

A STOCHASTIC MAXIMUM PRINCIPLE FOR FORWARD–BACKWARD STOCHASTIC CONTROL SYSTEMS WITH QUADRATIC GENERATORS AND SAMPLE-WISE CONSTRAINTS

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Abstract. This paper examines the stochastic maximum principle (SMP) for a forward–backward stochastic control system where the backward state equation is characterized by the backward stochastic differential equation (BSDE) with quadratic growth and the forward state at the terminal time is constrained in a convex set with probability one. With the help of the theory of BSDEs with quadratic growth and the bounded mean oscillation (BMO) martingales, we employ the terminal perturbation approach and Ekeland’s variational principle to obtain a dynamic stochastic maximum principle. The main result has a wide range of applications in mathematical finance and we investigate a robust recursive utility maximization problem with bankruptcy prohibition as an example.

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1. INTRODUCTION

The class of backward stochastic differential equations (BSDEs), with generators having a quadratic growth in the state variable z , has attracted much attention in the past two decades. Besides the increasingly developed and enriched existence and uniqueness theory [1–7], BSDEs with quadratic growth have found applications in stochastic control and mathematical finance, say, stochastic linear-quadratic control with random coefficients [8], utility maximization problems [9, 10] *etc.*

In this paper, motivated especially by their applications in the risk-sensitive optimal control problems [11–13] and the robust portfolio-consumption optimization model under model uncertainty [14] together with the related recursive utility maximization problems [15] with the bankruptcy prohibition, we are encountering with the following stochastic recursive optimal control problems involving BSDEs with quadratic growth and state constraints with sample-wise type (a sample-wise constraint requires that the state at certain time or at all times be in a prescribed set with probability 1). Denote by $\mathcal{U}[0, T]$ the set of all the admissible controls and the cost functional is defined by the following mixed initial-terminal type (see [16])

$$J(u(\cdot)) := \mathbb{E}_{\mathbb{P}} [h(X_T^u, Y_0^u)], \quad (1.1)$$

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where h is any given smooth function, and $X^u(\cdot)$, $Y^u(\cdot)$ are the solutions to the controlled forward–backward stochastic differential equation (FBSDE, see [17–19])

$$\begin{cases} dX_t^u = b(t, X_t^u, u_t)dt + \sigma(t, X_t^u, u_t)dW_t, \\ dY_t^u = -f(t, X_t^u, Y_t^u, Z_t^u, u_t)dt + (Z_t^u)^\top dW_t, \quad t \in [0, T], \\ X_0^u = x_0, \quad Y_T^u = \Phi(X_T^u), \end{cases} \quad (1.2)$$

where W is a standard Brownian motion, the coefficients b , σ , f , Φ are deterministic, measurable functions in suitable sizes, and f is quadratic growth in z . The objective is to find $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ (if it ever exists) such that

$$J(\bar{u}(\cdot)) := \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot)), \quad (1.3)$$

and the terminal state X_T^u of the stochastic differential equation (SDE) in (1.2) is required to take values in a given convex set $K \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}_+$) with probability one. On the one hand, when $K = \mathbb{R}^n$, $h(x, y) = y$, $f(t, x, y, z, u) = \frac{\gamma}{2}|z|^2 + F(t, x, u)$ with some $\gamma > 0$ and measurable function F , if the coefficients admit enough integrability then (1.2)–(1.3) is closely related to the classical risk-sensitive control problems [11, 12] by using exponential transformation and Itô’s formula. After that Moon [13] studied the generalized case if g depends on (y, z) through the dynamic programming approach. On the other hand, under the setting of Brownian filtration and for any given $u(\cdot) \in \mathcal{U}[0, T]$, when $n = 1$, $K = [0, +\infty)$, $h(x, y) = -y$, $f(t, x, y, z, u) = U(u) - \beta y - \frac{1}{2\theta}|z|^2$, and if $\mathcal{U}[0, T]$ represents the set of consumption–portfolio strategies $u(\cdot)$ feasible for the initial wealth $x_0 \geq 0$, it follows from the main result in [14] that $-J(u(\cdot))$ is optimal for the minimization part of the sup–inf problem proposed in [15] thanks to the method of dual representation (see also [20]), where θ is the risk-averse parameter and U is the utility function. Furthermore, this means that the objective (1.3) is equivalent to the maximization part of the sup–inf problem in [15].

The existence of constraints with sample-wise type as above is more a rule than an exception in reality, for example, in the continuous-time mean-variance portfolio selection problem [21] and the recursive utility maximization problems [22] with bankruptcy prohibition. Another important example is the study of the Neyman–Pearson lemma for hypothesis tests under a class of nonlinear probability measures – g -probabilities [23], where the setting $K = [0, 1]$ plays a role as a criterion to exclude the tests that make the g -probability of Type I error beyond the given acceptable significance level.

Based on the above motivation, we are aimed at deducing the necessary condition of the optimality – stochastic maximum principle (SMP) for problem (1.2)–(1.3). Since Peng [24] obtained the general SMP for the classical stochastic control systems, researchers have made progress in the SMP for coupled forward–backward stochastic control systems (see [16, 25–27] and references therein) driven by FBSDEs when $K = \mathbb{R}^n$. For the case $K \subsetneq \mathbb{R}^n$, (1.2)–(1.3) is well studied [28, 29] when f is globally Lipschitz continuous in (x, y, z) and it is generalized to the fully coupled case [30] and to the mean-field case [31]. In the existing literature, there are two main approaches to getting the SMP, one is based on the pure BSDE approach [22] and the other is based on the Ekeland variational principle [28–30]. Particularly, adopting the BSDE method, the authors in [15] establish a comparison theorem for specific BSDEs with quadratic growth and derive a dynamic SMP in the semimartingale context. However, the comparison theorem may not hold since we do not require f have special structures or convexity/concavity in z , and therefore we will resort to the Ekeland variational principle to achieve this goal under our framework.

We encountered three difficulties in deducing the SMP for (1.2)–(1.3). The first one is that the BSDE in (1.2) is no longer Lipschitz but quadratic growth in z , which leads to the derivative f_z being unbounded. The unboundedness of f_z brings much trouble in obtaining the following estimate, for example, when f depends only on z ,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T \left| \int_0^1 f_z(\bar{Z}_t + \lambda(Z_t^\varepsilon - \bar{Z}_t))d\lambda - f_z(\bar{Z}_t) \right|^2 dt \right)^p \right] = 0 \quad (1.4)$$

for some $p > 1$, where $\bar{Z}(\cdot)$ represents the optimal trajectory and $Z^\varepsilon(\cdot)$ represents the state trajectory after perturbation, because one can deduce (1.4) when f is Lipschitz in z by using the dominated convergence

theorem. The second one is that when the family of approximate controls produced by Ekeland's variational principle converges to the optimal one, in which appropriate space can we obtain the convergence of the solutions of the approximate variational equations to the one solving the original variational equation? In the classical case, such an issue can be solved by applying the continuous dependence of the solutions to the Lipschitz BSDEs on the parameters, but it entails estimating the difference between the solutions from two different linear BSDEs with unbounded coefficients under our framework. Furthermore, to this end, we first need to ensure that the approximate state trajectories converge to the optimal one, which essentially involves the convergence of solutions of a family of quadratic BSDEs. The third one is that the adjoint equation is a linear SDE with unbounded coefficients due to the unboundedness of f_z . When we deduce the SMP, the solution to it will appear as a component of the integrands of stochastic integrals with respect to the Brownian motion (see (3.24)–(3.25)). Such stochastic integrals are only local martingales whose mathematical expectation at the terminal time T may not exist. So we must check all these stochastic integrals are true martingales on the time interval $[0, T]$ before taking the expectation.

To overcome the aforementioned difficulties, we deduce the desired convergence (1.4) by applying the energy-type inequality of the bounded mean oscillation (BMO) martingales together with the generalized dominated convergence theorem. Using the estimate in [32] for the linear BSDEs with stochastic Lipschitz coefficients involving BMO martingales, the convergence of both the approximate state trajectories and the approximate variational equations are attained, and the former convergence is stronger than the latter one. To tackle the last difficulty, we note that the solution of the adjoint equation is the Doléans–Dade exponential of a certain BMO martingale, which satisfies the reverse Hölder inequality (R_p) as long as $p \in [1, \bar{p})$ for some $\bar{p} > 1$ (see [33], Chap. 3, Defi. 3.1). On the other hand, the most complicated term to estimate, in those integrands of stochastic integrals, is a product including the solutions of the variational equation and the adjoint equation. Based on this observation, we choose a proper $p \in (1, \bar{p})$ together with its conjugate p^* , such that the (R_p) condition holds and the solution of the variational equation admits a p^* -moment. Then we can apply Hölder's inequality with the couple (p, p^*) to the square root of the quadratic variation of that stochastic integral to verify it is a true martingale.

The rest of the paper is organized as follows. In Section 2, preliminaries and the formulation of our problem are given. We use a pure backward formulation of (1.2) in which the terminal state X_T^u is regarded as the control variable. Unlike the formulation in [23, 28], such a reformulated set of admissible controls no longer includes all square-integrable random variables but higher-order ones because of the quadratic growth in z . In , employing the analysis of BMO martingales, we guarantee the well-posedness of both the variational equation and the adjoint equation. Then, applying Ekeland's variational principle, we obtain a dynamic SMP that characterizes the optimal terminal state. In Section 4, to illustrate the established SMP, we study its applications to a robust recursive utility maximization problem with bankruptcy prohibition.

2. PRELIMINARIES AND PROBLEM FORMULATION

Let $T \in (0, +\infty)$, $n, d \in \mathbb{N}_+$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a standard d -dimensional Brownian motion $W = (W_t^1, W_t^2, \dots, W_t^d)_{t \in [0, T]}^\top$ is defined. $\mathbb{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$ is the \mathbb{P} -augmentation of the natural filtration of W . Denote by \mathbb{R}^n the n -dimensional real Euclidean space and $\mathbb{R}^{n \times d}$ the set of $n \times d$ real matrices. The scalar product (resp. norm) of any two $n \times d$ matrices A, B is denoted by $\langle A, B \rangle = \text{tr}\{AB^\top\}$ (resp. $|A| = \sqrt{\text{tr}\{AA^\top\}}$), where the superscript \top denotes the transpose of vectors or matrices. Particularly in this paper, for any measurable and bounded function f defined on (or on any subset of) any Euclidean space, we simply denote the bound of f by $\sup|f|$.

For any given $p, q \geq 1$, we introduce the following spaces and notation.

- $L_{\mathcal{F}_T}^p(\Omega; \mathbb{R}^n)$: the space of \mathcal{F}_T -measurable, \mathbb{R}^n -valued random variables ξ such that

$$\|\xi\|_{L^p} := (\mathbb{E}_{\mathbb{P}} [|\xi|^p])^{\frac{1}{p}} < +\infty.$$

- $L_{\mathcal{F}_T}^\infty(\Omega; \mathbb{R}^n)$: the space of \mathcal{F}_T -measurable, \mathbb{R}^n -valued random variables ξ such that

$$\|\xi\|_{L^\infty} := \mathbb{P} - \text{ess sup}_{\omega \in \Omega} |\xi(\omega)| < +\infty.$$

- $\mathcal{M}_{\mathcal{F}}^{p,q}([0, T]; \mathbb{R}^n)$: the space of \mathbb{F} -adapted, \mathbb{R}^n -valued processes $\varphi(\cdot)$ on $[0, T]$ such that

$$\|\varphi(\cdot)\|_{p,q} := \left\| \left(\int_0^T |\varphi_t|^p dt \right)^{\frac{1}{p}} \right\|_{L^q} < +\infty.$$

In particular, we denote by $\mathcal{M}_{\mathcal{F}}^p([0, T]; \mathbb{R}^n)$ the above space when $p = q$.

- $L_{\mathcal{F}}^\infty([0, T]; \mathbb{R}^n)$: the space of \mathbb{F} -adapted, \mathbb{R}^n -valued processes $\varphi(\cdot)$ on $[0, T]$ such that

$$\|\varphi(\cdot)\|_\infty := \lambda \otimes \mathbb{P} - \text{ess sup}_{(t,\omega) \in [0,T] \times \Omega} |\varphi_t(\omega)| < +\infty,$$

where λ denotes the Lebesgue measure on $[0, T]$.

