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DISPERSIVE PROBLEM AND CONTROL SETS OF LINEAR CONTROL SYSTEMS ON LIE GROUPS

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Abstract. This paper is dedicated to the study of dispersiveness and controllability of linear control systems on Lie groups. Dispersiveness means absence of recursiveness, that is contrary to the existence of control set. A linear control system on a Lie group associates with a derivation operator. For a linear system with stable derivation, it is shown that the system is dispersive if and only if the trajectories through the neutral element have no limit at infinity. As a consequence, a nonempty limit set at the neutral element is a necessary condition for the linear system to admit a control set or to be controllable. In the nondispersive case, the control sets are described by the Lie subgroup of all recurrent points of the automorphism flow of the system. If the derivation operator is asymptotically stable, the central limit set at the neutral element is the unique control set of the system.

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1. Introduction

Dispersiveness is a dynamic property characterized by the absence of recursiveness. A dynamical system is parallelizable if its orbit space forms a trivial bundle, which implies dispersiveness and orbital stability. J. Dugundji and H. Antosiewicz [1] have shown that a dynamical system on a locally compact separable metric space is dispersive if and only if it is parallelizable. O. Hájek [2] extended the Dugundji–Antosiewicz theorem by omitting the separability. These results show that the dispersiveness is suitable for problems concerning the behavior of close trajectories.

The control theoretical model of dispersiveness was introduced in [3, 4]. The prolongational limit criterion was reproduced, assuring that the vacancy of the prolongational limit sets is a necessary and sufficient condition for the dispersiveness. The paper [5] established sufficient conditions for dispersiveness of invariant control systems on the Heisenberg group. In [6], one of the main results shown that a dispersive control affine system has absolutely stable orbits (all-order stable orbits). The paper [7] proved that a control affine system with piecewise constant controls is dispersive if and only if its associated control flow is parallelizable. Studies of dispersiveness became important in control theory due to the fact that dispersive systems are completely uncontrollable, indeed a dispersive control system admits no control set. If controllability is required, as in the most theoretical studies or applications in engineering, it is then important to know any family of dispersive systems. The present

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paper shows sufficient conditions for dispersiveness of linear control systems on Lie groups, exhibiting a class of uncontrollable systems, and necessary conditions for controllability.

The concept of linear control system on Lie group is a generalization of linear system on Euclidean space ([8, 9]). It is determined by a family of differential equations on a connected Lie group G

$$\dot{x} = \mathcal{X}(x) + \sum_{i=1}^{m} u_i(t) X_i(x), \qquad x \in G,$$

with a linear vector field $\mathcal{X}$, invariant vector fields $X_1, ..., X_m$, and piecewise constant control functions valued in a control range $U \subset \mathbb{R}^m$. The vector field $\mathcal{X}$ associates to a derivation $\mathcal{D}$ of the Lie algebra $\mathfrak{g}$. In studies of controllability, the spectral analysis of $\mathcal{D}$ plays a fundamental role (*e.g.* [10–15]). Assuming the condition $0 \in \operatorname{int}(U)$, V. Ayala *et al.* [10] showed that a linear control system on solvable Lie group has only one control set with nonempty interior, and A. Silva [12] proved that the system is controllable, if both the positive and negative orbits of the neutral element are open and the derivation $\mathcal{D}$ has only eigenvalues with zero real part.

In the present paper, we discuss the dispersive problem and control sets of a linear control system on Lie group, by taking into consideration the relationship between dispersiveness and uncontrollability. A fundamental fact about a linear control system is that the neutral element $1 \in G$ is an equilibrium of the linear vector field $\mathcal{X}$. This is a recurrent point of the control system, if $0 \in U$, which implies that the system is not dispersive. Thus, in the case of a linear control system, the dispersive problem makes sense when the control range does not contain the zero. In this case, the control vectors $X_1, ..., X_m$ may either favor the controllability or induce the dispersiveness. We combine the prolongational limit criterion and the spectral analysis of the derivation $\mathcal{D}$ into the discussion of this problem. In the case of $\mathcal{D}$ having stable equilibria, the dynamics of the system depends on the relation between the invariant vector fields X_i and the derivation $\mathcal{D}$. The recurrence set of the automorphism flow $\mathcal{X}_t$, $\mathrm{Re}^+(\mathcal{X})$, is a Lie subgroup of G. The central result shows that the recurrence set of the entire control system $\mathrm{Re}^+(\Sigma)$ coincides with the nonwandering set $\mathrm{NW}(\Sigma)$, and the following decomposition holds:

$$\mathrm{Re}^+(\Sigma) = \mathrm{Re}^+(\mathcal{X})\, \omega_\Sigma^+(1)$$

where $\omega_\Sigma^+(1)$ is the positive limit set at the neutral element. The main result of dispersiveness assures that the linear control system is dispersive if, and only if, $0 \notin U$ and $\omega_\Sigma^+(1)$ is empty. If $0 \in U$, the system is not dispersive, $\mathrm{Re}^+(\mathcal{X}) \subset \mathrm{Re}^+(\Sigma)$, and the control sets are described in the following form:

$$C^+(q) = \left\{x \in \mathrm{Re}^+(\Sigma) : \omega_\Sigma^+(x) = \omega_\Sigma^+(q)\right\}, \qquad q \in \mathrm{Re}^+(\mathcal{X}).$$

In particular, if the derivation $\mathcal{D}$ has only eigenvalues with negative real part, $C^+(1) = \omega_\Sigma^+(1)$ is the unique control set of the linear system.

The paper is organized as follows. Section 2 contains the basic definitions of linear vector fields on Lie groups. In Section 3, we explain the relations of recursive and dispersive concepts in the general framework of linear control systems. Section 4 contains the main results of the paper, where both the dispersive and control problems are simultaneously investigated. In Section 5, we study the special case of linear control system with commutative control vectors. Illustrating examples are exhibited in Section 6.

2. Preliminaries

This section contains the basic notations of linear vector fields on Lie groups. We refer to [8, 9] for the technical information.

Let G be a finite dimensional connected Lie group and $\mathfrak{X}(G)$ the Lie algebra of the C^∞ vector fields on G. The neutral element of G is indicated by 1. The corresponding Lie algebra $\mathfrak{g}$ of G is identified with the Lie algebra of left invariant vector fields in $\mathfrak{X}(G)$. Let $\exp : \mathfrak{g} \to G$ be the exponential map. For a given $X \in \mathfrak{X}(G)$, the associated vector field X^* on $\mathfrak{g}$ satisfies $\mathrm{d}\exp_Y(X^*(Y)) = X(\exp(Y))$, with Y in a neighborhood V of the origin 0 in $\mathfrak{g}$. The exponential map conjugates the flows X_t^* in $\mathfrak{g}$ and X_t in G, indeed $\exp(X_t^*(Y)) = X_t(\exp(Y))$ for all $Y \in V$.

The Lie algebra $\mathfrak{g}$ is often endowed with a Lie bracket $[\cdot,\cdot]$ and an inner product $\langle\cdot,\cdot\rangle$. For a given element $X \in \mathfrak{g}$, the adjoint map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}(X)(Y) = [X,Y]$. The Lie group G is endowed with a right invariant distance d. This means that $\mathrm{d}(xg, yg) = \mathrm{d}(x,y)$ for all $x, y, g \in G$.

The normalizer of $\mathfrak{g}$ is by definition the space

$$\mathrm{norm}_{\mathfrak{X}(G)}(\mathfrak{g}) = \{\mathcal{X} \in \mathfrak{X}(G) : [\mathcal{X}, X] \in \mathfrak{g} \text{ for all } X \in \mathfrak{g}\}$$

where $[\mathcal{X}, X]$ is the Lie bracket of $\mathcal{X}$ and X. A vector field $\mathcal{X}$ on G is said to be **linear** if $\mathcal{X} \in \mathrm{norm}_{\mathfrak{X}(G)}(\mathfrak{g})$ and $\mathcal{X}(1) = 0$.

A vector field $\mathcal{X}$ is linear if, and only if, its flow $\mathcal{X}_t$ define a group of automorphism of G, that is, $\mathcal{X}_t(gh) = \mathcal{X}_t(g)\mathcal{X}_t(h)$, for all $t \in \mathbb{R}$ and $g, h \in G$ ([13], Thm. 1). The associated vector field $\mathcal{X}^*$ on $\mathfrak{g}$ determines a derivation $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ that satisfies $\mathcal{D}(X) = \mathrm{d}\mathcal{X}_1(X) = -[\mathcal{X}, X]$ for all $X \in \mathfrak{g}$. We then have

$$\mathcal{X}_t(\exp(X)) = \exp\left(\mathrm{e}^{t\mathcal{D}}(X)\right), \quad \mathcal{X}(\exp(X)) = \mathrm{d}\exp_X(\mathcal{D}(X)), \quad \mathrm{d}(\mathcal{X}_t)_1(X) = \mathrm{e}^{t\mathcal{D}}(X)$$

for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$. The details for all these properties can be found in [12, 13].

The generalized eigenspaces associated to the derivation $\mathcal{D}$ are given by

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : (\mathcal{D} - \lambda I)^n(X) = 0 \text{ for some } n \geq 1\},$$

where λ is an eigenvalue of $\mathcal{D}$. We can decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^- \oplus \mathfrak{g}^0$, where

$$\mathfrak{g}^+ = \bigoplus_{\mathrm{Re}(\lambda)>0} \mathfrak{g}_\lambda, \qquad \mathfrak{g}^- = \bigoplus_{\mathrm{Re}(\lambda)<0} \mathfrak{g}_\lambda, \qquad \mathfrak{g}^0 = \bigoplus_{\mathrm{Re}(\lambda)=0} \mathfrak{g}_\lambda.$$

These vector spaces are Lie algebras (see *e.g.* [16], Prop. 3.1). If $\mathfrak{g}$ is a real Lie algebra, this decomposition is obtained by considering the complexification $\mathfrak{g}_{\mathbb{C}}$ and the linear extension $\mathcal{D}_{\mathbb{C}}$. We define $\mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$ and $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$. It should be observed that the origin $0 \in \mathfrak{g}^-$ is a global attractor for the flow $\mathrm{e}^{t\mathcal{D}}\big|_{\mathfrak{g}^-}$, or in other words, 0 is exponentially stable in positive time for the flow $\mathrm{e}^{t\mathcal{D}}\big|_{\mathfrak{g}^-}$. This means that there are numbers $\lambda < 0$ and $B > 0$ such that $\left\|\mathrm{e}^{t\mathcal{D}}(X)\right\| \leq B\mathrm{e}^{\lambda t}\|X\|$ for all $X \in \mathfrak{g}^-$ and $t \geq 0$. The same is true for $\mathfrak{g}^+$ in negative time, that is $0 \in \mathfrak{g}^+$ is a global repeller for the flow $\mathrm{e}^{t\mathcal{D}}\big|_{\mathfrak{g}^+}$.

For a given point $x \in G$ and a vector field $\mathcal{X}$, the *positive trajectory* and the *negative trajectory* of x are respectively defined by the sets

$$\mathcal{X}^+(x) = \{\mathcal{X}_t(x) : t \geq 0\}, \qquad \mathcal{X}^-(x) = \{\mathcal{X}_t(x) : t \leq 0\}.$$

The *positive limit set*, *positive prolongation,* and *positive prolongational limit set* of x are respectively defined by

$$\omega_{\mathcal{X}}^+(x) = \{y \in G : \text{there is a sequence } t_n \to +\infty \text{ such that } \mathcal{X}_{t_n}(x) \to y\},$$
$$D_{\mathcal{X}}^+(x) = \left\{y \in M : \text{there are sequences } (t_n) \text{ in } \mathbb{R}^+ \text{ and } x_n \to x \text{ such that } \mathcal{X}_{t_n}(x_n) \to y\right\},$$
$$J_{\mathcal{X}}^+(x) = \{y \in G : \text{there are sequences } t_n \to +\infty \text{ and } x_n \to x \text{ such that } \mathcal{X}_{t_n}(x_n) \to y\}.$$

The *negative limit set* $\omega_{\mathcal{X}}^{-}(x)$, *negative prolongation* $D_{\mathcal{X}}^{-}(x)$, and *positive prolongational limit set* $J_{\mathcal{X}}^{-}(x)$ are similarly defined in negative time. All these limit sets are invariant and satisfy the relations

$$\operatorname{cl}\left(\mathcal{X}^{\pm}(x)\right)=\mathcal{X}^{\pm}(x)\cup\omega_{\mathcal{X}}^{\pm}(x), \qquad D_{\mathcal{X}}^{\pm}(x)=\mathcal{X}^{\pm}(x)\cup J_{\mathcal{X}}^{\pm}(x)$$
$$\omega_{\mathcal{X}}^{\pm}(x)\subset J_{\mathcal{X}}^{\pm}(x)=\bigcap\nolimits_{t>0}D_{\mathcal{X}}^{\pm}\left(\mathcal{X}_{t}(x)\right).$$

The positive limit sets are nonempty, if the flow $\mathcal{X}_t$ is asymptotically compact. We recall that $\mathcal{X}_t$ is asymptotically compact if for any bounded sequence (x_n) in G and $t_n\to+\infty$ there is a convergent subsequence of $(\mathcal{X}_{t_n}(x_n))$.

