be confirmed by our numerical simulations. Specially, for $a = -1, b = -2, \lambda = 1, L = 0.9$, Figure 4b shows that the system will be stabilized only when the

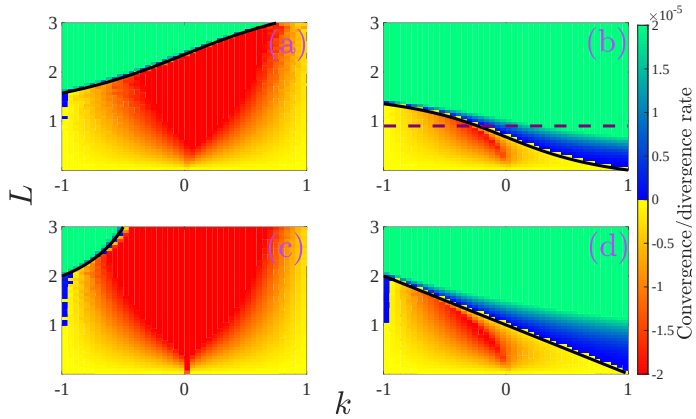


FIGURE 4. Black curves are depicted by the analytical results according to Appendix.A, below which are the stability region for k, L. The colors represent the exponential rates of the convergence or divergence of the trajectory numerically generated by equation (1.2). The parameters are (a) $a = 1, b = 1, \lambda = 1$ (b) $a = -1, b = -2, \lambda = 1$ (c) $a = 0, b = 1, \lambda = 1$ (d) $a = -1, b = 0, \lambda = 1$. Purple dashed line in (b) represents $L = 0.9$, which will be investigated more clearly in Figure 5.

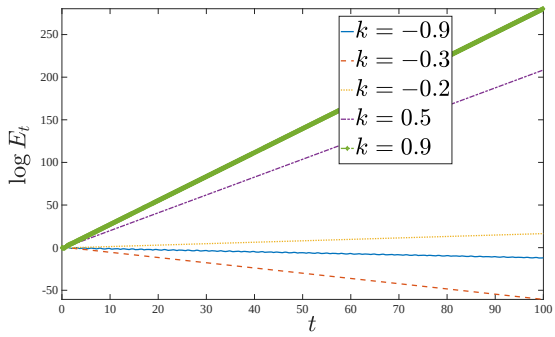


FIGURE 5. For different values of coupling gain k, the energy E_t converges or diverges exponentially with different rates. The parameters are $a = -1, b = -2, \lambda = 1, L = 0.9$.

coupling gain $k \in (-1, k_*)$ for $k_* \approx -0.23$. Numerical confirmation is shown in Figure 5. For different values of coupling gain k, the energy converges or diverges exponentially with different rates.

5. Backstepping control

In this section, we want to use Backstepping method combined with the observer design to stabilize the system with the case that cannot be stabilized by the proportional feedback control. We first make a scaling of space variable $x \to L - x$, then the control could be on the right side and the boundary condition be on the left. Theorem 1.3 can apply to the following system:

$$\begin{cases} \partial_t y_1 - \partial_x y_1 + a y_2 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ \partial_t y_2 + \lambda \partial_x y_2 + b y_1 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ y_2(t,0) = y_1(t,0), & t \in (0,+\infty), \\ y_1(t,L) = U(t), & t \in (0,+\infty). \end{cases} \tag{5.1}$$

The output is

$$Y(t) = y_2(t, L). \tag{5.2}$$

Applying the results in Anfinsen *et al.* [23], we design the following observer:

$$\begin{cases} \partial_t \hat{y}_1 - \partial_x \hat{y}_1 + a\hat{y}_2 = \Gamma_1(x)(Y(t) - \hat{y}_2(t, L)), \quad (t, x) \in (0, +\infty) \times (0, L), \\ \partial_t \hat{y}_2 + \lambda \partial_x \hat{y}_2 + b\hat{y}_1 = \Gamma_2(x)(Y(t) - \hat{y}_2(t, L)), \quad (t, x) \in (0, +\infty) \times (0, L), \\ \hat{y}_2(t, 0) = \hat{y}_1(t, 0), \quad t \in (0, +\infty), \\ \hat{y}_1(t, L) = U(t), \quad t \in (0, +\infty), \end{cases} \tag{5.3}$$

with $\Gamma_1(x), \Gamma_2(x)$ are injection gains to be designed.

