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AN APPROXIMATION METHOD FOR EXACT CONTROLS OF VIBRATING SYSTEMS WITH NUMERICAL VISCOSITY

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Abstract. We analyze a method for the approximation of exact controls of a second order infinite dimensional system with bounded input operator. The algorithm combines Russell's "stabilizability implies controllability" principle and a finite elements method of order θ with vanishing numerical viscosity. We show that the algorithm is convergent for any initial data in the energy space and that the error is of order θ for sufficiently smooth initial data. Both results are consequences of the uniform exponential decay of the discrete solutions guaranteed by the added viscosity and improve previous estimates obtained in the literature. Several numerical examples for the wave and the beam equations are presented to illustrate the method analyzed in this article.

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1. Introduction

The numerical approximation of exact controls for linear evolution equations has been one of the topics of interest in control theory since the beginning of the 90s when, in a series of articles due to R. Glowinski, J.-L. Lions and collaborators (see, for example, [1–3]), conjugate gradient type algorithms were proposed for finding controls characterized by the property of having the minimum L^2 norm. These are the so called HUM controls which inherit the name of the systematic method proposed by J.-L. Lions for their study.

These pioneering articles led to the development of an important specialized literature that contributed to a deeper understanding of this type of controls (see [4–6] and the references therein). Although most of the efforts were directed towards the proposal of new discretization schemes capable of ensuring the convergence of the discrete HUM controls, there were also other (more direct) approaches to the approximation of the controls. We mention among them the methods based on Huygens principle [7] (for hyperbolic equations) or flatness outputs [8, 9] (for parabolic equations).

Keywords and phrases: Infinite dimensional systems, exact control, approximation, error estimate, finite elements, vanishing viscosity.

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In this work we analyze an alternative numerical method for computing exact controls for a class of infinite dimensional systems modeling elastic vibrations. This method combines three main ideas: Russell's "stabilizability implies controllability" principle, error estimates for finite element-type approximations of the considered infinite dimensional systems and the technique of vanishing viscosity. We focus on the case of bounded input operators which excludes boundary control for systems governed by PDEs.

Our study has two main aims. On one hand, we show that the rate of convergence of our approximations to an exact control has the same order as the finite-element method, if the initial data to be controlled are sufficiently smooth. On the other hand, we prove that the method still converges in the case of finite energy initial data. Let us briefly indicate the structure of the work, its main ideas and the most notable results obtained in it.

In Section 2 we describe Russell's principle which states that backwards and forwards exponential stabilizability of a dynamical system implies its exact controllability in some time $\tau > 0$. This principle was used to show the exact boundary controllability property for the linear wave equation [10], Theorem 5.3. An abstract version of it has been proposed in [11] for the case of bounded control operators and further generalized in [12]. A similar idea stays at the origin of the concept of back and forth observers for linear infinite dimensional systems in [13]. A detailed discussion and several new applications of the principle are presented in [14]. For other general details, see also [15]. In the hypotheses in which it is verified, Russell's principle has the possibility of providing an exact control $u \in C([0,\tau];U)$ of the dynamic system under study for each initial data Q_0, with two remarkable properties: it is (in principle) easy to compute and it preserves the regularity of Q_0.

In the following Section 3, we describe how approximations u_h of the control u provided by Russell's principle can be given. This is done in Algorithm 1 below, which consists in solving a finite number $N(h)$ of forward and backward space discrete systems obtained by using a numerical scheme combining finite elements of order θ and vanishing viscosity. A first algorithm based on Russell's principle has been used to compute an exact boundary control for a class of second order evolution equations in [16] where the case $N(h) = 1$ is analyzed. This choice is convenient for implementation purposes but it does not yield the convergence of u_h to u. In [17] the full algorithm is developed and it is proved that, with an appropriate choice of $N(h)$, the convergence is ensured for sufficiently regular initial data and the error is of order slightly lower than θ. The main difference between Algorithm 1 and the one proposed in [17] consists in the addition of the viscosity term which, as we shall see, has a few interesting consequences.

After some important preliminaries developed in Section 4 and dealing with the evaluation of the error introduced in the dynamical system by the discretization scheme, the main results are presented in Section 5. As mentioned before, there are two main results proved in this section:

- In Theorem 5.4, we show that, if the initial data Q_0 to be controlled is in the energy space $H_{\frac{1}{2}} \times H_0$, the family $(u_h)_h$ provided by Algorithm 1 in Section 3 converges to u as h tends to zero.
- In Theorem 5.1, we show that, if the initial data Q_0 to be controlled is sufficiently smooth (belonging to $H_{\frac{3}{2}} \times H_{\frac{1}{2}}$), the order of convergence of the family $(u_h)_h$ to u is precisely θ, the order of the finite-element method.

The spaces H, U, $H_{\frac{1}{2}}$ and $H_{\frac{3}{2}}$ mentioned above are introduced at the beginning of the following section.

It is known that, after spatial discretization, the decay of the semigroup corresponding to an exponentially stable dynamical system may not be uniform with respect to the mesh size h. This phenomenon has been remarked by Banks *et al.* [18], where the use of a mixed finite element method is proposed to restore the uniform decay rate. Later on, several studies have confirmed that this defect is due to the spurious high frequencies introduced by the numerical scheme traveling at arbitrarily small velocities which, therefore, show a lack of propagation in space (see, for instance, the recent article [19] and the references therein). In order to cure this defect, an approach consisting in adding a correcting numerical viscous term in the discrete system, vanishing in the limit, has been proposed (see [20–26]). The vanishing viscosity takes charge of the spurious high oscillations and leads to a uniform (with respect to the mesh size) exponential decay of the discrete semigroup. This property is used in this paper in order to obtain our main results mentioned above. Indeed, both Theorem 5.1, which improves the convergence rate of the controls obtained in [17], and Theorem 5.4, which shows the convergence

of the algorithm for initial data in the energy space, use in a fundamental way the uniform exponential decay of the discrete semigroup. This can be explained by recalling that Algorithm 1 solves a finite number $N(h)$ of forward and backward space discrete systems. The uniform exponential decay ensured by the viscosity term enables us to deduce better error estimates which do not degenerate when the number of computed solutions $N(h)$ tends to infinity when h goes to zero.

To illustrate the efficiency of this approach, we apply it to several systems governed by PDEs and describe the associated numerical simulations in the last Section 6.

2. Russel's principle and construction of exact controls

In order to give the precise statement of our results we need some notation. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space with the induced norm $\|\cdot\|$, and assume that the unbounded operator $A_0 : \mathcal{D}(A_0) \to H$ is self-adjoint, strictly positive and with compact resolvent. Then, according to classical results, the operator A_0 is diagonalizable with an orthonormal basis $(\varphi_k)_{k\geqslant 1}$ of eigenvectors, and the corresponding family of positive eigenvalues, in ascending order, $(\lambda_k)_{k\geqslant 1}$ satisfies $\lim\limits_{k\to\infty} \lambda_k = \infty$. Moreover, we have

$$\mathcal{D}(A_0) = \left\{ z \in H \middle| \sum_{k\geqslant 1} \lambda_k^2 \left|\langle z, \varphi_k\rangle\right|^2 < \infty \right\}, \qquad A_0 z = \sum_{k\geqslant 1} \lambda_k \langle z, \varphi_k\rangle \varphi_k \quad (z \in \mathcal{D}(A_0)).$$

For $\alpha \geqslant 0$, the operator A_0^α is defined by

$$\mathcal{D}(A_0^\alpha) = \left\{ z \in H \middle| \sum_{k\geqslant 1} \lambda_k^{2\alpha} \left|\langle z, \varphi_k\rangle\right|^2 < \infty \right\}, \qquad A_0^\alpha z = \sum_{k\geqslant 1} \lambda_k^\alpha \langle z, \varphi_k\rangle \varphi_k \quad (z \in \mathcal{D}(A_0^\alpha)). \tag{2.1}$$

For every $\alpha \geqslant 0$ we denote by H_α the space $\mathcal{D}(A_0^\alpha)$ endowed with the inner product

$$\langle \varphi, \psi\rangle_\alpha = \langle A_0^\alpha \varphi, A_0^\alpha \psi\rangle \qquad (\varphi, \psi \in H_\alpha).$$

The induced norm is denoted by $\|\cdot\|_\alpha$. From the above facts it follows that for every $\alpha \geqslant 0$ the operator A_0 is unitary from $H_{\alpha+1}$ onto H_α, and strictly positive on H_α.

Let $(U, \langle\cdot,\cdot\rangle_U)$ be another Hilbert space with the corresponding norm $\|\cdot\|_U$, and let $B_0 \in \mathcal{L}(U, H)$ be an input operator. We consider the system

$$\ddot{q}(t) + A_0 q(t) + B_0 u(t) = 0 \qquad (t \geqslant 0), \tag{2.2}$$

$$q(0) = q_0,\ \dot{q}(0) = q_1. \tag{2.3}$$

We assume that the above system is *exactly controllable in time* $\tau_0 > 0$, *i.e.* for every $q_0 \in H_{\frac{1}{2}}$, $q_1 \in H$ there exists a control $u \in L^2(0, \tau_0; U)$ such that

$$q(\tau_0) = \dot{q}(\tau_0) = 0. \tag{2.4}$$

Now, we consider the second order differential equation

$$\ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0 \qquad (t \geqslant 0), \tag{2.5}$$

$$w(0) = w_0,\ \dot{w}(0) = w_1. \tag{2.6}$$

It is well known that the above equation defines a well posed dynamical system in the state space $X = H_{\frac{1}{2}} \times H$. More precisely, the solution $\begin{bmatrix} w \\ \dot{w} \end{bmatrix}$ of (2.5)−(2.6) is given by

$$\begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix} = \mathbb{T}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in X, \quad t \geqslant 0 \right), \tag{2.7}$$

where $\mathbb{T}$ is the contraction semigroup on X generated by $\mathcal{A} - \mathcal{B}\mathcal{B}^*$ and the matriceal operators $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to X$, $\mathcal{B} \in \mathcal{L}(U, X)$ are defined by

$$\mathcal{D}(\mathcal{A}) = H_1 \times H_{\frac{1}{2}}, \quad \mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}.$$

Let $\tau > 0$ and consider the backwards system

$$\ddot{w}_b(t) + A_0 w_b(t) - B_0 B_0^* \dot{w}_b(t) = 0 \qquad (t \leqslant \tau), \tag{2.8}$$

$$w_b(\tau) = w(\tau),\ \dot{w}_b(\tau) = \dot{w}(\tau). \tag{2.9}$$

It is not difficult to check that the solution $\begin{bmatrix} w_b \\ \dot{w}_b \end{bmatrix}$ of (2.8)−(2.9) is given by

$$\begin{bmatrix} w_b(t) \\ \dot{w}_b(t) \end{bmatrix} = \mathbb{S}_{\tau - t} \begin{bmatrix} w(\tau) \\ \dot{w}(\tau) \end{bmatrix} \qquad (t \in [0, \tau]), \tag{2.10}$$

where $\mathbb{S}$ is the contraction semigroup in X generated by $-\mathcal{A} - \mathcal{B}\mathcal{B}^*$.

We define $L_\tau \in \mathcal{L}(X)$ by

$$L_\tau \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} w_b(0) \\ \dot{w}_b(0) \end{bmatrix} \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in X \right). \tag{2.11}$$

With the above notation, the operator L_τ clearly satisfies $L_\tau = \mathbb{S}_\tau \mathbb{T}_\tau$.

In the following we present a useful result given in [17], Proposition 2.5.