A point $x\in G$ is said to be *positively recurrent*, *negatively recurrent,* or *nonwandering* with respect to $\mathcal{X}$ if, respectively, $x\in\omega_{\mathcal{X}}^{+}(x)$, $x\in\omega_{\mathcal{X}}^{-}(x)$, or $x\in J_{\mathcal{X}}^{+}(x)$. The set of all positive recurrent points with respect to $\mathcal{X}$ is indicated by $\operatorname{Re}^{+}(\mathcal{X})$, the set of all negative recurrent points with respect to $\mathcal{X}$ is indicated by $\operatorname{Re}^{-}(\mathcal{X})$, and the set of all nonwandering points with respect to $\mathcal{X}$ is denoted by $\operatorname{NW}(\mathcal{X})$. It is well-known that $\bigcup_{x\in G}\omega_{\mathcal{X}}^{+}(x)\cup\omega_{\mathcal{X}}^{-}(x)\subset\operatorname{NW}(\mathcal{X})$.

We often indicate by $\operatorname{Eq}(\mathcal{X})$ the set of the equilibria of $\mathcal{X}$, and by $\operatorname{Pe}(\mathcal{X})$ the set of all periodic points of $\mathcal{X}$. In general, we have the inclusions

$$\operatorname{Eq}(\mathcal{X})\subset\operatorname{Pe}(\mathcal{X})\subset\operatorname{Re}^{+}(\mathcal{X})\cap\operatorname{Re}^{-}(\mathcal{X})\subset\operatorname{cl}\left\{\operatorname{Re}^{+}(\mathcal{X})\cup\operatorname{Re}^{-}(\mathcal{X})\right\}\subset\operatorname{NW}(\mathcal{X}).$$

Remark 2.1. We recall that an equilibrium x of a vector field $\mathcal{X}$ is *(Lyapunov) stable* if every neighborhood U of x contains a positively invariant neighborhood of x. If $\mathcal{X}$ is linear and the neutral element is stable, the flow $\mathcal{X}_t$ is asymptotically compact, all the equilibria of $\mathcal{X}$ are stable, and $J_{\mathcal{X}}^{+}(x)=\omega_{\mathcal{X}}^{+}(x)$ for all $x\in G$ ([17], Thm. 5.1). In this case, we have $D_{\mathcal{X}}^{+}(x)=\operatorname{cl}\left(\mathcal{X}^{+}(x)\right)$ for all $x\in G$, and $\operatorname{Re}^{+}(\mathcal{X})=\operatorname{Re}^{-}(\mathcal{X})=\operatorname{NW}(\mathcal{X})$.

Remark 2.2. For all $x\in G$, $\omega_{\mathcal{X}}^{+}(x)\subset J_{\mathcal{X}}^{+}(w)$ for every $w\in\omega_{\mathcal{X}}^{+}(x)$. Indeed, take $y,w\in\omega_{\mathcal{X}}^{+}(x)$ and divergent nets $t_n\to+\infty$ and $s_n\to+\infty$ such that $\mathcal{X}_{t_n}(x)\to y$ and $\mathcal{X}_{s_n}(x)\to w$. For each $k\in\mathbb{N}$, take $t_{n_k}>2s_k$. We have $\mathcal{X}_{t_{n_k}-s_k}\left(\mathcal{X}_{s_k}(x)\right)=\mathcal{X}_{t_{n_k}}(x)\to y$, with $t_{n_k}-s_k\to+\infty$ and $\mathcal{X}_{s_k}(x)\to w$. Thus $y\in J_{\mathcal{X}}^{+}(w)$. In the case $J_{\mathcal{X}}^{+}=\omega_{\mathcal{X}}^{+}$, we have $\omega_{\mathcal{X}}^{+}(x)=\omega_{\mathcal{X}}^{+}(w)$ for all $w\in\omega_{\mathcal{X}}^{+}(x)$.

In the following, we show the main properties of the recurrence set of an automorphism flow.

Proposition 2.3. *Assume that the neutral element* 1 *of* G *is a stable equilibrium of a linear vector field* $\mathcal{X}$*.*

1. *The recurrence set* $\operatorname{Re}^{+}(\mathcal{X})$ *is a Lie subgroup of* G*.*
2. *For each* $x\in G$*, there is a* $q\in\operatorname{Re}^{+}(\mathcal{X})$ *such that* $\omega_{\mathcal{X}}^{+}(x)=\omega_{\mathcal{X}}^{+}(q)$*.*

Proof. (1) We have $\operatorname{Re}^{+}(\mathcal{X})=\operatorname{NW}(\mathcal{X})$. For $x,y\in\operatorname{Re}^{+}(\mathcal{X})$, there are sequences $t_n\to+\infty$ and $s_n\to+\infty$ such that $\mathcal{X}_{t_n}(x)\to x$ and $\mathcal{X}_{s_n}(y)\to y$. As $\mathcal{X}_t$ is an automorphism of G, we have $\mathcal{X}_{t_n}\left(x^{-1}\right)=\mathcal{X}_{t_n}(x)^{-1}\to x^{-1}$. Hence $x^{-1}\in\operatorname{Re}^{+}(\mathcal{X})$. For each $k\in\mathbb{N}$, choose s_{n_k} such that $s_{n_k}>t_k$. We then have

$$xy=\lim_{k\to\infty}\mathcal{X}_{t_k}(x)\,\mathcal{X}_{s_{n_k}}(y)=\lim_{k\to\infty}\mathcal{X}_{t_k}(x)\,\mathcal{X}_{t_k}\left(\mathcal{X}_{s_{n_k}-t_k}(y)\right)=\lim_{k\to\infty}\mathcal{X}_{t_k}\left(x\mathcal{X}_{s_{n_k}-t_k}(y)\right).$$

We may assume that $x\mathcal{X}_{s_{n_k}-t_k}(y)\to z$, by the asymptotic compactness. Hence, $xy\in J_{\mathcal{X}}^{+}(z)=\omega_{\mathcal{X}}^{+}(z)$, and therefore xy is nowandering. This proves that $\operatorname{Re}^{+}(\mathcal{X})$ is a subgroup of G. Now, let (x_n) be a sequence in $\operatorname{Re}^{+}(\mathcal{X})$ such that $x_n\to x$. For each n fixed, there is a divergent sequence $t_{(n.k)}\to+\infty$ such that $x_n=\lim_{k\to\infty}\mathcal{X}_{t_{(n,k)}}(x_n)$. By the Cantor diagonal argument, we can find a sequence (k_n) such that $t_{(n,k_n)}>n$ and $\mathcal{X}_{t_{(n,k_n)}}(x_n)\to x$. This means that x is nonwandering, and therefore $\operatorname{Re}^{+}(\mathcal{X})$ is a closed subgroup. By the Cartan theorem, $\operatorname{Re}^{+}(\mathcal{X})$ is a Lie subgroup of G.

(2) We have $\emptyset\neq\omega_{\mathcal{X}}^{+}(x)\subset\operatorname{NW}(\mathcal{X})=\operatorname{Re}^{+}(\mathcal{X})$. For any $q\in\omega_{\mathcal{X}}^{+}(x)$, we have $\omega_{\mathcal{X}}^{+}(x)=\omega_{\mathcal{X}}^{+}(q)$ (Rem. 2.2). □

3. Dispersive and recursive concepts of control systems

This section contains the basic results on dispersiveness and recursiveness in the control framework. Throughout, G is a connected Lie group with Lie algebra $\mathfrak{g}$ and right invariant distance d.

We consider a linear control system on G given by the family of differential equations

$$\begin{gathered}\dot{x}=\mathcal{X}\left(x\right)+\sum_{i=1}^{m}u_i\left(t\right)X_i\left(x\right), \qquad x\in G,\\ u\in\mathcal{U}_{pc}=\{u:\mathbb{R}\to U: u \text{ piecewise constant}\},\end{gathered} \tag{Σ}$$

with compact and convex control range $U\subset\mathbb{R}^m$, linear vector field $\mathcal{X}$ associated to a derivation $\mathcal{D}:\mathfrak{g}\to\mathfrak{g}$, and left invariant control vectors $X_1,\dots,X_m$. For each $u\in\mathcal{U}_{pc}$ and $x\in G$, the preceding equation has a unique solution $\varphi\left(t,x,u\right)$, $t\in\mathbb{R}$, with $\varphi\left(0,x,u\right)=x$. This solution satisfies the formula

$$\varphi\left(t,x,u\right)=\mathcal{X}_t\left(x\right)\varphi\left(t,1,u\right), \qquad t\in\mathbb{R}, x\in G, u\in\mathcal{U}_{pc}. \tag{$P1$}$$

The closure $\mathcal{U}=\mathrm{cl}\left(\mathcal{U}_{pc}\right)$ with respect to the weak* topology of $L_\infty\left(\mathbb{R},\mathbb{R}^m\right)$ is a compact metric space, and the phase map $\varphi:\mathbb{R}\times G\times\mathcal{U}\to G$ is continuous. We refer to [18], Chapter 4 for related information.

For all $x\in G$ and $\tau\in\mathbb{R}$, we define the sets

$$\begin{aligned}\mathcal{O}^+\left(x\right)&=\left\{\varphi\left(t,x,u\right):t\geq 0,u\in\mathcal{U}\right\}, &\qquad \mathcal{O}^+_{>\tau}\left(x\right)&=\left\{\varphi\left(t,x,u\right):t>\tau,u\in\mathcal{U}\right\},\\ \mathcal{O}^-\left(x\right)&=\left\{\varphi\left(t,x,u\right):t\leq 0,u\in\mathcal{U}\right\}, &\qquad \mathcal{O}^-_{>\tau}\left(x\right)&=\left\{\varphi\left(t,x,u\right):t<-\tau,u\in\mathcal{U}\right\}.\end{aligned}$$

The set $\mathcal{O}^+\left(x\right)$ is called the *positive orbit* of x; $\mathcal{O}^-\left(x\right)$ is called the *negative orbit* of x. A set $X\subset G$ is said to be *positively invariant*, if $\mathcal{O}^+\left(x\right)\subset X$ for all $x\in X$; it is called *negatively invariant*, if $\mathcal{O}^-\left(x\right)\subset X$ for all $x\in X$.

Remark 3.1. The following properties are easily checked by using the formula $P1$:

1. $\mathcal{O}^+_{>t}\left(x\right)=x\mathcal{O}^+_{>t}\left(1\right)$ and $\mathcal{O}^-_{>t}\left(x\right)=x\mathcal{O}^-_{>t}\left(1\right)$, for all $x\in\mathrm{Eq}\left(\mathcal{X}\right)$.
2. If $\mathcal{O}^+_{>t}\left(1\right)$ is open then $\mathcal{O}^+_{>t}\left(x\right)$ is open for all $x\in\mathrm{Eq}\left(\mathcal{X}\right)$. One may replace $\mathcal{O}^+_{>t}\left(1\right)$ by $\mathcal{O}^-_{>t}\left(1\right)$, and open by closed, compact, *etc.*

For each $x\in G$, we define the *positive limit set, positive prolongation*, and *positive prolongational limit set* of x respectively by

$$\begin{aligned}\omega^+_\Sigma\left(x\right)&=\bigcap_{t>0}\mathrm{cl}\left(\mathcal{O}^+_{>t}\left(x\right)\right)\\ &=\left\{\begin{array}{c}y\in G:\text{there are sequences }\left(t_n\right)\text{ in }\mathbb{R}\text{ and }\left(u_n\right)\text{ in }\mathcal{U}\\ \text{such that }t_n\to+\infty\text{ and }\varphi\left(t_n,x,u_n\right)\to y\end{array}\right\},\\ D^+_\Sigma\left(x\right)&=\bigcap_{\varepsilon>0}\mathrm{cl}\left(\mathcal{O}^+_{>0}\left(\mathrm{B}(x,\varepsilon)\right)\right)\\ &=\left\{\begin{array}{c}y\in G:\text{there are sequences }\left(t_n\right)\text{ in }\mathbb{R}^+,\ \left(u_n\right)\text{ in }\mathcal{U}\text{, and }\left(x_n\right)\text{ in }G\\ \text{such that }x_n\to x\text{ and }\varphi\left(t_n,x_n,u_n\right)\to y\end{array}\right\}.\\ J^+_\Sigma\left(x\right)&=\bigcap_{t>0}\bigcap_{\varepsilon>0}\mathrm{cl}\left(\mathcal{O}^+_{>t}\left(\mathrm{B}(x,\varepsilon)\right)\right)\\ &=\left\{\begin{array}{c}y\in G:\text{there are sequences }\left(t_n\right)\text{ in }\mathbb{R},\ \left(u_n\right)\text{ in }\mathcal{U}\text{, and }\left(x_n\right)\text{ in }G\\ \text{such that }x_n\to x,\ t_n\to+\infty\text{, and }\varphi\left(t_n,x_n,u_n\right)\to y\end{array}\right\}.\end{aligned}$$

The *negative limit set* $\omega_{\Sigma}^{-}(x)$, *negative prolongation* $D_{\Sigma}^{-}(x)$, and *positive prolongational limit set* $J_{\Sigma}^{-}(x)$ are similarly defined in negative time.

