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SEMI-CLASSICAL OBSERVATION SUFFICES FOR OBSERVABILITY: WAVE AND SCHRÖDINGER EQUATIONS

Nicolas Burq[1], Belhassen Dehman[2] and Jérôme Le Rousseau[3,*]

Abstract. For the linear wave and Schrödinger equations we show how observability can be deduced from the observability of solutions localized in frequency with a dyadic scale.

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1. Waves and observability

On a bounded smooth open set Ω of $\mathbb{R}^d$, consider the operator $A = -\Delta = -\sum_{1\le j\le d} \partial_j^2$. The associated wave equation in the case of homogeneous Dirichlet boundary conditions is

$$\begin{cases} (\partial_t^2 + A)\, u = 0 & \text{in } \mathbb{R}\times\Omega,\\ u = 0 & \text{in } \mathbb{R}\times\partial\Omega,\\ u_{|t=0} = \underline{u}^0,\ \partial_t u_{|t=0} = \underline{u}^1 & \text{in } \Omega. \end{cases} \tag{1.1}$$

1.1. Strong and weak solutions

For $\underline{u}^0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $\underline{u}^1 \in H_0^1(\Omega)$, there exists a unique

$$u \in \mathscr{C}^0\big(\mathbb{R}; H^2(\Omega)\cap H_0^1(\Omega)\big) \cap \mathscr{C}^1\big(\mathbb{R}; H_0^1(\Omega)\big) \cap \mathscr{C}^2\big(\mathbb{R}; L^2(\Omega)\big)$$

solution to (1.1). Such a solution is called a strong solution as $(\partial_t^2 + A)\, u = 0$ holds in $L^2_{\text{loc}}\big(\mathbb{R}; L^2(\Omega)\big)$. One denotes by

$$\mathcal{E}_2(u)(t) = \frac{1}{2}\big(\|Au(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{H_0^1(\Omega)}\big)$$

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[1] Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, Université Paris-Saclay, Bâtiment 307, 91405 Orsay Cedex & CNRS UMR 8628 & Institut Universitaire de France, France.
[2] Université de Tunis El Manar, Faculté des Sciences de Tunis, 2092 El Manar & Ecole Nationale d'Ingénieurs de Tunis, ENIT-LAMSIN, B.P. 37, 1002 Tunis, Tunisia.
[3] Université Sorbonne Paris Nord, Laboratoire Analyse, Géométrie et Applications, LAGA, CNRS, UMR 7539, F-93430, Villetaneuse, France.

[*] Corresponding author: jlr@math.univ-paris13.fr

the (strong) energy of u at time t. Since the equation (1.1) is homogeneous this energy is independent of time t that is,

$$\mathcal{E}_2(u)(t) = \mathcal{E}_2(u)(0) = \frac{1}{2}\big(\|A\underline{u}^0\|^2_{L^2(\Omega)} + \|\underline{u}^1\|^2_{H^1_0(\Omega)}\big).$$

One thus simply writes $\mathcal{E}_2(u)$. In particular this conservation of the energy states the continuity of the map

$$\begin{aligned}\big(H^2(\Omega)\cap H^1_0(\Omega)\big)\oplus H^1_0(\Omega) &\to \mathscr{C}^0\big(\mathbb{R}; H^2(\Omega)\cap H^1_0(\Omega)\big)\cap\mathscr{C}^1\big(\mathbb{R}; H^1_0(\Omega)\big)\cap\mathscr{C}^2\big(\mathbb{R}; L^2(\Omega)\big) \\ (\underline{u}^0, \underline{u}^1) &\mapsto u.\end{aligned} \tag{1.2}$$

For less regular intitial data one uses a notion of weak solution. For instance, if $\underline{u}^0 \in H^1_0(\Omega)$ and $\underline{u}^1 \in L^2(\Omega)$, there exists a unique

$$u \in \mathscr{C}^0\big(\mathbb{R}; H^1_0(\Omega)\big)\cap\mathscr{C}^1\big(\mathbb{R}; L^2(\Omega)\big),$$

that is a weak solution of (1.1), meaning $u_{|t=0} = \underline{u}^0$ and $\partial_t u_{|t=0} = \underline{u}^1$ and $(\partial_t^2 + A)u = 0$ holds in $\mathscr{D}'\big(\mathbb{R}\times\Omega\big)$. For such a solution one considers the following energy

$$\mathcal{E}_1(u)(t) = \frac{1}{2}\big(\|u(t)\|^2_{H^1_0(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Omega)}\big)$$

independent of time t as above, that is,

$$\mathcal{E}_1(u) = \mathcal{E}_1(u)(t) = \mathcal{E}_1(u)(0) = \frac{1}{2}\big(\|\underline{u}^0\|^2_{H^1_0(\Omega)} + \|\underline{u}^1\|^2_{L^2(\Omega)}\big).$$

With the density of $H^2(\Omega)\cap H^1_0(\Omega)$ in $H^1_0(\Omega)$ and of $H^1_0(\Omega)$ in $L^2(\Omega)$, one can approach $\underline{u}^0$ and $\underline{u}^1$ by smoother data, and thus approach $u(t)$ by strong solutions.

1.2. Observation operator, admissiblity and observability

An observation operator is an operator L on $L^2(\Omega)$, possibly unbounded, with values in a Hilbert space K. Basic examples in the framework of the present introduction are the following ones.

Example 1.1.

1. If ω is an open subset of Ω one can define $\mathsf{L} : v \mapsto \mathbf{1}_\omega v$, yielding a bounded operator on $L^2(\Omega)$.
2. If Γ is an open set of $\partial\Omega$ one can define $\mathsf{L} : v \mapsto \mathbf{1}_\Gamma \partial_{\mathsf{n}} v_{|\partial\Omega}$, where n is the outgoing normal vector at $\partial\mathcal{M}$, yielding an unbounded operator on $L^2(\Omega)$.

The observation operator is said to satisfy an admissibility condition if an estimate of the following form holds

$$\int_0^S \|\,\mathsf{L}\, u(t)\|^2_K \mathrm{d}t \le C\mathcal{E}_j(u),$$

for some $S > 0$, $C > 0$ and an energy level $j = 1$ or 2 (other energy levels are considered in the abstract development in what follows).

For example, let us assume here that $j = 2$, that is, admissibility is given at the level of strong solutions. One says that observability holds with the operator L in time $T > 0$ if one has

$$\mathcal{E}_\ell(u) \leq C_{\text{obs}} \int_0^T \| \mathsf{L}\, u(t) \|_K^2 \mathrm{d}t,$$

with $\ell = 1$ or 2, for some $C_{\text{obs}} > 0$ for any strong solution to (1.1). If $\ell = 1$ one says that observability holds with some loss of derivative, or some loss of energy, here, a loss of one energy level.

Observability estimates are important in applications such as inverse problems or controllability issues. In particular, for waves, observability is equivalent to exact controllability; see *e.g.* [8]. For more aspects on admissibility, observability and their connections with controllability, we refer the reader to the book of M. Tucsnak et G. Weiss [23].

1.3. Derivation of an observability estimate

There are various methods to derive observability estimates for the wave equation. Some rely on a multiplier approach going back to the seminal work of J.-L. Lions [20]. Others rely on microlocal methods following the celebrated article of C. Bardos, G. Lebeau, and J. Rauch [2].

The purpose of the present article is not the derivation of observability *per se*. We are rather interested in showing that observability, be it with energy loss or not, can be deduced from the observation of very particular types of waves. The waves we shall consider are localized in a frequency band making them easier to handle than general waves (in particular when applying microlocal techniques). The frequency band is indexed by an integer k and ranges from $\alpha \rho^{|k|}$ to $\rho^{|k|}/\alpha$ for $0 < \alpha < 1$ and some $\rho > 1$. This framework is given a semi-classical aspect by using the small parameter $h_k = \rho^{-|k|}$.

Wave equation solutions can be decomposed as a sum of two half-wave equation solutions corresponding respectively to positive and negative time frequencies. For a wave u^k, localized in frequency as described above, a very pleasant property states that it fulfills the half-wave equation

$$(\partial_t - \operatorname{sgn}(k) i A^{1/2}) u^k = 0. \tag{1.3}$$

This is justified in Section 2. This can greatly simplify the analysis necessary for the derivation of an observation inequality as compared to treating all solutions to the wave equation. Also, the frequency localization of u^k allows one to use powerful tools from semi-classical analysis that are often easier to handle than the analogous tools from microlocal analysis. The use of such tools can allow one to treat the case of coefficients with limited regularity; see for instance [5] for this last point. Having in mind the analysis of the HUM control operator carried out in [9] the introduction of waves with frequencies limited to a narrow band is very natural. In [9], the authors show that the control operator acts microlocally with a highly separated treatment of frequency bands similar to those considered here.

The starting point of the present article is to assume that a uniform observability estimate holds for frequency localized waves like $u^k(t)$, that is, for some $C_{\text{obs}} > 0$ one has

$$\mathcal{E}_\ell(u^k) \leq C_{\text{obs}} \int_0^T \| \mathsf{L}\, u^k(t) \|_K^2 \,\mathrm{d}t, \tag{1.4}$$

for all k sufficiently large. Our main result, under a unique-continuation property to be described below, is the derivation from (1.4) of the observability inequality for general waves $u(t)$ in the considered energy level

$$\mathcal{E}_\ell(u) \leq C'_{\text{obs}} \int_0^{T'} \| \mathsf{L}\, u(t) \|_K^2 \,\mathrm{d}t,$$

for any $T' > T$ and some $C'_{\text{obs}} > 0$. We shall also show that an admissibility condition can be used to give the proper energy level where this inequality holds.

To allow for a general use of this result, we present it in an abstract framework.

1.4. Schrödinger equation

In the same geometrical setting as above, the Schrödinger equation, in the case of Dirichlet boundary conditions reads

$$\begin{cases} (i\partial_t + A)\, u = 0 & \text{in } \mathbb{R} \times \Omega, \\ u = 0 & \text{in } \mathbb{R} \times \partial\Omega, \\ u_{|t=0} = \underline{u}^0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

For $\underline{u}^0 \in D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ there exists a unique solution in

$$u \in \mathscr{C}^0\big(\mathbb{R}; H^2(\Omega) \cap H^1_0(\Omega)\big) \cap \mathscr{C}^1\big(\mathbb{R}; L^2(\Omega)\big),$$

solution to (1.1) and $(i\partial_t + A)\, u = 0$ holds in $L^2_{\text{loc}}\big(\mathbb{R}; L^2(\Omega)\big)$. In fact, the norm

$$\|u(t)\|_{D(A)} = \|Au(t)\|_{L^2(\Omega)},$$

is independent of t. As for the wave equation, other levels of regularity are possible. If $\underline{u}^0 \in H^1_0(\Omega)$ there exists a unique solution to (1.5) in

$$u \in \mathscr{C}^0\big(\mathbb{R}; H^1_0(\Omega)\big) \cap \mathscr{C}^1\big(\mathbb{R}; H^{-1}(\Omega)\big),$$

and the norm $\|u(t)\|_{H^1_0(\Omega)}$ remains constant. If $\underline{u}^0 \in L^2(\Omega)$ there exists a unique solution in

$$u \in \mathscr{C}^0\big(\mathbb{R}; L^2(\Omega)\big) \cap \mathscr{C}^1\big(\mathbb{R}; D(A)'\big),$$

and the norm $\|u(t)\|_{L^2(\Omega)}$ remains constant.

For an observation operator as above, observability takes the form

$$\|\underline{u}^0\|_{D(A^\ell)} \le C_{\text{obs}} \int_0^T \|\mathsf{L}\, u\|_K^2 \, \mathrm{d}t,$$

here at the regularity given by $D(A^\ell)$, for $\ell = 0, 1/2$ or 1 in the above levels of solutions. As for the wave equation, under a unique-continuation property, we shall derive such an observability inequality from a similar inequality holding for solutions localized in frequency.

The Schrödinger equation can be seen sometimes as a half-wave equation; compare (1.5) and (1.3). With respect to the analysis we carry out in the present paper, this comparison is very relevant and the analysis is more involved for the wave equation. In what follows, we shall thus cover the wave equation first and cover the case of the Schrödinger on a second pass, yet with all necessary details.

1.5. Other settings

In this introductory section we have concentrated our attention on the case of the wave and the Schrödinger equations stated on a bounded smooth open set Ω of $\mathbb{R}^d$, along with homogeneous Dirichlet boundary conditions, that is, $Bu = 0$ with $Bu = u_{|\partial\Omega}$. This is done for the purpose of motivation. However, the abstract framework we present in what follows allows one to consider more general settings. We give a nonexhaustive list of such settings.

1. One can consider the elliptic operator A to be the Lapace-Beltrami (up to principal part with the requirement that A be selfadjoint and nonnegative) on a smooth Riemannian manifold $\mathcal{M}$ without boundary. If viewed as an unbounded operator on $L^2(\mathcal{M})$, one sees that 0 is an eigenvalue associated with constant functions. Considering the operator acting on $L^2(\mathcal{M})/\mathbb{C}$ one then obtains the setting developed in what follows.
2. On a bounded smooth open set or on a smooth Riemannian manifold $\mathcal{M}$ with boundary, one can consider Neumann boundary conditions, that is, $Bu = 0$ with $Bu = \partial_{\mathsf{n}} u_{|\partial\mathcal{M}}$, with n the outgoing normal vector at $\partial\mathcal{M}$. The operator A can be the Laplace(-Beltrami) operator. Similarly to the case without boundary, 0 is an eigenvalue of the elliptic operator A associated with constant functions. The same quotient procedure yields a setting compatible with the analysis developed in what follows. More generally, one can consider a boundary operator B that fulfills the more general Lopatinskiĭ-Šapiro boundary condition that encompasses both Dirichlet and Neumann conditions, with the requirement that the considered elliptic operator be selfadjoint and nonnegative; we refer for instance to [19], Chapters 2 and 4. Then, one has to consider a quotient with respect to the kernel of the resulting unbounded operator if this kernel is not trivial.
3. Above, the coefficients of the elliptic operator are considered smooth. This can be relaxed down to Lipschitz regularity, yet preserving the properties needed in what follows. Similarly, the regularity of the open set Ω or the manifold $\mathcal{M}$ (and its boundary $\partial\mathcal{M}$) can be chosen as low as $W^{2,\infty}$. Then, spectral properties of the elliptic operator remain unchanged and solutions to the wave and the Schrödinger equations can be defined similarly, for instance through a semigroup formulation and the Hille-Yosida theorem.
   Observability of waves was obtained for $\mathscr{C}^2$-coefficients ($\mathscr{C}^3$-boundary) in [4]. This was recently extended by the authors [5] to the case of $\mathscr{C}^1$-coefficients ($\mathscr{C}^2$-boundary). Both articles rely on the method presented here to obtain observability from semiclassical observability. In case of Lipschitz coefficents ($W^{2,\infty}$-boundary), observability is an open question. With the present article, proving such a result amounts to achieving a semiclassical observability result.

1.6. Notation

Often to avoid introducing constants explicitely we use the notation $a \lesssim b$, with the meaning $a \leq Cb$ for some constant $C > 0$. Similarly, the notation $a \eqsim b$ reads $Cb \leq a \leq C'b$ for some constants $C > 0$ and $C' > 0$.

2. Abstract equations and semi-classical reduction

Let E be a Hilbert space. Consider a positive unbounded selfadjoint operator A on E with dense domain $D(\mathsf{A})$. Assume that there exists a real Hibert basis $(e_\nu)_{\nu\in\mathbb{N}}$ of E, associated with a nondecreasing sequence of eigenvalues, $(\lambda_\nu)_{\nu\in\mathbb{N}}$, with $\lambda_\nu \to +\infty$ as $\nu \to +\infty$, for instance if A has a compact resolvent map. In the example of the introduction, one has $E = L^2(\Omega)$ and $\mathsf{A} = -\Delta$ with $D(\mathsf{A}) = H^2(\Omega) \cap H^1_0(\Omega)$.

