ESAIM: COCV 30 (2024) 27
https://doi.org/10.1051/cocv/2024016

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

UNIQUE CONTINUATION AND OBSERVABILITY FOR A SEMI-LINEAR PARABOLIC SYSTEM

Guojie Zheng[1,*] and Taige Wang[2]

Abstract. This article is concerned with the unique continuation and observability of a semi-linear parabolic system in a bounded domain Ω with homogeneous Dirichlet boundary conditions. We first study the existence and uniqueness of the L^∞-solution to the system when the initial value belongs to $L^\infty(\Omega)^2$. We then build up some observation estimates for this system. As a consequence, we can conclude if two solutions are equal over a nonempty open subset $\omega \subset \Omega$ at some time $T > 0$, then they coincide over $\Omega \times [0, T]$.

Mathematics Subject Classification. 35K40, 93B07, 35K45.

Received August 22, 2023. Accepted February 27, 2024.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a C^2 boundary $\partial\Omega$, and let $T > 0$. This paper studies the following semi-linear parabolic system:

$$\begin{cases} \partial_t u(x,t) - \triangle u(x,t) + f_1(u(x,t), v(x,t)) = 0, & \text{in } \Omega \times (0,T], \\ \partial_t v(x,t) - \triangle v(x,t) + f_2(u(x,t), v(x,t)) = 0, & \text{in } \Omega \times (0,T], \\ u(x,t) = 0,\ v(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $(u_0(x), v_0(x))$ is the initial value, and $f_i(\cdot,\cdot) : \mathbb{R}^2 \mapsto \mathbb{R}$ $(i = 1, 2)$ are two real value functions.

The model presented in this manuscript identifies a class of coupled parabolic equations, which have broad and proficient applications popularly in applied PDEs modeling ground, such as reaction-diffusion equations with nonlinearity and their applications (see [1–4]). This paper aims to study the unique continuation and observability of the parabolic system (1.1). Unique continuation is one of the most important subjects in PDEs, while observability can be regarded as a special unique continuation (see [5–10]). Simply speaking, observability means that the state of a system can be inferred by the knowledge of its external observation. The observability and controllability of linear systems are dual in mathematics (see Sect. 2.3 of [11]). However, it is not correct for nonlinear systems. Although these two concepts are both important in the control theory, the study of

Keywords and phrases: Observability, unique continuation, semi-linear parabolic system, frequency function.

[1] College of Digital Technology and Engineering Ningbo University of Finance & Economics, Ningbo 315175, PR China.
[2] Department of Mathematical Sciences University of Cincinnati, Cincinnati, OH 45221, USA.

[*] Corresponding author: guojiezheng@yeah.net

observability is more significant in most cases. The observability of differential equations have also been the object of numerous studies. Related references can be found in [9, 12–16] and the rich works cited therein.

In this paper, we will first study the well-posedness of the system (1.1). To this end, we impose an assumption on the functions $f_i(\cdot,\cdot)$ $(i=1,2)$ as follows.

(A_1) The functions $f_i(\cdot,\cdot):\mathbb{R}^2\mapsto\mathbb{R}$ $(i=1,2)$ are locally Lipschitz.

With the aid of (A_1), we will establish the existence and uniqueness of the L^∞-solution to the system (1.1) when the initial value $\big(u_0,v_0\big)\in L^\infty(\Omega)^2$, *i.e.*, the unique solution (u,v) belongs to $L^\infty((0,T);L^\infty(\Omega)^2)$ for some positive number T (see Thm. 3.1 in Sect. 3). When the well-posedness is achieved, we will study the unique continuation and observability of the system (1.1) by the frequency function method. The first result is stated as follows:

Theorem 1.1. *Suppose that (A_1) holds. Let ω be a nonempty open subset of Ω, let $(u_i^0,v_i^0)\in L^\infty(\Omega)^2$ $(i=1,2)$, let $(u_i,v_i)\in L^\infty((0,T);L^\infty(\Omega)^2)$ $(i=1,2)$ be the solution to the system (1.1) with the initial value (u_i^0,v_i^0), and let $L_M>0$ be the Lipschitz constant of f_i $(i=1,2)$ on the domain $\mathcal{D}_M:=\{(s_1,s_2)\in\mathbb{R}^2\mid |s_1|+|s_2|\le M\}$, where*

$$M:=\max\{\|(u_i,v_i)\|_{L^\infty(0,T;L^\infty(\Omega)^2)}\mid i=1,2\}. \tag{1.2}$$

Then, the following estimates are valid:
(i) There exist two positive numbers $\beta=\beta(\Omega,\omega)\in(0,1)$ and $C=C(\Omega,\omega)$ such that

$$\begin{aligned}\int_\Omega(|u_1-u_2|^2+|v_1-v_2|^2)(x,T)\mathrm{d}x\le\mathrm{e}^{C(1+L_M^2)(\frac{1}{T}+T)}\Big(\int_\Omega\big(|u_1^0-u_2^0|^2+|v_1^0-v_2^0|^2\big)(x)\mathrm{d}x\Big)^{1-\beta}\\ \times\Big(\int_\omega\big(|u_1-u_2|^2+|v_1-v_2|^2\big)(x,T)\mathrm{d}x\Big)^{\beta}.\end{aligned} \tag{1.3}$$

(ii) If $(u_1^0,v_1^0)\neq(u_2^0,v_2^0)$, then there exists a positive number $C=C(\Omega,\omega)$ such that

$$\begin{aligned}\int_\Omega\big(|u_1^0-u_2^0|^2+|v_1^0-v_2^0|^2\big)(x)\mathrm{d}x\le\mathrm{e}^{C\Big((1+L_M^2)(\frac{1}{T}+T)+(1+T\mathrm{e}^{(L_M+2L_M^2T)})\frac{\|u_1^0-u_2^0\|^2_{L^2(\Omega)}+\|v_1^0-v_2^0\|^2_{L^2(\Omega)}}{\|u_1^0-u_2^0\|^2_{H^{-1}(\Omega)}+\|v_1^0-v_2^0\|^2_{H^{-1}(\Omega)}}\Big)}\\ \times\int_\omega\big(|u_1-u_2|^2+|v_1-v_2|^2\big)(x,T)\mathrm{d}x.\end{aligned} \tag{1.4}$$

Remark 1.2. Several notes on Theorem 1.1 are given in order.