- $\mathcal{S}_{\mathcal{F}}^p([0, T]; \mathbb{R}^n)$: the space of continuous processes $\varphi(\cdot) \in \mathcal{M}_{\mathcal{F}}^p([0, T]; \mathbb{R}^n)$ such that

$$\|\varphi\|_{\mathcal{S}^p} := \left\| \sup_{t \in [0, T]} |\varphi_t| \right\|_{L^p} < +\infty.$$

- $\mathcal{S}_{\mathcal{F}}^\infty([0, T]; \mathbb{R}^n)$: the space of continuous processes $\varphi(\cdot) \in L_{\mathcal{F}}^\infty([0, T]; \mathbb{R}^n)$ such that

$$\|\varphi\|_{\mathcal{S}^\infty} := \left\| \sup_{t \in [0, T]} |\varphi_t| \right\|_{L^\infty} < +\infty.$$

- BMO_p : the space of real-valued, continuous \mathbb{F} -martingales M with $M_0 = 0$ such that

$$\|M\|_{\text{BMO}_p} := \sup_{\tau} \left\| \left(\mathbb{E}_{\mathbb{P}} [|M_T - M_\tau|^p | \mathcal{F}_\tau] \right)^{\frac{1}{p}} \right\|_\infty < +\infty, \quad p \in [1, +\infty), \quad (2.1)$$

where the supremum is taken over all stopping times $\tau \in [0, T]$. By Corollary 2.1 in [33], M is a BMO_p martingale if and only if it is a BMO_q martingale for every $q \geq 1$. Therefore, it is simple to write BMO to represent BMO_p .

- $\mathcal{E}(M)$: the Doléans–Dade exponential of a continuous \mathbb{F} -martingales M , that is,

$$\mathcal{E}(M_t) = \exp \left\{ M_t - \frac{1}{2} \langle M \rangle_t \right\}, \quad t \in [0, T].$$

- p_M^* : the conjugate exponent of p_M , i.e. $(p_M)^{-1} + (p_M^*)^{-1} = 1$, where $M \in \text{BMO}$, p_M is the positive constant defined by the following function:

$$\Psi(x) = \sqrt{1 + \frac{1}{x^2} \ln \frac{2x-1}{2(x-1)}} - 1, \quad x \in (1, +\infty) \quad (2.2)$$

with $\Psi(p_M) = \|M\|_{\text{BMO}_2}$. By Theorem 3.1 in [33], if $p \in (1, p_M)$, then, for any stopping time $\tau \in [0, T]$,

$$\mathbb{E}_{\mathbb{P}}[(\mathcal{E}(M_T))^p / (\mathcal{E}(M_\tau))^p \mid \mathcal{F}_\tau] \leq C_0, \quad \mathbb{P}\text{-a.s.}, \quad (2.3)$$

where C_0 is positive constant depending only on p and BMO_2 , and (2.3) is called the reverse Hölder inequality.

- $H \cdot W$: H is an \mathbb{F} -adapted process and $H \cdot W$ is the stochastic integral of H with respect to W . If $H \cdot W \in \text{BMO}$, then we write simply p_H for $p_{H \cdot W}$ and p_H^* for $p_{H \cdot W}^*$ without ambiguity.

To curb the proliferation of indices and avoid the complexity, we unify the definitions of (partial) derivatives of functions of several variables by adopting the following notation ($\forall m \in \mathbb{N}_+$, each element of $\mathbb{R}^m = \mathbb{R}^{m \times 1}$ is understood as a column vector throughout this paper): Let $d_1, d_2, D \in \mathbb{N}_+$.

- (i) For any function $\mathbb{R}^{d_1} \ni x \mapsto \varphi(x) \in \mathbb{R}$,

$$\varphi_x(\cdot) := (\varphi_{x_1}(\cdot), \dots, \varphi_{x_{d_1}}(\cdot)) \in \mathbb{R}^{1 \times d_1} \text{ (row vector)}.$$

- (ii) For any function $\mathbb{R}^{d_1} \ni x \mapsto \varphi(x) = (\varphi^1(\cdot), \dots, \varphi^D(\cdot))^\top \in \mathbb{R}^D$,

$$\varphi_x(\cdot) := \begin{pmatrix} \varphi_x^1(\cdot) \\ \vdots \\ \varphi_x^D(\cdot) \end{pmatrix} = \begin{pmatrix} \varphi_{x_1}^1(\cdot) & \cdots & \varphi_{x_{d_1}}^1(\cdot) \\ \vdots & \ddots & \vdots \\ \varphi_{x_1}^D(\cdot) & \cdots & \varphi_{x_{d_1}}^D(\cdot) \end{pmatrix} \in \mathbb{R}^{D \times d_1} = \mathcal{L}(\mathbb{R}^{d_1}; \mathbb{R}^D).$$

The operation $\mathbb{R}^{d_1} \ni y \mapsto \varphi_x(\cdot)y \in \mathbb{R}^D$ is given by

$$\varphi_x(\cdot)y = \begin{pmatrix} \langle \varphi_x^1(\cdot)^\top, y \rangle_{\mathbb{R}^{d_1}} \\ \vdots \\ \langle \varphi_x^D(\cdot)^\top, y \rangle_{\mathbb{R}^{d_1}} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{d_1} \varphi_{x_i}^1(\cdot), y_i \\ \vdots \\ \sum_{i=1}^{d_1} \varphi_{x_i}^D(\cdot), y_i \end{pmatrix} \in \mathbb{R}^D.$$

Note that the adjoint $\varphi_x(\cdot)^\top \in \mathbb{R}^{d_1 \times D} = \mathcal{L}(\mathbb{R}^D; \mathbb{R}^{d_1})$ of the operator $\varphi_x(\cdot)$ is given by

$$\varphi_x(\cdot)^\top = (\varphi_x^1(\cdot)^\top, \dots, \varphi_x^D(\cdot)^\top) \in \mathbb{R}^{d_1 \times D} = \mathcal{L}(\mathbb{R}^D; \mathbb{R}^{d_1}).$$

The operation $\mathbb{R}^D \ni z \mapsto \varphi_x(\cdot)^\top z \in \mathbb{R}^{d_1}$ is given by

$$\varphi_x(\cdot)^\top z = \sum_{i=1}^D \varphi_x^i(\cdot)^\top z_i \in \mathbb{R}^{d_1}.$$

- (iii) For any function $\mathbb{R}^{d_1 \times d_2} \ni x \mapsto \varphi(x) \in \mathbb{R}$,

$$\varphi_x(\cdot) := \begin{pmatrix} \varphi_{x_{11}}(\cdot) & \cdots & \varphi_{x_{d_1 1}}(\cdot) \\ \vdots & \ddots & \vdots \\ \varphi_{x_{1 d_2}}(\cdot) & \cdots & \varphi_{x_{d_1 d_2}}(\cdot) \end{pmatrix} \in \mathbb{R}^{d_2 \times d_1}.$$

(This is consistent to the case of $d_2 = 1$.)

(iv) For any function $\mathbb{R}^{d_1 \times d_2} \ni x \mapsto \varphi(x) = (\varphi^1(x), \dots, \varphi^D(x))^\top \in \mathbb{R}^D$,

$$\varphi_x(\cdot) := \begin{pmatrix} \varphi_x^1(\cdot) \\ \vdots \\ \varphi_x^D(\cdot) \end{pmatrix} \in \mathbb{R}^{D \times (d_2 \times d_1)} = \mathcal{L}(\mathbb{R}^{d_1 \times d_2}; \mathbb{R}^D).$$

The operation $\mathbb{R}^{d_1 \times d_2} \ni y \mapsto \varphi_x(\cdot)y \in \mathbb{R}^D$ is given by

$$\varphi_x(\cdot)y = \begin{pmatrix} \langle \varphi_x^1(\cdot)^\top, y \rangle_{\mathbb{R}^{d_1 \times d_2}} \\ \vdots \\ \langle \varphi_x^D(\cdot)^\top, y \rangle_{\mathbb{R}^{d_1 \times d_2}} \end{pmatrix} \in \mathbb{R}^D.$$

Note that the adjoint $\varphi_x(\cdot)^\top \in \mathbb{R}^{(d_1 \times d_2) \times D} = \mathcal{L}(\mathbb{R}^D; \mathbb{R}^{d_1 \times d_2})$ of the operator $\varphi_x(\cdot)$ is given by

$$\varphi_x(\cdot)^\top = (\varphi_x^1(\cdot)^\top, \dots, \varphi_x^D(\cdot)^\top) \in \mathbb{R}^{(d_1 \times d_2) \times D} = \mathcal{L}(\mathbb{R}^D; \mathbb{R}^{d_1 \times d_2}).$$

The operation $\mathbb{R}^D \ni z \mapsto \varphi_x(\cdot)^\top z \in \mathbb{R}^{d_1 \times d_2}$ is given by

$$\varphi_x(\cdot)^\top z = \sum_{i=1}^D \varphi_x^i(\cdot)^\top z_i \in \mathbb{R}^{d_1 \times d_2}.$$

(This is consistent to the case of $d_2 = 1$.)

2.1. Classical formulation

Let $\bar{p}^* > 1$ be a number which will be determined later. Consider the following forward–backward stochastic control system: over the set of admissible controls

$$\mathcal{U}[0, T] := \{u(\cdot) \mid u(\cdot) \in \mathcal{M}_{\mathcal{F}}^{2,4\bar{p}^*}([0, T]; \mathbb{R}^{n \times d})\},$$

minimizing the cost functional

$$J(u(\cdot)) := \mathbb{E}_{\mathbb{P}} [h(X_T^u, Y_0^u)] \tag{2.4}$$

subject to the controlled FBSDE

$$\begin{cases} dX_t^u = b(t, X_t^u, u_t)dt + \sigma(t, X_t^u, u_t)dW_t, \\ dY_t^u = -f(t, X_t^u, Y_t^u, Z_t^u, u_t)dt + (Z_t^u)^\top dW_t, \\ X_0^u = x_0, Y_T^u = \Phi(X_T^u), \end{cases} \tag{2.5}$$

and an additional convex constraint

$$X_T \in K, \quad \mathbb{P}\text{-a.s.} \tag{2.6}$$

with K being a nonempty subset of \mathbb{R}^n , where $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} b &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \mapsto \mathbb{R}^n, & \sigma &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \mapsto \mathbb{R}^{n \times d}, \\ f &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{n \times d} \mapsto \mathbb{R}, & \Phi &: \mathbb{R}^n \mapsto \mathbb{R}, \\ h &: \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}, \end{aligned}$$

are deterministic, measurable functions. Throughout the paper we make the following assumptions.

Assumption 2.1. Let $L > 0$ be given.

- (i) b, σ, f, h, Φ are continuous in their arguments; Φ is continuously differentiable; b, σ are continuously differentiable in (x, u) ; f is continuously differentiable in (x, y, z, u) ; h is continuously differentiable in (x, y) .
- (ii) $\Phi, \Phi_x, b_x, \sigma_x, b_u, \sigma_u, f_y$ are bounded; h_x, h_y are bounded by $L(1 + |x| + |y|)$.
- (iii)

$$\begin{aligned} |f(t, x, 0, 0, u)| &\leq L, & |f_x(t, x, y, z, u)| &\leq L(1 + |y| + |z|^2), \\ |f_z(t, x, y, z, u)| &\leq L(1 + |z|), & |f_u(t, x, y, z, u)| &\leq L(1 + |y| + |z|). \end{aligned}$$

Assumption 2.2. The set K is closed and convex.

Let $A = e^{6L \sup|\Phi| + LT} [\sup|\Phi| + LT + 2(L^{-2} + T)]$ and \bar{p} be the constant such that

$$\Psi(\bar{p}) = \sqrt{3L^2(T + 2A)}, \quad (2.7)$$

where the function $\Psi(\cdot)$ is defined by (2.2). We assign the value $\bar{p}(\bar{p} - 1)^{-1}$ to \bar{p}^* . Obviously, \bar{p}^* is the conjugate of \bar{p} .

2.2. Backward formulation

In this subsection, we give an equivalent backward formulation of the above stochastic optimal control problem (2.4)–(2.5). To do so we need an additional assumption:

Assumption 2.3. There exists $\alpha > 0$ such that

$$|\sigma(t, x, u_1) - \sigma(t, x, u_2)| \geq \alpha |u_1 - u_2|$$

for all $x \in \mathbb{R}^n$, $t \in [0, T]$ and $u_1, u_2 \in \mathbb{R}^{n \times d}$.