Proposition 2.1. *With the above notation, assume that (2.2)−(2.3) is exactly controllable in time $\tau_0 > 0$ and $B_0 B_0^* \in \mathcal{L}\left(H_1, H_{\frac{1}{2}}\right)$. Then, the restrictions of $\mathbb{T}$ and $\mathbb{S}$ to $H_1 \times H_{\frac{1}{2}}$ and $H_{\frac{3}{2}} \times H_1$ are contractions semigroups on these spaces with generators that are restrictions of $\mathcal{A} - \mathcal{B}\mathcal{B}^*$ and $-\mathcal{A} - \mathcal{B}\mathcal{B}^*$ to $H_{\frac{3}{2}} \times H_1$ and $H_2 \times H_{\frac{3}{2}}$, respectively. Moreover, for any $\tau \geqslant \tau_0$ there exists a norm on $\mathcal{L}\left(H_{\frac{3}{2}} \times H_1\right)$, equivalent to the standard norm, such that*

$$\|\mathbb{T}_\tau\|_{\mathcal{L}\left(H_{\frac{3}{2}} \times H_1\right)} < 1, \qquad \|\mathbb{S}_\tau\|_{\mathcal{L}\left(H_{\frac{3}{2}} \times H_1\right)} < 1. \tag{2.12}$$

In addition, the semigroups $\mathbb{T}$ and $\mathbb{S}$ are exponentially stable verifying

$$\|\mathbb{T}_\tau\|_{\mathcal{L}(X)} < 1, \qquad \|\mathbb{S}_\tau\|_{\mathcal{L}(X)} < 1,$$

and $I - L_\tau$ is invertible in $\mathcal{L}(X)$ and $\mathcal{L}\left(H_{\frac{3}{2}} \times H_1\right)$, where the inverse is given as follows

$$(I - L_\tau)^{-1} = \sum_{n \geqslant 0} L_\tau^n. \tag{2.13}$$

In the above hypothesis, Russel's principle (see, for instance [10]) can be used to construct an explicit control u for (2.2)−(2.3). More precisely, we have the following result.

Proposition 2.2. *In the hypothesis of Proposition 2.1 we assume that (2.2)−(2.3) is exactly controllable in time $\tau_0 > 0$. Then for any $\tau \geqslant \tau_0$ a control $u \in C([0,\tau], U)$ for (2.2)−(2.3), steering the initial state $\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in X$ to rest in time τ, is given by*

$$u(t) = B_0^* \dot{w}(t) + B_0^* \dot{w}_b(t) = \mathcal{B}^* \mathbb{T}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \mathcal{B}^* \mathbb{S}_{\tau - t} \mathbb{T}_\tau \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \qquad (t \in [0,\tau]), \tag{2.14}$$

where $\begin{bmatrix} w \\ \dot{w} \end{bmatrix}$ and $\begin{bmatrix} w_b \\ \dot{w}_b \end{bmatrix}$ are the solutions of (2.5)−(2.6) and (2.8)−(2.9), respectively, with initial data

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = (I - L_\tau)^{-1} \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}. \tag{2.15}$$

Remark 2.3. Two of the properties of the above control u are very important in our study:

- From (2.13) and (2.15) we deduce that the initial data $\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$ needed to construct the control u can be approximated by solving a finite number N of forward and backward equations of (2.5)−(2.6) and (2.8)−(2.9), respectively:

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \approx \sum_{n=0}^{N} L_\tau^n \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}. \tag{2.16}$$

- From Proposition 2.1 and (2.15) it follows that the regularity assumptions on $\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$ are inherited by $\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$. More precisely, $\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in H_{\frac{3}{2}} \times H_1$ implies that $\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in H_{\frac{3}{2}} \times H_1$. This implies that (2.14) provides smoother controls u for more regular initial data.

3. The semi-discrete problem and control approximation

In order to provide a numerical method to approximate the control u given in Proposition 2.2, we need more assumptions and notation. Assume that there exists a family $(V_h)_{h>0}$ of finite dimensional subspaces of $H_{\frac{1}{2}}$ and $\theta > 0$, $h^* > 0$, $C_0 > 0$ such that for every $h \in (0, h^*)$,

$$\|\pi_h \varphi - \varphi\|_{\frac{1}{2}} \leqslant C_0 h^\theta \|\varphi\|_1 \qquad (\varphi \in H_1), \tag{3.1}$$

$$\|\pi_h \varphi - \varphi\| \leqslant C_0 h^\theta \|\varphi\|_{\frac{1}{2}} \qquad \left(\varphi \in H_{\frac{1}{2}}\right), \tag{3.2}$$

where π_h is the orthogonal projector from $H_{\frac{1}{2}}$ onto V_h. Assumptions (3.1)−(3.2) are, in particular, satisfied when finite elements are used for the approximation of Sobolev spaces. For instance, these types of estimates are proved to hold in the case of Lagrange finite elements in [27], Theorem 4.4-4.

Moreover, we introduce $\widetilde{\pi}_h$ the orthogonal projection of H onto V_h. As is well known, since π_h and $\widetilde{\pi}_h$ are orthogonal projectors, they are self adjoint operators. From (3.2) and the fact that $H_{\frac{1}{2}}$ is dense in H it follows that

$$\|\varphi - \widetilde{\pi}_h\varphi\| \leqslant \|\varphi - \pi_h\varphi\| \leqslant C_0 h^\theta \|\varphi\|_{\frac{1}{2}} \qquad \left(\varphi \in H_{\frac{1}{2}}\right), \tag{3.3}$$

and

$$\lim_{h\to 0} \|\varphi - \widetilde{\pi}_h\varphi\| = 0 \qquad (\varphi \in H). \tag{3.4}$$

We define the linear operator $A_{0h} \in \mathcal{L}(V_h)$ by

$$\langle A_{0h}\varphi_h, \psi_h\rangle = \langle A_0^{\frac{1}{2}}\varphi_h, A_0^{\frac{1}{2}}\psi_h\rangle \qquad (\varphi_h, \psi_h \in V_h). \tag{3.5}$$

The operator A_{0h} is clearly symmetric and strictly positive.

Denote $U_h = B_0^* V_h \subset U$ and define the operators $B_{0h} \in \mathcal{L}(U, H)$ by

$$B_{0h}u = \widetilde{\pi}_h B_0 u \qquad (u \in U). \tag{3.6}$$

Note that Ran $B_{0h} \subset V_h$. The adjoint $B_{0h}^* \in \mathcal{L}(H, U)$ of B_{0h} is

$$B_{0h}^*\varphi = B_0^*\widetilde{\pi}_h\varphi \qquad (\varphi \in H). \tag{3.7}$$

Since $U_h = B_0^* V_h$, from (3.7), it follows that Ran $B_{0h}^* = U_h$ and that

$$\langle B_{0h}^*\varphi_h, B_{0h}^*\psi_h\rangle_U = \langle B_0^*\varphi_h, B_0^*\psi_h\rangle_U \qquad (\varphi_h, \psi_h \in V_h). \tag{3.8}$$

From (3.6) we have that the family $(\|B_{0h}\|_{\mathcal{L}(U,H)})_{h\in(0,h^*)}$ is bounded.

In order to approximate the exact control u from Proposition 2.2, given by (2.14), we shall use two discrete damped equations, one forward and one backward, corresponding to (2.5)−(2.6) and (2.8)−(2.9), respectively. More precisely, we introduce the semi-discrete equations

$$\ddot{w}_h(t) + A_{0h}w_h(t) + B_{0h}B_{0h}^*\dot{w}_h(t) + \zeta h^\eta A_{0h}\dot{w}_h(t) = 0 \qquad (t \geqslant 0), \tag{3.9}$$

$$w_h(0) = w_{0h},\ \dot{w}_h(0) = w_{1h}, \tag{3.10}$$

and

$$\ddot{w}_{b,h}(t) + A_{0h}w_{b,h}(t) - B_{0h}B_{0h}^*\dot{w}_{b,h}(t) - \zeta h^\eta A_{0h}\dot{w}_{b,h}(t) = 0 \qquad (t \leqslant \tau). \tag{3.11}$$

$$w_{b,h}(\tau) = w_{b,0h},\ \dot{w}_{b,h}(\tau) = w_{b,1h}. \tag{3.12}$$

Notice that, in each of the equation (3.9) and (3.11), a numerical viscosity term has been introduced: $\zeta h^\eta A_{0h}\dot{w}_h(t)$ and $-\zeta h^\eta A_{0h}\dot{w}_{b,h}(t)$, respectively. In the above equations ζ and η are positive real numbers

which will be conveniently chosen later on. As we shall see in the following section, these terms reinforce the dissipation in each equation in order to ensure the uniform stability (in h) of both discrete systems. These properties will allow us to obtain better error estimates and convergence results for the discrete approximations $u_h \in C([0,\tau]; U_h)$ of the control u.

We consider the following algorithm to compute the approximations u_h:

Algorithm 1:

1. Take $\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$ in $H_{\frac{1}{2}} \times H$. Let $q_{0h} = \pi_h q_0$ and $q_{1h} = \pi_h q_1$ if $q_1 \in H_{\frac{1}{2}}$ and $q_{1h} = \widetilde{\pi}_h q_1$ if $q_1 \notin H_{\frac{1}{2}}$.
2. For any $h > 0$ chose $N(h) \in \mathbb{N}$.
3. For $n = 1, 2, \ldots, N(h)$ let $\begin{bmatrix} w_h^n \\ \dot{w}_h^n \end{bmatrix}$ be the solution of (3.9)−(3.10) with initial data

$$w_h(0) = \begin{cases} q_{0h} & \text{if } n = 1 \\ w_{b,h}^{n-1}(0) & \text{if } 1 < n \leqslant N(h), \end{cases} \tag{3.13}$$

$$\dot{w}_h(0) = \begin{cases} q_{1h} & \text{if } n = 1 \\ \dot{w}_{b,h}^{n-1}(0) & \text{if } 1 < n \leqslant N(h), \end{cases} \tag{3.14}$$

and $\begin{bmatrix} w_{b,h}^n \\ \dot{w}_{b,h}^n \end{bmatrix}$ be the solution of (3.11)−(3.12) with initial data

$$w_{b,h}(\tau) = w_h^n(\tau), \quad \dot{w}_{b,h}(\tau) = \dot{w}_h^n(\tau). \tag{3.15}$$

4. Compute $\begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix}$ as follows:

$$\begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix} = \begin{bmatrix} q_{0h} \\ q_{1h} \end{bmatrix} + \sum_{n=1}^{N(h)} \begin{bmatrix} w_{b,h}^n(0) \\ \dot{w}_{b,h}^n(0) \end{bmatrix} = \sum_{n=1}^{N(h)} \begin{bmatrix} w_h^n(0) \\ \dot{w}_h^n(0) \end{bmatrix} + \begin{bmatrix} w_{b,h}^{N(h)}(0) \\ \dot{w}_{b,h}^{N(h)}(0) \end{bmatrix}. \tag{3.16}$$

5. Compute the control u_h,

$$u_h = B_{0h}^* \dot{w}_h + B_{0h}^* \dot{w}_{b,h}, \tag{3.17}$$

where $\begin{bmatrix} w_h \\ \dot{w}_h \end{bmatrix}$ is the solution of (3.9) with initial data

$$w_h(0) = w_{0h}, \dot{w}_h(0) = w_{1h}, \tag{3.18}$$

and $\begin{bmatrix} w_{b,h} \\ \dot{w}_{b,h} \end{bmatrix}$ is the solution of (3.11) with initial data

$$w_{b,h}(\tau) = w_h(\tau), \dot{w}_{b,h}(\tau) = \dot{w}_h(\tau). \tag{3.19}$$

Remark 3.1. Step 4 of the above algorithm provides an approximation $\begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix}$ of the initial data $\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$ from (2.15) which allows to construct the exact control u according to (2.14). We remark that, in the case $\zeta = 0$, the series $\begin{bmatrix} q_{0h} \\ q_{1h} \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} w_{b,h}^n(0) \\ \dot{w}_{b,h}^n(0) \end{bmatrix}$ would give us an exact control of the discrete problem. The approximation $\begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix}$ is obtained by turning this series into a finite sum. Therefore, our algorithm firstly uses Russel's principle as described for the continuous context in Proposition 2.2 and combines it with the truncation idea already mentioned in (2.16). In the end it provides us a computable family of approximations $\left(\begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix} \right)_{h>0}$ which, as shown below (see (5.7) in the proof of Thm. 5.1), converges to $\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$ as h tends to zero. In order to use the above scheme, at step 2, we have to choose the truncation range $N(h)$. If the data $\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$ to be controlled belong to the regular space $H_{\frac{3}{2}} \times H_1$, a value $N(h) = \left[\frac{\theta}{\ln \alpha^{-1}} \ln(h^{-1}) \right]$ is provided in Theorem 5.1 which ensures the desired error of order θ. However, there are no such estimates for $N(h)$ in the case of initial data $\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$ in $H_{\frac{1}{2}} \times H$. In practice the values of $N(h)$ are provided by the stopping criterion (6.1) used in the final Section 6 devoted to numerical experiments. In Figure 7 we compare the values of $N(h)$ given by (6.1) with the ones obtained in Theorem 5.1.