The objects $\mathcal{O}^{\pm}(x)$, $\omega_{\Sigma}^{\pm}(x)$, $D_{\Sigma}^{\pm}(x)$, and $J_{\Sigma}^{\pm}(x)$ define set valued functions $F: G \to 2^{G}$, with $F(X) = \bigcup_{x \in X} F(x)$ for all subset $X \subset G$. These sets satisfy the relations

$$D_{\Sigma}^{\pm}(x) = \mathcal{O}^{\pm}(x) \cup J_{\Sigma}^{\pm}(x) \qquad \text{and} \qquad \mathrm{cl}\left(\mathcal{O}^{\pm}(x)\right) = \mathcal{O}^{\pm}(x) \cup \omega_{\Sigma}^{\pm}(x).$$

A point $x \in G$ is said to be *positively (Poincaré) recurrent*, *negatively recurrent, recurrent,* or *nonwandering* if, respectively, $x \in \omega_{\Sigma}^{+}(x)$, $x \in \omega_{\Sigma}^{-}(x)$, $x \in \omega_{\Sigma}^{+}(x) \cap \omega_{\Sigma}^{-}(x)$, or $x \in J_{\Sigma}^{+}(x)$. If $x \notin \omega_{\Sigma}^{+}(x)$ for all $x \in G$, the control system is called *Poincaré unstable* (or *Poisson unstable*); if $x \notin J_{\Sigma}^{+}(x)$ for all $x \in G$, the control system is called *completely unstable*; if $J_{\Sigma}^{+}(x) = \emptyset$ for all $x \in G$, the control system is called *dispersive.*

It is well-known that $x \in J_{\Sigma}^{+}(y)$ if, and only if, $y \in J_{\Sigma}^{-}(x)$. Thus, we may replace $J_{\Sigma}^{+}(x) = \emptyset$ by $J_{\Sigma}^{-}(x) = \emptyset$ in the definition of dispersiveness.

The set of all positive recurrent points, and the set of all negative recurrent points of the control system Σ, are respectively indicated by $\mathrm{Re}^{+}(\Sigma)$ and $\mathrm{Re}^{-}(\Sigma)$. The closure of $\mathrm{Re}^{+}(\Sigma)$ is usually called the *Birkhoff center* of the control system, and it is indicated by $\mathrm{Bi}(\Sigma)$. The set of all nonwandering points is denoted by $\mathrm{NW}(\Sigma)$. It is well-known that $\bigcup\limits_{x \in G} \omega_{\Sigma}^{+}(x) \cup \omega_{\Sigma}^{-}(x) \subset \mathrm{NW}(\Sigma)$, which implies $\mathrm{cl}\left\{\mathrm{Re}^{+}(\Sigma) \cup \mathrm{Re}^{-}(\Sigma)\right\} \subset \mathrm{NW}(\Sigma)$ ([3]).

Remark 3.2. The nonwandering set $\mathrm{NW}(\Sigma)$ is closed in G. Indeed, let (x_n) be a sequence in $\mathrm{NW}(\Sigma)$ such that $x_n \to x$. For each x_n, there are sequences $t_n^k \to +\infty$, $x_n^k \to x_n$, and $\left(u_n^k\right) \subset \mathcal{U}$ such that $\varphi\left(t_n^k, x_n^k, u_n^k\right) \to x_n$. As $x_n \to x$, we can find sequences $t_n^{k_n} \to +\infty$, $x_n^{k_n} \to x$, and $\varphi\left(t_n^{k_n}, x_n^{k_n}, u_n^{k_n}\right) \to x$. Hence $x \in J_{\Sigma}^{+}(x)$, proving that $x \in \mathrm{NW}(\Sigma)$.

It should be remembered that a point $x \in G$ is *periodic* in the control sense if there is a control function $u \in \mathcal{U}$ and a time $\tau > 0$ such that $\varphi(\tau, x, u) = x$. In other words, a point x is periodic if $x \in \mathcal{O}_{>0}^{+}(x)$, or equivalently, $x \in \mathcal{O}_{>0}^{-}(x)$. In this case, the set $H_x = \mathcal{O}_{>0}^{+}(x) \cap \mathcal{O}_{>0}^{-}(x)$ is called the *holding set* of x. For instance, if the positive orbit $\mathcal{O}^{+}(1)$ is open, the neutral element is a periodic point of the control system. It is clear that every periodic point is recurrent. The converse holds, only if the positive orbit is a closed set ([19]). We often indicate by $\mathrm{Pe}(\Sigma)$ the set of all periodic points of the system. In general, we have the following set inclusions:

$$\mathrm{Pe}(\Sigma) \subset \mathrm{Re}^{+}(\Sigma) \cap \mathrm{Re}^{-}(\Sigma) \subset \mathrm{cl}\left\{\mathrm{Re}^{+}(\Sigma) \cup \mathrm{Re}^{-}(\Sigma)\right\} \subset \mathrm{NW}(\Sigma).$$

In the case of complete instability, all these sets are empty (absence of recursiveness). Thus, a completely unstable control system has no periodic trajectory. On the other hand, a control system is dispersive (and then completely unstable), if it satisfies $\mathrm{Pe}(\Sigma) = \emptyset$ and $D_{\Sigma}^{+}(x) = \mathcal{O}^{+}(x)$ for all x ([3, 19]).

Remark 3.3. All the dynamical properties at the neutral element are transferred to the equilibria of the linear vector field $\mathcal{X}$. The following properties are easily checked by using the formula $P1$:

1. $J_{\Sigma}^{\pm}(x) = xJ_{\Sigma}^{\pm}(1)$ for all $x \in \mathrm{Eq}(\mathcal{X})$;
2. $\omega_{\Sigma}^{\pm}(x) = x\omega_{\Sigma}^{\pm}(1)$ for all $x \in \mathrm{Eq}(\mathcal{X})$;
3. $\mathrm{Eq}(\mathcal{X}) \subset \mathrm{NW}(\Sigma)$, if $1 \in \mathrm{NW}(\Sigma)$.
4. $\mathrm{Eq}(\mathcal{X}) \subset \mathrm{Re}^{\pm}(\Sigma)$, if $1 \in \mathrm{Re}^{\pm}(\Sigma)$;
5. $\mathrm{Eq}(\mathcal{X}) \subset \mathrm{Pe}(\Sigma)$, if $1 \in \mathrm{Pe}(\Sigma)$. In this case, $H_x = xH_1$ for every $x \in \mathrm{Eq}(\mathcal{X})$.

For $x \in \mathrm{Eq}(\mathcal{X})$ and $y \in J_{\Sigma}^{+}(x)$, there are sequences (t_n) in $\mathbb{R}$, (u_n) in $\mathcal{U}$, and (x_n) in G such that $x_n \to x$, $t_n \to +\infty$, and $\varphi(t_n, x_n, u_n) \to y$. Hence $x^{-1}x_n \to 1$ and $\varphi\left(t_n, x^{-1}x_n, u_n\right) = x^{-1}\varphi(t_n, x_n, u_n) \to x^{-1}y$, and therefore $x^{-1}y \in J_{\Sigma}^{+}(1)$. This proves the equality $J_{\Sigma}^{+}(x) = xJ_{\Sigma}^{+}(1)$. We may analogously prove all the relations in items (1) and (2). The items $(3)-(5)$ are consequences from $(1), (2)$.

We now recall the notions of controllability and show their relations with the recursive concepts.

Definition 3.4. A nonempty subset $C \subset G$ is called a *control set* of the control system Σ if it satisfies the following conditions:

1. For each $x \in C$, there is a control function $u \in \mathcal{U}_{pc}$ with $\varphi(t, x, u) \in C$ for all $t \geq 0$.
2. $C \subset \mathrm{cl}(\mathcal{O}^+(x))$ for every $x \in C$.
3. C is maximal satisfying both properties (1) and (2).

The control system is said to be *controllable* if $G = \mathrm{cl}(\mathcal{O}^+(x))$ for every $x \in G$.

This notion of control set is a usual definition in control theory, which means a region of the state space where complete approximate controllability holds (see *e.g.* [18, 20, 21]). It is related to the Poincaré recurrence and the Hoxin holding sets. Indeed, all the points inside a control set are positively recurrent. If C is a control set and $x \in C$, there is a control function $u \in \mathcal{U}_{pc}$ with $\varphi(t, x, u) \in C$ for all $t \geq 0$. This implies that $C \subset \mathrm{cl}\left(\mathcal{O}^+_{>0}(\varphi(t, x, u))\right) \subset \mathrm{cl}\left(\mathcal{O}^+_{>t}(x)\right)$ for all $t \geq 0$. Thus, $C \subset \omega^+_\Sigma(x)$ for every $x \in C$, and $C \subset \mathrm{Re}^+(\Sigma)$. In particular, if the system is controllable, we have $G = \omega^+_\Sigma(x)$ for every $x \in G$ (property also known as topological transitivity).

For every $x \in \mathrm{Pe}(\Sigma)$, it is clear that the holding set H_x is contained in a control set. More generally, every positively recurrent point is contained in a control set ([7]). Thus, the system is Poincaré unstable if, and only if, it has no control set. A control system that has no control set is also known as *completely uncontrollable.*

Definition 3.5. A nonempty set $P \subset G$ is called a *prolongational control set* if it satisfies the following conditions:

1. For each $x \in P$, there is a control function $u \in \mathcal{U}_{pc}$ such that $\varphi(t, x, u) \in P$ for all $t \geq 0$;
2. $P \subset D^+(x)$ for all $x \in P$;
3. P is maximal satisfying both properties (1) and (2).

The control system is said to be controllable by prolongations if $G = D^+(x)$ for all $x \in G$.

All the points inside a prolongational control set are nonwandering. In fact, if $P \subset G$ is a prolongational control set, we have $P \subset J^+_\Sigma(x)$ for every $x \in P$ ([22], Prop. 5.1). Thus $P \subset \mathrm{NW}(\Sigma)$. In particular, if the system is controllable by prolongation, we have $G = J^+_\Sigma(x)$ for every $x \in G$.

The positive prolongation $D^+(x)$ is interpreted as an extension of the positive orbit $\mathcal{O}^+(x)$. The concept of prolongational control set is then an extension of the notion of control set. Indeed, every control set is contained in a prolongational control set. In particular, a controllable system is controllable by prolongation.

Summarizing, we may distinguish the notions of dispersive system, controllable system, and prolongational controllable system, by using the metric structure of G. The control system Σ is:

Dispersive if for every pair $x, y \in G$ there are numbers $\delta, T > 0$ such that the conditions $\mathrm{d}(z, y) < \delta$ and $|t| > T$ imply $\mathrm{d}(\varphi(t, z, u), x) \geq \delta$ for all $u \in \mathcal{U}_{pc}$ ([3], Thm. 3.1).

Controllable if for every pair of states $x, y \in G$ and a number $\delta > 0$ there is a $t > 0$ and a $u \in \mathcal{U}_{pc}$ such that $\mathrm{d}(\varphi(t, y, u), x) < \delta$.

Controllable by prolongation if for every pair of states $x, y \in G$ and a number $\delta > 0$ there is a $t > 0$, a $u \in \mathcal{U}_{pc}$, and a $z \in G$ such that $\mathrm{d}(z, y) < \delta$ and $\mathrm{d}(\varphi(t, z, u), x) < \delta$.

It is important to note that a dispersive control system is completely unstable with the stronger property $J^+_\Sigma(x) = \emptyset$ for all $x \in G$. However, a complete unstable system need not be dispersive, unless the equality $\omega^+_\Sigma = J^+_\Sigma$ holds. The following result was proved in [7], Theorem 3.1 for a general control affine system

Theorem 3.6. *Assume that $\omega^+_\Sigma(x) = J^+_\Sigma(x)$ for all $x \in G$. The following statements are equivalent:*

1. *The control system is dispersive.*
2. *The control system is completely unstable (there is no nonwandering point).*
3. *The control system is completely uncontrollable (there is no control set).*

In the following, we show a sufficient condition for the equality $\omega_{\Sigma}^{+} = J_{\Sigma}^{+}$.