We have the following proposition:

Proposition 5.1. *Suppose the system (5.1) and the observer (5.3) with $\lambda, a, b \in \mathbb{R}$, $T_{opt1} \triangleq \frac{(\lambda+1)}{\lambda} L$. There exists suitable injection designs $\Gamma_1(x), \Gamma_2(x)$ such that for all $t > T_{opt1}$, we have:*

$$\begin{cases} y_1(t, x) = \hat{y}_1(t, x), \quad (t, x) \in [T_{opt1}, +\infty) \times [0, L], \\ y_2(t, x) = \hat{y}_2(t, x), \quad (t, x) \in [T_{opt1}, +\infty) \times [0, L]. \end{cases} \tag{5.4}$$

Proof of Proposition 5.1. The state estimation errors $\widetilde{y}_1 \triangleq y_1 - \hat{y}_1, \widetilde{y}_2 \triangleq y_2 - \hat{y}_2$ satisfy the dynamics:

$$\begin{cases} \partial_t \widetilde{y}_1 - \partial_x \widetilde{y}_1 + a\widetilde{y}_2 = -\Gamma_1(x)\widetilde{y}_2(L), & (t, x) \in (0, +\infty) \times (0, L), \\ \partial_t \widetilde{y}_2 + \lambda \partial_x \widetilde{y}_2 + b\widetilde{y}_1 = -\Gamma_2(x)\widetilde{y}_2(L), & (t, x) \in (0, +\infty) \times (0, L), \\ \widetilde{y}_2(t, 0) = \widetilde{y}_1(t, 0), & t \in (0, +\infty), \\ \widetilde{y}_1(t, L) = 0, & t \in (0, +\infty). \end{cases} \tag{5.5}$$

Design the backstepping transformation:

$$\begin{cases} \widetilde{y}_1(t, x) = \alpha(t, x) + \int_x^L P^1(x, \xi)\beta(t, \xi)\mathrm{d}\xi, \\ \widetilde{y}_2(t, x) = \beta(t, x) + \int_x^L P^2(x, \xi)\beta(t, \xi)\mathrm{d}\xi, \end{cases} \tag{5.6}$$

which makes system (5.5) become the target system as follows.

$$\begin{cases} \partial_t \alpha - \partial_x \alpha + \int_x^L Q^1(x, \xi)\alpha(\xi)\mathrm{d}\xi = 0, & (t, x) \in (0, +\infty) \times (0, L), \\ \partial_t \beta + \lambda \partial_x \beta + \int_x^L Q^2(x, \xi)\alpha(\xi)\mathrm{d}\xi + b\alpha = 0, & (t, x) \in (0, +\infty) \times (0, L), \\ \alpha(t, 0) = \beta(t, 0), & t \in (0, +\infty), \\ \alpha(t, L) = 0, & t \in (0, +\infty). \end{cases} \tag{5.7}$$

The proof of existence of Q^1, Q^2, P^1, P^2 can be found in [23], Section B. The solution of target system (5.7) will vanish in finite time for $t > T_{opt1}$. If we denote by $\Gamma_1(x) \triangleq \lambda P^1(x, L), \Gamma_2(x) \triangleq \lambda P^2(x, L)$, by the transformation (5.6), we obtain that the solution $[\tilde{y}_1, \tilde{y}_2]^{\mathrm{T}}$ of equation (5.5) will vanish when $t > T_{opt1}$. □