(a) From (1.3) or (1.4), we can observe if two solutions $\big(u_1(\cdot,T),v_1(\cdot,T)\big)=\big(u_2(\cdot,T),v_2(\cdot,T)\big)$ over ω, then $\big(u_1,v_1\big)=\big(u_2,v_2\big)$ over $\Omega\times[0,T]$. It means we can identify the states of the system (1.1) over $\Omega\times[0,T]$ by the observation in a small nonempty open subset $\omega\subset\Omega$ at the terminal time T. From the perspective of control theory, it can be viewed as observability of this semi-linear system, where $\omega\times\{T\}$ is the observation region. In most publications about observability, the observation region is $\omega\times E$, where $E=[0,T]$ (see [14, 15]) or E is a measurable subset of $[0,T]$ with a positive measure (see [16]). From the mathematical point of view, (1.3) or (1.4) can be regarded as a quantitative unique continuation property for the system (1.1).

(b) We present the strategy to study Theorem 1.1 as follows:
Inspired by the idea in [10], we first establish an L^∞-estimate for the solution (u,v) to the system (1.1). By (A_1) and the L^∞-estimate, we have the following inequalities:

$$\begin{cases}|\partial_t(u_1-u_2)-\triangle(u_1-u_2)|\le\Lambda(|u_1-u_2|+|v_1-v_2|), & \text{in }\Omega\times(0,T],\\ |\partial_t(v_1-v_2)-\triangle(v_1-v_2)|\le\Lambda(|u_1-u_2|+|v_1-v_2|), & \text{in }\Omega\times(0,T],\end{cases} \tag{1.5}$$

where (u_i, v_i) $(i = 1, 2)$ are two solutions to (1.1), and Λ is a positive number. Then, we prove Theorem 1.1 by using (1.5), the localization process (see [17]), and the frequency function method (see [7, 8, 16, 18–20]). In this article, we construct a new frequency function to resolve this problem (see Sect. 2).

(c) The difference between this paper and [10] is as follows:
In [10], the domain Ω is convex, and a global frequency function is employed to solve the problem. However, Ω is a general bounded domain in this article. We solve this problem by a localized frequency function.

Next, we give a further discussion for this problem, and introduce the second assumption on the functions $f_i(\cdot,\cdot)$ $(i = 1, 2)$ as follows.

(A_2) The functions $f_i(\cdot,\cdot) : \mathbb{R}^2 \mapsto \mathbb{R}$ $(i = 1, 2)$ belong to $C^1(\mathbb{R}^2)$, and $f_i(0, 0) = 0$ $(i = 1, 2)$. Furthermore, there are positive numbers $C > 0$ and $p > 1$ such that for each $(x, y) \in \mathbb{R}^2$,

$$|\nabla f_i(x, y)| \le C(1 + |x|^{p-1} + |y|^{p-1}), \ \ i = 1, 2.$$

The proof of Theorem 1.1 depends on the initial condition $(u_i^0, v_i^0) \in L^\infty(\Omega)^2$ $(i = 1, 2)$. It is natural to ask if these conclusions still hold when $(u_i^0, v_i^0) \in L^q(\Omega)^2$ (where $1 < q < +\infty$, and $i = 1, 2$). Indeed, the solution (u_i, v_i) $(i = 1, 2)$ may not belong to $L^\infty((0, T); L^\infty(\Omega)^2)$, and inequalities (1.3) and (1.4) may be invalid in this case. However, under the condition (A_2) and $\max\{\frac{n(p-1)}{2}, p\} < q < +\infty$, we obtain that $(u_i, v_i) \in C([0, T]; L^q(\Omega)^2) \cap L^\infty_{loc}((0, T); L^\infty(\Omega)^2)$ for some positive number T (see Thm. 3.2 in Section 3). Then, we can also get the unique continuation and observability of the parabolic system (1.1). The second result is presented as follows:

Theorem 1.3. *Suppose that (A_2) holds, and $\max\{\frac{n(p-1)}{2}, p\} < q < +\infty$. Let ω be a nonempty open subset of Ω, let $(u_i^0, v_i^0) \in L^q(\Omega)^2$ $(i = 1, 2)$, and let $(u_i, v_i) \in C([0, T]; L^q(\Omega)^2) \cap L^\infty_{loc}((0, T); L^\infty(\Omega)^2)$ be the solution to the system (1.1) corresponding to the initial value (u_i^0, v_i^0). Then, there exist positive numbers $\beta = \beta(\Omega, \omega) \in (0, 1)$ and $C = C(\Omega, \omega, \|(u_i^0, v_i^0)\|_{L^q(\Omega)^2}, T)$ $(i = 1, 2)$ such that*

$$\int_\Omega (|u_1 - u_2|^2 + |v_1 - v_2|^2)(x, T)\mathrm{d}x \le C\Big(\int_\omega (|u_1 - u_2|^2 + |v_1 - v_2|^2)(x, T)\mathrm{d}x\Big)^\beta. \tag{1.6}$$

If we further assume that $\big(u_1(\cdot, T), v_1(\cdot, T)\big) = \big(u_2(\cdot, T), v_2(\cdot, T)\big)$ over ω, then $\big(u_1, v_1\big) = (u_2, v_2)$ over $\Omega \times [0, T]$.

We organize the paper as follows: In Section 2, some preliminary results are presented. Section 3 is devoted to the well-posedness of the system (1.1). In Section 4, we will give some estimates for an auxiliary system. In Section 5, we will prove our main results. Finally, in the Appendix, we prove a Lemma.