Proposition 3.7. *Assume that the neutral element* 1 *of G is a stable equilibrium of the linear vector field $\mathcal{X}$. One has $J_{\Sigma}^{+}(x) = \omega_{\Sigma}^{+}(x)$ for every $x \in G$.*

Proof. It is enough to show that $J_{\Sigma}^{+}(x) \subset \omega_{\Sigma}^{+}(x)$. Let $y \in J_{\Sigma}^{+}(x)$ and take sequences $t_n \to +\infty$, $x_n \to x$, and $(u_n) \subset \mathcal{U}_{pc}$ such that $\varphi(t_n, x_n, u_n) \to y$. By the right invariance of the distance d, we have

$$\mathrm{d}(\varphi(t_n, x_n, u_n), \varphi(t_n, x, u_n)) = \mathrm{d}(\mathcal{X}_{t_n}(x_n)\varphi(t_n, 1, u_n), \mathcal{X}_{t_n}(x)\varphi(t_n, 1, u_n)) = \mathrm{d}(\mathcal{X}_{t_n}(x_n), \mathcal{X}_{t_n}(x)).$$

Since the flow $\mathcal{X}_t$ is asymptotically compact, we may assume that $\mathcal{X}_{t_n}(x_n) \to z$. As $xx_n^{-1} \to 1$ and $J_{\mathcal{X}}^{+}(1) = \{1\}$, we also have $\mathcal{X}_{t_n}(xx_n^{-1}) \to 1$. Hence $\mathcal{X}_{t_n}(x) = \mathcal{X}_{t_n}(xx_n^{-1})\mathcal{X}_{t_n}(x_n) \to z$, and therefore

$$\mathrm{d}(\varphi(t_n, x_n, u_n), \varphi(t_n, x, u_n)) = \mathrm{d}(\mathcal{X}_{t_n}(x_n), \mathcal{X}_{t_n}(x)) \longrightarrow 0.$$

It follows that

$$\mathrm{d}(\varphi(t_n, x, u_n), y) \leq \mathrm{d}(\varphi(t_n, x, u_n), \varphi(t_n, x_n, u_n)) + \mathrm{d}(\varphi(t_n, x_n, u_n), y) \longrightarrow 0$$

and thus $y \in \omega_{\Sigma}^{+}(x)$. □

4. Main results

In this section, we now the main results of the paper. We study the dynamics of a linear control system that has a stable linear part. We discuss the dispersive problem and characterize the control sets of a nondispersive system. It should be observed that all the properties of the control system Σ proved for positive time may be similarly proved for negative time, by using the corresponding time-reversed control system.

The control system has nonempty limit sets if it is asymptotically compact, as the following.

Definition 4.1. The control system Σ is said to be *asymptotically compact at a point* $x \in G$ if for any sequences $t_n \to +\infty$ and $(u_n) \subset \mathcal{U}_{pc}$ there exists a convergent subsequence of $(\varphi(t_n, x, u_n))$. The control system is said to be *asymptotically compact* if for any bounded sequence (x_n) in G, $t_n \to +\infty$, and $(u_n) \subset \mathcal{U}_{pc}$ there exists a convergent subsequence of $(\varphi(t_n, x_n, u_n))$.

An asymptotically compact control system is asymptotically compact at every point. It is clear that $\omega_{\Sigma}^{+}(x) \neq \emptyset$ if the system is asymptotically compact at x.

Proposition 4.2. *Assume that the flow $\mathcal{X}_t$ is asymptotically compact. The linear control system Σ is asymptotically compact if, and only if, it is asymptotically compact at the neutral element.*

Proof. Suppose the system is asymptotically compact at the neutral element, and consider any bounded sequence (x_n) in G, $t_n \to +\infty$, and $(u_n) \subset \mathcal{U}_{pc}$. By the hypothesis, we can find subsequences $(x_{n_k}), (t_{n_k}), (u_{n_k})$ and points $x, y \in G$ such that $\mathcal{X}_{t_{n_k}}(x_{n_k}) \to x$ and $\varphi(t_{n_k}, 1, u_{n_k}) \to y$. Hence, $\varphi(t_{n_k}, x_{n_k}, u_{n_k}) = \mathcal{X}_{t_{n_k}}(x_{n_k})\varphi(t_{n_k}, 1, u_{n_k}) \to xy$, and therefore the control system is asymptotically compact. The converse is clear. □

Recall that the flow $\mathcal{X}_t$ is asymptotically compact, if the neutral element is a stable equilibrium of $\mathcal{X}$. If in addition the control system is asymptotically compact at the neutral element, the positive limit sets are nonempty. In the following, we show that the limit set at the neutral element plays a fundamental role in the dynamics of the linear control system.

Proposition 4.3. *Assume that the flow $\mathcal{X}_t$ is asymptotically compact. For every $x \in G$, one has the following relations:*

1. $D_{\Sigma}^{+}(x) \subset D_{\mathcal{X}}^{+}(x)\,\mathrm{cl}(\mathcal{O}^{+}(1))$, $\omega_{\Sigma}^{+}(x) \subset \omega_{\mathcal{X}}^{+}(x)\,\omega_{\Sigma}^{+}(1)$, *and* $J_{\Sigma}^{+}(x) \subset J_{\mathcal{X}}^{+}(x)\,\omega_{\Sigma}^{+}(1)$;

2. $\omega_{\Sigma}^{+}\left(\mathrm{cl}\left(\mathcal{X}^{+}\left(x\right)\right)\right)\subset\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)\subset J_{\Sigma}^{+}\left(\omega_{\mathcal{X}}^{+}\left(x\right)\right)$;
3. $J_{\Sigma}^{+}\left(\mathcal{X}^{+}\left(x\right)\right)\subset J_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)\subset J_{\Sigma}^{+}\left(J_{\mathcal{X}}^{+}\left(x\right)\right)=J_{\Sigma}^{+}\left(D_{\mathcal{X}}^{+}\left(x\right)\right)$.

Proof. Take $y\in D_{\Sigma}^{+}\left(x\right)$, and sequences $\left(t_{n}\right)$ in $\mathbb{R}^{+}$, $\left(u_{n}\right)$ in $\mathcal{U}$, and $\left(x_{n}\right)$ in G such that $x_{n}\to x$ and $\varphi\left(t_{n},x_{n},u_{n}\right)\to y$. Since the flow $\mathcal{X}_{t}$ is asymptotically compact by hypothesis, we may assume that the sequence $\mathcal{X}_{t_{n}}\left(x_{n}\right)$ converges to some $z\in G$. This means that $z\in D_{\mathcal{X}}^{+}\left(x\right)$. We then have

$$\varphi\left(t_{n},1,u_{n}\right)=\mathcal{X}_{t_{n}}\left(x_{n}\right)^{-1}\mathcal{X}_{t_{n}}\left(x_{n}\right)\varphi\left(t_{n},1,u_{n}\right)=\mathcal{X}_{t_{n}}\left(x_{n}\right)^{-1}\varphi\left(t_{n},x_{n},u_{n}\right)\to z^{-1}y$$

which means $z^{-1}y\in\mathrm{cl}\left(\mathcal{O}^{+}\left(1\right)\right)$. Thus, $y\in z\mathrm{cl}\left(\mathcal{O}^{+}\left(1\right)\right)\subset D_{\mathcal{X}}^{+}\left(x\right)\mathrm{cl}\left(\mathcal{O}^{+}\left(1\right)\right)$. This proves the first inclusion $D_{\Sigma}^{+}\left(x\right)\subset D_{\mathcal{X}}^{+}\left(x\right)\mathrm{cl}\left(\mathcal{O}^{+}\left(1\right)\right)$. Now, take $y\in J_{\Sigma}^{+}\left(x\right)$, and sequences $\left(t_{n}\right)$ in $\mathbb{R}$, $\left(u_{n}\right)$ in $\mathcal{U}$, and $\left(x_{n}\right)$ in G such that $x_{n}\to x$, $t_{n}\to+\infty$, and $\varphi\left(t_{n},x_{n},u_{n}\right)\to y$. Similarly, we may assume that $\mathcal{X}_{t_{n}}\left(x_{n}\right)$ converges to some $z\in G$, which means $z\in J_{\mathcal{X}}^{+}\left(x\right)$. We then have $\varphi\left(t_{n},1,u_{n}\right)\to z^{-1}y$, hence $z^{-1}y\in\omega_{\Sigma}^{+}\left(1\right)$. This proves the inclusion $J_{\Sigma}^{+}\left(x\right)\subset J_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$. Analogously, we may take a constant sequence $x_{n}=x$, and prove the inclusion $\omega_{\Sigma}^{+}\left(x\right)\subset\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$.

We now show the item (2). For any $y\in\mathrm{cl}\left(\mathcal{X}^{+}\left(x\right)\right)$, we have $\omega_{\mathcal{X}}^{+}\left(y\right)\subset\omega_{\mathcal{X}}^{+}\left(x\right)$, and then $\omega_{\Sigma}^{+}\left(y\right)\subset\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$, by the item (1). This means $\omega_{\Sigma}^{+}\left(\mathrm{cl}\left(\mathcal{X}^{+}\left(x\right)\right)\right)\subset\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$. On the other hand, take $y\in\omega_{\mathcal{X}}^{+}\left(x\right)$ and $z\in\omega_{\Sigma}^{+}\left(1\right)$. There are sequences $t_{n},s_{n}\to+\infty$ and $\left(u_{n}\right)\subset\mathcal{U}_{pc}$ such that $\mathcal{X}_{t_{n}}\left(x\right)\to y$ and $\varphi\left(s_{n},1,u_{n}\right)\to z$. We can find a subsequence $\left(t_{n_{k}}\right)$ of $\left(t_{n}\right)$ such that $t_{n_{k}}>s_{k}+k$ for all $k>0$. We then have

$$yz=\lim_{k\to\infty}\mathcal{X}_{t_{n_{k}}}\left(x\right)\varphi\left(s_{k},1,u_{n}\right)=\varphi\left(s_{k},\mathcal{X}_{t_{n_{k}}-s_{k}}\left(x\right),u_{n}\right).$$

By the asymptotic compactness, we may assume that $\left(\mathcal{X}_{t_{n_{k}}-s_{k}}\left(x\right)\right)$ converges to a point w, and then $w\in\omega_{\mathcal{X}}^{+}\left(x\right)$ as $t_{n_{k}}-s_{k}\to+\infty$. It follows that $yz\in J_{\Sigma}^{+}\left(w\right)\subset J_{\Sigma}^{+}\left(\omega_{\mathcal{X}}^{+}\left(x\right)\right)$, proving the inclusion $\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)\subset J_{\Sigma}^{+}\left(\omega_{\mathcal{X}}^{+}\left(x\right)\right)$.

We finally prove the item (3). For any $y\in\mathcal{X}^{+}\left(x\right)$, we have $J_{\mathcal{X}}^{+}\left(y\right)=J_{\mathcal{X}}^{+}\left(x\right)$, and then $J_{\Sigma}^{+}\left(y\right)\subset J_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$, by the item (1). This means $J_{\Sigma}^{+}\left(\mathcal{X}^{+}\left(x\right)\right)\subset J_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$. Take $y\in J_{\mathcal{X}}^{+}\left(x\right)$, $z\in\omega_{\Sigma}^{+}\left(1\right)$, and sequences $t_{n},s_{n}\to+\infty$, $x_{n}\to x$, and $\left(u_{n}\right)\subset\mathcal{U}_{pc}$ such that $\mathcal{X}_{t_{n}}\left(x_{n}\right)\to y$ and $\varphi\left(s_{n},1,u_{n}\right)\to z$. By taking subsequences if necessary, we may assume that $t_{n}-s_{n}\to+\infty$ and $\mathcal{X}_{t_{n}-s_{n}}\left(x_{n}\right)\to w$. As $yz=\lim_{n\to\infty}\varphi\left(s_{n},\mathcal{X}_{t_{n}-s_{n}}\left(x_{n}\right),u_{n}\right)$, it follows that $yz\in J_{\Sigma}^{+}\left(w\right)\subset J_{\Sigma}^{+}\left(J_{\mathcal{X}}^{+}\left(x\right)\right)$, proving the inclusion $J_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)\subset J_{\Sigma}^{+}\left(J_{\mathcal{X}}^{+}\left(x\right)\right)$. By the relation $D_{\mathcal{X}}^{+}\left(x\right)=\mathcal{X}^{+}\left(x\right)\cup J_{\mathcal{X}}^{+}\left(x\right)$, we have

$$J_{\Sigma}^{+}\left(D_{\mathcal{X}}^{+}\left(x\right)\right)=J_{\Sigma}^{+}\left(\mathcal{X}^{+}\left(x\right)\right)\cup J_{\Sigma}^{+}\left(J_{\mathcal{X}}^{+}\left(x\right)\right)\subset J_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)\cup J_{\Sigma}^{+}\left(J_{\mathcal{X}}^{+}\left(x\right)\right)=J_{\Sigma}^{+}\left(J_{\mathcal{X}}^{+}\left(x\right)\right)$$

hence $J_{\Sigma}^{+}\left(J_{\mathcal{X}}^{+}\left(x\right)\right)=J_{\Sigma}^{+}\left(D_{\mathcal{X}}^{+}\left(x\right)\right)$. □

Corollary 4.4. *Assume that the neutral element* 1 *of* G *is a stable equilibrium of the linear vector field* $\mathcal{X}$. *The following equalities hold for every* $x\in G$:

$$\omega_{\Sigma}^{+}\left(\mathrm{cl}\left(\mathcal{X}^{+}\left(x\right)\right)\right)=\omega_{\Sigma}^{+}\left(\omega_{\mathcal{X}}^{+}\left(x\right)\right)=\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right).$$

Proof. By the hypothesis, we have $J_{\mathcal{X}}^{+}\left(x\right)=\omega_{\mathcal{X}}^{+}\left(x\right)$ and $J_{\Sigma}^{+}\left(x\right)=\omega_{\Sigma}^{+}\left(x\right)$ for all $x\in G$ (Rem. 2.1 and Prop. 3.7). For a given $y\in\omega_{\mathcal{X}}^{+}\left(x\right)$, we have $\omega_{\Sigma}^{+}\left(y\right)\subset\omega_{\mathcal{X}}^{+}\left(y\right)\omega_{\Sigma}^{+}\left(1\right)\subset\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$, by the item (1) of Proposition 4.3. Hence, $\omega_{\Sigma}^{+}\left(\omega_{\mathcal{X}}^{+}\left(x\right)\right)\subset\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$, and the equalities $\omega_{\Sigma}^{+}\left(\mathrm{cl}\left(\mathcal{X}^{+}\left(x\right)\right)\right)=\omega_{\Sigma}^{+}\left(\omega_{\mathcal{X}}^{+}\left(x\right)\right)=\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$ follow by the item (2) of Proposition 4.3. □

We now show the main theorems.

