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ASYMPTOTIC BEHAVIOR OF NULL CONTROLLABILITY COST FOR PARABOLIC EQUATIONS WITH VANISHING DIFFUSIVITY UNDER ROBIN AND NEUMANN BOUNDARY CONDITIONS

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Abstract. In this paper we study the null controllability cost of a transport-diffusion system under Robin boundary conditions with distributed control and in which the transport coefficient is a gradient field. First, we provide some conditions on transport coefficient and boundary potential to show that the control cost decays exponentially when the viscosity vanishes and the control time is sufficiently large. On the other hand, if the range of the control region by the transport flow does not cover that of Ω, we prove that the control cost explodes exponentially for the Neumann boundary conditions case with vanishing viscosity and all control time.

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1. Introduction and main results

Let Ω be a domain (bounded connected open subset) of $\mathbb{R}^N$, $N \geq 1$ with Lipschitz boundary Γ (which is required to be C^2 for certain results) and ν the outer unit normal field on Γ and $\omega \subset \Omega$ be a nonempty open subset. We set

$$\Omega_T := \Omega \times (0,T), \quad \omega_T := \omega \times (0,T) \quad \text{and} \quad \Gamma_T := \Gamma \times (0,T),$$

where $T > 0$ is the control time.

We consider the following controlled linear transport-diffusion system with viscosity $\varepsilon > 0$ and autonomous Robin (or Fourier) boundary conditions:

$$\begin{cases} \partial_t y - \varepsilon \Delta y + X \cdot \nabla y + q\, y = v(x,t) \mathbb{1}_\omega & \text{in } \Omega_T, \\ \varepsilon \partial_\nu y + \beta\, y = 0 & \text{on } \Gamma_T, \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Keywords and phrases: Carleman estimates, uniform controllability, transport equation, singular limits, cost control, spectral decomposition.

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In this paper, we will assume that, the potential terms $q \in L^\infty(\Omega)$ and $\beta \in L^\infty(\Gamma)$ are bounded. The functions $y = y(x,t)$, $\mathbb{1}_\omega$ are the state and the characteristic function of ω and the function $v \in L^2(\omega_T)$ acts as a distributed control and it is used to drive the state to 0 at time T from the initial state $y_0 \in L^2(\Omega)$. The vector X is a gradient vector field *i.e.*, there is $f \in W^{2,\infty}(\Omega)$ a scalar field, such that $X := \nabla f$.

The field of gradient vectors is very important in theoretical physics and mathematics, especially in differential topology, and it appears in Witten–Helffer–Sjöstrand theory [1, 2]. As an application, the spectral properties of the Witten Laplacian provide a means of deducing the topological properties of the couple $(\Omega; f)$.

By rescaling in time, we get an upper bound of the cost on which the system is null controllable. That means, we show that there exists $C = C(\Omega, \omega, T, \varepsilon) > 0$, such that for all initial state $y_0 \in L^2(\Omega)$, there exists a control $v \in L^2(\omega_T)$ such that the solution of (1.1) verifies $y(\cdot, T) = 0$ and

$$\|v\|_{L^2(\omega_T)} \leqslant C\|y_0\|_{L^2(\Omega)}. \tag{1.2}$$

This is proved in Proposition 3.1 in Section 3. The novelty is the cost, as the null controllability is a classical result (see [3, 4]). Note that Proposition 3.1 does not have any hypothesis related with f, so it provides a universal upper bound.

Various researchers have studied null controllability results of the heat equation under different scenarios, including Dirichlet, Neumann, or Robin boundary conditions [5, 6]. For dynamic boundary conditions, [7, 8] offer relevant information.

The best constant which satisfies (1.2), called the null controllability cost, is defined by

$$\mathcal{K}(\Omega, \omega, T, \varepsilon) := \sup_{y_0 \in L^2(\Omega)\setminus\{0\}} \inf_{v \in \mathcal{C}(y_0)} \frac{||v||_{L^2(\omega_T)}}{||y_0||_{L^2(\Omega)}}, \tag{1.3}$$

where

$$\mathcal{C}(y_0) := \{v \in L^2(\omega_T) : \text{ the solution of (1.1) satisfies } y(\cdot, T) = 0\}.$$

In this paper we study its asymptotic behavior when the viscosity vanishes. In general, the cost of control is obtained by studying the observability of the following adjoint system of (1.1)

$$\begin{cases} -\partial_t \varphi - \varepsilon\Delta\varphi - \nabla f \cdot \nabla\varphi + (q - \Delta f)\varphi = 0 & \text{in } \Omega_T, \\ \varepsilon\partial_\nu\varphi + (\partial_\nu f + \beta)\,\varphi = 0 & \text{on } \Gamma_T, \\ \varphi(x,T) = \varphi_T(x) & \text{in } \Omega, \end{cases} \tag{1.4}$$

when $X = \nabla f$ and $f \in W^{2,\infty}(\Omega)$. In fact, by the Hilbert Uniqueness Method (we refer to [9, 10]), we have

$$\mathcal{K}(\Omega, \omega, T, \varepsilon) = \sup_{\varphi_T \in L^2(\Omega)\setminus\{0\}} \frac{||\varphi(\cdot, 0)||_{L^2(\Omega)}}{||\varphi||_{L^2(\omega_T)}}, \tag{1.5}$$

for φ the solution of (1.4).

The important geometric quantities involved in the present work are the potential associated with the function f, defined by

$$\mathcal{V}(x) := \frac{|\nabla f(x)|^2}{4}, \text{ for all } x \in \overline{\Omega}$$

and its minimum, noted $E_0 := \min_{\overline{\Omega}} \mathcal{V}$. We also define the quantity

$$\beta_f(x) := \frac{\partial_\nu f(x)}{2} + \beta(x), \text{ for all } x \in \Gamma.$$

The first main outcome of this paper is to prove that if $\beta_f \geq 0$ and $E_0 > 0$, then the cost of the null controllability of (1.1) decays exponentially when the viscosity vanishes and the control time is sufficiently large. Precisely, we will prove the following result:

Theorem 1.1. *We suppose that:*

(1) Ω is a $\mathcal{C}^2$ domain and $\omega \subset\subset \Omega$ is a nonempty open subset,
(2) for all $x \in \Gamma$, $\beta_f(x) = \frac{\partial_\nu f(x)}{2} + \beta(x) \geq 0$ and $E_0 = \min_{x\in\overline{\Omega}} \frac{|\nabla f(x)|^2}{4} > 0$.

Then, there are T_1, C_1, $C_2 > 0$ depending only on Ω, ω, f and q such that, for all $T \geq T_1$ and $\varepsilon \in (0,1)$, the null controllability cost of (1.1) *satisfies*

$$\mathcal{K}(\Omega, \omega, T, \varepsilon) \leqslant C_1 \exp\left(\frac{-C_2}{\varepsilon}\right). \tag{1.6}$$

Two approaches are examined in the literature: the spectral approach, illustrated in [11], and the approach based on Agmon inequalities, shown in [12]. In both cases, Carleman and dissipation estimates are used.

Remark 1.2. A dissipation estimate can be proved by Agmon inequality for a general transport $X = X(x,t)$ as in [12, 13]. But the difficulty arises in the proof of the Carleman estimate as explained in Remark 4.1. This is the reason why we have imposed to X to be a gradient field, which allows us to prove Carleman's estimate while going through a self-adjoint system (system without transport term). In this paper, we adopt the spectral approach, which is based on very explicit hypotheses about the transport term and the boundary potential, as specified in Item (2) of Theorem 1.1, unlike the approach based on Agmon's inequalities, see for example property 2 in [12].

The second main outcome of this paper is to show that if there exists $h > 0$ such that $f(\omega) \subset (m_f + h, M_f)$, where

$$m_f := \min_{\overline{\Omega}} f \quad \text{and} \quad M_f := \max_{\overline{\Omega}} f,$$

then the control cost explodes exponentially when the viscosity vanishes and all control time for the Neumann boundary conditions. Precisely:

Theorem 1.3. *Let $h > 0$ such that $f(\omega) \subset (m_f + h, M_f)$. Then, there exist a constant $C > 0$ depending only on Ω, ω, f and h such that, for all $\varepsilon > 0$ and $T > 0$, we have the following estimate:*

$$\mathcal{K}(\Omega, \omega, T, \varepsilon) \geq \frac{C}{T^{\frac{1}{2}}} \exp\left(\frac{C}{\varepsilon}\right), \tag{1.7}$$

where $\mathcal{K}$ is the null controllability cost of (1.1) *with $q = 0$ and $\beta = 0$.*

Remark 1.4. The proof is done to study the behaviour of the system with respect to ε, to see clear contrast with respect to Theorem 1.1. We are not seeking to obtain the optimal bound when $T \to 0$. In fact, the proof consists on estimating how much it takes to control the first frequency to 0, so the optimal bound for the time variable cannot be obtained by using this method.

These results are the answers to the open questions presented in [12], Remark 3 and [11]. In [12], Theorems 1 and 2, Sergio Guerrero and Gilles Lebeau established these results for a general speed belonging to $W^{1,\infty}(\mathbb{R}^N \times (0,+\infty))$ and Dirichlet conditions, while in [11], Theorems 2.7 and 2.8, Jon Asier Bárcena–Petisco proved the same results with $X = (1, 0, \dots, 0)$, $q = 0$, and Robin boundary conditions.

Regarding the state of the art, Coron and Guerrero initiated the study of the null controllability cost when the viscosity vanishes on the 1-D problem with constant speed in [14]. Over the past years, researchers have focused on the controllability problem in one-dimensional and explored uniform controllability and associated minimal time in [14–18]. The work [19] examined uniform controllability when the speed is expressed as a gradient and viscosity vanishes, and obtained upper and lower bounds on the minimal time needed to control to zero, uniformly in the vanishing viscosity limit. As in this paper, they also considered a spectral decomposition. However, unlike in this paper, they focus on one-dimensional domains with Dirichlet boundary conditions. Similarly, [20] presented a numerical method to estimate the cost of controllability as a solution of a generalized eigenvalue problem involving the control operator, and [21] dealt with estimating the perturbed solution. In higher dimensions, there are few results regarding the cost of null controllability when viscosity vanishes for non-constant speed, except for [12], which explored the problem with Dirichlet conditions and general transport belonging to $W^{1,\infty}(\Omega)$ and [22], which studied the uniform observability of gradient flows with vanishing viscosity. For the case of Neumann or Robin boundary conditions, [11] analysed the problem with constant speed $X = (1, 0, \dots, 0)$. Moreover, we would like to remark that in our work [13] we have considered a flow depending on the time variable, but on the expense of restricting ourself to very specific boundary conditions. In that paper the proof of the dissipation estimate is considerably different, as in this paper we use spectral techniques, whereas in [13] we use Agmon inequality.

The null controllability cost is treated in several types of evolution equations, namely the Stokes system [23], an artificial advection-diffusion problem [24, 25], the Burgers equation [26], the KdV equation [27], the heat equation in the networks [28] and a parabolic system of fourth order [29–31]. For the approximate-controllability cost with a dynamic boundary, we refer to [32].

The motivation for studying this concept comes from various fields of mathematics and physics. A first motivation for studying singular limits in control problems is the search of controllability properties for the perturbed system itself and to establish the controllability of limit system, as illustrated in the paper [33] which shows the null controllability of the heat equation as singular limit of the exact controllability of dissipative wave equations and [34] which studies the null controllability of a degenerated reaction–diffusion system in cardiac electro-physiology. Another important motivation appears in the theory of conservation law when the velocity is a gradient (conservative force), the determination of a physical solution (called entropy) is based on the vanishing viscosity, see [35, 36] for more details.

Our paper is structured as follows. In Section 2, we introduce the functional framework and some results. We symmetrize the system (1.4) to find a self adjoint operator and study the spectral properties of this operator. In Section 3 we prove an upper bound of the null controllability cost. Sections 4 and 5 are devoted to the proof of our main results, Theorem 1.1 and Theorem 1.3.

2. Conjugation of gradient flows and spectral analysis

In this section, we will symmetrize the system (1.4) and examine some spectral properties of the found operator.

2.1. Notations and function spaces

Let Ω be a domain of $\mathbb{R}^N$, $N \geq 1$ with Lipschitz boundary Γ and ν is the outer unit normal field on Γ. For $x, y \in \mathbb{R}^N$, $x \cdot y$ denotes the canonical scalar product of x and y and $|x|$ is the Euclidean norm of x. If A is a Lebesgue-measurable part of $\mathbb{R}^N$, we will note $|A|$ its measure. We take $L^2(\Omega)$ and $L^2(\Gamma)$ the classical Hilbert spaces over $\mathbb{R}$ with respect to the Lebesgue measure $\mathrm{d}x$ on Ω and the $(N-1)$-dimensional Hausdorff measure $\mathrm{d}\sigma$ on Γ. We will note $\mathcal{D}(\Omega)$ the space of the test functions on Ω, $H^1(\Omega)$ and $W^{2,\infty}(\Omega)$ are the usual Sobolev spaces over Ω. We recall that there exists a unique linear bounded operator $\gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma)$ such that $\gamma_0(u) = u_{|\Gamma}$ if $u \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$, see [37]. The function $\gamma_0(u)$ is called the trace of u and one can also use the notation $u_{|\Gamma}$ for $u \in H^1(\Omega)$ (to simplify, we denote u instead of $u_{|\Gamma}$). The dual of $H^{1/2}(\Gamma)$ is noted by $H^{-1/2}(\Gamma)$ and $\langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$ is the duality product. We will employ the following $H^1(\Omega)-$trace estimate

$$\int_\Gamma |u|^2 \mathrm{d}\sigma \leqslant C \|u\|_{H^1(\Omega)} \|u\|_{L^2(\Omega)}, \tag{2.1}$$

where $C > 0$ depending only on Ω. For the proof of the inequality (2.1), we refer to [38], Theorem 1.5.1.10. Here, we use the definition of the Laplacian as a weak derivative. Let $u \in H^1(\Omega)$, we say that $\Delta u \in L^2(\Omega)$ if there exists a function $g \in L^2(\Omega)$ such that, for all $v \in \mathcal{D}(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x = - \int_\Omega g \, v \mathrm{d}x. \tag{2.2}$$

In this case, the function $g \in L^2(\Omega)$ verifying (2.2) is unique, we denote g by Δu.

Let $H_\Delta(\Omega) := \{u \in H^1(\Omega), \ \ \Delta u \in L^2(\Omega)\}$, there exists a unique linear bounded operator $\gamma_1 : H_\Delta(\Omega) \longrightarrow H^{-1/2}(\Gamma)$ such that $\gamma_1(u) = \partial_\nu u$ is the normal derivative of u if $u \in \mathcal{C}^1(\overline{\Omega})$. This operator satisfies the generalized Green's formula

$$\int_\Omega \Delta u \, v \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x = \langle \gamma_1(u), \gamma_0(v) \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}, \tag{2.3}$$

for all $u \in H_\Delta(\Omega)$ and $v \in H^1(\Omega)$ (to simplify, we denote $\partial_\nu u$ instead of $\gamma_1(u)$).

2.2. Some preliminary spectral results

It is classical that systems with a transport term in the form of gradient and small diffusion can be symmetrized (symmetrized problems have self-adjoint operators). To symmetrize system (1.4), we consider the change

$$\varphi(\cdot, t) \longmapsto \Phi(\cdot, t) := \exp\left(\frac{f(\cdot)}{2\varepsilon}\right) \varphi(\cdot, t). \tag{2.4}$$

Then φ is the solution of (1.4) if and only if Φ is the solution of the following system:

$$\begin{cases} -\partial_t \Phi - \varepsilon \Delta \Phi + q_{f,\varepsilon} \Phi = 0 & \text{in } \Omega_T, \\ \varepsilon \partial_\nu \Phi + \beta_f \Phi = 0 & \text{on } \Gamma_T, \\ \Phi(x, T) = \Phi_T(x) & \text{in } \Omega, \end{cases} \tag{2.5}$$

where $q_{f,\varepsilon} := q + \frac{\mathcal{V}}{\varepsilon} - \frac{\Delta f}{2}$, $\mathcal{V} := \frac{|\nabla f|^2}{4}$, $\beta_f := \frac{\partial_\nu f}{2} + \beta$ and $\Phi_T := \exp\left(\frac{f(\cdot)}{2\varepsilon}\right) \varphi_T$.