Note that Assumptions 2.1 and 2.3 imply the mapping $u \mapsto \sigma(t, x, u)$ is a bijection from $\mathbb{R}^{n \times d}$ onto itself for any (t, x) . Therefore, let $\theta = \sigma(t, x, u)$ and denote the inverse function by $u = \tilde{\sigma}(t, x, \theta)$. Then system (2.5) can be rewritten as

$$\begin{cases} dX_t = -l(t, X_t, \theta_t)dt + \theta_t dW_t, \\ dY_t = -g(t, X_t, Y_t, Z_t, \theta_t)dt + (Z_t)^\top dW_t, \\ X_0 = x_0, Y_T = \Phi(X_T), \quad t \in [0, T], \end{cases}$$

where $l(t, x, \theta) = -b(t, x, \tilde{\sigma}(t, x, \theta))$ and $g(t, x, y, z, \theta) = f(t, x, y, z, \tilde{\sigma}(t, x, \theta))$. Since $u \mapsto \sigma(t, x, u)$ is a bijection, we may regard $\theta(\cdot)$ as the control variable. Due to the well-posedness of the BSDEs with Lipschitz generators, selecting $\theta(\cdot)$ is equivalent to selecting the terminal state X_T . Then we obtain the following purely backward control system:

$$\begin{cases} dX_t^\xi = -l(t, X_t^\xi, \theta_t^\xi)dt + \theta_t^\xi dW_t, \\ X_T^\xi = \xi, \\ dY_t^\xi = -g(t, X_t^\xi, Y_t^\xi, Z_t^\xi, \theta_t^\xi)dt + (Z_t^\xi)^\top dW_t, \\ Y_T^\xi = \Phi(\xi), \quad t \in [0, T], \end{cases} \quad (2.8)$$

where ξ is the control variable to be chosen from

$$\mathcal{U}_{ad} = \{\xi \in L^4_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) : \xi(\omega) \in K, \mathbb{P}\text{-a.s. } \omega \in \Omega\}.$$

The equivalent cost functional is

$$J(\xi) := \mathbb{E}_{\mathbb{P}} \left[h(\xi, Y_0^\xi) \right]. \quad (2.9)$$

Thus, the original problem is equivalent to minimizing $J(\xi)$ over \mathcal{U}_{ad} , subject to the controlled system (2.8) and the initial constraint $X_0^\xi = x_0$.

Remark 2.4. According to the definitions of l, g and Assumption 2.1, one can verify that l and g satisfy similar conditions in Assumption 2.1.

From the existence result (Thm. 2.3) in [5] and the uniqueness result (Lem. 2.1) in [4], we have:

Theorem 2.5. *Let Assumptions 2.1 and 2.3 hold. Then, for any $\xi \in \mathcal{U}_{ad}$, (2.8) admits a unique solution $(X^\xi(\cdot), Y^\xi(\cdot), Z^\xi(\cdot), \theta^\xi(\cdot)) \in \mathcal{S}_{\mathcal{F}}^{4\bar{p}^*}([0, T]; \mathbb{R}^n) \times \mathcal{S}_{\mathcal{F}}^\infty([0, T]; \mathbb{R}) \times \mathcal{M}_{\mathcal{F}}^2([0, T]; \mathbb{R}^d) \times \mathcal{M}_{\mathcal{F}}^{2, 4\bar{p}^*}([0, T]; \mathbb{R}^{n \times d})$ such that $Z^\xi \cdot W \in \text{BMO}$. Furthermore, we have the following estimate:*

$$\begin{cases} \|X^\xi\|_{\mathcal{S}_{\mathcal{F}}^{4\bar{p}^*}} + \|\theta^\xi\|_{\mathcal{S}_{\mathcal{F}}^{2, 4\bar{p}^*}} \leq C \mathbb{E}_{\mathbb{P}} \left[|\xi|^{4\bar{p}^*} + \left(\int_0^T |l(t, 0, 0)| dt \right)^{4\bar{p}^*} \right], \\ \|Y^\xi\|_{\mathcal{S}_\infty} + \|Z^\xi \cdot W\|_{\text{BMO}_2}^2 < A, \end{cases} \quad (2.10)$$

where C depends on $T, \bar{p}^*, \sup |l_x|, \sup |l_u|$.

Corollary 2.6. *From the energy-type inequality ([33], page 26) and the second inequality in (2.10), applying Hölder's inequality yields*

$$\sup_{\xi \in \mathcal{U}_{ad}} \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |Z_t^\xi|^2 dt \right)^p \right] < ([p] + 1)! A^{2p}, \quad \forall p > 0.$$

3. STOCHASTIC MAXIMUM PRINCIPLE

Applying Ekeland's variational principle, we derive the stochastic maximum principle for the optimization problem (2.8)–(2.9) in this section. The proposition below, which will be used frequently, follows from Corollary 9 and Theorem 10 in [32].

Proposition 3.1. *Assume $\lambda(\cdot) \in L_{\mathcal{F}}^\infty([0, T]; \mathbb{R})$, $\mu \cdot W \in \text{BMO}$ and $(\xi, \varphi(\cdot)) \in L_{\mathcal{F}_T}^{\beta_0}(\Omega; \mathbb{R}) \times \mathcal{M}_{\mathcal{F}}^{1, \beta_0}([0, T]; \mathbb{R})$ for some $\beta_0 > p_\mu^*$. Then there exists a unique solution $(Y(\cdot), Z(\cdot)) \in \bigcap_{1 < \beta < \beta_0} (\mathcal{S}_{\mathcal{F}}^\beta([0, T]; \mathbb{R}) \times \mathcal{M}_{\mathcal{F}}^{2, \beta}([0, T]; \mathbb{R}^d))$ to the following BSDE*

$$Y_t = \xi + \int_t^T (\lambda_s Y_s + \mu_s^\top Z_s + \varphi_s) ds - \int_t^T Z_s^\top dW_s, \quad t \in [0, T].$$

Moreover, for any $\beta \in (1, \beta_0)$, we have

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |Y_t|^\beta + \left(\int_0^T |Z_t|^2 dt \right)^{\frac{\beta}{2}} \right] \leq C \left(\mathbb{E}_{\mathbb{P}} \left[|\xi|^{\beta_0} + \left(\int_0^T |\varphi_t| dt \right)^{\beta_0} \right] \right)^{\frac{\beta}{\beta_0}}, \quad (3.1)$$

where $C > 0$ depends on $\beta, \beta_0, T, \|\lambda\|_\infty, \|\mu \cdot W\|_{\text{BMO}_2}$, and increases with respect to $\|\mu \cdot W\|_{\text{BMO}_2}$.

3.1. Variational equation

In this subsection, the constant C will change from line to line in our proof.

Let $\bar{\xi}$ be optimal and $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{\theta}(\cdot))$ be the corresponding state trajectory to (2.8). Recall the notation and conventions in Section 2 and denote

$$l_x(t) = l_x(t, \bar{X}_t, \bar{\theta}_t), \quad l_\theta(t) = l_\theta(t, \bar{X}_t, \bar{\theta}_t), \quad g_w(t) = g_w(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{\theta}_t) \quad (3.2)$$

for simplicity, where $w = x, y, z, \theta$.

Using the convexity of K and taking an arbitrary $\xi \in \mathcal{U}_{ad}$, we know, for $\varepsilon \in [0, 1]$,

$$\xi^\varepsilon := \bar{\xi} + \varepsilon(\xi - \bar{\xi}) \in \mathcal{U}_{ad}.$$

Let $(X^\varepsilon(\cdot), Y^\varepsilon(\cdot), Z^\varepsilon(\cdot), \theta^\varepsilon(\cdot))$ be the state trajectory of (2.8) associated with ξ^ε . To derive the first-order necessary condition in terms of small ε , we consider the following two BSDEs:

$$\begin{cases} d\hat{X}_t = - \left[l_x(t)\hat{X}_t + l_\theta(t)\hat{\theta}_t \right] dt + \hat{\theta}_t dW_t, & t \in [0, T], \\ \hat{X}_T = \xi - \bar{\xi}, \end{cases} \quad (3.3)$$

and

$$\begin{cases} d\hat{Y}_t = - \left[g_x(t)\hat{X}_t + g_y(t)\hat{Y}_t + g_z(t)\hat{Z}_t + g_\theta(t)\hat{\theta}_t \right] dt \\ \quad + \hat{Z}_t^\top dW_t, & t \in [0, T], \\ \hat{Y}_T = (\Phi_x(\bar{\xi}))^\top (\xi - \bar{\xi}). \end{cases} \quad (3.4)$$

Under Assumptions 2.1 and 2.3, note that (3.3) is a linear BSDE with bounded coefficients. According to the existence and uniqueness result of solution of BSDE ([34], Them. 5.1), we obtain:

Lemma 3.2. *Let Assumptions 2.1 and 2.3 hold. Then (3.3) admits a unique solution $(\hat{X}(\cdot), \hat{\theta}(\cdot)) \in \mathcal{S}_{\mathcal{F}}^{4\bar{p}^*}([0, T]; \mathbb{R}^n) \times \mathcal{M}_{\mathcal{F}}^{2, 4\bar{p}^*}([0, T]; \mathbb{R}^{n \times d})$.*

From the *a priori* estimate for BSDEs ([34], Prop. 5.1), we can obtain the following estimates by using the method in [28] similarly.

Lemma 3.3. *Let Assumptions 2.1 and 2.3 hold. Then, for any $\beta \in (1, 4\bar{p}^*]$, we have*

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t|^\beta + \left(\int_0^T |\theta_t^\varepsilon - \bar{\theta}_t|^2 dt \right)^{\frac{\beta}{2}} \right] = O(\varepsilon^\beta), \quad (3.5)$$

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t - \varepsilon \hat{X}_t|^\beta + \left(\int_0^T |\theta_t^\varepsilon - \bar{\theta}_t - \varepsilon \hat{\theta}_t|^2 dt \right)^{\frac{\beta}{2}} \right] = o(\varepsilon^\beta). \quad (3.6)$$

For $t \in [0, T]$, we set

$$\begin{aligned} (\chi_t^{1, \varepsilon}, \eta_t^{1, \varepsilon}, \zeta_t^{1, \varepsilon}, \Theta_t^{1, \varepsilon}) &:= (X_t^\varepsilon - \bar{X}_t, Y_t^\varepsilon - \bar{Y}_t, Z_t^\varepsilon - \bar{Z}_t, \theta_t^\varepsilon - \bar{\theta}_t), \\ \Lambda_t^\varepsilon &:= (\bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{\theta}_t), \quad \Lambda_t^\varepsilon := (X_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon, \theta_t^\varepsilon). \end{aligned}$$

Then we have

$$\begin{cases} d\eta_t^{1,\varepsilon} = - \left[\tilde{g}_x^\varepsilon(t) \chi_t^{1,\varepsilon} + \tilde{g}_y^\varepsilon(t) \eta_t^{1,\varepsilon} + \tilde{g}_z^\varepsilon(t) \zeta_t^{1,\varepsilon} + \tilde{g}_\theta^\varepsilon(t) \Theta_t^{1,\varepsilon} \right] dt \\ \quad + \left(\zeta_t^{1,\varepsilon} \right)^\top dW_t, \quad t \in [0, T], \\ \eta_T^{1,\varepsilon} = \Phi(\xi^\varepsilon) - \Phi(\bar{\xi}), \end{cases} \quad (3.7)$$

where $\tilde{g}_z^\varepsilon(t) = \int_0^1 g_z(t, \Lambda_t + \nu(\Lambda_t^\varepsilon - \Lambda_t)) d\nu$; $\tilde{g}_x^\varepsilon(t)$, $\tilde{g}_y^\varepsilon(t)$, $\tilde{g}_\theta^\varepsilon(t)$ are defined similarly.

Remark 3.4. Due to the second inequality in (2.10) and $|\tilde{g}_z^\varepsilon(t)| \leq L(1 + |\bar{Z}_t| + |Z_t^\varepsilon|)$, it can be verified that $\sup_{\varepsilon \in [0,1]} \|\tilde{g}_z^\varepsilon \cdot W\|_{\text{BMO}_2}^2 < 3L^2(T + 2A)$. Then, recalling (2.2), by definition of $p_{\tilde{g}_z^\varepsilon}$ and \bar{p} , we have $\sup_{\varepsilon \in [0,1]} p_{\tilde{g}_z^\varepsilon}^* < \bar{p}^*$.

The result below follows from Proposition 3.1, which provides the estimate for $(\eta^{1,\varepsilon}(\cdot), \zeta^{1,\varepsilon}(\cdot))$.