It follows from Proposition 5.1 that the right side of equation (5.3) vanishes when $t > T_{opt1}$, which makes it become a homogeneous linear hyperbolic system. By [1], Sections 2.2 and 2.3, we know that there exists an

invertible backstepping transformation

$$\begin{cases} z(t,x) = \hat{y}_1(t,x) - \int_0^x [K^{11}(x,\xi)\hat{y}_1(t,\xi) + K^{12}(x,\xi)\hat{y}_2(t,\xi)]\mathrm{d}\xi, \\ w(t,x) = \hat{y}_2(t,x) - \int_0^x [K^{21}(x,\xi)\hat{y}_1(t,\xi) + K^{22}(x,\xi)\hat{y}_2(t,\xi)]\mathrm{d}\xi, \end{cases} \tag{5.8}$$

with its inverse transformation

$$\begin{cases} \hat{y}_1(t,x) = z(t,x) + \int_0^x [L^{11}(x,\xi)z(t,\xi) + L^{12}(x,\xi)w(t,\xi)]\mathrm{d}\xi, \\ \hat{y}_2(t,x) = w(t,x) + \int_0^x [L^{21}(x,\xi)z(t,\xi) + L^{22}(x,\xi)w(t,\xi)]\mathrm{d}\xi, \end{cases} \tag{5.9}$$

which transforms system (5.3) into the target system as follows.

$$\begin{cases} \partial_t z - \partial_x z = 0, & (t,x) \in (T_{opt1}, +\infty) \times (0,L), \\ \partial_t w + \lambda \partial_x w = g(x)w(t,0), & (t,x) \in (T_{opt1}, +\infty) \times (0,L), \\ w(t,0) = z(t,0), & t \in (T_{opt1}, +\infty), \\ z(t,L) = 0, & t \in (T_{opt1}, +\infty). \end{cases} \tag{5.10}$$

From transformation (5.9) evaluated at $x = L$, noting that $z(t,L) \equiv 0$, we obtain the following feedback control laws for the system (5.3)

$$U(t) \triangleq \int_0^L [L^{11}(L,\xi)z(t,\xi) + L^{12}(L,\xi)w(t,\xi)]\mathrm{d}\xi.$$

This yields that $\hat{y}_1, \hat{y}_2$ vanishes in finite time $t \geq T_{opt} \triangleq T_{opt1} + T_{opt2}$ with $T_{opt1} = T_{opt2} = \dfrac{(\lambda+1)L}{\lambda}$. Thus we get the main theorem in this section.

Theorem 5.2. *There exists a boundary feedback control law $U(t)$ for the system (5.1) with $\lambda, a, b \in \mathbb{R}$ such that, for every $Y_0 \in L^2(0,L)$, the solution $Y \in C^0([0,+\infty); L^2(0,L))$ to (5.1) satisfies $Y(t) = 0, \forall t \geq T_{opt}$, where $T_{opt} = \frac{2(\lambda+1)}{\lambda}L$.*

Remark 5.3. Proposition 5.1 and Theorem 5.2 have designed boundary feedback controls that stabilize the cases which cannot be stabilized by the proportional boundary feedback control mentioned in Theorem 1.3.

Remark 5.4. For the following system:

$$\begin{cases} \partial_t y_1 - \lambda(x)\partial_x y_1 + a(x)y_2 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ \partial_t y_2 + \mu(x)\partial_x y_2 + b(x)y_1 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ y_2(t,0) = y_1(t,0), & t \in (0,+\infty), \\ y_1(t,L) = U(t), & t \in (0,+\infty). \end{cases} \tag{5.11}$$

with $\lambda(x), \mu(x) \in C^1([0,L]), a(x), b(x) \in C^0([0,1])$ are known functions that satisfied $\lambda(x), \mu(x) > 0$ and $k \in \mathbb{R}$.

The basic ideas of designing the observer in Proposition 5.1 and the backstepping control in Theorem 5.2 can be applied to this system. Therefore, the system could be stabilized to zero in finite time by a boundary control $U(t)$ depending on $y_2(\tau, L)(\tau \in (0,t))$.