Any $u \in E$ reads $u = \sum_{\nu\in\mathbb{N}} u_\nu \, e_\nu$ with $u_\nu = (u, e_\nu)_E$ and $(u_\nu)_\nu \in \ell^2(\mathbb{C})$. For $s \geq 0$ one has

$$D(\mathsf{A}^s) = \{u \in E;\ (\lambda_\nu^s u_\nu) \in \ell^2(\mathbb{C})\}.$$

For $s < 0$, $D(\mathsf{A}^s)$ denotes the dual of $D(\mathsf{A}^{|s|})$ using E as a pivot space, and if $u \in D(\mathsf{A}^s)$ then $u = \sum_{\nu\in\mathbb{N}} u_\nu e_\nu$ with convergence for the natural dual norm on $D(\mathsf{A}^s)$ and $\mathsf{A}^s u = \sum_{\nu\in\mathbb{N}} \lambda_\nu^s u_\nu e_\nu \in E$. In all cases, a norm on $D(\mathsf{A}^s)$ is given by

$$\|u\|^2_{D(\mathsf{A}^s)} = \|\mathsf{A}^s u\|^2_E = \big\|(\lambda_\nu^s u_\nu)_\nu\big\|^2_{\ell^2(\mathbb{C})} = \sum_{\nu\in\mathbb{N}} \lambda_\nu^{2s} |u_\nu|^2,$$

with the associated inner product $(u,v)_{D(\mathsf{A}^s)} = (\mathsf{A}^s u, \mathsf{A}^s v)_E = \big((\lambda_\nu^s u_\nu)_\nu, (\lambda_\nu^s v_\nu)_\nu\big)_{\ell^2(\mathbb{C})}$. One has the continuous and dense injection $D(\mathsf{A}^s) \hookrightarrow D(\mathsf{A}^{s'})$ if $s \geq s'$, moreover compact if $s > s'$. In fact, one defines $D(\mathsf{A}^\infty) = \bigcap_{s\in\mathbb{R}} D(\mathsf{A}^s)$. If $u = \sum_{\nu\in\mathbb{N}} u_\nu e_\nu \in D(\mathsf{A}^s)$, one sees that $U_n = \sum_{\lambda_\nu \leq n} u_\nu e_\nu \in D(\mathsf{A}^\infty)$ and $U_n \to u$ in $D(\mathsf{A}^s)$ as $n \to \infty$. Hence the injection $D(\mathsf{A}^\infty) \hookrightarrow D(\mathsf{A}^s)$ is dense for any $s \in \mathbb{R}$.

2.1. Abstract wave equation and energy levels

The wave equation reads

$$\partial_t^2 u + \mathsf{A} u = 0, \qquad u_{|t=0} = \underline{u}^0,\ \partial_t u_{|t=0} = \underline{u}^1. \tag{2.1}$$

With the initial conditions $\underline{u}^0 \in E = D(\mathsf{A}^0)$ and $\underline{u}^1 \in D(\mathsf{A}^{-1/2})$, the unique solution to (2.1) in $\mathscr{C}^0(\mathbb{R}; E) \cap \mathscr{C}^1\big(\mathbb{R}; D(\mathsf{A}^{-1/2})\big)$ is given by

$$u(t) = \sum_{\nu\in\mathbb{N}} \Big(\cos(t\sqrt{\lambda_\nu})\underline{u}_\nu^0 + \frac{1}{\sqrt{\lambda_\nu}}\sin(t\sqrt{\lambda_\nu})\underline{u}_\nu^1\Big)e_\nu = \sum_{\nu\in\mathbb{N}} \big(e^{it\sqrt{\lambda_\nu}}u_\nu^+ + e^{-it\sqrt{\lambda_\nu}}u_\nu^-\big)e_\nu, \tag{2.2}$$

with $u_\nu^\pm = (\underline{u}_\nu^0 \mp i\underline{u}_\nu^1/\sqrt{\lambda_\nu})/2$. Moreover, one has $u \in \cap_k \mathscr{C}^k(\mathbb{R}; D(\mathsf{A}^{-k/2}))$. Note that $(u_\nu^\pm)_{\nu\in\mathbb{N}} \in \ell^2(\mathbb{C})$. In turn the r.h.s. of (2.2) is solution to the wave equation (2.1) with $\underline{u}^0$ and $\underline{u}^1$ given by

$$\underline{u}_\nu^0 = u_\nu^+ + u_\nu^- \ \text{ and }\ \underline{u}_\nu^1 = i\sqrt{\lambda_\nu}(u_\nu^+ - u_\nu^-). \tag{2.3}$$

Note that $u \in L^2_{\mathrm{loc}}(\mathbb{R}; E) \cap H^1_{\mathrm{loc}}\big(\mathbb{R}; D(\mathsf{A}^{-1/2})\big) \cap H^2_{\mathrm{loc}}\big(\mathbb{R}; D(\mathsf{A}^{-1})\big)$ and the equation in (2.1) is fulfilled in $L^2_{\mathrm{loc}}\big(\mathbb{R}; D(\mathsf{A}^{-1})\big)$. The energy of the solution is given by

$$\mathcal{E}_0(u)(t) = \frac{1}{2}\big(\|u(t)\|_E^2 + \|\partial_t u(t)\|^2_{D(\mathsf{A}^{-1/2})}\big) = \frac{1}{2}\big(\|u(t)\|_E^2 + \|\mathsf{A}^{-1/2}\partial_t u(t)\|_E^2\big).$$

It is constant with respect to t, that is,

$$\begin{aligned}\mathcal{E}_0(u)(t) = \mathcal{E}_0(u)(0) &= \frac{1}{2}\big(\|\underline{u}^0\|_E^2 + \|\underline{u}^1\|^2_{D(\mathsf{A}^{-1/2})}\big) = \frac{1}{2}\sum_{\nu\in\mathbb{N}}\big(|\underline{u}_\nu^0|^2 + \lambda_\nu^{-1}|\underline{u}_\nu^1|^2\big) \\ &= \sum_{\nu\in\mathbb{N}}\big(|u_\nu^+|^2 + |u_\nu^-|^2\big).\end{aligned} \tag{2.4}$$

We thus simply write $\mathcal{E}_0(u)$ and one has

$$\mathcal{E}_0(u) = \frac{1}{2}(t_2 - t_1)^{-1}\int_{t_1}^{t_2} \big(\|u(t)\|_E^2 + \|\partial_t u(t)\|^2_{D(\mathsf{A}^{-1/2})}\big)\,dt, \tag{2.5}$$

for any time interval $[t_1, t_2]$, leading to a well defined energy if only considering the solution u in $L^2_{\mathrm{loc}}(\mathbb{R}; E) \cap H^1_{\mathrm{loc}}\big(\mathbb{R}; D(\mathsf{A}^{1/2})\big) \cap H^2_{\mathrm{loc}}\big(\mathbb{R}; D(\mathsf{A}^{-1})\big)$.

More generally, if $s \in \mathbb{R}$ and $\underline{u}^0 \in D(\mathsf{A}^{s/2})$ and $\underline{u}^1 \in D(\mathsf{A}^{(s-1)/2})$, the unique solution to (2.1) in $\cap_k \mathscr{C}^k(\mathbb{R}; D(\mathsf{A}^{(s-k)/2})$ is given by (2.2), and one can define the energy

$$\mathcal{E}_s(u)(t) = \frac{1}{2}\big(\|u(t)\|^2_{D(\mathsf{A}^{s/2})} + \|\partial_t u(t)\|^2_{D(\mathsf{A}^{(s-1)/2})}\big) = \frac{1}{2}\big(\|\mathsf{A}^{s/2}u(t)\|_E^2 + \|\mathsf{A}^{(s-1)/2}\partial_t u(t)\|_E^2\big),$$

that is also constant with respect to t. Note that if $u(t)$ is such a solution then $\mathsf{A}^{s/2}u(t)$ is a solution to (2.1) in $\cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{-k/2})\big)$ as above, with

$$\mathcal{E}_s(u) = \mathcal{E}_0\big(\mathsf{A}^{s/2}u\big) = \sum_{\nu\in\mathbb{N}} \lambda_\nu^s\big(|u_\nu^+|^2 + |u_\nu^-|^2\big). \tag{2.6}$$

We shall say that such a solution to the wave equation lies in the s-energy level. Similarly to (2.5) one has

$$\mathcal{E}_s(u) = \frac{1}{2}(t_2-t_1)^{-1}\int_{t_1}^{t_2} \big(\|u(t)\|^2_{D(\mathsf{A}^{s/2})} + \|\partial_t u(t)\|^2_{D(\mathsf{A}^{(s-1)/2})}\big)\, \mathrm{d}t, \tag{2.7}$$

for any time interval $[t_1, t_2]$.

If $\underline{u}^0, \underline{u}^1 \in D(\mathsf{A}^\infty)$ the associated solution $u(t)$ is such that $u \in \mathscr{C}^k(\mathbb{R}; \mathsf{A}^s)$ for any $k\in\mathbb{N}$ and $s\in\mathbb{R}$. One has $\mathcal{E}_s(u) < \infty$ and one says that $u(t)$ lies in all energy levels.

If $\underline{u}^0 \in D(\mathsf{A}^{\ell/2})$ and $\underline{u}^1 \in D(\mathsf{A}^{(\ell-1)/2})$ and if one denotes by $u(t)$ the unique solution to the wave equation (2.1) that lies in the ℓ-energy level, there exists a sequence $u_n(t)$ of solutions that lie in all energy levels and such that

$$\mathcal{E}_\ell(u-u_n) \to 0 \qquad \text{as } n\to+\infty, \tag{2.8}$$

from the density of $D(\mathsf{A}^\infty)$ in $D(\mathsf{A}^s)$ for any $s\in\mathbb{R}$. It suffices to consider two sequences $(\underline{u}_n^0)_n$ and $(\underline{u}_n^1)_n$ both in $D(\mathsf{A}^\infty)$ such that $\|\underline{u}^0 - \underline{u}_n^0\|_{D(\mathsf{A}^{\ell/2})} \to 0$ and $\|\underline{u}^1 - \underline{u}_n^1\|_{D(\mathsf{A}^{(\ell-1)/2})} \to 0$ and let $u_n(t)$ be the associated solution to the wave equation.

2.2. Dyadic decomposition for waves

Let $0<\alpha<1$, $\varrho \in]1, 1/\alpha[$ and set

$$J_k = \{\nu;\ \alpha\varrho^{|k|} \le \sqrt{\lambda_\nu} < \varrho^{|k|}/\alpha\}, \qquad k\in\mathbb{Z}^*.$$

Note that $\#J_k < \infty$ from the assumed properties of the eigenvalues. Set also $h_k = \varrho^{-|k|}$. Introduce

$$E_k = \operatorname{span}\{e_\nu;\ \nu\in J_k\},$$

equipped with the norm $\|u\|_E = \sum_{\nu\in J_k}|u_\nu|^2$ for $u = \sum_{\nu\in J_k} u_\nu e_\nu \in E_k$. Observe that if $u\in E_k$ then $\mathsf{A}^n u\in E_k$, using that $\#J_k<\infty$. Hence, E_k is a subspace of $D(\mathsf{A}^\infty)$.

At this stage it is important to note that $J_{-k} = J_k$ implying $E_{-k} = E_k$. However, we shall identify $u\in E_k$ with the following solution of the wave equation

$$u = \sum_{\nu\in J_k} \mathrm{e}^{\operatorname{sgn}(k) it\sqrt{\lambda_\nu}} u_\nu e_\nu. \tag{2.9}$$

The sign of k here becomes important. Yet, note that $u\in E_k$ if and only if $\bar u \in E_{-k}$ through this identification since the eigenfunctions e_ν are assumed real.

Following up, we identify $\partial_t^\ell u$ with $u = \sum_{\nu\in J_k}\big(i\operatorname{sgn}(k)\big)^\ell \lambda_\nu^{\ell/2} u_\nu e_\nu \in E_k$, that is, its value at $t=0$. Similarly, one identifies $\mathsf{A}^s u$ with $\sum_{\nu\in J_k}\lambda_\nu^s u_\nu e_\nu \in E_k$.

Lemma 2.1. *For $u\in E_k$, the norms*

$$h_k^{2s+r}\|\partial_t^r \mathsf{A}^s u\|_E, \quad r\in\mathbb{N},\ s\in\mathbb{R},$$

are equivalent to $\|u\|_E$*, uniformly with respect to* $k \in \mathbb{Z}^*$*.*

Proof. One writes

$$h_k^{2(2s+r)} \|\partial_t^r \mathsf{A}^s u\|_E^2 = \sum_{\nu \in J_k} |h_k^2 \lambda_\nu|^{2s+r} \, |u_\nu|^2 \eqsim \sum_{\nu \in J_k} |u_\nu|^2 = \|u\|_E^2,$$

as $h_k^2 \lambda_\nu \eqsim 1$ for $\nu \in J_k$. ☐

For $u \in E_k$, the identified solution to the wave equation given in (2.9) lies in all energy level. One has

$$\mathcal{E}_s(u) = \frac{1}{2}\big(\|\mathsf{A}^{s/2} u(t)\|_E^2 + \|\mathsf{A}^{(s-1)/2} \partial_t u(t)\|_E^2\big) \eqsim h_k^{-2s} \|u\|_E^2.$$

In particular, note that for $u \in E_k$ both terms in the energy coincide; this is not the case in general for a solution of the wave equation for fixed time (while it is true in time average). The reason is that $u \in E_k$ is in fact solution to the following half-wave equation

$$(\partial_t - \operatorname{sgn}(k) i \mathsf{A}^{1/2}) u = 0.$$

We introduce the following sets of sequences of functions

$$\begin{aligned} B &= \big\{ (u^k)_{k \in \mathbb{Z}^*};\ u^k \in E_k \text{ and } \|u^k\|_{L^2(\Omega)} \leq 1 \big\}, \\ B^\pm &= \big\{ (u^k)_{k \in \pm\mathbb{N}^*};\ u^k \in E_k \text{ and } \|u^k\|_{L^2(\Omega)} \leq 1 \big\}. \end{aligned} \tag{2.10}$$

2.3. Abstract Schrödinger equation and dyadic decomposition

The Schrödinger equation associated with the operator A reads

$$\partial_t u - i\mathsf{A} u = 0, \qquad u_{|t=0} = \underline{u}^0. \tag{2.11}$$

With the initial conditions $\underline{u}^0 \in D(\mathsf{A}^p)$, for some $p \in \mathbb{R}$, the unique solution to (2.11) in $\mathscr{C}^0\big(\mathbb{R}; D(\mathsf{A}^p)\big) \cap \mathscr{C}^1\big(\mathbb{R}; D(\mathsf{A}^{p-1})\big)$ is given by

$$u(t) = \sum_{\nu \in \mathbb{N}} \mathrm{e}^{it\lambda_\nu} \underline{u}_\nu^0 \, e_\nu. \tag{2.12}$$

One has

$$\|u(t)\|_{D(\mathsf{A}^p)}^2 = \sum_{\nu \in \mathbb{N}} \lambda_\nu^{2p} |\underline{u}_\nu^0|^2 = \|\underline{u}^0\|_{D(\mathsf{A}^p)}^2.$$

Moreover, one has $u \in \cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{p-k})\big)$.

As above let $0 < \alpha < 1$, $\varrho \in]1, 1/\alpha[$ and set $h_k = \varrho^{-k}$ and

$$J_k^S = \{\nu;\ \alpha \varrho^k \leq \lambda_\nu < \varrho^k / \alpha\}, \qquad k \in \mathbb{N}^*.$$

Note that $\# J_k^S < \infty$ from the assumed properties of the eigenvalues. Introduce

$$E_k^S = \operatorname{span}\{e_\nu;\ \nu \in J_k^S\},$$

equipped with the norm $\|u\|_E^2 = \sum_{\nu \in J_k^S} |u_\nu|^2$ for $u = \sum_{\nu \in J_k^S} u_\nu e_\nu \in E_k^S$. Note that E_k^S is a subspace of $D(\mathsf{A}^\infty)$. We shall identify $u \in E_k^S$ with the following solution to the Schrödinger equation

$$u = \sum_{\nu \in J_k^S} \mathrm{e}^{it\lambda_\nu} u_\nu e_\nu. \tag{2.13}$$

The counterpart to Lemma 2.1 is the following lemma.

Lemma 2.2. *For $u \in E_k^S$, and $r \in \mathbb{N}$ and $s \in \mathbb{R}$ the norm*

$$h_k^{s+r} \|\partial_t^r \mathsf{A}^s u\|_E,$$

is equivalent to $\|u\|_E$, uniformly with respect to $k \in \mathbb{N}^*$.

We introduce the following set of sequences of functions

$$B^S = \big\{ (u^k)_{k \in \mathbb{N}^*};\ u^k \in E_k^S \text{ and } \|u^k\|_{L^2(\Omega)} \leq 1 \big\}. \tag{2.14}$$

If $\underline{u}^0 \in D(\mathsf{A}^\infty)$ the associated solution $u(t)$ is such that $u \in \mathscr{C}^k(\mathbb{R}; \mathsf{A}^s)$ for any $k \in \mathbb{N}$ and $s \in \mathbb{R}$. If $\underline{u}^0 \in D(\mathsf{A}^p)$, denote by $u(t)$ the unique solution to the Schrödinger equation (2.11) that lies in $\mathscr{C}^0\big(\mathbb{R}; D(\mathsf{A}^p)\big)$. From the density of $D(\mathsf{A}^\infty)$ in $D(\mathsf{A}^s)$ for any $s \in \mathbb{R}$ one can consider a sequence $(\underline{u}_n^0)_n \subset D(\mathsf{A}^\infty)$ such that $\|\underline{u}^0 - \underline{u}_n^0\|_{D(\mathsf{A}^p)} \to 0$. The associated solutions $u_n(t)$ to the Schrödinger equation are such that

$$\sup_{t \in \mathbb{R}} \|u(t) - u_n(t)\|_{D(\mathsf{A}^p)} = \|\underline{u}^0 - \underline{u}_n^0\|_{D(\mathsf{A}^p)} \ \to 0 \qquad \text{as } n \to +\infty. \tag{2.15}$$

3. Main results

3.1. Observation operator and unique-continuation assumption

For some Hilbert space K consider an observation operator $\mathsf{L} : E \to K$, possibly unbounded, with domain given by $D(\mathsf{L}) = D(\mathsf{A}^{m_0})$ for some $m_0 \in \mathbb{R}$, with

$$\| \mathsf{L}\, u\|_K \leq C_0 \|u\|_{D(\mathsf{A}^{m_0})}. \tag{3.1}$$

We introduce the following assumption.