Theorem 4.5. *If the neutral element* 1 *of* G *is a stable equilibrium of the linear vector field* $\mathcal{X}$*, one has the set equalities*

$$\mathrm{NW}\left(\Sigma\right)=\mathrm{Re}^{+}\left(\Sigma\right)=\mathrm{Re}^{+}\left(\mathcal{X}\right)\omega_{\Sigma}^{+}\left(1\right)=\omega_{\Sigma}^{+}\left(G\right).$$

Proof. As $J_{\Sigma}^{+}=\omega_{\Sigma}^{+}$, we have $\mathrm{NW}\left(\Sigma\right)=\mathrm{Re}^{+}\left(\Sigma\right)$. If $\mathrm{Re}^{+}\left(\Sigma\right)=\emptyset$ (completely unstable system), we have $\omega_{\Sigma}^{+}\left(1\right)=\emptyset$, hence the equality $\mathrm{Re}^{+}\left(\Sigma\right)=\mathrm{Re}^{+}\left(\mathcal{X}\right)\omega_{\Sigma}^{+}\left(1\right)$ holds in this case. Suppose that $\mathrm{Re}^{+}\left(\Sigma\right)\neq\emptyset$. By Corollary 4.4, we have $\mathrm{Re}^{+}\left(\Sigma\right)\subset\bigcup_{x\in G}\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$. As $\mathrm{Re}^{+}\left(\mathcal{X}\right)=\bigcup_{x\in G}\omega_{\mathcal{X}}^{+}\left(x\right)$, it follows that $\mathrm{Re}^{+}\left(\Sigma\right)\subset\mathrm{Re}^{+}\left(\mathcal{X}\right)\omega_{\Sigma}^{+}\left(1\right)$. On the other hand, take any $x\in G$ and $z\in\omega_{\Sigma}^{+}\left(1\right)$. There are sequences $t_{n}\rightarrow+\infty$ and $\left(u_{n}\right)\subset\mathcal{U}$ such that $\varphi\left(t_{n},1,u_{n}\right)\rightarrow z$. By the asymptotic stability of the flow $\mathcal{X}_{t}$, there is a subsequence $t_{n_{k}}\rightarrow+\infty$ such that $\mathcal{X}_{t_{n}}\left(x\right)\rightarrow y$. Hence, $y\in\omega_{\mathcal{X}}^{+}\left(x\right)$ and

$$\varphi\left(t_{n_{k}},x,u_{n_{k}}\right)=\mathcal{X}_{t_{n}}\left(x\right)\varphi\left(t_{n_{k}},1,u_{n_{k}}\right)\longrightarrow yz.$$

This means that $yz\in\omega_{\Sigma}^{+}\left(x\right)$, and therefore yz is nonwandering, indeed $yz\in\mathrm{Re}^{+}\left(\Sigma\right)$. Now, for a given $w\in\omega_{\mathcal{X}}^{+}\left(x\right)$, there is a sequence $s_{n}\rightarrow+\infty$ such that $\mathcal{X}_{s_{n}}\left(y\right)\rightarrow w$ (Rem. 2.1). For any n fixed, we have

$$\varphi\left(t_{n_{k}},\mathcal{X}_{s_{n}}\left(x\right),u_{n_{k}}\right)=\mathcal{X}_{s_{n}}\left(\mathcal{X}_{t_{n}}\left(x\right)\right)\varphi\left(t_{n_{k}},1,u_{n_{k}}\right)\longrightarrow\mathcal{X}_{s_{n}}\left(y\right)z$$

as $k\rightarrow+\infty$, hence $\mathcal{X}_{s_{n}}\left(y\right)z\in\omega_{\Sigma}^{+}\left(\mathcal{X}_{s_{n}}\left(x\right)\right)$, and therefore $\mathcal{X}_{s_{n}}\left(y\right)z$ is nonwandering. Since $\mathrm{NW}\left(\Sigma\right)$ is a closed set and $\mathcal{X}_{s_{n}}\left(y\right)z\rightarrow wz$, it follows that wz is nonwandering, and therefore $\omega_{\mathcal{X}}^{+}\left(x\right)z\subset\mathrm{Re}^{+}\left(\Sigma\right)$. This proves the inclusion $\mathrm{Re}^{+}\left(\mathcal{X}\right)\omega_{\Sigma}^{+}\left(1\right)\subset\mathrm{Re}^{+}\left(\Sigma\right)$, and therefore the equality $\mathrm{Re}^{+}\left(\Sigma\right)=\mathrm{Re}^{+}\left(\mathcal{X}\right)\omega_{\Sigma}^{+}\left(1\right)$ holds.

Now, for any $x\in G$, we have $\omega_{\Sigma}^{+}\left(x\right)\subset\omega_{\mathcal{X}}^{+}\left(x\right)\omega_{\Sigma}^{+}\left(1\right)$, by Corollary 4.4, hence $\omega_{\Sigma}^{+}\left(x\right)\subset\mathrm{Re}^{+}\left(\Sigma\right)$. Thus $\omega_{\Sigma}^{+}\left(G\right)\subset\mathrm{Re}^{+}\left(\Sigma\right)$. The other inclusion $\mathrm{Re}^{+}\left(\Sigma\right)\subset\omega_{\Sigma}^{+}\left(G\right)$ is clear, and the proof is finished. □

Since the inclusions $\bigcup_{x\in G}\omega_{\Sigma}^{+}\left(x\right)\cup\omega_{\Sigma}^{-}\left(x\right)\subset\mathrm{NW}\left(\Sigma\right)$ and $\mathrm{cl}\left\{\mathrm{Re}^{+}\left(\Sigma\right)\cup\mathrm{Re}^{-}\left(\Sigma\right)\right\}\subset\mathrm{NW}\left(\Sigma\right)$ hold in general, the sameness $J_{\Sigma}^{+}=\omega_{\Sigma}^{+}$ also implies the following equalities of closed sets

$$\mathrm{Bi}\left(\Sigma\right)=\mathrm{NW}\left(\Sigma\right)=\mathrm{Re}^{+}\left(\Sigma\right)\cup\mathrm{Re}^{-}\left(\Sigma\right)=\bigcup_{x\in G}\omega_{\Sigma}^{+}\left(x\right)\cup\omega_{\Sigma}^{-}\left(x\right).$$

Assume that the recurrence set $\mathrm{Re}^{+}\left(\mathcal{X}\right)$ is a Lie subgroup of G (Prop. 2.3). and consider the standard projection $\pi:G\rightarrow\mathrm{Re}^{+}\left(\mathcal{X}\right)\backslash G$ of G onto the homogeneous space $\mathrm{Re}^{+}\left(\mathcal{X}\right)\backslash G$. Since $\mathrm{Re}^{+}\left(\mathcal{X}\right)$ is invariant by the automorphism flow $\mathcal{X}_{t}$, we can define the induced flow $\overline{\mathcal{X}}_{t}$ and the induced control system $\overline{\varphi}_{t}^{u}$ on $\mathrm{Re}^{+}\left(\mathcal{X}\right)\backslash G$ respectively by

$$\overline{\mathcal{X}}_{t}\left(\pi\left(x\right)\right):=\pi\left(\mathcal{X}_{t}\left(x\right)\right),\qquad\overline{\varphi}\left(t,\pi\left(x\right),u\right):=\pi\left(\varphi\left(t,x,u\right)\right),\qquad x\in G.$$

Let $\overline{\omega}_{\Sigma}^{+},\overline{\omega}_{\Sigma}^{-},\overline{J}_{\Sigma}^{+},\overline{J}_{\Sigma}^{-}$ be the corresponding limit sets of the induced control system $\overline{\varphi}$ on $\mathrm{Re}^{+}\left(\mathcal{X}\right)\backslash G$. The following relations are clear:

$$\begin{aligned}\pi\left(\omega_{\Sigma}^{+}\left(x\right)\right)&=\overline{\omega}_{\Sigma}^{+}\left(\pi\left(x\right)\right),\qquad\pi\left(\omega_{\Sigma}^{-}\left(x\right)\right)=\overline{\omega}_{\Sigma}^{-}\left(\pi\left(x\right)\right),\\\pi\left(J_{\Sigma}^{+}\left(x\right)\right)&=\overline{J}_{\Sigma}^{+}\left(\pi\left(x\right)\right),\qquad\pi\left(J_{\Sigma}^{-}\left(x\right)\right)=\overline{J}_{\Sigma}^{-}\left(\pi\left(x\right)\right),\qquad x\in G.\end{aligned}$$

By Theorem 4.5, we have the fibration $\mathrm{Re}^{+}\left(\mathcal{X}\right)\dashrightarrow\mathrm{Re}^{+}\left(\Sigma\right)\rightarrow\overline{\omega}_{\Sigma}^{+}\left(1\right)$.

Theorem 4.6. *Assume that the neutral element* 1 *of* G *is a stable equilibrium of the linear vector field* $\mathcal{X}$*. If the linear control system* Σ *is not dispersive, the positive limit set* $\overline{\omega}_{\Sigma}^{+}\left(1\right)$ *is the unique control set of the induced control system* $\overline{\varphi}$ *on the homogeneous space* $\mathrm{Re}^{+}\left(\mathcal{X}\right)\backslash G$*.*

Proof. Let $x, y \in \omega_\Sigma^+(1)$. We claim there are $a, b \in \mathrm{Re}^+(\mathcal{X})$ such that $ax \in \omega_\Sigma^+(y)$ and $by \in \omega_\Sigma^+(x)$. Indeed, take sequences $(t_n), (s_n)$ in $\mathbb{R}$ and $(u_n), (v_n)$ in $\mathcal{U}_{pc}$ such that $t_n, s_n \to +\infty$, $\varphi(t_n, 1, u_n) \to x$, and $\varphi(s_n, 1, v_n) \to y$. By the asymptotic stability of the flow $\mathcal{X}_t$, we may assume that $\mathcal{X}_{t_n}(\varphi(s_n, 1, v_n)) \to a$, which means $a \in J_\mathcal{X}^+(y) = \omega_\mathcal{X}^+(y)$. We then have $a \in \mathrm{Re}^+(\mathcal{X})$ and

$$\varphi(t_n, \varphi(s_n, 1, v_n), u_n) = \mathcal{X}_{t_n}(\varphi(s_n, 1, v_n))\, \varphi(t_n, 1, u_n) \to ax$$

hence $ax \in J_\Sigma^+(y) = \omega_\Sigma^+(y)$. Similarly, we can find $b \in \omega_\mathcal{X}^+(x)$ such that $by \in \omega_\Sigma^+(x)$. We then have $\pi(x) \in \overline{\omega}_\Sigma^+(\pi(y))$ and $\pi(y) \in \overline{\omega}_\Sigma^+(\pi(x))$. This proves that $\overline{\omega}_\Sigma^+(1) \subset \overline{\omega}_\Sigma^+(\pi(x))$ for all $x \in \omega_\Sigma^+(1)$. As the positive limit set $\overline{\omega}_\Sigma^+(1)$ is positively invariant, the induced system $\overline{\varphi}$ is controllable on $\overline{\omega}_\Sigma^+(1)$. Now, by Theorem 4.5, we have $\overline{\omega}_\Sigma^+(1) = \overline{\omega}_\Sigma^+(\pi(G))$, hence all the recurrent points of the induced system $\overline{\varphi}$ are contained in $\overline{\omega}_\Sigma^+(1)$. Thus $\overline{\omega}_\Sigma^+(1)$ is the unique control set of $\overline{\varphi}$ on $\mathrm{Re}^+(\mathcal{X}) \backslash G$. □

In special, if the limit sets of the flow $\mathcal{X}_t$ are periodic trajectories, we have $\mathrm{Re}^+(\mathcal{X}) = \mathrm{Pe}(\mathcal{X})$. In this case, $\mathrm{Re}^+(\Sigma) = \mathrm{Pe}(\mathcal{X})\,\omega_\Sigma^+(1)$. If these limit sets reduce to the equilibria of $\mathcal{X}$, we have $\mathrm{Re}^+(\Sigma) = \mathrm{Eq}(\mathcal{X})\,\omega_\Sigma^+(1) = \bigcup_{e \in \mathrm{Eq}(\mathcal{X})} \omega_\Sigma^+(e)$. This is the case $\mathfrak{g}^0 = \ker \mathcal{D}$ and $\mathfrak{g} = \mathfrak{g}^- \oplus \ker \mathcal{D}$, in which the control sets of the control system are the transitivity classes of the equilibria, as follows.