Appendix A. Calculation on $\mathcal{A}_{a,b,\lambda}$

Recall that the characteristic equation (3.10) is

$$(k-1)\cosh(\eta L) - \big[(k+1)\frac{\lambda+1}{2\lambda}\sigma + (\frac{kb}{\lambda}+a)\big]\frac{\sinh(\eta L)}{\eta} = 0.$$

with

$$\eta^2 = \frac{(\lambda+1)^2\sigma^2 - 4\lambda ab}{4\lambda^2}.$$

and the definition of $\mathcal{A}_{a,b,\lambda}$ is

$$\mathcal{A}_{a,b,\lambda} \triangleq \big\{(k,L)||k|<1, L\geq 0, \text{ there exists } \beta\in\mathbb{R} \text{ such that } F_{a,b,\lambda,k,L}(\mathrm{i}\beta)=0\big\},$$

First, if $L=0$, we have $F_{a,b,\lambda,k,L}(\sigma) = k-1 \neq 0$ for $k\in(-1,1)$, which means

$$(k,0)\notin \mathcal{A}_{a,b,\lambda}. \tag{A.1}$$

We discuss $\mathcal{A}_{a,b,\lambda}$ with $(k,L)\in(-1,1)\times(0,+\infty)$. We first observe that if $\sigma=\mathrm{i}\beta$, we have

$$\eta^2\in\mathbb{R},\ \cosh(\eta L)\in\mathbb{R},\ \frac{\sinh(\eta L)}{\eta}\in\mathbb{R}.$$

Thus, the imaginary part of equation (3.10) is:

$$(k+1)\frac{\lambda+1}{2\lambda}\beta\frac{\sinh(\eta L)}{\eta} = 0, \tag{A.2}$$

and the real part of equation (3.10) is:

$$(k-1)\cosh(\eta L) - (\frac{kb}{\lambda}+a)\frac{\sinh(\eta L)}{\eta} = 0. \tag{A.3}$$

If $\frac{\sinh(\eta L)}{\eta}=0$ (motivated by equation (A.2)), the left side of equation (A.3) is $k-1\neq 0$, thus $\frac{\sinh(\eta L)}{\eta}\neq 0$. Recall that $\lambda>0$, from equation (A.2), we obtain

$$\beta = 0.$$

Therefore,

$$\mathcal{A}_{a,b,\lambda} = \big\{(k,L)||k|<1, L>0, F_{a,b,\lambda,k,L}(0)=0\big\},$$

and the characteristic equation can be simplified:

$$(k-1)\cosh(\eta L) - (\frac{kb}{\lambda}+a)\frac{\sinh(\eta L)}{\eta} = 0, \tag{A.4}$$

with

$$\eta^2 = -\frac{ab}{\lambda}.$$

We now divide into three cases:

Case I. $ab = 0$, the corresponding $\eta = 0$. equation (A.4) yields:

$$(k-1) - (\frac{kb}{\lambda} + a)L = 0.$$

Define $L_k \triangleq \frac{k-1}{\frac{kb}{\lambda}+a}$, we obtain

$$\mathcal{A}_{a,b,\lambda} = \begin{cases} \big\{(k, L_k) | k \in (-1, 0]\big\}, & \text{if } a = 0, b > 0. \\ \big\{(k, L_k) | k \in [0, 1)\big\}, & \text{if } a = 0, b < 0. \\ \emptyset, & \text{if } a \geq 0, b = 0. \\ \big\{(k, L_k) | k \in (-1, 1)\big\}, & \text{if } a < 0, b = 0. \end{cases} \tag{A.5}$$

As shown in Figure A.1, if $\mathscr{A}_{a,b,\lambda}$ is not an empty set, $k - L$ plane is separated into two blocks. That is

$$L_c = \begin{cases} +\infty, & \text{if } a = 0. \\ +\infty, & \text{if } a \geq 0, b = 0. \\ L_{-1} = -\frac{2}{a}, & \text{if } a < 0, b = 0. \end{cases} \tag{A.6}$$