The next result shows some properties of the linear operator A_ε on $L^2(\Omega)$ defined by

$$\begin{cases} D(A_\varepsilon) & := \{y \in H^1(\Omega), \ \ \Delta y \in L^2(\Omega) \ \text{ and } \ \varepsilon \partial_\nu y + \beta_f y = 0\}, \\ A_\varepsilon y & := -\varepsilon \Delta y + q_{f,\varepsilon} y. \end{cases}$$

Proposition 2.1. *Let $\varepsilon > 0$.*

(1) The operator $-A_\varepsilon$ generates a quasi-contractive C_0-semigroup $(\mathcal{T}_\varepsilon(t))_{t\geq 0}$ on $L^2(\Omega)$; that is, there is a constant w such that $\|\mathcal{T}_\varepsilon(t)\|_{\mathcal{L}(L^2(\Omega);L^2(\Omega))} \leq \exp(wt)$ for all $t \geq 0$.
(2) The operator A_ε is self-adjoint on $L^2(\Omega)$ and has compact resolvents.

Proof. We consider the bilinear form $\mathfrak{a}_\varepsilon : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\mathfrak{a}_\varepsilon(y, z) := \varepsilon \int_\Omega \nabla y \cdot \nabla z \mathrm{d}x + \int_\Omega q_{f,\varepsilon} y z \mathrm{d}x + \int_\Gamma \beta_f y z \mathrm{d}\sigma. \tag{2.6}$$

Clearly, the form $\mathfrak{a}_\varepsilon$ is symmetric and continuous. We claim that $\mathfrak{a}_\varepsilon$ is $L^2(\Omega)$-elliptic, *i.e.*, that, there are constants $w \geq 0$, $\alpha > 0$ depend on ε such that, for all $y \in H^1(\Omega)$, we have

$$\mathfrak{a}_\varepsilon(y, y) + w\|y\|^2_{L^2(\Omega)} \geq \alpha\|y\|^2_{H^1(\Omega)}. \tag{2.7}$$

By the trace estimate (2.1) and Young's inequality, we obtain

$$\left|\int_\Gamma \beta_f |y|^2 \mathrm{d}\sigma\right| \leqslant \frac{\varepsilon}{2}\|y\|^2_{H^1(\Omega)} + \frac{C^2\|\beta_f\|^2_\infty}{2\varepsilon}\|y\|^2_{L^2(\Omega)}.$$

Then

$$\mathfrak{a}_\varepsilon(y, y) \geq \frac{\varepsilon}{2}\|y\|^2_{H^1(\Omega)} - \left(\|q_{f,\varepsilon}\|_\infty + \frac{C^2\|\beta_f\|^2_\infty}{2\varepsilon} + \varepsilon\right)\|y\|^2_{L^2(\Omega)}. \tag{2.8}$$

Hence $\mathfrak{a}_\varepsilon$ is $L^2(\Omega)$-elliptic. Using [39], we obtain $\mathfrak{a}_\varepsilon$ induces a self-adjoint and quasi-accretive operator B_ε on $L^2(\Omega)$, *i.e.*, that, there is $w \in \mathbb{R}$ such that $B_\varepsilon + w$ is accretive (or monotone). The operator B_ε is given as follows, a function $y \in H^1(\Omega)$ belongs to $D(B_\varepsilon)$ if and only if there is $g \in L^2(\Omega)$ such that $\mathfrak{a}_\varepsilon(y, z) = \langle g, z\rangle_{L^2(\Omega)}$ for all $z \in H^1(\Omega)$ and in this case $B_\varepsilon y = g$. The proof of $A_\varepsilon = B_\varepsilon$ is standard and based on the generalized Green's formula (2.3). We then conclude that $A_\varepsilon = B_\varepsilon$ is self-adjoint and quasi-accretive operator on $L^2(\Omega)$. In particular $-A_\varepsilon$ generates a quasi-contractive C_0-semigroup $(\mathcal{T}_\varepsilon(t))_{t\geq 0}$ on $L^2(\Omega)$. The compactness of the injection of $H^1(\Omega)$ into $L^2(\Omega)$ leads to that of the resolvents. □

The eigenvalue problem of the operator A_ε is given by $A_\varepsilon y = \lambda y$, this leading to the following spectral problem for the Laplacian with Robin boundary conditions.

$$\begin{cases} -\varepsilon\Delta y + q_{f,\varepsilon} y = \lambda y & \text{in } \Omega, \\ \varepsilon\partial_\nu y + \beta_f y = 0 & \text{on } \Gamma. \end{cases}$$

The eigenvalues of self-adjoint operators with compact resolvents such that the eigenvalues are bounded from below can be characterized by the following min–max principle, called the Courant–Fischer Theorem.

Theorem 2.2. *Let $\varepsilon > 0$. There exist an orthonormal basis $\left(\phi_n^{(\varepsilon)}\right)_{n\geq 1}$ of $L^2(\Omega)$ and a sequence of numbers*

$$\lambda_1^{(\varepsilon)} \leqslant \lambda_2^{(\varepsilon)} \leqslant \ldots \leqslant \lambda_n^{(\varepsilon)} \leqslant \lambda_{n+1}^{(\varepsilon)} \leqslant \ldots.$$

whose limit is $+\infty$ such that $A_\varepsilon\phi_n^{(\varepsilon)} = \lambda_n^{(\varepsilon)}\phi_n^{(\varepsilon)}$, for all $n \geq 1$. In addition, we have

$$\lambda_n^{(\varepsilon)} = \min_{\substack{V \text{ subspace of } D(A_\varepsilon) \\ \dim(V)=n}} \ \max_{\substack{y\in V \\ \|y\|_{L^2(\Omega)=1}}} \mathfrak{a}_\varepsilon(y, y) \qquad \forall n \geq 1. \tag{2.9}$$

The following proposition concerns the existence and uniqueness of weak solutions of (1.4), (2.5) and the spectral decomposition of the solutions of (1.4).

Proposition 2.3. *Let $\varepsilon > 0$.*

(1) Let $\Phi_T \in L^2(\Omega)$, then the system (2.5) has a unique weak solution $\Phi \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega))$ that satisfies the estimate

$$\|\Phi\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\varepsilon}\|\Phi\|_{L^2(0,T;H^1(\Omega))} \leqslant C\exp\left(CT\left(\varepsilon + \frac{\|\beta_f\|_\infty^2}{\varepsilon} + \|q_{f,\varepsilon}\|_\infty\right)\right)\|\Phi_T\|_{L^2(\Omega)}, \quad (2.10)$$

where $C > 0$ depending only on Ω.

(2) Let $\varphi_T \in L^2(\Omega)$, there exists a unique solution $\varphi \in C([0,T];L^2(\Omega))$ to (1.4) given by

$$\varphi(\cdot,t) = \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n^{(\varepsilon)}(T-t)}\left(\int_\Omega \exp\left(\frac{f(\xi)}{2\varepsilon}\right)\varphi_T(\xi)\phi_n^{(\varepsilon)}(\xi)\mathrm{d}\xi\right)\phi_n^{(\varepsilon)}(\cdot)\exp\left(\frac{-f(\cdot)}{2\varepsilon}\right). \quad (2.11)$$

Proof. (1) The system (2.5) is known to be well posed. Additionally, a standard method can be used to derive the following equality for every $t \in (0,T)$:

$$\begin{aligned} -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Phi|^2\mathrm{d}x &= -\varepsilon\int_\Omega |\nabla\Phi|^2\mathrm{d}x - \int_\Gamma \beta_f|\Phi|^2\mathrm{d}\sigma - \int_\Omega q_{f,\varepsilon}|\Phi|^2\mathrm{d}x \\ &= -\mathfrak{a}_\varepsilon(\Phi(\cdot,t),\Phi(\cdot,t)). \end{aligned}$$

From (2.8), we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Phi|^2\mathrm{d}x \leqslant -\frac{\varepsilon}{2}\|\Phi\|_{H^1(\Omega)}^2 + \left(\|q_{f,\varepsilon}\|_\infty + \frac{C^2\|\beta_f\|_\infty^2}{2\varepsilon} + \varepsilon\right)\|\Phi\|_{L^2(\Omega)}^2.$$

An integration of this inequality on (t,T), gives

$$\int_\Omega |\Phi(x,t)|^2\mathrm{d}x + \varepsilon\|\Phi\|_{L^2(t,T;H^1(\Omega))}^2 \leqslant \int_\Omega |\Phi_T|^2\mathrm{d}x + 2\left(\|q_{f,\varepsilon}\|_\infty + \frac{C^2\|\beta_f\|_\infty^2}{2\varepsilon} + \varepsilon\right)\int_t^T\int_\Omega |\Phi(x,s)|^2\mathrm{d}x\mathrm{d}s.$$

By the Grönwall's lemma, we have

$$\begin{aligned} \int_\Omega |\Phi(x,t)|^2\mathrm{d}x + \varepsilon\|\Phi\|_{L^2(t,T;H^1(\Omega))}^2 &\leqslant \exp\left(2\left(\varepsilon + \frac{C^2\|\beta_f\|_\infty^2}{2\varepsilon} + \|q_{f,\varepsilon}\|_\infty\right)(T-t)\right)\int_\Omega |\Phi_T|^2\mathrm{d}x \\ &\leqslant \exp\left(2\left(\varepsilon + \frac{C^2\|\beta_f\|_\infty^2}{2\varepsilon} + \|q_{f,\varepsilon}\|_\infty\right)T\right)\int_\Omega |\Phi_T|^2\mathrm{d}x. \end{aligned}$$

In particular,

$$\sup_{0\leqslant t\leqslant T}\int_\Omega |\Phi(x,t)|^2\mathrm{d}x \leqslant \exp\left(2\left(\varepsilon + \frac{C^2\|\beta_f\|_\infty^2}{2\varepsilon} + \|q_{f,\varepsilon}\|_\infty\right)T\right)\int_\Omega |\Phi_T|^2\mathrm{d}x \quad (2.12)$$

and

$$\varepsilon\|\Phi\|_{L^2(0,T;H^1(\Omega))}^2 \leqslant \exp\left(2\left(\varepsilon + \frac{C^2\|\beta_f\|_\infty^2}{2\varepsilon} + \|q_{f,\varepsilon}\|_\infty\right)T\right)\int_\Omega |\Phi_T|^2\mathrm{d}x. \quad (2.13)$$

From (2.12) and (2.13), we easily obtain (2.10).

(2) We have seen that φ is a solution of (1.4), if and only if

$$\Phi(\cdot,t) = \exp\left(\frac{f(\cdot)}{2\varepsilon}\right)\varphi(\cdot,t)$$

is a solution of (2.5). Hence (1.4) has a unique solution $\varphi \in \mathcal{C}([0,T];L^2(\Omega))$.
Since $-A_\varepsilon$ generates a C_0-semigroup $(\mathcal{T}_\varepsilon(t))_{t\geq 0}$ on $L^2(\Omega)$, then the weak solution of (2.5) is given by

$$\Phi(\cdot,t) = \mathcal{T}_\varepsilon(T-t)\Phi_T.$$

From Theorem 2.2, as $\left(\phi_n^{(\varepsilon)}\right)_{n\geq 1}$ is an orthonormal basis of $L^2(\Omega)$, then

$$\Phi_T = \sum_{n=1}^{\infty}\langle\Phi_T,\phi_n^{(\varepsilon)}\rangle_{L^2(\Omega)}\phi_n^{(\varepsilon)} = \sum_{n=1}^{\infty}\left(\int_\Omega \Phi_T(\xi)\phi_n^{(\varepsilon)}(\xi)\mathrm{d}\xi\right)\phi_n^{(\varepsilon)}$$

Since $A_\varepsilon\phi_n^{(\varepsilon)} = \lambda_n^{(\varepsilon)}\phi_n^{(\varepsilon)}$, then $\mathcal{T}_\varepsilon(T-t)\phi_n^{(\varepsilon)} = \exp\left(-\lambda_n^{(\varepsilon)}(T-t)\right)\phi_n^{(\varepsilon)}$. Thus

$$\Phi(\cdot,t) = \sum_{n=1}^{\infty}\left(\int_\Omega \Phi_T(\xi)\phi_n^{(\varepsilon)}(\xi)\mathrm{d}\xi\right)\exp\left(-\lambda_n^{(\varepsilon)}(T-t)\right)\phi_n^{(\varepsilon)}.$$

Hence

$$\begin{aligned}
\varphi(\cdot,t) &= \exp\left(\frac{-f(\cdot)}{2\varepsilon}\right)\Phi(\cdot,t)\\
&= \exp\left(\frac{-f(\cdot)}{2\varepsilon}\right)\sum_{n=1}^{\infty}\left(\int_\Omega \Phi_T(\xi)\phi_n^{(\varepsilon)}(\xi)\mathrm{d}\xi\right)\exp\left(-\lambda_n^{(\varepsilon)}(T-t)\right)\phi_n^{(\varepsilon)}\\
&= \exp\left(\frac{-f(\cdot)}{2\varepsilon}\right)\sum_{n=1}^{\infty}\left(\int_\Omega \exp\left(\frac{f(\xi)}{2\varepsilon}\right)\varphi_T(\xi)\phi_n^{(\varepsilon)}(\xi)\mathrm{d}\xi\right)\exp\left(-\lambda_n^{(\varepsilon)}(T-t)\right)\phi_n^{(\varepsilon)}.
\end{aligned}$$

□

Notation. We denote by

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_n \leqslant \lambda_{n+1} \leqslant ....$$

the sequence of eigenvalues of the Laplacian operator with Neumann boundary conditions. By the min–max principle, we have

$$\lambda_n = \min_{\substack{V \text{ subspace of } H^1(\Omega)\\ \dim(V)=n}} \max_{\substack{y\in V\\ \|y\|_{L^2(\Omega)=1}}} \|\nabla y\|^2_{L^2(\Omega)} \quad \forall n \geq 1. \tag{2.14}$$

Similarly, if A is a set $\#A$ denotes the number of elements of A.

The following result enables us to compare the eigenvalues of the operator A_ε and the ones of the Laplacian operator with Neumann boundary conditions.

Proposition 2.4. *We assume that* $\beta_f \geq 0$*. Then, for all* $\varepsilon > 0$

$$\lambda_n^{(\varepsilon)} \geq \varepsilon\lambda_n - \|q\|_\infty - \frac{\|\Delta f\|_\infty}{2} + \frac{E_0}{\varepsilon} \quad \forall n \geq 1.$$

Proof. Let $n \geq 1$, from (2.14) and (2.6), we have

$$\begin{aligned}
\varepsilon\lambda_n &\leqslant \varepsilon \max_{\substack{y\in V\\ \|y\|_{L^2(\Omega)=1}}} \|\nabla y\|^2_{L^2(\Omega)} \\
&= \max_{\substack{y\in V\\ \|y\|_{L^2(\Omega)=1}}} \left(\mathfrak{a}_\varepsilon(y,y) - \int_\Omega q_{f,\varepsilon}|y|^2 \mathrm{d}x - \int_\Gamma \beta_f |y|^2 \mathrm{d}\sigma\right),
\end{aligned}$$

for all V subspace of $H^1(\Omega)$ such that $\dim(V) = n$.
Since $\beta_f \geq 0$ and $q_{f,\varepsilon} \geq -\|q\|_\infty - \frac{\|\Delta f\|_\infty}{2} + \frac{E_0}{\varepsilon}$, then

$$\varepsilon\lambda_n \leqslant \max_{\substack{y\in V\\ \|y\|_{L^2(\Omega)=1}}} \left(\mathfrak{a}_\varepsilon(y,y) + \|q\|_\infty + \frac{\|\Delta f\|_\infty}{2} - \frac{E_0}{\varepsilon}\right).$$

In particular, for all V subspace of $D(A_\varepsilon)$ such that $\dim(V) = n$, we have

$$\varepsilon\lambda_n - \|q\|_\infty - \frac{\|\Delta f\|_\infty}{2} + \frac{E_0}{\varepsilon} \leqslant \max_{\substack{y\in V\\ \|y\|_{L^2(\Omega)=1}}} \mathfrak{a}_\varepsilon(y,y).$$

Hence, by (2.9), we obtain

$$\varepsilon\lambda_n - \|q\|_\infty - \frac{\|\Delta f\|_\infty}{2} + \frac{E_0}{\varepsilon} \leqslant \min_{\substack{V \text{ subspace of } D(A_\varepsilon)\\ \dim(V)=n}} \max_{\substack{y\in V\\ \|y\|_{L^2(\Omega)=1}}} \mathfrak{a}_\varepsilon(y,y) = \lambda_n^{(\varepsilon)}.$$

□

We end this section with a reminder about the **Weyl's law** satisfied by the sequence of eigenvalues associated to the Laplacian operator with Neumann boundary conditions.