2. Preliminary lemmas

In this section, we start with introducing some notations. We use $\|\cdot\|_q$ $(q \in [1, +\infty])$ to denote the norm of the Banach space $L^q(\Omega)$. For each $(\xi, \eta) \in L^q(\Omega)^2$ $(q \in [1, +\infty])$, we denote by $\|(\xi, \eta)\|_q := \|\xi\|_q + \|\eta\|_q$ the norm of (ξ, η) in $L^q(\Omega)^2$. We use $C(\dots)$ to stand for a positive constant which depends on what is enclosed in the brackets.

Now, we denote by $\mathrm{e}^{t\triangle}$ the semigroup generated by $\triangle$ on $L^q(\Omega)$, where $q \in [1, +\infty)$. The next standard smoothing effect of $\mathrm{e}^{t\triangle}$ can be found in [21], and it will be used later.

Lemma 2.1. *Let $1 \le q \le p \le +\infty$ and $f \in L^q(\Omega)$,*

$$\|\mathrm{e}^{t\triangle} f\|_p \le t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_q, \ \ \textit{for} \ \ t > 0. \tag{2.1}$$

Next, we arbitrarily take $T > 0$ and consider the following linear parabolic system:

$$\begin{cases} \partial_t u(x,t) - \triangle u(x,t) + a_1(x,t)u(x,t) + b_1(x,t)v(x,t) = 0, & \text{in } \Omega \times (0,T], \\ \partial_t v(x,t) - \triangle v(x,t) + a_2(x,t)u(x,t) + b_2(x,t)v(x,t) = 0, & \text{in } \Omega \times (0,T], \\ u(x,t) = 0, \ v(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x), & \text{in } \Omega. \end{cases} \tag{2.2}$$

Then, we have

Lemma 2.2. *Let σ be a number with $\sigma > \frac{n}{2}$ and $\sigma \geq 1$, and let $a_i(x,t), b_i(x,t) \in L^\infty((0,T); L^\sigma(\Omega))$ $(i = 1,2)$. If $(u_0, v_0) \in L^\gamma(\Omega)^2$ $(1 \leq \gamma < +\infty)$, then, there exists a unique solution $(u,v) \in C([0,T]; L^\gamma(\Omega)^2 \cap L^\infty_{loc}((0,T); L^\infty(\Omega)^2)$ to the system (2.2). Moreover, there exists a positive number $C = C(n,\sigma,\gamma,\Omega)$ such that for $t \in (0,T]$,*

$$\|(u(t), v(t))\|_\infty \leq C e^{CL^\vartheta t} t^{-\frac{n}{2\gamma}} \|(u_0, v_0)\|_\gamma, \tag{2.3}$$

where

$$L = \sum_{i=1,2} \left(\|a_i\|_{L^\infty((0,T);L^\sigma(\Omega))} + \|b_i\|_{L^\infty((0,T);L^\sigma(\Omega))} \right), \ \vartheta = \frac{2\sigma}{2\sigma - n}. \tag{2.4}$$

This lemma will be proved in the Appendix. Next, we will introduce the definition of the frequency function. Let $x_0 \in \Omega$ and $\bar{R} > 0$. Here and in what follows, we denote by $B_{\bar{R}} := B(x_0, \bar{R})$ the open ball centered at x_0 and of radius $\bar{R}$. Arbitrarily take $(y,z) \in H^1((0,T); (L^2(\Omega \cap B_{\bar{R}}))^2) \cap L^2((0,T); (H^2(\Omega \cap B_{\bar{R}}) \cap H^1_0(\Omega \cap B_{\bar{R}}))^2)$, then for each $\lambda > 0$, we define functions

$$G_\lambda(x,t) := \frac{1}{(T - t + \lambda)^{n/2}} e^{-\frac{|x - x_0|^2}{4(T - t + \lambda)}}, \ (x,t) \in \mathbb{R}^n \times [0,T] \tag{2.5}$$

and

$$N_\lambda(t) := \frac{\int_{\Omega \cap B_{\bar{R}}} \left(|\nabla y|^2 + |\nabla z|^2 \right)(x,t) \cdot G_\lambda(x,t) dx}{\int_{\Omega \cap B_{\bar{R}}} \left(|y|^2 + |z|^2 \right)(x,t) \cdot G_\lambda(x,t) dx}, \tag{2.6}$$

when $t \in (0,T]$ and $\int_{\Omega \cap B_{\bar{R}}} \left(|y|^2 + |z|^2 \right)(x,t) \cdot G_\lambda(x,t) dx \neq 0$.

Remark 2.3. The above function $N_\lambda(\cdot)$ can be viewed as a localized frequency function. We simply call it the *frequency function.*

Lemma 2.4. *The frequency function $N_\lambda(\cdot)$ given in (2.6) has the following properties:*
(i) If $t \in (0,T]$, $\lambda > 0$, and $\int_{\Omega \cap B_{\bar{R}}} \left(|y|^2 + |z|^2 \right)(x,t) \cdot G_\lambda(x,t) dx \neq 0$, then

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega \cap B_{\bar{R}}} \left(|y|^2 + |z|^2 \right)(x,t) \cdot G_\lambda(x,t) dx + N_\lambda(t) \int_{\Omega \cap B_{\bar{R}}} \left(|y|^2 + |z|^2 \right)(x,t) \cdot G_\lambda(x,t) dx \\ = &\int_{\Omega \cap B_{\bar{R}}} \left[\left(y \cdot (\partial_t y - \triangle y) \right) + \left(z \cdot (\partial_t z - \triangle z) \right) \right](x,t) \cdot G_\lambda(x,t) dx. \end{aligned}$$

(ii) If $\Omega\cap B_{\bar R}$ *is star-shaped with* x_0, $t\in(0,T]$, $\lambda>0$, *and* $\int_{\Omega\cap B_{\bar R}}\big(|y|^2+|z|^2\big)(x,t)\cdot G_\lambda(x,t)\mathrm{d}x\neq 0$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}t}N_\lambda(t)\le\frac{1}{T-t+\lambda}N_\lambda(t)+\frac{\int_{\Omega\cap B_{\bar R}}\big[(\partial_t y-\triangle y)^2+(\partial_t z-\triangle z)^2\big](x,t)\cdot G_\lambda(x,t)\mathrm{d}x}{\int_{\Omega\cap B_{\bar R}}\big(|y|^2+|z|^2\big)(x,t)\cdot G_\lambda(x,t)\mathrm{d}x}.$$

Proof. We can prove it by the same method used in Lemma 2.3 of [16]. Here, we omit the details. □

The following interpolation inequality is borrowed from [17].