Case II. $ab > 0$, equation (A.4) yields:

$$(k-1)\cos\left(\sqrt{\frac{ab}{\lambda}}L\right) - (\frac{kb}{\lambda} + a)\frac{\sin\left(\sqrt{\frac{ab}{\lambda}}L\right)}{\sqrt{\frac{ab}{\lambda}}} = 0.$$

It can be written as

$$\cot\left(\sqrt{\frac{ab}{\lambda}}L\right) = \frac{\frac{kb}{\lambda} + a}{(k-1)\sqrt{\frac{ab}{\lambda}}}.$$

We define

$$L_{k,n} \triangleq \sqrt{\frac{\lambda}{ab}}\left(\operatorname{arccot}\left(\frac{\frac{kb}{\lambda} + a}{(k-1)\sqrt{\frac{ab}{\lambda}}}\right) + n\pi\right) \ (n \in \mathbb{N}). \tag{A.7}$$

Then we obtain

$$\mathcal{A}_{a,b,\lambda} = \big\{(k, L_{k,n}) | k \in (-1, 1)\big\}.$$

As shown in Figure A.2, $\mathscr{A}_{a,b,\lambda}$ consists of a sequences of curves, separating $k - L$ plane into several blocks.

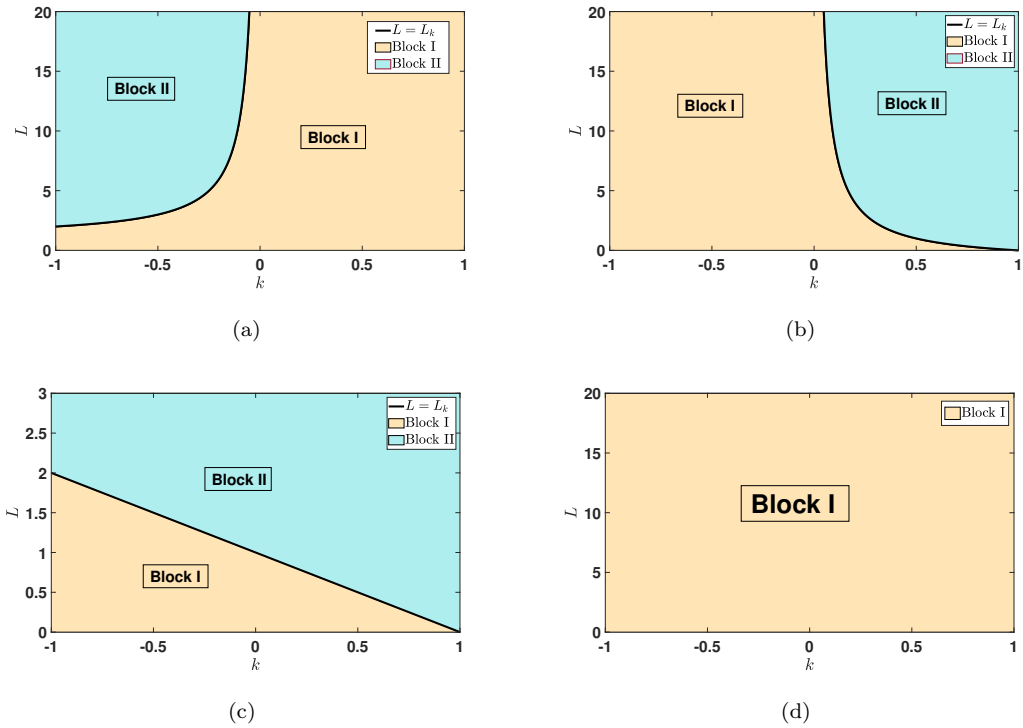


FIGURE A.1. $k - L$ plane is separated by marginal curves determined by $\mathcal{A}_{a,b,\lambda}$ for the case $ab = 0$. Black marginal curves are determined by equation (A.5). The parameters are (a) $a = 0, b = 1, \lambda = 1$,(b) $a = 0, b = -1, \lambda = 1$, (c) $a = -1, b = 0, \lambda = 1$, (d) $a = 1, b = 0, \lambda = 1$.