4. Convergence of the discrete solutions

The aim of this section is to analyse the convergence of the approximate solutions corresponding to the following numerical scheme with viscosity

$$\ddot{w}_h(t) + A_{0h} w_h(t) + B_{0h} B_{0h}^* \dot{w}_h(t) + \zeta h^\eta A_{0h} \dot{w}_h(t) = 0 \qquad (t \geqslant 0)\,, \tag{4.1}$$

$$w_h(0) = w_{0h},\ \dot{w}_h(0) = w_{1h}, \tag{4.2}$$

where we have used the notation from Section 3 for the families of operators $(A_{0h})_{h>0}$ and $(B_{0h})_{h>0}$.

Concerning the convergence of the approximate solutions given by (4.1)−(4.2) we recall the following result which gives an error estimate in the case of sufficiently regular initial data.

Proposition 4.1. *Let $w_0 \in H_{\frac{3}{2}}$, $w_1 \in H_1$, $\zeta \in [0,1]$ and let w, w_h be the corresponding solutions of (2.5)−(2.6) and (4.1)−(4.2), respectively. Moreover, assume that $B_0 B_0^* \in \mathcal{L}\left(H_1, H_{\frac{1}{2}}\right)$, $w_{0h} = \pi_h(w_0), w_{1h} = \pi_h(w_1)$ and $\eta \geqslant \theta$. Then there exist two constants $K, h^* > 0$ such that for every $h \in (0, h^*)$, we have*

$$\|\dot{w}(t) - \dot{w}_h(t)\| + \|w(t) - w_h(t)\|_{\frac{1}{2}} \leqslant K t h^\theta \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1 \right) \qquad (t \geqslant 0)\,. \tag{4.3}$$

Proof. See [28], Proposition 5.1. □

For $h > 0$ we denote $X_h = V_h \times V_h$ and $\mathcal{W}_h(t) = \begin{bmatrix} w_h(t) \\ \dot{w}_h(t) \end{bmatrix}$, and we consider the operators

$$\mathcal{A}_h = \begin{bmatrix} 0 & I \\ -A_{0h} & -\zeta h^\eta A_{0h} \end{bmatrix}, \quad \mathcal{B}_h = \begin{bmatrix} 0 \\ B_{0h} \end{bmatrix}. \tag{4.4}$$

The discrete analogues of the semigroups $\mathbb{T}, \mathbb{S}$ and of the operator L_t, denoted by $\mathbb{T}_h, \mathbb{S}_h$, and $L_{h,t}$, respectivly, are defined, for every $h > 0$, by

$$\mathbb{T}_{h,t} = e^{t(\mathcal{A}_h - \mathcal{B}_h \mathcal{B}_h^*)}, \ \mathbb{S}_{h,t} = e^{t(-\mathcal{A}_h - \mathcal{B}_h \mathcal{B}_h^*)}, \ L_{h,t} = \mathbb{S}_{h,t}\mathbb{T}_{h,t} \qquad (t \geqslant 0). \tag{4.5}$$

For every $h > 0$ we define $\Pi_h \in \mathcal{L}\left(H_{\frac{1}{2}} \times H_{\frac{1}{2}}, X_h\right)$ and $\widetilde{\Pi}_h \in \mathcal{L}\left(H_{\frac{1}{2}} \times H, X_h\right)$

$$\Pi_h = \begin{bmatrix} \pi_h & 0 \\ 0 & \pi_h \end{bmatrix}, \qquad \widetilde{\Pi}_h = \begin{bmatrix} \pi_h & 0 \\ 0 & \widetilde{\pi}_h \end{bmatrix}. \tag{4.6}$$

The following result is a direct consequence of Proposition 4.1.

Corollary 4.2. *In the hypotheses of Proposition 4.1, there exist two constants $K_1, h^* > 0$, such that for every $h \in (0, h^*)$ and $t > 0$, we have (recall that $L_t = \mathbb{S}_t\mathbb{T}_t$, for every $t \geqslant 0$)*

$$\|\mathbb{T}_t Z_0 - \mathbb{T}_{h,t}\Pi_h Z_0\|_X \leqslant K_1 t h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} \qquad \left(Z_0 \in H_{\frac{3}{2}} \times H_1\right), \tag{4.7}$$

$$\|\mathbb{S}_t Z_0 - \mathbb{S}_{h,t}\Pi_h Z_0\|_X \leqslant K_1 t h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} \qquad \left(Z_0 \in H_{\frac{3}{2}} \times H_1\right), \tag{4.8}$$

$$\|L_t Z_0 - L_{h,t}\Pi_h Z_0\|_X \leqslant K_1 t h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} \qquad \left(Z_0 \in H_{\frac{3}{2}} \times H_1\right). \tag{4.9}$$

Proof. Relation (4.7) represents an equivalent version of (4.3). By taking into account that

$$P\mathbb{S}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \mathbb{T}_t P \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in H_1 \times H_{\frac{1}{2}}\right),$$

where

$$P \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} w_0 \\ -w_1 \end{bmatrix} \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in X\right),$$

it follows that (4.8) is a consequence of (4.7). To prove (4.9), we remark that

$$\|L_t Z_0 - L_{h,t}\Pi_h Z_0\|_X \leqslant \|\mathbb{S}_t\mathbb{T}_t Z_0 - \mathbb{S}_{h,t}\Pi_h\mathbb{T}_t Z_0\|_X + \|\mathbb{S}_{h,t}\left(\Pi_h\mathbb{T}_t Z_0 - \mathbb{T}_{h,t}\Pi_h Z_0\right)\|_X,$$

and we use (4.7) and (4.8).

□

From (2.12) in Proposition 2.1 we deduce that

$$\|L_\tau\|_{\mathcal{L}\left(H_{\frac{3}{2}} \times H_1\right)} < 1 \qquad (\tau \geqslant \tau_0). \tag{4.10}$$

In fact, relation (4.10) is a consequence of fact that the infinite dimensional linear systems (2.5)−(2.6) and (2.8)−(2.9) are exponentially stable. It is by now well-known that, if $\zeta = 0$, the approximation (4.1)−(4.2) may not be uniformly stable with respect to the discretisation parameter (see, for instance, [4, 6, 18, 19, 22]). However, if $\zeta > 0$, the parameter η can be chosen such that the added numerical viscosity ensures the uniform

stability of (4.1)−(4.2). In the following we suppose that the family of systems (4.1)−(4.2) is uniformly stable, in the sense that exist constants M, ν, $h^* > 0$ (independent of h, w_{0h} and w_{1h}) such that for all $h \in (0, h^*)$ we have

$$\|\dot{w}_h(t)\|^2 + \left\|A_{0h}^{\frac{1}{2}} w_h(t)\right\|^2 \leqslant M e^{-\nu t} \left(\|w_{1h}\|^2 + \left\|A_{0h}^{\frac{1}{2}} w_{0h}\right\|^2 \right) \qquad (t \geqslant 0). \tag{4.11}$$

For details concerning the context in which (4.11) holds we refer the interested reader to [20, 21, 24–26]. From (4.11) it follows that there exist τ_1 and $\alpha_1 \in (0,1)$, independent of h, such that the following relation holds

$$\|L_{h,t}\|_{\mathcal{L}(X_h)} < \alpha_1 \qquad (t \geqslant \tau_1)\,. \tag{4.12}$$

As a consequence of the uniform estimate (4.12) we can prove the following important estimate which will allow us to obtain the desired error estimates for the approximate control given by the algorithm introduced in Section 3.

Corollary 4.3. *There exists $h^* > 0$ such that, for every $t \geqslant \max\{\tau_0, \tau_1\}$, $h \in (0, h^*)$ and $k \in \mathbb{N}$, we have*

$$\left\|L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\right\|_X \leqslant \alpha^{k-1} \left(C_0 \alpha + k K_1 t\right) h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} \qquad \left(Z_0 \in H_{\frac{3}{2}} \times H_1\right), \tag{4.13}$$

where $\alpha = \max\{\|L_t\|_{\mathcal{L}\left(H_{\frac{3}{2}} \times H_1\right)}, \|L_{h,t}\|_{\mathcal{L}(X_h)}\} \in (0,1)$. In (4.13) the constant C_0 is the one in (3.1)−(3.2) and the constant K_1 is given in (4.9).

Proof. Taking into account the following inequality

$$\left\|L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\right\|_X \leqslant \left\|L_t^k Z_0 - \Pi_h L_t^k Z_0\right\|_X + \left\|\Pi_h L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\right\|_X, \tag{4.14}$$

we evaluate each of the right-hand side terms from above. By using Proposition 2.1 we get, for every $t \geqslant 0$, the invariance of the space $H_{\frac{3}{2}} \times H_1$ with respect to L_t. In addition, using (3.1) and (3.2) we infer the existence of a constant $C_0 > 0$ such that

$$\left\|L_t^k Z_0 - \Pi_h L_t^k Z_0\right\|_X \leqslant C_0 h^\theta \|L_t\|^k_{\mathcal{L}\left(H_{\frac{3}{2}} \times H_1\right)} \|Z_0\|_{H_{\frac{3}{2}} \times H_1}. \tag{4.15}$$

In order to evaluate the second right-hand term, by denoting $A_k := \left\|\Pi_h L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\right\|_X$ and using (4.9), we remark that

$$\begin{aligned}
A_k &\leqslant \left\|\Pi_h L_t(L_t^{k-1}) Z_0 - L_{h,t} \Pi_h L_t^{k-1} Z_0\right\|_X + \left\|L_{h,t}\left(\Pi_h L_t^{k-1} Z_0 - L_{h,t}^{k-1} \Pi_h Z_0\right)\right\|_X \\
&\leqslant K_1 t h^\theta \|L_t\|^{k-1}_{\mathcal{L}\left(H_{\frac{3}{2}} \times H_1\right)} \|Z_0\|_{H_{\frac{3}{2}} \times H_1} + \alpha A_{k-1} \\
&\leqslant 2 K_1 t h^\theta \alpha^{k-1} \|Z_0\|_{H_{\frac{3}{2}} \times H_1} + \alpha^2 A_{k-2} \leqslant \dots \\
&\leqslant K_1 t h^\theta k \alpha^{k-1} \|Z_0\|_{H_{\frac{3}{2}} \times H_1}.
\end{aligned}$$

Finally, using the last inequality combined with (4.14) and (4.15) we get (4.13). □

Remark 4.4. Let us compare estimate (4.13) with the corresponding one proved in [17], Corollary 3.3, where the right hand side term is replaced by $(C_0 + k K_1 t)\, h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1}$. Notice that, whereas in [17], Corollary 3.3

the error is estimated by a quantity depending on k, in (4.13) the error term is bounded independently of k. This is achieved by the introduction of the vanishing viscosity term and the uniform estimate (4.12) and will offer us the possibility to improve all the error estimates for the approximate controls proved in [17].