Theorem 4.7. *Assume that the neutral element* 1 *of* G *is a stable equilibrium of the linear vector field* $\mathcal{X}$*, and the positive limit sets of the flow* $\mathcal{X}_t$ *are the equilibria. There are two possibilities for the linear control system* Σ*:*

1. *The system is dispersive.*
2. *The sets* $C^+(e) = \{x \in \mathrm{Re}^+(\Sigma) : \omega_\Sigma^+(x) = \omega_\Sigma^+(e)\}$ *(*$e \in \mathrm{Eq}(\mathcal{X})$*) are all the control sets of the system.*

Proof. Suppose that the control system is not dispersive. For each $x \in G$, there is $e \in \mathrm{Eq}(\mathcal{X})$ such that $\omega_\mathcal{X}^+(x) = \{e\}$, and then $\omega_\Sigma^+(x) = \omega_\Sigma^+(e) = e\omega_\Sigma^+(1)$, by Corollary 4.4. As $\mathrm{Re}^+(\Sigma) = \mathrm{Eq}(\mathcal{X})\,\omega_\Sigma^+(1)$ is nonempty, the limit set $\omega_\Sigma^+(1)$ is nonempty, and therefore all the sets $C^+(e)$ are nonempty. We claim that each $C^+(e)$ is a control set. Indeed, for a given $x \in C^+(e)$, we have $x \in \mathrm{Re}^+(\Sigma)$, and then there is a control function u such that $\omega_\Sigma^+(\varphi(t, x, u)) = \omega_\Sigma^+(x)$ for all $t > 0$ (see Thm. 3.1 in [7]). Thus, $\omega_\Sigma^+(\varphi(t, x, u)) = \omega_\Sigma^+(x) = \omega_\Sigma^+(e)$ for all $t > 0$, which means $\varphi(t, x, u) \in C^+(e)$ for all $t > 0$. For any $x, y \in C^+(e)$, we have $y \in \omega_\Sigma^+(y) = \omega_\Sigma^+(e) = \omega_\Sigma^+(x)$, and therefore $C^+(e) \subset \omega_\Sigma^+(x)$ for all $x \in C^+(e)$. Thus $C^+(e)$ satisfies the properties (1) and (2) of Definition 3.4. This means that $C^+(e)$ is contained in a control set C. We then have $C \subset \mathrm{Re}^+(\Sigma)$, and for a given $x \in C^+(e)$, $C \subset \omega_\Sigma^+(x) = \omega_\Sigma^+(e)$. Hence, $C = C^+(e)$, and therefore $C^+(e)$ is a control set. Now, suppose that C' is any control set of the system. For a given $z \in C'$, there is an equilibrium $\mathrm{e}' \in \mathrm{Eq}(\mathcal{X})$ such that $\omega_\Sigma^+(z) = \omega_\Sigma^+(\mathrm{e}')$, hence $z \in C^+(\mathrm{e}')$. As $C^+(\mathrm{e}')$ is a control set, it must be $C^+(\mathrm{e}') = C'$. □

Thus, in the case $\mathfrak{g} = \mathfrak{g}^- \oplus \ker \mathcal{D}$ without dispersiveness, we have the equality $\mathrm{Re}^+(\Sigma) = \bigcup_{e \in \mathrm{Eq}(\mathcal{X})} C^+(e)$. In the stronger case $\mathfrak{g} = \mathfrak{g}^-$, $C^+(1) = \omega_\Sigma^+(1)$ is the unique control set of the linear system. Besides, $\omega_\Sigma^+(x) = \omega_\Sigma^+(1)$ for all $x \in G$, which means that the region of attraction of $\omega_\Sigma^+(1)$ is the entire group G. Thus the central limit set $\omega_\Sigma^+(1)$ is the global attractor of the system.

It should be observed that $0 \in U$ is not a necessary condition for the results of Theorems 4.5, 4.6, and 4.7. On the other hand, $0 \notin U$ is a necessary condition for the dispersiveness. In fact, if $0 \in U$, the flow $\mathcal{X}_t$ defines a particular solution (input $u_0 \equiv 0$) for the control system. This implies that $\emptyset \neq \mathrm{Re}^+(\mathcal{X}) \subset \mathrm{Re}^+(\Sigma)$, and then the system is not dispersive. By Theorem 3.6 and Propositions 3.7 and 4.3, we have the following characterization of dispersiveness.

Corollary 4.8. *Assume that the neutral element* 1 *is a stable equilibrium of the linear vector field* $\mathcal{X}$*, and* $0 \notin U$*. The following conditions on the linear control system* Σ *are equivalent:*

1. $\omega_\Sigma^+(1) = \emptyset$*.*
2. *The system is dispersive.*

3. *The system is completely uncontrollable.*
4. *The system is completely unstable.*

If the control range contains the zero, we can describe the control sets of the linear control system by the recurrent points of its linear vector field, as follows.

Theorem 4.9. *Assume that the control range U of the linear control system Σ contains the zero, and the neutral element 1 of G is a stable equilibrium of the linear vector field $\mathcal{X}$. The system Σ is not dispersive, and the sets $C^{+}(q)=\left\{x\in \mathrm{Re}^{+}(\Sigma):\omega_{\Sigma}^{+}(x)=\omega_{\Sigma}^{+}(q)\right\}$ ($q\in \mathrm{Re}^{+}(\mathcal{X})$) are all the control sets of the system.*

Proof. We claim that for a given $x\in \mathrm{Re}^{+}(\Sigma)$ there is a $q\in \mathrm{Re}^{+}(\mathcal{X})$ such that $\omega_{\Sigma}^{+}(x)=\omega_{\Sigma}^{+}(q)$. Indeed, take a sequence $\varphi(t_n,x,u_n)\to x$ with $t_n\to+\infty$. We may assume that $\mathcal{X}_{t_n}(x)\to q$, and then $q\in \mathrm{Re}^{+}(\mathcal{X})$. Hence, $q\in\omega_{\mathcal{X}}^{+}(x)\subset\omega_{\Sigma}^{+}(x)$, and therefore $\omega_{\Sigma}^{+}(q)\subset\omega_{\Sigma}^{+}(x)$. On the other hand, as $\mathcal{X}_{t_n}(x)\varphi(t_n,1,u_n)\to x$, we have $x=qz$, with $z\in\omega_{\Sigma}^{+}(1)$, $z=\lim_n\varphi(t_n,1,u_n)$. Take a sequence $\mathcal{X}_{s_n}(q)\to q$ with $s_n\to+\infty$. For each $k\in\mathbb{N}$, take s_{n_k} such that $s_{n_k}>2t_k$. We may assume that $\mathcal{X}_{s_{n_k}-t_k}(q)\to q'\in\omega_{\mathcal{X}}^{+}(q)$, and then

$$\varphi\left(t_k,\mathcal{X}_{s_{n_k}-t_k}(q),u_k\right)=\mathcal{X}_{s_{n_k}}(q)\,\varphi(t_k,1,u_k)\to qz=x$$

as $k\to+\infty$, which means $x\in J_{\Sigma}^{+}(q')=\omega_{\Sigma}^{+}(q')\subset\omega_{\Sigma}^{+}(q)$. This proves that $\omega_{\Sigma}^{+}(x)=\omega_{\Sigma}^{+}(q)$.

We now claim that $C^{+}(q)$ is a control set for all $q\in \mathrm{Re}^{+}(\mathcal{X})$. Indeed, for a given $x\in C^{+}(q)$, we have $x\in \mathrm{Re}^{+}(\Sigma)$, and then there is a control function u such that $\omega_{\Sigma}^{+}(\varphi(t,x,u))=\omega_{\Sigma}^{+}(x)$ for all $t>0$ (proof for Thm. 3.1 in [7]). Thus, $\omega_{\Sigma}^{+}(\varphi(t,x,u))=\omega_{\Sigma}^{+}(x)=\omega_{\Sigma}^{+}(q)$ for all $t>0$, which means $\varphi(t,x,u)\in C^{+}(q)$ for all $t>0$. For $x,y\in C^{+}(q)$, we have $y\in\omega_{\Sigma}^{+}(y)=\omega_{\Sigma}^{+}(q)=\omega_{\Sigma}^{+}(x)$, and therefore $C^{+}(q)\subset\omega_{\Sigma}^{+}(x)$ for all $x\in C^{+}(q)$. Thus $C^{+}(q)$ is contained in a control set C. For any $x\in C^{+}(q)$ and $z\in C$, we have $\omega_{\Sigma}^{+}(z)=\omega_{\Sigma}^{+}(x)=\omega_{\Sigma}^{+}(q)$, hence $C=C^{+}(q)$, proving that $C^{+}(q)$ is a control set.

Finally, we suppose that C' is a control set of the system. For a given $z\in C'$, there is $q'\in \mathrm{Re}^{+}(\mathcal{X})$ such that $\omega_{\Sigma}^{+}(z)=\omega_{\Sigma}^{+}(q')$, hence $z\in C^{+}(q')$. As $C^{+}(q')$ is a control set, and $C'\cap C^{+}(q')\neq\emptyset$, it must be $C^{+}(q')=C'$, finishing the proof. □

5. A special case

In this section, we study the special case of linear control system with commutative control vectors. Throughout, the linear control system Σ has control vectors $X_1,\ldots,X_m$ in the center $\mathfrak{z}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. Their associated vector fields X_i^{*} are constant, $X_i^{*}=X_i(1)$, since $X_i(\exp(X))=\mathrm{d}\exp_X(X_i(1))$, for all $X\in\mathfrak{g}$. As $\mathcal{X}^{*}=\mathcal{D}$, we can associate the system Σ with the linear control system on the Lie algebra $\mathfrak{g}$ given by

$$\dot{X}=\mathcal{D}(X)+\sum_{i=1}^{m}u_i(t)\,X_i(1)\,,\qquad X\in\mathfrak{g}. \tag{$\Sigma_{\mathfrak{g}}$}$$

For each $u\in\mathcal{U}_{pc}$ and $X\in\mathfrak{g}$, we denote by $\varphi^{*}(t,X,u)$ the solution of $\Sigma_{\mathfrak{g}}$ with $\varphi^{*}(0,X,u)=X$. By the variation of parameters, we have

$$\varphi^{*}(t,X,u)=\mathrm{e}^{t\mathcal{D}}(X)+\sum_{i=1}^{m}\int_0^t u_i(s)\,\mathrm{e}^{(t-s)\mathcal{D}}(X_i(1))\,\mathrm{d}s.$$

The exponential map is a semiconjugation of the systems Σ and $\Sigma_{\mathfrak{g}}$, indeed $\exp(\varphi^{*}(t,X,u))=\varphi(t,\exp(X),u)$ for all $t\in\mathbb{R}$, $X\in\mathfrak{g}$, and $u\in\mathcal{U}_{pc}$. This implies the following set inclusions:

$$\exp\left(\omega_{\Sigma_{\mathfrak{g}}}^{\pm}(X)\right)\subset\omega_{\Sigma}^{\pm}(\exp(X))\,,\quad \exp\left(J_{\Sigma_{\mathfrak{g}}}^{\pm}(X)\right)\subset J_{\Sigma}^{\pm}(\exp(X))\,,\quad \text{for all } X\in\mathfrak{g}.$$

In this case of linear control system, the derivation $\mathcal{D}$ has more incisive influence on the dynamics. If the origin $0 \in \mathfrak{g}$ is an asymptotically stable equilibrium of $\mathcal{D}$ ($\mathfrak{g} = \mathfrak{g}^+$) the positive limit set of the neutral element is ever nonempty and compact, as the following.

