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# GLOBAL EXACT CONTROLLABILITY OF THE VISCOUS AND RESISTIVE MHD SYSTEM IN A RECTANGLE THANKS TO THE LATERAL SIDES AND TO DISTRIBUTED PHANTOM FORCES

Jiajiang Liao*

**Abstract.** We consider the 2-D incompressible viscous and resistive magnetohydrodynamics (MHD) system in a rectangle, with controls on the lateral sides. The velocity satisfies Dirichlet boundary conditions, while the magnetic field follows perfectly conducting wall boundary conditions on the remaining, uncontrolled part of the boundary. We extend the small-time global exact null controllability result of Coron *et al.* in [*Ann PDE* **5** (2019) 1–49] from Navier–Stokes equations to MHD equations, with a little help of distributed phantom forces, which can be chosen arbitrarily small in any given Sobolev spaces. Our analysis relies on Coron's return method, the well-prepared dissipation method, long-time nonlinear Cauchy–Kovalevskaya estimates and Badra's local exact controllability result.

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## 1. Introduction and the main result

### 1.1. Setting

In this paper, we investigate an incompressible viscous and diffusive magnetohydrodynamic (MHD) system in a 2-D rectangular domain. The velocity, pressure and magnetic field are denoted by $u = {}^t(u_1, u_2), p$ and $B = {}^t(B_1, B_2)$, respectively. Let $\Omega = (0, L) \times (-1, 1), L > 0$, and let $\Gamma = (\{0\} \times [-1, 1]) \cup (\{L\} \times [-1, 1])$ the union of the left and right boundary of $\Omega$. Assume that we can act some controls on $\Gamma$ and on the remaining part of the boundary $\partial\Omega\backslash\Gamma$, $u$ satisfies Dirichlet boundary conditions and $B$ satisfies perfectly conducting wall boundary conditions. Hence $(u, p, B)$ satisfies:

$$\begin{cases} \partial_t u + u \cdot \nabla u - B \cdot \nabla B - \Delta u + \nabla p = 0 & \text{in } \Omega, \\ \partial_t B + u \cdot \nabla B - B \cdot \nabla u - \Delta B = 0 & \text{in } \Omega, \\ \operatorname{div} u = \operatorname{div} B = 0 & \text{in } \Omega, \\ u = 0, \quad B_2 = 0 \quad \text{and} \quad \partial_y B_1 = 0 & \text{on } \partial\Omega\backslash\Gamma, \end{cases} \tag{1.1}$$

*Keywords and phrases:* Controllability, magnetohydrodynamics, boundary layers, multi-scales asymptotic expansion, well-prepared dissipation method, long-time nonlinear Cauchy–Kovalevskaya estimates.

School of Mathematical Sciences, Beihang University, 102206 Beijing, PR China.

* Corresponding author: jjliao@buaa.edu.cn

where we assumed the viscosity and the resistivity coefficient to be one for simplicity purpose. One controls the part $\Gamma$ on the boundary, so that no boundary condition is required there.

## 1.2. Main result

Let us introduce the space

$$L^2_{\mathrm{div}}(\Omega) := \{u = {}^t(u_1, u_2) \in L^2(\Omega;\mathbb{R}^2) | \operatorname{div} u = 0 \text{ in } \Omega, u_2 = 0 \text{ on } \partial\Omega\backslash\Gamma\}.$$

For a vector field $u \in L^2(\Omega;\mathbb{R}^2)$ with $\operatorname{div} u = 0$ in the sense of distribution, we get by Theorem III.2.2 of [14] that $u \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\partial\Omega)$ for the outward unit normal vector $\mathbf{n}$. The condition $u_2 = 0$ on $\partial\Omega\backslash\Gamma$ holds in the following sense:

$$\langle u \cdot \mathbf{n}, \gamma(\varphi)\rangle_{\partial\Omega} = \int_\Omega u \cdot \nabla\varphi = 0, \quad \forall\, \varphi \in H^1(\Omega) \text{ with } \gamma(\varphi) = 0 \text{ on } \Gamma,$$

where $\gamma(\varphi)$ is the trace of $\varphi$ at $\partial\Omega$.

For a Banach space $X$ and $f_1, f_2 \in X$, throughout this paper we define, for simplicity,

$$\|(f_1, f_2)\|_X := \left(\|f_1\|_X^2 + \|f_2\|_X^2\right)^{\frac{1}{2}}. \tag{1.2}$$

Our main result is the following theorem.

**Theorem 1.1.** *Let $T > 0$, assume that $u_0, B_0 \in L^2_{\mathrm{div}}(\Omega), u_0 = {}^t(u_{0,1}, u_{0,2}), B_0 = {}^t(B_{0,1}, B_{0,2})$ and satisfy*

$$\int_\Omega u_{0,1}(x, y)\mathrm{d}x\mathrm{d}y = \int_\Omega B_{0,1}(x, y)\mathrm{d}x\mathrm{d}y = 0. \tag{1.3}$$

*For any $k \in \mathbb{N}$ and $\delta > 0$, there exist two phantom forces $f, g \in L^1((0, T); H^k(\Omega)$ satisfying $\|(f, g)\|_{L^1((0,T);H^k(\Omega))} \le \delta$, a Leray weak solution $(u, B) \in C_w([0, T]; L^2_{\mathrm{div}}(\Omega)) \cap L^2((0, T); H^1(\Omega))$ of*

$$\begin{cases} \partial_t u + u \cdot \nabla u - B \cdot \nabla B - \Delta u + \nabla p = f & \text{in } (0, T) \times \Omega, \\ \partial_t B + u \cdot \nabla B - B \cdot \nabla u - \Delta B = g & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = \operatorname{div} B = 0 & \text{in } (0, T) \times \Omega, \\ u = 0, \quad B_2 = 0 \quad \text{and} \quad \partial_y B_1 = 0 & \text{on } (0, T) \times \partial\Omega\backslash\Gamma, \end{cases} \tag{1.4}$$

*satisfying $(u(0, \cdot), B(0, \cdot)) = (u_0, B_0)$ and $(u(T, \cdot), B(T, \cdot)) = (0, 0)$.*

**Remark 1.2.** The zero-integral condition (1.3) is a necessary and sufficient condition such that the initial data $(u_0, B_0)$ can be extended to $L^2_{\mathrm{div}}(\mathcal{B})$ with $\mathcal{B} = \mathbb{R} \times (-1, 1)$, see (1.8) the definition of $L^2_{\mathrm{div}}(\mathcal{B})$ and Lemma C.1 for more explanation.

## 1.3. Notation of controlled Leray weak solution

In Theorem 1.1, the notation of Leray weak solution corresponds to the following weak formulation:

$$\begin{aligned}
&-\int_0^T\int_\Omega (u\cdot\partial_t\varphi + B\cdot\partial_t\psi) - \int_0^T\int_\Omega \big((u\otimes u - B\otimes B):\nabla\varphi + (B\otimes u - u\otimes B):\nabla\psi\big)\\
&+2\int_0^T\int_\Omega D(u):D(\varphi) + 2\int_0^T\int_\Omega D(B):D(\psi)\\
&=\int_\Omega \big(u_0\cdot\varphi(0,\cdot) + B_0\cdot\psi(0,\cdot)\big) + \int_0^T\int_\Omega (f\cdot\varphi + g\cdot\psi),
\end{aligned} \tag{1.5}$$

for any test functions $\varphi = {}^t(\varphi_1,\varphi_2), \psi = {}^t(\psi_1,\psi_2) \in C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^2)$ which are divergence free, vanish both on the boundary $\Gamma$ and at time $t=T$, and satisfy $\varphi_1 = \psi_2 = 0$ on $\partial\Omega\backslash\Gamma$. The notation $D(\cdot)$ stands for the symmetric part of the gradient and for $2\times 2$ matrix $M_1, M_2$, $M_1 : M_2$ denotes the trace of $M_1M_2$. Thus (1.5) encodes the no-slip conditions for $u$ and Navier boundary conditions for $B$ on $\partial\Omega\backslash\Gamma$ only and one has control on $\Gamma$.

## 1.4. Related literature

Theorem 1.1 establishes the small-time global exact null controllability of the incompressible MHD equations in a 2-D rectangular domain in the case where no-slip Dirichlet boundary conditions is imposed on the upper and lower boundary for the velocity and perfectly conducting wall boundary conditions is imposed in the same place for the magnetic field, while we are able to act on the velocity and magnetic field on the left and right boundary of the rectangular domain, as well as in the interior of the rectangle through a distributed phantom force which can be chosen arbitrarily small for any given Sobolev regularity space. As far as I know, this is the first global exact controllability result for the incompressible viscous and resistive MHD equations.

The proof of Theorem 1.1 relies on a combination of Coron's return method, the well-prepared dissipation method, long-time nonlinear Cauchy–Kovalevskaya estimates and Badra's local exact controllability result. The return method was initially introduced by Coron [7] to prove the controllability of the 2-D Euler equation (also see [8, 9, 15]). The well-prepared dissipation method was introduced by Marbach [22] for the 1-D Burgers equation and has been used in [10, 11, 20, 21] for the Navier stokes equations.

Similar to the Navier–Stokes equations, the presence of no-slip Dirichlet boundary conditions for the velocity generates a large boundary layer near the uncontrolled boundary for the velocity field, leading to derivative losses during the estimate of the remainder. To overcome this difficulty, we utilize a long-time nonlinear Cauchy-Kovalevskaya estimate to analyze the remainder as in [11, 21], after we regularized the initial data into horizontally analytic functions with arbitrarily large analytic radius. This idea can be traced back to the influential works of Sammartino and Caflisch [26, 27].

For the controllability results of the MHD equation, Badra achieved local exact controllability to trajectories by employing two internal controls with no-slip boundary conditions for velocity and perfectly conducting wall boundary conditions for the magnetic field [1]. This result will be utilized once we reduce the size of the initial data through a well-prepared dissipation method. Rissel *et al.* demonstrated small-time global approximate controllability for the incompressible MHD in a 2-D simply-connected domain with coupled Navier boundary conditions for velocity and Navier boundary conditions for the magnetic field [24]. In the case of a 3-D domain and general coupled Navier boundary conditions, they obtained a similar conclusion by introducing a pressure term and an additional force in the equation of the magnetic field.

Kukavica *et al.* proposed a sufficient and necessary condition on the initial data in order for the ideal MHD equations to be null controllable in a rectangular domain, with controls on the side boundary [18]. If this condition is not satisfied, a simple shear external magnetic force can assist in transitioning from one state to another. Rissel *et al.* proved the global exact controllability of the idea incompressible MHD equations in a rectangle, with controls on the vertical boundary and with the assistance of a pressure term in the equation of the magnetic field [25].

The existence of global weak solutions with finite energy (the counterpart of the celebrated result by Leray for the incompressible Navier–Stokes system) and local strong solutions to the resistive and viscous MHD system in 2-D or 3-D have been demonstrated in [12]. Furthermore, for smooth initial data, they proved the smoothness and uniqueness of global weak solutions in the 2-D case. On the other hand, in [29], the authors proved the uniqueness of the local strong solutions in 3-D, together with some regularity criteria. In [19], the authors demonstrated the existence and uniqueness for both the local strong solution with large initial data and the global strong solution with small initial data, as well as the weak-strong uniqueness of solutions. In Appendix A, we will provide a specific theorem regarding the smoothness and uniqueness of Leray weak solution for the 2-D incompressible MHD equations with $L^2$ initial data for completeness.

### 1.5. Strategy of the proof of theorem 1.1

The proof of Theorem 1.1 is divided into three steps and will be explained in the following paragraphs. Firstly, we regularize the initial data into a horizontally analytic function with an arbitrary analyticity radius. Then we establish that such a function can be driven approximately to zero. Finally, we utilize Badra's local exact controllability result.

**Step 1.** We extend the rectangle into the horizontal band

$$\mathcal{B} := \mathbb{R} \times (-1, 1). \tag{1.6}$$

Consider the MHD equations in the extended domain

$$\begin{cases} \partial_t u + u \cdot \nabla u - B \cdot \nabla B - \Delta u + \nabla p = \xi + f & \text{in } (0,T) \times \mathcal{B}, \\ \partial_t B + u \cdot \nabla B - B \cdot \nabla u - \Delta B = \eta + g & \text{in } (0,T) \times \mathcal{B}, \\ \operatorname{div} u = \operatorname{div} B = 0 & \text{in } (0,T) \times \mathcal{B}, \\ u = 0, \quad B_2 = 0 \quad \text{and} \quad \partial_y B_1 = 0 & \text{on } (0,T) \times \partial\mathcal{B}, \end{cases} \tag{1.7}$$

where $\xi, \eta$ are control functions supported in $\overline{\mathcal{B}\backslash\Omega}$, and $f, g$ are forces supported in $\overline{\Omega}$. We introduce the domain

$$\mathcal{G} := [-4L, -3L] \times [-1, 1],$$

and the space

$$L^2_{\operatorname{div}}(\mathcal{B}) = \{u = {}^t(u_1, u_2) \in L^2(\mathcal{B}; \mathbb{R}^2) | \operatorname{div} u = 0 \text{ in } \mathcal{B}, u_2 = 0 \text{ on } \partial\mathcal{B}\}. \tag{1.8}$$

**Proposition 1.3.** *Let $T > 0, \rho_b > 0$ and $(u_0, B_0) \in L^2_{\operatorname{div}}(\Omega)$ which satisfies $\int_\Omega u_{0,1}(x,y)\mathrm{d}x\mathrm{d}y = \int_\Omega B_{0,1}(x,y)\mathrm{d}x\mathrm{d}y = 0$. For any $k \in \mathbb{N}$ and $\delta > 0$, there exists an extension $(u_a, B_a) \in L^2(\mathcal{B})$ of $(u_0, B_0)$ to the domain $\mathcal{B}$, control forces $\xi$ and $\eta$ in $C^\infty([0,T] \times \overline{\mathcal{B}\backslash\Omega})$, forces $f$ and $g$ in $C^\infty([0,T] \times \overline{\Omega})$ satisfying*

$$\|(f, g)\|_{L^1(0,T;H^k(\Omega))} \le \delta, \tag{1.9}$$

*a constant $C_b$ and a Leray weak solution $(u, B) \in C_w([0,T]; L^2_{\operatorname{div}}(\mathcal{B})) \cap L^2((0,T); H^1(\mathcal{B}))$ to (1.7) associated with the initial data $(u_a, B_a)$ and $(u_b, B_b) := (u(T, \cdot), B(T, \cdot))$ satisfies*

$$\|(u_b, B_b)|_{\mathcal{G}}\|_{H^k(\mathcal{G})} \le \delta, \tag{1.10}$$

$$\forall m \ge 0, \quad \|\partial_x^m (u_b, B_b)\|_{H^3(\mathcal{B})} \le \frac{m!}{\rho_b^m} C_b, \tag{1.11}$$

$$\|(u_b, B_b)\|_{L^1_x(H^2_y)(\mathcal{B})} \leq C_b. \tag{1.12}$$

The proof of Proposition 1.3 will be presented in Appendix B. With this proposition, we can regularize the initial data (after extending it to $\mathcal{B}$) into a horizontally analytic function with arbitrary analyticity radius.

**Step 2.** We prove that the new initial data can be driven approximately to zero, which is the key proposition for constructing the main theorem.

**Proposition 1.4.** *Let $T > 0$, there exists a $\rho_b > 0$ such that for any $\sigma > 0$ and $(u_b, B_b) \in L^2_{\text{div}}(\mathcal{B})$ which satisfies (1.11) and (1.12) for a constant $C_b$, for every $k \in \mathbb{N}$, there exist forces $\xi, \eta \in C^\infty([0,T] \times \overline{\mathcal{B}\backslash\Omega})$ and $f, g \in C^\infty([0,T] \times \overline{\Omega})$ satisfying*

$$\|(f,g)\|_{L^1((0,T);H^k(\mathcal{B}))} \leq C_k \|(u_b, B_b)|_{\mathcal{G}}\|_{H^k(\mathcal{G})}, \tag{1.13}$$

*where constant $C_k$ only depends on $k$, and a Leray weak solution $(u, B) \in C_w([0,T]; L^2_{\text{div}}(\mathcal{B})) \cap L^2((0,T); L^2(\mathcal{B}))$ to (1.7) associated with the initial data $(u_b, B_b)$ such that $(u_c, B_c) := (u(T, \cdot),\ B(T, \cdot))$ satisfies*

$$\|(u_c, B_c)|_\Omega\|_{L^2(\Omega)} \leq \sigma. \tag{1.14}$$

**Step 3.** The final step is to utilize Badra's local exact controllability result [1] to drive the $L^2$ small data to zero. In Badra's original result [1], the control functions are internal forces localized in a non-empty, simply connected subset of the entire domain. Additionally, the domain in Badra's result is smooth and bounded. Therefore, we need to find an alternative method to extend $\Omega$.

Let $\Omega' = (-L, 2L) \times (-1, 1)$ and let $\mathcal{O}$ be a smooth annular-type domain of $\mathbb{R}^2$ which satisfies

$$\Omega' \subset \mathcal{O}, \{-L, 2L\} \times (-1,1) \subset \mathcal{O}, (-L, 2L) \times \{-1, 1\} \subset \partial\mathcal{O}.$$

We still denote by $\mathbf{n} = {}^t(n_1, n_2)$ the outward unit normal vector on $\partial\mathcal{O}$ and set

$$L^2_{\text{div}}(\mathcal{O}) = \{u \in L^2(\mathcal{O}; \mathbb{R}^2) | \operatorname{div} u = 0 \text{ in } \mathcal{O}, u \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}. \tag{1.15}$$

Note that $\mathcal{O}$ is multiply-connected and only one cut is required to make $\mathcal{O}$ simply-connected. Let

$$\mathcal{X} = \{u \in L^2(\mathcal{O}; \mathbb{R}^2) | \operatorname{div} u = 0 \text{ in } \mathcal{O}, \operatorname{curl} u = 0 \text{ in } \mathcal{O}, u \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}, \tag{1.16}$$

where $\operatorname{div} u := \partial_x u_1 + \partial_y u_2, \operatorname{curl} u := \partial_x u_2 - \partial_y u_1$. From [1] and the references therein we know that $\mathcal{X}$ is a one-dimensional space (or see Appendix I of [30]). We denote by g the basis of $\mathcal{X}$, so that $\mathcal{X} = \mathbb{R}\mathrm{g}$.

We take a subset $\omega = (-L, -\frac{L}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ which satisfies $\overline{\omega} \subset \mathcal{O}, \overline{\omega} \cap \overline{\Omega} = \emptyset$. Let $\mathbf{1}_\omega : L^2(\omega) \to L^2(\mathcal{O})$ denotes the expansion operator defined by $\mathbf{1}_\omega(f)(x,y) = f(x,y)$ if $(x,y) \in \omega$ and $\mathbf{1}_\omega(f)(x,y) = 0$ else. By taking the target trajectory to be zero and taking control functions supported in $\overline{\omega}$, we can directly get the following theorem from Theorem 1.3 of [1].

**Theorem 1.5.** *Let $T > 0$, there exists constants $C_0 > 0$ and $\sigma > 0$ such that for all $(u_0, B_0) \in L^2_{\text{div}}(\mathcal{O})$ satisfying*

$$\int_{\mathcal{O}} B_0 \cdot \mathrm{g} dx dy = 0, \tag{1.17}$$

$$\|(u_0, B_0)\|_{L^2(\mathcal{O})} \leq \sigma, \tag{1.18}$$

*there exist control functions* $(\xi,\eta)\in L^2((0,T)\times\omega)$ *satisfying*

$$\operatorname{div}(\mathbf{1}_\omega\eta)=0 \text{ in } (0,T)\times\mathcal{O} \quad \text{and} \quad (\mathbf{1}_\omega\eta)\cdot\mathbf{n}=0 \text{ on } (0,T)\times\partial\mathcal{O}, \tag{1.19}$$

*such that the following system*

$$\begin{cases} \partial_t u+u\cdot\nabla u-B\cdot\nabla B-\Delta u+\nabla p=\mathbf{1}_\omega\xi & \text{in } (0,T)\times\mathcal{O},\\ \partial_t B+u\cdot\nabla B-B\cdot\nabla u-\Delta B=\mathbf{1}_\omega\eta & \text{in } (0,T)\times\mathcal{O},\\ \operatorname{div}u=\operatorname{div}B=0 & \text{in } (0,T)\times\mathcal{O},\\ u=0, \quad B\cdot\mathbf{n}=0 \quad \text{and} \quad curl\, B=0 & \text{on } (0,T)\times\partial\mathcal{O},\\ \int_{\mathcal{O}}B(t)\cdot g\mathrm{d}x\mathrm{d}y=0 & \forall\, t\in(0,T),\\ u(0,\cdot)=u_0, \quad B(0,\cdot)=B_0 & \text{in } \mathcal{O}, \end{cases} \tag{1.20}$$

*admits a solution* $(u,B)\in C([0,T];L^2(\mathcal{O}))\cap L^2((0,T);H^1(\mathcal{O}))$ *satisfying* $(u(T,\cdot),B(T,\cdot))=(0,0)$ *as well as estimate*

$$\|(u,B)\|_{C([0,T];L^2(\mathcal{O}))}+\|(u,B)\|_{L^2((0,T);H^1(\mathcal{O}))}+\|(\xi,\eta)\|_{L^2((0,T)\times\omega)}\le C_0\sigma. \tag{1.21}$$

**Remark 1.6.** Condition (1.19) and the perfectly conduction wall boundary conditions for the magnetic field $B$ make sure that there will be no pressure term in the equation of $B$. It can be obtained by multiplying the equation for $B$ by $\nabla\varphi$ for any $\varphi\in C^\infty(\overline{\mathcal{O}})$, integrating by parts and note that $\operatorname{div}B=0$, $-\Delta B\cdot\mathbf{n}=\nabla^\perp\operatorname{curl}B\cdot\mathbf{n}=-\mathbf{n}^\perp\cdot\nabla\operatorname{curl}B=0$ on the boundary $\partial\mathcal{O}$, since $\mathbf{n}^\perp\cdot\nabla$ is a tangential derivative and $\operatorname{curl}B=0$ on the boundary, where $\nabla^\perp:={}^t(\partial_y,-\partial_x),\mathbf{n}^\perp:={}^t(n_2,-n_1)$.