Lemma 3.5. *Let Assumptions 2.1 and 2.3 hold. Then, for any $\beta \in (1, 2\bar{p}^*)$, we have*

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} \left| \eta_t^{1,\varepsilon} \right|^\beta + \left(\int_0^T \left| \zeta_t^{1,\varepsilon} \right|^2 dt \right)^{\frac{\beta}{2}} \right] = O(\varepsilon^\beta). \quad (3.8)$$

Proof. Under Assumptions 2.1 and 2.3, we get

$$\begin{aligned} \left| \tilde{g}_x^\varepsilon(t) \chi_t^{1,\varepsilon} \right| &\leq C \left(1 + |Y_t^\varepsilon| + |\bar{Y}_t| + |Z_t^\varepsilon|^2 + |\bar{Z}_t|^2 \right) \left| \chi_t^{1,\varepsilon} \right|, \\ \left| \tilde{g}_\theta^\varepsilon(t) \Theta_t^{1,\varepsilon} \right| &\leq C \left(1 + |Y_t^\varepsilon| + |\bar{Y}_t| + |Z_t^\varepsilon| + |\bar{Z}_t| \right) \left| \Theta_t^{1,\varepsilon} \right|. \end{aligned}$$

Due to (2.10), $Y^\varepsilon(\cdot)$, $\bar{Y}(\cdot)$ are both bounded by A . Therefore, for any $\beta \in (1, 2\bar{p}^*)$, by Proposition 3.1 and Remark 3.4, we obtain

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} \left| \eta_t^{1,\varepsilon} \right|^\beta + \left(\int_0^T \left| \zeta_t^{1,\varepsilon} \right|^2 dt \right)^{\frac{\beta}{2}} \right] \\ &\leq C \left(\mathbb{E}_{\mathbb{P}} \left[\left| \Phi(\xi^\varepsilon) - \Phi(\bar{\xi}) \right|^{2\bar{p}^*} + \left(\int_0^T \left| \tilde{g}_x^\varepsilon(t) \chi_t^{1,\varepsilon} + \tilde{g}_\theta^\varepsilon(t) \Theta_t^{1,\varepsilon} \right| dt \right)^{2\bar{p}^*} \right] \right)^{\frac{\beta}{2\bar{p}^*}} \\ &\leq C \left\{ \left(\mathbb{E}_{\mathbb{P}} \left[\left| (\xi - \bar{\xi}) \right|^{2\bar{p}^*} \right] \right)^{\frac{\beta}{2\bar{p}^*}} \varepsilon^\beta + \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T \left(1 + |Z_t^\varepsilon|^2 + |\bar{Z}_t|^2 \right) \left| \chi_t^{1,\varepsilon} \right| dt \right)^{2\bar{p}^*} \right] \right)^{\frac{\beta}{2\bar{p}^*}} \right. \\ &\quad \left. + \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T \left(1 + |Z_t^\varepsilon| + |\bar{Z}_t| \right) \left| \Theta_t^{1,\varepsilon} \right| dt \right)^{2\bar{p}^*} \right] \right)^{\frac{\beta}{2\bar{p}^*}} \right\}, \end{aligned} \quad (3.9)$$

where the constant C is independent of ε . For the second term in the last inequality of (3.9), by (3.5) and Corollary 2.6, it follows from Hölder's inequality that

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T \left(1 + |Z_t^\varepsilon|^2 + |\bar{Z}_t|^2 \right) |\chi_t^{1,\varepsilon}| dt \right)^{2\bar{p}^*} \right] \\ & \leq \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\chi_t^{1,\varepsilon}|^{2\bar{p}^*} \left(\int_0^T \left(1 + |Z_t^\varepsilon|^2 + |\bar{Z}_t|^2 \right) dt \right)^{2\bar{p}^*} \right] \\ & \leq \left(\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\chi_t^{1,\varepsilon}|^{4\bar{p}^*} \right] \right)^{\frac{1}{2}} \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T \left(1 + |Z_t^\varepsilon|^2 + |\bar{Z}_t|^2 \right) dt \right)^{4\bar{p}^*} \right] \right)^{\frac{1}{2}} \\ & \leq C\varepsilon^{2\bar{p}^*}; \end{aligned}$$

similarly, for the third term in the last inequality of (3.9), we get

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T \left(1 + |Z_t^\varepsilon| + |\bar{Z}_t| \right) |\Theta_t^{1,\varepsilon}| dt \right)^{2\bar{p}^*} \right] \\ & \leq 3^{\bar{p}^*} \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T \left(1 + |Z_t^\varepsilon|^2 + |\bar{Z}_t|^2 \right) dt \right)^{\bar{p}^*} \left(\int_0^T |\Theta_t^{1,\varepsilon}|^2 dt \right)^{\bar{p}^*} \right] \\ & \leq 3^{\bar{p}^*} \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T \left(1 + |Z_t^\varepsilon|^2 + |\bar{Z}_t|^2 \right) dt \right)^{2\bar{p}^*} \right] \right)^{\frac{1}{2}} \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |\Theta_t^{1,\varepsilon}|^2 dt \right)^{2\bar{p}^*} \right] \right)^{\frac{1}{2}} \\ & \leq C\varepsilon^{2\bar{p}^*}. \end{aligned}$$

Consequently, we have

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} \left| \eta_t^{1,\varepsilon} \right|^\beta + \left(\int_0^T |\zeta_t^{1,\varepsilon}|^2 dt \right)^{\frac{\beta}{2}} \right] \leq C\varepsilon^\beta,$$

where C is independent of ε . The proof is complete. \square

Corollary 3.6. *For any $p \in [1, \bar{p}^*)$, (3.8) implies that*

$$\lim_{\varepsilon \rightarrow 0^+} \mathbb{E}_{\mathbb{P}} \left[\left| \left(\int_0^T |Z_t^\varepsilon|^2 dt \right)^p - \left(\int_0^T |\bar{Z}_t|^2 dt \right)^p \right| \right] = 0.$$

Now we prove the well-posedness of (3.4).

Lemma 3.7. *Let Assumptions 2.1 and 2.3 hold. Then there exists a unique solution $(\hat{Y}(\cdot), \hat{Z}(\cdot)) \in \mathcal{S}_{\mathcal{F}}^\beta([0, T]; \mathbb{R}) \times \mathcal{M}_{\mathcal{F}}^{2,\beta}([0, T]; \mathbb{R}^d)$ to (3.4) for all $\beta \in (1, 2\bar{p}^*)$.*

Proof. Under Assumption 2.1, we get

$$\begin{aligned} |g_x(t)\hat{X}_t| & \leq C \left(1 + |\bar{Y}_t| + |\bar{Z}_t|^2 \right) |\hat{X}_t|, \\ |g_\theta(t)\hat{\theta}_t| & \leq C \left(1 + |\bar{Y}_t| + |\bar{Z}_t| \right) |\hat{\theta}_t|. \end{aligned}$$

For any $\beta \in (1, 2\bar{p}^*)$, by Theorem 2.5 and Lemma 3.2, one can check that (3.4) verifies the conditions in Proposition 3.1. So it admits a unique solution $(\hat{Y}(\cdot), \hat{Z}(\cdot)) \in \bigcap_{1 < \beta < 2\bar{p}^*} \left(\mathcal{S}_{\mathcal{F}}^\beta([0, T]; \mathbb{R}) \times \mathcal{M}_{\mathcal{F}}^{2,\beta}([0, T]; \mathbb{R}^d) \right)$ to

(3.4). Moreover, from (3.1), since $\bar{Y}(\cdot) \in L^\infty([0, T]; \mathbb{R}^n)$, we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\hat{Y}_t|^\beta + \left(\int_0^T |\hat{Z}_t|^2 dt \right)^{\frac{\beta}{2}} \right] \\ & \leq C \left(\mathbb{E}_{\mathbb{P}} \left[|(\Phi_x(\bar{\xi}))^\top (\xi - \bar{\xi})|^{2\bar{p}^*} + \left(\int_0^T |g_x(t)\hat{X}_t + g_\theta(t)\hat{\theta}_t| dt \right)^{2\bar{p}^*} \right] \right)^{\frac{\beta}{2\bar{p}^*}} \\ & \leq C \left\{ \left(\mathbb{E}_{\mathbb{P}} \left[|(\xi - \bar{\xi})|^{2\bar{p}^*} \right] \right)^{\frac{\beta}{2\bar{p}^*}} \right. \\ & \quad \left. + \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T (1 + |\bar{Z}_t|^2) |\hat{X}_t| dt \right)^{2\bar{p}^*} + \left(\int_0^T (1 + |\bar{Z}_t|) |\hat{\theta}_t| dt \right)^{2\bar{p}^*} \right] \right)^{\frac{\beta}{2\bar{p}^*}} \right\}. \end{aligned}$$

Recall $(\hat{X}(\cdot), \hat{\theta}(\cdot)) \in \mathcal{S}_{\mathcal{F}}^{4\bar{p}^*}([0, T]; \mathbb{R}^n) \times \mathcal{M}_{\mathcal{F}}^{2, 4\bar{p}^*}([0, T]; \mathbb{R}^{n \times d})$ from Lemma 3.2. Then, similarly to the proof of the estimate (3.9), we can obtain

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\hat{Y}_t|^\beta + \left(\int_0^T |\hat{Z}_t|^2 dt \right)^{\frac{\beta}{2}} \right] < +\infty$$

for all $\beta \in (1, 2\bar{p}^*)$, which accomplishes the proof. \square

Now we give the main result of this subsection.

Lemma 3.8. *Let Assumptions 2.1 and 2.3 hold. Then, for any $\beta \in (1, \bar{p}^*)$,*

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |Y_t^\varepsilon - \bar{Y}_t - \varepsilon \hat{Y}_t|^\beta + \left(\int_0^T |Z_t^\varepsilon - \bar{Z}_t - \varepsilon \hat{Z}_t|^2 dt \right)^{\frac{\beta}{2}} \right] = o(\varepsilon^\beta). \quad (3.10)$$

Proof. We use the notation $(\chi_t^{1, \varepsilon}, \eta_t^{1, \varepsilon}, \zeta_t^{1, \varepsilon}, \Theta_t^{1, \varepsilon})$, $\tilde{g}_x^\varepsilon(t)$, $\tilde{g}_y^\varepsilon(t)$, $\tilde{g}_z^\varepsilon(t)$, $\tilde{g}_\theta^\varepsilon(t)$ introduced in the proof of Lemma 3.5. Setting $(\chi_t^{2, \varepsilon}, \eta_t^{2, \varepsilon}, \zeta_t^{2, \varepsilon}, \Theta_t^{2, \varepsilon}) := (\chi_t^{1, \varepsilon} - \varepsilon \hat{X}_t, \eta_t^{1, \varepsilon} - \varepsilon \hat{Y}_t, \zeta_t^{1, \varepsilon} - \varepsilon \hat{Z}_t, \Theta_t^{1, \varepsilon} - \varepsilon \hat{\theta}_t)$, we have

$$\begin{cases} d\eta_t^{2, \varepsilon} = - \left[g_x(t)\chi_t^{2, \varepsilon} + g_y(t)\eta_t^{2, \varepsilon} + g_z(t)\zeta_t^{2, \varepsilon} + g_\theta(t)\Theta_t^{2, \varepsilon} + R^\varepsilon(t) \right] dt \\ \quad + (\zeta_t^{2, \varepsilon})^\top dW_t, \quad t \in [0, T], \\ \eta_T^{2, \varepsilon} = \left[\int_0^1 (\Phi_x(\bar{\xi} + \lambda(\xi^\varepsilon - \bar{\xi})) - \Phi_x(\bar{\xi})) d\lambda \right]^\top (\xi^\varepsilon - \bar{\xi}) \end{cases} \quad (3.11)$$

where

$$\begin{aligned} R^\varepsilon(t) = & [\tilde{g}_x^\varepsilon(t) - g_x(t)] \chi_t^{1, \varepsilon} + [\tilde{g}_y^\varepsilon(t) - g_y(t)] \eta_t^{1, \varepsilon} \\ & + [\tilde{g}_z^\varepsilon(t) - g_z(t)] \zeta_t^{1, \varepsilon} + [\tilde{g}_\theta^\varepsilon(t) - g_\theta(t)] \Theta_t^{1, \varepsilon}, \end{aligned}$$

For any $\beta \in (1, \bar{p}^*)$ and $\beta_0 \in (\beta \vee p_{g_z}^*, \bar{p}^*)$, by Proposition 3.1, we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\eta_t^{2, \varepsilon}|^\beta + \left(\int_0^T |\zeta_t^{2, \varepsilon}|^2 dt \right)^{\frac{\beta}{2}} \right] \\ & \leq C \left(\mathbb{E}_{\mathbb{P}} \left[|\eta_T^{2, \varepsilon}|^{\beta_0} + \left(\int_0^T |g_x(t) \chi_t^{2, \varepsilon} + g_\theta(t) \Theta_t^{2, \varepsilon} + R^\varepsilon(t)| dt \right)^{\beta_0} \right] \right)^{\frac{\beta}{\beta_0}}. \end{aligned} \quad (3.12)$$

We only estimate the most difficult terms in (3.12) as follows. The other terms are similar.