Lemma 2.5. *(**Weyl's law** [40]). For any real* x*, we denote* $N(x)$ *the number of eigenvalues (counting repetitions) of the Neumann Laplacian which are smaller than* x*:*

$$N(x) := \#\{n \geq 1 : \quad \lambda_n \leqslant x\}.$$

Then, we have

$$\lim_{x\to\infty} \frac{N(x)}{x^{\frac{N}{2}}} = \frac{|B(0,1)|\,|\Omega|}{(2\pi)^N}.$$

As a consequence of this lemma, using the definition of the limit, there is $C > 0$ depending on Ω such that for all $x > 0$,

$$N(x) \leqslant C\left(1 + x^{\frac{N}{2}}\right). \tag{2.15}$$

3. An upper bound of the null controllability cost

In this section, we prove an upper bound for the cost of the null controllability of (1.1) for a general transport $X \in L^\infty(\Omega)$. In this case, the adjoint system of (1.1) is given by

$$\begin{cases} -\partial_t \varphi - \varepsilon \Delta \varphi - \nabla \cdot (\varphi X) + q\,\varphi = 0 & \text{in } \Omega_T, \\ (\varepsilon \nabla \varphi + \varphi X) \cdot \nu(x) + \beta\,\varphi = 0 & \text{on } \Gamma_T, \\ \varphi(x,T) = \varphi_T(x) & \text{in } \Omega. \end{cases} \tag{3.1}$$

By Lions' Theorem [41, 42], the system (3.1) has a unique weak solution

$$\varphi \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)).$$

The following proposition gives an upper bound on the controllability cost of the system (1.1). Note that the Hypothesis of Proposition 3.1 are more general than those of Theorem 1.1:

Proposition 3.1. *Let Ω be a $\mathcal{C}^2$ domain, $\omega \subset\subset \Omega$ is a nonempty open subset and $X \in L^\infty(\Omega)$. Then, for all $T > 0$ and $\varepsilon > 0$, the system (1.1) is null controllable at time T. Moreover there exists $C > 0$ depending only on Ω and ω such that, we have the estiamte:*

$$\mathcal{K}(\Omega,\omega,T,\varepsilon) \leqslant \varepsilon \exp\left(C \left(1 + \frac{1}{\varepsilon T} + \frac{\|q\|_\infty^{\frac{2}{3}}}{\varepsilon^{\frac{2}{3}}} + \frac{\|X\|_\infty^2}{\varepsilon^2} + \frac{\|\beta\|_\infty^2}{\varepsilon^2} + T \left(\|q\|_\infty + \frac{\|X\|_\infty^2}{\varepsilon} + \frac{\|\beta\|_\infty^2}{\varepsilon} \right) \right) \right), \tag{3.2}$$

for $\mathcal{K}$ the cost of the null controllability of (1.1).

It is indeed known that the system (1.1) is null controllable, and thus the novelty is on the bound.

Proof. For any time $T > 0$ and for all general vector fields $X \in L^\infty(\Omega)^N$, $q \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma)$ and $\varepsilon = 1$, we refer to [6], Theorem 2. In order to return to this case, we make the change of state

$$z(x,t) := y\left(x, \frac{t}{\varepsilon}\right),$$

where y is the weak solution of (1.1). Then z satisfies the following system:

$$\begin{cases} \partial_t z - \Delta z + X_\varepsilon \cdot \nabla z + q_\varepsilon z = u_\varepsilon(x,t) \mathbb{1}_\omega & \text{in } \Omega_{\varepsilon T}, \\ \partial_\nu z + \beta_\varepsilon z = 0 & \text{on } \Gamma_{\varepsilon T}, \\ z(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{3.3}$$

for $\varepsilon X_\varepsilon = X$, $\varepsilon q_\varepsilon = q$, $\varepsilon \beta_\varepsilon = \beta$ and $\varepsilon\, u_\varepsilon(\cdot,t) = v\left(\cdot, \frac{t}{\varepsilon}\right)$, for all $t \in [0, \varepsilon T]$. Applying [6], Theorem 2, we obtain, for all $T > 0$ and $\varepsilon > 0$, the system (3.3) is null controllable at time T with controls $u_\varepsilon \in L^2(\omega \times (0, \varepsilon T))$. Moreover, one can find u such that

$$\|u_\varepsilon\|_{L^2(\omega \times (0,\varepsilon T))} \leqslant H \|y_0\|_{L^2(\Omega)}$$

with a constant H of the form

$$\begin{aligned} H &= \exp\left(C \left(1 + \frac{1}{\varepsilon T} + \|q_\varepsilon\|_\infty^{\frac{2}{3}} + \|X_\varepsilon\|_\infty^2 + \|\beta_\varepsilon\|_\infty^2 + \varepsilon T \left(\|q_\varepsilon\|_\infty + \|X_\varepsilon\|_\infty^2 + \|\beta_\varepsilon\|_\infty^2 \right) \right) \right) \\ &= \exp\left(C \left(1 + \frac{1}{\varepsilon T} + \frac{\|q\|_\infty^{\frac{2}{3}}}{\varepsilon^{\frac{2}{3}}} + \frac{\|X\|_\infty^2}{\varepsilon^2} + \frac{\|\beta\|_\infty^2}{\varepsilon^2} + T \left(\|q\|_\infty + \frac{\|X\|_\infty^2}{\varepsilon} + \frac{\|\beta\|_\infty^2}{\varepsilon} \right) \right) \right). \end{aligned}$$

We turn to the system (1.1) by the change of variable $t = \varepsilon\,\tau$, we obtain, for all $T > 0$ and $\varepsilon > 0$, the system (1.1) is null controllable at time T with the control $v := v_\varepsilon \in L^2(\omega \times (0,T))$ $(v_\varepsilon(\cdot,t) = \varepsilon u_\varepsilon(\cdot,\varepsilon t),\ 0 \leqslant t \leqslant T)$ such that

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant \varepsilon\, H \|y_0\|_{L^2(\Omega)}. \tag{3.4}$$

Finally, we obtain (3.2) form (1.3) and (3.4). □

Remark 3.2. The proposition 3.1 remains valid if the coefficients of the system (1.1) also depend on the time variable and are bounded.

4. Exponential decay of null controllability cost

In this section, we prove Theorem 1.1. Firstly, we show the validity of the following observability inequality

$$\iint_{\Omega\times\left(\frac{T}{4},\frac{3T}{4}\right)} |\varphi(x,t)|^2 \mathrm{d}x\mathrm{d}t \leqslant C \exp\left(\frac{C}{\varepsilon}\left(1+T^{-1}\right)\right) \iint_{\omega\times(0,T)} |\varphi(x,t)|^2 \mathrm{d}x\mathrm{d}t, \tag{4.1}$$

for φ the solution of the system adjoint (1.4) and the constant $C > 0$ independent of ε and T. Secondly, using the spectral analysis, we prove an important dissipation estimate (see (4.49) below) which is crucial to prove our first main result Theorem 1.1.

4.1. Carleman estimates for the system (2.5)

The objective of this section is to prove a Carleman estimate for the system (2.5).

Let $\varepsilon > 0$ and $T > 0$, we consider the following positive weight functions $\alpha_\pm$ and $\xi_\pm$ which depend only on Ω and ω:

$$\xi_\pm(x,t) := \frac{\exp(4\lambda \pm \lambda\eta(x))}{t(T-t)} \quad \text{and} \quad \alpha_\pm(x,t) := \frac{\exp(6\lambda) - \exp(4\lambda \pm \lambda\eta(x))}{t(T-t)}.$$

Here, $\lambda \geq 1$ and $\eta = \eta(x)$ is a function in $\mathcal{C}^2(\overline{\Omega})$ satisfying

$$\eta > 0 \text{ in } \Omega, \quad \eta = 0 \text{ on } \Gamma, \quad \inf_{\Omega\setminus\omega'} |\nabla\eta(x)| = \delta > 0 \quad \text{and} \quad \|\eta\|_\infty = 1, \tag{4.2}$$

where ω' is a nonempty open subset of ω. The existence of such a function η satisfying (4.2) is proved in [3] if Ω is a $\mathcal{C}^2$ domain, as well as the whole approach on how to prove the controllability of parabolic equations. Also, the approach of using two weights for dealing with Neumann boundary conditions dates back to [4], where they prove the null controllability for Neumann boundary conditions. The reason why the two weights are necessary is to cancel some boundary terms with Neumann boundary conditions that appears while doing the Carleman inequality, as we do in Step 5.

The functions $\xi_\pm$ and $\alpha_\pm$ verify the following elementary properties:

$$\begin{aligned} &\xi_\pm \geq \frac{4}{T^2}, \quad \xi_- \leqslant \xi_+, \quad \alpha_+ \leqslant \alpha_-, \quad |\partial_t \xi_\pm| \leqslant T\xi_\pm^2, \quad |\partial_t \alpha_\pm| \leqslant T\xi_\pm^2, \\ &|\nabla(\partial_t \alpha_\pm)| \leqslant \|\nabla\eta\|_\infty \lambda T \xi_\pm^2 \quad \text{and} \quad |\partial_t^2 \alpha_\pm| \leqslant 2T^2 \xi_\pm^3. \end{aligned}$$

Since $\eta = 0$ on Γ, we have the following equations on Γ_T:

$$\alpha_+ = \alpha_- \quad \text{and} \quad \xi_+ = \xi_-.$$

To simplify, on Γ_T we will note these common values by α and ξ.

Remark 4.1. Realizing the observability inequality (4.1) requires finding a Carleman estimate for the solutions of the adjoint system (1.4) while respecting the constraint $s \geq \frac{C}{\varepsilon}$ and $\lambda \geq C$, for $C > 0$ independent of ε. However, the presence of a transport term and the boundary conditions makes this task particularly difficult. Indeed, if X is a general function instead of a gradient field, in the computation of the Carleman estimate, we obtain the following term

$$\varepsilon \iint_{\Gamma_T} (\nabla\psi_\pm \cdot X(x,t))\partial_\nu \psi_\pm \mathrm{d}\sigma\mathrm{d}t - \frac{\varepsilon}{2}\iint_{\Gamma_T} (X(x,t)\cdot\nu(x))|\nabla\psi_\pm|^2 \mathrm{d}\sigma\mathrm{d}t,$$

which is not easily treatable, where $\psi_\pm := \exp(-s\alpha_\pm)\varphi$ and φ the solution of the adjoint system (1.4). To remedy this, we will establish a Carleman estimate for the solutions of the system (2.5) (system without transport). Then, thanks to the change (2.4) and a well-known optimization arguments, we will obtain the observability inequality (4.1). The techniques used are inspired by [6, 11]. Another difficulty comes from the Robin boundary conditions, the smoothness of solutions is worse than the Dirichlet conditions, and as the solutions of the adjoint system may not belong to $L^2(0,T;H^2(\Omega))$. To overcome this difficulty, we will approximate the solution using smooth functions preserving the Robin boundary conditions.

Notation. Note the following energy spaces

$$\begin{aligned} \mathcal{Z} &:= \{y \in \mathcal{C}([0,T]; D(A_\varepsilon)\cap\mathcal{C}^2(\overline{\Omega})) \ : \ \partial_t y \in L^\infty(0,T;\mathcal{C}^2(\overline{\Omega}))\},\\ \mathcal{W} &:= \{y \in L^2(0,T;H^1(\Omega)) \ : \ \partial_t y \in L^2(0,T;L^2(\Omega))\}. \end{aligned}$$

Lemma 4.2. *Let $T > 0$, $\varepsilon > 0$ and Ω is a $\mathcal{C}^2$ domain. Then, for all $\Phi_T \in D(A_\varepsilon^2)$, there exists (Φ_n) a sequence of $\mathcal{Z}$ which converges to Φ in $\mathcal{W}$ where Φ is the solution of (2.5) with data Φ_T. In particular:*

$$-\partial_t\Phi_n - \varepsilon\Delta\Phi_n + q_{f,\varepsilon}\Phi_n \longrightarrow 0 \ \ \textit{in} \ \ L^2(0,T;L^2(\Omega)).$$

Proof. Let $\Phi_T \in D(A_\varepsilon^2)$. Using rescaling in time and Theorem VII.5 in [43], we obtain $\Phi \in \mathcal{C}^1([0,T], D(A_\varepsilon))$. In particular, for all $\mu > 0$, there exists $d > 0$ such that, for all $t, s \in [0,T]$, we have

$$|t-s| < d \Longrightarrow \|\partial_t\Phi(\cdot,t) - \partial_t\Phi(\cdot,s)\|_{D(A_\varepsilon)} < \mu. \tag{4.3}$$

We consider $0 = t_0 < t_1 < \cdots < t_k = T$ a partition of $[0,T]$ with norm (or mesh)

$$\max_{0\leq i\leq k-1}(t_{i+1}-t_i) < d.$$

Let $i = 1,\cdots,k$ and $t \in (t_{i-1}, t_i]$. From (4.3), we have

$$\|\partial_t\Phi(\cdot,t) - \partial_t\Phi(\cdot,t_i)\|_{D(A_\varepsilon)} < \mu.$$

Let us choose $\phi \in \mathcal{C}^2(\overline{\Omega})$ and $\psi_i \in D(A_\varepsilon)\cap\mathcal{C}^2(\overline{\Omega})$ such that

$$\|\phi - \Phi_T\|_{D(A_\varepsilon)} < \mu \ \text{ and } \ \|\psi_i - \partial_t\Phi(\cdot,t_i)\|_{D(A_\varepsilon)} < \mu. \tag{4.4}$$

Then

$$\|\partial_t\Phi(\cdot,t) - \psi_i\|_{D(A_\varepsilon)} < 2\mu. \tag{4.5}$$

Consider $\psi = \psi(x,t)$ and $\tilde{\Phi}_\mu = \tilde{\Phi}_\mu(x,t)$ the functions defined by

$$\psi(\cdot,t) := \psi_i \quad t \in (t_{i-1}, t_i], \tag{4.6}$$

$$\tilde{\Phi}_\mu(\cdot,t) := \phi - \int_t^T \psi(\cdot,s)\mathrm{d}s \quad t \in [0,T]. \tag{4.7}$$

Using (4.4)–(4.7) and $\Phi(\cdot,t) = \Phi_T - \displaystyle\int_t^T \partial_t \Phi(\cdot,s)\mathrm{d}s$, we obtain $\tilde{\Phi}_\mu \in \mathcal{Z}$ and

$$\begin{cases} \|\tilde{\Phi}_\mu - \Phi\|_{\mathcal{C}([0,T],D(A_\varepsilon))} < (1+2T)\mu, \\ \|\partial_t \tilde{\Phi}_\mu - \partial_t \Phi\|_{L^\infty(0,T,D(A_\varepsilon))} \le 2\mu. \end{cases} \tag{4.8}$$

On the other hand, from the quasi ellipticity of the form $\mathfrak{a}_\varepsilon$ in (2.7), we obtain

$$\langle A_\varepsilon \Phi, \Phi\rangle_{L^2(\Omega)} + w\|\Phi\|^2_{L^2(\Omega)} \ge \alpha \|\Phi\|^2_{H^1(\Omega)} \ \ \forall \Phi \in D(A_\varepsilon). \tag{4.9}$$

Hence by picking $\mu = 1/n$, based on the estimates (4.8) and (4.9) and the definition of $D(A_\varepsilon)$ the result follows. ☐

Now, we are in position to establish the desired Carleman estimate.