**Remark 1.7.** The proof of Theorem 1.5 relies on the Carleman inequality for the adjoint system of the linearized system of (1.20) (refer to Thm. 3.1 of [1]). It should be note that the constant $c_0$ in Theorem 3.1 of [1] is actually dependent on $\lambda$. As demonstrated in [1], the expected Carleman inequality can be established by utilizing the Carleman inequality for the adjoint Stokes equations with Dirichlet boundary conditions (refer to Thm. 4.1 in [17] or Thm. 3.4 in [23]) and the Carleman inequality for the parabolic equations with Dirichlet boundary conditions (refer to Rem. 1.2 in [13]). The first result relies on the Carleman inequality for the parabolic equations with nonhomogeneous boundary conditions. Although [17] has never been published, we may employ a weaker form (refer to Thm. 2.1 in [16]) as an alternative, at the cost that the constant on the right-hand-side depends on $\lambda$, which is acceptable in our case. Consequently, Theorem 1.5 can be derived by combining the Carleman inequality, the classic duality method and a fix-point argument.

Note that in Step 2, we only get the smallness of $(u_c,B_c)$ in $L^2(\Omega)$. Besides, $\mathcal{B}$ is not a bounded domain, thus we cannot use Theorem 1.5 directly. We need to find another way to extend $(u_c,B_c)|_\Omega$ to $\mathcal{O}$ such that the $L^2(\mathcal{O})$ norm of the extended functions are small. We have the following Lemma, which will be proved in Appendix C.

**Lemma 1.8.** *There exists a constant* $c_0$*, such that for any* $(u_0,B_0)\in L^2_{\mathrm{div}}(\Omega)\cap C^\infty(\overline{\Omega})$ *which satisfies*

$$\int_\Omega u_{0,1}(x,y)\mathrm{d}x\mathrm{d}y=\int_\Omega B_{0,1}(x,y)\mathrm{d}x\mathrm{d}y=0, \tag{1.22}$$

$$u_0=0 \quad B_{0,2}=0 \quad \text{and} \quad \partial_y B_{0,1}=0 \quad \text{on } \partial\Omega\backslash\Gamma, \tag{1.23}$$

*there exists an extension* $(\tilde u_0,\tilde B_0)\in L^2_{\mathrm{div}}(\mathcal{O})$ *satisfying*

$$(\tilde u_0,\tilde B_0)|_\Omega=(u_0,B_0), \tag{1.24}$$

$$||(\tilde u_0,\tilde B_0)\|_{L^2(\mathcal{O})}\le c_0\|(u_0,B_0)\|_{L^2(\Omega)}, \tag{1.25}$$

$$\tilde{u}_0 = 0 \quad \tilde{B}_0 \cdot \mathbf{n} = 0 \quad \textit{and} \quad \textit{curl}\,\tilde{B}_0 = 0 \qquad \textit{on } \partial\mathcal{O}, \tag{1.26}$$

$$\int_{\mathcal{O}} \tilde{B}_0 \cdot \mathrm{g} \mathrm{d}x\mathrm{d}y = 0. \tag{1.27}$$

Combine the above propositions, we can prove the main theorem.

*Proof of Theorem 1.1.* Let $T > 0, u_0, B_0 \in L^2_{\mathrm{div}}(\Omega), k \in \mathbb{N}$ and $\delta > 0$. Let $\rho_b$ and $C_k$ be given by Proposition 1.4 for a time interval of length $\frac{T}{3}$, let $\sigma$ be given by Theorem 1.5 for a time interval of length $\frac{T}{3}$. Let $c_0$ be the constant in Lemma 1.8.

Firstly, by applying Proposition 1.3 with a time interval length $\frac{T}{3}$, $\rho, k, \frac{\delta}{1+C_k}$ and $(u_0, B_0)$, there exists a solution $(u, B)$ defined on $[0, \frac{T}{3}]$ with $(u, B)|_\Omega = (u_0, B_0)$ and $(u_b, B_b) = (u(\frac{T}{3}, \cdot), B(\frac{T}{3}, \cdot))$ which satisfies (1.10)–(1.12), control forces $\xi, \eta \in C^\infty([0, \frac{T}{3}] \times \overline{\mathcal{B}\backslash\Omega})$ and forces $f, g \in C^\infty([0, \frac{T}{3}] \times \overline{\Omega})$ with $\|(f, g)\|_{L^1(0, \frac{T}{3}; H^k(\Omega))} \leq \frac{\delta}{1+C_k}$.

Then we apply Proposition 1.4 in $[\frac{T}{3}, \frac{2T}{3}]$ with $\rho_b, k, \frac{\sigma}{c_0}$ and $(u_b, B_b)$. This yields a solution $(u, B)$ defined on $[\frac{T}{3}, \frac{2T}{3}]$ with $(u_c, B_c) = (u(\frac{2T}{3}, \cdot), B(\frac{2T}{3}, \cdot))$ satisfies $\|(u_c, B_c)|_\Omega\|_{L^2(\Omega)} \leq \frac{\sigma}{c_0}$ and $(f, g)$ satisfies $\|(f, g)\|_{L^1([\frac{T}{3}, \frac{2T}{3}]; H^k(\Omega))} \leq C_k \|(u_b, B_b)\|_{H^k(\mathcal{G})} \leq \frac{C_k \delta}{1+C_k}$.

Without loss of generality we may assume that $(u_c, B_c) \in L^2_{\mathrm{div}}(\mathcal{B}) \cap C^\infty(\overline{\mathcal{B}})$. Indeed, we can use Theorem A.1 again for time interval $[\frac{2T}{3}, \frac{5T}{6}]$ associated with data $(u_c, B_c)$ at time $t = \frac{2T}{3}$ and replace the $\sigma$ above by $\sigma' > 0$ which satisfies $E_0(\frac{\sigma'}{c_0}) \leq \frac{\sigma}{c_0}$, then $(u'_c, B'_c) \in L^2_{\mathrm{div}}(\mathcal{B}) \cap C^\infty(\overline{\mathcal{B}})$ meets our requirements. Since $(u_c, B_c) \in L^2_{\mathrm{div}}(\mathcal{B})$ is naturally an extension of $(u_c, B_c)|_\Omega \in L^2_{\mathrm{div}}(\Omega)$, hence by using Lemma C.1 (i),

$$\int_\Omega u_{c,1}(x, y)\mathrm{d}x\mathrm{d}y = \int_\Omega B_{c,1}(x, y)\mathrm{d}x\mathrm{d}y = 0.$$

Then by using Lemma 1.8, we can get another extension $(\tilde{u}_c, \tilde{B}_c) \in L^2_{\mathrm{div}}(\mathcal{O})$ of $(u_c, B_c)|_\Omega$ which satisfies $\|(\tilde{u}_c, \tilde{B}_c)\|_{L^2(\mathcal{O})} \leq \sigma$. Finally we apply Theorem 1.5 in $[\frac{2T}{3}, T]$ with initial data $(\tilde{u}_c, \tilde{B}_c)$ to find a solution $(u, B)$ which vanishes at $t = T$. By stitching in time, we have found a solution $(u, B)$ on $[0, T]$ which drive the initial data $(u_0, B_0)$ into zero and the force $(f, g)$ satisfies (1.9), which finishes the proof of Theorem 1.1. ☐

### 1.6. Organization of the rest of the paper

The remainder of this paper is organized as follows. In Section 2, we will prove Proposition 1.4. After scaling the time, we construct a family of approximate solutions using the return method and well-prepared dissipation method, such that the $L^2$ norm of the velocity and magnetic field is sufficiently small at the endpoint time. In Section 3, we estimate the remainders and provide a proof of Proposition 2.6 which is the crucial estimate for proving Proposition 1.4. In Section 4, we present a proof of Lemma 3.2, which plays a important role in establishing estimates for the remainders. In Section 5, we prove Proposition 3.3 which finalizes the estimates of the remainder. In Appendix A, we offer a smoothness result for the uncontrolled 2-D MHD equations with $L^2$ initial data. In Appendix B, we prove the regularization Proposition 1.3. In order to employ Badra's local exact controllability result, we prove Lemma 1.8 in Appendix C, which extends small data in $L^2_{\mathrm{div}}(\Omega)$ into a small data in $L^2_{\mathrm{div}}(\mathcal{O})$ on the extended domain.

## 2. Global approximate controllability and proof of Proposition 1.4

In this section, we prove Proposition 1.4. We will construct a family of approximate solutions depending on a small parameter $0 < \varepsilon \ll 1$ which drive the horizontally analytic data $(u_b, B_b)$ approximately to zero.

## 2.1. Time scaling

For $T>0, \varepsilon \in (0,1), t \in [0, \frac{T}{\varepsilon}]$, we introduce the following time scaling

$$
\begin{aligned}
(u^\varepsilon, B^\varepsilon)(t,x,y) &= \varepsilon (u,B)(\varepsilon t, x, y),\\
(p^\varepsilon, \xi^\varepsilon, \eta^\varepsilon, f^\varepsilon, g^\varepsilon)(t,x,y) &= \varepsilon^2 (p,\xi,\eta,f,g)(\varepsilon t, x, y).
\end{aligned}
$$

Then $(u,p,B,\xi,\eta,f,g)$ define solutions to (1.7) with initial data $(u_b, B_b)$ if and only if the new unknowns $(u^\varepsilon, p^\varepsilon, B^\varepsilon, \xi^\varepsilon, \eta^\varepsilon, f^\varepsilon, g^\varepsilon)$ are solutions to the following rescaled system:

$$
\begin{cases}
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - B^\varepsilon \cdot \nabla B^\varepsilon - \varepsilon \Delta u^\varepsilon + \nabla p^\varepsilon = \xi^\varepsilon + f^\varepsilon & \text{in } (0, \frac{T}{\varepsilon}) \times \mathcal{B},\\
\partial_t B^\varepsilon + u^\varepsilon \cdot \nabla B^\varepsilon - B^\varepsilon \cdot \nabla u^\varepsilon - \varepsilon \Delta B^\varepsilon = \eta^\varepsilon + g^\varepsilon & \text{in } (0, \frac{T}{\varepsilon}) \times \mathcal{B},\\
\operatorname{div} u^\varepsilon = \operatorname{div} B^\varepsilon = 0 & \text{in } (0, \frac{T}{\varepsilon}) \times \mathcal{B},\\
u^\varepsilon = 0, \quad B_2^\varepsilon = 0 \quad \text{and} \quad \partial_y B_1^\varepsilon = 0 & \text{on } (0, \frac{T}{\varepsilon}) \times \partial\mathcal{B},\\
(u^\varepsilon, B^\varepsilon) = \varepsilon (u_b, B_b) & \text{on } \{0\} \times \mathcal{B}.
\end{cases}
\tag{2.1}
$$

Thus Proposition 1.4 is equivalent to finding $\xi^\varepsilon, \eta^\varepsilon$ supported outside $\Omega$, $f^\varepsilon, g^\varepsilon$ supported in $\overline{\Omega}$ with

$$
\|(f^\varepsilon, g^\varepsilon)\|_{L^1((0,\frac{T}{\varepsilon});L^2(\Omega))} \le C_k \varepsilon \|(u_b, B_b)\|_{H^k(\mathcal{G})},
$$

such that

$$
\|(u^\varepsilon, B^\varepsilon)(\frac{T}{\varepsilon}, \cdot)\|_{L^2(\Omega)} = o(\varepsilon).
$$

## 2.2. Ansatz

Let's take a smooth function $\phi(y) \in C^\infty([-1,1])$ satisfying $\phi(y) = 1 - |y|$ when $\frac14 \le |y| \le 1$ and $\phi(y) \ge \frac34$ when $|y| \le \frac14$. For function $V(t,z), t, z \ge 0$, we introduce the notation $\{V\}_\varepsilon$ as follows,

$$
\{V\}_\varepsilon(t,y) := V(t, \frac{\phi(y)}{\sqrt{\varepsilon}}) \qquad \text{for } y \in [-1,1].
$$

Take a smooth function $\chi(y) \in C^\infty([-1,1])$ satisfying $\chi(y) = 1$ when $\frac23 \le |y| \le 1$ and $\chi(y) = 0$ when $|y| \le \frac13$. We introduce the following ansatz to (2.1), for $t \ge 0, (x,y) \in \mathcal{B}$,

$$
\begin{cases}
u^\varepsilon(t,x,y) := u^0(t) + \chi(y) v^0(t, \frac{\phi(y)}{\sqrt{\varepsilon}}) + \varepsilon u^1(t,x,y) + \varepsilon w^\varepsilon(t,y) + \varepsilon r^\varepsilon(t,x,y),\\
B^\varepsilon(t,x,y) := \varepsilon B^1(t,x,y) + \varepsilon R^\varepsilon(t,x,y),\\
p^\varepsilon(t,x,y) := p^0(t) + \varepsilon \pi^\varepsilon(t,x,y),\\
\xi^\varepsilon(t,x,y) := \varepsilon f^1|_{\mathcal{B}\setminus\Omega}(t,x,y),\\
f^\varepsilon(t,x,y) := \varepsilon f^1|_{\Omega}(t,x,y),\\
\eta^\varepsilon(t,x,y) := \varepsilon g^1|_{\mathcal{B}\setminus\Omega}(t,x,y),\\
g^\varepsilon(t,x,y) := \varepsilon g^1|_{\Omega}(t,x,y).
\end{cases}
\tag{2.2}
$$

We will define each terms involved in (2.2) later.

### 2.3. Return method and base Euler flow: the construction of $u^0$

Let $n \in \mathbb{N}$, let $h \in C^\infty((0,T);\mathbb{R})$ satisfies

$$\operatorname{supp} h \subset (0, \tfrac{T}{3}] \cup [\tfrac{2T}{3}, T), \tag{2.3}$$

$$\int_0^{\frac{T}{3}} h(t)\mathrm{d}s = 4L, \tag{2.4}$$

$$\int_0^T t^k h(t)\mathrm{d}t = 0 \quad \text{ for } 0 \le k < n. \tag{2.5}$$

The index $n$ will be chosen later, as well as the function $h(t)$. We define

$$u^0(t) := h(t)e_x \quad \text{ and } \quad p^0(t) = -\dot{h}(t)x,$$

where $\{e_x, e_y\}$ is the unit vector of the canonical basis of $\mathbb{R}^2$. Then $(u^0, p^0)$ satisfies

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = 0 & \text{in } \mathbb{R}_+ \times \mathcal{B}, \\ \operatorname{div} u^0 = 0 & \text{in } \mathbb{R}_+ \times \mathcal{B}, \\ u^0 \cdot e_y = 0 & \text{in } \mathbb{R}_+ \times \partial\mathcal{B}, \end{cases} \tag{2.6}$$

with initial data $u^0(0) = 0$ and $u^0(t) = 0$ for $t \ge T$.

### 2.4. Boundary layer profile: the construction of $v^0$

**Definition 2.1.** For $z \in \mathbb{R}$, we denote $\langle z \rangle := \sqrt{1+z^2}$ and for $s, q \in \mathbb{N}$, we set

$$H^s_q(\mathbb{R}_+) := \Big\{ \ f \in H^s(\mathbb{R}_+) : \ \sum_{j=0}^{s} \int_{\mathbb{R}_+} \langle z \rangle^{2q} |\partial_z^j f(z)|^2 \mathrm{d}z < +\infty \ \Big\},$$

endowed with its natural associated norm.

**Definition 2.2.** Let $k \in \mathbb{N}, \gamma > 0$ and $X$ a Banach space with norm $\|\cdot\|_X$. We define the space $C^k_\gamma(\mathbb{R}_+; X)$ of the functions $f \in C^k(\mathbb{R}_+; X)$ such that

$$\|f\|_{C^k_\gamma(\mathbb{R}_+;X)} := \sup_{t \ge 0, 0 \le j \le k} \left( \|\partial_t^j f(t)\|_X \langle t \rangle^\gamma \right) < +\infty,$$

where

$$C^k(\mathbb{R}_+; X) := \big\{ \ f \in C(\mathbb{R}_+; X) | \partial_t^j f \in C(\mathbb{R}_+; X), \ 0 \le j \le k \ \big\}.$$

**Proposition 2.3** (Propo. 2.4 of [21])**.** *Let $\gamma > 0$, $k, s, q, n \in \mathbb{N}$ satisfy*

$$n \ge \frac{q}{2} + \gamma - 1. \tag{2.7}$$

*and we define*

$$\tilde{\gamma} := 2n + 3, \quad \tilde{s} := s + 2k + 2n, \quad \tilde{q} := 2n + 3. \tag{2.8}$$

*Given $f \in C^0_{\tilde\gamma}(\mathbb{R}_+; H^{\tilde s}_{\tilde q}(\mathbb{R}_+))$ when $k = 0$ and $f \in C^{k-1}_{\tilde\gamma}(\mathbb{R}_+; H^{\tilde s}_{\tilde q}(\mathbb{R}_+))$ when $k \geq 1$, we can find a nonzero function $h \in C_0^\infty(0,T)$, supported in $(0, \frac{T}{3}] \cup [\frac{2T}{3}, T)$, such that the following system*

$$\begin{cases} \partial_t v - \partial_z^2 v = f & t \geq 0, z \geq 0, \\ v(t,0) = h(t) & t \geq 0, \\ v(0,z) = 0 & z \geq 0, \end{cases} \tag{2.9}$$

*has a unique solution $v \in C^k_\gamma(\mathbb{R}_+; H^s_q(\mathbb{R}_+))$. Moreover, if $f$ is supported away from $t = 0$ as a function of time $t$, so does $v$.*

For any $N \in \mathbb{N}$, by utilising Proposition 2.3 and taking $f \equiv 0, k = 0, \gamma = 2, s = 2, q = 2N + 3$, we can find a smooth function $h(t) \in C_0^\infty(0,T)$ satisfying (2.3)–(2.5), such that there is a unique solution $V(t,z) \in C^0_2(\mathbb{R}_+; H^2_{2N+3}(\mathbb{R}_+))$ to

$$\begin{cases} \partial_t V - \partial_z^2 V = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ V(t,0) = h(t) & \text{on } \mathbb{R}_+, \\ V(0,z) = 0 & \text{in } \mathbb{R}_+. \end{cases} \tag{2.10}$$

We define

$$v^0(t,z) := -V(t,z)e_x.$$

## 2.5. Linearized Euler flow profile: the construction of $(u^1, B^1, f^1, g^1)$

We take a cut-off function $\nu(t) \in C^\infty(\mathbb{R}; [0,1])$ with $\nu(t) = 1$ when $t \leq \frac{T}{3}$ and $\nu(t) = 0$ when $t \geq \frac{2T}{3}$. We set, for $\geq 0, (x,y) \in \mathcal{B}$,

$$(u^1, B^1)(t,x,y) = \nu(t)(u_b, B_b)(t, x - \int_0^t h(s)ds, y). \tag{2.11}$$

Since $(u_b, B_b)$ satisfies

$$u_b|_{\partial\mathcal{B}} = 0, \quad B_{b,2}|_{\partial\mathcal{B}} = 0 \quad \text{and} \quad \partial_y B_{b,1}|_{\partial\mathcal{B}} = 0,$$

where $B_b = {}^t(B_{b,1}, B_{b,2})$. Then $(u^1, B^1)$ satisfies

$$\begin{cases} \partial_t u^1 + h(t)\partial_x u^1 = f^1 & \text{in } [0,T] \times \mathcal{B}, \\ \operatorname{div} u^1 = 0 & \text{in } [0,T] \times \mathcal{B}, \\ u^1 = 0 & \text{on } [0,T] \times \partial\mathcal{B}, \\ u^1 = u_b & \text{on } \{0\} \times \mathcal{B}, \end{cases}$$

$$\begin{cases} \partial_t B^1 + h(t)\partial_x B^1 = g^1 & \text{in } [0,T] \times \mathcal{B}, \\ \operatorname{div} B^1 = 0 & \text{in } [0,T] \times \mathcal{B}, \\ B^1_2 = 0 \quad \text{and} \quad \partial_y B^1_1 = 0 & \text{on } [0,T] \times \partial\mathcal{B}, \\ B^1 = B_b & \text{on } \{0\} \times \mathcal{B}, \end{cases}$$

and

$$u^1(t,x,y) = B^1(t,x,y) = 0 \qquad \text{when } t \geq T.$$

Since $\dot{\nu}(t)$ is supported in $[\frac{T}{3}, \frac{2T}{3}]$, and $h(t)$ satisfies (2.3) and (2.4),

$$(f^1, g^1)(t,x,y) = \dot{\nu}(t)(u_b, B_b)(t, x-4L, y).$$

We take

$$\xi^\varepsilon = \varepsilon f^1|_{\mathcal{B}\setminus\Omega}, \quad f^\varepsilon = \varepsilon f^1|_\Omega, \quad \eta^\varepsilon = \varepsilon g^1|_{\mathcal{B}\setminus\Omega} \quad \text{and} \quad g^\varepsilon = \varepsilon g^1|_\Omega.$$