Since $|g_\theta(t)| \leq L(1 + |\bar{Y}_t| + |\bar{Z}_t|)$ and $\|\bar{Y}\|_{\mathcal{S}^\infty} < +\infty$, by Corollary 2.6 and (3.6), it follows from Hölder's inequality that

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |g_\theta(t) \Theta_t^{2, \varepsilon}| dt \right)^{\beta_0} \right] \\ & \leq C \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T (1 + |\bar{Z}_t|) |\Theta_t^{2, \varepsilon}| dt \right)^{\beta_0} \right] \\ & \leq C \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |\Theta_t^{2, \varepsilon}|^2 dt \right)^{\beta_0} \right] \right)^{\frac{1}{2}} \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T (1 + |\bar{Z}_t|^2) dt \right)^{\beta_0} \right] \right)^{\frac{1}{2}} \\ & = o(\varepsilon^{\beta_0}). \end{aligned}$$

On the one hand, as g_z is continuous with (x, y, z, θ) , from (3.5) and (3.8), we have $\tilde{g}_z^\varepsilon(\cdot)$ converges to $g_z(\cdot)$ in the product measure $\lambda \otimes \mathbb{P}$, where λ denotes the Lebesgue measure on $[0, T]$. On the other hand, taking $p = 1$ in Corollary 3.6, we deduce the uniform integrability of $\left\{ |Z^\varepsilon(\cdot)|^2 \right\}_\varepsilon$ under $\lambda \otimes \mathbb{P}$. Then, since $|\tilde{g}_z^\varepsilon(t) - g_z(t)|^2 \leq C(1 + |\bar{Z}_t|^2 + |Z_t^\varepsilon|^2)$, applying the Vitali convergence theorem yields

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_{\mathbb{P}} \left[\int_0^T |\tilde{g}_z^\varepsilon(t) - g_z(t)|^2 dt \right] = 0,$$

which implies that $\left(\int_0^T |\tilde{g}_z^\varepsilon(t) - g_z(t)|^2 dt \right)^{\beta_0} \rightarrow 0$ in \mathbb{P} as $\varepsilon \rightarrow 0$. Similarly to the above, taking $p = \beta_0$ in Corollary 3.6 and then applying the Vitali convergence theorem again, we obtain

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |\tilde{g}_z^\varepsilon(t) - g_z(t)|^2 dt \right)^{\beta_0} \right] = 0. \quad (3.13)$$

Consequently, due to (3.8) and (3.13), we obtain

$$\begin{aligned}
& \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |\tilde{g}_z^\varepsilon(t) - g_z(t)| |\zeta^{1,\varepsilon}(t)| dt \right)^{\beta_0} \right] \\
& \leq C \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |\tilde{g}_z^\varepsilon(t) - g_z(t)|^2 dt \right)^{\frac{\beta_0}{2}} \left(\int_0^T |\zeta_t^{1,\varepsilon}|^2 dt \right)^{\frac{\beta_0}{2}} \right] \\
& \leq C \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |\tilde{g}_z^\varepsilon(t) - g_z(t)|^2 dt \right)^{\beta_0} \right] \right)^{\frac{1}{2}} \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |\zeta_t^{1,\varepsilon}|^2 dt \right)^{\beta_0} \right] \right)^{\frac{1}{2}} \\
& = o(\varepsilon^{\beta_0}).
\end{aligned} \tag{3.14}$$

Consequently, we have

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\eta_t^{2,\varepsilon}|^\beta + \left(\int_0^T |\zeta_t^{2,\varepsilon}|^2 dt \right)^{\frac{\beta}{2}} \right] = o(\varepsilon^\beta),$$

which accomplishes the proof. \square

3.2. Variational inequality

In this subsection, we employ Ekeland's variational principle [35] to deal with the initial constraint $X_0^\xi = x_0$. For $\xi_1, \xi_2 \in \mathcal{U}_{ad}$, define a metric in \mathcal{U}_{ad} by

$$d(\xi_1, \xi_2) := \left(\mathbb{E}_{\mathbb{P}} \left[|\xi_1 - \xi_2|^{4\bar{p}^*} \right] \right)^{\frac{1}{4\bar{p}^*}}.$$

Obviously, $(\mathcal{U}_{ad}, d(\cdot, \cdot))$ is a complete metric space.

Remark 3.9. According to Assumption 2.3, the closedness of K implies that \mathcal{U}_{ad} is closed in the Banach space $(L_{\mathcal{F}_T}^{4\bar{p}^*}(\Omega; \mathbb{R}^n), \|\cdot\|_{4\bar{p}^*})$, and hence complete. The completeness of $(\mathcal{U}_{ad}, d(\cdot, \cdot))$ is crucial (only) for Ekeland's variational principle.

Given the optimal $\bar{\xi}$, we introduce a mapping $J_\delta : \mathcal{U}_{ad} \mapsto \mathbb{R}$ by

$$J_\delta(\xi) := \sqrt{|X_0^\xi - x_0|^2 + (\max\{0, J(\xi) - J(\bar{\xi}) + \delta\})^2},$$

where x_0 is the given initial state constraint and δ is an arbitrary positive constant. Let us check that the mappings $\xi \mapsto |X_0^\xi - x_0|^2$, $\xi \mapsto J(\xi)$, both from \mathcal{U}_{ad} to \mathbb{R} , are continuous functional on \mathcal{U}_{ad} .

Lemma 3.10. *Let Assumptions 2.1 and 2.3 hold. Then $|X_0^\xi - x_0|^2$ and $J(\xi)$ are both continuous functional on \mathcal{U}_{ad} .*

Proof. Under Assumptions 2.1 and 2.3, since Y_0^ξ is bounded from Theorem 2.5 and $d(\xi_m, \xi) \rightarrow 0$ implies that $\mathbb{E}_{\mathbb{P}}[|\xi_m|^{4\bar{p}^*}] \rightarrow \mathbb{E}_{\mathbb{P}}[|\xi|^{4\bar{p}^*}]$ as $m \rightarrow \infty$ for any $\{\xi_m\}_{m \in \mathbb{N}_+}$, ξ in \mathcal{U}_{ad} , we need only to show that X_0^ξ and Y_0^ξ are continuous on \mathcal{U}_{ad} . To do this, for any given $\xi_1, \xi_2 \in \mathcal{U}_{ad}$, let $(X_1(\cdot), Y_1(\cdot), Z_1(\cdot), \theta_1(\cdot))$, $(X_2(\cdot), Y_2(\cdot), Z_2(\cdot), \theta_2(\cdot))$ are respectively corresponding state trajectories to ξ_1, ξ_2 , satisfying (2.8). Then, by Proposition 5.1 in [34],

we have

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |X_1(t) - X_2(t)|^{4\bar{p}^*} + \left(\int_0^T |\theta_1(t) - \theta_2(t)|^2 dt \right)^{2\bar{p}^*} \right] \leq C \mathbb{E}_{\mathbb{P}} \left[|\xi_1 - \xi_2|^{4\bar{p}^*} \right], \quad (3.15)$$

which implies the continuity of X_0^ξ , where C is a positive constant depending on T, L, \bar{p}^* . On the other hand, for $t \in [0, T]$, note that

$$\begin{aligned} & g(t, X_1(t), Y_1(t), Z_1(t), \theta_1(t)) - g(t, X_2(t), Y_2(t), Z_2(t), \theta_2(t)) \\ &= \lambda_t [Y_1(t) - Y_2(t)] + \mu_t^\top [Z_1(t) - Z_2(t)] + \varphi_t, \end{aligned}$$

where

$$\begin{aligned} \lambda_t &= \frac{g(t, X_1(t), Y_1(t), Z_1(t), \theta_1(t)) - g(t, X_1(t), Y_2(t), Z_1(t), \theta_1(t))}{Y_1(t) - Y_2(t)} \mathbf{1}_{\{Y_1 \neq Y_2\}}(t), \\ \mu_t &= \frac{g(t, X_1(t), Y_2(t), Z_1(t), \theta_1(t)) - g(t, X_1(t), Y_2(t), Z_2(t), \theta_1(t))}{|Z_1(t) - Z_2(t)|^2} [Z_1(t) - Z_2(t)] \mathbf{1}_{\{Z_1 \neq Z_2\}}(t), \\ \varphi_t &= g(t, X_1(t), Y_2(t), Z_2(t), \theta_1(t)) - g(t, X_2(t), Y_2(t), Z_2(t), \theta_2(t)). \end{aligned}$$

Under Assumptions 2.1 and 2.3, one can verify that $\lambda(\cdot) \in L^\infty([0, T]; \mathbb{R})$, $\mu \cdot W \in \text{BMO}$ with $\|\mu \cdot W\|_{\text{BMO}_2}^2 < \Psi(\bar{p})$. Moreover, by Theorem 2.5, Corollary 2.6 and (3.15), we deduce $\varphi(\cdot) \in \mathcal{M}_{\mathcal{F}}^{1, 2\bar{p}^*}([0, T]; \mathbb{R})$. Then, by Proposition 3.1, for any $\beta \in (1, 2\bar{p}^*)$, we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |Y_1(t) - Y_2(t)|^\beta + \left(\int_0^T |Z_1(t) - Z_2(t)|^2 dt \right)^\beta \right] \\ & \leq C \left(\mathbb{E}_{\mathbb{P}} \left[|\Phi(\xi_1) - \Phi(\xi_2)|^{2\bar{p}^*} + \left(\int_0^T |\varphi_t| dt \right)^{2\bar{p}^*} \right] \right)^{\frac{\beta}{2\bar{p}^*}}, \end{aligned}$$

where C is a positive constant depending on $T, L, \sup |g_y|, \bar{p}^*, \beta, A$. Recall that Φ is Lipschitz continuous and observe that

$$|\varphi_t| \leq L \left[(1 + |Y_2(t)| + |Z_2(t)|^2) |X_1(t) - X_2(t)| + (1 + |Y_2(t)| + |Z_2(t)|) |\theta_1(t) - \theta_2(t)| \right].$$

Using (3.15), similarly to the proof of (3.9), we obtain

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |Y_1(t) - Y_2(t)|^\beta + \left(\int_0^T |Z_1(t) - Z_2(t)|^2 dt \right)^\beta \right] \leq C \left(\mathbb{E}_{\mathbb{P}} \left[|\xi_1 - \xi_2|^{4\bar{p}^*} \right] \right)^{\frac{\beta}{4\bar{p}^*}}, \quad (3.16)$$

which implies the continuity of Y_0^ξ . The proof is complete. \square

Theorem 3.11. *Let Assumptions 2.1 and 2.3 hold, $\bar{\xi}$ be an optimal control to (2.8)–(2.9). Then there exist a real number $a_0 \geq 0$ and $a_1 \in \mathbb{R}^n$, with $|a_0|^2 + |a_1|^2 \neq 0$, such that the following variational inequality holds*

$$\mathbb{E}_{\mathbb{P}} \left[\left\langle a_1, \hat{X}_0 \right\rangle + a_0 \langle h_x(\bar{\xi}, \bar{Y}_0), \xi - \bar{\xi} \rangle + a_0 h_y(\bar{\xi}, \bar{Y}_0) \hat{Y}_0 \right] \geq 0, \quad (3.17)$$

where \hat{X}_0, \hat{Y}_0 are solutions to (3.3), (3.4) respectively.