The uniform exponential decay (4.12) allows us to prove convergence of the discrete controls given by the Algorithm 1 in Section 3 in the case of initial data $\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in H_{\frac{1}{2}} \times H$. The following lemma is needed to give an analogue of Corollary 4.2 with less regular initial data.

Lemma 4.5. *Let* $Q_0 = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in H_{\frac{1}{2}} \times H$. *For each* $h > 0$ *there exists* $Q_{0h} = \begin{bmatrix} q_{0h} \\ q_{1h} \end{bmatrix} \in H_{\frac{3}{2}} \times H_1$ *such that*

$$\lim_{h \to 0} \|Q_0 - Q_{0h}\|_X = 0, \tag{4.16}$$

$$\lim_{h \to 0} h^{\theta} \|Q_{0h}\|_{H_{\frac{3}{2}} \times H_1} = 0. \tag{4.17}$$

Proof. Let $(\varphi_k)_{k \geqslant 1}$ be the orthonormal basis of eigenvectors of the operator A_0, with the corresponding family of positive eigenvalues $(\lambda_k)_{k \geqslant 1}$ that satisfies $\lim\limits_{k \to \infty} \lambda_k = \infty$. Recall that the family $(\lambda_k)_{k \geqslant 1}$ is nondecreasing. There exist two sequences $(q_{0k})_{k \geqslant 1}$ and $(q_{1k})_{k \geqslant 1}$ of scalars such that

$$q_j = \sum_{k \geqslant 1} q_{jk} \varphi_k \qquad (j = 0, 1). \tag{4.18}$$

Let us define

$$q_{jh} = \sum_{k=1}^{I_h} q_{jk} \varphi_k \qquad (j = 0, 1), \tag{4.19}$$

where $I_h \in \mathbb{N}^*$ will be chosen later on. We remark that

$$\|q_{0h}\|^2_{H_{\frac{3}{2}}} = \left\langle \sum_{k=1}^{I_h} q_{0k} \lambda_k^{\frac{3}{2}} \varphi_k, \sum_{k=1}^{I_h} q_{0k} \lambda_k^{\frac{3}{2}} \varphi_k \right\rangle = \sum_{k=1}^{I_h} |q_{0k}|^2 \lambda_k^3 \leqslant \lambda_{I_h}^2 \sum_{k=1}^{I_h} |q_{0k}|^2 \lambda_k \leqslant \lambda_{I_h}^2 \|q_0\|^2_{H_{\frac{1}{2}}},$$

and

$$\|q_{1h}\|^2_{H_1} = \sum_{k=1}^{I_h} |q_{1k}|^2 \lambda_k^2 \leqslant \lambda_{I_h}^2 \sum_{k=1}^{I_h} |q_{1k}|^2 \leqslant \lambda_{I_h}^2 \|q_1\|^2_H .$$

If we denote by $Q_{0h} = \begin{bmatrix} q_{0h} \\ q_{1h} \end{bmatrix}$, we obtain that

$$\|Q_{0h}\|_{H_{\frac{3}{2}} \times H_1} \leqslant \lambda_{I_h} \|Q_0\|_{H_{\frac{1}{2}} \times H_1}. \tag{4.20}$$

It follows that (4.16) and (4.17) hold if we can choose I_h such that

$$\lim_{h \to 0} I_h = \infty \text{ and } \quad \lim_{h \to 0} h^{\theta} \lambda_{I_h} = 0. \tag{4.21}$$

For $h > 0$, we define the following family of sets

$$P_h = \left\{ k \in \mathbb{N}^* \middle| \lambda_k \leqslant \frac{1}{h^{\frac{\theta}{2}}} \right\}.$$

It is not difficult to see that there exists $h_0 > 0$ such that for every $h \in (0, h_0)$, P_h is nonempty. Moreover, we have $\lim_{h\to 0} card(P_h) = \infty$. Finally, by choosing $I_h = card(P_h)$ it follows that (4.21) holds and the proof is finished. □

Corollary 4.6. *Let* $Q_0 = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in H_{\frac{1}{2}} \times H$ *and* $t > 0$. *The following assertions hold*

- $\lim_{h\to 0} \left\| \mathbb{T}_t Q_0 - \mathbb{T}_{h,t} \widetilde{\Pi}_h Q_0 \right\|_{C([0,\tau];X)} = 0;$
- $\lim_{h\to 0} \left\| \mathbb{S}_t Q_0 - \mathbb{S}_{h,t} \widetilde{\Pi}_h Q_0 \right\|_{C([0,\tau];X)} = 0;$
- $\lim_{h\to 0} \left\| \mathbb{L}_t Q_0 - \mathbb{L}_{h,t} \widetilde{\Pi}_h Q_0 \right\|_{X} = 0.$

Proof. For simplicity, we prove only the first assertion, the other ones being similar. Let $(Q_{0h})_{h>0}$ be the family constructed in Lemma 4.5. Since $Q_{0h} \in H_{\frac{3}{2}} \times H_1$, from Corollary 4.2 it follows that

$$\left\| \mathbb{T}_t Q_{0h} - \mathbb{T}_{h,t} \Pi_h Q_{0h} \right\|_X \leqslant C(\tau) h^\theta \left\| Q_{0h} \right\|_{H_{\frac{3}{2}} \times H_1} \quad (t \in [0,\tau]). \tag{4.22}$$

By using (4.22), the fact that $\|\mathbb{T}_{h,t}\|_{\mathcal{L}(X_h)} \leqslant 1$ and $\|\mathbb{T}_t\|_{\mathcal{L}(X)} \leqslant 1$, we deduce that

$$\begin{aligned} \left\| \mathbb{T}_t Q_0 - \mathbb{T}_{h,t} \widetilde{\Pi}_h Q_0 \right\|_X &\leqslant \left\| \mathbb{T}_t \left(Q_0 - Q_{0h} \right) \right\|_X + \left\| \mathbb{T}_{h,t} \left(\Pi_h Q_{0h} - \widetilde{\Pi}_h Q_0 \right) \right\|_X + \left\| \mathbb{T}_t Q_{0h} - \mathbb{T}_{h,t} \Pi_h Q_{0h} \right\|_X \\ &\leqslant \left\| Q_0 - Q_{0h} \right\|_X + \left\| \Pi_h Q_{0h} - \widetilde{\Pi}_h Q_0 \right\|_X + C(\tau) h^\theta \left\| Q_{0h} \right\|_{H_{\frac{3}{2}} \times H_1}. \end{aligned} \tag{4.23}$$

In order to evaluate the second right-hand term in (4.23), using (3.1)−(3.2) and (3.3) we get the existence of a constant $C_0 > 0$ such that

$$\begin{aligned} \left\| \widetilde{\Pi}_h Q_0 - \Pi_h Q_{0h} \right\|_X &\leqslant \left\| \widetilde{\Pi}_h Q_0 - \widetilde{\Pi}_h Q_{0h} \right\|_X + \left\| \widetilde{\Pi}_h Q_{0h} - \widetilde{\Pi}_h \Pi_h Q_{0h} \right\|_X + \left\| \widetilde{\Pi}_h \Pi_h Q_{0h} - \Pi_h Q_{0h} \right\|_X \\ &\leqslant \left\| Q_0 - Q_{0h} \right\|_X + \left\| Q_{0h} - \Pi_h Q_{0h} \right\|_X + \left\| \widetilde{\Pi}_h \Pi_h Q_{0h} - \Pi_h Q_{0h} \right\|_X \\ &\leqslant \left\| Q_0 - Q_{0h} \right\|_X + 2C_0 h^\theta \left\| Q_{0h} \right\|_{H_{\frac{3}{2}} \times H_1}. \end{aligned} \tag{4.24}$$

Finally, combining (4.23)−(4.24) and using Lemma 4.5, the first assertion is proved. □

5. Convergence of the approximate controls

In this section we show the convergence of the approximate controls u_h given by (3.17) to the exact control u for (2.2)−(2.3) introduced in (2.14). We analyze separately two different cases depending on the regularity assumptions for the initial data Q_0 to be controlled. Let us begin with the case of more regularly initial data.

Theorem 5.1. *Suppose that system* (2.2)−(2.3) *is exactly controllable in some time* $\tau_0 > 0$, $B_0 B_0^* \in \mathcal{L}\left(H_1, H_{\frac{1}{2}}\right)$ *and* $Q_0 = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in H_{\frac{3}{2}} \times H_1$. *Moreover, assume that the discrete system with numerical viscosity* (4.1)−(4.2) *is uniformly stable, i.e.* (4.11) *holds, and let* τ_1 *be given such that* (4.12) *is verified. Then for any* $\tau \geqslant \max\{\tau_0, \tau_1\}$

the family $(u_h)_{h>0}$ *from* $C([0,\tau];U_h)$*, defined by* (3.17) *in Algorithm 1 with* $N(h)=\left[\frac{\theta}{\ln\alpha^{-1}}\ln(h^{-1})\right]$*, converges when* $h\to 0$ *to the exact control* u *in time* τ *of* (2.2)−(2.3) *given by* (2.14)*. Moreover, there exist positive constants* h^* *and* C_τ *such that we have*

$$\|u-u_h\|_{C([0,\tau];U)}\leqslant C_\tau h^\theta\,\|Q_0\|_{H_{\frac{3}{2}}\times H_1}\qquad (0<h<h^*)\,. \tag{5.1}$$

Proof. Firstly, let us remark that u_h given by (3.17) can be written as

$$u_h(t)=\mathcal{B}_h^*\mathbb{T}_{h,t}\begin{bmatrix}w_{0h}\\ w_{1h}\end{bmatrix}+\mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\begin{bmatrix}w_{0h}\\ w_{1h}\end{bmatrix}\qquad (t\in[0,\tau])\,, \tag{5.2}$$

where

$$W_{0h}=\begin{bmatrix}w_{0h}\\ w_{1h}\end{bmatrix}=\sum_{n=0}^{N(h)}L_{h,\tau}^n\Pi_h\begin{bmatrix}q_0\\ q_1\end{bmatrix}. \tag{5.3}$$

We define $v_h:[0,\tau]\to U_h$ as follows

$$v_h(t)=\mathcal{B}_h^*\mathbb{T}_{h,t}\Pi_hW_0+\mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_hW_0\qquad (t\in[0,\tau])\,, \tag{5.4}$$

where $W_0=\begin{bmatrix}w_0\\ w_1\end{bmatrix}$ is given by (2.15). From (2.14), (5.4) and (3.7) we have

$$\begin{aligned}\|(u-v_h)(t)\|_U&=\|\mathcal{B}^*\mathbb{T}_tW_0+\mathcal{B}^*\mathbb{S}_{\tau-t}\mathbb{T}_\tau W_0-\mathcal{B}_h^*\mathbb{T}_{h,t}\Pi_hW_0-\mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_hW_0\|_U\\ &\leqslant\left\|\mathcal{B}^*\left(\mathbb{T}_tW_0-\widetilde{\Pi}_h\mathbb{T}_{h,t}\Pi_hW_0\right)\right\|_U+\left\|\mathcal{B}^*\left(\mathbb{S}_{\tau-t}\mathbb{T}_\tau W_0-\widetilde{\Pi}_h\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_hW_0\right)\right\|_U\\ &\leqslant C\left(\|\mathbb{T}_tW_0-\widetilde{\Pi}_h\mathbb{T}_{h,t}\Pi_hW_0\|_X+\|\mathbb{S}_{\tau-t}\mathbb{T}_\tau W_0-\widetilde{\Pi}_h\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_hW_0\|_X\right)\\ &\leqslant C\left(\left\|\mathbb{T}_tW_0-\widetilde{\Pi}_h\mathbb{T}_tW_0\right\|_X+\|\mathbb{T}_tW_0-\mathbb{T}_{h,t}\Pi_hW_0\|_X\right)\\ &+C\left(\left\|\mathbb{S}_{\tau-t}\mathbb{T}_\tau W_0-\widetilde{\Pi}_h\mathbb{S}_{\tau-t}\mathbb{T}_\tau W_0\right\|_X+\|\mathbb{S}_{\tau-t}\mathbb{T}_\tau W_0-\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_hW_0\|_X\right).\end{aligned}$$