The control functions $\xi^\varepsilon, \eta^\varepsilon$ are supported in $\overline{\mathcal{B}\setminus\Omega}$, $f^\varepsilon, g^\varepsilon$ are supported in $\overline{\Omega}$. Recall that $\mathcal{G} = [-4L, -3L]$, we find that, for any $k \in \mathbb{N}$,

$$\|(f^\varepsilon, g^\varepsilon)\|_{L^1((0,T);H^k(\Omega))} \leq \varepsilon T \|(u_b, B_b)\|_{H^k(\mathcal{G})}.$$

## 2.6. The construction of $w^\varepsilon$

For any $N \in \mathbb{N}$, and for the profile $V \in C_2^0(\mathbb{R}_+; H^2_{2N+3}(\mathbb{R}_+))$ defined in (2.10), we set

$$q^\varepsilon := -\varepsilon^{N+1}\Big(\frac{\nu\chi''}{\phi^{2N+2}}\{z^{2N+2}V\}_\varepsilon + \frac{2\nu\chi'\phi'}{\phi^{2N+3}}\{z^{2N+3}\partial_z V\}_\varepsilon\Big).$$

**Lemma 2.4.** *For $N \in \mathbb{N}$, there exists a unique solution $W^\varepsilon$ to*

$$\begin{cases} \partial_t W^\varepsilon - \varepsilon\partial_y^2 W^\varepsilon = q^\varepsilon & \text{in } \mathbb{R}_+ \times [-1,1], \\ W^\varepsilon = 0 & \text{on } \mathbb{R}_+ \times \{-1,1\}, \\ W^\varepsilon = 0 & \text{on } \{0\} \times [-1,1]. \end{cases} \tag{2.12}$$

*Moreover there exists a constant $C_N$ such that $W^\varepsilon$ satisfies*

$$\|W^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2(-1,1))} + \varepsilon\|W^\varepsilon\|_{L^2(\mathbb{R}_+;H^2(-1,1))} \leq C_N\varepsilon^{N+\frac{5}{4}}\|V\|_{C_2^0(\mathbb{R}_+;H^1_{2N+3}(\mathbb{R}_+))}, \tag{2.13}$$

$$\|W^\varepsilon\|_{L^2(\mathbb{R}_+;L^\infty([-1,1]))} \leq C_N\varepsilon^{N+\frac{1}{4}}\|V\|_{C_2^0(\mathbb{R}_+;H^1_{2N+3}(\mathbb{R}_+))}. \tag{2.14}$$

We define the technical term $w^\varepsilon$ by

$$w^\varepsilon(t,y) := W^\varepsilon(t,y)e_x.$$

**Lemma 2.5** (Lem. 4.3 of [11])**.** *Let $\gamma \in C^0([-1,1])$ with $\gamma \equiv 0$ on $(-\frac{1}{3}, \frac{1}{3})$. For $\mathcal{V} \in L^2(\mathbb{R}_+)$ and $\varepsilon > 0$,*

$$\|\gamma\{\mathcal{V}\}_\varepsilon\|_{L^2(-1,1)} \leq 2\varepsilon^{\frac{1}{4}}\|\gamma\|_{L^\infty(-1,1)}\|\mathcal{V}\|_{L^2(\mathbb{R}_+)}.$$

*Proof of Lemma 2.4.* Recall the definition of $\chi$ and $\phi$ in Section 2.2 and $0 \le \nu \le 1$ , by utilizing Lemma 2.5, there is a constant $C_N$ such that

$$
\begin{aligned}
\|q^\varepsilon(t)\|_{L^2(-1,1)} \le C\varepsilon^{N+\frac{5}{4}} \Big( \|\frac{\chi''}{\phi^{2N+2}}\|_{L^\infty(-1,1)} \|V(t)\|_{H^0_{2N+2}(\mathbb{R}_+)} \\
+ \|\frac{\chi'\phi'}{\phi^{2N+3}}\|_{L^\infty(-1,1)} \|V(t)\|_{H^1_{2N+3}(\mathbb{R}_+)} \Big) \\
\le C_N \varepsilon^{N+\frac{5}{4}} \|V(t)\|_{H^1_{2N+3}(\mathbb{R}_+)},
\end{aligned}
$$

which implies

$$
\|q^\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(-1,1))} \le C_N \varepsilon^{N+\frac{5}{4}} \|V\|_{C^0_2(\mathbb{R}_+;H^1_{2N+3}(\mathbb{R}_+))}.
$$

By the energy estimate of (2.12),

$$
\frac{1}{2}\partial_t \|W^\varepsilon\|^2_{L^2(-1,1)} + \varepsilon \|\partial_y W^\varepsilon\|^2_{L^2(-1,1)} \le \|q^\varepsilon\|_{L^2(-1,1)} \|W^\varepsilon\|_{L^2(-1,1)}.
$$

Hence

$$
\|W^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2(-1,1))} \le \|q^\varepsilon\|_{L^1(\mathbb{R}_+;L^2(-1,1))} \le C_N \varepsilon^{N+\frac{5}{4}} \|V\|_{C^0_2(\mathbb{R}_+;H^1_{2N+3}(\mathbb{R}_+))}. \tag{2.15}
$$

Multiplying the first equation of (2.12) by $\partial_t W^\varepsilon$ and integrating by parts, we find that

$$
\|\partial_t W^\varepsilon\|^2_{L^2(-1,1)} + \frac{\varepsilon}{2}\partial_t \|\partial_y W^\varepsilon\|^2_{L^2(-1,1)} \le \|q^\varepsilon\|_{L^2(-1,1)} \|\partial_t W^\varepsilon\|_{L^2(-1,1)}.
$$

Since $W^\varepsilon|_{t=0} = 0$ implies $\partial_y W^\varepsilon|_{t=0} = 0$, we have

$$
\|\partial_t W^\varepsilon\|_{L^2(\mathbb{R}_+\times(-1,1))} \le \|q^\varepsilon\|_{L^2(\mathbb{R}_+\times(-1,1))} \le C_N \varepsilon^{N+\frac{5}{4}} \|V\|_{C^0_2(\mathbb{R}_+;H^1_{2N+3}(\mathbb{R}_+))}.
$$

Thus

$$
\begin{aligned}
\|\partial_y^2 W^\varepsilon\|_{L^2(\mathbb{R}_+\times(-1,1))} &\le \frac{1}{\varepsilon}(\|\partial_t W^\varepsilon\|_{L^2(\mathbb{R}_+\times(-1,1))} + \|q^\varepsilon\|_{L^2(\mathbb{R}_+\times(-1,1))}) \\
&\le C_N \varepsilon^{N+\frac{1}{4}} \|V\|_{C^0_2(\mathbb{R}_+;H^1_{2N+3}(\mathbb{R}_+))}.
\end{aligned}
$$

Since $W^\varepsilon|_{y=\pm 1} = 0$, therefore

$$
\|W^\varepsilon\|_{L^2(\mathbb{R}_+;H^2(-1,1))} \le C_N \varepsilon^{N+\frac{1}{4}} \|V\|_{C^0_2(\mathbb{R}_+;H^1_{2N+3}(\mathbb{R}_+))}. \tag{2.16}
$$

By the Sobolev imbedding inequality,

$$
\|W^\varepsilon\|_{L^2(\mathbb{R}_+;L^\infty([-1,1]))} \le C\|W^\varepsilon\|_{L^2(\mathbb{R}_+;H^2([-1,1]))} \le C_N \varepsilon^{N+\frac{1}{4}} \|V\|_{C^0_2(\mathbb{R}_+;H^1_{2N+3}(\mathbb{R}_+))}. \tag{2.17}
$$

Then the proof of Lemma 2.4 has been finished by combining (2.15)–(2.17). □

### 2.7. The equation for the remainder and proof of Proposition 1.4

By inserting the ansatz (2.2) into (2.1), and by the construction of $u^0, v^0, u^1, B^1$ and $w^\varepsilon$, we find the equation for $(r^\varepsilon, R^\varepsilon)$:

$$\begin{cases} \partial_t r^\varepsilon + u^\varepsilon\cdot\nabla r^\varepsilon + r^\varepsilon\cdot\nabla u^\varepsilon_{app} - B^\varepsilon\cdot\nabla R^\varepsilon - \varepsilon R^\varepsilon\cdot\nabla B^1 - \varepsilon\Delta r^\varepsilon + \nabla\pi^\varepsilon = F^\varepsilon & \text{in } \mathcal{B},\\ \partial_t R^\varepsilon + u^\varepsilon\cdot\nabla R^\varepsilon + \varepsilon r^\varepsilon\cdot\nabla B^1 - B^\varepsilon\cdot\nabla r^\varepsilon - R^\varepsilon\cdot\nabla u^\varepsilon_{app} - \varepsilon\Delta R^\varepsilon = G^\varepsilon & \text{in } \mathcal{B},\\ \operatorname{div} r^\varepsilon = \operatorname{div} R^\varepsilon = 0 & \text{in } \mathcal{B},\\ r^\varepsilon = 0, \quad R^\varepsilon_2 = 0 \quad \text{and} \quad \partial_y R^\varepsilon_1 = 0 & \text{on } \partial\mathcal{B},\\ (r^\varepsilon, R^\varepsilon)|_{t=0} = (0,0) & \text{in } \mathcal{B}, \end{cases} \tag{2.18}$$

where

$$u^\varepsilon_{app}(t,x,y) := u^0(t) + \chi(y) v^0(t, \frac{\phi(y)}{\sqrt{\varepsilon}}) + \varepsilon u^1(t,x,y) + \varepsilon w^\varepsilon(t,y),$$

$$F^\varepsilon := (\chi\{V\}_\varepsilon + \varepsilon W^\varepsilon)\partial_x u^1 - \varepsilon u^1\cdot\nabla u^1 + \varepsilon B^1\cdot\nabla B^1 + \varepsilon\Delta u^1 \tag{2.19}$$
$$-\varepsilon u^1_2 \partial_y W^\varepsilon e_x + u^1_2\chi'\{V\}_\varepsilon e_x + \frac{u^1_2\phi'}{\phi}\chi\{z\partial_z V\}_\varepsilon e_x,$$

$$G^\varepsilon := (\chi\{V\}_\varepsilon - \varepsilon W^\varepsilon)\partial_x B^1 - \varepsilon u^1\cdot\nabla B^1 + \varepsilon B^1\cdot\nabla u^1 + \varepsilon\Delta B^1 \tag{2.20}$$
$$+\varepsilon B^1_2\partial_y W^\varepsilon e_x - B^1_2\chi'\{V\}_\varepsilon e_x - \frac{B^1_2\phi'}{\phi}\chi\{z\partial_z V\}_\varepsilon e_x.$$

By the construction of $u^1, B^1$ in Section 2.5, they are both supported in $[0,T]$, so does $F^\varepsilon$ and $G^\varepsilon$.

**Proposition 2.6.** *There is a constant $C > 0$, such that the remainder $(r^\varepsilon, R^\varepsilon)$ satisfies, for all $t \in [0, T/\varepsilon]$,*

$$\|(r^\varepsilon, R^\varepsilon)(t)\|_{L^2(\mathcal{B})} + \Big(\int_0^t \varepsilon\|\nabla(r^\varepsilon, R^\varepsilon)(s)\|^2_{L^2(\mathcal{B})} ds\Big)^{\frac12} \le C\varepsilon^{\frac14}, \tag{2.21}$$

*where we used the notation (1.2).*

### 2.8. Proof of Proposition 1.4

For our perpose, it would be enough to take $N = 0$ in the construction of $V$ in Section 2.4 and $W^\varepsilon$ in Section 2.6. Hence $V \in C^0_2(\mathbb{R}_+; H^2_3(\mathbb{R}_+))$. By construction, $u^0, u^1, B^1$ are supported in $[0,T]$, we evaluate the ansatz (2.2) at $t = \frac{T}{\varepsilon}$,

$$\|(u^\varepsilon, B^\varepsilon)(\frac{T}{\varepsilon},\cdot)\|_{L^2(\Omega)} \le C\big(\|\chi\{V(\frac{T}{\varepsilon},\cdot)\}_\varepsilon\|_{L^2(-1,1)} + \varepsilon\|W^\varepsilon(\frac{T}{\varepsilon},\cdot)\|_{L^2(-1,1)}$$
$$+\varepsilon\|(r^\varepsilon, R^\varepsilon)(\frac{T}{\varepsilon},\cdot)\|_{L^2(\mathcal{B})}\big).$$

By Lemma 2.5,

$$\|\chi\{V(\frac{T}{\varepsilon},\cdot)\}_\varepsilon\|_{L^2(-1,1)} \le C\varepsilon^{\frac14}\|V(\frac{T}{\varepsilon},\cdot)\|_{L^2(\mathbb{R}_+)} \le C\varepsilon^{\frac54}\|V\|_{C^0_1(\mathbb{R}_+;L^2(\mathbb{R}_+))}.$$

Combine Lemma 2.4 and Proposition 2.6,

$$\|(u^\varepsilon, B^\varepsilon)(\frac{T}{\varepsilon},\cdot)\|_{L^2(\Omega)} \leq C\varepsilon^{\frac{5}{4}},$$

which finishes the proof of Proposition 1.4.

## 3. Estimates of the remainder and proof of Proposition 2.6

In this section, we will prove Proposition 2.6 by a nonlinear long-time version of Cauchy–Kovalevskaya estimates, we follow the method initialed by Chemin in [5] (see also [6, 31]), which makes use of Fourier thoeory and Besov spaces. For $s = 0, \frac{1}{2}$, we introduce the Besov spaces $\dot{B}^s$ respectively endowed with the norm

$$\|a\|_{\dot{B}^s} := \sum_{k\in\mathbb{Z}} 2^{ks}\|\dot{\Delta}_k a\|_{L^2(\mathcal{B})},$$

where the dyadic operator $\dot{\Delta}_k$ is defined by

$$\dot{\Delta}_k a \overset{\text{def}}{=} \mathcal{F}^{-1}_{\xi\to x}(\varphi(2^{-k}|\xi|)\hat{a}(\xi,y)) \quad \text{with}$$
$$\operatorname{supp}\varphi \subset \{\tau\in\mathbb{R} : \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\} \quad \text{and} \quad \forall\,\tau>0, \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau) = 1,$$

where $\mathcal{F}^{-1}_{\xi\to x}a$ denotes the inverse Fourier transform of the distribution a with respect to the first variable and $\hat{a}(\xi,y) = \mathcal{F}_{x\to\xi}(a)(\xi,y)$. For more derails on Littlewood–Paley theory, we refer to [2].

We define

$$\rho_0 := 2 + C_*\int_0^\infty \|z\partial_z V(t,\cdot)\|_{L^\infty(\mathbb{R}_+)}\mathrm{d}t, \tag{3.1}$$

where the constant $C_*$ will be determined later. By the definition of weighted Sobolev spaces in Section 2.4, it's easy to bound $\rho_0$ by $2 + CC_*\|V\|_{C^0_2(\mathbb{R}_+;H^2_1(\mathbb{R}_+))}$ for an absolute constant $C>0$. Then we define $\beta(t)$ by

$$\begin{cases} \dot{\beta}(t) = C_*\Big(\chi_{[0,T]}(t) + \varepsilon\|\langle\partial_y\rangle \mathrm{e}^{\rho_0|\partial_x|}(u^1,B^1)(t,\cdot)\|^2_{\dot{B}^{\frac{1}{2}}} \\ \qquad\qquad + \|V(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} + \varepsilon\|W^\varepsilon(t,\cdot)\|^2_{L^\infty([-1,1])}\Big) \qquad t>0, \\ \beta(0) = 0, \end{cases} \tag{3.2}$$

where we denote by

$$\|\langle\partial_y\rangle a\|_{\dot{B}^s} := \|a\|_{\dot{B}^s} + \|\partial_y a\|_{\dot{B}^s}, \tag{3.3}$$

and we use the notation (1.2).

**Proposition 3.1.** *For $u^1, B^1, V, W^\varepsilon$ constructed in the last section, there exists a constant $\beta_*>0$, such that*

$$\sup_{t\in[0,T/\varepsilon]}\beta(t) = \beta(T/\varepsilon) \leq \beta_*.$$

*Proof.* Since $V \in C^0_2(\mathbb{R}_+; H^2_3(\mathbb{R}_+))$,

$$\int_0^{\frac{T}{\varepsilon}} \|V(s,\cdot)\|_{L^\infty(\mathbb{R}_+)} \mathrm{d}s \le C\|V\|_{C^0_2(\mathbb{R}_+;H^1(\mathbb{R}_+))} \le C\|V\|_{C^0_2(\mathbb{R}_+;H^2_3(\mathbb{R}_+))}.$$

By Lemma 2.4 and note that we have took $N = 0$,

$$\int_0^{\frac{T}{\varepsilon}} \varepsilon\|W^\varepsilon(s,\cdot)\|^2_{L^\infty([-1,1])} \mathrm{d}s \le C\varepsilon^{\frac{3}{2}}\|V\|^2_{C^0_2(\mathbb{R}_+;H^1_3(\mathbb{R}_+))}.$$

By Cauchy's inequality,

$$\begin{aligned}
&\|\langle\partial_y\rangle \mathrm{e}^{\rho_0|\partial_x|}(u^1,B^1)\|_{\dot{B}^{\frac{1}{2}}} \\
&= \sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}}\|\dot{\Delta}_k \mathrm{e}^{\rho_0|\partial_x|}(u^1,B^1)\|_{H^1(\mathcal{B})} \\
&\le \Big(\sum_{k<0}\|\dot{\Delta}_k \mathrm{e}^{\rho_0|\partial_x|}(u^1,B^1)\|^2_{H^1(\mathcal{B})}\Big)^{\frac{1}{2}} + \Big(\sum_{k\ge 0} 2^{2k}\|\dot{\Delta}_k \mathrm{e}^{\rho_0|\partial_x|}(u^1,B^1)\|^2_{H^1(\mathcal{B})}\Big)^{\frac{1}{2}} \\
&\le \|\mathrm{e}^{\rho_0|\partial_x|}(u^1,B^1)\|_{H^2(\mathcal{B})}.
\end{aligned}$$

By the definition of $u^1, B^1$ in Section 2.5 and (1.11)

$$\|\mathrm{e}^{\rho_0|\partial_x|}(u^1,B^1)\|_{H^2(\mathcal{B})} \le \|\mathrm{e}^{\rho_0|\partial_x|}(u_b,B_b)\|_{H^2(\mathcal{B})} \le C_b.$$

Hence

$$\begin{aligned}
\beta(\frac{T}{\varepsilon}) &\le C_*\int_0^{\frac{T}{\varepsilon}} \Big(\chi_{[0,T](s)} + \varepsilon\|\langle\partial_y\rangle \mathrm{e}^{\rho_0|\partial_x|}(u^1,B^1)\|^2_{\dot{B}^{\frac{1}{2}}} + \|V\|_{L^\infty_z} + \varepsilon\|W^\varepsilon\|^2_{L^\infty_y}\Big)\mathrm{d}s \\
&\le C_*(T + TC_b^2 + C\|V\|_{C^0_2(\mathbb{R}_+;H^2_3(\mathbb{R}_+))} + C\varepsilon^{\frac{3}{2}}\|V\|^2_{C^0_2(\mathbb{R}_+;H^2_3(\mathbb{R}_+))}).
\end{aligned}$$

Therefore, there exists a constant $\beta_* > 0$ such that for all $t \in [0, \frac{T}{\varepsilon}]$,

$$\beta(t) \le \beta(\frac{T}{\varepsilon}) \le \beta_*.$$

□

We define $\rho(t)$ by the following nolinear ODE:

$$\begin{cases}\dot{\rho}(t) = -C_*(\|z\partial_z V(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} + \varepsilon\|\langle\partial_y\rangle(r^\varepsilon_\Phi, R^\varepsilon_\Phi)\|^2_{\dot{B}^0}) \qquad t>0,\\ \rho(0) = \rho_0,\end{cases} \tag{3.4}$$

where $C_*$ is the same constant as in (3.1) and (3.2). We set

$$T^* := \sup\{t \in [0, \frac{T}{\varepsilon}] : \rho(t) \geq 1\}.$$

For $t \in [0, T^*]$, we take

$$\Phi(t,\xi) = \rho(t)|\xi| - \beta(t),$$

and define

$$v_\Phi := \mathcal{F}^{-1}_{\xi\to x}(\mathrm{e}^{\Phi(t,\xi)}\hat{v}(\xi,y)).$$

Then for $t \in [0, T^*]$, $(r^\varepsilon_\Phi, R^\varepsilon_\Phi)$ satisfies

$$\begin{cases} \partial_t r^\varepsilon_\Phi - \dot{\rho}|\partial_x| r^\varepsilon_\Phi + \dot{\beta} r^\varepsilon_\Phi - \varepsilon\Delta r^\varepsilon_\Phi + \nabla\pi^\varepsilon_\Phi \\ \qquad = F^\varepsilon_\Phi - (u^\varepsilon\cdot\nabla r^\varepsilon + r^\varepsilon\cdot\nabla u^\varepsilon_{app})_\Phi + (B^\varepsilon\cdot\nabla R^\varepsilon + \varepsilon R^\varepsilon\cdot\nabla B^1)_\Phi, \\ \partial_t R^\varepsilon_\Phi - \dot{\rho}|\partial_x| R^\varepsilon_\Phi + \dot{\beta} R^\varepsilon_\Phi - \varepsilon\Delta B^\varepsilon_\Phi \\ \qquad = G^\varepsilon_\Phi - (u^\varepsilon\cdot\nabla R^\varepsilon + \varepsilon r^\varepsilon\cdot\nabla B^1)_\Phi + (B^\varepsilon\cdot\nabla r^\varepsilon + R^\varepsilon\cdot\nabla u^\varepsilon_{app})_\Phi, \\ \operatorname{div} r^\varepsilon_\Phi = \operatorname{div} R^\varepsilon_\Phi = 0, \\ r^\varepsilon_\Phi|_{y=\pm1} = 0, \quad R^\varepsilon_{\Phi,2}|_{y=\pm1} = 0 \quad \text{and} \quad \partial_y R^\varepsilon_{\Phi,1}|_{y=\pm1} = 0, \\ (r^\varepsilon_\Phi, R^\varepsilon_\Phi)|_{t=0} = (0,0), \end{cases} \tag{3.5}$$

where $F^\varepsilon, G^\varepsilon$ are defined in (2.19) and (2.20).