Assumption 3.1 (unique-continuation)**.** If u is an eigenvector of A such that $\mathsf{L}(u) = 0$, then $u = 0$.

Observe that an eigenvector of A lies in $D(\mathsf{A}^\infty)$ and thus lies in $D(\mathsf{L})$.

3.2. From semi-classical observation to observability for waves

Our starting point will be the following property.

Semi-classical observability property (wave equation)**.** For some $\ell_1 \in \mathbb{R}$, $C > 0$, $k^0 \in \mathbb{N}$ and some $T > 0$ one has

$$\mathcal{E}_{\ell_1}(u^k) \leq C \int_0^T \| \mathsf{L}\, u^k(t)\|_K^2 \mathrm{d}t, \qquad (u^k)_{k \in \mathbb{Z}} \in B,\ |k| \geq k_0. \tag{3.2}$$

Our main result in the case of the wave equation is the following theorem.

Theorem 3.2. *Let $\ell_1 \in \mathbb{R}$ with $\ell_1 \leq 2m_0$. Assume that there exists $C > 0$, $k_0 > 0$, and $T > 0$ such that* (3.2) *holds for any $U = (u^k)_{k\in\mathbb{Z}} \in B$ and any $|k| \geq k_0$. Under the unique-continuation Assumption 3.1, for any $T' > T$ there exists $C' > 0$ such that for any $(\underline{u}^0, \underline{u}^1) \in D(\mathsf{A}^{m_0}) \times D(\mathsf{A}^{m_0 - 1/2})$ the solution to* (2.1) *given by* (2.2) *satisfies*

$$\mathcal{E}_{\ell_1}(u) \leq C' \int_0^{T'} \| \mathsf{L}\, u(t) \|_K^2 \, \mathrm{d}t. \tag{3.3}$$

Note that the r.h.s. in (3.3) makes sense because of (3.1) and $u(t) \in L^2_{\text{loc}}\big(\mathbb{R}; D(\mathsf{A}^{m_0})\big)$. Note that the requirement $\ell_1 \leq 2m_0$ is natural since $u(t)$ lies in the $(2m_0)$-energy level.

Remark 3.3. The theorem states that semi-classical observation on an interval of length T implies classical observation on any interval of greater length. In Remark 4.3 below (see Sect. 4, proof of Lem. 4.2), we explain how our proof does not *a priori* provide classical observation at $T' = T$. However, this calls for the following comment. With microlocal techniques in mind, the time $T > 0$ that appears in the semi-classical observability inequality (3.2) is any time $T > T_{GCC}$ where $T_{GCC} > 0$ is the time given by the celebrated *geometrical control condition* (GCC); see for instance [2]. With the only condition on T' being $T' > T$ one sees that one can also obtain the observability result for any $T' > T_{GCC}$. Having $T' > T$ is not a drawback.

Remark 3.4. Let $\bar{u}$ denote the complex conjugate. In many cases one has

$$\| \mathsf{L}(\bar{u}) \|_K = \| \mathsf{L}(u) \|_K. \tag{3.4}$$

As $\overline{u^k} \in E_{-k}$ if $u^k \in E_k$ note that having

$$\mathcal{E}_{\ell_1}(u^k) \leq C \int_0^T \| \mathsf{L}\, u^k(t) \|_K^2 \mathrm{d}t, \qquad (u^k)_{k\in\mathbb{N}} \in B^+, \ k \geq k_0, \tag{3.5}$$

implies (3.2). Consequently, if (3.4) does hold, then assuming (3.5) suffices to reach the conclusion of Theorem 3.2.

In the case $\ell_1 = 2m_0$, the argument we develop leading to a proof of Theorem 3.2 is based on [17] (see also [4]), yet with more details provided here. The proof of Theorem 3.2 in this first case is given in Section 5.1. The argument is further refined to treat the case $\ell_1 < 2m_0$, that is, the case of an observability estimate with some energy loss. The proof of Theorem 3.2 in this second case is carried out in Section 5.3. Even though the second case contains the first one, we chose to provide a simpler proof in the first case for the benefit of the reader.

3.3. Comments and comparison with existing results

The splitting of high- and low-frequencies for observability and controllability issues is now classical. It appeared for instance in the work of Bardos-Lebeau-Rauch [2]. There, high frequencies are observed by means of microlocal techniques and low frequencies are handled by means of a unique-continuation property. Here, the semi-classical observability inequality (3.2) states that one can *uniformly* observe high-frequency data: the constant C is uniform for $|k| \geq k_0$. The general observability constant in (3.3) takes moreover low-frequency phenomena into account. Thus, there is no *a priori* estimation of this second constant by that in (3.2). In the proof, low-frequencies are handled by a unique-continuation property within a contradiction argument. In this context, the approach is very similar to that of [2]. However, in [2] the notion of high-frequency observability is different. Low frequencies are not removed but their contribution appears somehow in a compact term that is removed by the unique-continuation argument.

To obtain an estimation of the general observability constant in (3.3), a quantified version of the unique-continuation property is needed. This approach is used in Theorem 1 in [14] where the observability constant is estimated by a high-frequency observation and a low-frequency spectral property. The latter is precisely a quantified unique-continuation property for eigenmodes. However, the notion of high-frequency observability is also different in [14]. There, high-frequency means that low frequencies are removed, whereas, here, high-frequency *bands* are considered. The present approach yields wave 'packets' that are much easier to handle for the derivation of the semi-classical observability inequality. This turns out important when one faces further technical difficulties if for instance considering coefficients with low regularity [5]. Compared with the result in [14], the present result allows one to treat the case of observability inequalities with losses of derivatives.

One important aspect of the present article lies in its abstract form. An abstract treatment of observability can also be found in [23], Section 6.9. There, a decomposition in wave packets is introduced and is quite similar to what is done here in the high-frequency regime. In [23] observability is characterized through a *Hautus test*, that is, an estimation that involves the resolvent operator of the semigroup generator. The property assumed on wave-packets is different from the semi-classical observability inequality considered here. A further study of the connections between the two approches would be of interest. In particular, in Proposition 6.9.5 in [23] the property on wave-packets is allowed to be only assumed for high-frequency wave packets. Note that an estimation of the observation time is given in [23], whereas here the observation time is any time T' such that $T' > T$ where T is the time that appears in the semi-classical observation inequality. In this approach we have in mind a time T greater than a time given by a geometrical control condition. Such time is then *not estimated* through some spectral properties as in [23].

Finally, for the purpose of a quantification of approximate controllability, splitting of low- and high-frequencies is also a key ingredient of the result in [16].

3.4. Admissibility condition for waves

In the introduction we also considered admissibility conditions. Such conditions are useful in cases where $\mathsf{L}\, u$ makes sense in energy levels lower than $2m_0$. Note that the $(2m_0)$-level is given by the boundedness of L on $D(\mathsf{A}^{m_0})$; see (3.1). Yet, since $\| \mathsf{L}\, u(t)\|_K$ appears in a time-integrated form in the sought observability estimates, in some cases, one can expect some improvement as formulated with the following additional assumption. Denote by $\mathcal{F}_{\ell_0}$ the space of solutions to (2.1) that lie in $L^2_{\text{loc}}\big(\mathbb{R}; D(\mathsf{A}^{\ell_0/2})\big) \cap H^1_{\text{loc}}\big(\mathbb{R}; D(\mathsf{A}^{(\ell_0-1)/2})\big) \cap H^2_{\text{loc}}\big(\mathbb{R}; D(\mathsf{A}^{\ell_0/2-1})\big)$ equipped with the norm $\mathcal{E}_{\ell_0}(u)^{1/2}$.

Assumption 3.5 (admissibility condition for waves at the ℓ_0-energy level)**.** For some $\ell_0 \leq 2m_0$, the operator L extends as an unbounded operator on $\mathcal{F}^{\ell_0}$ into $L^2_{\text{loc}}(\mathbb{R}; K)$, also denoted by L, and for some $S > 0$ and $C_S > 0$ one has

$$\int_0^S \| \mathsf{L}\, u(t)\|_K^2 \, \mathrm{d}t \leq C_S \, \mathcal{E}_{\ell_0}(u), \qquad u \in \mathcal{F}_{\ell_0}. \tag{3.6}$$

In other words, Assumption 3.5 states that L is bounded on the space of solutions that lie in the ℓ_0-energy level. Considering only $\ell_0 \leq 2m_0$ is natural since (3.6) holds for $\ell_0 = 2m_0$ by (3.1).

Example 3.6. A basic example where Assumption (3.5) is useful, meaning $\ell_0 < 2m_0$, is the case of the Dirichlet Laplace operator Δ_D as in the introduction and the observation operator L given by the Neumann trace operator localized in an open subset Γ of $\partial\Omega$, $\mathsf{L}\, u = \mathbf{1}_\Gamma \partial_{\mathsf{n}} u_{|\partial\Omega}$, as in Example 1.1-(2). With the trace map $H^{1/2+\varepsilon}(\Omega) \to H^\varepsilon(\partial\Omega)$, one can use $D(\mathsf{L}) = H^{3/2+\varepsilon} \cap H^1_0(\Omega) = D(A^{m_0})$ with $m_0 = 3/4 + \varepsilon/2$, for any $\varepsilon > 0$. If $\underline{u}^0 \in H^1_0(\Omega)$ and $\underline{u}^1 \in L^2(\Omega)$ the associated weak solution to the wave equation lies in $\mathscr{C}^0\big(\mathbb{R}; H^1_0(\Omega)\big) \cap \mathscr{C}^1\big(\mathbb{R}; L^2(\Omega)\big)$. One thus has $\nabla u \in \mathscr{C}^0\big(\mathbb{R}; L^2(\Omega)\big)$, a regularity too low to allow one to apply the trace theorem to define $\partial_{\mathsf{n}} u_{|\partial\Omega} = (\mathsf{n} \cdot \nabla u)_{|\partial\Omega}$. However, because of the so-called hidden regularity for such a solution to the wave equation, one finds that the trace $\partial_{\mathsf{n}} u_{|\partial\Omega}$ makes sense and lies in $L^2_{\text{loc}}(\mathbb{R}; L^2(\partial\Omega))$; see for example [15]. A weak solution lies in the 1-energy level we have defined and moreover one has, for any $S > 0$,

$$\int_0^S \|\mathbf{1}_\Gamma \partial_{\mathsf{n}} u_{|\partial\Omega}\|^2_{L^2(\partial\Omega)}\, \mathrm{d}t \lesssim \mathcal{E}_1(u).$$

In this case, one has $1 = \ell_0 < 2m_0 = 3/2 + \varepsilon$.

From the time invariance of the energy with (3.6) one finds

$$\int_J \|\mathsf{L}\, u(t)\|^2_K\, \mathrm{d}t \leq C_S\, \mathcal{E}_{\ell_0}(u),$$

for any interval J of length $|J| = S$. Moreover, for any bounded interval I one has

$$\int_I \|\mathsf{L}\, u(t)\|^2_K\, \mathrm{d}t \leq C_{|I|}\, \mathcal{E}_{\ell_0}(u), \tag{3.7}$$

for some $C_{|I|} > 0$ only function of $|I|$.

With Assumption 3.5 one obtains the following corollary to Theorem 3.2.

Corollary 3.7. *Let $\ell_1 \leq \ell_0 \leq 2m_0$. Assume that there exists $C > 0$, $k_0 > 0$, and $T > 0$ such that (3.2) holds for any $U = (u^k)_{k\in\mathbb{N}} \in B$ and any $k \geq k_0$. Assume also that (3.4) holds. Under the unique-continuation Assumption 3.1 and the admissibility Assumption 3.5, for any $T' > T$ there exists $C' > 0$ such that for any $(\underline{u}^0, \underline{u}^1) \in D(\mathsf{A}^{\ell_0/2}) \times D(\mathsf{A}^{(\ell_0-1)/2})$ the solution to (2.1) given by (2.2) satisfies*

$$\mathcal{E}_{\ell_1}(u) \leq C' \int_0^{T'} \|\mathsf{L}\, u(t)\|^2_K\, \mathrm{d}t. \tag{3.8}$$

The proof simply uses the density of solutions in the $(2m_0)$-energy level in the space of solution in the ℓ_0-energy level and that both sides of the inequality (3.8) are continuous with respect to the ℓ_0-energy; continuity of the r.h.s. is precisely (3.7) that follows from Assumption 3.5.

A remark similar to Remark 3.4 can be made for the result of Corollary 3.7.

3.5. Main result for the Schrödinger equation

We first state what is meant by semi-classical observability in the case of the Schrödinger equation.

Semi-classical observability property (Schrödinger equation)**.** For some $p_1 \in \mathbb{R}$, $C > 0$, $k_0 \in \mathbb{N}$ and some $T > 0$ one has

$$\|u^k\|_{D(\mathsf{A}^{p_1})} \leq C \int_0^T \|\mathsf{L}\, u^k(t)\|_K \mathrm{d}t, \qquad (u^k)_{k\in\mathbb{N}} \in B^S,\ k \geq k_0. \tag{3.9}$$

Our main result in the case of a the Schrödinger equation is the following theorem.

Theorem 3.8. *Let $p_1 \leq m_0$. Assume that there exists $C > 0$, $k_0 > 0$, and $T > 0$ such that (3.9) holds for any $U = (u^k)_{k\in\mathbb{N}} \in B^S$ and any $k \geq k_0$. Under the unique-continuation Assumption 3.1, for any $T' > T$ there exists $C' > 0$ such that for any $\underline{u}^0 \in D(\mathsf{A}^{m_0})$ the solution to (2.11) given by (2.12) satisfies*

$$\|\underline{u}^0\|_{D(\mathsf{A}^{p_1})} \leq C' \int_0^{T'} \|\mathsf{L}\, u(t)\|_K\, \mathrm{d}t. \tag{3.10}$$

Recall that m_0 is as given by the continuity property (3.1) for L.

Denote by $\mathcal{F}_{\ell_0}^S$ the space of solutions to (2.11) that lie in $L^2_{\text{loc}}\big(\mathbb{R}; D(\mathsf{A}^{p_0})\big) \cap H^1_{\text{loc}}\big(\mathbb{R}; D(\mathsf{A}^{p_0-1})\big)$ equipped with the norm $\|.\|_{D(\mathsf{A}^{p_0})}$. Similarly to waves an admissibility assumption reads as follows.

Assumption 3.9 (admissibility condition the Schrödinger equation in $D(\mathsf{A}^{p_0})$)**.** For some $p_0 \leq m_0$, the operator L extends as an unbounded operator on $\mathcal{F}_{\ell_0}^S$ into $L^2_{\text{loc}}(\mathbb{R}; K)$, also denoted by L, and for some $S > 0$ and $C_S > 0$ one has

$$\int_0^S \|\,\mathsf{L}\, u(t)\|_K \,\mathrm{d}t \leq C_S\, \|\underline{u}^0\|_{D(\mathsf{A}^{p_0})}, \quad u \in \mathcal{F}_{\ell_0}^S.$$

Then, for any bounded interval I one has

$$\int_I \|\,\mathsf{L}\, u(t)\|_K \,\mathrm{d}t \leq C_{|I|}\, \|\underline{u}^0\|_{D(\mathsf{A}^{p_0})}, \tag{3.11}$$

for some $C_{|I|} > 0$ only function of $|I|$.

With Assumption 3.9 one obtains the following corollary to Theorem 3.8.