Proposition 4.3. *Let $T > 0$, $\varepsilon \in (0,1)$, Ω is a $\mathcal{C}^2$ domain, $\omega \subset\subset \Omega$ is a nonempty open subset and assume that $q \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma)$, $f \in W^{2,\infty}(\Omega)$ such that $\beta_f \ge 0$. Then there are constants $C > 0$ and $\lambda_1, s_1 \ge 1$ depend only on ω and Ω such that*

$$\begin{aligned} &s^3\lambda^4 \iint_{\Omega_T} \exp(-2s\alpha_+)\xi_+^3 |\Phi|^2 \mathrm{d}x\mathrm{d}t + s\lambda^2 \iint_{\Omega_T} \exp(-2s\alpha_+)\xi_+ |\nabla\Phi|^2 \mathrm{d}x\mathrm{d}t \\ &+ s\lambda^2 \iint_{\Gamma_T} \beta_f |\partial_\nu \eta|^2 \left(\xi + s\xi^2\right) \exp(-2s\alpha)|\Phi|^2 \mathrm{d}\sigma\mathrm{d}t \leqslant Cs^3\lambda^4 \iint_{\omega\times(0,T)} \exp(-2s\alpha_+)\xi_+^3 |\Phi|^2 \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.10}$$

for any Φ solution of (2.5) with data $\Phi_T \in L^2(\Omega)$, $\lambda \ge \lambda_1$, $s \ge s_1 C(T,\varepsilon,f,q)$ and

$$C(T,\varepsilon,f,q) := \left(\frac{T}{\varepsilon} + \frac{T^2}{\varepsilon}(\|\nabla f\|_\infty + \|\nabla^2 f\|_\infty + \|\Delta f\|_\infty^{\frac{2}{3}} + \|q\|_\infty^{\frac{2}{3}}) + T^2 \right).$$

We will begin by proving an estimate for functions in $\mathcal{Z}$, the Lemma 4.2, has enabled us to obtain the estimate (4.10) for solutions of (2.5) with data $\Phi_T \in D(A_\varepsilon^2)$. This is necessary because the second order derivatives appear in some intermediary computations, but they are removed with proper integration by parts and using the positivity of some terms. Thus, by the density of $D(A_\varepsilon^2)$ in $L^2(\Omega)$ (for the density of $D(A_\varepsilon^2)$ in $D(A_\varepsilon)$ see Lemma VII.2 in [43]) and the estimate (2.10) for the solutions of (2.5), we obtain (4.10) with general data $\Phi_T \in L^2(\Omega)$. Indeed, the terms in (4.10) are continuous with respect to the $L^2(0,T;H^1(\Omega))$ norm.

Proof. In this proof s_1, λ_1, C and c are generic positive constants depending only on Ω and ω. For simplicity, we will divide the proof in five steps. In the first step, we make a change of variables to functions that decay when $t = 0$ and $t = T$. In the second step, we estimate the scalar product that we naturally obtain in the change of variables. In the third step, we obtain some first conclusions with the boundary terms in the left hand-side of the inequality. In the fourth step, we estimate the local term of the gradient. In the fifth step, we revert the change of variables and simplify the boundary terms, to obtain an estimate for functions in $\mathcal{Z}$.

Step 1: Change of variables. Let $\Phi \in \mathcal{Z}$, $\lambda \geq 1$, $s \geq 1$ parameters to be specified. Define

$$\psi_\pm := \exp(-s\alpha_\pm)\Phi \quad \text{and} \quad F_\pm := \exp(-s\alpha_\pm)(\partial_t \Phi + \varepsilon\Delta\Phi - q_{f,\varepsilon}\Phi). \tag{4.11}$$

We recall the definition of the tangential derivative ∇_Γ of a regular function $h \in \mathcal{C}^1(\overline{\Omega})$ is given by $\nabla_\Gamma h := \nabla h - (\partial_\nu h)\nu$ and that this definition depends only on the image of h on Γ. Since $\alpha_+ = \alpha_-$ on Γ_T, then

$$\psi_+ = \psi_- \quad \text{and} \quad \nabla_\Gamma \psi_+ = \nabla_\Gamma \psi_- \;\text{ on }\; \Gamma_T, \tag{4.12}$$

on Γ_T we will note respectively ψ and $\nabla_\Gamma \psi$ instead of $\psi_\pm$ and $\nabla_\Gamma \psi_\pm$.

We determine the problem solved by $\psi_\pm$. We first expand the spatial derivatives of $\alpha_\pm$ by the chain rule to bring η into play, but we do not expand $\partial_t \alpha_\pm$. We calculate

$$\begin{aligned}
\nabla\alpha_\pm &= -\nabla\xi_\pm = \mp\lambda\xi_\pm\nabla\eta \\
\Delta\alpha_\pm &= -\lambda^2\xi_\pm|\nabla\eta|^2 \mp \lambda\xi_\pm\Delta\eta \\
\partial_t\psi_\pm &= \exp(-s\alpha_\pm)\partial_t\Phi - s\partial_t\alpha_\pm\psi_\pm && (4.13)\\
\nabla\psi_\pm &= \exp(-s\alpha_\pm)\nabla\Phi - s\psi_\pm\nabla\alpha_\pm = \exp(-s\alpha_\pm)\nabla\Phi \pm s\lambda\xi_\pm\psi_\pm\nabla\eta && (4.14)\\
\partial_\nu\psi_\pm &= \exp(-s\alpha_\pm)\partial_\nu\Phi \pm s\lambda\xi_\pm\psi_\pm\partial_\nu\eta = \exp(-s\alpha)\partial_\nu\Phi \pm s\lambda\xi\psi\partial_\nu\eta && (4.15)\\
\Delta\psi_\pm &= \exp(-s\alpha_\pm)\Delta\Phi + \nabla(\exp(-s\alpha_\pm))\cdot\nabla\Phi - s\psi_\pm\Delta\alpha_\pm - s(\nabla\psi_\pm\cdot\nabla\alpha_\pm)\\
&= \exp(-s\alpha_\pm)\Delta\Phi - s^2\psi_\pm|\nabla\alpha_\pm|^2 - 2s(\nabla\psi_\pm\cdot\nabla\alpha_\pm) - s\psi_\pm\Delta\alpha_\pm\\
&= \exp(-s\alpha_\pm)\Delta\Phi - s^2\lambda^2\xi_\pm^2\psi_\pm|\nabla\eta|^2 \pm 2s\lambda\xi_\pm(\nabla\eta\cdot\nabla\psi_\pm)\\
&\quad + s\lambda^2\xi_\pm\psi_\pm|\nabla\eta|^2 \pm s\lambda\xi_\pm\psi_\pm\Delta\eta.
\end{aligned}$$

On Ω_T this yields transformed evolution equations

$$\begin{aligned}
\partial_t\psi_\pm + \varepsilon\Delta\psi_\pm - q_{f,\varepsilon}\psi_\pm = F_\pm - s\partial_t\alpha_\pm\psi_\pm - \varepsilon s^2\lambda^2\xi_\pm^2|\nabla\eta|^2\psi_\pm \pm 2\varepsilon s\lambda\xi_\pm(\nabla\eta\cdot\nabla\psi_\pm)\\
+\varepsilon s\lambda^2\xi_\pm|\nabla\eta|^2\psi_\pm \pm \varepsilon s\lambda\xi_\pm\Delta\eta\psi_\pm.
\end{aligned}$$

We rewrite this equality as

$$L_1\psi_\pm + L_2\psi_\pm = L_3\psi_\pm, \tag{4.16}$$

where

$$\begin{aligned}
L_1\psi_\pm &:= -2\varepsilon s\lambda^2\xi_\pm|\nabla\eta|^2\psi_\pm \mp 2\varepsilon s\lambda\xi_\pm(\nabla\eta\cdot\nabla\psi_\pm) + \partial_t\psi_\pm,\\
L_2\psi_\pm &:= \varepsilon s^2\lambda^2\xi_\pm^2|\nabla\eta|^2\psi_\pm + \varepsilon\Delta\psi_\pm + s\partial_t\alpha_\pm\psi_\pm - \frac{|\nabla f(x)|^2}{4\varepsilon}\psi_\pm, && (4.17)\\
L_3\psi_\pm &:= F_\pm \pm \varepsilon s\lambda\xi_\pm\Delta\eta\psi_\pm - \varepsilon s\lambda^2\xi_\pm|\nabla\eta|^2\psi_\pm + \left(q - \frac{\Delta f}{2}\right)\psi_\pm. && (4.18)
\end{aligned}$$

Remark 4.4. In this decomposition, we have split the potential term $q_{f,\varepsilon}$ into two parts $\left(q - \frac{\Delta f}{2}\right)$ and $\frac{|\nabla f(x)|^2}{4\varepsilon}$ in order to absorb the terms associated with constraint $s \geq \frac{C}{\varepsilon}$.

Applying $\|\cdot\|^2_{L^2(\Omega_T)}$ to the equation (4.16), we obtain

$$\|L_1\psi_\pm\|^2_{L^2(\Omega_T)} + 2(L_1\psi_\pm, L_2\psi_\pm)_{L^2(\Omega_T)} + \|L_2\psi_\pm\|^2_{L^2(\Omega_T)} = \|L_3\psi_\pm\|^2_{L^2(\Omega_T)}. \tag{4.19}$$

Step 2. Estimating the mixed terms in (4.19) from below with the control domain and some boundary terms. The principle of this step is to expand the second term of (4.19) and make some terms of this scalar product largely positive. For this, we use some properties of $\alpha_\pm$ and let s be large enough. Denoting by $(L_i\psi_\pm)_j$ the j-th term of the expression $L_i\psi_\pm$, we obtain

$$(L_1\psi_\pm, L_2\psi_\pm)_{L^2(\Omega_T)} = \sum_{\substack{1\leqslant i\leqslant 3\\ 1\leqslant j\leqslant 4}} ((L_1\psi_\pm)_i, (L_2\psi_\pm)_j)_{L^2(\Omega_T)}.$$

Let us compute each of these terms.

Step 2a. Estimate from below of $((L_1\psi_\pm),(L_2\psi_\pm)_1)_{L^2(\Omega_T)}$.

First, we have

$$((L_1\psi_\pm)_1, (L_2\psi_\pm)_1)_{L^2(\Omega_T)} = -2\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \xi_\pm^3 |\nabla\eta|^4 |\psi_\pm|^2 \mathrm{d}x\mathrm{d}t := A_\pm$$

and

$$\begin{aligned}
((L_1\psi_\pm)_2, (L_2\psi_\pm)_1)_{L^2(\Omega_T)} &= \mp 2\varepsilon^2 s^3\lambda^3 \iint_{\Omega_T} (\nabla\eta\cdot\nabla\psi_\pm)|\nabla\eta|^2 \xi_\pm^3 \psi_\pm \mathrm{d}x\mathrm{d}t\\
&= 3\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \xi_\pm^3 |\nabla\eta|^4 |\psi_\pm|^2 \mathrm{d}x\mathrm{d}t\\
&\quad \pm\varepsilon^2 s^3\lambda^3 \iint_{\Omega_T} \xi_\pm^3 \nabla\cdot(|\nabla\eta|^2\nabla\eta)|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t\\
&\quad \mp\varepsilon^2 s^3\lambda^3 \iint_{\Gamma_T} \xi^3 |\nabla\eta|^2 \partial_\nu\eta |\psi|^2 \mathrm{d}\sigma\mathrm{d}t\\
&:= B_\pm^1 + B_\pm^2 + B_\pm^3.
\end{aligned}$$

We can notice that $A_\pm + B_\pm^1 = \varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \xi_\pm^3 |\nabla\eta|^4 |\psi_\pm|^2 \mathrm{d}x\mathrm{d}t \geq 0$, from the properties (4.2) of η, we obtain

$$\begin{aligned}
A_\pm + B_\pm^1 &\geq c\,\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \xi_\pm^3 |\psi_\pm|^2 \mathrm{d}x\mathrm{d}t - C\varepsilon^2 s^3\lambda^4 \iint_{\omega'\times(0,T)} \xi_\pm^3 |\psi_\pm|^2 \mathrm{d}x\mathrm{d}t\\
&:= \tilde{A}_\pm - \tilde{B}_\pm,
\end{aligned}$$

for all $c\in(0,\delta^4)$. The term $B_\pm^2$ can be absorbed by $\tilde{A}_\pm$ if $\lambda \geq \lambda_1$ for large λ_1. Then

$$((L_1\psi_\pm)_1, (L_2\psi_\pm)_1)_{L^2(\Omega_T)} + ((L_1\psi)_2, (L_2\psi)_1)_{L^2(\Omega_T)} \geq \tilde{A}_\pm - \tilde{B}_\pm + B_\pm^3.$$

By integration by parts in time and $\psi_\pm(\cdot,0) = \psi_\pm(\cdot,T) = 0$, we obtain

$$((L_1\psi_\pm)_3, (L_2\psi_\pm)_1)_{L^2(\Omega_T)} = -\varepsilon s^2\lambda^2 \iint_{\Omega_T} |\nabla\eta|^2 \xi_\pm \partial_t\xi_\pm |\psi_\pm|^2 \mathrm{d}x\mathrm{d}t.$$

Since $|\partial_t\xi_\pm| \leqslant T\xi_\pm^2$, then

$$|((L_1\psi_\pm)_3, (L_2\psi_\pm)_1)_{L^2(\Omega_T)}| \leqslant C\varepsilon s^2\lambda^2 T \iint_{\Omega_T} \xi_\pm^3 |\psi_\pm|^2 \mathrm{d}x\mathrm{d}t,$$

this integral is absorbed by $\tilde{A}_\pm$ if we take $s \geq s_1\frac{T}{\varepsilon}$.

Consequently, we obtain

$$\begin{aligned}(L_1\psi_\pm,(L_2\psi_\pm)_1)_{L^2(\Omega_T)} &= \sum_{1\leqslant j\leqslant 3}((L_1\psi_\pm)_j,(L_2\psi_\pm)_1)_{L^2(\Omega_T)}\\ &\geq c\,\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T}\xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t - C\varepsilon^2 s^3\lambda^4\iint_{\omega'\times(0,T)}\xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ &\quad\mp\varepsilon^2 s^3\lambda^3\iint_{\Gamma_T}\xi^3|\nabla\eta|^2\partial_\nu\eta|\psi|^2\mathrm{d}\sigma\mathrm{d}t, \end{aligned}\tag{4.20}$$

for any $\lambda \geq \lambda_1$ and $s \geq s_1\frac{T}{\varepsilon}$.

Step 2b. Estimate from below of $((L_1\psi_\pm),(L_2\psi_\pm)_2)_{L^2(\Omega_T)}$**.**
By integration by parts, we have

$$\begin{aligned}((L_1\psi_\pm)_1,(L_2\psi_\pm)_2)_{L^2(\Omega_T)} &= -2\varepsilon^2 s\lambda^2\iint_{\Omega_T}\xi_\pm|\nabla\eta|^2\psi_\pm\Delta\psi_\pm\mathrm{d}x\mathrm{d}t\\ &= -2\varepsilon^2 s\lambda^2\iint_{\Gamma_T}\xi|\nabla\eta|^2\psi\partial_\nu\psi_\pm\mathrm{d}\sigma\mathrm{d}t + 2\varepsilon^2 s\lambda^2\iint_{\Omega_T}\xi_\pm(\nabla(|\nabla\eta|^2)\cdot\nabla\psi_\pm)\psi_\pm\mathrm{d}x\mathrm{d}t\\ &\quad\pm 2\varepsilon^2 s\lambda^3\iint_{\Omega_T}\xi_\pm|\nabla\eta|^2(\nabla\eta\cdot\nabla\psi_\pm)\psi_\pm\mathrm{d}x\mathrm{d}t + 2\varepsilon^2 s\lambda^2\iint_{\Omega_T}\xi_\pm|\nabla\eta|^2|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ &:= C_\pm^1 + C_\pm^2 + C_\pm^3 + C_\pm^4.\end{aligned}$$

We will keep $C_\pm^1$ and $C_\pm^4$ in the left hand side. For $C_\pm^2$ and $C_\pm^3$, applying Hölder's inequality, we obtain

$$\begin{aligned}|C_\pm^2| &\leqslant C\varepsilon^2 s\lambda^2\iint_{\Omega_T}\xi_\pm|\nabla\psi_\pm||\psi_\pm|\mathrm{d}x\mathrm{d}t\\ &\leqslant C\varepsilon^2 s\lambda^4\iint_{\Omega_T}\xi_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + C\varepsilon^2 s\iint_{\Omega_T}\xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Since $\xi_\pm \geq \frac{4}{T^2}$, then for all $s \geq s_1T^2$

$$|C_\pm^2| \leqslant C\varepsilon^2 s^2\lambda^4\iint_{\Omega_T}\xi_\pm^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + C\varepsilon^2 s\iint_{\Omega_T}\xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t.$$

We also obtain

$$\begin{aligned}|C_\pm^3| &\leqslant C\varepsilon^2 s\lambda^3\iint_{\Omega_T}\xi_\pm|\nabla\psi_\pm||\psi_\pm|\mathrm{d}x\mathrm{d}t\\ &\leqslant C\varepsilon^2 s^2\lambda^4\iint_{\Omega_T}\xi_\pm^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + C\varepsilon^2\lambda^2\iint_{\Omega_T}|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