*Proof of Proposition 2.6.* In order to distinguish the inner product from the notation 1.2, throughout the proof, we denote

$$(f|g)_{L^2(\mathcal{B})} := \int_{\mathcal{B}} f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y$$

the inner product of two functions $f$ and $g$ defined in $\mathcal{B}$. Apply the operator $\dot{\Delta}_k$ to the first two equations of (3.5), by using the energy estimates, the divergence free condition $(3.5)_3$, the boundary conditions $(3.5)_4$ and $\dot{\Delta}_k\Delta = \Delta\dot{\Delta}_k$, we find that for $0 \leq t \leq T^*$,

$$\begin{aligned} &\frac{1}{2}\partial_t\|\dot{\Delta}_k(r^\varepsilon_\Phi, R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})} + |\dot{\rho}|2^k\|\dot{\Delta}_k(r^\varepsilon_\Phi, R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})} + \dot{\beta}\|\dot{\Delta}_k(r^\varepsilon_\Phi, R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})} \\ &+\varepsilon\|\nabla\dot{\Delta}_k r^\varepsilon_\Phi\|^2_{L^2(\mathcal{B})} + \varepsilon\|\nabla\dot{\Delta}_k R^\varepsilon_\Phi\|^2_{L^2(\mathcal{B})} \\ &\leq |(\dot{\Delta}_k F^\varepsilon_\Phi|\dot{\Delta}_k r^\varepsilon_\Phi)_{L^2(\mathcal{B})}| + |(\dot{\Delta}_k G^\varepsilon_\Phi|\dot{\Delta}_k R^\varepsilon_\Phi)_{L^2(\mathcal{B})}| \\ &\quad + \big|\big(\dot{\Delta}_k(u^\varepsilon\cdot\nabla r^\varepsilon + r^\varepsilon\cdot\nabla u^\varepsilon_{app} - B^\varepsilon\cdot\nabla R^\varepsilon - \varepsilon R^\varepsilon\cdot\nabla B^1)_\Phi|\dot{\Delta}_k r^\varepsilon_\Phi\big)_{L^2(\mathcal{B})}\big| \\ &\quad + \big|\big(\dot{\Delta}_k(u^\varepsilon\cdot\nabla R^\varepsilon + \varepsilon r^\varepsilon\cdot\nabla B^1 - B^\varepsilon\cdot\nabla r^\varepsilon - R^\varepsilon\cdot\nabla u^\varepsilon_{app})_\Phi|\dot{\Delta}_k R^\varepsilon_\Phi\big)_{L^2(\mathcal{B})}\big|. \end{aligned} \tag{3.6}$$

Note that $F^\varepsilon, G^\varepsilon$ are supported in $[0,T]$, then by integrating about the time over $[0,t]$ for $t\in[0,T^*]$, taking the square root and taking the summation about $k\in\mathbb{Z}$, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|_{L^2(\mathcal{B})}+\sum_{k\in\mathbb{Z}}\Big(\int_0^t|\dot\rho|2^k\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}|\mathrm{d}s\Big)^{\frac12}\\
&+\sum_{k\in\mathbb{Z}}\Big(\int_0^t|\dot\beta|\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}|\mathrm{d}s\Big)^{\frac12}+\sum_{k\in\mathbb{Z}}\Big(\int_0^t\varepsilon\|\nabla\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}|\mathrm{d}s\Big)^{\frac12}\\
&\le C\Bigg(\sum_{k\in\mathbb{Z}}\Big(\int_0^t\chi_{[0,T]}(s)\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}+\sum_{k\in\mathbb{Z}}\Big(\int_0^T\|\dot\Delta_k(F^\varepsilon_\Phi,G^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}ds\Big)^{\frac12}\\
&+\sum_{k\in\mathbb{Z}}\Big(\int_0^t\Big|\big(\dot\Delta_k(u^\varepsilon\cdot\nabla r^\varepsilon+r^\varepsilon\cdot\nabla u^\varepsilon_{app}-B^\varepsilon\cdot\nabla R^\varepsilon-\varepsilon R^\varepsilon\cdot\nabla B^1)_\Phi|\dot\Delta_k r^\varepsilon_\Phi\big)_{L^2(\mathcal{B})}\Big|\,ds\Big)^{\frac12}\\
&+\sum_{k\in\mathbb{Z}}\Big(\int_0^t\Big|\big(\dot\Delta_k(u^\varepsilon\cdot\nabla R^\varepsilon+\varepsilon r^\varepsilon\cdot\nabla B^1-B^\varepsilon\cdot\nabla r^\varepsilon-R^\varepsilon\cdot\nabla u^\varepsilon_{app})_\Phi|\dot\Delta_k R^\varepsilon_\Phi\big)_{L^2(\mathcal{B})}\Big|\,\mathrm{d}s\Big)^{\frac12}\Bigg).
\end{aligned}\tag{3.7}$$

In order to estimate the last two terms of the right-hand-side of (3.6), we need the following Lemma, which is analogous to Lemma 5.3 of [21].

**Lemma 3.2.** *For functions $a,b,c$ defined in $\mathcal{B}$, for any small constant $c_0>0$, there exists $C>0$ such that*

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\Big(\int_0^t\Big|(\dot\Delta_k(ab)_\Phi|\dot\Delta_k(c)_\Phi)_{L^2(\mathcal{B})}\Big|\,\mathrm{d}s\Big)^{\frac12}&\le c_0\sum_{k\in\mathbb{Z}}\Big(\int_0^t\|\dot\Delta_k c_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}\\
&+C\sum_{k\in\mathbb{Z}}\Big(\int_0^t\|\langle\partial_y\rangle a_\Phi\|^2_{\dot B^{\frac12}}\|\dot\Delta_k b_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12},
\end{aligned}\tag{3.8}$$

*and*

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\Big(\int_0^t\Big|(\dot\Delta_k(ab)_\Phi|\dot\Delta_k(c)_\Phi)_{L^2(\mathcal{B})}\Big|\,ds\Big)^{\frac12}&\le c_0\sum_{k\in\mathbb{Z}}\Big(\int_0^t\|\dot\Delta_k c_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}\\
&+C\sum_{k\in\mathbb{Z}}\Big(\int_0^t 2^k\|\langle\partial_y\rangle(a_\Phi,b_\Phi)\|^2_{\dot B^0}\|\dot\Delta_k(a_\Phi,b_\Phi)\|^2_{L^2(\mathcal{B})}ds\Big)^{\frac12}.
\end{aligned}\tag{3.9}$$

We postpone the proof Lemma 3.2 to the Section 4. Now we continue to prove Proposition 2.6. By construction, $u^0+\chi\{v^0\}_\varepsilon+\varepsilon w^\varepsilon$ is independent of $x$ variable and divergence free, we find by integration by parts that

$$(\dot\Delta_k((u^0+\chi\{v^0\}_\varepsilon+\varepsilon w^\varepsilon)\cdot\nabla r^\varepsilon)_\Phi|\dot\Delta_k r^\varepsilon_\Phi)_{L^2(\mathcal{B})}=0.$$

Due to $\operatorname{div}u^1=\operatorname{div}r^\varepsilon=\operatorname{div}B^1=\operatorname{div}R^\varepsilon=0$, we have

$$\begin{aligned}
&(\dot\Delta_k(u^\varepsilon\cdot\nabla r^\varepsilon+r^\varepsilon\cdot\nabla u^\varepsilon_{app}-B^\varepsilon\cdot\nabla R^\varepsilon-\varepsilon R^\varepsilon\cdot\nabla B^1)_\Phi|\dot\Delta_k r^\varepsilon_\Phi)_{L^2(\mathcal{B})}\\
&=(\dot\Delta_k r^\varepsilon_\Phi\cdot\nabla(\chi\{v^0\}_\varepsilon)|\dot\Delta_k r^\varepsilon_\Phi)_{L^2(\mathcal{B})}-\varepsilon(w^\varepsilon\otimes\dot\Delta_k r^\varepsilon_\Phi|\nabla\dot\Delta_k r^\varepsilon_\Phi)_{L^{\mathcal{B}}}\\
&\quad-\varepsilon\big(\dot\Delta_k(r^\varepsilon\otimes(u^1+r^\varepsilon)+u^1\otimes r^\varepsilon-R^\varepsilon\otimes B^\varepsilon-B^1\otimes R^\varepsilon)_\Phi|\nabla\dot\Delta_k r^\varepsilon_\Phi\big)_{L^2(\mathcal{B})}.
\end{aligned}\tag{3.10}$$

For the second term of the right hand side of (3.10), obviously for any $c_0>0$, there is a constant $C>0$ such that

$$\sum_{k\in\mathbb{Z}}\Big(\int_0^t\Big|\varepsilon(w^\varepsilon\otimes\dot\Delta_k r^\varepsilon_\Phi|\nabla\dot\Delta_k r^\varepsilon_\Phi)_{L^{\mathcal{B}}}\Big|\mathrm{d}s\Big)^{\frac12}\le c_0\sum_{k\in\mathbb{Z}}\Big(\int_0^t\varepsilon\|\nabla\dot\Delta_k r^\varepsilon_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}$$
$$+\sum_{k\in\mathbb{Z}}\Big(\int_0^t\varepsilon\|W^\varepsilon\|^2_{L^\infty_z}\|\dot\Delta_k r^\varepsilon_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}. \tag{3.11}$$

For the last term of the right hand side of (3.10), we can use Lemma 3.2. For any $c_0>0$, there is a constant $C>0$ such that

$$\sum_{k\in\mathbb{Z}}\Big(\int_0^t\Big|\varepsilon\big(\dot\Delta_k(r^\varepsilon\otimes(u^1+r^\varepsilon)+u^1\otimes r^\varepsilon-R^\varepsilon\otimes B^\varepsilon-B^1\otimes R^\varepsilon)_\Phi|\nabla\dot\Delta_k r^\varepsilon_\Phi\big)_{L^2(\mathcal{B})}\Big|\mathrm{d}s\Big)^{\frac12}$$
$$\le c_0\sum_{k\in\mathbb{Z}}\Big(\int_0^t\varepsilon\|\nabla\dot\Delta_k r^\varepsilon_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}+C\sum_{k\in\mathbb{Z}}\Big(\int_0^t\varepsilon\|\langle\partial_y\rangle(u^1_\Phi,B^1_\Phi)\|^2_{\dot B^{\frac12}}\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12} \tag{3.12}$$
$$+C\sum_{k\in\mathbb{Z}}\Big(\int_0^t\varepsilon 2^k\|\langle\partial_y\rangle(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{\dot B^0}\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}.$$

For the first term of the right hand side of (3.10),

$$(\dot\Delta_k r^\varepsilon_\Phi\cdot\nabla(\chi\{v^0\}_\varepsilon)|\dot\Delta_k r^\varepsilon_\Phi)_{L^2(\mathcal{B})} \tag{3.13}$$
$$=-(\dot\Delta_k r^\varepsilon_{\Phi,2}\chi'\{V\}_\varepsilon|\dot\Delta_k r^\varepsilon_{\Phi,1})_{L^2(\mathcal{B})}-\big(\frac{\dot\Delta_k r^\varepsilon_{\Phi,2}}{\phi}\chi\phi'\{z\partial_z V\}_\varepsilon|\dot\Delta_k r^\varepsilon_{\Phi,1}\big)_{L^2(\mathcal{B})},$$

$$\sum_{k\in\mathbb{Z}}\Big(\int_0^t\Big|(\dot\Delta_k r^\varepsilon_{\Phi,2}\chi'\{V\}_\varepsilon|\dot\Delta_k r^\varepsilon_{\Phi,1})_{L^2(\mathcal{B})}\Big|\mathrm{d}s\Big)^{\frac12}\le C\sum_{k\in\mathbb{Z}}\Big(\int_0^t\|V\|_{L^\infty_z}\|\dot\Delta_k r^\varepsilon_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}, \tag{3.14}$$

where $\chi$ is supported in $\{|y|\ge\frac13\}$ in where $|\phi'|=1,\phi=1-|y|$. Since $r^\varepsilon_\Phi$ vanishes on the boundary,

$$\frac{\dot\Delta_k r^\varepsilon_{\Phi,2}}{1-|y|}(t,x,y)=\begin{cases}-\int_0^1\dot\Delta_k\partial_y r^\varepsilon_{\Phi,2}(t,x,1-(1-y)s)\mathrm{d}s & \text{if } y<0,\\ \int_0^1\dot\Delta_k\partial_y r^\varepsilon_{\Phi,2}(t,x,-1+(1+y)s)\mathrm{d}s & \text{if } y<0.\end{cases} \tag{3.15}$$

So that

$$\big|\big(\frac{\dot\Delta_k r^\varepsilon_{\Phi,2}}{\phi}\chi\phi'\{z\partial_z V\}_\varepsilon|\dot\Delta_k r^\varepsilon_{\Phi,1}\big)_{L^2(\mathcal{B})}\big|\le C\|z\partial_z V\|_{L^\infty_z}\|\dot\Delta_k\partial_y r^\varepsilon_{\Phi,2}\|_{L^2(\mathcal{B})}\|\dot\Delta_k r^\varepsilon_{\Phi,2}\|_{L^2(\mathcal{B})}.$$

By the divergence free condition $\operatorname{div} r^\varepsilon_\Phi=0$ and Bernstein's inequality,

$$\|\dot\Delta_k\partial_y r^\varepsilon_{\Phi,2}\|_{L^2(\mathcal{B})}=\|\dot\Delta_k\partial_x r^\varepsilon_{\Phi,1}\|_{L^2(\mathcal{B})}\le C2^k\|\dot\Delta_k r^\varepsilon_\Phi\|_{L^2(\mathcal{B})}.$$

Thus

$$\sum_{k\in\mathbb{Z}}\Big(\int_0^t \big|\big(\frac{\dot{\Delta}_k r^\varepsilon_{\Phi,2}}{\phi}\chi\phi'\{z\partial_z V\}_\varepsilon|\dot{\Delta}_k r^\varepsilon_{\Phi,1}\big)_{L^2(\mathcal{B})}\big|\mathrm{d}s\Big)^{\frac12} \le C\sum_{k\in\mathbb{Z}}\Big(\int_0^t 2^k\|z\partial_z V\|_{L^\infty_z}\|\dot{\Delta}_k r^\varepsilon_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}. \tag{3.16}$$

Combine (3.10), (3.11),(3.12), (3.13), (3.14) and (3.16), for any small constant $c_0>0$, there exists $C>0$ such that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\Big(\int_0^t \Big|\big(\dot{\Delta}_k(u^\varepsilon\cdot\nabla r^\varepsilon + r^\varepsilon\cdot\nabla u^\varepsilon_{app} - B^\varepsilon\cdot\nabla R^\varepsilon - \varepsilon R^\varepsilon\cdot\nabla B^1)_\Phi|\dot{\Delta}_k r^\varepsilon_\Phi\big)_{L^2(\mathcal{B})}\Big|\mathrm{d}s\Big)^{\frac12}\\
&\le c_0\sum_{k\in\mathbb{Z}}\Big(\int_0^t \varepsilon\|\nabla\dot{\Delta}_k r^\varepsilon_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}\\
&\quad + C\sum_{k\in\mathbb{Z}}\Big(\int_0^t 2^k\big(\varepsilon\|\langle\partial_y\rangle(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{\dot{B}^0} + \|z\partial_z V\|_{L^\infty_z}\big)\|\dot{\Delta}_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}\\
&\quad + C\sum_{k\in\mathbb{Z}}\Big(\int_0^t \big(\varepsilon\|W^\varepsilon\|^2_{L^\infty_z} + \|V\|_{L^\infty_z} + \varepsilon\|\langle\partial_y\rangle(u^1_\Phi,B^1_\Phi)\|^2_{\dot{B}^{\frac12}}\big)\|\dot{\Delta}_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}.
\end{aligned}\tag{3.17}$$

By similar analysis, we can prove that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\Big(\int_0^t \Big|\big(\dot{\Delta}_k(u^\varepsilon\cdot\nabla R^\varepsilon + \varepsilon r^\varepsilon\cdot\nabla B^1 - B^\varepsilon\cdot\nabla r^\varepsilon - R^\varepsilon\cdot\nabla u^\varepsilon_{app})_\Phi|\dot{\Delta}_k R^\varepsilon_\Phi\big)_{L^2(\mathcal{B})}\Big|\mathrm{d}s\Big)^{\frac12}\\
&\le c_0\sum_{k\in\mathbb{Z}}\Big(\int_0^t \varepsilon\|\nabla\dot{\Delta}_k R^\varepsilon_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}\\
&\quad + C\sum_{k\in\mathbb{Z}}\Big(\int_0^t 2^k\big(\varepsilon\|\langle\partial_y\rangle(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{\dot{B}^0} + \|z\partial_z V\|_{L^\infty_z}\big)\|\dot{\Delta}_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}\\
&\quad + C\sum_{k\in\mathbb{Z}}\Big(\int_0^t \big(\varepsilon\|W^\varepsilon\|^2_{L^\infty_z} + \|V\|_{L^\infty_z} + \varepsilon\|\langle\partial_y\rangle(u^1_\Phi,B^1_\Phi)\|^2_{\dot{B}^{\frac12}}\big)\|\dot{\Delta}_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}.
\end{aligned}\tag{3.18}$$

By combining (3.7), (3.17) and (3.18) and taking $c_0$ small enough, we find there exists a constant $C_*$ such that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|_{L^2(\mathcal{B})} + \sum_{k\in\mathbb{Z}}\Big(\int_0^t |\dot{\rho}|2^k\|\dot{\Delta}_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}|\mathrm{d}s\Big)^{\frac12}\\
&+\sum_{k\in\mathbb{Z}}\Big(\int_0^t |\dot{\beta}|\|\dot{\Delta}_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}|\mathrm{d}s\Big)^{\frac12} + \sum_{k\in\mathbb{Z}}\Big(\int_0^t \varepsilon\|\nabla\dot{\Delta}_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}|\mathrm{d}s\Big)^{\frac12}\\
&\le C_*\sum_{k\in\mathbb{Z}}\Big(\int_0^T \|\dot{\Delta}_k(F^\varepsilon_\Phi,G^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}
\end{aligned}$$

$$+\sum_{k\in\mathbb{Z}}\Big(\int_0^t C_*2^k\big(\varepsilon\|\langle\partial_y\rangle(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{\dot B^0}+\|z\partial_zV\|_{L^\infty_z}\big)\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}$$
$$+\sum_{k\in\mathbb{Z}}\Big(\int_0^t C_*\big(\chi_{[0,T]}(s)+\varepsilon\|W^\varepsilon\|^2_{L^\infty_z}+\|V\|_{L^\infty_z}+\varepsilon\|\langle\partial_y\rangle(u^1_\Phi,B^1_\Phi)\|^2_{\dot B^{\frac12}}\big)\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}.$$

By the definition of $\beta(t)$ in (3.2) and $\rho(t)$ in (3.4) and the fact that $\Phi(t,\xi)\le\rho_0|\xi|$, we have

$$\sum_{k\in\mathbb{Z}}\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|_{L^2(\mathcal{B})}+\sum_{k\in\mathbb{Z}}\Big(\int_0^t\varepsilon\|\nabla\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}|\mathrm{d}s\Big)^{\frac12}$$
$$\le C_*\sum_{k\in\mathbb{Z}}\Big(\int_0^T\|\dot\Delta_k(F^\varepsilon_\Phi,G^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}.$$

We have the following estimates for the source term $F^\varepsilon_\Phi$ and $G^\varepsilon_\Phi$, which will be proved in the Section 5.