Theorem 5.1. *For the linear control system Σ with commutative control vectors, the positive limit set $\omega_\Sigma^+(1)$ is nonempty and compact, if the neutral element 1 is an attractor of the linear vector field $\mathcal{X}$.*

Proof. We only prove the item (1). Suppose that the neutral element is an attractor of $\mathcal{X}$. By Theorem 4.7, $\omega_\Sigma^+(x) = \omega_\Sigma^+(1)$, for all $x \in G$. Consider $\epsilon > 0$ such that $\max\{|u_1|, \ldots, |u_m|\} \leq \epsilon$ for all $u = (u_1, \ldots, u_m) \in U$. Take numbers $\lambda < 0$ and $B > 0$ such that $\left\|\mathrm{e}^{t\mathcal{D}}(X)\right\| \leq B\mathrm{e}^{\lambda t}\|X\|$ for all $X \in \mathfrak{g}$ and $t > 0$. As $\varphi^*(t,0,u) = \sum\limits_{i=1}^{m}\int_0^t u_i(s)\,\mathrm{e}^{(t-s)\mathcal{D}}(X_i(1))\,\mathrm{d}s$, we have

$$\begin{aligned}
\|\varphi^*(t,0,u)\| &= \left\|\sum_{i=1}^{m}\int_0^t u_i(s)\,\mathrm{e}^{(t-s)\mathcal{D}}(X_i(1))\,\mathrm{d}s\right\| \\
&\leq \sum_{i=1}^{m}\int_0^t \epsilon B\mathrm{e}^{\lambda(t-s)}\|X_i(1)\|\,\mathrm{d}s \\
&= \sum_{i=1}^{m}\lambda\epsilon B\left(\mathrm{e}^{\lambda t}-1\right)\|X_i(1)\|
\end{aligned}$$

where $\lim\limits_{t\to+\infty}\sum\limits_{i=1}^{m}\lambda\epsilon B\left(\mathrm{e}^{\lambda t}-1\right)\|X_i(1)\| = -\sum\limits_{i=1}^{m}\lambda\epsilon B\|X_i(1)\|$. This means that the positive orbit $\mathcal{O}^+(0)$ is bounded, and therefore $\omega_{\Sigma_\mathfrak{g}}^+(0)$ is nonempty and compact. It follows that $\exp\left(\omega_{\Sigma_\mathfrak{g}}^+(0)\right)$ is nonempty and compact, and $\exp\left(\omega_{\Sigma_\mathfrak{g}}^+(0)\right) \subset \omega_\Sigma^\pm(1)$. Moreover, since $\omega_{\Sigma_\mathfrak{g}}^+(0)$ is positively invariant, the set $\exp\left(\omega_{\Sigma_\mathfrak{g}}^+(0)\right)$ is positively invariant. For a given $x \in \exp\left(\omega_{\Sigma_\mathfrak{g}}^+(0)\right)$, it follows that $\omega_\Sigma^+(1) = \omega_\Sigma^+(x) \subset \exp\left(\omega_{\Sigma_\mathfrak{g}}^+(0)\right)$. This means that $\omega_\Sigma^+(1) = \exp\left(\omega_{\Sigma_\mathfrak{g}}^+(0)\right)$, proving the item (1). □

When the derivation $\mathcal{D}$ is not a stable operator, same additional conditions on the control vectors assure the compactness of the limit set of the origin and the existence of control sets. For the following, $X_i(1)^{-,0}$ means the component of $X_i(1)$ in $\mathfrak{g}^{-,0}$.

Theorem 5.2. *The linear control system Σ with commutative control vectors admits a prolongational control set in the following cases:*

1. *$X_i(1) \in \mathfrak{g}^-$ for all $i = 1, \ldots, m$. In this case, the associated system $\Sigma_\mathfrak{g}$ is asymptotically compact at 0, and $\omega_{\Sigma_\mathfrak{g}}^+(0)$ is nonempty and compact.*
2. *$\left\{\sum\limits_{i=1}^{m}u_iX_i(1)^{-,0} : u \in U\right\} \cap \operatorname{Im}\mathcal{D}|_{\mathfrak{g}^{-,0}} \neq \emptyset$. In this case, there is a bounded negative trajectory of $\Sigma_\mathfrak{g}$ starting from a point $Z \in \mathfrak{g}^{-,0}$.*
3. *$\left\{\sum\limits_{i=1}^{m}u_iX_i(1) : u \in U\right\} \cap \operatorname{Im}\mathcal{D} \neq \emptyset$. In this case, there is a stationary trajectory of $\Sigma_\mathfrak{g}$.*
4. *$\mathcal{D}$ is surjective. In this case, there is a stationary trajectory of $\Sigma_\mathfrak{g}$ through each point $-\sum_{i=1}^{m}u_i\mathcal{D}^{-1}(X_i(1))$, for $u \in U$.*

Proof. Suppose that the condition (1) holds. There are numbers $\lambda > 0$ and $B > 0$ such that $\left\|\mathrm{e}^{t\mathcal{D}}(Y)\right\| \leq B\mathrm{e}^{-\lambda t}\|Y\|$ for all $Y \in \mathfrak{g}^-$ and $t \geq 0$. Consider $\epsilon > 0$ such that $\max\{|u_1|, \ldots, |u_m|\} \leq \epsilon$ for all $u = (u_1, \ldots, u_m) \in$

U. We then have

$$\left\|\varphi^{*}\left(t,0,u\right)\right\|\leq\sum_{i=1}^{m}\lambda\epsilon B\left(1-\mathrm{e}^{-\lambda t}\right)\left\|X_{i}\left(1\right)\right\|<\lambda\epsilon B\sum_{i=1}^{m}\left\|X_{i}\left(1\right)\right\|$$

for all $t>0$ and $u\in\mathcal{U}_{pc}$. Hence the positive orbit $\mathcal{O}^{+}\left(0\right)$ is bounded, which implies that the control system is asymptotically compact at the origin, and $\omega_{\Sigma_{\mathfrak{g}}}^{+}\left(0\right)$ is nonempty and compact.

We now assume that the condition (2) holds. There are elements $Z^{-,0}\in\mathfrak{g}^{-,0}$ and $u\in U$ such that $\mathcal{D}\left(Z^{-,0}\right)=\sum\limits_{i=1}^{m}u_{i}X_{i}\left(1\right)^{-,0}$. We then have

$$\begin{aligned}\varphi^{*}\left(t,0^{-,0},u\right)&=\mathrm{e}^{t\mathcal{D}}\int_{0}^{t}\mathrm{e}^{-s\mathcal{D}}\left(\sum_{i=1}^{m}u_{i}X_{i}\left(1\right)^{-,0}\right)\mathrm{d}s\\&=\mathrm{e}^{t\mathcal{D}}\int_{0}^{t}\mathrm{e}^{-s\mathcal{D}}\left(\mathcal{D}\left(Z^{-,0}\right)\right)\mathrm{d}s\\&=\mathrm{e}^{t\mathcal{D}}\left(Z^{-,0}\right)-Z^{-,0}\end{aligned}$$

hence

$$\mathrm{e}^{t\mathcal{D}}\left(-Z^{-,0}\right)+\varphi^{*}\left(t,0^{-,0},u\right)=-Z^{-,0}$$

which means $\varphi^{*}\left(t,-Z^{-,0},u\right)=-Z^{-,0}$ for all $t\in\mathbb{R}$, a stationary trajectory in $\mathfrak{g}^{-,0}$. Define $Z=-Z^{-,0}+0^{+}$. Since $\varphi^{*}\left(t,0^{+},u\right)$ is bounded at $t<0$, it follows that $\varphi^{*}\left(t,Z,u\right)$ is bounded at $t<0$.

If the condition (3) holds, there are elements $Z\in\mathfrak{g}$ and $u\in U$ such that $\mathcal{D}\left(Z\right)=\sum\limits_{i=1}^{m}u_{i}X_{i}\left(1\right)$, and then $\varphi^{*}\left(t,-Z,u\right)$ is a stationary trajectory. If the condition (4) holds, $\varphi^{*}\left(t,-\sum_{i=1}^{m}u_{i}\mathcal{D}^{-1}\left(X_{i}\left(1\right)\right),u\right)$ is a stationary trajectory, for all $u\in U$. □

If the condition (1) of Theorem 5.2 holds, $\exp\left(\omega_{\Sigma_{\mathfrak{g}}}^{+}\left(0\right)\right)$ is a nonempty, compact, and progressively invariant set into the central limit set $\omega_{\Sigma}^{+}\left(1\right)$, hence there is a precompact control set in $\omega_{\Sigma}^{+}\left(1\right)$.

6. Examples

This section contains illustrative examples for the results of the paper.

Example 6.1. Consider the neutral element component G of the affine group $\mathrm{Aff}\left(\mathbb{R}\right)$:

$$G=\left\{\begin{pmatrix}a&b\\0&1\end{pmatrix}:a>0,b\in\mathbb{R}\right\}.$$

The exponential map $\exp:\mathfrak{aff}\left(\mathbb{R}\right)\rightarrow G$ is given by $\exp\begin{pmatrix}a&b\\0&0\end{pmatrix}=\begin{pmatrix}\mathrm{e}^{a}&\frac{\mathrm{e}^{a}-1}{a}b\\0&1\end{pmatrix}$, $\exp\begin{pmatrix}0&b\\0&0\end{pmatrix}=\begin{pmatrix}1&b\\0&1\end{pmatrix}$. Consider the linear control system on G given by

$$\begin{gathered}\dot{x}=\mathcal{X}\left(x\right)+u\left(t\right)X_{1}\left(x\right),\\u\in\mathcal{U}_{pc}=\{u:\mathbb{R}\rightarrow[2,5]:u\text{ piecewise constant}\},\end{gathered}$$

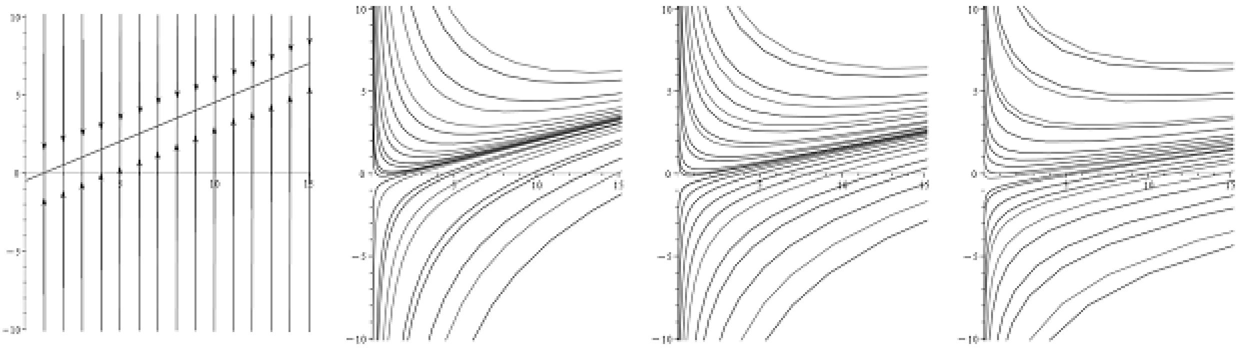

FIGURE 1. On the left, trajectories of the linear vector field $\mathcal{X}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a-2b-1 \\ 0 & 0 \end{pmatrix}$ on the neutral element component G of the affine group $\mathrm{Aff}(\mathbb{R})$, with the line of equilibria $a-2b-1=0$. On the right, trajectories of the dispersive linear control system $\dot{x} = \mathcal{X}(x) + u(t) X_1(x)$, with control vector $X_1(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and control range $U = [2,5]$; plots for the constant controls $u \equiv 2, 3, 5$. Phase portrait represented in $\mathbb{R}_*^+ \times \mathbb{R}$.

where

$$\mathcal{X}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a-2b-1 \\ 0 & 0 \end{pmatrix}, \qquad X_1(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We have $\mathcal{X}^* = \mathcal{D} = \mathrm{ad}\begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix}$, an internal derivation. Its standard matrix form is given by $[\mathcal{D}] = \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}$. The group of the equilibria $\mathrm{Eq}(\mathcal{X})$ coincides with the line $a-2b-1=0$, and the trajectories of $\mathcal{X}$ are vertical lines (considering the identification of G with $\mathbb{R}_*^+ \times \mathbb{R}$). All the equilibria of $\mathcal{X}$ are stable. The general solution through $x = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ is of the form

$$\begin{aligned} \varphi(t,x,u) &= \begin{pmatrix} a\mathrm{e}^{\int_0^t u(s)\mathrm{d}s} & b\mathrm{e}^{-2t} + a\int_0^t \mathrm{e}^{\left(2s-2t+\int_0^s u(\tau)\mathrm{d}\tau\right)}\mathrm{d}s + \frac{\mathrm{e}^{-2t}-1}{2} \\ 0 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} a & b\mathrm{e}^{-2t} + (a-1)\frac{1-\mathrm{e}^{-2t}}{2} \\ 0 & 1 \end{pmatrix}}_{\mathcal{X}_t(x)} \underbrace{\begin{pmatrix} \mathrm{e}^{\int_0^t u(s)\mathrm{d}s} & \int_0^t \mathrm{e}^{\left(2s-2t+\int_0^s u(\tau)\mathrm{d}\tau\right)}\mathrm{d}s + \frac{\mathrm{e}^{-2t}-1}{2} \\ 0 & 1 \end{pmatrix}}_{\varphi(t,1,u)}. \end{aligned}$$

We have $2t \leq \int_0^t u(s)\,\mathrm{d}s$ for all $u \in \mathcal{U}_{pc}$ and $t > 0$. For any sequences $t_n \to +\infty$ and $(u_n) \subset \mathcal{U}_{pc}$, it follows that $\lim_{n\to+\infty} \mathrm{e}^{\int_0^t u_m(s)\mathrm{d}s} = +\infty$, and therefore $\omega_\Sigma^+(1) = \emptyset$. By Proposition 4.3, this system is dispersive. See illustrations for the trajectories of this system in Figure 1.