Lemma 2.5. *Let* $\lambda>0$, *let* $0<S<T$, *and let* $f(\cdot),N(\cdot)\in C^1[S,T]$ *be two positive functions satisfying*

$$\begin{cases}\left|\frac{1}{2}f'(t)+N(t)f(t)\right|\le\tilde C f(t),\\ N'(t)\le\frac{1}{T-t+\lambda}N_\lambda(t)+\bar C,\end{cases}$$

where $\tilde C$ *and* $\bar C$ *are two positive constants. Then for* $S\le t_1<t_2<t_3\le T$, *we have*

$$f(t_2)^{1+D}\le f(t_3)\cdot f(t_1)^D\cdot\mathrm{e}^K,$$

where

$$D=\frac{\int_{t_2}^{t_3}\frac{\mathrm{d}t}{T-t+\lambda}}{\int_{t_1}^{t_2}\frac{\mathrm{d}t}{T-t+\lambda}},$$

and

$$K=2D(t_2-t_1)\big[\tilde C+\bar C(t_2-t_1)\big]+2(t_3-t_2)\Big[\tilde C+\bar C\int_{t_2}^{t_3}\frac{T-t_2+\lambda}{T-t+\lambda}\mathrm{d}t\Big].$$

Proof. The proof of this lemma can be found in Lemma 2.1 of [17]. We omit the details. □

3. Well-posedness on the system (1.1)

In this section, we will study the well-posedness of the system (1.1).

Theorem 3.1. *If* (A_1) *holds, and* $(u_0,v_0)\in L^\infty(\Omega)^2$, *then (1.1) admits a unique solution* $(u,v)\in L^\infty((0,T^*);L^\infty(\Omega)^2)$ *for some positive constant* T^*.

Proof. Let $T^*>0$, which will be determined later. We define $\mathcal{A}:=L^\infty((0,T^*);L^\infty(\Omega)^2)$ and

$$\mathcal{B}:=\{(\xi,\eta)\in\mathcal{A}\mid\|(\xi,\eta)\|_{\mathcal{A}}\le 2\|(u_0,v_0)\|_\infty+1\}.$$

The following proof will be split into two steps.
Step 1. We first prove that for some $T^*>0$, *the system (1.1) has a solution* $(u,v)\in\mathcal{A}$.

For each $(\xi,\eta)\in\mathcal{B}$, we define functions $(u(\cdot;\xi,\eta),v(\cdot;\xi,\eta))$ as follows:

$$\begin{cases}u(t;\xi,\eta):=\mathrm{e}^{t\triangle}u_0+\int_0^t\mathrm{e}^{(t-s)\triangle}f_1(\xi(s),\eta(s))\mathrm{d}s,\\ v(t;\xi,\eta):=\mathrm{e}^{t\triangle}v_0+\int_0^t\mathrm{e}^{(t-s)\triangle}f_2(\xi(s),\eta(s))\mathrm{d}s,\end{cases}$$

where $t\in[0,T^*]$. Clearly, $(u(\cdot;\xi,\eta),v(\cdot;\xi,\eta))\in\mathcal{A}$. Now, we define a map $\Psi:\mathcal{B}\mapsto\mathcal{A}$ by setting

$$\Psi(\xi,\eta):=(u(\cdot;\xi,\eta),v(\cdot;\xi,\eta)),\quad\text{for }(\xi,\eta)\in\mathcal{B}.$$

We claim that $\Psi(\mathcal{B}) \subseteq \mathcal{B}$ for some $T^* > 0$. Applying Lemma 2.1, we have that for $t \in [0, T^*]$,

$$\begin{aligned} &\|\big(u(t;\xi,\eta), v(t;\xi,\eta)\big)\|_\infty = \|u(t;\xi,\eta)\|_\infty + \|v(t;\xi,\eta)\|_\infty \\ &\le \|\mathrm{e}^{t\triangle} u_0\|_\infty + \|\mathrm{e}^{t\triangle} v_0\|_\infty + \sum_{i=1,2} \int_0^t \|\mathrm{e}^{(t-s)\triangle} f_i(\xi(s),\eta(s))\|_\infty \mathrm{d}s \\ &\le \|u_0\|_\infty + \|v_0\|_\infty + \sum_{i=1,2} \int_0^t \|f_i(\xi(s),\eta(s))\|_\infty \mathrm{d}s. \end{aligned} \tag{3.1}$$

Meanwhile, by (A_1), we observe that for $s \in [0, T^*]$, and $i = 1, 2$,

$$\begin{aligned} \|f_i(\xi(s),\eta(s))\|_\infty &\le \|f_i(\xi(s),\eta(s)) - f_i(0,0)\|_\infty + |f_i(0,0)| \\ &\le K_i(\|\xi\|_\infty + \|\eta\|_\infty) + |f_i(0,0)| \\ &\le 2K_i(\|u_0\|_\infty + \|v_0\|_\infty) + (K_i + |f_i(0,0)|), \end{aligned}$$

where K_i $(i = 1, 2)$ is the Lipschitz constant of f_i on the set $\{(x_1, x_2) \mid |x_1| + |x_2| \le 2\|(u_0, v_0)\|_\infty + 1\}$. It, combining with (3.1), leads to

$$\begin{aligned} \|\big(u(t;\xi,\eta), v(t;\xi,\eta)\big)\|_\infty \le\ & (1 + 2(K_1 + K_2)T^*) \cdot (\|u_0\|_\infty + \|v_0\|_\infty) \\ &+ \sum_{i=1,2} (K_i + |f_i(0,0)|)T^*, \end{aligned} \tag{3.2}$$

where $t \in [0, T^*]$. By choosing T^* small enough so that

$$2(K_1 + K_2)T^* \le 1, \quad \text{and} \quad \sum_{i=1,2} (K_i + |f_i(0,0)|)T^* \le 1. \tag{3.3}$$