**Proposition 3.3.** *There exists a constant $C>0$, such that the source term $(F^\varepsilon,G^\varepsilon)$ defined in (2.19) and (2.20) satisfies*

$$\sum_{k\in\mathbb{Z}}\Big(\int_0^T\|\dot\Delta_k(F^\varepsilon_\Phi,G^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}\le C\varepsilon^{\frac14}.$$

Therefore, for $t\in[0,T^*]$, there exists a constant $\overline{C}>0$ such that

$$\sum_{k\in\mathbb{Z}}\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|_{L^2(\mathcal{B})}+\sum_{k\in\mathbb{Z}}\Big(\int_0^t\varepsilon\|\nabla\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}|\mathrm{d}s\Big)^{\frac12}\le\overline{C}\varepsilon^{\frac14}.$$

For $t\le T^*$, by Minkowski's inequality,

$$\begin{aligned}\int_0^t\varepsilon\|\langle\partial_y\rangle(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{\dot B^0}\mathrm{d}s&=\int_0^t\varepsilon\big(\|(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|_{\dot B^0}+\|\partial_y(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|_{\dot B^0}\big)^2\mathrm{d}s\\&=\int_0^t\varepsilon\Big(\sum_{k\in\mathbb{Z}}\big(\|\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|_{L^2(\mathcal{B})}+\|\partial_y\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|_{L^2(\mathcal{B})}\big)\Big)^2\mathrm{d}s\\&\le T\overline{C}^2\varepsilon^{\frac12}+\Big(\sum_{k\in\mathbb{Z}}\Big(\int_0^t\varepsilon\|\partial_y\dot\Delta_k(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}\Big)^2\\&\le(T+1)\overline{C}^2\varepsilon^{\frac12}.\end{aligned}$$

Therefore

$$\rho(T^*)\ge 2-(T+1)C_*\overline{C}^2\varepsilon^{\frac12}.$$

Thus, for $\varepsilon$ small enough, $\rho(T^*)>1$ and thus $T^*=\frac{T}{\varepsilon}$ and we have for $t\in[0,\frac{T}{\varepsilon}]$,

$$\|(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|_{L^2(\mathcal{B})}+\Big(\int_0^t\varepsilon\|\nabla(r^\varepsilon_\Phi,R^\varepsilon_\Phi)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\Big)^{\frac12}\le C\varepsilon^{\frac14},$$

which implies (2.21) since $\beta(t) \le \beta_*$ by Proposition 3.1 and $\rho(t) \ge 1$ for $t \in [0, \frac{T}{\varepsilon}]$. This completes the proof of Proposition 2.6.

□

## 4. Proof of Lemma 3.2

This section is devoted to the proof of Lemma 3.2, which is quite similar to the 3-D sys-metric case in [21]. By using Bony's decomposition in [3] in the $x$ variable,

$$ab = T^h_a b + R^h(a,b) + T^h_b a,$$

where

$$T^h_a b := \sum_{k\in\mathbb{Z}} \dot{S}_{k-1} a \dot{\Delta}_k b \quad \text{and} \quad R^h(a,b) := \sum_{k\in\mathbb{Z}} \dot{\Delta}_k a \tilde{\dot{\Delta}}_k b \quad \text{with} \quad \tilde{\dot{\Delta}}_k := \sum_{|k'-k|\le 1} \dot{\Delta}_{k'}.$$

For a function $a \in L^2(\mathcal{B})$, we introduce the notation as in [11],

$$a^+ := \mathcal{F}^{-1}_{\xi\to x}|\hat{a}|.$$

By Bernstein's inequality and the notation (3.3), for function $f \in H^1(\mathcal{B})$,

$$\|\dot{\Delta}_k f\|_{L^\infty(\mathcal{B})} \le C\|\langle\partial_y\rangle \dot{\Delta}_k f\|_{L^\infty(\mathbb{R}_+;L^2([-1,1]))} \le C2^{\frac{k}{2}}\|\langle\partial_y\rangle\dot{\Delta}_k f\|_{L^2(\mathcal{B})}. \tag{4.1}$$

Thus

$$\begin{aligned}\|\dot{S}_{k-1}a^+_\Phi\|_{L^\infty(\mathcal{B})} &\le C\sum_{k'\le k-2}\|\dot{\Delta}_{k'}a^+_\Phi\|_{L^\infty(\mathcal{B})} \\ &\le C\sum_{k'\le k-2} 2^{\frac{k'}{2}}\|\langle\partial_y\rangle\dot{\Delta}_{k'}a_\Phi\|_{L^2(\mathcal{B})} \le C\|\langle\partial_y\rangle a_\Phi\|_{\dot{B}^{\frac12}}.\end{aligned} \tag{4.2}$$

By utilizing a similar proof of Lemma 5.7 of [11] and Proposition 3.1, we get

$$\begin{aligned}|(\dot{\Delta}_k(T^h_a b)_\Phi|\dot{\Delta}_k c_\Phi)_{L^2(\mathcal{B})}| &\le C\sum_{|k'-k|\le 1}\|\dot{S}_{k'-1}a^+_\Phi\|_{L^\infty(\mathcal{B})}\|\dot{\Delta}_{k'}b_\Phi\|_{L^2(\mathcal{B})}\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})} \\ &\le C\|\langle\partial_y\rangle a_\Phi\|_{\dot{B}^{\frac12}}\sum_{|k'-k|\le 1}\|\dot{\Delta}_{k'}b_\Phi\|_{L^2(\mathcal{B})}\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})}\end{aligned}$$

Hence for any $c_0 > 0$, there exists $C > 0$ such that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot{\Delta}_k(T^h_a b)_\Phi|\dot{\Delta}_k c_\Phi)_{L^2(\mathcal{B})}|\mathrm{d}s\right)^{\frac12} &\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot{\Delta}_k c_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12} \\ &+C\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\langle\partial_y\rangle a_\Phi\|^2_{\dot{B}^{\frac12}}\|\dot{\Delta}_k b_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}.\end{aligned} \tag{4.3}$$

By Bernstein's inequality, we find that

$$\begin{aligned}
&|(\dot{\Delta}_k(R^h(a,b))_\Phi|\dot{\Delta}_k c_\Phi)_{L^2(\mathcal{B})}| \\
&\leq C2^{\frac{k}{2}}\sum_{k'\geq k-3}\|\tilde{\dot{\Delta}}_{k'}a_\Phi^+\|_{L^2_x(L^\infty_y)}\|\dot{\Delta}_{k'}b_\Phi\|_{L^2(\mathcal{B})}\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})} \\
&\leq C2^{\frac{k}{2}}\sum_{k'\geq k-3}\|\langle\partial_y\rangle\tilde{\dot{\Delta}}_{k'}a_\Phi\|_{L^2(\mathcal{B})}\|\dot{\Delta}_{k'}b_\Phi\|_{L^2(\mathcal{B})}\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})} \\
&\leq C\sum_{k'\geq k-3}2^{\frac{k-k'}{2}}\|\langle\partial_y\rangle a_\Phi\|_{\dot{B}^{\frac{1}{2}}}\|\dot{\Delta}_{k'}b_\Phi\|_{L^2(\mathcal{B})}\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})}.
\end{aligned} \tag{4.4}$$

Then we can use Minkowvski's inequality and Hölder's inequality to get

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(\int_0^t|(\dot{\Delta}_k(R^h(a,b))_\Phi|\dot{\Delta}_k c_\Phi)_{L^2(\mathcal{B})}|\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac{1}{2}}+C\sum_{k\in\mathbb{Z}}\left(\int_0^t\Big(\sum_{k'\geq k-3}2^{\frac{k-k'}{2}}\|\langle\partial_y\rangle a_\Phi\|_{\dot{B}^{\frac{1}{2}}}\|\dot{\Delta}_{k'}b_\Phi\|_{L^2(\mathcal{B})}\Big)^2\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac{1}{2}}+C\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\langle\partial_y\rangle a_\Phi\|^2_{\dot{B}^{\frac{1}{2}}}\|\dot{\Delta}_k b_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.5}$$

For the last term, we observe that

$$\begin{aligned}
|(\dot{\Delta}_k(T^h_b a)_\Phi|\dot{\Delta}_k c_\Phi)_{L^2(\mathcal{B})}| &\leq C\sum_{|k'-k|\leq 1}\|\dot{S}_{k'-1}b_\Phi^+\|_{L^2_y(L^\infty_x)}\|\dot{\Delta}_{k'}a_\Phi\|_{L^\infty_y(L^2_x)}\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})} \\
&\leq\sum_{|k'-k|\leq 1}\sum_{l\leq k'-2}2^{\frac{l}{2}}\|\dot{\Delta}_l b_\Phi\|_{L^2(\mathcal{B})}\|\langle\partial_y\rangle\dot{\Delta}_{k'}a_\Phi\|_{L^2(\mathcal{B})}\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})} \\
&\leq C\sum_{l\leq k-1}2^{\frac{l-k}{2}}\|\dot{\Delta}_l b_\Phi\|_{L^2(\mathcal{B})}\|\langle\partial_y\rangle a_\Phi\|_{\dot{B}^{\frac{1}{2}}}\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(\int_0^t|(\dot{\Delta}_k(T^h_b a)_\Phi|\dot{\Delta}_k c_\Phi)_{L^2(\mathcal{B})}|\mathrm{d}s\right)^{\frac{1}{2}} \\
&c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac{1}{2}}+C\sum_{k\in\mathbb{Z}}\left(\int_0^t\Big(\sum_{l\leq k-1}2^{\frac{l-k}{2}}\|\langle\partial_y\rangle a_\Phi\|_{\dot{B}^{\frac{1}{2}}}\|\dot{\Delta}_l b_\Phi\|_{L^2(\mathcal{B})}\Big)^2\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot{\Delta}_k c_\Phi\|_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac{1}{2}}+C\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\langle\partial_y\rangle a_\Phi\|^2_{\dot{B}^{\frac{1}{2}}}\|\dot{\Delta}_k b_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.6}$$

Combine (4.3), (4.5) and (4.6), we get (3.8).

From (4.2), we can also get

$$\|\dot{S}_{k-1}a_\Phi^+\|_{L^\infty(\mathcal{B})}\leq C2^{\frac{k}{2}}\|\langle\partial_y\rangle a_\Phi\|_{\dot{B}^0},$$

which implies another estimate analogous to (4.3),

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(T_a^h b)_\Phi|\dot\Delta_k c_\Phi)_{L^2(\mathcal{B})}|\mathrm{d}s\right)^{\frac12} \le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\\ +C\sum_{k\in\mathbb{Z}}\left(\int_0^t 2^k\|\langle\partial_y\rangle a_\Phi\|^2_{\dot B^0}\|\dot\Delta_k b_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}.\end{aligned}\tag{4.7}$$

Similarly

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(T_b^h a)_\Phi|\dot\Delta_k c_\Phi)_{L^2(\mathcal{B})}|\mathrm{d}s\right)^{\frac12} \le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\\ +C\sum_{k\in\mathbb{Z}}\left(\int_0^t 2^k\|\langle\partial_y\rangle b_\Phi\|^2_{\dot B^0}\|\dot\Delta_k a_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}.\end{aligned}\tag{4.8}$$

We can get from (4.4) that

$$|(\dot\Delta_k(R^h(a,b))_\Phi|\dot\Delta_k c_\Phi)_{L^2(\mathcal{B})}| \le C\sum_{k'\ge k-3} 2^k\|\langle\partial_y\rangle a_\Phi\|_{\dot B^0}\|\dot\Delta_{k'} b_\Phi\|_{L^2(\mathcal{B})}\|\dot\Delta_k c_\Phi\|_{L^2(\mathcal{B})},\tag{4.9}$$

which implies

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(T_b^h a)_\Phi|\dot\Delta_k c_\Phi)_{L^2(\mathcal{B})}|\mathrm{d}s\right)^{\frac12}\\ &\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12} + C\sum_{k\in\mathbb{Z}}\left(\int_0^t 2^k\|\langle\partial_y\rangle a_\Phi\|^2_{\dot B^0}\|\dot\Delta_k b_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}.\end{aligned}\tag{4.10}$$

By combining (4.7), (4.8) and (4.10) we get (3.9) and complete the proof of Lemma 3.2.

## 5. Proof of Proposition 3.3

In this section, we prove Proposition 3.3.

We recall that $F^\varepsilon, G^\varepsilon$ are defined in (2.19) and (2.20). We estimate them term by term.

• $\underline{\chi\{V\}_\varepsilon\partial_x u^1}$: Note that $\chi\{V\}_\varepsilon$ is independent of variable $x$,

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T \|\dot\Delta_k(\chi\{V\}_\varepsilon\partial_x u^1)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12} \le \sum_{k\in\mathbb{Z}}\|\dot\Delta_k\partial_x u^1_\Phi\|_{L^\infty_T(L^\infty_y(L^2_x))}\left(\int_0^T \|\chi\{V\}_\varepsilon\|^2_{L^2_y}\mathrm{d}s\right)^{\frac12}.\tag{5.1}$$

By Lemma 2.5 and the fact that $V\in C^0_2(\mathbb{R}_+;H^2_3(\mathbb{R}_+))$,

$$\left(\int_0^T \|\chi\{V\}_\varepsilon\|^2_{L^2_y}ds\right)^{\frac12} \le C\varepsilon^{\frac14}\|V\|_{L^2([0,T]\times\mathbb{R}_+)} \le C\varepsilon^{\frac14}\|V\|_{C^0_2(\mathbb{R}_+;L^2(\mathbb{R}_+))}.$$

By the definition of $u^1$ in (2.11) and Bernstein's inequality,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\|\dot\Delta_k\partial_x u^1_\Phi\|_{L^\infty_T(L^\infty_y(L^2_x))}&\le C\sum_{k\in\mathbb{Z}}\|\dot\Delta_k\partial_x u_{b,\Phi}\|_{L^\infty_y(L^2_x)}\\&\le C\sum_{k\in\mathbb{Z}}2^k\|\langle\partial_y\rangle\dot\Delta_k u_{b,\Phi}\|_{L^2(\mathcal{B})}\\&\le C\Big(\sum_{k<0}\|\langle\partial_y\rangle\dot\Delta_k u_{b,\Phi}\|^2_{L^2(\mathcal{B})}\Big)^{\frac12}+C\Big(\sum_{k\ge0}2^{4k}\|\langle\partial_y\rangle\dot\Delta_k u_{b,\Phi}\|^2_{L^2(\mathcal{B})}\Big)^{\frac12}\\&\le C\|\mathrm{e}^{\rho_0|\partial_x|}u_b\|_{H^3(\mathcal{B})}.\end{aligned}\tag{5.2}$$

By taking $\rho_b=2\rho_0$, in (1.11), we find that

$$\|\mathrm{e}^{\rho_0|\partial_x|}u_b\|_{H^3(\mathcal{B})}\le CC_b.\tag{5.3}$$

Thus, there exists a constant $C>0$ such that

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(\chi\{V\}_\varepsilon\partial_x u^1)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\le C\varepsilon^{\frac14}.$$

- $\underline{\varepsilon W^\varepsilon\partial_x u^1}$: Similar to (5.1), combine (5.2), (5.3) and Lemma 2.4, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(\varepsilon W^\varepsilon\partial_x u^1)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\\&\le C\sum_{k\in\mathbb{Z}}\|\dot\Delta_k\partial_x u^1_\Phi\|_{L^\infty_T(L^\infty_y(L^2_x))}\Big(\int_0^T\|\varepsilon W^\varepsilon\|^2_{L^2_y}\mathrm{d}s\Big)^{\frac12}\le C\varepsilon^{\frac54},\end{aligned}$$

- $\underline{\frac{u^1_2\phi'}{\phi}\chi\{z\partial_z V\}_\varepsilon e_x}$: Note that $\chi(y)$ is supported in $\{y\ge\frac13\}$ and $|\phi'|=1,\phi(y)=1-|y|$, when $\chi\neq0$, therefore

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(\frac{u^1_2\phi'}{\phi}\chi\{z\partial_z V\}_\varepsilon e_x)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\\&\le C\sum_{k\in\mathbb{Z}}\|\frac{\dot\Delta_k u^1_{\Phi,2}}{1-|y|}\|_{L^\infty_T(L^\infty_y(L^2_x))}\Big(\int_0^T\|\chi\{z\partial_z V\}_\varepsilon\|^2_{L^2_y}\mathrm{d}s\Big)^{\frac12}\end{aligned}$$

By Lemma 2.5 and the fact that $V\in C^0_2(\mathbb{R}_+;H^2_3(\mathbb{R}_+))$,

$$\Big(\int_0^T\|\chi\{z\partial_z V\}_\varepsilon\|^2_{L^2_y}\mathrm{d}s\Big)^{\frac12}\le C\varepsilon^{\frac14}\|V\|_{C^0_2(\mathbb{R}_+;H^1_1(\mathbb{R}_+))}.$$

Similar to (3.15)

$$\frac{\dot\Delta_k u^1_{\Phi,2}}{1-|y|}(t,x,y)=\begin{cases}-\int_0^1\dot\Delta_k\partial_y u^1_{\Phi,2}(t,x,1-(1-y)s)\mathrm{d}s & \text{if}\, y<0,\\ \int_0^1\dot\Delta_k\partial_y u^1_{\Phi,2}(t,x,-1+(1+y)s)\mathrm{d}s & \text{if}\, y<0.\end{cases}\tag{5.4}$$

By the divergence free condition $\partial_y u^1_{\Phi,2} = -\partial_x u^1_{\Phi,1}$, the we can deduce similar to (5.2) that

$$\sum_{k\in\mathbb{Z}} \|\frac{\dot{\Delta}_k u^1_{\Phi,2}}{1-|y|}\|_{L^\infty_T(L^\infty_y(L^2_x))} \leq C\|\mathrm{e}^{\rho_0|\partial_x|}u_b\|_{H^3(\mathcal{B})} \leq CC_b.$$

Hence

$$\sum_{k\in\mathbb{Z}} \left(\int_0^T \|\dot{\Delta}_k(\frac{u^1_2\phi'}{\phi}\chi\{z\partial_z V\}_\varepsilon e_x)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac{1}{2}} \leq C\varepsilon^{\frac{1}{4}}$$

- $\underline{\varepsilon\Delta u^1:}$ By the construction of $u^1$ in (2.11),

$$\sum_{k\in\mathbb{Z}} \left(\int_0^T \|\dot{\Delta}_k(\varepsilon\Delta u^1)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac{1}{2}} \leq C\varepsilon\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k\mathrm{e}^{\rho_0|\partial_x|}\Delta u_b\|_{L^2(\mathcal{B})}.$$

Similar to (5.2), we can get

$$\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k\mathrm{e}^{\rho_0|\partial_x|}\partial_x^2 u_b\|_{L^2(\mathcal{B})} \leq C\|\mathrm{e}^{\rho_0|\partial_x|}u_b\|_{H^3(\mathcal{B})} \leq CC_b.$$

Thanks to the Cauchy inequality and the properties of operator $\dot{\Delta}_k$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k\mathrm{e}^{\rho_0|\partial_x|}\partial_y^2 u_b\|_{L^2(\mathcal{B})} &\leq \left(\sum_{k\in\mathbb{Z}}2^{-\frac{|k|}{2}}\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}2^{\frac{|k|}{2}}\|\dot{\Delta}_k\mathrm{e}^{\rho_0|\partial_x|}\partial_y^2 u_b\|^2_{L^2(\mathcal{B})}\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{\mathbb{R}}(|\xi|^{\frac{1}{2}}+|\xi|^{-\frac{1}{2}})2^{2\rho_0|\xi|}\|\mathcal{F}(\partial_y^2 u_b)(\xi)\|_{L^2_y}\mathrm{d}\xi\right)^{\frac{1}{2}}.\end{aligned} \tag{5.5}$$

For the low frequencies, by (1.12),

$$\begin{aligned}&\int_{|\xi|\leq 1}(|\xi|^{\frac{1}{2}}+|\xi|^{-\frac{1}{2}})\mathrm{e}^{2\rho_0|\xi|}\|\mathcal{F}(\partial_y^2 u_b)(\xi)\|_{L^2_y}\mathrm{d}\xi\\ &\leq \mathrm{e}^{2\rho_0}\|\mathcal{F}(\partial_y^2 u_b)\|_{L^\infty_\xi(L^2_y)}\int_{|\xi|\leq 1}(|\xi|^{\frac{1}{2}}+|\xi|^{-\frac{1}{2}})\mathrm{d}\xi\\ &\leq C\|\partial_y^2 u_b\|^2_{L^1_x(L^2_y)} \leq CC_b.\end{aligned} \tag{5.6}$$

For the high frequencies, by (1.11),

$$\begin{aligned}&\int_{|\xi|\geq 1}(|\xi|^{\frac{1}{2}}+|\xi|^{-\frac{1}{2}})\mathrm{e}^{2\rho_0|\xi|}\|\mathcal{F}(\partial_y^2 u_b)(\xi)\|_{L^2_y}\mathrm{d}\xi\\ &\leq \int_{|\xi|\geq 1}(|\xi|+1)^2\mathrm{e}^{2\rho_0|\xi|}\|\mathcal{F}(\partial_y^2 u_b)(\xi)\|_{L^2_y}\mathrm{d}\xi\\ &\leq \|\mathrm{e}^{\rho_0|\partial_x|}u_b\|^2_{H^3(\mathcal{B})} \leq CC_b.\end{aligned} \tag{5.7}$$

Gather the above inequalities

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(\varepsilon\Delta u^1)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\le C\varepsilon.$$

- $\underline{\varepsilon u_2^1\partial_y W^\varepsilon e_x:}$ Similar to (5.1), we have

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(\varepsilon u_2^1\partial_y W^\varepsilon e_x)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\le\sum_{k\in\mathbb{Z}}\|\dot\Delta_k u^1_{\Phi,2}\|_{L^\infty_T(L^\infty_y(L^2_x))}\Big(\int_0^T\|\varepsilon\partial_y W^\varepsilon\|^2_{L^2_y}\mathrm{d}s\Big)^{\frac12}.$$