Corollary 3.10. *Let $p_1 \leq p_0 \leq m_0$. Assume that there exists $C > 0$, $k_0 > 0$, and $T > 0$ such that (3.9) holds for any $U = (u^k)_{k\in\mathbb{N}} \in B^S$ and any $k \geq k_0$. Under the unique-continuation Assumption 3.1 and the admissibility Assumption 3.9, for any $T' > T$ there exists $C' > 0$ such that for any $\underline{u}^0 \in D(\mathsf{A}^{p_0})$ the solution to (2.11) given by (2.12) satisfies*

$$\|\underline{u}^0\|_{D(\mathsf{A}^{p_1})} \leq C' \int_0^{T'} \|\,\mathsf{L}\, u(t)\|_K \,\mathrm{d}t. \tag{3.12}$$

3.6. Existing and potential applications

In the introduction, we considered the wave equation on an open set of $\mathbb{R}^d$. This can be generalized to the manifold setting. Consider a compact connected Riemannian manifold $\mathcal{M}$ of dimension d with boundary endowed with a metric $g = (g_{ij})$. Introduce the elliptic operator $A = A_{\kappa,g} = \kappa^{-1} \operatorname{div}_g(\kappa \nabla_g)$, that is, in local coordinates

$$Af = \kappa^{-1} (\det g)^{-1/2} \sum_{1\leq i,j\leq d} \partial_{x_i} \big(\kappa (\det g)^{1/2} g^{ij}(x) \partial_{x_j} f\big). \tag{3.13}$$

where κ is a positive function on $\mathcal{M}$. The metric g and the function κ can be assumed $\mathscr{C}^k$ with $k \geq 1$ or Lipschitz. Recall that $(g^{ij}(x))$ is the inverse of $(g_{ij}(x))$. The operator A is unbounded on $E = L^2(\mathcal{M})$. With the domain $D(A) = H^2(\mathcal{M}) \cap H^1_0(\mathcal{M})$ one finds that A is selfadjoint, with respect to the L^2-inner product, and A is negative. With the elliptic operator A one also defines the wave operator

$$P = P_{\kappa,g} = \partial_t^2 - A_{\kappa,g}, \tag{3.14}$$

and one can consider the associated homogeneous wave equation

$$\begin{cases} P\, u = 0 & \text{in } \mathbb{R} \times \mathcal{M}, \\ u = 0 & \text{in } \mathbb{R} \times \partial\mathcal{M}, \\ u_{|t=0} = \underline{u}^0, \ \partial_t u_{|t=0} = \underline{u}^1 & \text{in } \mathcal{M}. \end{cases}$$

For an open set $\omega \subset \mathcal{M}$ one can consider the observation operator with $K = L^2(\Omega)$ and the action on a solution to the wave equation given by $\mathsf{L}_\omega\, u = \mathbf{1}_{\mathbb{R}\times\omega}\partial_t u$. It maps a weak-solution as above into $L^2_{\mathrm{loc}}(\mathbb{R}\times\omega)$. For an open set $\Gamma \subset \partial\mathcal{M}$ one can consider the observation operator with $K = L^2(\Gamma)$ and the action on a solution to the wave equation given by $\mathsf{L}_\Gamma\, u = \mathbf{1}_{\mathbb{R}\times\Gamma}\partial_{\mathsf{n}} u_{|\partial\Omega}$, where ∂_{n} is normal derivative at the boundary. In both cases the admissibility Assumption 3.5 holds as one has

$$\int_0^S \|\,\mathsf{L}_\omega\, u(t)\|^2_{L^2(\omega)}\mathrm{d}t \lesssim \mathcal{E}_1(u), \quad \text{and} \quad \int_0^S \|\,\mathsf{L}_\Gamma\, u(t)\|^2_{L^2(\Gamma)}\mathrm{d}t \lesssim \mathcal{E}_1(u),$$

for a weak solution and for some $S > 0$. The second property is in fact the so-called hidden regularity property of waves; see *e.g.* [15]. In both cases Assumption 3.1 holds with classical unique-continuation results for elliptic operators; see for instance Theorem 2.4 in [11] and Theorems 5.11 and 5.13 in [18]. The result of Corollary 3.7 thus applies. It is used without loss of energy, that is, in the case $\ell_1 = 1$, in [4] for a boundary observation in the case of $\mathscr{C}^2$-coefficients and in [5] for both types of observations in the case of $\mathscr{C}^1$-coefficients with also result for Lipschitz coefficients by a perturbation argument. In these references, powerful tools of semi-classical analysis and semi-classical measures are key to prove a semi-classical observability estimate as in (3.2).

Here, we also treat the case of the loss of derivatives, that is, if $\ell_1 < \ell_0$ in the assumed semi-classical observability estimate (3.2) and in the resulting observability estimate in Theorem 3.2 and Corollary 3.7. Estimates with such losses can be found in the literature. We refer for instance to the work of F. Fanelli and E. Zuazua [10]. Their result is in the case of very rough coefficients (log-Lipschitz) and only concerns the wave equation in one space dimension. Results in higher dimensions are open to our knowledge and the study of such cases could benefit from the use of simpler localized-in-frequency waves and their semi-classical setting. Though observation estimates with loss of derivatives are not so common for waves, they appear quite naturally for Schrödinger equations. See [3, 6] for such results in the presence of weak (hyperbolic) trapping, or Sections 6.4 and 6.5 in [7], and Section 4 in [21] for the observability of Schrödinger on the square with an observation in (say) the vertical boundary.

4. Time microlocalization

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{H} = D(\mathsf{A}^s)$ for some $s \in \mathbb{R}$ or $\mathcal{H} = K$ in what follows. For a function $F \in \mathscr{C}_c^\infty(\mathbb{R}_+^*)$, with $F \geq 0$, set $F_k(\tau) = F(\mathrm{sgn}(k)\tau)$ for $k \in \mathbb{Z}^*$, and consider the operator $F_k(h_k D_t)$ that simply acts as a Fourier multiplier on functions of time t with values in $\mathcal{H}$. Most often we shall write $F_k^{\mathcal{H}}(h_k D_t)$ to keep explicit on which space the operator acts. Since $F_k(h_k\tau)$ is bounded, $F_k^{\mathcal{H}}(h_k D_t)$ maps $L^2(\mathbb{R};\mathcal{H})$ into itself. It also maps $\mathscr{S}(\mathbb{R};\mathcal{H})$ (resp. $\mathscr{S}'(\mathbb{R};\mathcal{H})$) into itself. We shall choose F according to the following lemma.

Lemma 4.1. *One can choose F supported in $]\alpha, \alpha^{-1}[$ with $\sum_{k\in\mathbb{Z}^*} F_k(h_k\tau)^2 \geq 1$ if $|\tau| \geq 1$.*

Proof. Let $\alpha < a < \rho^{-1} < 1$ and $F \in \mathscr{C}_c^\infty(]\alpha, \alpha^{-1}[)$ such that $F = 1$ on $[a, a^{-1}]$. Let $|\tau| \geq 1$. With $h_k = \rho^{-|k|}$, one has $F_k(h_k\tau) = 1$ if $a < \rho^{-|k|}|\tau| < a^{-1}$ and $\mathrm{sgn}(k) = \mathrm{sgn}(\tau)$, or equivalently

$$\frac{\ln(|\tau|) + \ln(a)}{\ln(\rho)} \leq |k| \leq \frac{\ln(|\tau|) + \ln(a^{-1})}{\ln(\rho)} \quad \text{and} \quad \mathrm{sgn}(k) = \mathrm{sgn}(\tau).$$

The difference between the two bounds is $2\ln(a^{-1})/\ln(\rho) > 2$ and one has $\frac{\ln(|\tau|)+\ln(a^{-1})}{\ln(\rho)} > 1$. Hence, there is at least one value of $k \in \mathbb{Z}^*$ such that $F_k(h_k\tau) = 1$. ☐

Let $\mathsf{T} > 0$. For $j \in \mathbb{Z}$ set

$$I_j = [j\mathsf{T}, (j+1)\mathsf{T}[. \tag{4.1}$$

Define $H_{\mathcal{H}}$ as the space of functions $w \in L^2_{\text{loc}}(\mathbb{R};\mathcal{H})$ such that

$$\|w\|_{H_{\mathcal{H}}} := \sup_{j\in\mathbb{Z}} \|1_{I_j} w\|_{L^2(\mathbb{R};\mathcal{H})} < \infty, \tag{4.2}$$

that is, the space of uniformly locally L^2-bounded functions with values in $\mathcal{H}$.

One has $H_{\mathcal{H}} \subset \mathscr{S}'(\mathbb{R};\mathcal{H})$ and thus $F_k^{\mathcal{H}}(h_k D_t)w$ is a well defined tempered distribution in time t with values in $\mathcal{H}$. The following lemma improves upon this result.

Lemma 4.2. *The operator $F_k^{\mathcal{H}}(h_k D_t)$ fulfills the following properties.*

1. *One has*

$$F_k^{\mathcal{H}}(h_k D_t)w(t) = \sum_{j\in\mathbb{Z}} F_k^{\mathcal{H}}(h_k D_t)\big(1_{I_j} w\big)(t), \qquad w \in H_{\mathcal{H}}, \tag{4.3}$$

and for any $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$, there exists $C > 0$ such that

$$\|\phi F_k^{\mathcal{H}}(h_k D_t)w\|_{L^\infty(\mathbb{R};\mathcal{H})} \le C\|w\|_{H_{\mathcal{H}}}, \tag{4.4}$$

meaning that $F_k^{\mathcal{H}}(h_k D_t)$ maps $H_{\mathcal{H}}$ into $L^\infty_{\text{loc}}(\mathbb{R};\mathcal{H})$ continuously.

2. *There exists $C > 0$ such that*

$$\sum_{k\in\mathbb{Z}^*} \|F_k^{\mathcal{H}}(h_k D_t)(\psi w)\|^2_{L^2(\mathbb{R};\mathcal{H})} \le C\|\psi w\|^2_{L^2(\mathbb{R};\mathcal{H})},$$

for $w \in H_{\mathcal{H}}$ and $\psi \in L^\infty(\mathbb{R})$ with compact support.

3. *If $\varphi \in \mathscr{C}_c^\infty(]0,\mathsf{T}[)$ and $\psi \in L^\infty(\mathbb{R})$ is such that $\psi = 1$ in I_0, then for any $M \ge 1$, there exists $C_M > 0$ such that*

$$\big\|\varphi F_k^{\mathcal{H}}(h_k D_t)\big((1-\psi)w\big)\big\|_{L^2(\mathbb{R};\mathcal{H})} \le C_M h_k^M \|w\|_{H_{\mathcal{H}}}. \tag{4.5}$$

Proof. Let $w \in H_{\mathcal{H}}$ and set $w_j = 1_{I_j} w$. One has

$$F_k^{\mathcal{H}}(h_k D_t)w_j(t) = \frac{1}{2\pi}\int \mathrm{e}^{it\tau} F_k(h_k\tau)\hat{w}_j(\tau)\mathrm{d}\tau = \frac{1}{2\pi}\iint \mathrm{e}^{i(t-s)\tau} F_k(h_k\tau) w_j(s)\mathrm{d}\tau\mathrm{d}s.$$

Note that $F_k^{\mathcal{H}}(h_k D_t)w_j(t) \in \mathscr{S}(\mathbb{R};\mathcal{H})$ since its Fourier transform in t, $F_k(h_k\tau)\hat{w}_j(\tau)$, is in $\mathscr{C}_c^\infty(\mathbb{R};\mathcal{H})$. One finds

$$\begin{aligned}\|F_k^{\mathcal{H}}(h_k D_t)w_j(t)\|_{\mathcal{H}} &\lesssim \int_{\mathbb{R}} F_k(h_k\tau)\mathrm{d}\tau \int_{\mathbb{R}} \|w_j(s)\|_{\mathcal{H}}\mathrm{d}s \lesssim \mathsf{T}^{1/2} h_k^{-1}\|F\|_{L^1}\|w_j\|_{L^2(\mathbb{R};\mathcal{H})} \\ &\lesssim \mathsf{T}^{1/2} h_k^{-1}\|F\|_{L^1}\|w\|_{H_{\mathcal{H}}}, \qquad t \in \mathbb{R}.\end{aligned} \tag{4.6}$$

For the first part of the lemma we treat the case $k > 0$. The case $k < 0$ can be treated similarly. Consider $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$ and $j \in \mathbb{Z}$ such that $\gamma_j = \text{dist}(\text{supp}(\phi), I_j) > 0$. Using that

$$\frac{-i}{t-s}\partial_\tau \mathrm{e}^{i(t-s)\tau} = \mathrm{e}^{i(t-s)\tau},$$

for $t \neq s$, with N integrations by parts one writes

$$\phi(t)F_k^{\mathcal{H}}(h_kD_t)w_j(t) = \frac{i^N h_k^N}{2\pi}\iint \mathrm{e}^{i(t-s)\tau}F_k^{(N)}(h_k\tau)\frac{\phi(t)w_j(s)}{(t-s)^N}\mathrm{d}\tau\mathrm{d}s.$$

One finds

$$\|\phi(t)F_k^{\mathcal{H}}(h_kD_t)w_j(t)\|_{\mathcal{H}} \lesssim \mathsf{T}^{1/2}h_k^{N-1}\gamma_j^{-N}\|\phi\|_{L^\infty}\|F^{(N)}\|_{L^1}\|w\|_{H_{\mathcal{H}}}, \quad t\in\mathbb{R}. \tag{4.7}$$

Observe that there exists $j_0 \in \mathbb{N}$ such that $\gamma_j \geq (|j|-j_0)\mathsf{T}$ if $|j|\geq j_0$. If one chooses $N\geq 2$, one finds that the series $\sum_j F_k^{\mathcal{H}}(h_kD_t)w_j(t)$ converges in $L^\infty_{\mathrm{loc}}(\mathbb{R};\mathcal{H})$ thanks to the factor γ_j^{-N} in estimation (4.7). Taking into account that $\sum_j F_k^{\mathcal{H}}(h_kD_t)w_j$ converges to $F_k^{\mathcal{H}}(h_kD_t)w$ in $\mathscr{S}'(\mathbb{R};\mathcal{H})$, one concludes that $F_k^{\mathcal{H}}(h_kD_t)w \in L^\infty_{\mathrm{loc}}(\mathbb{R};\mathcal{H})$ and that (4.3) holds. Finally, using also (4.6) for a finite numbers of terms, with $|j|<j_0$, one concludes that estimate (4.4) holds

Let now $\psi\in L^\infty(\mathbb{R})$ have compact support. Then $\psi w\in L^2(\mathbb{R};\mathcal{H})$ and the Fourier transform of $F_k^{\mathcal{H}}(h_kD_t)(\psi w)$ is $F_k(h_k\tau)\widehat{\psi w}(\tau)$ giving

$$\|F_k^{\mathcal{H}}(h_kD_t)(\psi w)\|^2_{L^2(\mathbb{R};\mathcal{H})} = \int_{\mathbb{R}} F_k(h_k\tau)^2\|\widehat{\psi w}(\tau)\|^2_{\mathcal{H}}\,\mathrm{d}\tau.$$

Since $\mathrm{supp}(F)\subset[\alpha,\alpha^{-1}]$ and $h_k=\rho^{-|k|}$ one finds that $F_k(h_k\tau)\neq 0$ if $\tau\neq 0$ and

$$\frac{\ln(|\tau|)+\ln(\alpha)}{\ln(\rho)} \leq |k| \leq \frac{\ln(|\tau|)+\ln(\alpha^{-1})}{\ln(\rho)}.$$

The difference between the two bounds is $2\ln(\alpha^{-1})/\ln(\rho)$. Most important, it is constant. Hence, the sum $\sum_k F_k(h_k\tau)^2$ only involves a finite number m of terms that is independent of τ. Consequently

$$\sum_{k\in\mathbb{Z}^*}\|F_k^{\mathcal{H}}(h_kD_t)(\psi w)\|^2_{L^2(\mathbb{R};\mathcal{H})} \leq m\|F\|^2_{L^\infty}\|\widehat{\psi w}\|^2_{L^2(\mathbb{R};\mathcal{H})} \leq m\|F\|^2_{L^\infty}\|\psi w\|^2_{L^2(\mathbb{R};\mathcal{H})}.$$

Finally, consider $\varphi\in\mathscr{C}_c^\infty(]0,\mathsf{T}[)$ and $\psi\in L^\infty(\mathbb{R})$ such that $\psi=1$ in I_0. With (4.3) one has

$$F_k^{\mathcal{H}}(h_kD_t)\big((1-\psi)w\big) = \sum_{|j|\geq 1}F_k^{\mathcal{H}}(h_kD_t)\big((1-\psi)w_j\big), \qquad w_j = 1_{I_j}w.$$

Let $N\geq 2$. For $|j|=1$ with (4.7) one obtains,

$$\|\varphi(t)F_k^{\mathcal{H}}(h_kD_t)\big((1-\psi)w_j\big)(t)\|_{L^\infty(\mathbb{R};\mathcal{H})} \leq C_N\mathsf{T}^{1/2}h_k^{N-1}R^{-N}\|\varphi\|_{L^\infty}\|w\|_{H_{\mathcal{H}}}, \tag{4.8}$$

with $R=\mathrm{dist}\big(\mathrm{supp}(\varphi),I_0^c\big)$, using that $\|(1-\psi)w\|_{H_{\mathcal{H}}}\lesssim\|w\|_{H_{\mathcal{H}}}$. For $|j|\geq 2$ one finds in turn

$$\|\varphi(t)F_k^{\mathcal{H}}(h_kD_t)\big((1-\psi)w_j\big)(t)\|_{L^\infty(\mathbb{R};\mathcal{H})} \leq C_N\mathsf{T}^{1/2}h_k^{N-1}\big((|j|-1)\mathsf{T}\big)^{-N}\|\varphi\|_{L^\infty}\|w\|_{H_{\mathcal{H}}}. \tag{4.9}$$

Combining (4.8) and (4.9), one has

$$\|\varphi(t)F_k^{\mathcal{H}}(h_kD_t)\big((1-\psi)w\big)\|_{L^\infty(\mathbb{R};\mathcal{H})} \tag{4.10}$$

$$\leq C_N \mathsf{T}^{1/2} h_k^{N-1} \|\varphi\|_{L^\infty} \Big[\sum_{|j|\geq 2} \big(|j|-1)\mathsf{T}\big)^{-N} + 2R^{-N} \Big] \|w\|_{H_{\mathcal{H}}},$$

which yields (4.5) since $\sum_{|j|\geq 2} \big((|j|-1)\mathsf{T}\big)^{-N}$ converges. □

Remark 4.3. Having $\operatorname{supp}(\varphi) \subset]0, \mathsf{T}[$ implies $\psi = 1$ in a neighborhood of $\operatorname{supp}(\varphi)$, which is of importance in the proof and cannot be improved upon. Such localization is quite classical in microlocal techniques. In the proof of Theorem 3.2 in Section 5 one sets $\mathsf{T} = (T + T')/2$, giving $T < \mathsf{T} < T'$, leaving room for a time-localization function φ to be introduced. The use of the previous lemma therein is thus the technical reason for requiring $T < T'$ in Theorem 3.2. This is however not an important loss of generality, as explained in Remark 3.3.