Thus, we have $\Psi(\mathcal{B}) \subseteq \mathcal{B}$. Next, we claim that Ψ is a strict contraction for some $T^* > 0$. Given $(\xi_i, \eta_i) \in \mathcal{B}$ $(i = 1, 2)$, write $(u_i, v_i) = \Psi(\xi_i, \eta_i)$ $(i = 1, 2)$. Then, we have

$$\begin{cases} u_1(t) - u_2(t) = \int_0^t \mathrm{e}^{(t-s)\triangle}[f_1(\xi_1(s),\eta_1(s)) - f_1(\xi_2(s),\eta_2(s))]\mathrm{d}s, \\ v_1(t) - v_2(t) = \int_0^t \mathrm{e}^{(t-s)\triangle}[f_2(\xi_1(s),\eta_1(s)) - f_2(\xi_2(s),\eta_2(s))]\mathrm{d}s. \end{cases}$$

By direct computation, we obtain that for $t \in [0, T^*]$,

$$\begin{aligned} \|\big(u_1(t), v_1(t)\big) - \big(u_2(t), v_2(t)\big)\|_\infty &= \|u_1(t) - u_2(t)\|_\infty + \|v_1(t) - v_2(t)\|_\infty \\ &\le \sum_{i=1,2} \int_0^t \|f_i(\xi_1(s),\eta_1(s)) - f_i(\xi_2(s),\eta_2(s))\|_\infty \mathrm{d}s. \end{aligned}$$

This, along with (A_1), indicates for $t \in [0, T^*]$,

$$\|\big(u_1(t), v_1(t)\big) - \big(u_2(t), v_2(t)\big)\|_\infty \le (K_1 + K_2)T^* \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{A}}.$$

Choosing $T^* > 0$ satisfying (3.3) and

$$(K_1 + K_2)T^* < 1,$$

we obtain that Ψ is a strict contraction map from $\mathcal{B}$ to $\mathcal{B}$. Thus, Ψ has a unique fixed point (u,v) in $\mathcal{B}\subseteq\mathcal{A}$, which is a solution of (1.1).

Step 2. We show that the solution of (1.1) is unique in $\mathcal{A}$.

We suppose that $(u,v),(\tilde{u},\tilde{v})\in\mathcal{A}$ are both the solutions to the system (1.1) with the initial value (u_0,v_0). Write

$$\varrho_1 := \max\{\|(u,v)\|_{\mathcal{A}}, \|(\tilde{u},\tilde{v})\|_{\mathcal{A}}\}.$$

Using (A_1) and Lemma 2.1, we have that for $t\in(0,T^*]$,

$$\begin{aligned}\|\big(u(t),v(t)\big)-\big(\tilde{u}(t),\tilde{v}(t)\big)\|_\infty &= \sum_{i=1,2}\|\int_0^t \mathrm{e}^{(t-s)\triangle}[f_i\big(u(t),v(t)\big)-f_i\big(\tilde{u}(t),\tilde{v}(t)\big)]\mathrm{d}s\|_\infty\\ &\le (\tilde{K}_1+\tilde{K}_2)\int_0^t\|\big(u,v\big)-\big(\tilde{u},\tilde{v}\big)\|_\infty\mathrm{d}s\end{aligned}$$

where $\tilde{K}_i$ $(i=1,2)$ is respectively the Lipschitz constant of f_i on the set $\{(x_1,x_2)\,|\,|x_1|+|x_2|\le\varrho_1\}$. Uniqueness now is obtained by Gronwall's inequality. Hence, we complete the proof. □

Theorem 3.2. *If (A_2) holds, and $(u_0,v_0)\in L^q(\Omega)^2$, where $\max\{\frac{n(p-1)}{2},p\}<q<+\infty$, then (1.1) has a unique solution $(u,v)\in C([0,T^*];L^q(\Omega)^2)\cap L^\infty_{loc}((0,T^*);L^\infty(\Omega)^2)$ for some positive constant T^*, and the solution (u,v) satisfies*

$$\|(u(t),v(t))\|_q + t^{\frac{n}{2q}}\|(u(t),v(t))\|_\infty \le C\|(u_0,v_0)\|_q,\ 0<t\le T^*, \tag{3.4}$$

where the positive constant $C=C(\Omega,p,q,\|u_0\|_q,\|v_0\|_q,T^*)$.

Proof. Let $T^*>0$, which will be determined later. We write $\mathcal{E}:=C([0,T^*];L^q(\Omega)^2)$ and define its closed subset $\mathcal{K}$ as follows:

$$\mathcal{K}:=\{(\xi,\eta)\in\mathcal{E}\mid \sup_{0\le t\le T^*}\|(\xi,\eta)\|_q\le 2\|(u_0,v_0)\|_q\}.$$

The following proof will be split into three steps.

Step 1. We prove that for some $T^*>0$, the system (1.1) has a solution $(u,v)\in\mathcal{E}$.