By using (3.1), (3.3), (2.15) and Corollary 4.2, from the above estimate we deduce that there exist $h_1^*>0$ and $C_\tau^1>0$ such that for any $h\in(0,h_1^*)$ we have

$$\|u-v_h\|_{C([0,\tau];U)}\leqslant C_\tau^1h^\theta\,\|Q_0\|_{H_{\frac{3}{2}}\times H_1}\qquad (h\in(0,h_1^*)). \tag{5.5}$$

On the other hand we have that

$$\begin{aligned}\|(v_h-u_h)(t)\|_U&=\|\mathcal{B}_h^*\mathbb{T}_{h,t}W_{0h}+\mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}W_{0h}-\mathcal{B}_h^*\mathbb{T}_{h,t}\Pi_hW_0-\mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_hW_0\|_U\\ &\leqslant\|\mathcal{B}_h^*\mathbb{T}_{h,t}\left(W_{0h}-\Pi_hW_0\right)\|_U+\|\mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\left(W_{0,h}-\Pi_hW_0\right)\|_U\\ &\leqslant C\left(\|\mathbb{T}_{h,t}\left(W_{0h}-\Pi_hW_0\right)\|_X+\|\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\left(W_{0,h}-\Pi_hW_0\right)\|_X\right)\\ &\leqslant C\,\|W_{0h}-\Pi_hW_0\|_X\leqslant C\,\|W_{0h}-W_0\|_X\,,\end{aligned}$$

where for the last estimates we have used that $\|\mathbb{T}_{h,t}\|_{\mathcal{L}(X_h)} \leqslant 1$, $\|\mathbb{S}_{h,\tau-t}\|_{\mathcal{L}(X_h)} \leqslant 1$ and $\|\Pi_h\|_{\mathcal{L}\left(H_{\frac{1}{2}}\times H_{\frac{1}{2}},X_h\right)} \leqslant 1$. Consequently, we deduce that

$$\|u_h - v_h\|_{C([0,\tau];U)} \leqslant C\,\|W_{0h} - W_0\|_X\,. \tag{5.6}$$

We estimate the right-hand side of (5.6) as follows

$$\begin{aligned}||W_0 - W_{0h}||_X &= \left|\left|\sum_{n=0}^{\infty} L_\tau^n Q_0 - \sum_{n=0}^{N(h)} L_{h,\tau}^n \Pi_h Q_0\right|\right|_X \\ &\leqslant \sum_{n=N(h)+1}^{\infty} \|L_\tau\|_{\mathcal{L}(\mathcal{X})}^n \|Q_0\|_X + \sum_{n=0}^{N(h)} \|(L_\tau^n - L_{h,\tau}^n \Pi_h)Q_0\|_X.\end{aligned}$$

The above estimate and Corollary 4.3 imply that there exists h_2^* such that for any $h \in (0, h_2^*)$ the following inequalities are verified

$$\begin{aligned}||W_0 - W_{0h}||_X &\leqslant \left(\frac{\|L_\tau\|_{\mathcal{L}(\mathcal{X})}^{N(h)+1}}{1 - \|L_\tau\|_{\mathcal{L}(\mathcal{X})}} + \sum_{n=0}^{N(h)} \alpha^{n-1}\left(C_0\alpha + nK_1\tau\right)h^\theta\right)\|Q_0\|_{H_{\frac{3}{2}}\times H_1} \\ &\leqslant C_\tau^2\left(\|L_\tau\|_{\mathcal{L}(\mathcal{X})}^{N(h)+1} + h^\theta\right)\|Q_0\|_{H_{\frac{3}{2}}\times H_1} \leqslant C_\tau^2(\alpha^{N(h)+1} + h^\theta)\|Q_0\|_{H_{\frac{3}{2}}\times H_1}.\end{aligned}$$

Notice that the existence of a number $\alpha \in (0,1)$ independent of h (see (4.10) and (4.12)) allows us to use in the above estimates that

$$\sum_{n=0}^{\infty} \alpha^{n-1}\left(C_0\alpha + nK_1\tau\right) < C,$$

where C is a constant independent of h. By choosing $N(h) = \left[\frac{\theta}{\ln \alpha^{-1}}\ln(h^{-1})\right]$, we obtain

$$||W_0 - W_{0h}||_X \leqslant 2C_\tau^2 h^\theta\,\|Q_0\|_{H_{\frac{3}{2}}\times H_1}\,. \tag{5.7}$$

From (5.5) and (5.7) we obtain that (5.1) holds and the proof is complete. □

Remark 5.2. Estimate (5.1) from Theorem 5.1 shows that we can approximate the continuous control with an error bounded by h^θ, which is the error of the numerical scheme. This result is a consequence of the numerical viscosity added in the discrete equation, the error obtained in [17] where no viscosity is used, being bounded by the larger term $\ln\left(\frac{1}{h}\right)\,h^\theta$.

Notice that u_h given by (5.2) represents in fact an approximate control for the discrete equation. Theorem 5.1 allows us to estimate the norm of the solution of the controlled discrete equation at time τ. More precisely, we have the following result.

Corollary 5.3. *For each $h \in (0, h^*)$, let u_h be the discrete control given by Theorem 5.1 corresponding to the initial data $Q_0 = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in H_{\frac{3}{2}} \times H_1$, and let $(q_h, \dot{q}_h)$ solution of the equation*

$$\ddot{q}_h(t) + A_{0h}q_h(t) + B_{0h}u_h(t) = 0, \tag{5.8}$$

$$q_h(0) = \pi_h q_0, \dot{q}_h(0) = \pi_h q_1. \tag{5.9}$$

There exists a positive constant $C > 0$ independent of h such that we have

$$\|(q_h(\tau), \dot{q}_h(\tau))\|_X \leqslant Ch^\theta \|Q_0\|_{H_{\frac{3}{2}} \times H_1} \qquad (0 < h < h^*). \tag{5.10}$$

Proof. Let $\begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ be the controlled solution of (2.2)−(2.3) with the exact control given by (2.14). Since $B_0 B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and $\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in H_{\frac{3}{2}} \times H_1$ (see Rem. 2.3), from (2.14)−(2.15) we deduce that $B_0 u \in C([0,\tau]; H_{\frac{1}{2}})$ and

$$\|B_0 u\|_{C\left([0,\tau]; H_{\frac{1}{2}}\right)} \leqslant C \|Q_0\|_{H_{\frac{3}{2}} \times H_1}. \tag{5.11}$$

From (2.2)−(2.3) and (5.8)−(5.9) we have that the function $z_h = \pi_h q - q_h$ verifies

$$\ddot{z}_h(t) + A_{0h} z_h(t) + \pi_h B_0 u(t) - B_{0h} u_h(t) = 0, \tag{5.12}$$

$$z_h(0) = \dot{z}_h(0) = 0. \tag{5.13}$$

If we define

$$\xi_h(t) = \frac{1}{2} \|z_h(t)\|_{\frac{1}{2}}^2 + \frac{1}{2} \|\dot{z}_h(t)\|^2,$$

from (5.12)−(5.13) and (5.11) we get

$$\begin{aligned} \dot{\xi}_h(t) &= \langle \ddot{z}_h(t), \dot{z}_h(t) \rangle + \langle A_{0h} z_h(t), \dot{z}_h(t) \rangle = -\langle \pi_h B_0 u(t) - B_{0h} u_h(t), \dot{z}_h(t) \rangle \\ &\leqslant \sqrt{2} \|(\pi_h - \widetilde{\pi}_h) B_0 u(t) + B_{0h}(u(t) - u_h(t))\| \xi_h^{\frac{1}{2}}(t) \\ &\leqslant C \left(h^\theta \|Q_0\|_{H_{\frac{3}{2}} \times H_1} + \|u(t) - u_h(t)\|_U \right) \xi_h^{\frac{1}{2}}(t). \end{aligned}$$

Using the above estimate and Theorem 5.1, we get

$$\dot{\xi}_h(t) \leqslant Ch^\theta \|Q_0\|_{H_{\frac{3}{2}} \times H_1} \xi_h^{\frac{1}{2}}(t).$$

Hence, we have that

$$\xi_h^{\frac{1}{2}}(\tau) \leqslant C\tau h^\theta \|Q_0\|_{H_{\frac{3}{2}} \times H_1}. \tag{5.14}$$

Taking into account that u is an exact control for (2.2)−(2.3), we obtain

$$\|(q_h(\tau), \dot{q}_h(\tau))\|_X = \|\Pi_h (q(\tau), \dot{q}(\tau)) - (q_h(\tau), \dot{q}_h(\tau))\|_X = \sqrt{2} \xi_h^{\frac{1}{2}}(\tau).$$

Finally, (5.10) follows immediately from the last relation and (5.14). □

We pass to study the case of initial data Q_0 belonging to the space of finite energy X. We do not expect to obtain error estimates as in Theorem 5.1 but we shall be able to prove the convergence of the family $(u_h)_{h>0}$.

Theorem 5.4. *Suppose that system (2.2)−(2.3) is exactly controllable in some time $\tau_0 > 0$, $B_0B_0^* \in \mathcal{L}\left(H_1, H_{\frac{1}{2}}\right)$ and (4.11) holds. Let τ_1 be given in (4.12), $\tau \geqslant \max\{\tau_0, \tau_1\}$ and $Q_0 = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in H_{\frac{1}{2}} \times H$. Then, for any nondecreasing family $(N(h))_{h>0} \subset \mathbb{N}^*$ such that*

$$\lim_{h\to 0} N(h) = \infty, \tag{5.15}$$

the family $(u_h)_{h>0}$ defined by (3.17) in Algorithm 1 converges when $h \to 0$ to the exact control u in time τ of (2.2)−(2.3) given by (2.14).

Proof. Let $\varepsilon > 0$ and, as in Corollary 4.3, let $\alpha = \max\{\|L_\tau\|_{\mathcal{L}(H_{\frac{3}{2}}\times H_1)}, \|L_{h,\tau}\|_{\mathcal{L}(X_h)}\} \in (0,1)$. Moreover, let N_ε with the property

$$\sum_{n=N_\varepsilon+1}^{\infty} \alpha^n \|Q_0\|_X \leqslant \varepsilon. \tag{5.16}$$

From (5.15) if follows that there exists $h_1^* > 0$ such that $N_\varepsilon < N(h)$ for any $h < h_1^*$.