Proof. Due to Lemma 3.10, $J_\delta(\cdot)$ is continuous on \mathcal{U}_{ad} . In addition, it is easy to check the following properties hold:

$$\begin{aligned} J_\delta(\bar{\xi}) &= \delta; \\ J_\delta(\xi) &> 0, \quad \forall \xi \in \mathcal{U}_{ad}; \\ J_\delta(\bar{\xi}) &\leq \inf_{\xi \in \mathcal{U}_{ad}} J_\delta(\xi) + \delta. \end{aligned}$$

Thus, from Ekeland's variational principle [35], $\exists \xi_\delta \in \mathcal{U}_{ad}$ such that

$$\begin{aligned} \text{(i)} \quad & J_\delta(\xi_\delta) \leq J_\delta(\bar{\xi}); \\ \text{(ii)} \quad & d(\xi_\delta, \bar{\xi}) \leq \sqrt{\delta}; \\ \text{(iii)} \quad & J_\delta(\xi) + \sqrt{\delta}d(\xi, \xi_\delta) \geq J_\delta(\xi_\delta), \quad \forall \xi \in \mathcal{U}_{ad}. \end{aligned}$$

For any $\xi \in \mathcal{U}_{ad}$, set $\xi_\delta^\varepsilon = \xi_\delta + \varepsilon(\xi - \xi_\delta)$, $\varepsilon \in [0, 1]$. Let $(X_\delta^\varepsilon(\cdot), Y_\delta^\varepsilon(\cdot), Z_\delta^\varepsilon(\cdot), \theta_\delta^\varepsilon(\cdot))$ (resp. $(X_\delta(\cdot), Y_\delta(\cdot), Z_\delta(\cdot), \theta_\delta(\cdot))$) be the state trajectory corresponding to ξ_δ^ε (resp. ξ_δ), and $(\hat{X}_\delta(\cdot), \hat{\theta}_\delta(\cdot))$, $(\hat{Y}_\delta(\cdot), \hat{Z}_\delta(\cdot))$ be the solutions to (3.3), (3.4) respectively in which $\bar{\xi}$ is substituted by ξ_δ . From (iii) the above, we conclude

$$J_\delta(\xi_\delta^\varepsilon) - J_\delta(\xi_\delta) + \sqrt{\delta}d(\xi_\delta^\varepsilon, \xi_\delta) \geq 0. \quad (3.18)$$

On the other hand, similarly to (3.6) and (3.8), we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} \left| X_\delta^\varepsilon(t) - X_\delta(t) - \varepsilon \hat{X}_\delta(t) \right| \right] &= 0, \\ \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} \left| Y_\delta^\varepsilon(t) - Y_\delta(t) - \varepsilon \hat{Y}_\delta(t) \right| \right] &= 0. \end{aligned}$$

Thus we obtain $X_\delta^\varepsilon(0) - X_\delta(0) = \varepsilon \hat{X}_\delta(0) + o(\varepsilon)$ and $Y_\delta^\varepsilon(0) - Y_\delta(0) = \varepsilon \hat{Y}_\delta(0) + o(\varepsilon)$, which leads to the following expansions:

$$\begin{aligned} & |X_\delta^\varepsilon(0) - x_0|^2 - |X_\delta(0) - x_0|^2 = 2\varepsilon \left\langle X_\delta(0) - x_0, \hat{X}_\delta(0) \right\rangle + o(\varepsilon), \\ & (J(\xi_\delta^\varepsilon) - J(\bar{\xi}) + \delta)^2 - (J(\xi_\delta) - J(\bar{\xi}) + \delta)^2 \\ &= (\mathbb{E}_{\mathbb{P}} [h(\xi_\delta^\varepsilon, Y_\delta^\varepsilon(0)) - h(\bar{\xi}, \bar{Y}_0)] + \delta)^2 - (\mathbb{E}_{\mathbb{P}} [h(\xi_\delta, Y_\delta(0)) - h(\bar{\xi}, \bar{Y}_0)] + \delta)^2 \\ &= 2\varepsilon (\mathbb{E}_{\mathbb{P}} [h(\xi_\delta, Y_\delta(0)) - h(\bar{\xi}, \bar{Y}_0)] + \delta) \\ & \quad \cdot \mathbb{E}_{\mathbb{P}} \left[\langle h_x(\xi_\delta, Y_\delta(0)), \xi - \xi_\delta \rangle + h_y(\xi_\delta, Y_\delta(0)) \hat{Y}_\delta(0) \right] + o(\varepsilon). \end{aligned}$$

Now we consider the following two cases:

Case 1: There exists $\varepsilon_0 > 0$ such that $J(\xi_\delta^\varepsilon) - J(\bar{\xi}) + \delta > 0$ for all $\varepsilon \in (0, \varepsilon_0)$. In this case,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{J_\delta(\xi_\delta^\varepsilon) - J_\delta(\xi_\delta)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{J_\delta(\xi_\delta^\varepsilon) + J_\delta(\xi_\delta)} \cdot \frac{J_\delta^2(\xi_\delta^\varepsilon) - J_\delta^2(\xi_\delta)}{\varepsilon} \\ &= \frac{1}{J_\delta(\xi_\delta)} \left(\left\langle X_\delta(0) - x_0, \hat{X}_\delta(0) \right\rangle \right. \\ & \quad \left. + (J(\xi_\delta) - J(\bar{\xi}) + \delta) \mathbb{E}_{\mathbb{P}} \left[\langle h_x(\xi_\delta, Y_\delta(0)), \xi - \xi_\delta \rangle + h_y(\xi_\delta, Y_\delta(0)) \hat{Y}_\delta(0) \right] \right). \end{aligned}$$

Dividing (3.18) by ε and sending ε to 0, we obtain

$$\begin{aligned} & \left\langle a_1^\delta, \hat{X}_\delta(0) \right\rangle + a_0^\delta \mathbb{E}_{\mathbb{P}} \left[\langle h_x(\xi_\delta, Y_\delta(0)), \xi - \xi_\delta \rangle + h_y(\xi_\delta, Y_\delta(0)) \hat{Y}_\delta(0) \right] \\ & \geq -\sqrt{\delta} \left(\mathbb{E}_{\mathbb{P}} \left[|\xi - \xi_\delta|^{4\bar{p}^*} \right] \right)^{\frac{1}{4\bar{p}^*}}, \end{aligned} \quad (3.19)$$

where

$$a_1^\delta = \frac{1}{J_\delta(\xi_\delta)} \left(X_0^{\xi_\delta} - x_0 \right), \quad a_0^\delta = \frac{1}{J_\delta(\xi_\delta)} \left(J(\xi_\delta) - J(\bar{\xi}) + \delta \right).$$

Case 2: There exists a positive sequence $\{\varepsilon_n\}$ satisfying $\varepsilon_n \rightarrow 0$, such that $J(\xi_\delta^{\varepsilon_n}) - J(\bar{\xi}) + \delta \leq 0$. In this case, by its definition $J_\delta(\xi_\delta^{\varepsilon_n}) = \sqrt{|X_0^{\xi_\delta^{\varepsilon_n}} - x_0|^2}$ for sufficiently large n . Since $J_\delta(\cdot)$ is continuous on $\mathcal{U}_{ad}[0, T]$, we conclude $J_\delta(\xi_\delta) = \sqrt{|X_0^{\xi_\delta} - x_0|^2}$. Now we have

$$\lim_{n \rightarrow \infty} \frac{J_\delta(\xi_\delta^{\varepsilon_n}) - J_\delta(\xi_\delta)}{\varepsilon_n} = \lim_{n \rightarrow \infty} \frac{1}{J_\delta(\xi_\delta^{\varepsilon_n}) + J_\delta(\xi_\delta)} \cdot \frac{J_\delta^2(\xi_\delta^{\varepsilon_n}) - J_\delta^2(\xi_\delta)}{\varepsilon_n} = \frac{\left\langle X_0^{\xi_\delta} - x_0, \hat{X}_\delta(0) \right\rangle}{J_\delta(\xi_\delta)}.$$

Similar to Case 1, we derive $\left\langle a_1^\delta, \hat{X}_\delta(0) \right\rangle \geq -\sqrt{\delta} \left(\mathbb{E}_{\mathbb{P}} \left[|\xi - \xi_\delta|^{4\bar{p}^*} \right] \right)^{\frac{1}{4\bar{p}^*}}$, where $a_0^\delta = 0$, $a_1^\delta = \frac{X_0^{\xi_\delta} - x_0}{J_\delta(\xi_\delta)}$.

In summary, for both cases, we have $a_0^\delta \geq 0$, $|a_0^\delta|^2 + |a_1^\delta|^2 = 1$ and (3.19). Hence, there exist a convergent subsequence of (a_0^δ, a_1^δ) whose limit is denoted by (a_0, a_1) . Due to $d(\xi_\delta, \bar{\xi}) \leq \sqrt{\delta}$, we have $\xi_\delta \rightarrow \bar{\xi}$ in \mathcal{U}_{ad} , as $\delta \rightarrow 0$. So, in (3.15) and (3.16), substituting $\xi_\delta, \bar{\xi}$ for ξ_1, ξ_2 respectively, we deduce that

$$\lim_{\delta \rightarrow 0} \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |X_\delta(t) - \bar{X}_t|^{4\bar{p}^*} + \left(\int_0^T |\theta_\delta(t) - \bar{\theta}_t|^2 dt \right)^{2\bar{p}^*} \right] = 0, \quad (3.20)$$

$$\lim_{\delta \rightarrow 0} \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |Y_\delta(t) - \bar{Y}_t|^\beta + \left(\int_0^T |Z_\delta(t) - \bar{Z}_t|^2 dt \right)^\beta \right] = 0, \quad (3.21)$$

where $\beta \in (1, 2\bar{p}^*)$ are arbitrarily given. Then, from (3.20) and (3.21), for $\beta \in (1, \bar{p}^*)$, one can prove

$$\lim_{\delta \rightarrow 0} \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\hat{Y}_\delta(t) - \hat{Y}_t|^\beta + \left(\int_0^T |\hat{Z}_\delta(t) - \hat{Z}_t|^2 dt \right)^\beta \right] = 0$$

similarly to the proof of Lemma 3.8, which implies $\hat{Y}_\delta(0) \rightarrow \hat{Y}_0$. $\hat{X}_\delta(0) \rightarrow \hat{X}_0$ can be deduced by applying Proposition 5.1 in [34]. All in all, let $\delta \rightarrow 0$ in (3.19), we get (3.17). The proof is complete. \square

3.3. Maximum principle

In this subsection, we derive the stochastic maximum principle. To this end, we introduce the adjoint process $(p(\cdot), q(\cdot))$ associated with the optimal admissible control $\bar{\xi}$ to (2.8)–(2.9), which solves the following adjoint

system:

$$\begin{cases} dp_t = (l_x(t)^\top p_t + g_x(t)^\top q_t) dt + (l_\theta(t)^\top p_t + g_\theta(t)^\top q_t) dW_t, & t \in [0, T], \\ p_0 = a_1, \\ dq_t = g_y(t)q_t dt + g_z(t)q_t dW_t, & t \in [0, T], \\ q_0 = a_0 h_y(\bar{\xi}, \bar{Y}_0), \end{cases} \quad (3.22)$$

where $l_x(t)$, $l_\theta(t)$, $g_x(t)$, $g_y(t)$, $g_z(t)$ are defined by (3.2), and a_0 , a_1 are as in Theorem 3.11.

Now we prove the well-posedness of (3.22).

Lemma 3.12. *Let Assumptions 2.1 and 2.3 hold. Then (3.22) admits a unique strong solution $(p(\cdot), q(\cdot))$. Moreover, for any given $\beta \in (1, \bar{p})$, we have*

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |p_t|^\beta + \sup_{t \in [0, T]} |q_t|^\beta \right] < +\infty.$$

Proof. In (3.22), we first consider the SDE where $q(\cdot)$ satisfies because the coefficients involved in it are no longer bounded. Under Assumptions 2.1 and 2.3, one can check the coefficients satisfy the conditions in the basic theorem in [36] (pp. 756–757) to the underlying semi-martingale $((\overbrace{1, \dots, 1}^d)^\top t + W_t)_{t \in [0, T]}$ (see also Lem. 7.1 in [37]). So it admits a unique strong solution $q(\cdot)$ up to an evanescent set. Moreover, set

$$\tilde{q}_t = a_0 h_y(\bar{\xi}, \bar{Y}_0) \exp \left\{ \int_0^t g_y(s) ds \right\} \mathcal{E} \left(\int_0^t g_z(s) dW_s \right), \quad t \in [0, T].$$

Then, noting $\tilde{q}(\cdot)$ is continuous and applying Itô's lemma to \tilde{q}_t on $[0, T]$, one can verify that $\tilde{q}(\cdot) = q(\cdot)$, \mathbb{P} -a.s.. On the other hand, for any $\beta \in (1, \bar{p})$, recalling Remark 3.4 and using reverse Hölder's inequality, we obtain $\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\tilde{q}_t|^\beta \right] \leq C$, where $C > 0$ depends on L , T , $\sup |g_y|$, A , a_0 , β .