For each $(\xi,\eta)\in\mathcal{K}$, we define functions $(u(\cdot;\xi,\eta),v(\cdot;\xi,\eta))$ as follows:

$$\begin{cases} u(t;\xi,\eta):=\mathrm{e}^{t\triangle}u_0+\int_0^t\mathrm{e}^{(t-s)\triangle}f_1(\xi(s),\eta(s))\mathrm{d}s,\\ v(t;\xi,\eta):=\mathrm{e}^{t\triangle}v_0+\int_0^t\mathrm{e}^{(t-s)\triangle}f_2(\xi(s),\eta(s))\mathrm{d}s,\end{cases}$$

where $t\in[0,T^*]$. It is obvious that $(u(\cdot;\xi,\eta),v(\cdot;\xi,\eta))\in\mathcal{E}$. Now, we define a map $\Phi:\ \mathcal{K}\mapsto\mathcal{E}$ by setting

$$\Phi(\xi,\eta):=(u(\cdot;\xi,\eta),v(\cdot;\xi,\eta)),\ \text{ for }\ (\xi,\eta)\in\mathcal{K}.$$

We claim that $\Phi(\mathcal{K})\subseteq\mathcal{K}$ for some $T^*>0$. Indeed, by Lemma 2.1, we have that for $t\in[0,T^*]$,

$$\begin{aligned}&\|\big(u(t;\xi,\eta),v(t;\xi,\eta)\big)\|_q=\|u(t;\xi,\eta)\|_q+\|v(t;\xi,\eta)\|_q\\ &\le\|\mathrm{e}^{t\triangle}u_0\|_q+\|\mathrm{e}^{t\triangle}v_0\|_q+\sum_{i=1,2}\int_0^t\|\mathrm{e}^{(t-s)\triangle}f_i(\xi(s),\eta(s))\|_q\mathrm{d}s\\ &\le\|u_0\|_q+\|v_0\|_q+\sum_{i=1,2}\int_0^t(t-s)^{-\alpha}\|f_i(\xi(s),\eta(s))\|_{\frac{q}{p}}\mathrm{d}s,\end{aligned} \tag{3.5}$$

where

$$\alpha := \frac{n(p-1)}{2q} \in (0,1). \tag{3.6}$$

Meanwhile, it follows from the mean value theorem, (A_2), and Hölder inequality that for $s \in [0, T^*]$, and $i = 1, 2$,

$$\begin{aligned}\|f_i(\xi(s), \eta(s))\|_{\frac{q}{p}} &= \|f_i(\xi(s), \eta(s)) - f_i(0,0)\|_{\frac{q}{p}} \\ &\le C(1 + \|\xi\|_q^{p-1} + \|\eta\|_q^{p-1}) \cdot (\|\xi\|_q + \|\eta\|_q) \\ &\le C(1 + (\|u_0\|_q + \|v_0\|_q)^{p-1}) \cdot (\|u_0\|_q + \|v_0\|_q),\end{aligned}$$

where the constant C only depends on Ω. Combining the above inequality with (3.5) leads to

$$\|\big(u(t;\xi,\eta), v(t;\xi,\eta)\big)\|_q \le (1 + C(\Omega, p, q, \|u_0\|_q, \|v_0\|_q)T^{*1-\alpha}) \cdot (\|u_0\|_q + \|v_0\|_q), \tag{3.7}$$

where $t \in [0, T^*]$. By choosing T^* small enough so that

$$C(\Omega, p, q, \|u_0\|_q, \|v_0\|_q)T^{*1-\alpha} \le 1. \tag{3.8}$$

Thus, we have $\Phi(\mathcal{K}) \subseteq \mathcal{K}$. Next, we claim that Φ is a strict contraction for some $T^* > 0$. Given $(\xi_i, \eta_i) \in \mathcal{K}$ $(i = 1, 2)$, write $(u_i, v_i) = \Phi(\xi_i, \eta_i)$ $(i = 1, 2)$. Then, we have

$$\begin{cases} u_1(t) - u_2(t) = \int_0^t \mathrm{e}^{(t-s)\triangle}[f_1(\xi_1(s), \eta_1(s)) - f_1(\xi_2(s), \eta_2(s))]\mathrm{d}s, \\ v_1(t) - v_2(t) = \int_0^t \mathrm{e}^{(t-s)\triangle}[f_2(\xi_1(s), \eta_1(s)) - f_2(\xi_2(s), \eta_2(s))]\mathrm{d}s. \end{cases}$$

It, along with Lemma 2.1, indicates that for $t \in [0, T^*]$,

$$\begin{aligned}&\|\big(u_1(t), v_1(t)\big) - \big(u_2(t), v_2(t)\big)\|_q = \|u_1(t) - u_2(t)\|_q + \|v_1(t) - v_2(t)\|_q \\ \le &\sum_{i=1,2} \int_0^t (t-s)^{-\alpha} \|f_i(\xi_1(s), \eta_1(s)) - f_i(\xi_2(s), \eta_2(s))\|_{\frac{q}{p}} \mathrm{d}s.\end{aligned}$$

where α is the number given in (3.6). Then, by the mean value theorem, (A_2), and Hölder inequality, we obtain

$$\begin{aligned}&\|\big(u_1(t), v_1(t)\big) - \big(u_2(t), v_2(t)\big)\|_q \\ \le\, &C \sum_{i=1,2} \int_0^t (t-s)^{-\alpha} (1 + \sum_{i=1,2} \|\xi_i\|_q^{p-1} + \sum_{i=1,2} \|\eta_i\|_q^{p-1})(\|\xi_1 - \xi_2\|_q + \|\eta_1 - \eta_2\|_q)\mathrm{d}s \\ \le\, &\tilde{C}(\Omega, p, q, \|u_0\|_q, \|v_0\|_q)T^{*1-\alpha} \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{E}}.\end{aligned}$$

Thus, we have

$$\|\big(u_1, v_1\big) - \big(u_2, v_2\big)\|_{\mathcal{E}} \le \tilde{C}(\Omega, p, q, \|u_0\|_q, \|v_0\|_q)T^{*1-\alpha} \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{E}}. \tag{3.9}$$

Choosing $T^* > 0$ satisfying (3.8) and

$$\tilde{C}(\Omega, p, q, \|u_0\|_q, \|v_0\|_q) T^{*1-\alpha} < 1, \tag{3.10}$$

we obtain that Φ is a strict contraction map from $\mathcal{K}$ to $\mathcal{K}$. Thus, Φ has a unique fixed point (u, v) in $\mathcal{K}$, which is a solution of (1.1).

Step 2. We show that the solution of (1.1) is unique in $\mathcal{E}$.