4.1. Action on waves

We now consider the action of $F_k(h_k D_t)$ on a solution $u(t)$ of the abstract wave equation (2.1) as given by (2.2) that lies in the ℓ-energy level for some $\ell \in \mathbb{R}$. In such case, if one uses $\mathcal{H} = D(\mathsf{A}^{\ell/2})$, with (2.7) one sees that $u \in H_{D(\mathsf{A}^{\ell/2})}$ as defined in (4.2). Thus, $F_k^{D(\mathsf{A}^{\ell/2})}(h_k D_t)u(t)$ makes sense by Lemma 4.2. One has the following result.

Lemma 4.4. *Let $u(t)$ be a solution to (2.1) that lies in the ℓ-energy level for some $\ell \in \mathbb{R}$. Let $k \in \mathbb{Z}^*$. One has $F_k^{D(\mathsf{A}^{\ell/2})}(h_k D_t)u(t) \in E_k$ and*

$$F_k^{D(\mathsf{A}^{\ell/2})}(h_k D_t)u(t) = \sum_{\nu \in J_k} \begin{cases} F_k\big(h_k \sqrt{\lambda_\nu}\big) \mathrm{e}^{it\sqrt{\lambda_\nu}} u_\nu^+ e_\nu & \text{if } k > 0, \\ F_k\big(-h_k \sqrt{\lambda_\nu}\big) \mathrm{e}^{-it\sqrt{\lambda_\nu}} u_\nu^- e_\nu & \text{if } k < 0. \end{cases} \tag{4.11}$$

Proof. Observe that the series in (2.2) that defines $u(t)$ converges in the space $H_{D(\mathsf{A}^{\ell/2})}$. Hence, with the first part in Lemma 4.2 one finds

$$F_k^{D(\mathsf{A}^{\ell/2})}(h_k D_t)u(t) = \sum_{\nu \in \mathbb{N}} F_k^{D(\mathsf{A}^{\ell/2})}(h_k D_t)\big(\mathrm{e}^{it\sqrt{\lambda_\nu}} u_\nu^+ + \mathrm{e}^{-it\sqrt{\lambda_\nu}} u_\nu^-\big) e_\nu.$$

One has $F_k(h_k D_t)\mathrm{e}^{irt} = F_k(h_k r)\mathrm{e}^{irt}$, $r \in \mathbb{R}$; see for instance (18.1.27) in [12]. This gives

$$F_k^{D(\mathsf{A}^{\ell/2})}(h_k D_t)u(t) = \sum_{\nu \in J_k} \Big(F_k\big(h_k \sqrt{\lambda_\nu}\big) \mathrm{e}^{it\sqrt{\lambda_\nu}} u_\nu^+ + F_k\big(-h_k \sqrt{\lambda_\nu}\big) \mathrm{e}^{-it\sqrt{\lambda_\nu}} u_\nu^- \Big) e_\nu,$$

since $F_k(h_k \sqrt{\lambda_\nu}) = 0$ unless $\nu \in J_k$. This gives the result using the dependency of the support of F_k upon the sign of k. □

Because of the form of $u^k(t) = F_k^{D(\mathsf{A}^{\ell/2})}(h_k D_t)u(t)$ one sees that $u^k(t)$ is also solution to the wave equation. Yet, as the sum is finite in (4.11) one has

$$u^k(t) \in \mathscr{C}^m\big(\mathbb{R}; D(\mathsf{A}^r)\big), \quad \forall m \in \mathbb{N},\ r \in \mathbb{R}, \tag{4.12}$$

that is, the wave $u^k(t)$ lies in all energy levels.

We now consider the particular case of a solution $u(t)$ that lies in the $(2m_0)$-energy level with m_0 as appearing in the continuity property (3.1) of L.

Lemma 4.5. *Let $u(t)$ be a solution to (2.1) that lies in the m_0-energy level. One has $\mathsf{L}\, u^k = F_k^K(h_k D_t) \,\mathsf{L}\, u$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}; K)$ for $k \in \mathbb{Z}^*$.*

Proof. We treat the case $k > 0$; the proof for the case $k < 0$ is similar. With Lemma 4.4 one has $\mathsf{L}\, u^k = \sum_{\nu\in J_k} F_k\big(h_k\sqrt{\lambda_\nu}\big)\mathrm{e}^{it\sqrt{\lambda_\nu}}u_\nu^+\,\mathsf{L}\, e_\nu$, using that the sum is finite. For $n \in \mathbb{N}^*$ set

$$U_n(t) = \sum_{\lambda_\nu\leq n} \big(\mathrm{e}^{it\sqrt{\lambda_\nu}}u_\nu^+ + \mathrm{e}^{-it\sqrt{\lambda_\nu}}u_\nu^-\big)e_\nu.$$

One has

$$\|U_n(t) - u(t)\|^2_{D(\mathsf{A}^{m_0})} \lesssim \sum_{\lambda_\nu>n} \lambda_\nu^{2m_0}\big(|u_\nu^+|^2 + |u_\nu^-|^2\big)^2 = \mathcal{E}_{2m_0}(u - U_n), \qquad t\in\mathbb{R}.$$

One has $\mathcal{E}_{2m_0}(u - U_n) \to 0$ as $n \to +\infty$ since $\big(\lambda_\nu^{m_0}u_\nu^\pm\big)_\nu \in \ell^2(\mathbb{C})$. Hence,

$$\|U_n - u\|_{H_{D(\mathsf{A}^{m_0})}} \lesssim \|U_n - u\|_{L^\infty(\mathbb{R};D(\mathsf{A}^{m_0}))} \to 0, \quad \text{as} \quad n \to +\infty,$$

yielding in turn

$$F_k^K(h_kD_t)\,\mathsf{L}\,U_n \underset{\mathsf{n}\to\infty}{\longrightarrow} F_k^K(h_kD_t)\,\mathsf{L}\,u \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R};K), \tag{4.13}$$

by Lemma 4.2. As the sum defining U_n is finite one has

$$F_k^K(h_kD_t)\,\mathsf{L}\,U_n = \sum_{\lambda_\nu\leq n} F_k(h_k\sqrt{\lambda_\nu})\mathrm{e}^{it\sqrt{\lambda_\nu}}u_\nu^+\,\mathsf{L}\,e_\nu,$$

using the support property of F_k for $k > 0$ and that $F_k(h_kD_t)\mathrm{e}^{irt} = F_k(h_kr)\mathrm{e}^{irt}$, $r\in\mathbb{R}$; see for instance (18.1.27) in [12]. One observes that $F_k^K(h_kD_t)\,\mathsf{L}\,U_n = \mathsf{L}\,u^k$ for n chosen sufficiently large. The limit in (4.13) hence gives the result. ☐

4.2. Action on solutions to the Schrödinger equation

The counterpart results of Lemmata 4.4 and 4.5 for the Schrödinger equation are the following ones.

Lemma 4.6. *Let $u(t)$ be a solution to the Schrödinger equation (2.11) with $\underline{u}^0 \in D(\mathsf{A}^p)$ for $p\in\mathbb{R}$. Let $k\in\mathbb{N}^*$. One has $F_k^{D(\mathsf{A}^p)}(h_kD_t)u(t) \in E_k^S$ and*

$$F_k^{D(\mathsf{A}^p)}(h_kD_t)u(t) = \sum_{\nu\in J_k^S} F_k\big(h_k\lambda_\nu\big)\mathrm{e}^{it\lambda_\nu}\underline{u}_\nu^0 e_\nu. \tag{4.14}$$

Lemma 4.7. *Let $u(t)$ be a solution to (2.11) with $\underline{u}^0 \in D(\mathsf{A}^{m_0})$. One has $\mathsf{L}\,u^k = F_k^K(h_kD_t)\,\mathsf{L}\,u$ in $L^\infty_{\mathrm{loc}}(\mathbb{R};K)$ for $k\in\mathbb{N}^*$ and $u^k(t) = F_k^{D(\mathsf{A}^{m_0})}(h_kD_t)u(t)$.*

The proof of Lemmata 4.4 and 4.5 can be adapted *mutatis mutandis*.

One sees that $u^k(t)$ is also solution to the Schrödinger equation and

$$u^k(t) \in \mathscr{C}^m\big(\mathbb{R}; D(\mathsf{A}^r)\big), \quad \forall m\in\mathbb{N},\ r\in\mathbb{R}, \tag{4.15}$$

as the sum is finite in (4.14).

5. Proof of the main result for waves

As explained below Theorem 3.2, for the benefit of the reader, we have chosen to provide a proof for the case $\ell_1 = 2m_0$ and a proof for the case $\ell_1 \leq 2m_0$. Even though the second case contains the first one, the proof in the first case is less technical.

5.1. Case $\ell_1 = 2m_0$.

Let $\mathsf{T} = (T+T')/2$ and $\delta^0 = (\mathsf{T}-T)/2 = (T'-\mathsf{T})/2$. Because of the time invariance of the energy, the assumed semi-classical observation inequality (3.2) reads

$$\mathcal{E}_{\ell_1}(u^k) \leq C \int_{\delta^0}^{\mathsf{T}-\delta^0} \|\mathsf{L}\, u^k(t)\|_K^2 \mathrm{d}t, \qquad (u^k)_{k\in\mathbb{Z}} \in B,\ |k| \geq k_0, \tag{5.1}$$

and we aim to prove that, for any $\delta \in]0,\delta^0]$,

$$\mathcal{E}_{\ell_1}(u) \leq C' \int_{-\delta}^{\mathsf{T}+\delta} \|\mathsf{L}\, u(t)\|_K^2 \,\mathrm{d}t$$

holds for any solution u to the wave equation (2.1) written in (2.2) that lies in the ℓ_1-energy level, that is, $\underline{u}^0 \in D(\mathsf{A}^{m_0})$ and $\underline{u}^1 \in D(\mathsf{A}^{m_0-1/2})$ here. The simultaneous treatment of $0 < \delta \leq \delta^0$ is used for a technical argument in the proof of Lemma 5.1 below.

For such a solution u, one notes that $\mathsf{L}\, u \in H_K$ by (3.7) and one has

$$\|\mathsf{L}\, u\|_{H_K}^2 \lesssim \|\mathsf{L}\, u\|_{L^\infty(\mathbb{R};K)}^2 \lesssim \|u\|_{L^\infty(\mathbb{R};D(\mathsf{A}^{m_0}))}^2 \lesssim \mathcal{E}_{\ell_1}(u), \tag{5.2}$$

since $\ell_1 = 2m_0$. Let F be chosen as in Lemma 4.1. With Lemma 4.4, for $k \in \mathbb{Z}^*$, set

$$u^k(t) = F_k^{D(\mathsf{A}^{\ell_1/2})}(h_k D_t) u(t) = \sum_{\nu\in J_k} \begin{cases} F_k\big(h_k\sqrt{\lambda_\nu}\big)\mathrm{e}^{it\sqrt{\lambda_\nu}} u_\nu^+ e_\nu & \text{if } k>0,\\ F_k\big(-h_k\sqrt{\lambda_\nu}\big)\mathrm{e}^{-it\sqrt{\lambda_\nu}} u_\nu^- e_\nu & \text{if } k<0. \end{cases} \tag{5.3}$$

One has $u^k \in E_k$. With the semi-classical observation property (5.1) one has

$$\mathcal{E}_{\ell_1}(u^k) \lesssim \|\varphi\, \mathsf{L}\, u^k(t)\|_{L^2(\mathbb{R},K)}^2, \qquad \text{for } |k| \geq k_0, \tag{5.4}$$

where $\varphi \in \mathscr{C}_c^\infty(]0,\mathsf{T}[)$ with $\varphi = 1$ on a neighborhood of $[\delta^0, \mathsf{T}-\delta^0]$. One has

$$\mathcal{E}_{\ell_1}(u^k) = \sum_{\nu\in J_k} \lambda_\nu^{\ell_1} \begin{cases} F_k\big(h_k\sqrt{\lambda_\nu}\big)^2 |u_\nu^+|^2 & \text{if } k>0,\\ F_k\big(-h_k\sqrt{\lambda_\nu}\big)^2 |u_\nu^-|^2 & \text{if } k<0. \end{cases}$$

Set $u^0 = \sum_{\lambda_\nu \leq 1} \Big(\mathrm{e}^{it\sqrt{\lambda_\nu}} u_\nu^+ + \mathrm{e}^{-it\sqrt{\lambda_\nu}} u_\nu^-\Big) e_\nu$. With Lemma 4.1 one finds

$$\begin{aligned} \mathcal{E}_{\ell_1}(u-u^0) &= \mathcal{E}_{\ell_1}(u) - \mathcal{E}_{\ell_1}(u^0) = \sum_{\lambda_\nu>1} \lambda_\nu^{\ell_1}\big(|u_\nu^-|^2 + |u_\nu^+|^2\big) \\ &\lesssim \sum_{k\in\mathbb{Z}^*}\sum_{\lambda_\nu>1} \lambda_\nu^{\ell_1} F_k\big(-h_k\sqrt{\lambda_\nu}\big)^2 |u_\nu^-|^2 + \sum_{k\in\mathbb{Z}^*}\sum_{\lambda_\nu>1} \lambda_\nu^{\ell_1} F_k\big(h_k\sqrt{\lambda_\nu}\big)^2 |u_\nu^+|^2 \end{aligned}$$

$$\lesssim \sum_{k\in-\mathbb{N}^*}\sum_{\nu\in\mathbb{N}} \lambda_\nu^{\ell_1} F_k\big(-h_k\sqrt{\lambda_\nu}\big)^2 |u_\nu^-|^2 + \sum_{k\in\mathbb{N}^*}\sum_{\nu\in\mathbb{N}} \lambda_\nu^{\ell_1} F_k\big(h_k\sqrt{\lambda_\nu}\big)^2 |u_\nu^+|^2$$
$$\lesssim \sum_{k\in\mathbb{Z}^*} \mathcal{E}_{\ell_1}(u^k).$$

One thus obtains with (5.4)

$$\mathcal{E}_{\ell_1}(u) \lesssim \mathcal{E}_{\ell_1}(u^0) + \sum_{k\in\mathbb{Z}^*} \mathcal{E}_{\ell_1}(u^k) \tag{5.5}$$
$$\lesssim \mathcal{E}_{\ell_1}(u^0) + \sum_{1\le|k|<k_1} \mathcal{E}_{\ell_1}(u^k) + \sum_{|k|\ge k_1} \|\varphi \,\mathsf{L}\, u^k(t)\|^2_{L^2(\mathbb{R};K)},$$

for $k_1 \geq k_0$ to be chosen below.