Now let us focus on the SDE which $p(\cdot)$ satisfies. Under Assumptions 2.1 and 2.3, since l_x and l_θ are bounded, it admits a unique strong solution $p(\cdot)$ and, by using a standard estimate for SDEs, we get

$$\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |p_t|^\beta \right] \leq C \left\{ |a_1|^\beta + \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |g_x(t)q_t| dt \right)^\beta + \left(\int_0^T |g_\theta(t)q_t|^2 dt \right)^{\frac{\beta}{2}} \right] \right\}.$$

The right-hand side of the above inequality is finite since we can show that, for any given $\beta \in (1, \bar{p})$ and any $\beta_0 \in (\beta, \bar{p})$, by Corollary 2.6 and using Hölder's inequality,

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |g_\theta(t)q_t|^2 dt \right)^{\frac{\beta}{2}} \right] \\ & \leq C \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |q_t|^\beta \left(\int_0^T (1 + |\bar{Z}_t|)^2 dt \right)^{\frac{\beta}{2}} \right] \\ & \leq C \left(\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |q_t|^{\beta_0} \right] \right)^{\frac{\beta}{\beta_0}} \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T (1 + |\bar{Z}_t|)^2 dt \right)^{\frac{\beta\beta_0}{2(\beta_0 - \beta)}} \right] \right)^{\frac{\beta_0 - \beta}{\beta_0}} \\ & \leq C, \end{aligned}$$

where $C > 0$ depends on L , T , $\sup |g_y|$, A , a_0 , β , β_0 . The other term in that inequality can be estimated similarly. The proof is complete. \square

Theorem 3.13. *Let Assumptions 2.1 and 2.3 hold. If $\bar{\xi}$ is optimal to (2.8)–(2.9), then there exist $a_1 \in \mathbb{R}^n$ and $a_0 \in \mathbb{R}$ with $a_0 \geq 0$, $|a_0|^2 + |a_1|^2 \neq 0$, such that*

$$\langle p_T + q_T \Phi_x(\bar{\xi}) + a_0 h_x(\bar{\xi}, \bar{Y}_0), v - \bar{\xi} \rangle \geq 0, \quad \forall v \in K, \quad \mathbb{P}\text{-a.s.}, \quad (3.23)$$

where $(p(\cdot), q(\cdot))$ uniquely solves (3.22).

Proof. For $\xi \in \mathcal{U}_{ad}$, let $(\hat{X}(\cdot), \hat{\theta}(\cdot))$, $(\hat{Y}(\cdot), \hat{Z}(\cdot))$ be the solution to (3.3), (3.4) respectively. Applying Itô's lemma to $\langle p_t, \hat{X}_t \rangle + q_t \hat{Y}_t$ on $[0, T]$ yields

$$d \left[\langle p_t, \hat{X}_t \rangle + \langle q_t, \hat{Y}_t \rangle \right] = \Gamma_t dW_t, \quad (3.24)$$

where

$$\Gamma_t = p_t^\top \hat{\theta}_t + q_t \left(\hat{Z}_t^\top + g_z(t) \hat{Y}_t \right) + \hat{X}_t^\top (l_\theta(t)^\top p_t + g_\theta(t)^\top q_t), \quad t \in [0, T]. \quad (3.25)$$

We claim that $\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |\Gamma_t|^2 dt \right)^{\frac{1}{2}} \right] < +\infty$. Set $\beta_1 := 3\bar{p}(\bar{p} + 2)^{-1}$, $\beta_1^* := \beta_1(\beta_1 - 1)^{-1}$, $\beta_2 := \frac{7}{4}\bar{p}(\bar{p} - 1)^{-1}$, where \bar{p} is defined by (2.7). It is easy to check that $1 < \beta_1 < \bar{p}$ and $\bar{p}^* < \beta_1^* < \beta_2 < 2\bar{p}^*$. Actually, in (3.25), we can show that

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |q_t g_z(t) \hat{Y}_t|^2 dt \right)^{\frac{1}{2}} \right] \\ & \leq 2L \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |q_t| \left(\int_0^T (1 + |\bar{Z}_t|^2) |\hat{Y}_t|^2 dt \right)^{\frac{1}{2}} \right] \\ & \leq 2L \left(\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |q_t|^{\beta_1} \right] \right)^{\frac{1}{\beta_1}} \left(\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\hat{Y}_t|^{\beta_1^*} \left(\int_0^T (1 + |\bar{Z}_t|^2) dt \right)^{\frac{\beta_1^*}{2}} \right] \right)^{\frac{1}{\beta_1^*}} \\ & \leq 2L \left(\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |q_t|^{\beta_1} \right] \right)^{\frac{1}{\beta_1}} \left(\mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0, T]} |\hat{Y}_t|^{\beta_2} \right] \right)^{\frac{1}{\beta_2}} \left(\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T (1 + |\bar{Z}_t|^2) dt \right)^{\frac{\beta_1^* \beta_2}{2(\beta_2 - \beta_1^*)}} \right] \right)^{\frac{\beta_2 - \beta_1^*}{\beta_1^* \beta_2}} \end{aligned}$$

by using Hölder's inequality. Then it follows from reverse Hölder's inequality, Corollary 2.6 and Lemma 3.12 that $\mathbb{E}_{\mathbb{P}} \left[\left(\int_0^T |q_t g_z(t) \hat{Y}_t|^2 dt \right)^{\frac{1}{2}} \right] < +\infty$. The other terms in (3.25) can be estimated similarly, so $\int_0^\cdot \Gamma_t dW_t$ is a true martingale on $[0, T]$.

Integrating (3.24) from 0 to T , taking expectation and using the variational inequality (3.17), we obtain $\mathbb{E}_{\mathbb{P}} \left[\int_0^T \Gamma_t dW_t \right] = 0$ and

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[\langle p_T + q_T \Phi_x(\bar{\xi}) + a_0 h_x(\bar{\xi}, \bar{Y}_0), \xi - \bar{\xi} \rangle \right] \\ & = \mathbb{E}_{\mathbb{P}} \left[\langle a_1, \hat{X}_0 \rangle + a_0 h_y(\bar{\xi}, \bar{Y}_0) \hat{Y}_0 + a_0 \langle h_x(\bar{\xi}, \bar{Y}_0), \xi - \bar{\xi} \rangle \right] \\ & \geq 0. \end{aligned}$$

Since $\xi \in \mathcal{U}_{ad}$ is arbitrary, a standard argument yields (3.23). The proof is complete. \square

Denote the boundary of K by ∂K . Set $\Omega_0 := \{\omega \in \Omega \mid \bar{\xi}(\omega) \in \partial K\}$. According to Theorem 3.13, the following corollary holds.

Corollary 3.14. *Under assumptions of Theorem 3.13, for each $v \in K$,*

$$\begin{aligned} \langle p_T + q_T \Phi_x(\bar{\xi}) + a_0 h_x(\bar{\xi}, \bar{Y}_0), v - \bar{\xi} \rangle &\geq 0, \quad \mathbb{P}\text{-a.s. on } \Omega_0, \\ p_T + q_T \Phi_x(\bar{\xi}) + a_0 h_x(\bar{\xi}, \bar{Y}_0) &= 0, \quad \mathbb{P}\text{-a.s. on } \Omega \setminus \Omega_0. \end{aligned}$$

4. APPLICATIONS

In this section, we consider two applications separately, illustrating the obtained stochastic maximum principle.

4.1. Risk-sensitive optimal control problems with sample-wise constraints

Set $h(x, y) = y$ in (2.4) and let $f(t, x, y, z, u) = \frac{\gamma}{2} |z|^2 + F(t, x, u)$ in (2.5) with a positive number γ and a jointly measurable function F . Suppose that f verifies the corresponding conditions in Assumption 2.1. For any $u(\cdot) \in \mathcal{U}[0, T]$, under Assumptions 2.1, the (forward) SDE in (2.5) admits a unique solution $X^u(\cdot) \in \mathcal{S}_{\mathcal{F}}^{4\bar{p}^*}([0, T]; \mathbb{R}^n)$, and the BSDE in (2.5) admits a unique solution $(Y^u(\cdot), Z^u(\cdot)) \in \mathcal{S}_{\mathcal{F}}^{\infty}([0, T]; \mathbb{R}) \times \mathcal{M}_{\mathcal{F}}^2([0, T]; \mathbb{R}^d)$ according to Theorem 2.3 and Theorem 2.6 in [5]. Applying Itô's lemma to $e^{\gamma Y^u} + \int_0^{\cdot} F(s, X_s^u, u_s) ds$ on $[0, T]$ yields

$$\begin{aligned} \exp \left\{ \gamma \left(Y_t^u + \int_0^t F(s, X_s^u, u_s) ds \right) \right\} &= \exp \left\{ \gamma \left(\Phi(X_T^u) + \int_0^T F(s, X_s^u, u_s) ds \right) \right\} \\ &\quad - \gamma \int_t^T \exp \left\{ \gamma \left(Y_s^u + \int_0^s F(r, X_r^u, u_r) dr \right) \right\} Z_s^u dW_s, \quad t \in [0, T]. \end{aligned}$$

Since F is bounded and $Y^u(\cdot) \in \mathcal{S}_{\mathcal{F}}^{\infty}([0, T]; \mathbb{R})$, the local martingale $\left(\exp \left\{ \gamma Y_t^u + \int_0^t F(s, X_s^u, u_s) ds \right\} \right)_{t \in [0, T]}$ belongs to the class (D) and hence it is a true martingale. Putting $t = 0$ and taking $\mathbb{E}_{\mathbb{P}}[\cdot]$ on both sides of the above equation yield

$$e^{\gamma Y_0^u} = \mathbb{E}_{\mathbb{P}} \left[\exp \left\{ \gamma \left(\Phi(X_T^u) + \int_0^T F(s, X_s^u, u_s) ds \right) \right\} \right]. \quad (4.1)$$

The right-hand side of (4.1) is nothing but the cost in a risk-sensitive form studied in [12], where γ is the risk-sensitive parameter. Due to the monotonicity of exponential functions, minimizing Y_0^u over $\mathcal{U}[0, T]$ is equivalent to minimizing the right-hand side of (4.1) over $\mathcal{U}[0, T]$. Therefore, under the above setting, (2.4)–(2.6) degenerates into a risk-sensitive optimal control problem with sample-wise constraints as follows:

$$\begin{aligned} \text{minimize} \quad & J(u(\cdot)) := \mathbb{E}_{\mathbb{P}} \left[\exp \left\{ \gamma \left(\Phi(X_T^u) + \int_0^T F(s, X_s^u, u_s) ds \right) \right\} \right] \quad \text{over } \mathcal{U}[0, T] \\ \text{subject to} \quad & X_t^u = x_0 + \int_0^t b(s, X_s^u, u_s) ds + \int_0^t \sigma(s, X_s^u, u_s) dW_s, \quad t \in [0, T], \\ & X_T^u \in K, \quad \mathbb{P}\text{-a.s.}, \end{aligned} \quad (4.2)$$

where the constraint $K \subset \mathbb{R}^n$ is non-empty, closed, and convex.

Now we resort to the stochastic maximum principle obtained in Section 3 to deal with problem (4.2). The equivalently backward formulation of (4.2) is to

$$\begin{aligned} \text{minimize} \quad & J(\xi) := \mathbb{E}_{\mathbb{P}} \left[\exp \left\{ \gamma \left(\Phi(\xi) + \int_0^T \tilde{F}(s, X_s^{\xi}, \theta_s^{\xi}) ds \right) \right\} \right] \quad \text{over } \mathcal{U}_{ad} \\ \text{subject to} \quad & X_t^{\xi} = \xi + \int_t^T l(s, X_s^{\xi}, \theta_s^{\xi}) ds - \int_t^T \theta_s^{\xi} dW_s, \quad t \in [0, T], \\ & X_0^{\xi} = x_0, \quad \mathbb{P}\text{-a.s.}, \end{aligned} \quad (4.3)$$

where $l(t, x, \theta) = -b(t, x, \tilde{\sigma}(t, x, \theta))$ and $\tilde{F}(t, x, \theta) = F(t, x, \tilde{\sigma}(t, x, \theta))$ with $\tilde{\sigma}$ defined in Section 2. Let $\bar{\xi} \in \mathcal{U}_{ad}$ be optimal to (4.3) and $(\bar{X}(\cdot), \bar{\theta}(\cdot))$ be the unique solution to the BSDE in (4.3) corresponding to $\bar{\xi}$. For each $t \in [0, T]$, denote $l_x(t, \bar{X}_t, \bar{\theta}_t)$, $l_\theta(t, \bar{X}_t, \bar{\theta}_t)$, $\tilde{F}_x(t, \bar{X}_t, \bar{\theta}_t)$, $\tilde{F}_\theta(t, \bar{X}_t, \bar{\theta}_t)$ by $l_x(t)$, $l_\theta(t)$, $\tilde{F}_x(t)$, $\tilde{F}_\theta(t)$, respectively. Thanks to Theorem 3.13, we obtain the following stochastic maximum principle for (4.3).