By contradiction, we suppose that there is another solution $(\tilde{u}, \tilde{v}) \in \mathcal{E}$ to the system (1.1) (with the same initial value (u_0, v_0)). Write

$$\varrho := \max\{\|(u, v)\|_{\mathcal{E}}, \|(\tilde{u}, \tilde{v})\|_{\mathcal{E}}\},$$

and

$$t_0 := \inf\{t \in [0, T^*] \mid \big(u(t), v(t)\big) \neq \big(\tilde{u}(t), \tilde{v}(t)\big)\}. \tag{3.11}$$

It is obvious that $0 \leq t_0 < T^*$. By the continuity of $\big(u, v\big)$ and $\big(\tilde{u}, \tilde{v}\big)$, we have $\big(u(t_0), v(t_0)\big) = \big(\tilde{u}(t_0), \tilde{v}(t_0)\big)$. Now, by the mean value theorem and (A_2), we have that for $t \in (t_0, T^*]$,

$$\begin{aligned}\|\big(u(t), v(t)\big) - \big(\tilde{u}(t), \tilde{v}(t)\big)\|_q &= \|\sum_{i=1,2} \int_{t_0}^t \mathrm{e}^{(t-s)\triangle}[f_i\big(u(t), v(t)\big) - f_i\big(\tilde{u}(t), \tilde{v}(t)\big)]\mathrm{d}s\|_q \\ \leq C \int_{t_0}^t (t-s)^{-\alpha}(1 + \|u\|_q^{p-1} &+ \|v\|_q^{p-1} + \|\tilde{u}\|_q^{p-1} + \|\tilde{v}\|_q^{p-1}) \cdot \|\big(u, v\big) - \big(\tilde{u}, \tilde{v}\big)\|_q \mathrm{d}s,\end{aligned}$$

where α is the number given in (3.6). After some computations, we get

$$\|\big(u, v\big) - \big(\tilde{u}, \tilde{v}\big)\|_{C([t_0,t];L^q(\Omega)^2)} \leq C(\Omega, p, q, \varrho)(t - t_0)^{1-\alpha} \|\big(u, v\big) - \big(\tilde{u}, \tilde{v}\big)\|_{C([t_0,t];L^q(\Omega)^2)},$$

where $t \in (t_0, T^*]$. Let $\epsilon > 0$ satisfy that $C(\Omega, p, q, \varrho)\epsilon^{1-\alpha} < 1$ and $t_0 + \epsilon \leq T^*$. Then the above inequality indicates that

$$\big(u(s), v(s)\big) = \big(\tilde{u}(s), \tilde{v}(s)\big), \text{ when } s \in (t_0, t_0 + \epsilon),$$

which leads to a contradiction to (3.11). Hence, uniqueness is obtained.

Step 3. We prove (3.4).

Indeed, one can easily check that $\Phi(0, 0) = (\mathrm{e}^{t\triangle}u_0, \mathrm{e}^{t\triangle}v_0)$. Taking $(\xi_1, \eta_1) = (u, v)$ (which is the solution of (1.1) obtained in Step 1), and $(\xi_2, \eta_2) = (0, 0)$ in (3.9), we have

$$\begin{aligned}\|(u, v)\|_{\mathcal{E}} &= \|\Phi(u, v)\|_{\mathcal{E}} \leq \|\Phi(0, 0)\|_{\mathcal{E}} + \|\Phi(u, v) - \Phi(0, 0)\|_{\mathcal{E}} \\ &\leq \|(u_0, v_0)\|_q + \tilde{C}(\Omega, p, q, \|u_0\|_q, \|v_0\|_q) T^{*1-\alpha} \|(u, v)\|_{\mathcal{E}},\end{aligned}$$

which, together with (3.10), leads to

$$\|(u(t), v(t))\|_q \leq \frac{1}{1 - \tilde{C}(\Omega, p, q, \|u_0\|_q, \|v_0\|_q) T^{*1-\alpha}} \|(u_0, v_0)\|_q.$$

Define four functions as follows:

$$a_i(x,t) := \begin{cases} \frac{f_i(u,v)-f_i(0,v)}{u}, & u(x,t) \neq 0, \\ \frac{\partial f_i}{\partial u}(0,v), & \text{otherwise}, \end{cases}$$

and

$$b_i(x,t) := \begin{cases} \frac{f_i(0,v)-f_i(0,0)}{v}, & v(x,t) \neq 0, \\ \frac{\partial f_i}{\partial v}(0,0), & \text{otherwise}, \end{cases}$$

where $i = 1,2$, and $t \in [0,T^*]$. It follows from the mean value theorem and (A_2) that $a_i(x,t), b_i(x,t) \in L^\infty((0,T^*); L^{\frac{q}{p-1}}(\Omega))$ $(i=1,2)$. By direct computations, we have

$$f_i(u,v) = a_i(x,t)u + b_i(x,t)v, \;\; i = 1,2.$$

Then, taking $\gamma = q$ and $\sigma = \frac{q}{p-1}$ in Lemma 2.2, we obtain that for $t \in (0,T^*]$,

$$\|\big(u(t),v(t)\big)\|_\infty \leq C\mathrm{e}^{CL^\beta t} t^{-\frac{n}{2q}} \|(u_0,v_0)\|_q, \tag{3.12}$$

where $L = \sum_{i=1,2} \big(\|a_i\|_{L^\infty(0,T;L^\sigma(\Omega))} + \|b_i\|_{L^\infty(0,T;L^\sigma(\Omega))}\big)$ and $\beta = \frac{2\sigma}{2\sigma-n}$. This shows the solution $(u,v) \in L^\infty_{loc}((0,T^*); L^\infty(\Omega)^2)$. In summary, we get (3.4). Hence, we complete the proof. □

4. Analysis of an auxiliary system

Throughout this section, we suppose that (A_1) holds, and $(u_i^0, v_i^0) \in L^\infty(\Omega)^2$ $(i=1,2)$. Let $T > 0$, and let $(u_i,v_i) \in L^\infty((0,T); L^\infty(\Omega)^2)$ $(i=1,2)$ be the solution to the system (1.1) corresponding to the initial value (u_i^0, v_i^0). Write $(\phi,\varphi) = (u_1-u_2, v_1-v_2)$.