By Lemma 2.4,

$$\Big(\int_0^T\|\varepsilon\partial_y W^\varepsilon\|^2_{L^2_y}\mathrm{d}s\Big)^{\frac12}\le C\varepsilon^{\frac54}.$$

In the same way as (5.5), (5.6) and (5.7), we can get

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\|\dot\Delta_k u^1_{\Phi,2}\|_{L^\infty_T(L^\infty_y(L^2_x))}&\le\sum_{k\in\mathbb{Z}}\|\dot\Delta_k u^1_{\Phi,2}\|_{L^\infty_T(H^1_y(L^2_x))}\\&\le C(\|u_b\|_{L^1_x(H^1_y)}+\|\mathrm{e}^{\rho_0|\partial_x|}u_b\|_{H^2(\mathcal{B})})\le CC_b.\end{aligned}\tag{5.8}$$

Thus

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(\varepsilon u_2^1\partial_y W^\varepsilon e_x)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\le C\varepsilon^{\frac54}.$$

- $\underline{u_2^1\chi'\{V\}_\varepsilon e_x:}$ Similar to (5.1), we have

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(u_2^1\chi'\{V\}_\varepsilon e_x)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\le\sum_{k\in\mathbb{Z}}\|\dot\Delta_k u^1_{\Phi,2}\|_{L^\infty_T(L^\infty_y(L^2_x))}\Big(\int_0^T\|\chi'\{V\}_\varepsilon\|^2_{L^2_y}\mathrm{d}s\Big)^{\frac12}.$$

By Lemma 2.5,

$$\Big(\int_0^T\|\chi'\{V\}_\varepsilon\|^2_{L^2_y}\mathrm{d}s\Big)^{\frac12}\le C\varepsilon^{\frac14}\|V\|_{L^2([0,T]\times\mathbb{R}_+)}\le C\varepsilon^{\frac14}\|V\|_{C^0_2(\mathbb{R}_+;L^2(\mathbb{R}_+))}.$$

Combine (5.8), we get

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(u_2^1\chi'\{V\}_\varepsilon e_x)_\Phi\|^2_{L^2(\mathcal{B})}\mathrm{d}s\right)^{\frac12}\le C\varepsilon^{\frac14}.$$

• $\underline{\varepsilon u^1\cdot\nabla u^1 \text{ and } \varepsilon B^1\cdot\nabla B^1:}$ By the definition of $u^1$ in (2.11),

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot{\Delta}_k(\varepsilon u^1\cdot\nabla u^1)_\Phi\|_{L^2(\mathcal{B})}^2\mathrm{d}s\right)^{\frac{1}{2}}\le C\varepsilon\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k(u_b\cdot\nabla u_b)_\Phi\|_{L^2(\mathcal{B})}.$$

In the same way as (5.5), (5.6) and (5.7), we can get

$$\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k(u_b\cdot\nabla u_b)_\Phi\|_{L^2(\mathcal{B})}\le C\big(\|u_b\cdot\nabla u_b\|_{L^1_x(L^2_y)}+\|\langle\partial_x\rangle\mathrm{e}^{\rho_0|\partial_x|}(u_b\cdot\nabla u_b)\|_{L^2(\mathcal{B})}\big).$$

For the first term of the right hand side, from (1.11),

$$\begin{aligned}\|u_b\cdot\nabla u_b\|_{L^1_x(L^2_y)}&\le\|u_b\|_{L^2_x(L^\infty_y)}\|\nabla u_b\|_{L^2(\mathcal{B})}\\&\le C\|\langle\partial_y\rangle u_b\|_{L^2(\mathcal{B})}\|\nabla u_b\|_{L^2(\mathcal{B})}\le CC_b^2.\end{aligned}$$

For the second term, by the Plancherel theorem and the inequalities

$$|\xi|\le|\eta|+|\xi-\eta|\quad\text{and}\quad(|\xi|^2+1)\le2(|\eta|^2+1)(|\xi-\eta|^2+1)\qquad\forall\,\xi,\eta\in\mathbb{R},$$

Denote $\langle\xi\rangle=\sqrt{|\xi|^2+1},\xi\in\mathbb{R}$, by Minkowski's inequality, we arrive at

$$\begin{aligned}&\|\langle\partial_x\rangle\mathrm{e}^{\rho_0|\partial_x|}(u_b\cdot\nabla u_b)\|_{L^2(\mathcal{B})}^2\\&\le C\int_{\mathbb{R}}\langle\xi\rangle^2\mathrm{e}^{2\rho_0|\xi|}\|\mathcal{F}(u_b\cdot\nabla u_b)\|_{L^2_y}^2\mathrm{d}\xi\\&=C\int_{\mathbb{R}}\langle\xi\rangle^2\mathrm{e}^{2\rho_0|\xi|}\Big\|\int_{\mathbb{R}}\mathcal{F}u_b(\eta)\cdot\mathcal{F}(\nabla u_b)(\xi-\eta)\mathrm{d}\eta\Big\|_{L^2_y}^2\mathrm{d}\xi\\&\le C\int_{\mathbb{R}}\Big\|\int_{\mathbb{R}}\langle\eta\rangle\mathrm{e}^{\rho_0|\eta|}|\mathcal{F}(u_b)(\eta)|\langle\xi-\eta\rangle\mathrm{e}^{\rho_0|\xi-\eta|}|\mathcal{F}(\nabla u_b)(\xi-\eta)|\mathrm{d}\eta\Big\|_{L^2_y}^2\mathrm{d}\xi\\&\le C\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\langle\eta\rangle\mathrm{e}^{\rho_0|\eta|}\|\mathcal{F}(u_b)(\eta)\|_{L^\infty_y}\langle\xi-\eta\rangle\mathrm{e}^{\rho_0|\xi-\eta|}\|\mathcal{F}(\nabla u_b)(\xi-\eta)\|_{L^2_y}\mathrm{d}\eta\right)^2\mathrm{d}\xi\\&\le C\|\langle\eta\rangle\mathrm{e}^{\rho_0|\eta|}|\mathcal{F}(u_b)(\eta)|\|_{L^1_\eta(L^\infty_y)}^2\|\langle\xi\rangle\mathrm{e}^{\rho_0|\xi|}|\mathcal{F}(\nabla u_b)(\xi)|\|_{L^2_\xi(L^2_y)}^2.\end{aligned}$$

By the Plancherel theorem and (1.11),

$$\|\langle\xi\rangle\mathrm{e}^{\rho_0|\xi|}|\mathcal{F}(\nabla u_b)(\xi)|\|_{L^2_\xi(L^2_y)}^2\le C\|\mathrm{e}^{\rho_0|\partial_x|}u_b\|_{H^2(\mathcal{B})}\le CC_b,$$

$$\begin{aligned}\|\langle\eta\rangle\mathrm{e}^{\rho_0|\eta|}|\mathcal{F}(u_b)(\eta)|\|_{L^1_\eta(L^\infty_y)}^2&\le C\int_{\mathbb{R}}\langle\eta\rangle\mathrm{e}^{\rho_0|\eta|}\|\mathcal{F}(u_b)(\eta)\|_{H^1_y}\mathrm{d}\eta\\&\le C\left(\int_{\mathbb{R}}\langle\eta\rangle^4\mathrm{e}^{2\rho_0|\eta|}\|\mathcal{F}(u_b)(\eta)\|_{H^1_y}^2\mathrm{d}\eta\right)^{\frac{1}{2}}\\&\le C\|\mathrm{e}^{\rho_0|\partial_x|}u_b\|_{H^3(\mathcal{B})}\\&\le CC_b,\end{aligned}$$

Combine the above inequalities, we get

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T \|\dot{\Delta}_k(\varepsilon u^1\cdot\nabla u^1)_\Phi\|_{L^2(\mathcal{B})}^2\mathrm{d}s\right)^{\frac12}\le C\varepsilon.$$

Similarly,

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T \|\dot{\Delta}_k(\varepsilon B^1\cdot\nabla B^1)_\Phi\|_{L^2(\mathcal{B})}^2\mathrm{d}s\right)^{\frac12}\le C\varepsilon.$$

By combining the above estimates for eight terms of $F^\varepsilon$, there exists a constant $C>0$ such that

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T \|\dot{\Delta}_k F^\varepsilon_\Phi\|_{L^2(\mathcal{B})}^2\mathrm{d}s\right)^{\frac12}\le C\varepsilon^{\frac14}.$$

By the same analysis of $G^\varepsilon$ term by term, we can also get

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T \|\dot{\Delta}_k G^\varepsilon_\Phi\|_{L^2(\mathcal{B})}^2\mathrm{d}s\right)^{\frac12}\le C\varepsilon^{\frac14},$$

which completes the proof Proposition 3.3.

## Appendix A. On the smoothness of the 2-D MHD equations

In this appendix we prove the following smoothness and uniqueness result of the Leray weak solution to the uncontrolled 2-D MHD equations with $L^2$ initial data, that is to the following system:

$$\begin{cases}\partial_t u+u\cdot\nabla u-B\cdot\nabla B-\Delta u+\nabla p=0 & \text{in } (0,T)\times\mathcal{B},\\ \partial_t B+u\cdot\nabla B-B\cdot\nabla u-\Delta B=0 & \text{in } (0,T)\times\mathcal{B},\\ \operatorname{div} u=\operatorname{div} B=0 & \text{in } (0,T)\times\mathcal{B},\\ u=0,\quad B_2=0\quad \text{and}\quad \partial_y B_1=0 & \text{on } (0,T)\times\partial\mathcal{B},\\ (u,B)=(u_0,B_0) & \text{on } \{0\}\times\mathcal{B}.\end{cases}\tag{A.1}$$

**Theorem A.1.** *Let $T>0$, $u_0$ and $B_0$ in $L^2_{\mathrm{div}}(\mathcal{B})$. Then there exist a unique solution $(u,B)\in C^\infty((0,T]\times\mathcal{B})$ to (A.1) with initial data $(u_0,B_0)$. Moreover, for any $k\in\mathbb{N}$, there exists a continuous monotone increasing function $E_k(\cdot)$ with $E_k(0)=0$ and $E_k\le E_{k+1}$ such that*

$$\sum_{0\le j\le\frac{k}{2}}\|t^{\frac{k}{2}}\partial_t^j(u,B)\|^2_{L^\infty((0,T];H^{k-2j}(\mathcal{B}))}+\sum_{0\le j\le\frac{k+1}{2}}\|t^{\frac{k}{2}}\partial_t^j(u,B)\|^2_{L^2((0,T];H^{k+1-2j}(\mathcal{B}))}\le E_k(M),\tag{A.2}$$

*where $M=\|(u_0,B_0)\|_{L^2(\mathcal{B})}:=(\|u_0\|^2_{L^2(\mathcal{B})}+\|B_0\|^2_{L^2(\mathcal{B})})^{\frac12}$.*

*Proof.* The existence of the global Leray weak solution to (A.1) is proved in [12]. It can be obtained by analysis similar to the Navier–Stokes equations, see Chapter 5 of [28] for instance. Our goal is to prove (A.2) and the uniqueness. We can directly deduce from (A.2) that $(u,B)\in C^\infty((0,T]\times\mathcal{B})$. We follow the line of the proof of Theorem A.1 in [20].

We proceed by induction on $k$.

**The case $k = 0$.** We take

$$C_0 = \max\{T + 1, \frac{1}{2}, \frac{1}{2}\}, \quad E_0(M) = C_0 M^2.$$

By the energy inequality, we can directly get

$$\|(u, B)\|^2_{L^\infty([0,T];L^2(\mathcal{B}))} + \|(u, B)\|^2_{L^2([0,T];H^1(\mathcal{B}))} \le E_0(M). \tag{A.3}$$

**The case $k = 1$.** For any $\tau \in (0, T)$, from (A.3), we can find a $T_1 \in [\frac{\tau}{2}, \tau]$ such that $(u, B)(T_1) \in H^1(\mathcal{B})$ and

$$\|\nabla(u, B)(T_1)\|^2_{L^2(\mathcal{B})} \le \frac{2}{\tau} E_0. \tag{A.4}$$

We denote by $\mathbb{P}$ the Leray projection operator from $L^2(\mathcal{B})$ to $L^2_{\text{div}}(\mathcal{B})$ and $\tilde{\Delta} = \mathbb{P}\Delta$. Multiply the first two equations of (A.1) by $-t\tilde{\Delta}u$ and $-t\Delta B$ respectively, then we find that

$$\begin{aligned}
&\frac{1}{2}\partial_t(t\|\nabla(u, B)\|^2_{L^2(\mathcal{B})}) + t\|\tilde{\Delta}u\|^2_{L^2(\mathcal{B})} + t\|\Delta B\|^2_{L^2(\mathcal{B})} \\
&\le \frac{1}{2}\|\nabla(u, B)\|^2_{L^2(\mathcal{B})} + t(\|u \cdot \nabla u\|_{L^2(\mathcal{B})} + \|B \cdot \nabla B\|_{L^2(\mathcal{B})})\|\tilde{\Delta}u\|_{L^2(\mathcal{B})} \\
&\quad + t(\|u \cdot \nabla B\|_{L^2(\mathcal{B})} + \|B \cdot \nabla u\|_{L^2(\mathcal{B})})\|\Delta B\|_{L^2(\mathcal{B})},
\end{aligned}$$

where we use the divergence free condition and the boundary conditions of $u$ and $B$ in (A.1). Then by Cauchy's inequality,

$$\begin{aligned}
&\partial_t(t\|\nabla(u, B)\|^2_{L^2(\mathcal{B})}) + t\|\tilde{\Delta}u\|^2_{L^2(\mathcal{B})} + t\|\Delta B\|^2_{L^2(\mathcal{B})} \\
&\le C\big(\|\nabla(u, B)\|^2_{L^2(\mathcal{B})} + t(\|u \cdot \nabla u\|^2_{L^2(\mathcal{B})} + \|B \cdot \nabla B\|^2_{L^2(\mathcal{B})} + \|u \cdot \nabla B\|^2_{L^2(\mathcal{B})} + \|B \cdot \nabla u\|^2_{L^2(\mathcal{B})})\big) \\
&\le C\big(\|\nabla(u, B)\|^2_{L^2(\mathcal{B})} + t\|(u, B)\|^2_{L^4(\mathcal{B})}\|\nabla(u, B)\|^2_{L^4(\mathcal{B})}\big).
\end{aligned}$$

By the Gagliardo–Nirenberg interpolation inequality

$$\|(u, B)\|^2_{L^4(\mathcal{B})} \le C\|(u, B)\|_{L^2(\mathcal{B})}\|(u, B)\|_{H^1(\mathcal{B})}, \tag{A.5}$$

$$\|\nabla(u, B)\|^2_{L^4(\mathcal{B})} \le C\|\nabla(u, B)\|_{L^2(\mathcal{B})}\|\nabla(u, B)\|_{H^1(\mathcal{B})}. \tag{A.6}$$

We also need the Cattabriga–Solonnikov inequality

$$\|\nabla^2 u\|_{L^2(\mathcal{B})} \le C\|\tilde{\Delta}u\|_{L^2(\mathcal{B})}. \tag{A.7}$$

Note that the divergence free condition $\operatorname{div} B = 0$ implies $\partial_x\partial_y B_1 = -\partial_y^2 B_2$ and $\partial_x\partial_y B_2 = -\partial_x^2 B_1$, hence

$$\|\nabla^2 B\|_{L^2(\mathcal{B})} \le C\|\Delta B\|_{L^2(\mathcal{B})}. \tag{A.8}$$

Combine (A.5)–(A.8),

$$\begin{aligned}
&\partial_t(t\|\nabla(u, B)\|^2_{L^2(\mathcal{B})}) + t\|\nabla^2(u, B)\|^2_{L^2(\mathcal{B})} \le C\|\nabla(u, B)\|^2_{L^2(\mathcal{B})} \\
&+ Ct(E_0 + E_0^{\frac{1}{2}}\|\nabla(u, B)\|_{L^2(\mathcal{B})})(\|\nabla(u, B)\|^2_{L^2(\mathcal{B})} + \|\nabla(u, B)\|_{L^2(\mathcal{B})}\|\nabla^2(u, B)\|_{L^2(\mathcal{B})}).
\end{aligned}$$

Again, by Cauchy's inequality

$$\begin{aligned}&\partial_t(t\|\nabla(u,B)\|^2_{L^2(\mathcal{B})})+t\|\nabla^2(u,B)\|^2_{L^2(\mathcal{B})}\\&\le Ct\|\nabla(u,B)\|^2_{L^2(\mathcal{B})}+Ct(E_0+1)(E_0+\|\nabla(u,B)\|^2_{L^2(\mathcal{B})})\|\nabla(u,B)\|^2_{L^2(\mathcal{B})}.\end{aligned}\tag{A.9}$$

Note the definition of $T_1$ in (A.4), we integral the above inequality over $[T_1,t]$ for $t\in[T_1,T]$ and use (A.3),

$$\begin{aligned}&t\|\nabla(u,B)\|^2_{L^2(\mathcal{B})}+\int_{T_1}^t s\|\nabla^2(u,B)(s)\|^2_{L^2(\mathcal{B})}\mathrm{d}s\\&\le CE_0+\int_{T_1}^t Cs(E_0+1)(E_0+\|\nabla(u,B)(s)\|^2_{L^2(\mathcal{B})})\|\nabla(u,B)(s)\|^2_{L^2(\mathcal{B})}\mathrm{d}s.\end{aligned}$$

By utilizing Gronwall's inequality and (A.3), we get

$$\|t^{\frac12}\nabla(u,B)\|^2_{L^\infty([T_1,T];L^2(\mathcal{B}))}+\|t^{\frac12}\nabla^2(u,B)\|^2_{L^2([T_1,T];L^2(\mathcal{B}))}\le E_{11}(M),\tag{A.10}$$

where $E_{11}(M)=2E_0(M)\exp\big(C(E_0(M)+1)(E_0(M)T+E_0(M))\big)$.

By using the equation (A.1),

$$\begin{aligned}t\|\partial_t(u,B)\|^2_{L^2(\mathcal{B})}\le Ct\big(&\|\nabla^2(u,B)\|^2_{L^2(\mathcal{B})}+\|u\cdot\nabla u\|^2_{L^2(\mathcal{B})}+\|B\cdot\nabla B\|^2_{L^2(\mathcal{B})}\\&+\|u\cdot\nabla B\|^2_{L^2(\mathcal{B})}+\|B\cdot\nabla u\|^2_{L^2(\mathcal{B})}\big)\end{aligned}$$

Similar to (A.9), we find that, for $t\in[T_1,T]$,

$$t\|\partial_t(u,B)\|^2_{L^2(\mathcal{B})}\le C\big(t\|\nabla^2(u,B)\|^2_{L^2(\mathcal{B})}+(E_0+1)(E_0+\|\nabla(u,B)\|^2_{L^2(\mathcal{B})})E_{11}\big).$$

Thus

$$\|t^{\frac12}\partial_t(u,B)\|^2_{L^2([T_1,T];L^2(\mathcal{B}))}\le E_{12}:=C\big(E_{11}+(E_0+1)(E_0T+E_0)E_{11}\big)\tag{A.11}$$

Combine (A.10) and (A.11), we have proved (A.2) for $k=1$ with $E_1:=E_0+E_{11}+E_{12}$, since $E_1$ is independent of $\tau$. Moreover, $E_1(\cdot)$ is a monotone increasing continuous function with $E_1(0)=0$, and by construction, $E_1\ge E_0$.