Theorem 4.1. *Let Assumptions 2.1 and 2.3 hold. Then there exist $a_1 \in \mathbb{R}^n$ and $a_0 \in \mathbb{R}$ with $a_0 \geq 0$, $|a_0|^2 + |a_1|^2 \neq 0$, such that*

$$\langle p_T + q_T \Phi_x(\bar{\xi}), v - \bar{\xi} \rangle \geq 0, \quad \forall v \in K, \quad \mathbb{P}\text{-a.s.},$$

where $(p(\cdot), q(\cdot))$ uniquely solves the adjoint equation

$$\begin{cases} dp_t = \left(l_x(t)^\top p_t + \tilde{F}_x(t)^\top q_t \right) dt + \left(l_\theta(t)^\top p_t + \tilde{F}_\theta(t)^\top q_t \right) dW_t, & t \in [0, T], \\ dq_t = \gamma \bar{Z}_t^\top q_t dW_t, & t \in [0, T], \\ p_0 = a_1, \quad q_0 = a_0 \end{cases}$$

with $\bar{Z}(\cdot)$ being the second component of the unique solution of the BSDE

$$\begin{cases} d\bar{Y}_t = - \left[\frac{\gamma}{2} |\bar{Z}_t|^2 + \tilde{F}(t, \bar{X}_t, \bar{\theta}_t) \right] dt + (\bar{Z}_t)^\top dW_t, & t \in [0, T], \\ \bar{Y}_T = \Phi(\bar{\xi}). \end{cases}$$

4.2. A robust recursive utility maximization problem with bankruptcy prohibition

There are $d + 1$ investment instruments in the market. One of the instruments is a bank account (free risk); the others are stocks. The price processes are described by the following equations:

$$\begin{cases} dP_t^0 = P_t^0 r_t dt, \\ P_0^0 = \kappa_0 > 0, \\ dP_t^i = P_t^i \left[b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j \right], \\ P_0^i = \kappa_i > 0, \quad i = 1, \dots, d, \quad t \in [0, T]. \end{cases}$$

where the interest rate $r(\cdot)$, the stock-appreciation rate $b(\cdot) := (b^1(\cdot), \dots, b^d(\cdot))^\top$, the stock-volatility $\sigma(\cdot) := \{\sigma^{ij}(\cdot)\}_{1 \leq i, j \leq d}$ are all deterministic, bounded processes in suitable sizes. Moreover, $r(\cdot)$ is assumed to be nonnegative and $\sigma(\cdot)$ is assumed to be invertible whose inverse $\sigma^{-1}(\cdot)$ is also bounded.

An investor whose initial wealth is taken $x_0 \geq 0$ as a primitive, decides to invest in the i th stock ($i = 1, \dots, d$) with the amount $\theta^i(\cdot)$. Denote $X(\cdot)$ and $Y(\cdot)$ by the wealth process and the recursive utility of the investor, respectively. Let $B_t := (b_t^1 - r_t, \dots, b_t^d - r_t)$; $\theta(\cdot) = (\theta^1(\cdot), \dots, \theta^d(\cdot))^\top$ be the portfolio process and $\phi(\cdot) = \sigma^{-1}(\cdot) B^\top(\cdot)$ be the risk premium process. Here, we suppose that the instantaneous consumption rate $c(\cdot)$ depends only on the wealth process $X(\cdot)$. Thus, by the conventional calculation, the wealth process $X(\cdot)$ satisfies the following SDE:

$$\begin{cases} dX_t^\theta = \left[(r_t X_t^\theta + \theta_t^\top \sigma_t \phi_t) - c(X_t^\theta) \right] dt + \theta_t^\top \sigma_t dW_t, & t \in [0, T], \\ X_0^\theta = x_0, \end{cases} \quad (4.4)$$

where the consumption function c is nonnegative and continuous differentiable. The recursive utility of the investor's wealth $X^\theta(\cdot)$ is described by the following BSDE:

$$\begin{cases} dY_t^\theta = -g(t, X_t^\theta, Y_t^\theta, Z_t^\theta)dt + (Z_t^\theta)^\top dW_t, & t \in [0, T], \\ Y_T^\theta = \Phi(X_T^\theta), \end{cases} \quad (4.5)$$

where g and Φ satisfy Assumption 2.1.

Our problem is that an investor chooses portfolio $\theta(\cdot)$ so as to maximize the recursive utility Y_0^θ of his wealth $X^\theta(\cdot)$ with bankruptcy prohibition. Equivalently, we put $h(x, y) = -y$ since the control problem is to minimize the cost functional, that is,

$$\begin{aligned} & \text{minimize} && J(\theta(\cdot)) = -Y_0^\theta \\ & \text{subject to} && \theta(\cdot) \in \mathcal{M}_{\mathcal{F}}^{2,4\bar{p}^*}([0, T]; \mathbb{R}^d), \quad X_t^\theta \geq 0, \quad t \in [0, T], \end{aligned}$$

where \bar{p}^* is the exponential conjugate of \bar{p} introduced by (2.7). Using the method in , let $\theta_t = \sigma_t^\top \theta_t$, then we get the following equivalent control system

$$\begin{cases} dX_t^\xi = -l(t, X_t^\xi, \theta_t^\xi)dt + (\theta_t^\xi)^\top dW_t, & t \in [0, T], \\ X_T^\xi = \xi, \\ dY_t^\xi = -g(t, X_t^\xi, Y_t^\xi, Z_t^\xi)dt + (Z_t^\xi)^\top dW_t, & t \in [0, T], \\ Y_T^\xi = \Phi(\xi), \end{cases}$$

where $l(t, x, \theta) = -xr_t - \theta^\top \phi_t + c(x)$. As $l(t, 0, 0) \geq 0$, it ensures by the comparison theorem of BSDEs with Lipschitz generators ([34], Thm. 2.2) that if the terminal wealth $X_T^\theta \in K := [0, +\infty)$ then the wealth process $X_t^\theta \geq 0$, \mathbb{P} -a.s. $t \in [0, T]$. Therefore, the equivalent objective is

$$\begin{aligned} & \text{minimize} && J(\xi) = -Y_0^\xi \\ & \text{subject to} && \xi \in L_{\mathcal{F}_T}^{4\bar{p}^*}(\Omega; \mathbb{R}), \quad \xi \geq 0, \quad X_0^\xi = x_0. \end{aligned}$$

Let $\bar{\xi}$ be an optimal terminal wealth and $\bar{X}(\cdot)$, $\bar{Y}(\cdot)$ be the wealth process and the utility associated with $\bar{\xi}$, respectively. According to (3.22), the adjoint system is

$$\begin{cases} dp_t = [(c_x(\bar{X}_t) - r_t)p_t + g_x(t)q_t]dt - \phi_t^\top p_t dW_t, & t \in [0, T], \\ p_0 = a_1, \\ dq_t = g_y(t)q_t dt + g_z(t)q_t dW_t, & t \in [0, T], \\ q_0 = -a_0, \end{cases}$$

where $g_w(t) = g_w(t, \bar{Y}_t, \bar{Z}_t)$, $w = x, y, z$; $a_0, a_1 \in \mathbb{R}$ with $a_0 \geq 0$ and $|a_0|^2 + |a_1|^2 \neq 0$. Note that in this case the mapping $h(x, y) = -y$ which leads to $q_0 = -a_0$. The solution is

$$\begin{cases} p_t = \left[a_1 + \int_0^t g_x(s)q_s \Lambda_s ds \right] \Lambda_t^{-1}, \\ q_t = -a_0 \exp \left\{ \int_0^t \left(g_y(s) - \frac{1}{2} |g_z(s)|^2 \right) ds + \int_0^t g_z(s) dW_s \right\}, & t \in [0, T], \end{cases}$$

where $\Lambda_t = \exp \left\{ \int_0^t \left[r_s - c'(\bar{X}_s) + \frac{1}{2} |\phi_s|^2 \right] ds + \int_0^t \phi_s^\top dW_s \right\}$, $t \in [0, T]$.

Set $\Omega_0 = \{\omega \in \Omega \mid \bar{\xi}(\omega) = 0\}$ and suppose Assumptions 2.1 and 2.3. Then, by Theorem 3.13, we deduce that there exist constants $a_0, a_1 \in \mathbb{R}$ with $a_0 \geq 0$ and $|a_0|^2 + |a_1|^2 \neq 0$, such that

$$\begin{aligned} p_T + q_T \Phi_x(\bar{\xi}) &\geq 0, \quad \mathbb{P}\text{-a.s. on } \Omega_0, \\ p_T + q_T \Phi_x(\bar{\xi}) &= 0, \quad \mathbb{P}\text{-a.s. on } \Omega \setminus \Omega_0. \end{aligned} \quad (4.6)$$

Once c, g , and Φ are given, we will derive the expression of the optimal control $\bar{\xi}$. For example, we take $c(x) = \alpha x$, $\Phi(x) = \arctan(x)$, $g(t, x, y, z) = U(c(x)) - \beta y - \frac{\gamma}{2} |z|^2$, where $\alpha, \beta, \gamma > 0$, and $U(\cdot)$ is a bounded utility function that has a bounded and continuous derivative. We claim that the optimal terminal wealth can be represented as

$$\bar{\xi} = \sqrt{\left(-\frac{q_T}{p_T} - 1\right)^+}, \quad (4.7)$$

where

$$\begin{aligned} p_T &= \left[a_1 + \alpha \int_0^T U'(\alpha \bar{X}_s) q_s \Lambda_s ds \right] \Lambda_T^{-1}, \\ q_T &= -a_0 \exp \left\{ -\beta T - \frac{\gamma^2}{2} \int_0^T |\bar{Z}_s|^2 ds - \gamma \int_0^T \bar{Z}_s^\top dW_s \right\} \end{aligned}$$

with $\Lambda_t = \exp \left\{ \int_0^t \left[r_s - \alpha + \frac{1}{2} |\phi_s|^2 \right] ds + \int_0^t \phi_s^\top dW_s \right\}$, $t \in [0, T]$. We provide a sketch of the proof.

Case 1: $a_0 > 0$. In this case, we deduce that $q_T < 0$. Hence, from (4.6), on the one hand we have, on Ω_0 , \mathbb{P} -a.s.,

$$p_T + q_T \geq 0 \implies p_T \geq -q_T > 0 \implies \begin{cases} -\frac{q_T}{p_T} - 1 \leq 0, \\ a_1 + \alpha \int_0^T U'(\alpha \bar{X}_s) q_s \Lambda_s ds > 0. \end{cases}$$

On the other hand, on $\Omega \setminus \Omega_0$, \mathbb{P} -a.s.,

$$p_T + q_T \Phi_x(\bar{\xi}) = 0 \implies -q_T > p_T = -q_T \Phi_x(\bar{\xi}) > 0 \implies \begin{cases} |\bar{\xi}|^2 = -\frac{q_T}{p_T} - 1 > 0, \\ a_1 + \alpha \int_0^T U'(\alpha \bar{X}_s) q_s \Lambda_s ds > 0. \end{cases}$$

Case 2: $a_0 = 0$. In this case, we deduce that $a_1 \neq 0$ and $q_t = 0$, $t \in [0, T]$. Hence, from (4.6), we have $p_T \geq 0$ on Ω_0 and $p_T = 0$ on $\Omega \setminus \Omega_0$, \mathbb{P} -a.s.. But $a_1 \neq 0$ implies that $p_T > 0$. So we deduce that $\bar{\xi} = 0$ \mathbb{P} -a.s. and $a_1 > 0$.

In summary, for both cases, we have (4.7).

Remark 4.2. In view of the stochastic differential utility, the above example is closely related to the robust expected utility model studied in [20]. The generator $g(t, x, y, z) = U(c(x)) - \beta y - \frac{\gamma}{2} |z|^2$ can be interpreted as an intertemporal aggregator where $U(c(x)) - \beta y$ corresponds to the standard expected additive utility in continuous-time, and $\gamma > 0$ is the risk-averse parameter which reflects the issue of robustness in portfolio decision (see [14] for more details).

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