For all $s \geq s_1T^2$, we conclude that

$$\begin{aligned}((L_1\psi_\pm)_1,(L_2\psi_\pm)_2)_{L^2(\Omega_T)} &\geq -2\varepsilon^2 s\lambda^2\iint_{\Gamma_T}\xi|\nabla\eta|^2\psi\partial_\nu\psi_\pm\mathrm{d}\sigma\mathrm{d}t\\ &\quad - C\varepsilon^2 s^2\lambda^4\iint_{\Omega_T}\xi_\pm^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t - C\varepsilon^2\iint_{\Omega_T}(s\xi_\pm+\lambda^2)|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ &\quad + 2\varepsilon^2 s\lambda^2\iint_{\Omega_T}\xi_\pm|\nabla\eta|^2|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.21}$$

Integration by parts gives

$$\begin{aligned}((L_1\psi_\pm)_2,(L_2\psi_\pm)_2)_{L^2(\Omega_T)} &= \mp 2\varepsilon^2 s\lambda \iint_{\Omega_T} \xi_\pm(\nabla\eta\cdot\nabla\psi_\pm)\Delta\psi_\pm \mathrm{d}x\mathrm{d}t\\ &= \mp 2\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi(\nabla\eta\cdot\nabla\psi_\pm)\partial_\nu\psi_\pm \mathrm{d}\sigma\mathrm{d}t\\ &\quad \pm 2\varepsilon^2 s\lambda \iint_{\Omega_T} \xi_\pm(\nabla(\nabla\eta\cdot\nabla\psi_\pm)\cdot\nabla\psi_\pm)\mathrm{d}x\mathrm{d}t\\ &\quad \pm 2\varepsilon^2 s\lambda \iint_{\Omega_T} (\nabla\eta\cdot\nabla\psi_\pm)(\nabla\xi_\pm\cdot\nabla\psi_\pm)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using $\nabla(\nabla\eta\cdot\nabla\psi_\pm)\cdot\nabla\psi_\pm = \nabla^2\eta(\nabla\psi_\pm,\nabla\psi_\pm)+\frac{1}{2}\nabla\eta\cdot\nabla(|\nabla\psi_\pm|^2)$, where $\nabla^2\eta$ denotes the Hessian matrix of η (it is considered as a symmetrical bilinear form) and another integration by parts, we obtain

$$\begin{aligned}((L_1\psi_\pm)_2,(L_2\psi_\pm)_2)_{L^2(\Omega_T)} \;=\;& \mp 2\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi(\nabla\eta\cdot\nabla\psi_\pm)\partial_\nu\psi_\pm \mathrm{d}\sigma\mathrm{d}t\\ & \pm 2\varepsilon^2 s\lambda \iint_{\Omega_T} \xi_\pm\nabla^2\eta(\nabla\psi_\pm,\nabla\psi_\pm)\mathrm{d}x\mathrm{d}t \pm \varepsilon^2 s\lambda \iint_{\Omega_T} \xi_\pm\nabla\eta\cdot\nabla(|\nabla\psi_\pm|^2)\mathrm{d}x\mathrm{d}t\\ & +2\varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_\pm|\nabla\eta\cdot\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ =\;& \mp 2\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi(\nabla\eta\cdot\nabla\psi_\pm)\partial_\nu\psi_\pm \mathrm{d}\sigma\mathrm{d}t \pm 2\varepsilon^2 s\lambda \iint_{\Omega_T} \xi_\pm\nabla^2\eta(\nabla\psi_\pm,\nabla\psi_\pm)\mathrm{d}x\mathrm{d}t\\ & \pm\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta|\nabla\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t - \varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_\pm|\nabla\eta|^2|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ & \mp\varepsilon^2 s\lambda \iint_{\Omega_T} \xi_\pm\Delta\eta|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t + 2\varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_\pm|\nabla\eta\cdot\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ :=\;& D^1_\pm + D^2_\pm + D^3_\pm + D^4_\pm + D^5_\pm + D^6_\pm.\end{aligned}$$

Since $\eta = 0$ on Γ, we obtain $D^1_\pm = \mp 2\varepsilon^2 s\lambda \iint_{\Gamma_T} \partial_\nu\eta\xi|\partial_\nu\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t$. Moreover, $D^6_\pm \geq 0$ and $|D^2_\pm + D^5_\pm| \leqslant C\varepsilon^2 s\lambda \iint_{\Omega_T} \xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t$. Therefore

$$\begin{aligned}((L_1\psi_\pm)_2,(L_2\psi_\pm)_2)_{L^2(\Omega_T)} \geq& \mp 2\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta|\partial_\nu\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t\\ & \pm\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta|\nabla\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t - \varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_\pm|\nabla\eta|^2|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ & -C\varepsilon^2 s\lambda \iint_{\Omega_T} \xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.22}$$

Next, integration by parts in space and time, $\nabla\psi_\pm(\cdot,0) = \nabla\psi_\pm(\cdot,T) = 0$ and $\psi_\pm = \psi$ on Γ_T yield

$$((L_1\psi_\pm)_3,(L_2\psi_\pm)_2)_{L^2(\Omega_T)} = \varepsilon \iint_{\Gamma_T} \partial_t\psi\partial_\nu\psi_\pm \mathrm{d}\sigma\mathrm{d}t := E_\pm. \tag{4.23}$$

From (4.21), (4.22), (4.23) and $\lambda \geq 1$, we obtain

$$\begin{aligned}(L_1\psi_\pm,(L_2\psi_\pm)_2)_{L^2(\Omega_T)} \geq& -2\varepsilon^2 s\lambda^2 \iint_{\Gamma_T} \xi|\nabla\eta|^2\psi\partial_\nu\psi_\pm \mathrm{d}\sigma\mathrm{d}t \\ &-C\varepsilon^2 s^2\lambda^4 \iint_{\Omega_T} \xi_\pm^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t - C\varepsilon^2 \iint_{\Omega_T}(s\lambda\xi_\pm+\lambda^2)|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t \\ &+\varepsilon^2 s\lambda^2 \iint_{\Omega_T}\xi_\pm|\nabla\eta|^2|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t \mp 2\varepsilon^2 s\lambda \iint_{\Gamma_T}\xi\partial_\nu\eta|\partial_\nu\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t \\ &\pm\varepsilon^2 s\lambda \iint_{\Gamma_T}\xi\partial_\nu\eta|\nabla\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t + \varepsilon\iint_{\Gamma_T}\partial_t\psi\partial_\nu\psi_\pm\mathrm{d}\sigma\mathrm{d}t. \end{aligned} \tag{4.24}$$

for all $s \geq s_1T^2$. Thanks to the properties (4.2) of η, the fourth term in the right hand side of (4.24) can be reduced as follows

$$\varepsilon^2 s\lambda^2 \iint_{\Omega_T}\xi_\pm|\nabla\eta|^2|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t \geq c\,\varepsilon^2 s\lambda^2\iint_{\Omega_T}\xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t - C\varepsilon^2 s\lambda^2\iint_{\omega'\times(0,T)}\xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t,$$

for all $c \in (0,\delta^2)$.

Let us use $\xi_\pm \geq \frac{4}{T^2}$, the third term in the right hand side of (4.24) is absorbed by $c\varepsilon^2 s\lambda^2 \iint_{\Omega_T}\xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t$ if $\lambda \geq \lambda_1$ and $s \geq s_1T^2$ for λ_1 and s_1 are large enough.

Hence, for $\lambda \geq \lambda_1$ and $s \geq s_1T^2$, we have

$$\begin{aligned}(L_1\psi_\pm,(L_2\psi_\pm)_2)_{L^2(\Omega_T)} \geq& -2\varepsilon^2 s\lambda^2 \iint_{\Gamma_T} \xi|\nabla\eta|^2\psi\partial_\nu\psi_\pm \mathrm{d}\sigma\mathrm{d}t \\ &-C\varepsilon^2 s^2\lambda^4 \iint_{\Omega_T} \xi_\pm^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + c\,\varepsilon^2 s\lambda^2 \iint_{\Omega_T}\xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t \\ &-C\varepsilon^2 s\lambda^2 \iint_{\omega'\times(0,T)}\xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t \mp 2\varepsilon^2 s\lambda \iint_{\Gamma_T}\xi\partial_\nu\eta|\partial_\nu\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t \\ &\pm\varepsilon^2 s\lambda \iint_{\Gamma_T}\xi\partial_\nu\eta|\nabla\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t + \varepsilon\iint_{\Gamma_T}\partial_t\psi\partial_\nu\psi_\pm\mathrm{d}\sigma\mathrm{d}t. \end{aligned} \tag{4.25}$$

Step 2c. Estimate from below of $((L_1\psi_\pm),(L_2\psi_\pm)_3)_{L^2(\Omega_T)}$.
Next, we have

$$((L_1\psi_\pm)_1,(L_2\psi_\pm)_3)_{L^2(\Omega_T)} = -2\varepsilon s^2\lambda^2\iint_{\Omega_T}\xi_\pm|\nabla\eta|^2\partial_t\alpha_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t,$$

we obtain from $|\partial_t\alpha_\pm| \leqslant T\xi_\pm^2$ that

$$\begin{aligned}\left|((L_1\psi_\pm)_1,(L_2\psi_\pm)_3)_{L^2(\Omega_T)}\right| &\leqslant 2\varepsilon s^2\lambda^2\iint_{\Omega_T}\xi_\pm|\nabla\eta|^2|\partial_t\alpha_\pm||\psi_\pm|^2\mathrm{d}x\mathrm{d}t \\ &\leqslant C\varepsilon^2 s^3\lambda^2\iint_{\Omega_T}\xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.26}$$

for all $s \geq s_1 \frac{T}{\varepsilon}$. We also have

$$\begin{aligned}
((L_1\psi_\pm)_2, (L_2\psi_\pm)_3)_{L^2(\Omega_T)} &= \mp 2\varepsilon s^2\lambda \iint_{\Omega_T} \xi_\pm(\nabla\eta\cdot\nabla\psi_\pm)\partial_t\alpha_\pm\psi_\pm \mathrm{d}x\mathrm{d}t \\
&= \mp\varepsilon s^2\lambda \iint_{\Gamma_T} \xi\partial_t\alpha\partial_\nu\eta|\psi|^2\mathrm{d}\sigma\mathrm{d}t \pm \varepsilon s^2\lambda \iint_{\Omega_T} \nabla\cdot(\partial_t\alpha_\pm\xi_\pm\nabla\eta)|\psi_\pm|^2\mathrm{d}x\mathrm{d}t \\
&= \mp\varepsilon s^2\lambda \iint_{\Gamma_T} \xi\partial_t\alpha\partial_\nu\eta|\psi|^2\mathrm{d}\sigma\mathrm{d}t \pm \varepsilon s^2\lambda \iint_{\Omega_T} \xi_\pm(\nabla(\partial_t\alpha_\pm)\cdot\nabla\eta)|\psi_\pm|^2\mathrm{d}x\mathrm{d}t \\
&\quad +\varepsilon s^2\lambda^2 \iint_{\Omega_T} \xi_\pm\partial_t\alpha_\pm|\nabla\eta|^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t \pm \varepsilon s^2\lambda \iint_{\Omega_T} \xi_\pm\partial_t\alpha_\pm\Delta\eta|\psi_\pm|^2\mathrm{d}x\mathrm{d}t \\
&:= F^1_\pm + F^2_\pm + F^3_\pm + F^4_\pm.
\end{aligned}$$

From $|\partial_t\alpha_\pm| \leqslant T\xi_\pm^2$ and $|\nabla(\partial_t\alpha_\pm)| \leqslant \|\nabla\eta\|_\infty\lambda T\xi_\pm^2$, we obtain

$$|F^2_\pm + F^3_\pm + F^4_\pm| \leqslant C\varepsilon^2 s^3\lambda^2 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t, \text{ for all } \lambda \geq 1 \text{ and } s \geq s_1\frac{T}{\varepsilon},$$

it comes that

$$((L_1\psi_\pm)_2, (L_2\psi_\pm)_3)_{L^2(\Omega_T)} \geq \mp\varepsilon s^2\lambda \iint_{\Gamma_T} \xi\partial_t\alpha\partial_\nu\eta|\psi|^2\mathrm{d}x\mathrm{d}t - C\varepsilon^2 s^3\lambda^2 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t. \tag{4.27}$$

By time integration and $\psi_\pm(\cdot,0) = \psi_\pm(\cdot,T) = 0$ in an exponential way, we obtain

$$\begin{aligned}
((L_1\psi_\pm)_3, (L_2\psi_\pm)_3)_{L^2(\Omega_T)} &= s\iint_{\Omega_T} \partial_t\alpha_\pm\partial_t\psi_\pm\psi_\pm\mathrm{d}x\mathrm{d}t \\
&= \frac{s}{2}\iint_{\Omega_T} \partial_t\alpha_\pm\partial_t|\psi_\pm|^2\mathrm{d}x\mathrm{d}t = -\frac{s}{2}\iint_{\Omega_T} \partial_t^2\alpha_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Since $|\partial_t^2\alpha_\pm| \leqslant 2T^2\xi_\pm^3$, for all $\lambda \geq 1$ and $s_1\frac{T}{\varepsilon}$, we get

$$\left|((L_1\psi_\pm)_3, (L_2\psi_\pm)_3)_{L^2(\Omega_T)}\right| \leqslant C\varepsilon^2 s^3\lambda^2 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t. \tag{4.28}$$

From (4.26), (4.27) and (4.28), we deduce for $\lambda \geq 1$ and $s \geq s_1\frac{T}{\varepsilon}$ that

$$\begin{aligned}
(L_1\psi_\pm, (L_2\psi_\pm)_3)_{L^2(\Omega_T)} &= \sum_{1\leqslant j\leqslant 3} ((L_1\psi_\pm)_j, (L_2\psi_\pm)_3)_{L^2(\Omega_T)} \\
&\geq -C\varepsilon^2 s^3\lambda^2 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t \mp \varepsilon s^2\lambda \iint_{\Gamma_T} \xi\partial_t\alpha\partial_\nu\eta|\psi|^2\mathrm{d}\sigma\mathrm{d}t.
\end{aligned} \tag{4.29}$$

Step 2d. Estimate from below of $((L_1\psi_\pm),(L_2\psi_\pm)_4)_{L^2(\Omega_T)}$.
Let us now consider the scalar product

$$((L_1\psi_\pm)_1, (L_2\psi_\pm)_4)_{L^2(\Omega_T)} = \frac{s\lambda^2}{2}\iint_{\Omega_T} \xi_\pm|\nabla f|^2|\nabla\eta|^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t.$$

Since $\xi_\pm \geq \frac{4}{T^2}$, then

$$|((L_1\psi_\pm)_1,(L_2\psi_\pm)_4)_{L^2(\Omega_T)}| \leqslant C\|\nabla f\|_\infty^2 s\lambda^2T^4 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t. \tag{4.30}$$

We also have

$$\begin{aligned}((L_1\psi_\pm)_2,(L_2\psi_\pm)_4)_{L^2(\Omega_T)} &= \pm\frac{s\lambda}{2}\iint_{\Omega_T} \xi_\pm|\nabla f|^2(\nabla\eta\cdot\nabla\psi_\pm)\psi_\pm\mathrm{d}x\mathrm{d}t\\ &= \pm\frac{s\lambda}{4}\iint_{\Gamma_T}\xi|\nabla f|^2\partial_\nu\eta|\psi|^2\mathrm{d}\sigma\mathrm{d}t \mp\frac{s\lambda}{4}\iint_{\Omega_T}\xi_\pm(\nabla(|\nabla f|^2)\cdot\nabla\eta)|\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ &\quad -\frac{s\lambda^2}{4}\iint_{\Omega_T}\xi_\pm|\nabla f|^2|\nabla\eta|^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t\mp\frac{s\lambda}{4}\iint_{\Omega_T}\xi_\pm|\nabla f|^2\Delta\eta|\psi_\pm|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using $\nabla|\nabla f|^2\cdot\nabla\eta = 2\nabla^2 f(\nabla f,\nabla\eta)$, we obtain

$$\begin{aligned}((L_1\psi_\pm)_2,(L_2\psi_\pm)_4)_{L^2(\Omega_T)} &= \pm\frac{s\lambda}{4}\iint_{\Gamma_T}\xi|\nabla f|^2\partial_\nu\eta|\psi|^2\mathrm{d}\sigma\mathrm{d}t\mp\frac{s\lambda}{2}\iint_{\Omega_T}\xi_\pm\nabla^2 f(\nabla f,\nabla\eta)|\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ &\quad -\frac{s\lambda^2}{4}\iint_{\Omega_T}\xi_\pm|\nabla f|^2|\nabla\eta|^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t\mp\frac{s\lambda}{4}\iint_{\Omega_T}\xi_\pm|\nabla f|^2\Delta\eta|\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ &:= G_\pm^1+G_\pm^2+G_\pm^3+G_\pm^4.\end{aligned}$$