With Lemma 4.5, one can write

$$\mathcal{E}_{\ell_1}(u) \lesssim \sum_{0\le|k|<k_1} \mathcal{E}_{\ell_1}(u^k) + \sum_{|k|\ge k_1} \|\varphi F_k^K(h_k D_t)\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)}. \tag{5.6}$$

Set $\psi_\delta = 1_{[-\delta,\mathsf{T}+\delta]}$. With the third part of Lemma 4.2 and (5.2) one has

$$\|\varphi(t) F_k^K(h_k D_t)\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} \lesssim \|\varphi(t) F_k^K(h_k D_t)(\psi_\delta \,\mathsf{L}\, u)\|^2_{L^2(\mathbb{R};K)} + h_k^{2M} \|\,\mathsf{L}\, u\|^2_{H_K}$$
$$\lesssim \|\varphi(t) F_k^K(h_k D_t)(\psi_\delta \,\mathsf{L}\, u)\|^2_{L^2(\mathbb{R};K)} + h_k^{2M} \mathcal{E}_{\ell_1}(u).$$

With the second part of Lemma 4.2 one finds

$$\sum_{|k|\ge k_1} \|\varphi(t) F_k^K(h_k D_t)\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} \lesssim \|\psi_\delta \,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} + \sum_{|k|\ge k_1} h_k^{2M} \mathcal{E}_{\ell_1}(u)$$
$$\lesssim \|\psi_\delta \,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} + h_{k_1}^{2M} \mathcal{E}_{\ell_1}(u).$$

using that $h_k = \rho^{-|k|}$ with $\rho > 1$. With (5.6) one finds

$$\mathcal{E}_{\ell_1}(u) \lesssim \sum_{0\le|k|<k_1} \mathcal{E}_{\ell_1}(u^k) + \|\psi_\delta \,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} + h_{k_1}^{2M} \mathcal{E}_{\ell_1}(u).$$

For $k_1 \geq k_0$ chosen sufficiently large one obtains

$$\mathcal{E}_{\ell_1}(u) \lesssim \sum_{0\le|k|<k_1} \mathcal{E}_{\ell_1}(u^k) + \|\psi_\delta \,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)}. \tag{5.7}$$

To remove the first term on the r.h.s. of (5.7) we shall use the following lemma that states that only the trivial solution is invisible for the observation operator L.

Lemma 5.1 (absence of invisible waves)**.** *Let* $u \in \cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m_0-k/2})\big)$ *be solution to* (2.1) *and such that* $\psi_\delta \,\mathsf{L}\, u = 0$. *Then* $u = 0$.

Recall that writing $\psi_\delta \,\mathsf{L}\, u = 0$ makes sense by (5.2).

Proof. For $0 < \delta \leq \delta^0$ as above, set

$$\mathscr{N}_\delta = \big\{u \in \cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m_0-k/2})\big);\ u \text{ solution to (2.1) and } \mathsf{L}\, u(t) = 0 \text{ if } t \in]-\delta, \mathsf{T}+\delta[\big\},$$

that is, the space of invisible solutions in the sense of the observation operator $\psi_\delta\, \mathsf{L}$. We equip $\mathscr{N}_\delta$ with the norm associated with the energy $\mathcal{E}_{\ell_1}$. With (5.7) one has

$$\mathcal{E}_{\ell_1}(u) \lesssim \sum_{0\leq |k| < k_1} \mathcal{E}_{\ell_1}(u^k) \ \text{ on } \mathscr{N}_\delta, \quad 0 < \delta \leq \delta^0. \tag{5.8}$$

As the maps $u \mapsto u^k$ have a finite rank, they are compact. With (5.8) it follows that $\mathscr{N}_\delta$ has a compact unit ball and is thus finite dimensional by the Riesz theorem.

We claim that

$$u \in \mathscr{N}_\delta \quad \Rightarrow \quad u \in \bigcap_{m,r\in\mathbb{N}} \mathscr{C}^r\big(\mathbb{R}; D(\mathsf{A}^m)\big) \ \text{ and } \ \partial_t u \in \mathscr{N}_\delta. \tag{5.9}$$

The finite dimensional space $\mathscr{N}_\delta$ is thus stable under the action of the operator ∂_t. Consequently this operator has an eigenvector $\mathsf{v} \in \mathscr{N}_\delta$ with associated eigenvalue μ. One finds $\mathsf{A}\mathsf{v} = -\partial_t^2 \mathsf{v} = -\mu^2 \mathsf{v}$ meaning that $\mathsf{v}(t)$ is an eigenfunction for A for all $t \in \mathbb{R}$. As $\mathsf{L}\, \mathsf{v}(t) = 0$ if $t \in]-\delta, \mathsf{T}+\delta[$, with the unique-continuation Assumption 3.1 one obtains $\mathsf{v}(t) = 0$ for all $t \in]-\delta, \mathsf{T}+\delta[$. Hence, $\mathsf{v} = 0$ since the energy of this solution is zero and one concludes that $\mathscr{N}_\delta = \{0\}$.

We now prove our claim (5.9). Let $0 < \delta' < \delta$ and note that $\mathscr{N}_\delta \subset \mathscr{N}_{\delta'}$. Let $u \in \mathscr{N}_\delta$. For $0 < \varepsilon < \delta - \delta'$, observe that

$$w_\varepsilon(t) = \frac{u(t+\varepsilon) - u(t)}{\varepsilon} \in \mathscr{N}_{\delta'}.$$

On the one hand, as $u \in \mathscr{C}^1\big(\mathbb{R}; D(\mathsf{A}^{m_0-1/2})\big)$, one has

$$w_\varepsilon(t) \ \to \ \partial_t u(t) \quad \text{in } D(\mathsf{A}^{m_0-1/2}), \quad \forall t \in \mathbb{R}. \tag{5.10}$$

On the other hand, if one applies the operator $F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)$, one has

$$w_\varepsilon^k(t) = \frac{u^k(t+\varepsilon) - u^k(t)}{\varepsilon}.$$

Note indeed that $F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(u(.+\varepsilon))(t) = F_k^{D(\mathsf{A}^{m_0})}(h_k D_t) u(t+\varepsilon) = u^k(t+\varepsilon)$ since $F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)$ is a simple Fourier multiplier. Since $u^k \in \cap_{m,r\in\mathbb{N}} \mathscr{C}^r\big(\mathbb{R}, D(\mathsf{A}^m)\big)$, one finds that, for any k, $w_\varepsilon^k(t)$ converges to $\partial_t u^k(t)$ in $\mathscr{C}^r\big(J, D(\mathsf{A}^m)\big)$, for any $r, m \in \mathbb{N}$ and any bounded interval J. Hence, recalling that w_ε^k and $\partial_t u^k$ are solutions to the wave equation (2.1), and solving this equation from any initial time $t_0 \in J$, one finds $w_\varepsilon^k \to \partial_t u^k$ in the norm associated with the ℓ_1-energy. With (5.8) one finds that $(w_\varepsilon)_\varepsilon$ is of Cauchy type in $\mathscr{N}_{\delta'}$ for this latter norm, as $\varepsilon \to 0$. It thus converges to some $w \in \mathscr{N}_{\delta'}$, as $\mathscr{N}_{\delta'}$ is complete since finite dimensional. Then, one has

$$w_\varepsilon(t) \ \to w(t) \quad \text{in } D(\mathsf{A}^{m_0}), \qquad \text{uniformly for } t \in \text{any bounded interval of } \mathbb{R},$$

yielding $w = \partial_t u$ by (5.10). Consequently $\partial_t u \in \mathscr{N}_{\delta'}$ meaning $\partial_t u \in \cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m_0-k/2})\big)$ and $\mathsf{L}\, \partial_t u(t) = 0$ for $t \in]-\delta', \mathsf{T}+\delta'[$. Our choice of $\delta' \in]0, \delta[$ is however arbitrary. Hence, one obtains $\mathsf{L}\, \partial_t u(t) = 0$ for $t \in]-\delta, \mathsf{T}+\delta[$, meaning that $\partial_t u \in \mathscr{N}_\delta$.

Iterating the argument, one obtains $\mathsf{A}u = -\partial_t^2 u \in \mathscr{N}_\delta$, thus $\mathsf{A}u \in \cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m_0-k/2})\big)$ implying $u \in \cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m_0+1-k/2})\big)$. Iterations give $u \in \cap_{r,m\in\mathbb{N}} \mathscr{C}^r\big(\mathbb{R}; D(\mathsf{A}^m)\big)$. □

We now conclude the proof of Theorem 3.2 by a classical argument by contradiction, assuming that the observation inequality

$$\mathcal{E}_{\ell_1}(u) \lesssim \|\psi_\delta \,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} \tag{5.11}$$

does not hold. Then, there exists a sequence of initial conditions $(\underline{u}^{0,n}, \underline{u}^{1,n}) \in D(\mathsf{A}^{m_0}) \times D(\mathsf{A}^{m_0-1/2})$ with associated solutions $(u_n)_{n\in\mathbb{N}}$ to the wave equation such that $\mathcal{E}_{\ell_1}(u_n) = 1$ and $\|\psi_\delta \,\mathsf{L}\, u_n\|_{L^2(\mathbb{R};K)} \to 0$. Some subsequence, that we also write $(\underline{u}^{0,n}, \underline{u}^{1,n})$ for simplicity, weakly converges to some $(\underline{u}^0, \underline{u}^1) \in D(\mathsf{A}^{m_0}) \times D(\mathsf{A}^{m_0-1/2})$. Associated with $(\underline{u}^0, \underline{u}^1)$ is a solution u, also in the ℓ_1-energy level, and u_n converges weakly to u in $L^2\big(-\delta, \mathsf{T}+\delta; D(\mathsf{A}^{m_0})\big) \cap H^1\big(-\delta, \mathsf{T}+\delta; D(\mathsf{A}^{m_0-1/2})\big)$. Moreover one has $\psi_\delta \,\mathsf{L}\, u = 0$. In fact, one considers $\tilde{\mathsf{L}} : D(\mathsf{A}^{m_0}) \times D(\mathsf{A}^{m_0-1/2}) \to H_K$ with $\tilde{\mathsf{L}}(\underline{v}^0, \underline{v}^1) = \psi_\delta \,\mathsf{L}\, v$ where v is the linear wave with initial conditions $\underline{v}^0$ and $\underline{v}^1$ as given by (2.2). With (5.2) the map $\tilde{\mathsf{L}}$ is continuous. It is thus also continuous for the weak topologies; see for instance [22], Proposition 35.8. Since $\tilde{\mathsf{L}}(\underline{u}^{0,n}, \underline{u}^{1,n})$ converges strongly to 0, and thus also weakly, this gives $\tilde{\mathsf{L}}(\underline{u}^0, \underline{u}^1) = 0$, that is, $\psi_\delta \,\mathsf{L}\, u = 0$. With Lemma 5.1 one concludes that $u = 0$, and thus $\underline{u}^0 = \underline{u}^1 = 0$.

As above, for a linear wave v with initial conditions $(\underline{v}^0, \underline{v}^1) \in D(\mathsf{A}^{m_0}) \times D(\mathsf{A}^{m_0-1/2})$, one observes that $(\underline{v}^0, \underline{v}^1) \mapsto v^k(t) = F_k^{D(\mathsf{A}^{m_0})}(h_k D_t) v(t)$ is compact since with a finite dimensional range; see the expression in Lemma 4.4. As one has $(\underline{u}^0_n, \underline{u}^1_n) \rightharpoonup (0,0)$, one obtains that u_n^k converges strongly to 0 in the norm given by the $\mathcal{E}_{\ell_1}$-energy, for $0 \le |k| < k_1$. Here, one thus obtains

$$\lim_{n\to\infty} \sum_{0\le|k|<k_1} \mathcal{E}_{\ell_1}(u_n^k) = 0.$$

Estimate (5.7) applied to u_n thus leads to a contradiction since both terms on the r.h.s. converge to zero and the l.h.s. is equal to 1. This concludes the contradiction argument and the proof of Theorem 3.2 in the case $\ell_1 = 2m_0$. □

5.2. Refined time-microlocalization estimates

Here, we consider a solution $u(t)$ to the abstract wave equation (2.1) with

$$\underline{u}^0 \in D(\mathsf{A}^m) \;\text{ and }\; \underline{u}^1 \in D(\mathsf{A}^{m-1/2}),$$

for some $m \in \mathbb{R}$. Then, $u(t)$ lies in the $(2m)$-energy level and $u \in \cap_k \mathscr{C}^k(\mathbb{R}; D(\mathsf{A}^{m-k/2})$.

Let $F \in \mathscr{C}_c^\infty(]\alpha, \alpha^{-1}[)$ be as given by Lemma 4.1. Consider $\tilde{F} \in \mathscr{C}_c^\infty(]\alpha, \alpha^{-1}[)$ such that $\tilde{F} = 1$ in a neighborhood of $\operatorname{supp}(F)$. A first result we shall use is the following one.

Lemma 5.2. *Let $k \in \mathbb{Z}^*$. Let $\ell \in \mathbb{R}$. There exists $C = C_{m,\ell} > 0$ such that*

$$\big\|\tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t) u\big\|^2_{H_{D(\mathsf{A}^m)}} \le C h_k^{2(\ell-2m)} \mathcal{E}_\ell(u). \tag{5.12}$$

The definition of the Fourier multiplier $\tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)$ is as in the beginning of Section 4 for $F_k^{D(\mathsf{A}^m)}(h_k D_t)$. Combined with the third part of Lemma 4.2 one has the following corollary.

Corollary 5.3. *If $\varphi \in \mathscr{C}_c^\infty(]0,\mathsf{T}[)$ and $\psi \in L^\infty(\mathbb{R})$ is such that $\psi = 1$ in a neighborhood of $\overline{I_0}$, then for any $M \geq 1$ and $\ell \in \mathbb{R}$ there exists $C = C_{M,m,\ell} > 0$ such that*

$$\big\|\varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)\tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)u\big\|^2_{L^2(\mathbb{R};D(\mathsf{A}^m))} \leq C h_k^M \mathcal{E}_\ell(u). \tag{5.13}$$

Proof of Lemma 5.2. We consider the case $k > 0$. The case $k < 0$ is treated similarly. With Lemma 4.4 one has

$$\tilde{u}^k(t) = \sum_{\nu \in J_k} \tilde{F}_k\big(h_k\sqrt{\lambda_\nu}\big)\mathrm{e}^{it\sqrt{\lambda_\nu}}u_\nu^+ e_\nu \in E_k \subset D(\mathsf{A}^\infty).$$

By Lemma 2.1 one has

$$\begin{aligned} h_k^{4m}\|\tilde{u}^k(t)\|^2_{D(\mathsf{A}^m)} \eqsim h_k^{2\ell}\|\tilde{u}^k(t)\|^2_{D(\mathsf{A}^{\ell/2})} &\eqsim h_k^{2\ell}\sum_{\nu\in\tilde{J}_k}\lambda_\nu^\ell F_k\big(h_k\sqrt{\lambda_\nu}\big)^2\big|u_\nu^+\big|^2 \\ &\lesssim h_k^{2\ell}\sum_{\nu\in\tilde{J}_k}\lambda_\nu^\ell\big|u_\nu^+\big|^2, \end{aligned}$$

using that F_k is a bounded function since compactly supported. This gives

$$\|\tilde{u}^k(t)\|^2_{D(\mathsf{A}^m)} \lesssim h_k^{2(\ell-2m)}\mathcal{E}_\ell(u).$$

The result follows from the definition of $\|.\|_{H_{D(\mathsf{A}^m)}}$ in (4.2). ☐

A second important result is given by the following lemma.

Lemma 5.4. *Let $\varphi \in \mathscr{C}_c^\infty(]0,\mathsf{T}[)$ and $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$ be such that $\psi = 1$ in a neighborhood of $\overline{I_0}$, then for any $M \geq 1$ and $\ell \in \mathbb{R}$ there exists $C = C_{M,m,\ell} > 0$ such that*

$$\big\|\varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)\big(\operatorname{Id} - \tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)\big)u\big\|^2_{L^2(\mathbb{R};D(\mathsf{A}^m))} \leq C h_k^M \mathcal{E}_\ell(u). \tag{5.14}$$

Note that here one assumes the function ψ to be smooth as opposed to the results in Lemma 4.2 and Corollary 5.3. In fact, the proof of Lemma 5.4 is based on a kernel regularization argument that requires smoothness of the function ψ.

Proof. As in other proofs we treat the case $k > 0$. The case $k < 0$ can be treated similarly. The proof is short in the case $\ell \geq 2m$. In fact, one writes

$$\varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)\big(\operatorname{Id} - \tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)\big)u = w_1 + w_2,$$

with

$$w_1 = -\varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)\tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)u \quad\text{and}\quad w_2 = \varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)u.$$

The norm of w_1 is estimated by Corollary 5.3. An estimation of the norm of w_2 follows from the third part of Lemma 4.2, taking into account that $\ell \geq 2m$.