Example 6.2. Consider again the neutral element component G of the affine group $\mathrm{Aff}(\mathbb{R})$, and the linear control system on G given by

$$\begin{gathered} \dot{x} = \mathcal{X}(x) + u(t) X_1(x), \\ u \in \mathcal{U}_{pc} = \{u : \mathbb{R} \to [p,q] : u \text{ piecewise constant}\}, \end{gathered}$$

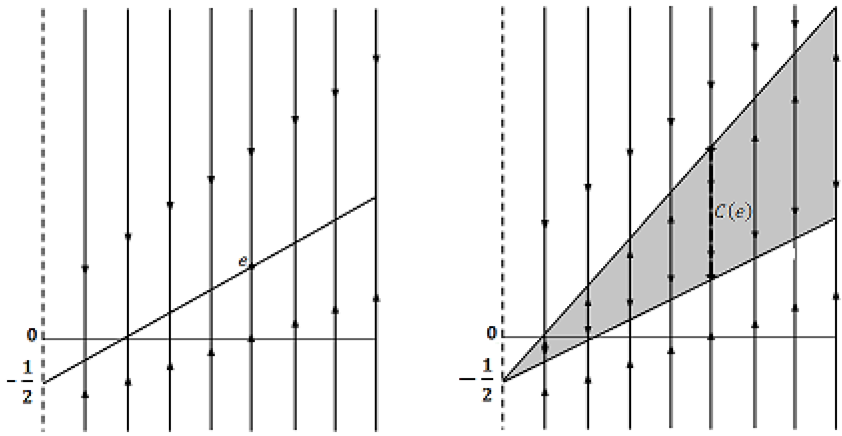

FIGURE 2. On the left, trajectories of the linear vector field $\mathcal{X}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a-2b-1 \\ 0 & 0 \end{pmatrix}$ on the neutral element component G of the affine group $\mathrm{Aff}(\mathbb{R})$, with the line of the equilibria $a-2b-1=0$. On the right, the phase portrait of the control system $\dot{x} = \mathcal{X}(x) + u(t) X_1(x)$, with control vector $X_1(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; nondispersive system with control sets $C(e)$ $(e \in \mathrm{Eq}(\mathcal{X}))$.

where

$$\mathcal{X}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a-2b-1 \\ 0 & 0 \end{pmatrix}, \qquad X_1(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This system is not dispersive, even if $0 \notin [p,q]$. Indeed, the general solution through $x = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ is of the form

$$\begin{aligned} \varphi(t,x,u) &= \begin{pmatrix} a & b\mathrm{e}^{-2t} + \frac{a-1}{2}\left(1-\mathrm{e}^{-2t}\right) + a\mathrm{e}^{-2t}\int_0^t u(s)\,\mathrm{e}^{2s}\mathrm{d}s \\ 0 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} a & b\mathrm{e}^{-2t} + (a-1)\,\frac{1-\mathrm{e}^{-2t}}{2} \\ 0 & 1 \end{pmatrix}}_{\mathcal{X}_t(x)} \underbrace{\begin{pmatrix} 1 & \mathrm{e}^{-2t}\int_0^t u(s)\,\mathrm{e}^{2s}\mathrm{d}s \\ 0 & 1 \end{pmatrix}}_{\varphi(t,1,u)}. \end{aligned}$$

Note that $\lim_{t\to+\infty} \mathcal{X}_t(x) = \begin{pmatrix} a & \frac{a-1}{2} \\ 0 & 1 \end{pmatrix} = e \in \mathrm{Eq}(\mathcal{X})$. Suppose that $0 < p$. Since $p \le u(s) \le q$, we have $p\dfrac{1-\mathrm{e}^{-2t}}{2} \le \mathrm{e}^{-2t}\int_0^t u(s)\,\mathrm{e}^{2s}\mathrm{d}s \le q\dfrac{1-\mathrm{e}^{-2t}}{2}$, and then $\dfrac{p}{2} \le \lim_{t\to+\infty} \mathrm{e}^{-2t}\int_0^t u(s)\,\mathrm{e}^{2s}\mathrm{d}s \le \dfrac{q}{2}$. Thus

$$\begin{aligned} \omega_\Sigma^+(1) &= \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} : v \in \left[\frac{p}{2}, \frac{q}{2}\right] \right\}, \\ \omega_\Sigma^+(x) = \omega_\Sigma^+(e) &= \left\{ \begin{pmatrix} a & av + \frac{a-1}{2} \\ 0 & 1 \end{pmatrix} : v \in \left[\frac{p}{2}, \frac{q}{2}\right] \right\}. \end{aligned}$$

By Theorem 4.7, the control sets of the system are the sets $C(e) = \omega_\Sigma^+(e)$, $e \in \mathrm{Eq}(\mathcal{X})$, and $\mathrm{Re}^+(\Sigma) = \mathrm{NW}(\Sigma) = \bigcup_{e\in\mathrm{Eq}(\mathcal{X})} \omega_\Sigma^+(e)$. See illustrations in Figure 2.

Example 6.3. Let G be the three-dimensional Heisenberg group:

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Consider the standard basis $\mathcal{B} = \{E_1, E_2, E_3\}$ for the Heisenberg algebra $\mathfrak{g}$:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $[E_1, E_3] = E_2$, and $[E_i, E_j] = 0$ in all the other cases of i, j. In this basis, a derivation $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ has matrix form

$$\mathcal{D} = \begin{pmatrix} a & 0 & b \\ c & d & e \\ f & 0 & d - a \end{pmatrix}$$

with eigenvalues

$$\lambda_1 = d, \qquad \lambda_2 = \frac{d + \sqrt{(2a - \mathrm{d})^2 + 4bf}}{2}, \qquad \lambda_3 = \frac{d - \sqrt{(2a - \mathrm{d})^2 + 4bf}}{2}.$$

Consider the linear control system on G given by

$$\begin{gathered} \dot{x} = \mathcal{X}(x) + u(t) E_1(x), \\ u \in \mathcal{U}_{pc} = \{u : \mathbb{R} \to [p, q] : u \text{ piecewise constant}\}, \end{gathered}$$

where

$$\mathcal{X}^* = \mathcal{D} = \begin{pmatrix} 0 & 0 & 0 \\ c & d & e \\ f & 0 & d \end{pmatrix}, \qquad d < 0.$$

This is a case $\mathfrak{g} = \mathfrak{g}^- \oplus \ker \mathcal{D}$. The associated vector field of E_1 is given by

$$E_1^*(X) = \frac{\mathrm{d}}{\mathrm{d}t} tE_1 * X|_{t=0} = E_1 + \frac{1}{2}[E_1, X]$$

where $tE_1 * X$ is the Baker–Campbell–Hausdorff product of tE_1 and X in $\mathfrak{g}$. The associated control system on the Lie algebra is given by

$$\dot{X} = \mathcal{D}(X) + \frac{u(t)}{2} \mathrm{ad}(E_1)(X) + u(t) E_1, \qquad X \in \mathfrak{g}.$$

As $\mathcal{D}(E_1) = 0$, it follows that $\mathcal{D}$ and $\mathrm{ad}(E_1)$ commute, and then the solution for this system in $\mathfrak{g}$ is of the form

$$\varphi^*(t, X, u) = \mathrm{e}^{t\mathcal{D}} \mathrm{e}^{\int_0^t \frac{u(s)}{2} \mathrm{ad}(E_1)\, \mathrm{d}s}(X) + \int_0^t u(\tau) E_1 \, \mathrm{d}\tau$$

where $\varphi^*(t,0,u) = \int_0^t u(\tau) E_1 \, d\tau$. Since the exponential map is a diffeomorphism, the linear control system is dispersive, if $\omega^+_{\Sigma_\mathfrak{g}}(0) = \emptyset$. Indeed, if $0 \notin [p,q]$, the sequence $\left(\int_0^{t_n} u(\tau)\right)$ diverges as $t_n \to +\infty$, hence $\omega^+_{\Sigma_\mathfrak{g}}(0) = \emptyset$.

7. Final comments and open problems

In this paper, both dispersive and control problems were studied in the framework of linear control systems on Lie groups. The results for the dispersive problem automatically furnished information for the control problem, and vice-versa. For a stable linear vector field, the dynamics of the control system depends on the limit behavior through the neutral element: the system is dispersive, if the trajectories through the neutral element have no limit at infinity; it admits a control set, if the limit set at the neutral element is nonempty. For the case of linear control system with commutative control vectors, the hypothesis of asymptotic stability on the linear part of the system entails nondispersiveness and the existence of a unique control set, that is the limit set at the neutral element. In the commutative case, a control set ever exists if the derivation operator is surjective, including the hyperbolic case. All these results contribute with the knowledge of control theory, furnishing a study on the opposite concepts of dispersiveness and controllability. We think the dispersive problem may be examined in particular cases of Lie groups, for instance, in the nilpotent and solvable cases, motivated by the decomposition theorems obtained in [12].

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References

[1] J. Dugundji and H.A. Antosiewicz, Parallelizable flows and Lyapunov's second method. *Ann. Math.* **73** (1961) 543–555.

[2] O. Hájek, Parallelizability revisited. *Proc. Amer. Math. Soc.* **27** (1971) 77–84.

[3] J.A. Souza and H.V.M. Tozatti, Prolongational limit sets of control systems, *J. Diff. Equ.* **254** (2013) 2183–2195.

[4] J.A. Souza and H.V.M. Tozatti, Some aspects of stability for semigroup actions and control systems, *J. Dyn. Diff. Equ.* **26** (2014) 631–654.

[5] J.A. Souza, Sufficient conditions for dispersiveness of invariant control affine systems on the Heisenberg group. *Syst. Control Lett.* **124** (2019) 68–74.

[6] A.L. Marques, H.M.V. Tozatti, J.A. Souza, Higher prolongations of control affine systems: absolute stability and generalized recurrence. *SIAM J. on Control Optim.* **58** (2020) 3019–3040.

[7] J.A. Souza, Parallelizability of control systems. *Math. Control Signals Syst.* **33** (2021) 259–278.

[8] V. Ayala and L.A.B. San Martin, *Controllability properties of a class of control systems on Lie groups*, Lecture Note in Control and Inform. Sci. 258. Springer, Berlin (2001) 83–92.

[9] V. Ayala and J. Tirao, *Linear Control Systems on Lie Groups and Controllability*, edited by G. Ferreyra *et al.* American Mathematical Society, Providence, RI (1999).

[10] V. Ayala, A.J. Da Silva and G. Zsigmond, Control sets of linear systems on Lie groups. *Nonlinear Differ. Equ. Appl.* **24** (2017) 1021–9722.

[11] A.J. Da Silva, Outer invariance entropy for linear systems on Lie groups. *J. Dyn. Control Syst.* **52** (2014) 3917–3934.

[12] A.J. Da Silva, Controllability of linear systems on solvable Lie groups. *SIAM J. Control Optim.* **54** (2016) 372–390.

[13] Ph. Jouan, Controllability of linear systems on Lie groups. *J. Dyn. Control Syst.* **17** (2011) 591–616.

[14] Ph. Jouan, Equivalence of control systems with linear systems on Lie groups and homogeneous spaces. *ESAIM Control Optim. Calc. Var.* **16** (2010) 956–973.

[15] Ph. Jouan and M. Dath, Controllability of linear systems on low dimensional nilpotent and solvable Lie groups. *J. Dyn. Control Syst.* **22** (2016) 207–225.

[16] L.A.B. San Martin, Álgebras de Lie, 2nd edn. Editora Unicamp, Campinas (2010).

[17] J.A. Souza, Decomposition of linear systems on disconnected Lie groups. *J. Dyn. Diff. Equ.* (2023). https://doi.org/10.1007/s10884-023-10287-x
[18] F. Colonius and W. Kliemann, The Dynamics of Control. Birkhäuser, Boston (2000).
[19] J.A. Souza, Poisson stability and periodicity of control affine systems. *J. Optim. Theory Appl.* **185** (2020) 72–79.
[20] V. Jurdjevic, Geometric Control Theory. Cambridge University Press, Cambridge (1997).
[21] E.E. Sontag, Mathematical Control Theory. Springer-Verlag, New York (1990).
[22] J.A. Souza, Prolongational controllability and weak attraction for control affine systems. *J. Dyn. Control Syst.* **27** (2021) 335–353.

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