By Young inequality, $\lambda\geq 1$ and $\xi_\pm\geq\frac{4}{T^2}$, we obtain

$$|G_\pm^2+G_\pm^3+G_\pm^4| \leqslant Cs\lambda^2T^4\left(\|\nabla^2 f\|_\infty^2+\|\nabla f\|_\infty^2\right)\iint_{\Omega_T}\xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t.$$

Thus,

$$\begin{aligned}((L_1\psi_\pm)_2,(L_2\psi_\pm)_4)_{L^2(\Omega_T)} &\geq \pm\frac{s\lambda}{4}\iint_{\Gamma_T}\xi|\nabla f|^2\partial_\nu\eta|\psi|^2\mathrm{d}\sigma\mathrm{d}t\\ &\quad -Cs\lambda^2T^4\left(\|\nabla^2 f\|_\infty^2+\|\nabla f\|_\infty^2\right)\iint_{\Omega_T}\xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.31}$$

By integration in time and $\psi_\pm(\cdot,0)=\psi_\pm(\cdot,T)=0$, we have

$$((L_1\psi_\pm)_3,(L_2\psi_\pm)_4)_{L^2(\Omega_T)} = -\frac{1}{4\varepsilon}\iint_{\Omega_T}|\nabla f|^2\partial_t\psi_\pm\psi_\pm\mathrm{d}x\mathrm{d}t = 0. \tag{4.32}$$

From (4.30), (4.31) and (4.32), we conclude that

$$\begin{aligned}(L_1\psi_\pm,(L_2\psi_\pm)_4)_{L^2(\Omega_T)} &= \sum_{1\leqslant j\leqslant 3}((L_1\psi_\pm)_j,(L_2\psi_\pm)_4)_{L^2(\Omega_T)}\\ &\geq \pm\frac{s\lambda}{4}\iint_{\Gamma_T}\xi|\nabla f|^2\partial_\nu\eta|\psi|^2\mathrm{d}\sigma\mathrm{d}t\\ &\quad -Cs\lambda^2T^4\left(\|\nabla^2 f\|_\infty^2+\|\nabla f\|_\infty^2\right)\iint_{\Omega_T}\xi_\pm^3|\psi_\pm|^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.33}$$

Step 3. First conclusion.
Taking in account (4.20), (4.25), (4.29) and (4.33), for any $\lambda \geq \lambda_1$ and $s \geq s_1\left(\frac{T}{\varepsilon}+T^2\right)$, we obtain

$$\begin{aligned}
(L_1\psi_\pm, L_2\psi_\pm)_{L^2(\Omega_T)} \geq\ & c\,\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t - C\varepsilon^2 s^3\lambda^4 \iint_{\omega'\times(0,T)} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t \\
& \mp\varepsilon^2 s^3\lambda^3 \iint_{\Gamma_T} \xi^3|\nabla\eta|^2\partial_\nu\eta\psi^2 \mathrm{d}\sigma\mathrm{d}t - 2\varepsilon^2 s\lambda^2 \iint_{\Gamma_T} \xi|\nabla\eta|^2\psi\partial_\nu\psi_\pm \mathrm{d}\sigma\mathrm{d}t \\
& -C\varepsilon^2 s^2\lambda^4 \iint_{\Omega_T} \xi_\pm^2|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t + c\,\varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_\pm|\nabla\psi_\pm|^2 \mathrm{d}x\mathrm{d}t \\
& -C\varepsilon^2 s\lambda^2 \iint_{\omega'\times(0,T)} |\nabla\psi_\pm|^2\xi_\pm \mathrm{d}x\mathrm{d}t \mp 2\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta|\partial_\nu\psi_\pm|^2 \mathrm{d}\sigma\mathrm{d}t \\
& \pm\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta|\nabla\psi_\pm|^2 \mathrm{d}\sigma\mathrm{d}t + \varepsilon \iint_{\Gamma_T} \partial_t\psi\partial_\nu\psi_\pm \mathrm{d}\sigma\mathrm{d}t \\
& -C\varepsilon^2 s^3\lambda^2 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t \pm \varepsilon s^2\lambda \iint_{\Gamma_T} \xi\partial_t\alpha\partial_\nu\eta|\psi|^2 \mathrm{d}\sigma\mathrm{d}t \\
& \pm\frac{s\lambda}{4} \iint_{\Gamma_T} \xi|\nabla f|^2\partial_\nu\eta|\psi|^2 \mathrm{d}\sigma\mathrm{d}t - Cs\lambda^2T^4\left(\|\nabla^2 f\|_\infty^2 + \|\nabla f\|_\infty^2\right) \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

From (4.19), we obtain

$$\begin{aligned}
&\|L_1\psi_\pm\|_{L^2(\Omega_T)}^2 + \|L_2\psi_\pm\|_{L^2(\Omega_T)}^2 + c\,\varepsilon^2 s^3 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t \\
&\quad + c\,\varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_\pm|\nabla\psi_\pm|^2 \mathrm{d}x\mathrm{d}t + 2\left(B_\pm^3 + C_\pm^1 + D_\pm^1 + D_\pm^3 + E_\pm + F_\pm^1 + G_\pm^1\right) \\
&\leqslant C\Bigg(\|L_3\psi_\pm\|_{L^3(\Omega_T)}^2 + \varepsilon^2 s^3\lambda^4 \iint_{\omega'\times(0,T)} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s\lambda^2 \iint_{\omega'\times(0,T)} \xi_\pm|\nabla\psi_\pm|^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon^2 s^2\lambda^4 \iint_{\Omega_T} \xi_\pm^2|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s^3\lambda^2 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t \\
&\quad + s\lambda^2T^4\left(\|\nabla^2 f\|_\infty^2 + \|\nabla f\|_\infty^2\right) \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t\Bigg).
\end{aligned} \tag{4.34}$$

The last integral in the right hand side of (4.34) can be absorbed by $\tilde{A}_\pm$ if $\lambda \geq 1$ and $s \geq s_1\frac{T^2}{\varepsilon}(\|\nabla^2 f\|_\infty + \|\nabla f\|_\infty)$. Also, one can see that $\varepsilon^2 s^2\lambda^4 \iint_{\Omega_T} \xi_\pm^2|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t$ and $\varepsilon^2 s^3\lambda^2 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t$ are absorbed by the same term if we take respectively $s \geq s_1 T^2$ and $\lambda \geq \lambda_1$.

Thus, for any $\lambda \geq \lambda_1$ and $s \geq s_1\left(\frac{T}{\varepsilon} + \frac{T^2}{\varepsilon}(\|\nabla^2 f\|_\infty + \|\nabla f\|_\infty) + T^2\right)$, we have

$$\begin{aligned}
&\|L_1\psi_\pm\|_{L^2(\Omega_T)}^2 + \|L_2\psi_\pm\|_{L^2(\Omega_T)}^2 + c\,\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t \\
&\quad + c\,\varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_\pm|\nabla\psi_\pm|^2 \mathrm{d}x\mathrm{d}t + 2\left(B_\pm^3 + C_\pm^1 + D_\pm^1 + D_\pm^3 + E_\pm + F_\pm^1 + G_\pm^1\right) \\
&\leqslant C\left(\|L_3\psi_\pm\|_{L^3(\Omega_T)}^2 + \varepsilon^2 s^3\lambda^4 \iint_{\omega'\times(0,T)} \xi_\pm^3|\psi_\pm|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s\lambda^2 \iint_{\omega'\times(0,T)} \xi_\pm|\nabla\psi_\pm|^2 \mathrm{d}x\mathrm{d}t\right).
\end{aligned}$$

From (4.18), we obtain

$$\|L_3\psi_\pm\|^2_{L^3(\Omega_T)} \leqslant C\left(\iint_{\Omega_T}|F_\pm|^2\mathrm{d}x\mathrm{d}t + \varepsilon^2s^2\lambda^2\iint_{\Omega_T}\xi_\pm^2|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + (\|q\|^2_\infty + \|\Delta f\|^2_\infty)\iint_{\Omega_T}|\psi_\pm|^2\mathrm{d}x\mathrm{d}t\right). \tag{4.35}$$

The second term in the right hand side of (4.35) is absorbed by $\tilde{A}_\pm$ for $s \geq s_1T^2$, the same goes for the last term, if we take $s \geq s_1\frac{T^2}{\varepsilon^{\frac{2}{3}}}\left(\|\Delta f\|^{\frac{2}{3}}_\infty + \|q\|^{\frac{2}{3}}_\infty\right)$ and $\lambda \geq 1$.

Finally, we obtain

$$\begin{aligned}&\|L_1\psi_\pm\|^2_{L^2(\Omega_T)} + \|L_2\psi_\pm\|^2_{L^2(\Omega_T)} + c\,\varepsilon^2s^3\lambda^4\iint_{\Omega_T}\xi^3_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + c\,\varepsilon^2s\lambda^2\iint_{\Omega_T}|\nabla\psi_\pm|^2\xi_\pm\mathrm{d}x\mathrm{d}t + 2\,I_\pm\\ &\leqslant C\left(\iint_{\Omega_T}|F_\pm|^2\mathrm{d}x\mathrm{d}t + \varepsilon^2s^3\lambda^4\iint_{\omega'\times(0,T)}\xi^3_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + \varepsilon^2s\lambda^2\iint_{\omega'\times(0,T)}\xi_\pm|\nabla\psi_\pm|^2\mathrm{d}x\mathrm{d}t\right),\end{aligned} \tag{4.36}$$

for any $\varepsilon \in (0,1)$, $\lambda \geq \lambda_1$, $s \geq s_1\left(\frac{T}{\varepsilon} + \frac{T^2}{\varepsilon}(\|\nabla f\|_\infty + \|\nabla^2 f\|_\infty + \|\Delta f\|^{\frac{2}{3}}_\infty + \|q\|^{\frac{2}{3}}_\infty) + T^2\right)$ and

$$I_\pm := B^3_\pm + C^1_\pm + D^1_\pm + D^3_\pm + E_\pm + F^1_\pm + G^1_\pm. \tag{4.37}$$

Step 4. Elimination of the integral of $|\nabla\psi_\pm|^2$ on the right-hand side of (4.36).
We start by adding integral of $|\Delta\psi_\pm|^2$ to the left-hand side of (4.36), so that we can eliminate the last term in the right-hand side of (4.36).

Using (4.17), $\xi_\pm \geq \frac{4}{T^2}$, $s \geq s_1T^2$ and $|\partial_t\alpha_\pm| \leqslant T\xi^2_\pm$, we obtain

$$\begin{aligned}\varepsilon^2s^{-1}\iint_{\Omega_T}\xi^{-1}_\pm|\Delta\psi_\pm|^2\mathrm{d}x\mathrm{d}t \leqslant C\Bigg(&\varepsilon^2s^3\lambda^4\iint_{\Omega_T}\xi^3_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + sT^2\iint_{\Omega_T}\xi^3_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t\\ &+ \|\nabla f\|^4_\infty\frac{s^{-1}}{\varepsilon^2}\iint_{\Omega_T}\xi^{-1}_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + \|L_2\psi_\pm\|^2_{L^2(\Omega_T)}\Bigg).\end{aligned}$$

Hence

$$\varepsilon^2s^{-1}\iint_{\Omega_T}\xi^{-1}_\pm|\Delta\psi_\pm|^2\mathrm{d}x\mathrm{d}t \leqslant C\left(\varepsilon^2s^3\lambda^4\iint_{\Omega_T}\xi^3_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + \|L_2\psi_\pm\|^2_{L^2(\Omega_T)}\right),$$

for all $\lambda \geq 1$ and $s \geq s_1\left(\frac{T}{\varepsilon} + \|\nabla f\|_\infty\frac{T^2}{\varepsilon} + T^2\right)$.

Consequently, we deduce from (4.36) that

$$\begin{aligned}&\varepsilon^2s^{-1}\iint_{\Omega_T}\xi^{-1}_\pm|\Delta\psi_\pm|^2\mathrm{d}x\mathrm{d}t + \varepsilon^2s^3\lambda^4\iint_{\Omega_T}\xi^3_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + \varepsilon^2s\lambda^2\iint_{\Omega_T}|\nabla\psi_\pm|^2\xi_\pm\mathrm{d}x\mathrm{d}t + c\,I_\pm\\ &\leqslant C\left(\iint_{\Omega_T}|F_\pm|^2\mathrm{d}x\mathrm{d}t + \varepsilon^2s^3\lambda^4\iint_{\omega'\times(0,T)}\xi^3_\pm|\psi_\pm|^2\mathrm{d}x\mathrm{d}t + \varepsilon^2s\lambda^2\iint_{\omega'\times(0,T)}|\nabla\psi_\pm|^2\xi_\pm\mathrm{d}x\mathrm{d}t\right),\end{aligned}$$

for any $\lambda \geq \lambda_1$, $s \geq s_1\left(\frac{T}{\varepsilon} + \frac{T^2}{\varepsilon}(\|\nabla f\|_\infty + \|\nabla^2 f\|_\infty + \|\Delta f\|^{\frac{2}{3}}_\infty + \|q\|^{\frac{2}{3}}_\infty) + T^2\right)$.

We are now ready to eliminate the last term in the right-hand side of (4.36), let us introduce a cut-off function, between ω' and ω. More precisely, let $\theta \in \mathcal{C}^2(\omega)$ be a positive function such that $\theta = 1$ in ω'. Integrating by parts and with Cauchy–Schwarz inequality we can obtain that:

$$\begin{aligned}&\varepsilon^2 s^{-1} \iint_{\Omega_T} \xi_\pm^{-1} |\Delta\psi_\pm|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \xi_\pm^3 |\psi_\pm|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_\pm |\nabla\psi_\pm|^2 \mathrm{d}x\mathrm{d}t + c\, I_\pm \\ &\leqslant C \left(\iint_{\Omega_T} |F_\pm|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s^3\lambda^4 \iint_{\omega\times(0,T)} \xi_\pm^3 |\psi_\pm|^2 \mathrm{d}x\mathrm{d}t \right),\end{aligned} \tag{4.38}$$

for any $\lambda \geq \lambda_1$, $s \geq s_1 \left(\frac{T}{\varepsilon} + \frac{T^2}{\varepsilon} (\|\nabla f\|_\infty + \|\nabla^2 f\|_\infty + \|\Delta f\|_\infty^{\frac{2}{3}} + \|q\|_\infty^{\frac{2}{3}}) + T^2 \right)$.