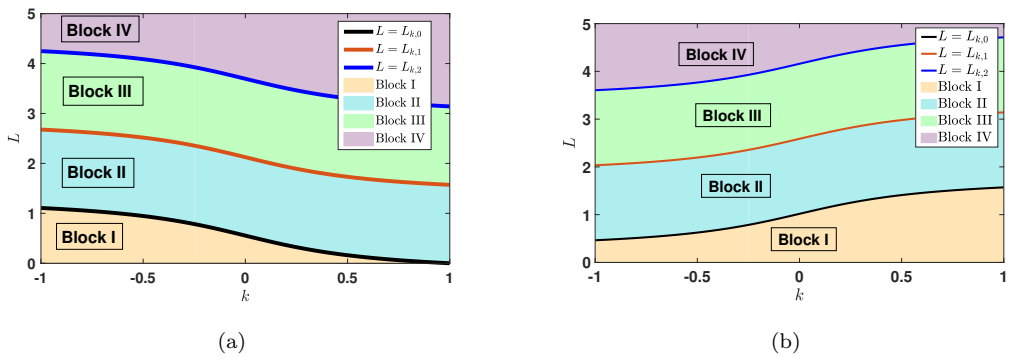


FIGURE A.2. $k - L$ plane is separated by marginal curves determined by $\mathcal{A}_{a,b,\lambda}$ for the case $ab > 0$. Black marginal curves are determined by equation (A.7). The parameters are (a) $a = -1, b = -4, \lambda = 1$,(b) $a = 1, b = 4, \lambda = 1$.

That is,

$$L_c = \begin{cases} L_{1,0} = \sqrt{\frac{\lambda}{ab}}\pi, & \text{if } a, b > 0. \\ L_{-1,0} = \sqrt{\frac{\lambda}{ab}} \operatorname{arccot}\left(\frac{b-\lambda a}{2\sqrt{\lambda ab}}\right), & \text{if } a, b < 0. \end{cases} \tag{A.8}$$

Case III. $ab < 0$, equation (A.4) yields:

$$\coth\left(\sqrt{-\frac{ab}{\lambda}}L\right) = \frac{\frac{kb}{\lambda}+a}{(k-1)\sqrt{-\frac{ab}{\lambda}}}.$$

Only if $\frac{\frac{kb}{\lambda}+a}{(k-1)\sqrt{-\frac{ab}{\lambda}}} > 1$ the marginal curves exists, which can be defined by

$$L_k \triangleq \sqrt{-\frac{\lambda}{ab}}\coth^{-1}\left(\frac{\frac{kb}{\lambda}+a}{(k-1)\sqrt{-\frac{ab}{\lambda}}}\right) > 0. \tag{A.9}$$

More precisely, let

$$h(k) \triangleq \frac{\frac{kb}{\lambda}+a}{(k-1)\sqrt{-\frac{ab}{\lambda}}},$$

which is continuous with respect to $k \in (-1, 1)$. Denote by

$$k_1 \triangleq \frac{-\lambda a - \sqrt{-\lambda ab}}{b - \sqrt{-\lambda ab}} = \sqrt{\frac{-\lambda a}{b}}\mathrm{Sgn}(a),$$

with $h(k_1) = 1$.

- $-\lambda a \geq b > 0$, $h'(k) > 0$. The range of $h(k)$ is $(h(-1), +\infty)$ with $h(-1) > 1$.

$$\mathcal{A}_{a,b,\lambda} = \big\{(k, L_k)|k \in (-1, 1)\big\}.$$

- $0 > -\lambda a > b$, $h'(k) > 0$. The range of $h(k)$ is $(h(-1), +\infty)$ with $h(-1) < -1$.

$$\mathcal{A}_{a,b,\lambda} = \big\{(k, L_k)|k \in (k_1, 1)\big\}.$$

- $b > -\lambda a > 0$, $h'(k) < 0$. The range of $h(k)$ is $(-\infty, h(-1))$ with $h(-1) > 1$.