As in the first part of the proof of Theorem 5.1, u_h will be given by (5.2), where this time W_{0h} is defined by

$$W_{0h} = \begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix} = \sum_{n=0}^{N(h)} L_{h,\tau}^n \widetilde{\Pi}_h \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}. \tag{5.17}$$

Moreover, we introduce $v_h : [0,\tau] \to U_h$ as follows

$$v_h(t) = \mathcal{B}_h^* \mathbb{T}_{h,t} \widetilde{\Pi}_h W_0 + \mathcal{B}_h^* \mathbb{S}_{h,\tau-t} \mathbb{T}_{h,\tau} \widetilde{\Pi}_h W_0 \qquad (t \in [0,\tau]), \tag{5.18}$$

where $W_0 = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$ is given by (2.15). As in the proof of Theorem 5.1 we obtain that

$$\begin{aligned} \|(u - v_h)(t)\|_U \leqslant\ & C\left(\left\|\mathbb{T}_t W_0 - \widetilde{\Pi}_h \mathbb{T}_t W_0\right\|_X + \left\|\mathbb{T}_t W_0 - \mathbb{T}_{h,t}\widetilde{\Pi}_h W_0\right\|_X\right) \\ & + C\left(\left\|\mathbb{S}_{\tau-t}\mathbb{T}_\tau W_0 - \widetilde{\Pi}_h \mathbb{S}_{\tau-t}\mathbb{T}_\tau W_0\right\|_X + \left\|\mathbb{S}_{\tau-t}\mathbb{T}_\tau W_0 - \mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\widetilde{\Pi}_h W_0\right\|_X\right). \end{aligned}$$

By using (3.4) and Corollary 4.6, from the above estimate we deduce that there exists $h_2^* > 0$ such that

$$\|u - v_h\|_{C([0,\tau];U)} \leqslant \varepsilon \qquad (h \in (0, h_2^*)). \tag{5.19}$$

On the other hand (5.6) holds and we have to estimate $\|W_{0h} - W_0\|_X$. By using (2.14), (2.15), (5.2), (5.3) and (5.4) we have

$$\begin{aligned} \|W_{0h} - W_0\|_X &\leqslant \left\|\sum_{n=0}^{\infty} L_\tau^n Q_0 - \sum_{n=0}^{N(h)} L_{h,\tau}^n \widetilde{\Pi}_h Q_0\right\|_X \\ &= \left\|\sum_{n=N_\varepsilon+1}^{\infty} L_\tau^n Q_0 + \sum_{n=0}^{N_\varepsilon} L_\tau^n Q_0 - \sum_{n=0}^{N_\varepsilon} L_{h,\tau}^n \widetilde{\Pi}_h Q_0 - \sum_{N_\varepsilon+1}^{N(h)} L_{h,\tau}^n \widetilde{\Pi}_h Q_0\right\|_X \end{aligned}$$

$$\leqslant \sum_{n=N_\varepsilon+1}^{\infty} \|L_\tau\|^n_{\mathcal{L}(\mathcal{X})} \|Q_0\|_X + \sum_{n=0}^{N_\varepsilon} \left\| L^n_\tau Q_0 - L^n_{h,\tau} \widetilde{\Pi}_h Q_0 \right\|_X$$
$$+ \sum_{n=N_\varepsilon+1}^{\infty} \|L_{h,\tau}\|^n_{\mathcal{L}(\mathcal{X})} \|Q_0\|_X . \tag{5.20}$$

From the above estimate and (5.16) we deduce that

$$\|W_{0h} - W_0\|_X \leqslant 2\varepsilon + \sum_{n=0}^{N_\varepsilon} \left\| L^n_\tau Q_0 - L^n_{h,\tau} \widetilde{\Pi}_h Q_0 \right\|_X . \tag{5.21}$$

Using Corollary 4.6, it follows that there exists h_3^* such that, for $h \in (0, h_3^*)$, we have that the following inequalities hold

$$\left\| L_\tau \left(L^k_\tau Q_0 \right) - L_{h,\tau} \left(\widetilde{\Pi}_h L^k_\tau Q_0 \right) \right\|_X < \frac{\epsilon}{N_\varepsilon} \qquad (0 \leqslant k \leqslant N_\varepsilon - 1).$$

From the above inequalities we deduce that, for any $1 \leqslant n \leqslant N_\varepsilon$, we have

$$\left\| L^n_\tau Q_0 - L^n_{h,\tau} \widetilde{\Pi}_h Q_0 \right\|_X$$

$$\leqslant \left\| L_\tau \left(L^{n-1}_\tau Q_0 \right) - L_{h,\tau} (\widetilde{\Pi}_h L^{n-1}_\tau Q_0 \right) \right\|_X + \left\| L_{h,\tau} \widetilde{\Pi}_h \left(L_\tau (L^{n-2}_\tau Q_0) - L_{h,\tau} \widetilde{\Pi}_h L^{n-2}_\tau Q_0 \right) \right\|_X$$

$$+ \left\| L^2_{h,\tau} \widetilde{\Pi}_h \left(L_\tau (L^{n-3}_\tau Q_0) - L_{h,\tau} \widetilde{\Pi}_h L^{n-3}_\tau Q_0 \right) \right\|_X + \ldots + \left\| L^{n-1}_{h,\tau} \widetilde{\Pi}_h \left(L_\tau Q_0 - L_{h,\tau} \widetilde{\Pi}_h Q_0 \right) \right\|_X$$

$$\leqslant \frac{\varepsilon}{N_\varepsilon} \left(1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} \right) \leqslant \frac{\epsilon}{N_\varepsilon (1-\alpha)}.$$

It follows that

$$\sum_{n=0}^{N_\varepsilon} \left\| L^n_\tau Q_0 - L^n_{h,\tau} \widetilde{\Pi}_h Q_0 \right\|_X \leqslant \frac{\varepsilon}{1-\alpha} + \left\| Q_0 - \widetilde{\Pi}_h Q_0 \right\| . \tag{5.22}$$

By taking into account (3.4), (5.20)−(5.22) and (5.6), we deduce that

$$\lim_{h \to 0} \|u_h - v_h\|_{C([0,\tau];U)} = 0. \tag{5.23}$$

Finally, combining (5.19) and (5.23), we have proved that the family $(u_h)_{h>0}$ converges in $C([0,\tau];U)$ to u, given by (2.14), which is an exact control in time τ of (2.2)−(2.3). □

Remark 5.5. Notice that the viscosity term $\zeta h^\eta A_{0h} \dot{w}^n_h(t)$ in systems (3.9)−(3.12) guarantees that $\alpha < 1$, uniformly with respect with the parameter h, which is an essential property in the proof of Theorem 5.4, as it can be seen in estimate (5.20). The convergence of the algorithm in the absence of the viscosity term remains an open problem.

Remark 5.6. In Theorems 5.1 and 5.4 we have considered a time τ different than the optimal control time τ_0 of the continuous equation. Indeed, τ should be greater and equal to τ_0 and, also, τ_1 given by (4.12). If τ_1 can be chosen equal to τ_0, as it is the case in the continuous equation (see [29], Lem. 2.2), is an interesting open question.

6. Numerical experiments

The aim of this section is to numerically illustrate the results obtained in the previous sections for the wave equation in dimension one and two in space and for Euler-Bernoulli beam equation. In this purpose we approximate these equations by finite elements in space and by a Newmark scheme of parameters $\beta = 0.25$ et $\gamma = 0.5$ in time. We also choose the parameter $N(h)$ appearing in (3.16) as the smallest positive integer for which we have

$$\left(\|w_{b,h}^{N(h)}(0)\|^2_{H_{\frac{1}{2}}} + \|\dot{w}_{b,h}^{N(h)}(0)\|^2_{H}\right)^{\frac{1}{2}} \leq h^{\theta}. \tag{6.1}$$

Remark 6.1. The relation (6.1) is justified by the following estimates:

$$\left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} - \sum_{n=0}^{N(h)} L_\tau^n \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \right\|_X \leqslant \sum_{k=1}^{\infty} \left\| L_\tau^k \begin{bmatrix} w_b^{N(h)}(0) \\ \dot{w}_b^{N(h)}(0) \end{bmatrix} \right\|_X \leqslant \frac{1}{1-\alpha} \left\| \begin{bmatrix} w_b^{N(h)}(0) \\ \dot{w}_b^{N(h)}(0) \end{bmatrix} \right\|_X . \tag{6.2}$$

6.1. One dimensional wave equation

We consider the following wave equation:

$$\begin{cases} \ddot{q}(t,x) - \partial_x^2 q(t,x) + \chi_{a,b}(x)u(t,x) = 0, & (t,x) \in (0,T) \times (0,1) \\ q(t,0) = q(t,1) = 0, & t \in (0,T) \\ q(0,x) = q_0(x), \quad \dot{q}(0,x) = q_1(x), \qquad x \in (0,1), \end{cases} \tag{6.3}$$

with the distributed control u. For every $0 \leq a < b \leq 1$ we denote by $\chi_{a,b}$ the C^∞ function given by

$$\chi_{a,b}(x) = \begin{cases} 0, & x \in [0,a] \cup [b,1] \\ 1, & x \in [a+\delta, b-\delta] \\ \exp\left(\alpha\left(\frac{1}{\delta^2} - \frac{1}{(x-a)(a+2\delta-x)}\right)\right), & x \in (a, a+\delta) \\ \exp\left(\alpha\left(\frac{1}{\delta^2} - \frac{1}{(x-b+2\delta)(b-x)}\right)\right), & x \in (b-\delta, b) \end{cases} \tag{6.4}$$

and α and δ are positive numbers to choose later. For the remaining part of this section we choose $a = 0.1$, $b = 0.5$, $\delta = 0.1$ and $\alpha = 0.02$. The final time in which we want to control to zero the solution of (6.3) is $T = 3$. The function $\chi_{a,b}$ corresponding to these parameters is displayed in Figure 1.

This is easy to see that wave equation (6.3) can be written using the formalism and notation in Section 2. More exactly, let $H = L^2(0,1)$, $H_1 = H^2(0,1) \cap H_0^1(0,1)$ and $A_0 : H_1 \to H$ be defined by $A_0\varphi = -\varphi''$ for every $\varphi \in H_1$. Therefore, the space $H_{\frac{1}{2}}$ is given by $H_{\frac{1}{2}} = H_0^1(0,1)$ and

$$H_{\frac{3}{2}} = \left\{\varphi \in H^3(0,1) \cap H_0^1(0,1) \mid \varphi_{xx}(0) = \varphi_{xx}(1) = 0\right\}.$$

For the remaining part of this section, we also set $U = L^2(a,b)$ and for every $\varphi \in U$ we define $B_0(\varphi) = \chi_{a,b}\varphi \in H$, with $\chi_{a,b}$ defined by (6.4).

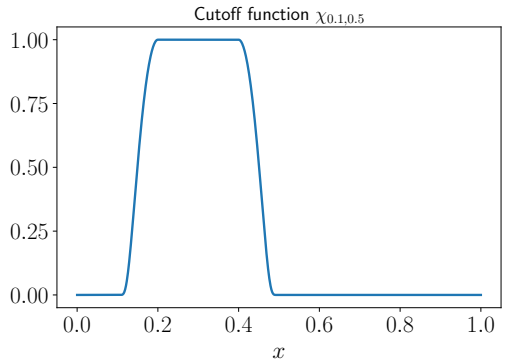


FIGURE 1. Function $\chi_{0.1,0.5}$ with $\delta = 0.1$ and $\alpha = 0.02$.