Step 5. Simplification of the boundary terms. By summing (4.38) for $i = +, -$, we obtain

$$\begin{aligned}&\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \xi_+^3 |\psi_+|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_+ |\nabla\psi_+|^2 \mathrm{d}x\mathrm{d}t + c\, (I_+ + I_-) \\ &\leqslant C \left(\iint_{\Omega_T} \left(|F_+|^2 + |F_-|^2\right) \mathrm{d}x\mathrm{d}t + \varepsilon^2 s^3\lambda^4 \iint_{\omega\times(0,T)} \left(\xi_+^3 |\psi_+|^2 + \xi_-^3 |\psi_-|^2\right) \mathrm{d}x\mathrm{d}t \right).\end{aligned} \tag{4.39}$$

From the definitions of $\xi_\pm$ and $\alpha_\pm$, we have $\xi_- \leqslant \xi_+$ and $\alpha_+ \leqslant \alpha_-$ in Ω_T. Then, the estimate (4.39) gives

$$\begin{aligned}&\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \xi_+^3 |\psi_+|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_+ |\nabla\psi_+|^2 \mathrm{d}x\mathrm{d}t + c\, (I_+ + I_-) \\ &\leqslant C \left(\iint_{\Omega_T} |F_+|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s^3\lambda^4 \iint_{\omega\times(0,T)} \xi_+^3 |\psi_+|^2 \mathrm{d}x\mathrm{d}t \right).\end{aligned}$$

Before simplifying $I_+ + I_-$, we turn back to our original function Φ. From (4.11), we deduce that

$$\begin{aligned}&\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \exp(-2s\alpha_+) \xi_+^3 |\Phi|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s\lambda^2 \iint_{\Omega_T} |\nabla\psi_+|^2 \xi_+ \mathrm{d}x\mathrm{d}t + c\, (I_+ + I_-) \\ &\leqslant C \left(\iint_{\Omega_T} \exp(-2s\alpha_+) |\partial_t \Phi + \varepsilon\Delta\Phi - q_{f,\varepsilon}\Phi|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s^3\lambda^4 \iint_{\omega\times(0,T)} \exp(-2s\alpha_+) \xi_+^3 |\Phi|^2 \mathrm{d}x\mathrm{d}t \right).\end{aligned} \tag{4.40}$$

For $\nabla\Phi$, we use the identity given in (4.14), we have

$$\exp(-s\alpha_+)\nabla\Phi = \nabla\psi_+ - s\lambda\xi_+\nabla\eta\psi_+,$$

Applying the triangular inequality to this identity, we find

$$\begin{aligned}&\varepsilon^2 s\lambda^2 \iint_{\Omega_T} \exp(-2s\alpha_+) \xi_+ |\nabla\Phi|^2 \mathrm{d}x\mathrm{d}t \\ &\leqslant C \left(\varepsilon^2 s\lambda^2 \iint_{\Omega_T} \xi_+ |\nabla\psi_+|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \exp(-2s\alpha_+) \xi_+^3 |\Phi|^2 \mathrm{d}x\mathrm{d}t \right).\end{aligned}$$

Consequently, we can add the previous integral of $|\nabla\Phi|^2$ to the left-hand side of (4.40):

$$\begin{aligned}&\varepsilon^2 s^3\lambda^4 \iint_{\Omega_T} \exp(-2s\alpha_+)\xi_+^3|\Phi|^2 \mathrm{d}x\mathrm{d}t + \varepsilon^2 s\lambda^2 \iint_{\Omega_T} \exp(-2s\alpha_+)\xi_+|\nabla\Phi|^2\mathrm{d}x\mathrm{d}t + c\,(I_+ + I_-)\\ &\leqslant C\left(\iint_{\Omega_T} \exp(-2s\alpha_+)|\partial_t\Phi + \varepsilon\Delta\Phi - q_{f,\varepsilon}\Phi|^2\mathrm{d}x\mathrm{d}t + \varepsilon^2 s^3\lambda^4 \iint_{\omega\times(0,T)} \exp(-2s\alpha_+)\xi_+^3|\Phi|^2\mathrm{d}x\mathrm{d}t\right).\end{aligned} \tag{4.41}$$

Next, we will simplify $I_+ + I_-$. It is clear that

$$B_+^3 + B_-^3 = 0, \quad F_+^1 + F_-^1 = 0 \quad \text{and} \quad G_+^1 + G_-^1 = 0. \tag{4.42}$$

From $|\nabla\psi_\pm|^2 = |\nabla_\Gamma\psi_\pm|^2 + |\partial_\nu\psi_\pm|^2$ and (4.12), we obtain

$$D_\pm^3 = \pm\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta|\nabla_\Gamma\psi|^2\mathrm{d}\sigma\mathrm{d}t \pm \varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta|\partial_\nu\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t,$$

then

$$D_\pm^1 + D_\pm^3 = \pm\varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta|\nabla_\Gamma\psi|^2\mathrm{d}\sigma\mathrm{d}t \mp \varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta|\partial_\nu\psi_\pm|^2\mathrm{d}\sigma\mathrm{d}t,$$

therefore, from (4.15) we obtain

$$\begin{aligned}\sum_{j\in\{+,-\}} D_j^1 + D_j^3 &= \varepsilon^2 s\lambda \iint_{\Gamma_T} \xi\partial_\nu\eta\left(|\partial_\nu\psi_-|^2 - |\partial_\nu\psi_+|^2\right)\mathrm{d}\sigma\mathrm{d}t\\ &= -4\varepsilon^2 s^2\lambda^2 \iint_{\Gamma_T} |\partial_\nu\eta|^2\xi^2\exp(-s\alpha)\psi\partial_\nu\Phi\mathrm{d}\sigma\mathrm{d}t\\ &= 4\varepsilon s^2\lambda^2 \iint_{\Gamma_T} \beta_f|\partial_\nu\eta|^2\xi^2\exp(-2s\alpha)|\Phi|^2\mathrm{d}\sigma\mathrm{d}t.\end{aligned} \tag{4.43}$$

From (4.15), we also obtain

$$\begin{aligned}C_+^1 + C_-^1 &= -2\varepsilon^2 s\lambda^2 \iint_{\Gamma_T} |\nabla\eta|^2\xi\psi\left(\partial_\nu\psi_+ + \partial_\nu\psi_-\right)\mathrm{d}\sigma\mathrm{d}t\\ &= -4\varepsilon^2 s\lambda^2 \iint_{\Gamma_T} |\nabla\eta|^2\xi\psi\exp(-s\alpha)\partial_\nu\Phi\mathrm{d}\sigma\mathrm{d}t\\ &= 4\varepsilon s\lambda^2 \iint_{\Gamma_T} \beta_f|\nabla\eta|^2\xi\exp(-2s\alpha)|\Phi|^2\mathrm{d}\sigma\mathrm{d}t.\end{aligned} \tag{4.44}$$

Since, $\eta = 0$ on Γ, from (4.43) and (4.44), we have

$$\sum_{j\in\{+,-\}} C_j^1 + D_j^1 + D_j^3 = 4\varepsilon s\lambda^2 \iint_{\Gamma_T} \beta_f|\partial_\nu\eta|^2\left(\xi + s\xi^2\right)\exp(-2s\alpha)|\Phi|^2\mathrm{d}\sigma\mathrm{d}t. \tag{4.45}$$

On the other hand, from (4.15) and (4.13), we get

$$\begin{aligned} E_+ + E_- &= \varepsilon \iint_{\Gamma_T} \partial_t \psi \left(\partial_\nu \psi_+ + \partial_\nu \psi_-\right) \mathrm{d}\sigma \mathrm{d}t \\ &= 2\varepsilon \iint_{\Gamma_T} \partial_t \psi \exp(-s\alpha) \partial_\nu \Phi \mathrm{d}\sigma \mathrm{d}t \\ &= -2 \iint_{\Gamma_T} \beta_f \partial_t \psi \psi \mathrm{d}\sigma \mathrm{d}t = 0, \end{aligned} \tag{4.46}$$

since β_f does not depend on the time variable and $\psi(\cdot, 0) = \psi(\cdot, T) = 0$. By consideration of (4.37), (4.42), (4.45) and (4.46), we obtain

$$I_+ + I_- = 4\varepsilon s \lambda^2 \iint_{\Gamma_T} \beta_f |\partial_\nu \eta|^2 \left(\xi + s\xi^2\right) \exp(-2s\alpha) |\Phi|^2 \mathrm{d}\sigma \mathrm{d}t.$$

The estimate (4.41) and $\beta_f \geq 0$ implies the following

$$\begin{aligned} &\varepsilon^2 s^3 \lambda^4 \iint_{\Omega_T} \exp(-2s\alpha_+) \xi_+^3 |\Phi|^2 \mathrm{d}x \mathrm{d}t + \varepsilon^2 s \lambda^2 \iint_{\Omega_T} \exp(-2s\alpha_+) \xi_+ |\nabla \Phi|^2 \mathrm{d}x \mathrm{d}t \\ &\quad + 4\varepsilon s \lambda^2 \iint_{\Gamma_T} \beta_f |\partial_\nu \eta|^2 \left(\xi + s\xi^2\right) \exp(-2s\alpha) |\Phi|^2 \mathrm{d}\sigma \mathrm{d}t \\ &\leqslant C \left(\iint_{\Omega_T} \exp(-2s\alpha_+) |\partial_t \Phi + \varepsilon \Delta \Phi - q_{f,\varepsilon} \Phi|^2 \mathrm{d}x \mathrm{d}t + \varepsilon^2 s^3 \lambda^4 \iint_{\omega \times (0,T)} \exp(-2s\alpha_+) \xi_+^3 |\Phi|^2 \mathrm{d}x \mathrm{d}t \right), \end{aligned} \tag{4.47}$$

for any $\Phi \in \mathcal{Z}$, $\lambda \geq \lambda_1$ and $s \geq s_1 \left(\frac{T}{\varepsilon} + \frac{T^2}{\varepsilon} (\|\nabla f\|_\infty + \|\nabla^2 f\|_\infty + \|\Delta f\|_\infty^{\frac{2}{3}} + \|q\|_\infty^{\frac{2}{3}}) + T^2 \right)$.

Finally, we obtain the estimate (4.10) by density as explained before its proof. □

Under the same conditions of the previous proposition, we have the following

Corollary 4.5. *There exists a constant $C > 0$ depending only on Ω, ω, such that*

$$\iint_{\Omega \times \left(\frac{T}{4}, \frac{3T}{4}\right)} |\varphi(x,t)|^2 \mathrm{d}x \mathrm{d}t \leqslant \kappa \iint_{\omega \times (0,T)} |\varphi(x,t)|^2 \mathrm{d}x \mathrm{d}t, \tag{4.48}$$

for a constant κ of the form

$$\kappa = C \exp\left\{ C \left(\frac{1}{\varepsilon T} + \frac{1}{\varepsilon} \left(\|f\|_\infty + \|\nabla f\|_\infty + \|\nabla^2 f\|_\infty + \|\Delta f\|_\infty^{\frac{2}{3}} + \|q\|_\infty^{\frac{2}{3}} \right) \right) \right\}.$$

Proof. From the Carleman estimate (4.10), we obtain

$$\iint_{\Omega \times (0,T)} \exp\left(-2s\alpha_+\right) \xi_+^3 |\Phi|^2 \mathrm{d}x \mathrm{d}t \leq C \iint_{\omega \times (0,T)} \exp\left(-2s\alpha_+\right) \xi_+^3 |\Phi|^2 \mathrm{d}x \mathrm{d}t,$$

where $\lambda = \lambda_1$ and $s = s_1 \left(\frac{T}{\varepsilon} + \frac{T^2}{\varepsilon} (\|\nabla f\|_\infty + \|\nabla^2 f\|_\infty + \|\Delta f\|_\infty^{\frac{2}{3}} + \|q\|_\infty^{\frac{2}{3}}) + T^2 \right)$ are fixed and $C > 0$ depending only on Ω and ω.

Taking lower and upper estimates with respect to x of the weight functions, we have

$$\iint_{\Omega\times(0,T)} F(t)|\Phi|^2\mathrm{d}x\mathrm{d}t \leq C\iint_{\omega\times(0,T)} G(t)|\Phi|^2\mathrm{d}x\mathrm{d}t,$$

where

$$\begin{aligned}
F(t) &= \exp\left(-2s\max_{x\in\overline{\Omega}}\alpha_+(x,t)\right)\left(\min_{x\in\overline{\Omega}}\xi_+(x,t)\right)^3\\
&= \exp\left(-2s\frac{\exp(6\lambda)-\exp(4\lambda)}{t(T-t)}\right)\frac{\exp(12\lambda)}{(t(T-t))^3},\\
G(t) &= \exp\left(-2s\min_{x\in\overline{\Omega}}\alpha_+(x,t)\right)\left(\max_{x\in\overline{\Omega}}\xi_+(x,t)\right)^3\\
&= \exp\left(-2s\frac{\exp(6\lambda)-\exp(5\lambda)}{t(T-t)}\right)\frac{\exp(15\lambda)}{(t(T-t))^3}.
\end{aligned}$$

It is easy to check that the function G admits a maximum on $[0,T]$ at $t=\frac{T}{2}$ and F admits a minimum on $\left[\frac{T}{4},\frac{3T}{4}\right]$ at $t=\frac{T}{4}$ or $\frac{3T}{4}$. Thus

$$\iint_{\Omega\times\left(\frac{T}{4},\frac{3T}{4}\right)} |\Phi|^2\mathrm{d}x\mathrm{d}t \leq C\frac{G\left(\frac{T}{2}\right)}{F\left(\frac{T}{4}\right)}\iint_{\omega\times(0,T)} |\Phi|^2\mathrm{d}x\mathrm{d}t.$$

From the expressions of F and G, we obtain

$$\iint_{\Omega\times\left(\frac{T}{4},\frac{3T}{4}\right)} |\Phi|^2\mathrm{d}x\mathrm{d}t \leq C\exp\left(C\left(\frac{1}{\varepsilon T}+\frac{1}{\varepsilon}\left(\|\nabla f\|_\infty+\|\nabla^2 f\|_\infty+\|\Delta f\|_\infty^{\frac{2}{3}}+\|q\|_\infty^{\frac{2}{3}}\right)\right)\right)\iint_{\omega\times(0,T)} |\Phi|^2\mathrm{d}x\mathrm{d}t,$$

where $C>0$ depending only on Ω and ω. Using $\Phi(\cdot,t)=\exp\left(\frac{f(\cdot)}{2\varepsilon}\right)\varphi(\cdot,t)$, we deduce the estimate (4.48). □

4.2. Dissipation result and proof of Theorem 1.1

In this subsection we will prove a very important dissipation result which will lead the proof of Theorem 1.1.

Proposition 4.6. *Let $\varepsilon>0$, $T>0$ and assume that $\beta_f\geq 0$. Then, there exists $C>0$ depends only on Ω such that for all $T_0\in(0,T]$ and $\varphi_T\in L^2(\Omega)$, the solution of the adjoint system (1.4) verify the following dissipation estimates:*

$$\|\varphi(\cdot,0)\|_{L^2(\Omega)} \leqslant C\exp\left(\left(\|q\|_\infty+\frac{\|\Delta f\|_\infty}{2}\right)T+\frac{\|f\|_\infty-E_0T_0+1/T_0}{\varepsilon}\right)\|\varphi(\cdot,t)\|_{L^2(\Omega)}, \tag{4.49}$$

for all $t\in[T_0,T]$.

Proof. According to the spectral decomposition (2.11) of φ, we obtain

$$\varphi(\cdot,0)=\sum_{n=1}^{\infty}\exp\left(-\lambda_n^{(\varepsilon)}t\right)\left(\int_\Omega\exp\left(\frac{f(\xi)}{2\varepsilon}\right)\varphi(\xi,t)\phi_n^{(\varepsilon)}(\xi)\mathrm{d}\xi\right)\phi_n^{(\varepsilon)}(\cdot)\exp\left(\frac{-f(\cdot)}{2\varepsilon}\right).$$

Using the triangle inequality, the Cauchy–Schwarz inequality and $\|\phi_n^{(\varepsilon)}\|_{L^2(\Omega)} = 1$, we obtain the following estimate

$$\|\varphi(\cdot,0)\|_{L^2(\Omega)} \leqslant \sum_{n=1}^{\infty} \exp\left(-\lambda_n^{(\varepsilon)} t\right) \exp\left(\frac{\|f\|_\infty}{\varepsilon}\right) \|\varphi(\cdot,t)\|_{L^2(\Omega)}.$$

From Proposition 2.4, $E_0 \geq 0$ and $\lambda_n \geq 0$, for all $n \geq 1$, we find

$$\begin{aligned} \|\varphi(\cdot,0)\|_{L^2(\Omega)} &\leqslant \sum_{n=1}^{\infty} \exp\left(-\varepsilon\lambda_n t + \left(\|q\|_\infty + \frac{\|\Delta f\|_\infty}{2}\right) t - \frac{E_0}{\varepsilon} t\right) \exp\left(\frac{\|f\|_\infty}{\varepsilon}\right) \|\varphi(\cdot,t)\|_{L^2(\Omega)} \\ &\leqslant \exp\left(\left(\|q\|_\infty + \frac{\|\Delta f\|_\infty}{2}\right) t + \frac{\|f\|_\infty - E_0 t}{\varepsilon}\right) \sum_{n=1}^{\infty} \exp\left(-\varepsilon\lambda_n t\right) \|\varphi(\cdot,t)\|_{L^2(\Omega)} \\ &\leqslant \exp\left(\left(\|q\|_\infty + \frac{\|\Delta f\|_\infty}{2}\right) T + \frac{\|f\|_\infty - E_0 T_0}{\varepsilon}\right) \sum_{n=1}^{\infty} \exp\left(-\varepsilon\lambda_n T_0\right) \|\varphi(\cdot,t)\|_{L^2(\Omega)}, \end{aligned} \tag{4.50}$$

for all $t \in [T_0, T]$.