**The case $k\ge2$.** Inductively, we assume (A.2) is true for index $0,1,\cdots,k-1$. Then, for any $\tau\in(0,T)$, we can find a $T_k\in(\frac{\tau}{2},\tau]$ such that

$$\sum_{0\le j\le\frac{k}{2}}\|t^{\frac{k-1}{2}}\partial_t^j(u,B)(T_k)\|^2_{H^{k-2j}(\mathcal{B})}\le\frac{2}{\tau}E_{k-1}.$$

If $k$ is even, we apply $\partial_t^{\frac{k}{2}}$ to (A.1),

$$\begin{cases}\partial_t^{\frac{k}{2}+1}u+\partial_t^{\frac{k}{2}}(u\cdot\nabla u-B\cdot\nabla B)-\Delta\partial_t^{\frac{k}{2}}u+\nabla\partial_t^{\frac{k}{2}}p=0 & \text{in }\mathcal{B},\\ \partial_t^{\frac{k}{2}+1}B+\partial_t^{\frac{k}{2}}(u\cdot\nabla B-B\cdot\nabla u)-\Delta\partial_t^{\frac{k}{2}}B=0 & \text{in }\mathcal{B},\\ \operatorname{div}\partial_t^{\frac{k}{2}}u=\operatorname{div}\partial_t^{\frac{k}{2}}B=0 & \text{in }\mathcal{B},\\ \partial_t^{\frac{k}{2}}u=0,\quad \partial_t^{\frac{k}{2}}B_2=0\quad\text{and}\quad \partial_y\partial_t^{\frac{k}{2}}B_1=0 & \text{on }\partial\mathcal{B}.\end{cases}\tag{A.12}$$

Multiply the first two equations of (A.12) by $t^k\partial_t^{\frac{k}{2}}u$ and $t^k\partial_t^{\frac{k}{2}}B$ respectively, use the divergence free condition, the boundary conditions and integrate by parts, we arrive at

$$\begin{aligned}
&\frac{1}{2}\partial_t\big(t^k\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2\big)+t^k\|\nabla\partial_t^{\frac{k}{2}}u\|_{L^2(\mathcal{B})}^2+t^k\|\nabla\partial_t^{\frac{k}{2}}B\|_{L^2(\mathcal{B})}^2\\
&=\frac{k}{2}t^{k-1}\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2-\int_{\mathcal{B}}t^k\partial_t^{\frac{k}{2}}(u\cdot\nabla u-B\cdot\nabla B)\cdot\partial_t^{\frac{k}{2}}u\\
&\quad-\int_{\mathcal{B}}t^k\partial_t^{\frac{k}{2}}(u\cdot\nabla B-B\cdot\nabla u)\cdot\partial_t^{\frac{k}{2}}B.
\end{aligned}$$

While

$$\begin{aligned}
&\int_{\mathcal{B}}t^k(u\cdot\nabla\partial_t^{\frac{k}{2}}u-B\cdot\nabla\partial_t^{\frac{k}{2}}B)\cdot\partial_t^{\frac{k}{2}}u+\int_{\mathcal{B}}t^k(u\cdot\nabla\partial_t^{\frac{k}{2}}B-B\cdot\nabla\partial_t^{\frac{k}{2}}u)\cdot\partial_t^{\frac{k}{2}}B\\
&=\int_{\mathcal{B}}\frac{1}{2}t^ku\cdot\nabla(|\partial_t^{\frac{k}{2}}u|^2+||\partial_t^{\frac{k}{2}}B|^2)-\int_{\mathcal{B}}t^kB\cdot\nabla(\partial_t^{\frac{k}{2}}u\cdot\partial_t^{\frac{k}{2}}B)\\
&=0,
\end{aligned}$$

thus we have

$$\begin{aligned}
&\partial_t\big(t^k\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2\big)+t^k\|\nabla\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2\\
&\le Ckt^{k-1}\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2+\sum_{\alpha+\beta=\frac{k}{2},\alpha>0}C_kt^k\left|\int_{\mathcal{B}}(\partial_t^{\alpha}u\cdot\nabla\partial_t^{\beta}u-\partial_t^{\alpha}B\cdot\nabla\partial_t^{\beta}B)\cdot\partial_t^{\frac{k}{2}}u\right|\\
&\quad+\sum_{\alpha+\beta=\frac{k}{2},\alpha>0}C_kt^k\left|\int_{\mathcal{B}}(\partial_t^{\alpha}u\cdot\nabla\partial_t^{\beta}B-\partial_t^{\alpha}B\cdot\nabla\partial_t^{\beta}u)\cdot\partial_t^{\frac{k}{2}}B\right|\\
&\le Ckt^{k-1}\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2\\
&\quad+\sum_{\alpha+\beta=\frac{k}{2},\alpha>0}C_kt^k\|\partial_t^{\alpha}(u,B)\|_{L^4(\mathcal{B})}\|\nabla\partial_t^{\beta}(u,B)\|_{L^2(\mathcal{B})}\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^4(\mathcal{B})}\\
&\le Ckt^{k-1}\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2+\sum_{\alpha+\beta=\frac{k}{2},\alpha>0}C_kt^{2\alpha+\beta}\|\nabla\partial_t^{\beta}(u,B)\|_{L^2(\mathcal{B})}\|\partial_t^{\alpha}(u,B)\|_{L^4(\mathcal{B})}^2\\
&\quad+\sum_{\beta\le\frac{k}{2}-1}C_kt^{k+\beta}\|\nabla\partial_t^{\beta}(u,B)\|_{L^2(\mathcal{B})}\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^4(\mathcal{B})}^2
\end{aligned}\tag{A.13}$$

By the Gagliardo–Nirenberg interpolation inequality and Cauchy's inequality, for any $\delta>0$ there exit constants $C_{\delta,\alpha}$ such that

$$\begin{aligned}
&t^{2\alpha+\beta}\|\nabla\partial_t^{\beta}(u,B)\|_{L^2(\mathcal{B})}\|\partial_t^{\alpha}(u,B)\|_{L^4(\mathcal{B})}^2\\
&\le t^{2\alpha+\beta}\|\nabla\partial_t^{\beta}(u,B)\|_{L^2(\mathcal{B})}\big(\|\partial_t^{\alpha}(u,B)\|_{L^2(\mathcal{B})}^2+\|\partial_t^{\alpha}(u,B)\|_{L^2(\mathcal{B})}\|\nabla\partial_t^{\alpha}(u,B)\|_{L^2(\mathcal{B})}\big)\\
&\le\delta t^{2\alpha}\|\nabla\partial_t^{\alpha}(u,B)\|_{L^2(\mathcal{B})}^2+C_{\delta,\alpha}t^{2\alpha}\|\partial_t^{\alpha}(u,B)\|_{L^2(\mathcal{B})}^2(1+t^{2\beta}\|\nabla\partial_t^{\beta}(u,B)\|_{L^2(\mathcal{B})}^2).
\end{aligned}$$

Inserting the above inequality into the last inequality of (A.13), taking $\delta$ small enough we find that

$$\begin{aligned}&\partial_t\big(t^k\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2\big)+t^k\|\nabla\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2\\&\le C_kt^{k-1}\|\partial_t^{\frac{k}{2}}(u,B)\|_{L^2(\mathcal{B})}^2+\sum_{0<\alpha\le\frac{k}{2}-1}C_kt^{2\alpha}\|\nabla\partial_t^{\alpha}(u,B)\|_{L^2(\mathcal{B})}^2\\&+\sum_{\alpha+\beta=\frac{k}{2},\alpha>0}C_kt^{2\alpha}\|\partial_t^{\alpha}(u,B)\|_{L^2(\mathcal{B})}^2\big(1+t^{2\beta}\|\nabla\partial_t^{\beta}(u,B)\|_{L^2(\mathcal{B})}^2\big)\end{aligned}$$

We can deduce from the inductive assumption that

$$\begin{gathered}t^{2\alpha}\|\partial_t^{\alpha}(u,B)(t)\|_{L^2(\mathcal{B})}^2\le E_{k-1}\qquad\forall\ t\in[0,T],\alpha\le\tfrac{k}{2},\\\int_0^T\Big(t^{k-1}\|\partial_t^{\frac{k}{2}}(u,B)(t)\|_{L^2(\mathcal{B})}^2+\sum_{\alpha\le\frac{k}{2}-1}t^{2\alpha}\|\nabla\partial_t^{\alpha}(u,B)\|_{L^2(\mathcal{B})}^2\Big)\mathrm{d}t\le E_{k-1}.\end{gathered}$$

Then by the definition of $T_k$ and Gronwall's inequality,

$$\|t^{\frac{k}{2}}\partial_t^{\frac{k}{2}}(u,B)\|_{L^\infty([T_k,T];L^2(\mathcal{B}))}^2+\|t^{\frac{k}{2}}\nabla\partial_t^{\frac{k}{2}}(u,B)\|_{L^2([T_k,T];L^2(\mathcal{B}))}^2\le E_{k,1},\tag{A.14}$$

with

$$E_{k,1}=C_k\big(E_{k-1}+E_{k-1}(T+E_{k-1})\big)\exp\big(C_k(T+E_{k-1})\big).$$

Next, we will gain space derivatives *via* the equation (A.1). For $2\alpha+\beta=k-2$, apply $\partial_t^{\alpha}\nabla^{\beta}$ to (A.1),

$$\begin{aligned}&t^k\|\partial_t^{\alpha}\nabla^{\beta+2}(u,B)\|_{L^2(\mathcal{B})}^2\\&\le Ct^k\big(\|\tilde{\Delta}\partial_t^{\alpha}\nabla^{\beta}u\|_{L^2(\mathcal{B})}^2+\|\Delta\partial_t^{\alpha}\nabla^{\beta}(u,B)\|_{L^2(\mathcal{B})}^2\|\big)\\&\le Ct^k\big(\|\partial_t^{\alpha+1}\nabla^{\beta}(u,B)\|_{L^2(\mathcal{B})}^2+\|\partial_t^{\alpha}\nabla^{\beta}(u\cdot\nabla u-B\cdot\nabla B)\|_{L^2(\mathcal{B})}^2\\&\quad+\|\partial_t^{\alpha}\nabla^{\beta}(u\cdot\nabla B-B\cdot\nabla u)\|_{L^2(\mathcal{B})}^2\big)\\&\le Ct^k\big(\|\partial_t^{\alpha+1}\nabla^{\beta}(u,B)\|_{L^2(\mathcal{B})}^2+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\\beta_1+\beta_2=\beta}}\|\partial_t^{\alpha_1}\nabla^{\beta_1}(u,B)\|_{L^4(\mathcal{B})}^2\|\partial_t^{\alpha_2}\nabla^{\beta_2+1}(u,B)\|_{L^4(\mathcal{B})}^2\big)\end{aligned}$$

Then use the Gagliardo–Nirenberg interpolation inequality, we have

$$\begin{aligned}&t^{2\alpha_1+\beta_1+\frac{1}{2}}\|\partial_t^{\alpha_1}\nabla^{\beta_1}(u,B)\|_{L^4(\mathcal{B})}^2\\&\le Ct^{2\alpha_1+\beta_1+\frac{1}{2}}(\|\partial_t^{\alpha_1}\nabla^{\beta_1}(u,B)\|_{L^2(\mathcal{B})}^2+\|\partial_t^{\alpha_1}\nabla^{\beta_1}(u,B)\|_{L^2(\mathcal{B})}\|\partial_t^{\alpha_1}\nabla^{\beta_1+1}(u,B)\|_{L^2(\mathcal{B})})\\&\le C(E_{k-1}T^{\frac{1}{2}}+E_{k-1}),\end{aligned}$$

$$\begin{aligned}&t^{2\alpha_2+\beta_2+\frac{3}{2}}\|\partial_t^{\alpha_2}\nabla^{\beta_2+1}(u,B)\|_{L^4(\mathcal{B})}^2\\&\le Ct^{2\alpha_2+\beta_2+\frac{3}{2}}(\|\partial_t^{\alpha_2}\nabla^{\beta_2+1}(u,B)\|_{L^2(\mathcal{B})}^2+\|\partial_t^{\alpha_2}\nabla^{\beta_2+1}(u,B)\|_{L^2(\mathcal{B})}\|\partial_t^{\alpha_2}\nabla^{\beta_2+2}(u,B)\|_{L^2(\mathcal{B})})\\&\le\begin{cases}C(E_{k-1}T^{\frac{1}{2}}+E_{k-1})\qquad\text{when }2\alpha_2+\beta_2\le k-2,\\C(E_{k-1}T^{\frac{1}{2}}+E_{k-1}^{\frac{1}{2}}t^{\frac{k}{2}}\|\partial_t^{\alpha}\nabla^{\beta+2}(u,B)\|_{L^2(\mathcal{B})})\qquad\text{when }2\alpha_2+\beta_2=k-2,\end{cases}\end{aligned}$$

where we used $2\alpha_1+\beta_1+1\le 2\alpha+\beta+1=k-1, 2\alpha_1+\beta_1+1\le k-1$ and the inductive assumption. Therefore

$$\begin{aligned} &t^k\|\partial_t^\alpha\nabla^{\beta+2}(u,B)\|^2_{L^2(\mathcal{B})}\\ &\le Ct^k\|\partial_t^{\alpha+1}\nabla^\beta(u,B)\|^2_{L^2(\mathcal{B})}+C(E_{k-1}^2+E_{k-1}^3)(T+1). \end{aligned} \tag{A.15}$$

Combine (A.14) and (A.15), we get

$$\sum_{0\le j\le\frac{k}{2}}\|t^{\frac{k}{2}}\partial_t^j\nabla^{k-2j}(u,B)\|^2_{L^\infty([T_k,T];L^2(\mathcal{B}))}\le E_{k,2}, \tag{A.16}$$

for $E_{k,2}=C_k\big(E_{k,1}+(T+1)(E_{k-1}^2+E_{k-1}^3)\big)$.

Similar to (A.15), we can get, for $2\alpha+\beta=k-2$,

$$\begin{aligned} &t^k\|\partial_t^\alpha\nabla^{\beta+3}(u,B)\|^2_{L^2(\mathcal{B})}\\ &\le Ct^k\|\partial_t^{\alpha+1}\nabla^{\beta+1}(u,B)\|^2_{L^2(\mathcal{B})}+C(E_{k,2}^2+E_{k,2}^3)(T+1). \end{aligned}$$

Hence, we can deduce from (A.14) that

$$\sum_{0\le j\le\frac{k}{2}}\|t^{\frac{k}{2}}\partial_t^j\nabla^{k+1-2j}(u,B)\|^2_{L^2([T_k,T]\times\mathcal{B})}\le E_{k,3}, \tag{A.17}$$

for $E_{k,3}=C_k\big(E_{k,1}+(T+1)(E_{k,2}^2+E_{k,2}^3)\big)$. By combining (A.14), (A.16) and (A.17), we arrive at (A.2) for $k$ is even with $E_k=E_{k-1}+E_{k,1}+E_{k,2}+E_{k,3}$, since $E_k$ is independent of $\tau$. Moreover $E_k(\cdot)$ is a monotone increasing continuous function with $E_k(0)=0$ and $E_k\ge E_{k-1}$.

If $k$ is odd, we apply $\partial_t^{\frac{k-1}{2}}$ to (A.1) and multiply by $-t^k\tilde{\Delta}\partial_t^{\frac{k-1}{2}}u$ and $-t^k\Delta\partial_t^{\frac{k-1}{2}}B$ respectively, the rest of the estimates is similar to the case when $k$ is even and we can prove that (A.2) holds true, we omit the details. Therefore we have finish the induction process.

**The uniqueness.** The proof of the uniqueness is quite classic by using energy method and Gronwall's inequality. Suppose $(u_1,B_1,p_1)$ and $(u_2,B_2,p_2)$ are both Leray weak solutions to (A.1) with the same initial data $(u_0,B_0)$. From the above analysis, $u_1,u_2\in C^\infty((0,T]\times\mathcal{B})$ and satisfy (A.2). Let $u=u_1-u_2, B=B_1-B_2, p=p_1-p_2$, then $(u,B,p)$ satisfies

$$\begin{cases} \partial_t u+u\cdot\nabla u_1+u_2\cdot\nabla u-B\cdot\nabla B_1-B_2\cdot\nabla B-\Delta u+\nabla p=0 & \text{in } \mathcal{B},\\ \partial_t B+u\cdot\nabla B_1+u_2\cdot\nabla B-B\cdot\nabla u_1-B_2\cdot\nabla u-\Delta B=0 & \text{in } \mathcal{B},\\ \operatorname{div}u=\operatorname{div}B=0 & \text{in } \mathcal{B},\\ u=0,\quad B_2=0\quad\text{and}\quad\partial_y B_1=0 & \text{on } \partial\mathcal{B},\\ (u,B)=(0,0) & \text{on } \{t=0\}. \end{cases}$$

By using energy method, we find that

$$\begin{aligned} &\frac{1}{2}\partial_t\|(u,B)\|^2_{L^2(\mathcal{B})}+\|\nabla u\|^2_{L^2(\mathcal{B})}+\|\nabla B\|^2_{L^2(\mathcal{B})}\\ &\le|\int_{\mathcal{B}}(u\cdot\nabla u_1-B\cdot\nabla B_1)\cdot u|+|\int_{\mathcal{B}}(u\cdot\nabla B_1-B\cdot\nabla B_1)\cdot B|\\ &\le\|\nabla(u_1,B_1)\|_{L^2(\mathcal{B})}\|(u,B)\|^2_{L^4(\mathcal{B})}\\ &\le C\|\nabla(u_1,B_1)\|_{L^2(\mathcal{B})}(\|(u,B)\|^2_{L^2(\mathcal{B})}+\|(u,B)\|_{L^2(\mathcal{B})}\|\nabla(u,B)\|_{L^2(\mathcal{B})}), \end{aligned}$$

where we used $\int_{\mathcal{B}} \nabla p \cdot u = 0, \int_{\mathcal{B}} u_2 \cdot \nabla u \cdot u = 0, \int_{\mathcal{B}} u_2 \cdot \nabla B \cdot B = 0$ and $\int_{\mathcal{B}} B_2 \cdot \nabla B \cdot u + \int_{\mathcal{B}} B_2 \cdot \nabla u \cdot B = \int_{\mathcal{B}} B_2 \cdot \nabla(u \cdot B) = 0$, since $u_1, u_2$ and $u$ are smooth therefore the integration by parts is allowed. Thus by Cauchy's inequality,

$$\begin{aligned}&\partial_t \|(u,B)\|^2_{L^2(\mathcal{B})} + \|\nabla(u,B)\|^2_{L^2(\mathcal{B})}\\&\le C(\|\nabla(u_1,B_1)\|_{L^2(\mathcal{B})} + \|\nabla(u_1,B_1)\|^2_{L^2(\mathcal{B})})\|(u,B)\|^2_{L^2(\mathcal{B})}.\end{aligned}$$

Since $(u_1, B_1)$ satisfies $\int_0^T \|\nabla(u_1,B_1)\|^2_{L^2(\mathcal{B})} \le E_0(M)$, with $M = \|(u_0,B_0)\|^2_{L^2(\mathcal{B})}$, we can obtain that $(u,B) \equiv (0,0)$ by using Gronwall's inequality.

□

## Appendix B. Proof of the regularization proposition 1.3

In this section, we prove the regularization Proposition 1.3.

Let $T > 0, \rho_b > 0, \delta > 0$ and $k \in \mathbb{N}$. Let $(u_0, B_0) \in L^2_{\text{div}}(\Omega)$ satisfies (1.3). By Lemma C.1, we can construct an extension $(u_a, B_a) \in L^2(\mathcal{B})$ of $(u_0, B_0)$ such that

$$\begin{gathered}u_a \text{ and } B_a \text{ are both supported in } [-L, 2L] \times [-1,1],\\(u_a, B_a)|_\Omega = (u_0, B_0),\\\|(u_a,B_a)\|_{L^2(\mathcal{B})} \le c_0 \|(u_0,B_0)\|_{L^2(\Omega)},\end{gathered}$$

for a constant $c_0$.

By Theorem A.1, there is a unique solution $(v, H) \in C^\infty((0,T] \times \mathcal{B})$ to

$$\begin{cases}\partial_t v + \mathbb{P}(v \cdot \nabla v - H \cdot \nabla H) - \tilde{\Delta} v = 0 & \text{in } (0,T) \times \mathcal{B},\\\partial_t H + v \cdot \nabla H - H \cdot \nabla v - \Delta H = 0 & \text{in } (0,T) \times \mathcal{B},\\\operatorname{div} v = \operatorname{div} H = 0 & \text{in } (0,T) \times \mathcal{B},\\v = 0, \quad H_2 = 0 \quad \text{and} \quad \partial_y H_1 = 0 & \text{on } (0,T) \times \partial\mathcal{B},\\(v,H) = (u_a, B_a) & \text{on } \{0\} \times \mathcal{B}.\end{cases} \tag{B.1}$$

Let $\beta(t) \in C^\infty([0,T];[0,1])$ with $\beta = 1$ on $[0, \frac{T}{3}]$,and $\beta = 0$ on $[\frac{2T}{3}, T]$. Let $\theta(x) \in C^\infty(\mathbb{R};[0,1])$ with $\theta = 1$ when $x \in [-L, 2L]$ and $\theta = 0$ when $x < -2L$ or $x > 3L$. We consider $u = \beta v + (1-\beta)\theta v, B = \beta H + (1-\beta)\theta H$. Then $(u,B) \in C^\infty((0,T] \times \mathcal{B})$ and is the solution to

$$\begin{cases}\partial_t u + \mathbb{P}(u \cdot \nabla u - B \cdot \nabla B) - \tilde{\Delta} u = F, & u(0) = u_a,\\\partial_t B + u \cdot \nabla B - B \cdot \nabla u - \Delta B = G, & B(0) = B_a,\end{cases}$$

where

$$\begin{aligned}F :=\ & \dot{\beta}(1-\theta)v - 2(1-\beta)\nabla\theta \cdot \nabla v - (1-\beta)(\Delta\theta)v\\&-\mathbb{P}(\beta(1-\beta)v \cdot \nabla((1-\theta)v) + (1-\beta)^2\theta v \cdot \nabla((1-\theta)v))\\&+\mathbb{P}(\beta(1-\beta)H \cdot \nabla((1-\theta)H) + (1-\beta)^2\theta H \cdot \nabla((1-\theta)H)),\\G :=\ & \dot{\beta}(1-\theta)H - 2(1-\beta)\nabla\theta \cdot \nabla H - (1-\beta)(\Delta\theta)H\\&-\beta(1-\beta)v \cdot \nabla((1-\theta)H) - (1-\beta)^2\theta v \cdot \nabla((1-\theta)H)\\&+\beta(1-\beta)H \cdot \nabla((1-\theta)v) + (1-\beta)^2\theta H \cdot \nabla((1-\theta)v),\end{aligned}$$