$$\mathcal{A}_{a,b,\lambda} = \big\{(k, L_k)|k \in (-1, k_1)\big\}.$$

- $0 > b \geq -\lambda a$, $h'(k) < 0$. The range of $h(k)$ is $(-\infty, h(-1))$ with $h(-1) < -1$. We obtain $L_k < 0$ for $k \in (-1, 1)$.

$$\mathcal{A}_{a,b,\lambda} = \emptyset.$$

As shown in Figure A.3, if $\mathscr{A}_{a,b,\lambda}$ is not an empty set, $k - L$ plane is separated into two blocks. It is not difficult to obtain while $ab < 0$:

$$L_c = \begin{cases} L_{-1} = \sqrt{-\frac{\lambda}{ab}}\coth^{-1}\left(\frac{b-\lambda a}{2\sqrt{-\lambda ab}}\right), & \text{if } -\lambda a > b > 0. \\ +\infty, & \text{else.} \end{cases} \tag{A.10}$$

We finish the discuss on the set $\mathcal{A}_{a,b,\lambda}$.

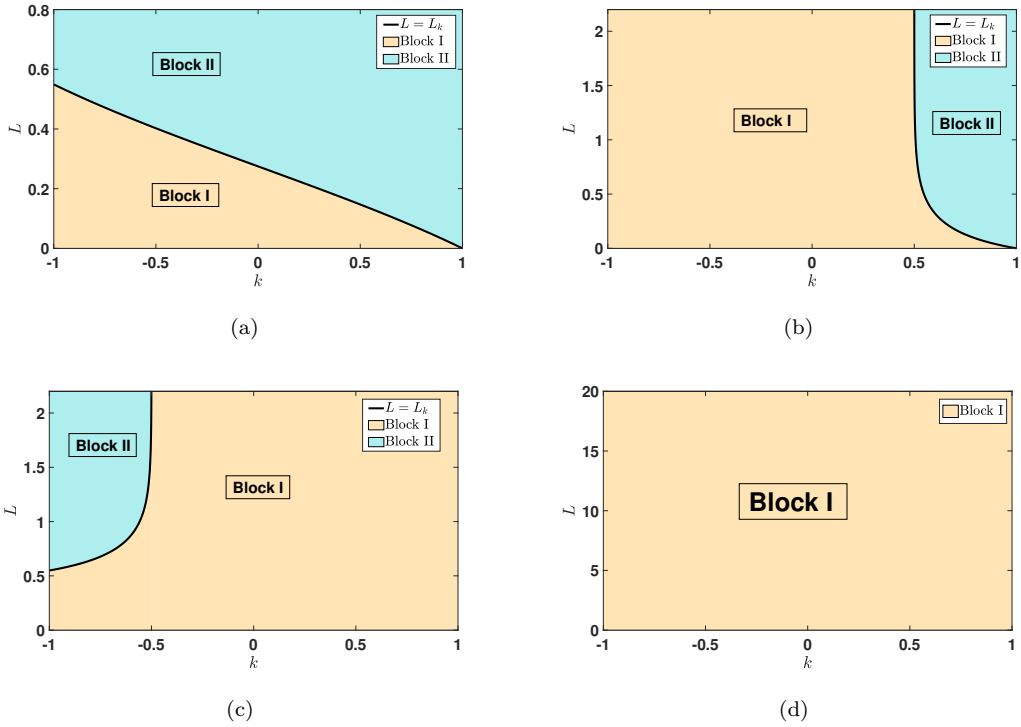


FIGURE A.3. $k - L$ plane is separated into several blocks by marginal curves determined by $\mathcal{A}_{a,b,\lambda}$ for the case $ab < 0$. Black marginal curves are determined by equation (A.9). The parameters are (a) $a = -4, b = 1, \lambda = 1$,(b) $a = 1, b = -4, \lambda = 1$, (c) $a = -1, b = 4, \lambda = 1$, (d) $a = 4, b = -1, \lambda = 1$.

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