In order to increase the series in the right hand side of (4.50), we consider the partition $(I_k)_{k\geq 1}$ of $\mathbb{N}^*$ defined by $I_k := \{n \geq 1 : \quad \lambda_n \in [k-1, k)\}$ and (2.15), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \exp\left(-\varepsilon\lambda_n T_0\right) &= \sum_{k=1}^{\infty} \sum_{n\in I_k} \exp\left(-\varepsilon\lambda_n T_0\right) \\ &\leqslant \sum_{k=1}^{\infty} \sum_{n\in I_k} \exp\left(-\varepsilon(k-1)T_0\right) = \sum_{k=1}^{\infty} \# I_k \exp\left(-\varepsilon(k-1)T_0\right) \\ &\leqslant C \sum_{k=1}^{\infty} k^N \exp\left(-\varepsilon(k-1)T_0\right). \end{aligned} \tag{4.51}$$

Considering that $z \mapsto z^N \mathrm{e}^{-cN}$ is increasing in $[0, N/c]$ and decreasing in $[N/c, +\infty)$, we have that:

$$\sum_{k=1}^{\infty} k^N \exp\left(-\varepsilon k T_0\right) \leq 2\int_0^{+\infty} x^N \mathrm{e}^{-\varepsilon x T_0} \mathrm{d}x = \frac{2}{(\varepsilon T_0)^{N-1}} \int_0^{+\infty} y^N \mathrm{e}^{-y} \mathrm{d}y = \frac{2(N!)}{(\varepsilon T_0)^{N-1}}. \tag{4.52}$$

From (4.50), (4.51) and (4.52), we obtain

$$\|\varphi(\cdot,0)\|_{L^2(\Omega)} \leqslant C \exp\left(\left(\|q\|_\infty + \frac{\|\Delta f\|_\infty}{2}\right) T + \frac{\|f\|_\infty - E_0 T_0 + 1/T_0}{\varepsilon}\right) \|\varphi(\cdot,t)\|_{L^2(\Omega)},$$

for some $C > 0$ depending only on Ω. Hence the requested estimate. □

Proof of Theorem 1.1. From the estimates (4.48) and (4.49), there exists a constant $C > 0$ depending only on Ω, ω, such that

$$\iint_{\Omega\times\left(\frac{T}{4},\frac{3T}{4}\right)} |\varphi(x,t)|^2 \mathrm{d}x\mathrm{d}t \leqslant C \exp\left(\frac{C\left(1/T + C(f,q)\right)}{\varepsilon}\right) \iint_{\omega\times(0,T)} |\varphi(x,t)|^2 \mathrm{d}x\mathrm{d}t$$

and

$$\|\varphi(\cdot,0)\|_{L^2(\Omega)} \leqslant C\exp\left(\mu T + \frac{\|f\|_\infty - E_0T/4 + 4/T}{\varepsilon}\right)\|\varphi(\cdot,t)\|_{L^2(\Omega)} \ \ \forall t\in\left[\frac{T}{4},T\right], \tag{4.53}$$

where $\mu := \|q\|_\infty + \frac{\|\Delta f\|_\infty}{2}$ and $C(f,q) := \|f\|_\infty + \|\nabla f\|_\infty + \|\nabla^2 f\|_\infty + \|\Delta f\|_\infty^{\frac{2}{3}} + \|q\|_\infty^{\frac{2}{3}}$.

Integrating (4.53) on $\left[\frac{T}{4},\frac{3T}{4}\right]$, we obtain

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant \frac{2C^3\exp(2\mu T)}{T}\exp\left(\frac{-C(f,q,T)}{\varepsilon}\right)\iint_{\omega\times(0,T)}|\varphi(x,t)|^2\mathrm{d}x\mathrm{d}t, \tag{4.54}$$

where $C(f,q,T) := -2\|f\|_\infty - C.C(f,q) + E_0T/2 - (8+C)/T$. Since $E_0>0$, then there exists $T_1>0$, such that, for all $T\geq T_1$, $C(f,q,T)>0$. Finally, we obtain (1.6) from (1.5) and (4.54). □

Remark 4.7. We can notice that the results of Sections 2 and 3 are still true, if we replace β by β_ε, assuming that $(\beta_\varepsilon)_{0<\varepsilon\leqslant\varepsilon_0} \subset L^\infty(\Gamma)$ for some $\varepsilon_0>0$. In particular the control cost decreases exponentially when the viscosity vanishes and the control time is sufficiently large of the following system

$$\begin{cases} \partial_t y - \varepsilon\Delta y + X\cdot\nabla y + q\,y = v(x,t)\mathbb{1}_\omega & \text{in } \Omega_T,\\ \partial_\nu y + a_\varepsilon(x)y = 0 & \text{on } \Gamma_T,\\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases}$$

where $X=\nabla f$, $f\in W^{2,\infty}(\Omega)$, $q\in L^\infty(\Omega)$, $a_\varepsilon\in L^\infty(\Gamma)$ such that $\frac{\partial_\nu f}{2\varepsilon} + a_\varepsilon \geq 0$ on Γ for all $0<\varepsilon\leqslant\varepsilon_0$ and $E_0 = \min_{\overline{\Omega}}|\nabla f| > 0$.

Remark 4.8. In [13], we proved uniform controllability in ε by Agmon's approach under Neumann boundary conditions, whereas in this work, the boundary conditions are more general (Robin boundary conditions), the assumed conditions are clear and the results found are more robust than uniform controllability.

5. Exponential explosion of the null controllability cost

The objective of this section is to prove our second main result Theorem 1.3. Our proof is based on the construction of a specific solution $\overline{\varphi}$ of the adjoint system (1.4), for which we obtain $\exp\left(\frac{C}{\varepsilon}\right) = \mathcal{O}_{\varepsilon\to 0}\left(\frac{\|\overline{\varphi}(\cdot,0)\|_{L^2(\Omega)}}{\|\overline{\varphi}\|_{L^2(\omega_T)}}\right)$.

5.1. A stationary solution of (2.5)

Using the change (2.4), we are looking for a stationary solution $\overline{\Phi}$ (independent of t) of (2.5). This solution will be an eigenfunction of the operator A_ε associated with the eigenvalue 0. The following proposition answers this question in the case $q\geq 0$ and $\beta\geq 0$.

Proposition 5.1. *Let $q\geq 0$ and $\beta\geq 0$, then*

(1) $\sigma(A_\varepsilon)\subset[0,+\infty)$. *i.e.* $\lambda_1^{(\varepsilon)}\geq 0$.

(2) $\lambda_1^{(\varepsilon)} = 0 \Longleftrightarrow q=0$ *and* $\beta=0$.
In this case, $\overline{\Phi} := \exp\left(-\frac{f}{2\varepsilon}\right)$ *is an eigenvector of* A_ε *associated to the eigenvalue* 0.

Proof. From (2.9), we have $\lambda_1^{(\varepsilon)} = \min\limits_{\substack{y\in D(A_\varepsilon)\\ \|y\|_{L^2(\Omega)}=1}} \mathfrak{a}_\varepsilon(y,y)$, where

$$\mathfrak{a}_\varepsilon(y,y) = \varepsilon\int_\Omega |\nabla y|^2 \mathrm{d}x + \int_\Omega q_{f,\varepsilon}|y|^2\mathrm{d}x + \int_\Gamma \beta_f |y|^2 \mathrm{d}\sigma.$$

Recall that $q_{f,\varepsilon} = q + \frac{\mathcal{V}}{\varepsilon} - \frac{\Delta f}{2}$, $\mathcal{V} = \frac{|\nabla f|^2}{4}$ and $\beta_f = \frac{\partial_\nu f}{2} + \beta$.

(1) Let $y \in D(A_\varepsilon)$, by positivity of q and β, an integration by parts, inequalities of Cauchy–Schwarz and the Young, we obtain:

$$\begin{aligned}
\mathfrak{a}_\varepsilon(y,y) &\geq \varepsilon\int_\Omega |\nabla y|^2\mathrm{d}x + \int_\Omega \left(\frac{\mathcal{V}}{\varepsilon} - \frac{\Delta f}{2}\right)|y|^2\mathrm{d}x + \int_\Gamma \frac{\partial_\nu f}{2}|y|^2\mathrm{d}\sigma \qquad (5.1)\\
&= \varepsilon\int_\Omega |\nabla y|^2\mathrm{d}x + \int_\Omega \frac{\mathcal{V}}{\varepsilon}|y|^2\mathrm{d}x - \int_\Omega \frac{\Delta f}{2}|y|^2\mathrm{d}x + \int_\Gamma \frac{\partial_\nu f}{2}|y|^2\mathrm{d}\sigma\\
&= \varepsilon\int_\Omega |\nabla y|^2\mathrm{d}x + \int_\Omega \frac{\mathcal{V}}{\varepsilon}|y|^2\mathrm{d}x + \int_\Omega y\nabla y\cdot\nabla f\mathrm{d}x\\
&= \varepsilon\int_\Omega |\nabla y|^2\mathrm{d}x + \int_\Omega \frac{\mathcal{V}}{\varepsilon}|y|^2\mathrm{d}x + \int_\Omega \left(\sqrt{2\varepsilon}\nabla y\right)\cdot\left(\frac{y\nabla f}{\sqrt{2\varepsilon}}\right)\mathrm{d}x\\
&\geq \varepsilon\int_\Omega |\nabla y|^2\mathrm{d}x + \int_\Omega \frac{\mathcal{V}}{\varepsilon}|y|^2\mathrm{d}x - \int_\Omega \left|\sqrt{2\varepsilon}\nabla y\right|\left|\frac{y\nabla f}{\sqrt{2\varepsilon}}\right|\mathrm{d}x \qquad (5.2)\\
&\geq \varepsilon\int_\Omega |\nabla y|^2\mathrm{d}x + \int_\Omega \frac{\mathcal{V}}{\varepsilon}|y|^2\mathrm{d}x - \varepsilon\int_\Omega |\nabla y|^2\mathrm{d}x - \int_\Omega \frac{|y\nabla f|^2}{4\varepsilon}\mathrm{d}x \qquad (5.3)\\
&= 0.
\end{aligned}$$

(2) We have

$$\lambda_1^{(\varepsilon)} = 0 \Longleftrightarrow \exists y \in D(A_\varepsilon)\setminus\{0\},\ \mathfrak{a}_\varepsilon(y,y) = 0.$$

Let $y \in D(A_\varepsilon)\setminus\{0\}$, from the first point, when $\mathfrak{a}_\varepsilon(y,y)=0$, the inequalities (5.1), (5.2) and (5.3) become equalities, then

$$\mathfrak{a}_\varepsilon(y,y) = 0 \Longleftrightarrow \begin{cases} \mathfrak{a}_\varepsilon(y,y) = \varepsilon\int_\Omega |\nabla y|^2\mathrm{d}x + \int_\Omega \left(\frac{\mathcal{V}}{\varepsilon} - \frac{\Delta f}{2}\right)|y|^2\mathrm{d}x \\ \qquad\qquad + \int_\Gamma \frac{\partial_\nu f}{2}|y|^2\mathrm{d}\sigma, \\ -\left(\sqrt{2\varepsilon}\nabla y\right)\cdot\left(\frac{y\nabla f}{\sqrt{2\varepsilon}}\right) = \left|\sqrt{2\varepsilon}\nabla y\right|\left|\frac{y\nabla f}{\sqrt{2\varepsilon}}\right| = \varepsilon|\nabla y|^2 + \frac{|y\nabla f|^2}{4\varepsilon}. \end{cases}$$

Using the equality case in Cauchy–Schwarz and Young inequality, we find

$$\mathfrak{a}_\varepsilon(y,y) = 0 \Longleftrightarrow \begin{cases} \int_\Omega q|y|^2\mathrm{d}x + \int_\Gamma \beta|y|^2\mathrm{d}\sigma = 0, \\ \nabla y = -\frac{y}{2\varepsilon}\nabla f. \end{cases}$$

Since $q \geq 0$, $\beta \geq 0$, $y \neq 0$ in the domain Ω (note that $y \in D(A_\varepsilon) \setminus \{0\}$ implies that $y \neq 0$ on Γ), then

$$\mathfrak{a}_\varepsilon(y,y) = 0 \Longleftrightarrow \begin{cases} q = 0 \text{ and } \beta = 0, \\ \exists c \in \mathbb{R} \setminus \{0\}, \quad y = c \exp\left(-\frac{f}{2\varepsilon}\right). \end{cases}$$

□

5.2. Proof of Theorem 1.3

We consider (1.1) with pure Neumann boundary conditions and without potential, that is $\beta = 0$ and $q = 0$. In this case (1.1) becomes

$$\begin{cases} \partial_t y - \varepsilon \Delta y + \nabla f(x) \cdot \nabla y = v(t,x) \mathbb{1}_\omega & \text{in } \Omega_T, \\ \partial_\nu y = 0 & \text{on } \Gamma_T, \\ y(x,0) = y^0(x) & \text{in } \Omega \end{cases} \tag{5.4}$$

and the associated adjoint system is given by

$$\begin{cases} -\partial_t \varphi - \varepsilon \Delta \varphi - \nabla f(x) \cdot \nabla \varphi + -\Delta f \, \varphi = 0 & \text{in } \Omega_T, \\ \varepsilon \, \partial_\nu \varphi + \partial_\nu f \, \varphi = 0 & \text{on } \Gamma_T, \\ \varphi(x,T) = \varphi_T(x) & \text{in } \Omega. \end{cases} \tag{5.5}$$

Remark 5.2. The condition $f(\omega) \subset (m_f + h, M_f)$ implies that there is a heat transfer by transport in the uncontrolled part $\Omega \setminus \omega$, the control acts on the region ω, then we cannot control this transfer, this explains the exponential explosion of the null controllability cost. It is necessary to have the exponential decrease of the controllability cost, for example when $\Omega = \omega$, we can show that the null controllability cost of (5.4) is increased by a quantity of the form $C\left(1 + \sqrt{\varepsilon} + \frac{1}{\sqrt{\varepsilon}} + \frac{1}{\sqrt{T}}\right)$ whatever $T > 0$ and $\varepsilon > 0$ for $C > 0$ independent of ε and T.

Proof of Theorem 1.3. Let $q = 0$ and $\beta = 0$, from Proposition 5.1, $\overline{\Phi} = \exp\left(-\frac{f}{2\varepsilon}\right)$, then $\overline{\varphi} = \exp\left(-\frac{f}{\varepsilon}\right)$ is a solution of (5.5).

On the one hand

$$\begin{aligned} \|\overline{\varphi}(\cdot,0)\|^2_{L^2(\Omega)} = \|\overline{\varphi}\|^2_{L^2(\Omega)} &= \int_\Omega |\overline{\varphi}|^2 \mathrm{d}x = \int_\Omega \exp\left(-\frac{2f}{\varepsilon}\right) \mathrm{d}x \\ &\geq \int_{\Omega \cap \{f \in (m_f, m_f + h/2)\}} \exp\left(-\frac{2f}{\varepsilon}\right) \mathrm{d}x \\ &\geq |\Omega \cap \{f \in (m_f, m_f + h/2)\}| \exp\left(-\frac{2m_f + h}{\varepsilon}\right). \end{aligned} \tag{5.6}$$

On the other hand, using $f(\omega) \subset (m_f + h, M_f)$, we obtain

$$\begin{aligned} \|\overline{\varphi}\|^2_{L^2(\omega \times (0,T))} &= \int_{\omega \times (0,T)} |\overline{\varphi}|^2 \mathrm{d}x\mathrm{d}t = T \int_\omega |\overline{\varphi}|^2 \mathrm{d}x \\ &= T \int_\omega \exp\left(-\frac{2f}{\varepsilon}\right) \mathrm{d}x \\ &\leqslant T|\omega| \exp\left(-\frac{2m_f + 2h}{\varepsilon}\right). \end{aligned} \tag{5.7}$$

From (1.5), (5.6) and (5.7), we obtain

$$\mathcal{K}(\Omega,\omega,T,\varepsilon) \geq \frac{\|\overline{\varphi}(\cdot,0)\|_{L^2(\Omega)}}{\|\overline{\varphi}\|_{L^2(\omega\times(0,T))}} \geq \frac{|\Omega\cap\{f\in(m_f,m_f+h/2)\}|^{\frac{1}{2}}}{T^{\frac{1}{2}}|\omega|^{\frac{1}{2}}}\exp\left(\frac{h}{2\varepsilon}\right).$$

Since $\min_{\overline{\Omega}} f = m_f$ and f is continuous on $\overline{\Omega}$, then $\Omega\cap f^{-1}(m_f, m_f+h/2)$ is a nonempty open subset of Ω, thus $|\Omega\cap f^{-1}(m_f,m_f+h/2)|>0$. □

Remark 5.3. (**Open problem**). We have used that the vector field X is a gradient vector field to symmetrize the system adjoint in order to use the spectral decomposition of the adjoint system solution, which implies a very strong dissipation result, and also to find stationary solutions. An interesting question arises, whether results similar to Theorems 1.1 and 1.3 remains true for a general vector fields X belonging to $L^\infty(\Omega)$.

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