Since the Newmark scheme used to discretize in time the equation is of order two we choose to discretize in space the wave equation using P_2 finite elements. In this purpose we consider $N+1$ points $x_i = ih$ with $0 \leq i \leq N$ and $h = 1/N$. For every N (and, therefore, for the corresponding h) we define a subspace V_h of $H_{\frac{1}{2}}$ as follows:

$$V_h = \{\varphi \in H_{\frac{1}{2}} \text{ such that } \varphi\,|_{[x_i,x_{i+1}]} \in \mathbb{R}^2[x] \text{ for every } 0 \leq i < N\}.$$

For this choice of V_h the estimate (3.3) holds with $\theta = 2$. For what follows we set $\eta = \theta = 2$. We chose the initial data to control $(q_0, q_1) \in H_{\frac{3}{2}} \times H_{\frac{1}{2}}$ displayed in Figure 2a. More precisely, these initial data have the limit regularity required by Theorem 5.1, which is $(q_0, q_1) \in \left(H_{\frac{3}{2}} \times H_{\frac{1}{2}}\right) \setminus (H_2 \times H_1)$, and are given by

$$\begin{aligned} q_0(x) = &\left(-\frac{7664}{3}x^7 - \frac{13412}{3}x^6 - 2682x^5 + \frac{1675}{3}x^4 + \frac{1}{6}x^3\right)\mathbb{1}_{[0,\frac{1}{2}]}(x) \\ &+ \left(\frac{7664}{3}x^7 - \frac{13412}{3}x^6 + 29506x^5 - 35205x^4 + \frac{29173}{2}x^3 - \frac{20115}{2}x^2 + \frac{13409}{6}x - \frac{415}{2}\right)\mathbb{1}_{[\frac{1}{2},1]}(x) \end{aligned}$$

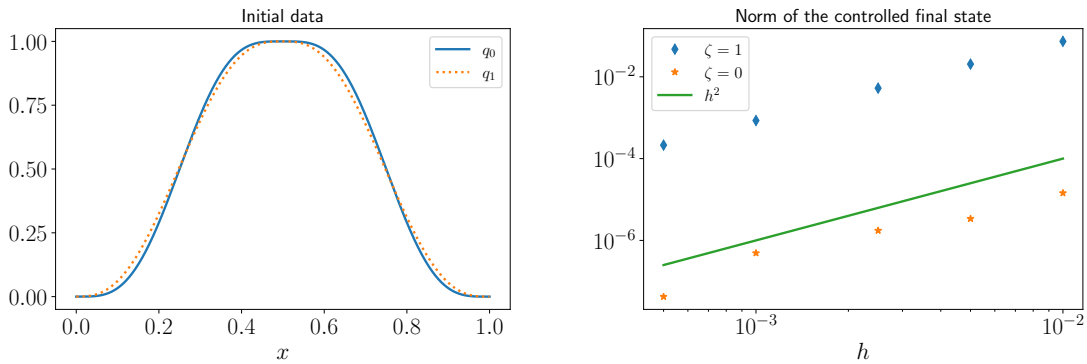
$$\begin{aligned} q_1(x) = &\left(192x^5 - 238x^4 + 78x^3 + \frac{1}{2}x^2\right)\mathbb{1}_{[0,\frac{1}{2}]}(x) \\ &+ \left(-192x^5 + 718x^4 - 1034x^3 + \frac{1427}{2}x^2 - 237x + \frac{63}{2}\right)\mathbb{1}_{[\frac{1}{2},1]}(x). \end{aligned}$$

We consider several values of h (listed in Tab. 1) and we take the discretization step in time $\Delta t = h$. The number of iterations needed to fulfill the criteria (6.1) is also reported in Table 1 for $\zeta \in \{0, 1\}$. We observe that the number of iterations is very similar regardless $\zeta = 0$ or $\zeta = 1$. As expected, $N(h)$ increases when h goes to zero.

For every value of h and for $\zeta \in \{0, 1\}$ we numerically compute a control using the proposed method. The evolution with respect to h of the norm in $H_{\frac{1}{2}} \times H$ of the corresponding controlled solution at the time T is displayed in Figure 2b. We observe a complete agreement with our theoretical results, for both $\zeta = 0$ and $\zeta = 1$ this norm behaving as h^θ. Nevertheless, for $\zeta = 1$ this norm is larger. In Figure 3 we display the contour map of the control obtained for $h = 5 \times 10^{-4}$, $\zeta = 0$ (left) and $\zeta = 1$ (right).

TABLE 1. Number of iterations $N(h)$ needed to fulfill (6.1) for the one-dimensional wave equation (6.3), initial data in Figure 2a, for different values of h and $\zeta \in \{0, 1\}$.

	$h = \frac{1}{100}$	$h = \frac{1}{200}$	$h = \frac{1}{400}$	$h = \frac{1}{1000}$	$h = \frac{1}{2000}$
$\zeta = 0$	15	17	19	21	24
$\zeta = 1$	14	16	18	21	24



(a) (b)

FIGURE 2. (a) Initial data $(q_0, q_1) \in H_{\frac{3}{2}} \times H_{\frac{1}{2}}$ for the one dimensional wave equation (6.3). (b) Norm of the final state of the corresponding final state equation for different values of h and $\zeta \in \{0, 1\}$.

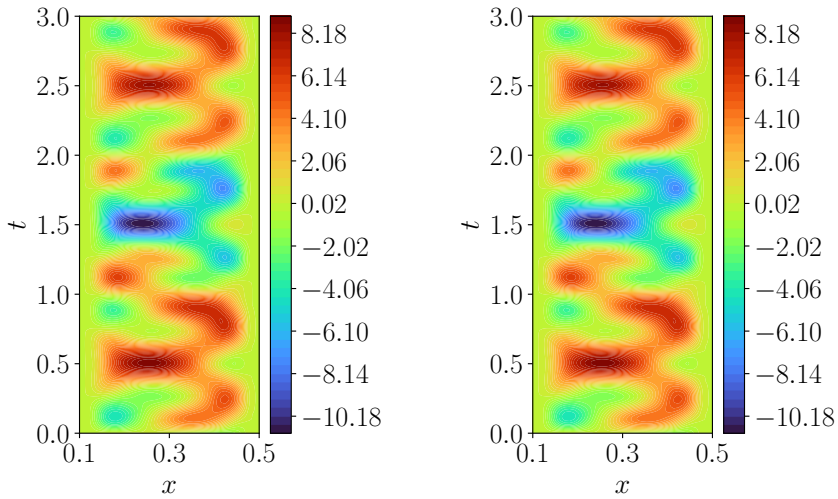


FIGURE 3. Control computed for $h = 5 \times 10^{-4}$, initial data in Figure 2, $\zeta = 0$ (left) and $\zeta = 1$ (right).

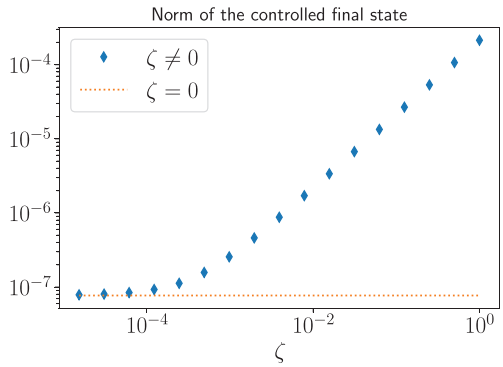


FIGURE 4. Norm of the controlled final state corresponding to initial data in Figure, $h = \frac{1}{2000}$ with respect to ζ.

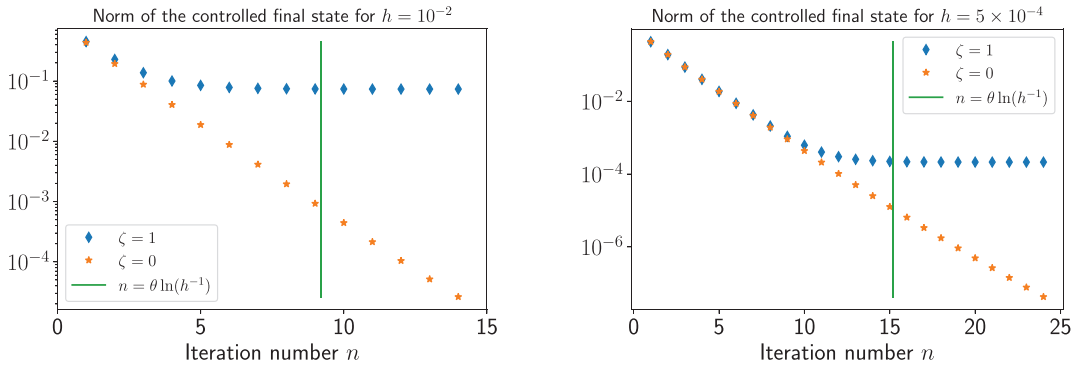


FIGURE 5. Norm of the controlled final state of the wave equation with initial data in Figure 2 and the control computed using n iterations.

In order to illustrate how the norm of the controlled final state depends on ζ we compute the control and the corresponding controlled solution for $\zeta \in \{2^{-i},\ 0 \leq i \leq 16\}$. In Figure 4 we observe that this norm decreases with a rate of the form $C\zeta$ until it reaches the norm of the final state of the controlled solution for $\zeta = 1$.

For better understanding the effect of the numerical viscosity term and the choice of $N(h)$, for each iteration in Algorithm 1 we computed the corresponding control. The evolution of norm of the final state of the corresponding controlled solution with respect to the number of iterations is displayed in Figure 5 in the case where $h = 10^{-2}$ and $h = 5 \times 10^{-4}$. For both choices of h we observe that after $n = \theta \ln(h^{-1})$ iterations the norm of the final state of the controlled solution remains constant when $\zeta = 1$. This is in perfect agreement with Theorem 5.1.

6.2. One-dimensional wave equation with initial data in $H_{\frac{1}{2}} \times H$

We consider here un intial data which is only in $H_{\frac{1}{2}} \times H$. More exactly, we take

$$q_0(x) = \begin{cases} 1 - 10|x - 0.6| & \text{for } x \in [0.5, 0.7] \\ 0 & \text{otherwise} \end{cases} \qquad q_1(x) = -\mathbb{1}_{(0.5,0.6]}(x) + \mathbb{1}_{(0.6,0.7)}(x). \tag{6.5}$$

The same numerical experiments as in Section 6.1 were done for this less regular initial data. We observe in Table 2 that Algorithm 1 does not coverge any more when $\zeta = 0$. For $\zeta = 1$ we observe the number of iterations needed for the converge of the Algorithm behaves similarly to the case studied in the previous section. The

TABLE 2. Number of iterations $N(h)$ needed to fulfill (6.1) for the one-dimensional wave equation (6.3), initial data in Figure 2a, for different values of h and $\zeta \in \{0, 1\}$.

	$h = \frac{1}{100}$	$h = \frac{1}{200}$	$h = \frac{1}{400}$	$h = \frac{1}{1000}$	$h = \frac{1}{2000}$
$\zeta = 0$	19	26	29	27	37
$\zeta = 1$	-	-	-	-	-

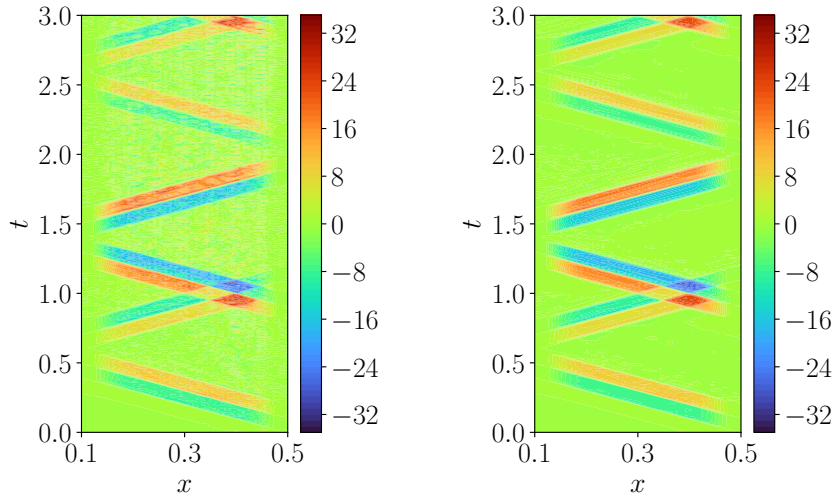


FIGURE 6. Control computed for $h = 5 \times 10^{-4}$, initial data (6.5), $\zeta = 0$ (left) and $\zeta = 1$ (right).

control obtained once the Algorithm converged or when the maximal number of iterations was reached is displayed in Figure 6. As exepected, the control obtained for $\zeta = 0$ has more high frequency oscilations than the control computed for $\zeta = 1$.

The evolution of norm of the controlled solution at the time $T = 3$ with respect to the discretization parameter h is illustrated in Figure 7b.