We shall thus only consider the case $\ell < 2m$. Let $r \in \mathbb{N}$ be such that $r \geq m - \ell/2 > 0$. With u solution to the wave equation one has $u = -\partial_t^2\mathsf{A}^{-1}u = D_t^2\mathsf{A}^{-1}u$ and thus $u = D_t^{2r}\mathsf{A}^{-r}u$. Set $w = \mathsf{A}^{-r}u \in$

$\cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m+r-k})\big)$. It is also a solution to the wave equation. One has

$$\mathcal{E}_{2m}(w) = \mathcal{E}_{2m-2r}(u) \lesssim \mathcal{E}_\ell(u). \tag{5.15}$$

One thus considers the action of the operator

$$P = \varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)\big(\operatorname{Id} - \tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)\big) D_t^{2r},$$

on w. Note that P maps $\mathscr{S}'\big(\mathbb{R}; D(\mathsf{A}^m)\big)$ into itself. Thus the action of P on $\sum_{j\in\mathbb{Z}} w_j$ yields $\sum_{j\in\mathbb{Z}} P w_j$ with convergence in $\mathscr{S}'\big(\mathbb{R}; D(\mathsf{A}^m)\big)$. Recall that $I_j = [j\mathsf{T}, (j+1)\mathsf{T}[$ and $w_j = 1_{I_j} w$.

The kernel of this operator is given by the following oscillatory integral

$$K(t,s) = (2\pi)^{-2}\varphi(t) \iiint e^{i(t-t')\tau' + i(t'-s)\tau}\big(1-\psi(t')\big) F_k(h_k\tau')\big(1-\tilde{F}_k(h_k\tau)\big)\tau^{2r}\, d\tau'\, dt'\, d\tau.$$

References on the subject of oscillatory integrals are [1, 13, 18]. In particular, usual operations such as integrations by parts are licit.

Since $\operatorname{supp}(F_k) \cap \operatorname{supp}(1-\tilde{F}_k) = \emptyset$ one has $\tau' \neq \tau$ in the integrand. In fact one has the following estimation.

Lemma 5.5. *There exists $C > 0$ such that for all $k \in \mathbb{Z}^*$ one has*

$$|\tau - \tau'|^{-1} \leq C \min(h_k, \tau^{-1}), \tag{5.16}$$

if $h_k\tau' \in \operatorname{supp}(F_k)$ and $h_k\tau \in \operatorname{supp}(1-\tilde{F}_k)$.

A proof of Lemma 5.5 is given below.

With $\frac{-i}{\tau-\tau'}\partial_{t'} e^{it'(\tau-\tau')} = e^{it'(\tau-\tau')}$, N integrations by parts give

$$K(t,s) = (-i)^N (2\pi)^{-2}\varphi(t) \iiint e^{i(t-t')\tau' + i(t'-s)\tau}\psi^{(N)}(t') \frac{\tau^{2r} F_k(h_k\tau')\big(1-\tilde{F}_k(h_k\tau)\big)}{(\tau-\tau')^N}\, d\tau'\, dt'\, d\tau.$$

This is the step of the proof where smoothness of the function ψ is used.

Observe that $t \neq t'$ if $t \in \operatorname{supp}(\varphi)$ and $t' \in \operatorname{supp}(1-\psi)$ or $\operatorname{supp}(\psi^{(N)})$. With $\frac{-i}{t-t'}\partial_{\tau'} e^{i\tau'(t-t')} = e^{i\tau'(t-t')}$, N' integrations by parts give

$$K(t,s) = (-i)^{N+N'} h_k^{N'} (2\pi)^{-2}\varphi(t) \iiint e^{i(t-t')\tau' + i(t'-s)\tau}\psi^{(N)}(t') \frac{\tau^{2r} F_k^{(N')}(h_k\tau')\big(1-\tilde{F}_k(h_k\tau)\big)}{(\tau-\tau')^N (t-t')^{N'}}\, d\tau'\, dt'\, d\tau.$$

Using $N \geq 2r+2$ and Lemma 5.5, with this form of the kernel of P, a first estimate one can write is the following

$$\big\|Pw_j(t)\big\|_{D(\mathsf{A}^m)} \lesssim \mathsf{T}^{1/2} h_k^{N'-1} \|\varphi\|_{L^\infty} \|\psi^{(N)}\|_{L^1} \|F_k^{(N')}\|_{L^1} \|1-\tilde{F}_k\|_{L^\infty} \|w_j\|_{L^2(\mathbb{R}; D(\mathsf{A}^m))}, \quad t \in \mathbb{R}. \tag{5.17}$$

For j such that $\operatorname{dist}(\operatorname{supp}(\psi), I_j) > 0$, we can proceed as in the proof of Lemma 4.2. Set $G_k(\sigma) = \sigma^{2r}(1-\tilde{F}_k)(\sigma)$. One has

$$K(t,s) = (-i)^{N+N'} h_k^{N'-2r} (2\pi)^{-2}\varphi(t) \iiint e^{i(t-t')\tau' + i(t'-s)\tau}\psi^{(N)}(t') \frac{F_k^{(N')}(h_k\tau') G_k(h_k\tau)}{(\tau-\tau')^N (t-t')^{N'}}\, d\tau'\, dt'\, d\tau.$$

Set $\gamma = \operatorname{dist}(\operatorname{supp}(\psi), I_j)$. One has $\frac{-i}{t'-s}\partial_\tau e^{i(t'-s)\tau} = e^{i(t-t')\tau'+i(t'-s)\tau}$. Thus, N'' integration by parts yield

$$Pw_j(t) = (-i)^{N+N'+N''} h_k^{N'+N''-2r}(2\pi)^{-2}\varphi(t) \iiiint e^{i(t-t')\tau'+i(t'-s)\tau}\psi^{(N)}(t') \times \frac{F_k^{(N')}(h_k\tau')G_k^{(N'')}(h_k\tau)}{(\tau-\tau')^N(t-t')^{N'}(t'-s)^{N''}} w_j(s)\, d\tau'\, dt'\, d\tau ds.$$

If $N'' \geq 2r+1$ then $\operatorname{supp}(G_k^{(N'')}) \subset \operatorname{supp}(\tilde{F}_k') \subset \operatorname{supp}(\tilde{F}_k)$. Using Lemma 5.5 one obtains the following estimate

$$\begin{aligned} &\big\|Pw_j(t)\big\|_{D(\mathsf{A}^m)} \\ &\quad\lesssim \mathsf{T}^{1/2} h_k^{N'+N''-2r-2}\gamma^{-N''}\|\varphi\|_{L^\infty}\|\psi^{(N)}\|_{L^1}\|F_k^{(N')}\|_{L^1}\|G_k^{(N'')}\|_{L^1}\|w_j\|_{L^2(\mathbb{R};D(\mathsf{A}^m))}, \quad t\in\mathbb{R}. \end{aligned} \tag{5.18}$$

If one chooses $N'' \geq 2$, with the $\gamma^{-N''}$ factor the sum with respect to j converges. Since $\sum_j w_j$ converges to w in $\mathscr{S}'\big(\mathbb{R}; D(\mathsf{A}^m)\big)$ one concludes that the action of P on w is equal to $\sum_j Pw_j$ in $\mathscr{S}'\big(\mathbb{R}; D(\mathsf{A}^m)\big)$ and thus in $L^\infty_{\text{loc}}\big(\mathbb{R}; D(\mathsf{A}^m)\big)$ by estimate (5.18) for $|j|$ sufficiently large and estimate (5.17) for the remaining finite number of terms. Moreover, one has

$$\big\|Pw(t)\big\|_{D(\mathsf{A}^m)} \lesssim C_M h_k^M \sup_{j\in\mathbb{Z}} \|w_j\|_{L^2(\mathbb{R};D(\mathsf{A}^m))} = C_M h_k^M \|w\|_{H_{D(\mathsf{A}^m)}} \lesssim C_M h_k^M \mathcal{E}_{2m}(w)^{1/2},$$

for any $M\in\mathbb{N}$. As $\varphi F_k^{D(\mathsf{A}^m)}(h_kD_t)(1-\psi)\big(\operatorname{Id} - \tilde{F}_k^{D(\mathsf{A}^m)}(h_kD_t)\big)u(t) = Pw(t)$, one concludes the proof with (5.15). □

Proof of Lemma 5.5. If $h_k\tau' \in \operatorname{supp}(F_k)$ then $\alpha \leq h_k\tau' \leq \alpha^{-1}$. If $h_k\tau \in \operatorname{supp}(1-\tilde{F}_k)$ then $h_k|\tau-\tau'| \gtrsim 1$ yielding

$$|\tau-\tau'|^{-1} \lesssim h_k. \tag{5.19}$$

First, consider the case $h_k\tau \leq 2\alpha^{-1}$. Then, $h_k \lesssim \tau^{-1}$. With (5.19) one obtains the result.
Second, consider the case $h_k\tau \geq 2\alpha^{-1}$. Then, one has

$$h_k|\tau-\tau'| = h_k\tau - h_k\tau' \geq \frac{1}{2}h_k\tau + \big(\frac{1}{2}h_k\tau - \alpha^{-1}\big) \geq \frac{1}{2}h_k\tau,$$

implying $|\tau-\tau'|^{-1} \lesssim \tau^{-1}$, yielding the result in this second case. □

5.3. General case: $\ell_1 \leq 2m_0$.

The assumed semi-classical observation inequality (3.2) reads

$$\mathcal{E}_{\ell_1}(u^k) \leq C\int_\delta^{\mathsf{T}-\delta} \|\mathsf{L}\, u^k(t)\|_K^2 dt, \qquad (u^k)_{k\in\mathbb{N}} \in B^+,\ k\geq k_0, \tag{5.20}$$

and we aim to prove that

$$\mathcal{E}_{\ell_1}(u) \leq C'\int_{-\delta}^{\mathsf{T}+\delta} \|\mathsf{L}\, u(t)\|_K^2\, dt, \tag{5.21}$$

holds for a solution u to the wave equation (2.1) written in (2.2) that lies in the $(2m_0)$-energy level. Thus, we consider $\underline{u}^0 \in D(\mathsf{A}^{m_0})$ and $\underline{u}^1 \in D(\mathsf{A}^{m_0-1/2})$. Then, $u \in \cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m_0-k/2})\big)$.

The beginning of the proof is similar to that given in Section 5.1 and one reaches the following estimate that is the counterpart to (5.6)

$$\mathcal{E}_{\ell_1}(u) \lesssim \sum_{0\leq |k| < k_1} \mathcal{E}_{\ell_1}(u^k) + \sum_{|k|\geq k_1} \|\varphi F_k^K(h_k D_t)\,\mathsf{L}\,u\|^2_{L^2(\mathbb{R};K)}, \tag{5.22}$$

for $k_1 \geq k_0$ to be chosen below. The treatment of the terms in the second sum is different from what is done in Section 5.1. Consider $\psi \in \mathscr{C}_c^\infty(]-\delta, \mathsf{T}+\delta[)$ such that $\psi = 1$ in a neighborhood of $\overline{I_0} = [0, \mathsf{T}]$. One writes

$$\|\varphi F_k^K(h_k D_t)\,\mathsf{L}\,u\|^2_{L^2(\mathbb{R};K)} \lesssim \|\varphi F_k^K(h_k D_t)\psi\,\mathsf{L}\,u\|^2_{L^2(\mathbb{R};K)} + \|\varphi F_k^K(h_k D_t)(1-\psi)\,\mathsf{L}\,u\|^2_{L^2(\mathbb{R};K)},$$

yielding, with the second part of Lemma 4.2

$$\mathcal{E}_{\ell_1}(u) \lesssim \sum_{0\leq |k| < k_1} \mathcal{E}_{\ell_1}(u^k) + \|\psi\,\mathsf{L}\,u\|^2_{L^2(\mathbb{R};K)} + \sum_{|k|\geq k_1} \|\varphi F_k^K(h_k D_t)(1-\psi)\,\mathsf{L}\,u\|^2_{L^2(\mathbb{R};K)}.$$

We now concentrate our attention on the terms in the last sum on the r.h.s.. First one writes

$$\|\varphi F_k^K(h_k D_t)(1-\psi)\,\mathsf{L}\,u\|_{L^2(\mathbb{R};K)} \lesssim \|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)u\|_{L^2(\mathbb{R};D(\mathsf{A}^{m_0}))},$$

using that L is bounded on $D(\mathsf{A}^{m_0})$; see (3.1). This gives

$$\mathcal{E}_{\ell_1}(u) \lesssim \sum_{0\leq |k| < k_1} \mathcal{E}_{\ell_1}(u^k) + \|\psi\,\mathsf{L}\,u\|^2_{L^2(\mathbb{R};K)} + \sum_{|k|\geq k_1} \|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)u\|^2_{L^2(\mathbb{R};D(\mathsf{A}^{m_0}))}. \tag{5.23}$$

Second, as in Section 5.2 consider $\tilde{F} \in \mathscr{C}_c^\infty(\mathbb{R}_+^*)$ such that $\tilde{F} = 1$ in a neighborhood of $\operatorname{supp}(F)$. With Corollary 5.3 and Lemma 5.4 one has

$$\begin{aligned}
&\|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)u\|^2_{L^2(\mathbb{R};D(\mathsf{A}^{m_0}))} \\
&\quad\lesssim \big\|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)\tilde{F}_k^{D(\mathsf{A}^{m_0})}(h_k D_t)u\big\|^2_{L^2(\mathbb{R};D(\mathsf{A}^{m_0})} \\
&\qquad + \big\|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)\big(\operatorname{Id} - \tilde{F}_k^{D(\mathsf{A}^{m_0})}(h_k D_t)\big)u\big\|^2_{L^2(\mathbb{R};D(\mathsf{A}^{m_0}))} \\
&\quad\lesssim h_k^{2M}\mathcal{E}_{\ell_1}(u),
\end{aligned}$$

for any $M \in \mathbb{N}$. From (5.23) using that $h_k = \rho^{-|k|}$ with $\rho > 1$ one obtains

$$\mathcal{E}_{\ell_1}(u) \lesssim \sum_{0\leq |k| < k_1} \mathcal{E}_{\ell_1}(u^k) + \|\psi\,\mathsf{L}\,u\|^2_{L^2(\mathbb{R};K)} + h_{k_1}^{2M}\mathcal{E}_{\ell_1}(u).$$

For $k_1 \geq k_0$ chosen sufficiently large one obtains

$$\mathcal{E}_{\ell_1}(u) \lesssim \sum_{0\leq |k| < k_1} \mathcal{E}_{\ell_1}(u^k) + \|\psi\,\mathsf{L}\,u\|^2_{L^2(\mathbb{R};K)}. \tag{5.24}$$

With (5.24) the result of Lemma 5.1 holds here too. Arguing as in the proof given in Section 5.1 one obtains the sought observability estimate. □

6. Proof of the main result for the Schrödinger equation

6.1. Refined time-microlocalization estimates

The results of Section 5.2 can be adapted to a solution $u(t)$ of a Schrödinger equation (2.11) with $\underline{u}^0 \in D(\mathsf{A}^m)$, for some $m \in \mathbb{R}$. Then, $u \in \cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m-k})\big)$.

Let $F \in \mathscr{C}_c^\infty(]\alpha, \alpha^{-1}[)$ be as given by Lemma 4.1. Consider $\tilde{F} \in \mathscr{C}_c^\infty(]\alpha, \alpha^{-1}[)$ such that $\tilde{F} = 1$ in a neighborhood of $\operatorname{supp}(F)$.

Lemma 6.1. *Let $p \in \mathbb{R}$. There exists $C = C_{m,p} > 0$ such that*

$$\big\|\tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t) u\big\|_{H_{D(\mathsf{A}^m)}} \leq C h_k^{p-m} \|\underline{u}^0\|_{D(\mathsf{A}^p)}, \qquad k \in \mathbb{N}^*. \tag{6.1}$$

The definition of the Fourier multiplier $\tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)$ is as in the beginning of Section 4.

Proof. With Lemma 4.6 one has

$$\tilde{u}^k(t) = \tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t) u(t) = \sum_{\nu \in J_k^S} \tilde{F}_k\big(h_k \lambda_\nu\big) \mathrm{e}^{it\lambda_\nu} \underline{u}_\nu^0 e_\nu \in E_k^S \in D(\mathsf{A}^\infty).$$

With Lemma 2.2 one writes

$$h_k^m \|\tilde{u}^k(t)\|_{D(\mathsf{A}^m)} \eqsim h_k^p \|\tilde{u}^k(t)\|_{D(\mathsf{A}^p)} \lesssim h_k^p \|u(t)\|_{D(\mathsf{A}^p)} \lesssim h_k^p \|\underline{u}^0\|_{D(\mathsf{A}^p)}.$$

The result follows from the definition of $\|.\|_{H_{D(\mathsf{A}^m)}}$ in (4.2). □

Combined with the third part of Lemma 4.2 one has the following corollary.