By the definition of $\beta$ and $\theta$, both $F$ and $G$ are smooth functions and are supported in $[\frac{T}{3}, T] \times \overline{\mathcal{B}\backslash\Omega'}$, where $\Omega' = (-L, 2L) \times (-1, 1)$. We define

$$u_N := \beta u + (1-\beta)\mathbf{P}_N u \quad \text{and} \quad B_N := \beta B + (1-\beta)\mathbf{P}_N B, \tag{B.2}$$

where

$$\mathbf{P}_N f(x,y) = \mathcal{F}^{-1}_{\xi\to x}(\mathcal{F}f(\xi,y)\chi(\frac{\xi}{N})),$$

for function $f(x,y)$, $\chi \in C_0^\infty(\mathbb{R})$ is a cut-off function with $\chi(\xi) = 1$ when $|\xi| \le 1$ and $\chi(\xi) = 0$ when $|\xi| \ge 2$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform operation and the inverse Fourier transform operator respectively in the direction of $x$. Then $(u_N, B_N)$ satisfies

$$\begin{cases} \partial_t u_N + \mathbb{P}(u_N\cdot\nabla u_N - B_N\cdot\nabla B_N) - \tilde{\Delta} u_N = F_N, & u_N(0) = u_a,\\ \partial_t B_N + u_N\cdot\nabla B_N - B_N\cdot\nabla u_N - \Delta B_N = G_N, & B_N(0) = B_a,\end{cases}$$

where

$$\begin{aligned} F_N :=\ & \dot{\beta}(u-\mathbf{P}_N u) + \beta F + (1-\beta)\mathbf{P}_N F\\ & -\mathbb{P}\big(\beta(1-\beta)u\cdot\nabla(u-\mathbf{P}_N u) + (1-\beta)^2\mathbf{P}_N u\cdot\nabla(u-\mathbf{P}_N u)\big)\\ & +\mathbb{P}\big(\beta(1-\beta)B\cdot\nabla(B-\mathbf{P}_N B) + (1-\beta)^2\mathbf{P}_N B\cdot\nabla(B-\mathbf{P}_N B)\big),\\ G_N :=\ & \dot{\beta}(B-\mathbf{P}_N B) + \beta G + (1-\beta)\mathbf{P}_N G\\ & -\beta(1-\beta)u\cdot\nabla(B-\mathbf{P}_N B) - (1-\beta)^2\mathbf{P}_N u\cdot\nabla(B-\mathbf{P}_N B)\\ & +\beta(1-\beta)B\cdot\nabla(u-\mathbf{P}_N u) + (1-\beta)^2\mathbf{P}_N B\cdot\nabla(u-\mathbf{P}_N u). \end{aligned}$$

By the definition of $\beta$ and the smoothness of $u, B, F$ and $G$, we can see that $F_N, G_N \in C^\infty([\frac{T}{3}, T]\times\mathcal{B})$. For $k \in \mathbb{N}$, since $v$ and $H$ belong to $C^0([\frac{T}{3}, T]; H^{k+1}(\mathcal{B}))$ from Theorem A.1, so does $u$ and $B$. Hence $(u - \mathbf{P}_N u, B - \mathbf{P}_N B)$ converges to zero in this space as $N$ goes to infinity. Note that $F$ and $G$ supported outside $\overline{\Omega'}$, so that

$$\|(F_N, G_N)|_\Omega\|_{L^1((0,T);H^k(\Omega'))} \to 0 \qquad \text{as } N\to\infty.$$

From (B.2),

$$u_N(T) = \mathbf{P}_N u(T) = \mathbf{P}_N(\theta v) \quad \text{and} \quad B_N(T) = \mathbf{P}_N B(T) = \mathbf{P}_N(\theta H).$$

By the definition of $\theta$ and $\mathcal{G} = [-4L, -3L]\times[-1,1]$,

$$\|(u_N(T), B_N(T)|_{\mathcal{G}}\|_{H^k(\mathcal{G})} \to 0 \qquad \text{as } N\to\infty.$$

For any $\delta > 0$, there exists a $N > 0$ such that

$$\|(F_N, G_N)|_\Omega\|_{L^1((0,T);H^k(\Omega))} + \|(u_N(T), B_N(T)|_{\mathcal{G}}\|_{H^k(\mathcal{G})} \le \delta.$$

We take $\xi = F_N|_{\mathcal{B}\backslash\Omega}, \eta = G_N|_{\mathcal{B}\backslash\Omega}, f = F_N|_\Omega, g = G_N|_\Omega$, $u_b = u_N(T,\cdot)$ and $B_b = u_N(T,\cdot)$, then (1.9) and (1.10) are satisfied. Moreover $u_b$ and $B_b$ are analytic in $x$, that is for any $\rho_b > 0$, there exits $C_b > 0$ such that (1.11) holds.

For any $i, j \in \mathbb{N}, i+j \leq 2$, since $\theta(x) \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}\|\partial_x^i \partial_y^j u_b\|_{L_x^1(L_y^2)(\mathcal{B})} &= \|\partial_x^i \partial_y^j \mathbf{P}_N(\theta v(T))\|_{L_x^1(L_y^2)(\mathcal{B})} \\ &\leq C\|\theta \partial_y^j v(T)\|_{L_x^1(L_y^2)(\mathcal{B})} \\ &\leq C\|v(T)\|_{H^2(\mathcal{B})} \\ &\leq CE_2(M),\end{aligned}$$

where $M = \|(u_0, B_0)\|_{L^2(\Omega)}$ and we used (A.2) for $k=2$. We can get a similar estimate for $\|B_b\|_{L_x^1(L_y^2)(\mathcal{B})}$, thus (1.12) holds true and we have finish the proof.

## Appendix C. Two extension lemmas

In this section, we will prove two extension lemmas, which extend data in $L^2_{\mathrm{div}}(\Omega)$ to $L^2_{\mathrm{div}}(\mathcal{B})$ and to $L^2_{\mathrm{div}}(\mathcal{O})$ respectively.

**Lemma C.1.** *(i) Assume that $(u_0, B_0) \in L^2_{\mathrm{div}}(\Omega)$, and there is an extension $(\tilde{u}_0, \tilde{B}_0) \in L^2_{\mathrm{div}}(\mathcal{B})$ such that*

$$(\tilde{u}_0, \tilde{B}_0)|_\Omega = (u_0, B_0), \tag{C.1}$$

*then*

$$\int_\Omega u_{0,1}(x,y)\mathrm{d}x\mathrm{d}y = \int_\Omega B_{0,1}(x,y)\mathrm{d}x\mathrm{d}y = 0. \tag{C.2}$$

*(ii) There exists a constant $c_0 > 0$, such that for any $(u_0, B_0) \in L^2_{\mathrm{div}}(\Omega)$ which satisfies (C.2), there is an extension $(\tilde{u}_0, \tilde{B}_0) \in L^2_{\mathrm{div}}(\mathcal{B})$ satisfies (C.1) and*

$$\tilde{u}_0 \text{ and } \tilde{B}_0 \text{ are both supported in } [-L, 2L] \times [-1, 1], \tag{C.3}$$

$$\|(\tilde{u}_0, \tilde{B}_0)\|_{L^2(\mathcal{B})} \leq c_0 \|(u_0, B_0)\|_{L^2(\Omega)}. \tag{C.4}$$

*(iii) In addition, if $(u_0, B_0) \in L^2_{\mathrm{div}}(\Omega) \cap C^\infty(\overline{\Omega})$ satisfies*

$$u_0 = 0 \quad B_{0,2} = 0 \quad \text{and} \quad \partial_y B_{0,1} = 0 \qquad \text{on } \partial\Omega \backslash \Gamma, \tag{C.5}$$

*then the extension we found in (ii) satisfies $(\tilde{u}_0, \tilde{B}_0) \in L^2_{\mathrm{div}}(\mathcal{B}) \cap C^\infty(\overline{\mathcal{B}})$ and*

$$\tilde{u}_0 = 0 \quad \tilde{B}_{0,2} = 0 \quad \text{and} \quad \partial_y \tilde{B}_{0,1} = 0 \qquad \text{on } \partial\mathcal{B}. \tag{C.6}$$

*Proof.* **For (i)**, if $(\tilde{u}_0, \tilde{B}_0) \in L^2_{\mathrm{div}}(\mathcal{B})$ is an extension of $(u_0, B_0) \in L^2_{\mathrm{div}}(\Omega)$. For all $x \in \mathbb{R}$, by using $\mathrm{div}\, \tilde{u}_0 = 0$ and $\tilde{u}_0|_{\partial\mathcal{B}} = 0$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^1 \tilde{u}_{0,1}(x,y)\mathrm{d}y = -\int_{-1}^1 \partial_y \tilde{u}_{0,2}(x,y)\mathrm{d}y = 0.$$

So that $\int_{-1}^1 \tilde{u}_{0,1}(x,y)\mathrm{d}y$ is independent of $x$. By Cauchy's inequality,

$$\int_\mathbb{R} \left| \int_{-1}^1 \tilde{u}_{0,1}(x,y)\mathrm{d}y \right|^2 \mathrm{d}x \leq 2 \int_\mathcal{B} |\tilde{u}_{0,1}(x,y)|^2 \mathrm{d}x\mathrm{d}y < \infty,$$

which implies $\int_{-1}^{1} \tilde{u}_{0,1}(x,y)\mathrm{d}y \equiv 0$ for all $x \in \mathbb{R}$. Therefore $\int_\Omega u_{0,1}(x,y)\mathrm{d}x\mathrm{d}y = 0$. Similarly we have $\int_\Omega B_{0,1}(x,y)\mathrm{d}x\mathrm{d}y = 0$.

**For (ii)**, firstly, we prove that smooth functions are dense in $L^2_{\mathrm{div}}(\Omega)$, thus it is suffice to prove (ii) when $(u_0, B_0) \in L^2_{\mathrm{div}}(\Omega) \cap C^\infty(\overline{\Omega})$. Indeed, assume that $u \in L^2_{\mathrm{div}}(\Omega)$, for $0 \le h \le 1$, we take

$$u_h(x,y) = \begin{cases} (\frac{1}{h}u_1(x,\frac{y}{h}), u_2(x,\frac{y}{h})) & \text{for } |y| < h, \\ (0,0) & \text{for } 1 \le |y| < 1. \end{cases}$$

Then its easy to check that $u_h \in L^2_{\mathrm{div}}(\Omega)$ and $\|u_h - u\|_{L^2(\Omega)} \to 0$ when $h \to 1^-$. We take a smooth function $j(x,y) \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$ which is supported un the unit ball of $\mathbb{R}^2$ and satisfies $\int_{\mathbb{R}^2} j(x,y)\mathrm{d}x\mathrm{d}y = 1$. Let $\varepsilon > 0$, then $j_\varepsilon(x,y) = \frac{1}{\varepsilon^2} j(\frac{x}{\varepsilon},\frac{y}{\varepsilon})$ be a smooth modifier. For $0 < h < 1$, $0 < \varepsilon < 1-h$, we take

$$u_{h,\varepsilon}(x,y) = (u_h * j_\varepsilon)|_\Omega(x,y).$$

Then $u_{h,\varepsilon} \in L^2_{\mathrm{div}}(\Omega) \cap C^\infty(\overline{\Omega})$ and satisfies $\|u_{h,\varepsilon} - u\|_{L^2(\Omega)} \to 0$ when $\varepsilon < 1-h, h \to 1^-, \varepsilon \to 0^+$.

Assume $(u_0, B_0) \in L^2_{\mathrm{div}}(\Omega) \cap C^\infty(\overline{\Omega})$, we define the stream functions by

$$\Phi(x,y) = -\int_{-1}^{y} u_{0,1}(x,s)\mathrm{d}s \quad \text{and} \quad \Psi(x,y) = -\int_{-1}^{y} B_{0,1}(x,s)\mathrm{d}s. \tag{C.7}$$

Then $(\Phi,\Psi) \in H^1(\Omega) \cap C^\infty(\overline{\Omega})$ and $(u_0, B_0) = (\nabla^\perp\Phi, \nabla^\perp\Psi)$, where $\nabla^\perp = (\partial_y, -\partial_x)$. By definition,

$$\|(\Phi,\Psi)\|_{H^1(\Omega)} \le 3\|(u_0,B_0)\|_{L^2(\Omega)}.$$

Since $u_0$ and $B_0$ are divergence free and are tangential to the upper and lower boundary of $\Omega$, for $x \in [0,L]$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi(x,1) = -\int_{-1}^{1} \partial_y u_{0,2}(x,y)\mathrm{d}y = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}\Psi(x,1) = -\int_{-1}^{1} \partial_y B_{0,2}(x,y)\mathrm{d}y = 0.$$

So that $\Phi(x,1)$ and $\Psi(x,1)$ are independent of $x$. Combine (C.2), we get

$$\Phi(x,-1) = \Phi(x,1) = \Psi(x,-1) = \Psi(x,1) = 0 \qquad \forall x \in [0,L]. \tag{C.8}$$

We define the extension $(\overline{\Phi},\overline{\Psi})$ to domain $\Omega' = (-L,2L)\times(-1,1)$ by

$$(\overline{\Phi}(x,y),\overline{\Psi}(x,y)) := \begin{cases} (\Phi(-x,y),\Psi(-x,y)) & \text{for } x \in [-L,0], \\ (\Phi(x,y),\Psi(x,y)) & \text{for } x \in [0,L], \\ (\Phi(2L-x,y),\Psi(2L-x,y)) & \text{for } x \in [L,2L]. \end{cases} \tag{C.9}$$

By definition, $(\overline{\Phi},\overline{\Psi}) \in H^1(\Omega') \cap C^\infty(\overline{\Omega'})$ and satisfies

$$(\overline{\Phi},\overline{\Psi})|_\Omega = (\Phi,\Psi) \tag{C.10}$$

$$\overline{\Phi}(x,-1) = \overline{\Phi}(x,1) = \overline{\Psi}(x,-1) = \overline{\Psi}(x,1) = 0 \quad \forall\, x \in [-L,2L], \tag{C.11}$$

$$\|(\overline{\Phi},\overline{\Psi})\|_{H^1(\Omega')} \le 3\|(\Phi,\Psi)\|_{H^1(\Omega)} \le 9\|(u_0,B_0)\|_{L^2(\Omega)}. \tag{C.12}$$

We take a cut-off function $\chi_c(x) \in C^\infty(\mathbb{R})$ with $\chi_c(x) = 1$ when $0 \le x \le L$ and $\chi_c(x) = 0$ when $x \le -\frac{L}{2}$ or $x \ge \frac{3}{2}L$. Let

$$\tilde{\Phi}(x,y) := \chi_c(x)\overline{\Phi}(x,y), \qquad \tilde{u} := \nabla^\perp \tilde{\Phi}, \tag{C.13}$$

$$\tilde{\Psi}(x,y) := \chi_c(x)\overline{\Psi}(x,y), \qquad \tilde{B} := \nabla^\perp \tilde{\Psi}. \tag{C.14}$$

Let's check that $(\tilde{u}, \tilde{B})$ satisfies all the requirements. Obviously, $\tilde{u}_0, \tilde{B}_0$ are divergence free and (C.1), (C.3) hold true, since $\chi_c(x) = 1$ in $\Omega$, $(\overline{\Phi}, \overline{\Psi})$ is an extension of $(\Phi, \Psi)$ and $(u_0, B_0) = \nabla^\perp(\Phi, \Psi)$. Thanks to (C.11) and $\chi_c(x) = 0$ outside $[-\frac{L}{2}, \frac{3L}{2}]$, $\tilde{u}_0, \tilde{B}_0$ are tangent to $\mathcal{B}$. By (C.12) and the definition of $(\tilde{u}_0, \tilde{B}_0)$, there is a constant $c_0 > 0$ such that

$$\|(\tilde{u}_0, \tilde{B}_0)\|_{L^2(\mathcal{B})} \le c_0 \|(u_0, B_0)\|_{L^2(\Omega)}.$$

Therefore $(\tilde{u}_0, \tilde{B}_0) \in L^2_{\text{div}}(\mathcal{B})$.

**For (iii)**, assume that $(u_0, B_0)$ satisfies the boundary conditions $u_0 = 0, B_{0,2} = 0$, and $\partial_y B_{0,1} = 0$ on $\partial\Omega \backslash \Gamma$, then

$$\nabla^\perp \Phi = 0, \quad \partial_x \Psi = 0 \quad \text{and} \quad \partial_y^2 \Psi = 0 \qquad \text{on } \partial\Omega\backslash\Gamma.$$

By definition (C.9),

$$\nabla^\perp \overline{\Phi} = 0, \quad \partial_x \overline{\Psi} = 0 \quad \text{and} \quad \partial_y^2 \overline{\Psi} = 0 \qquad \text{on } \partial\mathcal{B}.$$

By combining with (C.11) and the definition (C.13), (C.14), we get

$$\nabla^\perp \tilde{\Phi} = 0, \quad \partial_x \tilde{\Psi} = 0 \quad \text{and} \quad \partial_y^2 \tilde{\Psi} = 0 \qquad \text{on } \partial\mathcal{B},$$

which implies

$$\tilde{u}_0 = 0 \quad \tilde{B}_{0,2} = 0 \quad \text{and} \quad \partial_y B_{0,2} = 0 \qquad \text{on } \partial\mathcal{B},$$

which finishes the proof of Lemma C.1. □

*Proof of Lemma 1.8.* We define $(\Phi, \Psi)$ the stream functions of $(u_0, B_0)$ as in (C.7) and extend $\Phi$ to $\Omega'$ as in (C.9). We take the same cut-off function $\chi_c \in C_0^\infty(\mathbb{R})$ and define

$$\tilde{\Phi}(x,y) := \begin{cases} \chi_c(x)\overline{\Phi}(x,y) & \text{for } (x,y) \in \Omega', \\ 0 & \text{for } (x,y) \in \mathcal{O}\backslash\Omega' \end{cases} \quad \text{and} \quad \tilde{u}_0 = \nabla^\perp \tilde{\Phi}.$$

Its easy to check that $\tilde{u}_0 \in L^2_{\text{div}}(\mathcal{O}) \cap C^\infty(\overline{\mathcal{O}})$ meets all the requirements. That is $\tilde{u}_0|_\Omega = u_0, \tilde{u}_0$ vanishes on the boundary $\partial\mathcal{O}$ and there is a constant $c_1 > 0$ such that $\|\tilde{u}_0\|_{L^2(\mathcal{O})} \le c_1 \|u_0\|_{L^2(\Omega)}$.

We consider the following Laplace equation in $\Omega'' = \mathcal{O}\backslash\overline{\Omega}$,

$$\begin{cases} \Delta\Psi_1 = 0 & \text{in } \Omega'', \\ \Psi_1 = \Psi_{1b} & \text{on } \partial\Omega'', \end{cases}$$

where

$$\Psi_{1b} = \begin{cases} \Psi_b & \text{on } \Gamma, \\ 0 & \text{on } \partial\Omega''\backslash\Gamma, \end{cases} \qquad \text{and } \Psi_b = \gamma(\Psi)|_\Gamma \text{ is restriction of trace } \gamma(\Psi) \text{ on } \Gamma.$$

By standard elliptic theory (cf. Thm. III.4.1 of [4]), trace theorem and note that $\Psi_b = 0$ on $\partial\Omega\backslash\Gamma$, we get that

$$\|\Psi_1\|_{H^1(\Omega'')} \leq C\|\Psi_{1b}\|_{H^{\frac{1}{2}}(\partial\Omega'')} \leq C\|\Psi_b\|_{H^{\frac{1}{2}}(\Gamma)} \leq C\|\Psi\|_{H^1(\Omega)}. \tag{C.15}$$

We define

$$\tilde{\Psi} = \begin{cases} \Psi & \text{in } \overline{\Omega}, \\ \Psi_1 & \text{in } \Omega'', \end{cases} \qquad \text{and} \qquad \tilde{B}_0 = \nabla^\perp \tilde{\Psi}.$$

Obviously $\tilde{B}_0$ is an divergence free extension of $B_0$ and satisfies (1.25) due to (C.15). Let's check that $\tilde{B}_0$ satisfies condition (1.26)–(1.27). Let $\mathbf{n} = {}^t(n_1, n_2)$, then $\mathbf{n}^\perp = {}^t(n_2, -n_1)$ is a tangential vector on the boundary $\partial\mathcal{O}$. By (C.8) and the definition of $\Psi_1$, $\tilde{\Psi}$ vanishes on the boundary $\partial\mathcal{O}$, therefore

$$\tilde{B}_0 \cdot \mathbf{n} = \nabla^\perp \tilde{\Psi} \cdot \mathbf{n} = -\mathbf{n}^\perp \cdot \nabla\tilde{\Psi} = 0 \quad \text{on } \partial\mathcal{O}.$$

Thanks to (1.23), $\operatorname{curl} \tilde{B}_0 = \operatorname{curl} B_0 = \partial_x B_{0,2} - \partial_y B_{0,1} = 0$ on $\partial\Omega\backslash\Gamma$. On the boundary $\partial\Omega''\backslash\Gamma$, $\operatorname{curl}\tilde{B}_0 = -\Delta\Psi_1 = 0$, hence $\operatorname{curl}\tilde{B}_0 = 0$ holds on $\partial\mathcal{O}$. For a basis $\mathrm{g}$ of $\mathcal{X}$, $\operatorname{curl}\mathrm{g} = 0$, note that $\tilde{\Psi}$ vanishes on $\partial\mathcal{O}$, so we get

$$\int_\mathcal{O} \tilde{B}_0 \cdot \mathrm{g} = \int_\mathcal{O} \nabla^\perp\tilde{\Psi} \cdot \mathrm{g} = \int_{\partial\mathcal{O}} \tilde{\Psi}\tau \cdot \mathrm{g} - \int_\mathcal{O} \tilde{\Psi}\operatorname{curl}\mathrm{g} = 0.$$

In conclusion, we have found an extension $(\tilde{\Phi}, \tilde{\Psi}) \in L^2_{\text{div}}(\mathcal{O})$ satisfies (1.24)–(1.27).

□

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