6.3. Two-dimensional wave equation

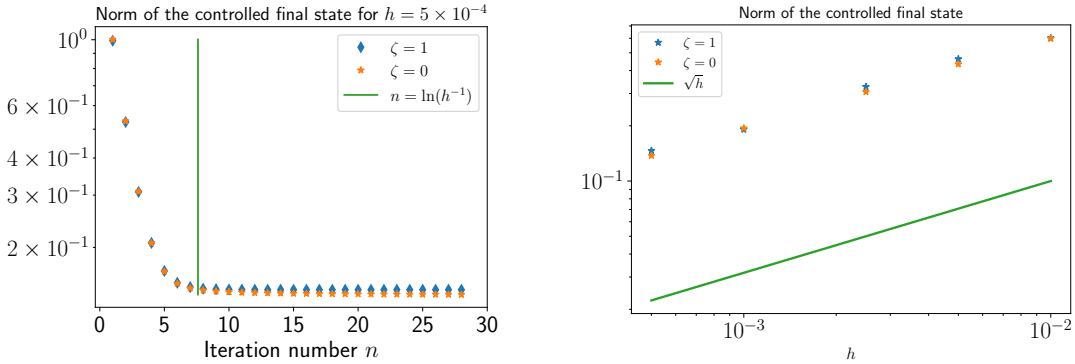
We consider the following two-dimensional wave equation:

$$\begin{cases} \ddot{q}(t,\boldsymbol{x}) - \Delta q(t,\boldsymbol{x}) + \chi_\omega(\boldsymbol{x}) u(t,\boldsymbol{x}) = 0, & (t,\boldsymbol{x}) \in (0,T) \times \Omega \\ q(t,\boldsymbol{x}) = 0, & (t,\boldsymbol{x}) \in (0,T) \times \partial\Omega \\ q(0,\boldsymbol{x}) = q_0(\boldsymbol{x}), \quad \dot{q}(0,\boldsymbol{x}) = q_1(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \end{cases} \tag{6.6}$$

with $\Omega = (0,1)^2$ and $\omega \subset \Omega$. In Figure 8a we represent a quadrangular $\mathcal{Q}_h$ mesh of Ω with the elements in ω colored in orange. The cutoff function χ_Ω is a regular function supported in ω with values between 0 and 1. This function is represented in Figure 8b.

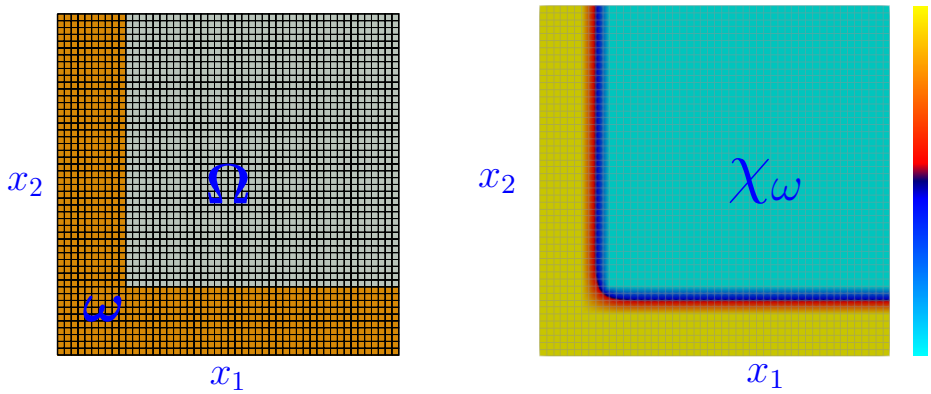
We employ quadratic elements in space and the Newmark scheme in time. More exactly, $V_h \subset H^{\frac{1}{2}}$ is defined by

$$V_h = \left\{ \varphi \in H_{\frac{1}{2}} \text{ such that } \varphi|_Q \in \mathbb{R}^2[x_1, x_2] \text{ for every } Q \in \mathcal{Q}_h \right\}.$$



(a) (b)

FIGURE 7. (a) Norm of the controlled solution at the time T controlled by the partial control obtained at each iteration of Algorithm 1 corresponding to initial data (6.5), $h = 5 \times 10^{-4}$ and $\zeta \in \{0, 1\}$. (b) Norm of the controlled solution at the time T controlled by the control obtained at the end of Algorithm 1 corresponding to initial data (6.5) and for different values of h and $\zeta \in \{0, 1\}$.



(a) (b)

FIGURE 8. (a) A quadrangular regular mesh $\mathcal{Q}_h$ of the unit square $\Omega = (0, 1)^2$ formed by 2500 square elements ($h = \frac{1}{50}$). (b) Level curves of the cutoff function χ_ω.

For this section we choose the following regular initial data:

$$u_0(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2), \quad u_1(\boldsymbol{x}) = 0, \qquad \text{for } \boldsymbol{x} = (x_1, x_2) \in \Omega. \tag{6.7}$$

We consider three meshes $\mathcal{Q}_h$ of Ω (h given in the first row of Tab. 3) and we take the discretization step in time $\Delta t = 10^{-2}$. The number of iterations needed to fulfill the criteria (6.1) is also reported in Table 3 for $\zeta \in \{0, 1\}$. We observe that the number of iterations is smaller for $\zeta = 0$ for large values of h. As expected, $N(h)$ increases when h goes to zero.

For each of the three meshes $\mathcal{Q}_h$ and for $\zeta \in \{0, 1\}$ we numerically compute a control using the proposed method. The evolution with respect to h of the norm in $H_{\frac{1}{2}} \times H$ of the corresponding controlled solution at the

TABLE 3. Number of iterations $N(h)$ needed to fulfill (6.1) for the two-dimensional wave equation (6.6), initial data (6.7), for different values of h and $\zeta \in \{0,1\}$.

	$h=\frac{1}{25}$	$h=\frac{1}{50}$	$h=\frac{1}{100}$
$\zeta=0$	31	51	66
$\zeta=1$	50	59	69

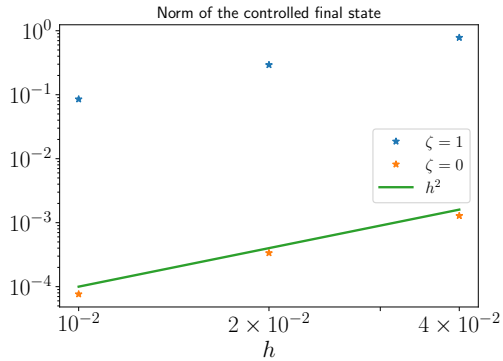


FIGURE 9. Norm of the final state of the controlled solution of the two dimensional wave equation (6.6) with initial data (6.7) for different values of h and $\zeta \in \{0,1\}$.

time $T=3$ is displayed in Figure 9. The results are similar to the one dimensional setting: for both $\zeta=0$ and $\zeta=1$ this norm behaving as h^θ. Nevertheless, for $\zeta=1$ this norm is slightly larger.

6.4. The Euler-Bernoulli beam equation

Another example of equation which can be handled in the framework described in Section 2 is the following Euler-Bernoulli beam equation with distributed control:

$$\begin{cases} \ddot{q}(t,x)+\partial_x^4 q(t,x)+\chi_{a,b}(x)u(t,x)=0, & (t,x)\in(0,T)\times(0,1) \\ q(t,0)=q(t,1)=\partial_x^2 q(t,0)=\partial_x^2 q(t,1)=0, & t\in(0,T) \\ q(0,x)=q_0(x), \quad \dot{q}(0,x)=q_1(x), \qquad x\in(0,1), & \end{cases} \tag{6.8}$$

with $\chi_{a,b}$ being the function given by (6.4). We set again $H=L^2(0,1)$. Then, the corresponding H_1 space is given by $H_1=\{\varphi\in H^4(0,1)\cap H_0^1(0,1)$ such that $\varphi''(0)=\varphi''(1)\}$ and $A_0:H_1\to H$ be defined by $A_0\varphi=-\partial_x^4\varphi$ for every $\varphi\in H_1$. Therefore, the space $H_{\frac{1}{2}}$ is given by $H_{\frac{1}{2}}=H^2\cap H_0^1(0,1)$ and

$$H_{\frac{3}{2}}=\left\{\varphi\in H^6(0,1)\cap H_1 \mid \partial_x^4\varphi(0)=\partial_x^4\varphi(1)=0\right\}.$$

As in the case of the wave equation, we set $U=L^2(a,b)$ and for every $\varphi\in U$ we define $B_0(\varphi)=\chi_{a,b}\varphi\in H$, with $\chi_{a,b}$ defined by (6.4).

In order to have a conform approximation, we discretize the equation (6.8) with respect to spatial variable by using Hermite finite elements. More precisely, we consider $N+1$ points $(x_i)_{0\le i\le N}$ equi-distributed in the interval $[0,1]$ and we set $h=1/N$. For every N (and, therefore, for the corresponding h) we define a subspace

TABLE 4. Number of iterations $N(h)$ needed to fulfill (6.1) for the Euler-Bernoulli beam equation (6.8), initial data (6.9), for different values of h and $\zeta \in \{0, 1\}$.

	$h = \frac{1}{100}$	$h = \frac{1}{200}$	$h = \frac{1}{400}$	$h = \frac{1}{1000}$
$\zeta = 0$	51	56	62	70
$\zeta = 1$	51	56	62	70

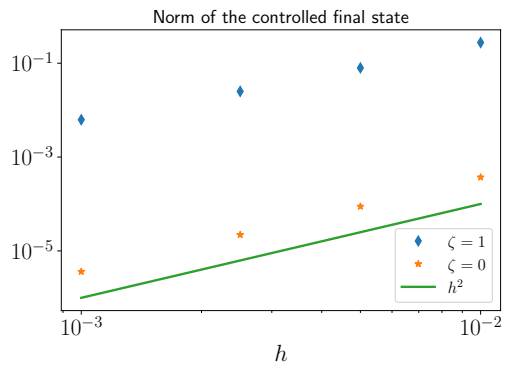


FIGURE 10. Norm of the final state of the controlled beam equation (6.8) for different values of h and $\zeta \in \{0, 1\}$.

V_h of $H_{\frac{1}{2}}$ as follows:

$$V_h = \{\varphi \in H_{\frac{1}{2}} \text{ such that } \varphi\,|_{[x_i, x_{i+1}]} \in \mathbb{R}^3[x] \text{ for every } 0 \leq i < N\}.$$

We recall that a function $\varphi \in V_h$ is uniquely determined by its values in the points x_i and the values of φ' evaluated on the same points. For the discretization in time we choose a discretization step $\Delta t = \frac{h}{10}$. The control u acts in the interval $(0.1, 0.4)$, *i.e.* $a = 0.1$, $b = 0.4$. We consider a very regular initial data

$$y_0(x) = \sin(\pi x), \qquad y_1(x) = 0, \qquad (x \in (0, 1)) \tag{6.9}$$

and a controllability time $T = 1$. We consider several values of h (listed in Tab. 4).

The number of iterations needed to fulfill the criteria (6.1) is also reported in Table 4 for $\zeta \in \{0, 1\}$. We observe, as for the wave equation, that the number of iterations is very similar regardless $\zeta = 0$ or $\zeta = 1$. As expected, $N(h)$ increases when h goes to zero.

For every value of h and for $\zeta \in \{0, 1\}$ we numerically compute a control using the proposed method. The evolution with respect to h of the norm in $H_{\frac{1}{2}} \times H$ of the corresponding controlled solution at the time T is displayed in Figure 10. We observe a complete agreement with our theoretical results, for both $\zeta = 0$ and $\zeta = 1$ this norm behaving as h^θ. Nevertheless, for $\zeta = 1$ this norm is larger.

We remark that, as noted in [30], in this particular case, the uniform exponential decay of the corresponding energy is preserved after discretisation. In fact, this property is illustrated in Figure 10, where no substantial improvement of the convergence rate is observed.

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