Corollary 6.2. *If $\varphi \in \mathscr{C}_c^\infty(]0, \mathsf{T}[)$ and $\psi \in L^\infty(\mathbb{R})$ is such that $\psi = 1$ in a neighborhood of $\overline{I_0}$, then for any $M \geq 1$ and $p \in \mathbb{R}$ there exists $C = C_{M,m,p} > 0$ such that*

$$\big\|\varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)\tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t) u\big\|_{L^2(\mathbb{R}; D(\mathsf{A}^m))} \leq C h_k^M \|\underline{u}^0\|_{D(\mathsf{A}^p)}. \tag{6.2}$$

Lemma 6.3. *Let $\varphi \in \mathscr{C}_c^\infty(]0, \mathsf{T}[)$ and $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$ be such that $\psi = 1$ in a neighborhood of $\overline{I_0}$, then for any $M \geq 1$ and $p \in \mathbb{R}$ there exists $C = C_{M,m,p} > 0$ such that*

$$\big\|\varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)\big(\operatorname{Id} - \tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)\big) u\big\|_{L^2(\mathbb{R}; D(\mathsf{A}^m))} \leq C h_k^M \|\underline{u}^0\|_{D(\mathsf{A}^p)}. \tag{6.3}$$

Proof. With Lemma 4.2 the proof is clear in the case $p \geq m$. We shall thus only consider the case $p < m$. Let $r \in \mathbb{N}$ be such that $r \geq m - p$. With u solution to the Schrödinger equation one has $u = D_t^r \mathsf{A}^{-r} u$. Set $w = \mathsf{A}^{-r} u$. It is also a solution of the Schrödinger equation that lies in $D(\mathsf{A}^\infty)$. One has

$$\|w(t)\|_{D(\mathsf{A}^m)} = \|u(t)\|_{D(\mathsf{A}^{m-r})} \lesssim \|u(t)\|_{D(\mathsf{A}^p)}, \quad t \in \mathbb{R}. \tag{6.4}$$

One thus considers the action of the operator

$$P = \varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)\big(\operatorname{Id} - \tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)\big) D_t^r,$$

on w. As in the proof of Lemma 5.4 one obtains

$$\big\|Pw(t)\big\|_{D(\mathsf{A}^m)} \lesssim C_M h_k^M \sup_{j\in\mathbb{Z}} \|w_j\|_{L^2(\mathbb{R};D(\mathsf{A}^m))} = C_M h_k^M \|w\|_{H_{D(\mathsf{A}^m)}} \lesssim C_M h_k^M \|w(0)\|_{D(\mathsf{A}^m)}.$$

As $\varphi F_k^{D(\mathsf{A}^m)}(h_k D_t)(1-\psi)\big(\operatorname{Id} - \tilde{F}_k^{D(\mathsf{A}^m)}(h_k D_t)\big)u(t) = Pw(t)$, one concludes the proof with (6.4). □

6.2. Proof of Theorem 3.8

Here, we provide only one proof that treats the general case $p_1 \le m_0$. Let $\mathsf{T} = (T+T')/2$ and $\delta = (\mathsf{T}-T)/2 = (T'-\mathsf{T})/2$. The assumed semi-classical observation inequality reads

$$\|u^k\|_{D(\mathsf{A}^{p_1})} \le C \int_\delta^{\mathsf{T}-\delta} \|\,\mathsf{L}\,u^k(t)\|_K \mathrm{d}t, \qquad (u^k)_{k\in\mathbb{N}} \in B^S,\ k \ge k_0, \tag{6.5}$$

and we aim to prove that

$$\|\underline{u}^0\|_{D(\mathsf{A}^{p_1})} \le C' \int_{-\delta}^{\mathsf{T}+\delta} \|\,\mathsf{L}\,u(t)\|_K \,\mathrm{d}t$$

holds for any solution u to the Schrödinger equation (2.11) written in (2.12) with $\underline{u}^0 \in D(\mathsf{A}^{m_0})$. We thus consider such a solution. One has $u \in \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m_0-k})\big)$.

One has

$$\|\,\mathsf{L}\,u\|_{H_K} \lesssim \|\,\mathsf{L}\,u\|_{L^\infty(\mathbb{R};K)} \lesssim \|u\|_{L^\infty(\mathbb{R};D(\mathsf{A}^{m_0}))} \lesssim \|\underline{u}^0\|_{D(\mathsf{A}^{m_0})}. \tag{6.6}$$

Let F be chosen as in Lemma 4.1. With Lemma 4.6, for $k \in \mathbb{N}^*$, set

$$u^k(t) = F_k^{D(\mathsf{A}^{p_1})}(h_k D_t)u(t) = \sum_{\nu\in J_k^S} F_k(h_k\lambda_\nu)\mathrm{e}^{it\lambda_\nu}\underline{u}_\nu^0 e_\nu. \tag{6.7}$$

One has $u^k \in E_k$. For $k \ge k_0$ with the semi-classical observation property (6.5) one has

$$\|u^k\|_{D(\mathsf{A}^{p_1})} \lesssim \|\varphi\,\mathsf{L}\,u^k(t)\|_{L^2(\mathbb{R},K)}, \qquad |k| \ge k_0, \tag{6.8}$$

where $\varphi \in \mathscr{C}_c^\infty(]0,\mathsf{T}[)$ with $\varphi = 1$ on a neighborhood of $[\delta, \mathsf{T}-\delta]$. In the cases $k > 0$ or $k < 0$, one has

$$\|u^k\|^2_{D(\mathsf{A}^{p_1})} = \sum_{\nu\in J_k^S} \lambda_\nu^{2p_1} F_k(h_k\lambda_\nu)^2 |\underline{u}_\nu^0|^2.$$

Set $u^0 = \sum_{\lambda_\nu\le 1} \mathrm{e}^{it\lambda_\nu}\underline{u}_\nu^0$. With Lemma 4.1 one finds

$$\begin{aligned}\|u-u^0\|^2_{D(\mathsf{A}^{p_1})} &= \|u\|^2_{D(\mathsf{A}^{p_1})} - \|u^0\|^2_{D(\mathsf{A}^{p_1})} = \sum_{\lambda_\nu>1} \lambda_\nu^{2p_1}|\underline{u}_\nu^0|^2 \lesssim \sum_{k\in\mathbb{N}^*}\sum_{\lambda_\nu>1} \lambda_\nu^{2p_1} F_k(h_k\lambda_\nu)^2|\underline{u}_\nu^0|^2 \\ &\lesssim \sum_{k\in\mathbb{N}^*} \|u^k\|^2_{D(\mathsf{A}^{p_1})}.\end{aligned}$$

One thus obtains with (6.8)

$$\begin{aligned}\|u\|^2_{D(\mathsf{A}^{p_1})} &\lesssim \|u^0\|^2_{D(\mathsf{A}^{p_1})} + \sum_{k\in\mathbb{N}^*} \|u^k\|^2_{D(\mathsf{A}^{p_1})} \\ &\lesssim \|u^0\|^2_{D(\mathsf{A}^{p_1})} + \sum_{1\le|k|<k_1} \|u^k\|^2_{D(\mathsf{A}^{p_1})} + \sum_{|k|\ge k_1} \|\varphi \,\mathsf{L}\, u^k(t)\|^2_{L^2(\mathbb{R};K)},\end{aligned} \tag{6.9}$$

for $k_1 \ge k_0$ to be chosen below.

With Lemma 4.7 one has

$$\|u\|^2_{D(\mathsf{A}^{p_1})} \lesssim \sum_{0\le|k|<k_1} \|u^k\|^2_{D(\mathsf{A}^{p_1})} + \sum_{|k|\ge k_1} \|\varphi F_k^K(h_k D_t)\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)}. \tag{6.10}$$

Consider $\psi \in \mathscr{C}_c^\infty(]-\delta, \mathsf{T}+\delta[)$ such that $\psi = 1$ in a neighborhood of $\overline{I_0} = [0, \mathsf{T}]$. One writes

$$\|\varphi F_k^K(h_k D_t)\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} \lesssim \|\varphi F_k^K(h_k D_t)\psi\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} + \|\varphi F_k^K(h_k D_t)(1-\psi)\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)},$$

yielding, with the second part of Lemma 4.2

$$\|u\|^2_{D(\mathsf{A}^{p_1})} \lesssim \sum_{0\le|k|<k_1} \|u^k\|^2_{D(\mathsf{A}^{p_1})} + \|\psi\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} + \sum_{|k|\ge k_1} \|\varphi F_k^K(h_k D_t)(1-\psi)\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)}.$$

One writes

$$\|\varphi F_k^K(h_k D_t)(1-\psi)\,\mathsf{L}\, u\|_{L^2(\mathbb{R};K)} \lesssim \|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)u\|_{L^2(\mathbb{R};D(\mathsf{A}^{m_0}))},$$

using that L is bounded on $D(\mathsf{A}^{m_0})$; see (3.1). This gives

$$\begin{aligned}\|u\|^2_{D(\mathsf{A}^{p_1})} \lesssim &\sum_{0\le|k|<k_1} \|u^k\|^2_{D(\mathsf{A}^{p_1})} + \|\psi\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} \\ &+ \sum_{|k|\ge k_1} \|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)u\|^2_{L^2(\mathbb{R};D(\mathsf{A}^{m_0}))}.\end{aligned} \tag{6.11}$$

Second, as in Section 5.2 consider $\tilde{F} \in \mathscr{C}_c^\infty(\mathbb{R}_+^*)$ such that $\tilde{F} = 1$ in a neighborhood of $\operatorname{supp}(F)$. With Corollary 6.2 and Lemma 6.3 one has

$$\begin{aligned}&\|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)u\|^2_{L^2(\mathbb{R};D(\mathsf{A}^{m_0}))} \\ &\quad\lesssim \big\|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)\tilde{F}_k^{D(\mathsf{A}^{m_0})}(h_k D_t)u\big\|^2_{L^2(\mathbb{R};D(\mathsf{A}^{m_0})} \\ &\qquad + \big\|\varphi F_k^{D(\mathsf{A}^{m_0})}(h_k D_t)(1-\psi)\big(\operatorname{Id} - \tilde{F}_k^{D(\mathsf{A}^{m_0})}(h_k D_t)\big)u\big\|^2_{L^2(\mathbb{R};D(\mathsf{A}^{m_0}))} \\ &\quad\lesssim h_k^{2M}\|u\|^2_{D(\mathsf{A}^{p_1})},\end{aligned}$$

for any $M \in \mathbb{N}$. From (6.11) using that $h_k = \rho^{-|k|}$ with $\rho > 1$ one obtains

$$\|u\|^2_{D(\mathsf{A}^{p_1})} \lesssim \sum_{0\le|k|<k_1} \|u^k\|^2_{D(\mathsf{A}^{p_1})} + \|\psi\,\mathsf{L}\, u\|^2_{L^2(\mathbb{R};K)} + h_{k_1}^{2M}\|u\|^2_{D(\mathsf{A}^{p_1})}.$$

For $k_1 \geq k_0$ chosen sufficiently large one obtains

$$\|u\|^2_{D(\mathsf{A}^{p_1})} \lesssim \sum_{0\leq|k|<k_1} \|u^k\|^2_{D(\mathsf{A}^{p_1})} + \|\psi \, \mathsf{L} \, u\|^2_{L^2(\mathbb{R};K)}. \tag{6.12}$$

The following lemma is the counterpart of Lemma 5.1.

Lemma 6.4 (absence of invisible solutions to the Schrödinger equation)**.** *Let* $u \in \cap_k \mathscr{C}^k\big(\mathbb{R}; D(\mathsf{A}^{m_0-k})\big)$ *be a solution to* (2.11) *such that* $\psi \, \mathsf{L} \, u = 0$. *Then* $u = 0$.

The proof is very similar to that of Lemma 5.1.

Proof. Set $\mathscr{N}_S$ as the space of such invisible solutions (in the sense of the observation operator $\psi \, \mathsf{L}$) equipped with the norm $\|\underline{u}^0\|_{D(\mathsf{A}^{m_0})}$. With (6.12) one has $\|\underline{u}^0\|_{D(\mathsf{A}^{m_0})} \lesssim \sum_{0\leq|k|<k_1} \|u^k\|^2_{D(\mathsf{A}^{p_1})}$ implying $\mathscr{N}_S = \operatorname{span}\{e_\nu; \nu \in \Upsilon\}$ with $\#\Upsilon < \infty$. Moreover, if $u \in \mathscr{N}_S$ then $u \in \mathscr{C}^m\big(\mathbb{R}, D(\mathsf{A}^r)\big)$ for any $m \geq 0$ and $r \geq 0$, similarly to what one has in (4.12). On this finite dimensional space one has $\psi \, \mathsf{L} \, \partial_t u = \psi \partial_t \, \mathsf{L} \, u = 0$. Thus ∂_t maps $\mathscr{N}_S$ into itself and consequently it has an eigenvector v with associated eigenvalue μ. One finds $\mathsf{A}\mathsf{v} = D_t \mathsf{v} = -i\mu \mathsf{v}$ meaning that $\mathsf{v}(t)$ is an eigenfunction for A for all t. With the unique-continuation Assumption 3.1 one obtains $\mathsf{v}(t) = 0$ for all t. Hence $\mathscr{N}_S = \{0\}$. □

We conclude the proof of Theorem 3.8 by an argument by contradiction similar to that in the proof of Theorem 3.2. Adaptation is left to the reader. □

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References

[1] S. Alinhac and P. Gérard, Opérateurs Pseudo-Différentiels et Théorème de Nash–Moser. Editions du CNRS (1991).

[2] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.* **30**(1992) 1024–1065.

[3] N. Burq, Contrôle de l'équation des plaques en présence d'obstacles strictement convexes. *Mém. Soc. Math. France (N.S.)* **55** (1993) 126.

[4] N. Burq, Contrôle de l'équation des ondes dans des ouverts peu réguliers. *Asymptotic Anal.* **14** (1997) 157–191.

[5] N. Burq, B. Dehman and J. Le Rousseau, Measure and continuous vector field at a boundary I: propagation equation and wave observability. Submitted (2023).

[6] N. Burq and G. Lebeau, Micro-local approach to the control for the plates equation, in Optimization, Optimal Control and Partial Differential Equations (Iaşi, 1992). Vol. 107 of *Internat. Ser. Numer. Math.* Birkhäuser, Basel (1992) 111–122.

[7] N. Burq and M. Zworski, Geometric control in the presence of a black box. *J. Am. Math. Soc.* **17** (2004) 443–471.

[8] J.-M. Coron, Control and Nonlinearity. Vol. 136 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI (2007).

[9] B. Dehman and G. Lebeau, Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time. *SIAM J. Control Optim.* **48** (2009) 521–550.

[10] F. Fanelli and E. Zuazua, Weak observability estimates for 1-D wave equations with rough coefficients. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32** (2015) 245–277.

[11] L. Hörmander, Uniqueness theorems for second order elliptic differential equations. *Commun. Part. Diff. Equ.* **8** (1983) 21–64.

[12] L. Hörmander, The Analysis of Linear Partial Differential Operators, Vol. III. Springer-Verlag (1985).

[13] L. Hörmander, The Analysis of Linear Partial Differential Operators, Vol. I, 2nd edn. Springer-Verlag (1990).

[14] E. Humbert, Y. Privat and E. Trélat, Observability properties of the homogeneous wave equation on a closed manifold. *Commun. Part. Diff. Equ.* **44** (2019) 749–772.

[15] I. Lasiecka, J.-L. Lions and R. Triggiani, Non homogeneous boundary value problems for second order hyperbolic operators. *J. Math. Pures Appl.* **65** (1986) 149–192.

[16] C. Laurent and M. Léautaud, Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves. *J. Eur. Math. Soc.* **21** (2019) 957–1069.

[17] G. Lebeau, Contrôle de équation de Schrödinger. *J. Math. Pures Appl.* **71** (1992) 267–291.

[18] J. Le Rousseau, G. Lebeau and L. Robbiano, Elliptic Carleman Estimates and Applications to Stabilization and Controllability, Vol. I: Dirichlet Boundary Conditions on Euclidean Space. PNLDE Subseries in Control. Birkhäuser (2022).

[19] J. Le Rousseau, G. Lebeau and L. Robbiano, Elliptic Carleman Estimates and Applications to Stabilization and Controllability, Vol. II: General Boundary Conditions on Riemnannian Manifolds. PNLDE Subseries in Control. Birkhäuser (2022).

[20] J.-L. Lions, Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués, Vol. 1. Masson, Paris (1988).

[21] G. Tenenbaum and M. Tucsnak, Fast and strongly localized observation for the Schrödinger equation. *Trans. AMS* **361** (2009) 951–977.

[22] F. Treves, Topological Vector Spaces, Distributions and Kernels. Academic Press, New York (1967).

[23] M. Tucsnak and G. Weiss, Observation and Control for Operator Semigroups. Birkhäuser Verlag, Basel (2009).