It is obvious that (ϕ,φ) solves the following equation:

$$\begin{cases} \partial_t\phi - \triangle\phi + F_1 = 0, & \text{in } \Omega\times(0,T], \\ \partial_t\varphi - \triangle\varphi + F_2 = 0, & \text{in } \Omega\times(0,T], \\ \phi(x,t) = 0,\; \varphi(x,t) = 0, & \text{on } \partial\Omega\times(0,T], \\ \phi(x,0) = u_1^0(x) - u_2^0(x), & \text{in } \Omega, \\ \varphi(x,0) = v_1^0(x) - v_2^0(x), & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $F_i = f_i(u_1,v_1) - f_i(u_2,v_2)$ $(i=1,2)$. It follows from (A_1) that

$$|F_i| \leq L_M(|\phi| + |\varphi|), \;\; i = 1,2, \tag{4.2}$$

where L_M is the Lipschitz constant of f_i $(i=1,2)$ on the bounded domain $\mathcal{D}_M$ (see Thm. 1.1).

4.1. Some estimates for (4.1)

We present some estimates for the solution of (4.1) in this subsection. First, we have

Lemma 4.1. *When $t \in (0,T]$, the following estimates are valid:*

$$\int_\Omega (|\phi|^2 + |\varphi|^2)(x,t)\mathrm{d}x \leq \mathrm{e}^{4L_M t} \int_\Omega (|\phi|^2 + |\varphi|^2)(x,0)\mathrm{d}x, \tag{4.3}$$

and

$$\int_\Omega (|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x+\int_\Omega (|\nabla\phi|^2+|\nabla\varphi|^2)(x,t)\mathrm{d}x \le L_1(1+\frac{1}{t})\mathbb{E}, \tag{4.4}$$

where $L_1 := 1+4L_M+2L_M^2T$ *(*L_M *is the constant given in (4.2)) and*

$$\mathbb{E} := \int_\Omega \big(|\phi|^2+|\varphi|^2\big)(x,0)\mathrm{d}x+\int_0^T\int_\Omega \big(|\phi|^2+|\varphi|^2\big)(x,s)\mathrm{d}x\mathrm{d}s. \tag{4.5}$$

Proof. Multiplying the first equation and the second equation of (4.1) by ϕ and φ respectively, and integrating them over Ω, we obtain when $t\in(0,T]$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x+\int_\Omega (|\nabla\phi|^2+|\nabla\varphi|^2)(x,t)\mathrm{d}x=-\int_\Omega (F_1\phi+F_2\varphi)\mathrm{d}x\mathrm{d}s.$$

It, together with (4.2), indicates

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\phi|^2+|\varphi|^2)\mathrm{d}x+\int_\Omega (|\nabla\phi|^2+|\nabla\varphi|^2)\mathrm{d}x\le 2L_M\int_\Omega (|\phi|^2+|\varphi|^2)\mathrm{d}x. \tag{4.6}$$

The estimate (4.3) can be deduced from (4.6).

Integrating (4.6) over $[0,t]$, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega (|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x+\int_0^t\int_\Omega (|\nabla\phi|^2+|\nabla\varphi|^2)\mathrm{d}x\mathrm{d}s\\ &\le 2L_M\int_0^t\int_\Omega (|\phi|^2+|\varphi|^2)\mathrm{d}x\mathrm{d}s+\frac{1}{2}\int_\Omega (|\phi|^2+|\varphi|^2)(x,0)\mathrm{d}x.\end{aligned} \tag{4.7}$$

Multiplying the first equation and the second equation of (4.1) by $t\partial_t\phi$ and $t\partial_t\varphi$ respectively, and integrating them over Ω, we obtain

$$\begin{aligned}&t\int_\Omega (|\partial_t\phi|^2+|\partial_t\varphi|^2)\mathrm{d}x-\int_\Omega (t\partial_t\phi\triangle\phi+t\partial_t\varphi\triangle\varphi)\mathrm{d}x\\ &=-\int_\Omega (t\partial_t\phi\cdot F_1+t\partial_t\varphi\cdot F_2)\mathrm{d}x.\end{aligned}$$

It follows from the integration by parts formula that

$$\begin{aligned}&t\int_\Omega (|\partial_t\phi|^2+|\partial_t\varphi|^2)\mathrm{d}x+\frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\nabla\phi|^2+|\nabla\varphi|^2)\mathrm{d}x\\ &\le\int_\Omega (t|\partial_t\phi|\cdot|F_1|+t|\partial_t\varphi|\cdot|F_2|)\mathrm{d}x.\end{aligned}$$

Combining the above with the Cauchy-Schwarz inequality yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[t\int_\Omega (|\nabla\phi|^2+|\nabla\varphi|^2)\mathrm{d}x]\le\frac{1}{2}\int_\Omega (|\nabla\phi|^2+|\nabla\varphi|^2)\mathrm{d}x+\frac{1}{4}\int_\Omega t(|F_1|^2+|F_2|^2)\mathrm{d}x.$$

Integrating it over $[0,t]$, we have

$$t\int_{\Omega}(|\nabla\phi|^2+|\nabla\varphi|^2)\mathrm{d}x \le \int_0^t\int_{\Omega}(|\nabla\phi|^2+|\nabla\varphi|^2)\mathrm{d}x\mathrm{d}s+\frac{1}{2}\int_0^t\int_{\Omega}s(|F_1|^2+|F_2|^2)\mathrm{d}x\mathrm{d}s.$$

It, together with (4.2) and (4.7), indicates

$$\begin{aligned} t\int_{\Omega}(|\nabla\phi|^2+|\nabla\varphi|^2)(x,t)\mathrm{d}x \le{}& 2(L_M+L_M^2T)\int_0^t\int_{\Omega}(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x\mathrm{d}s \\ &+\frac{1}{2}\int_{\Omega}(|\phi|^2+|\varphi|^2)(x,0)\mathrm{d}x. \end{aligned} \tag{4.8}$$

Combining (4.8) with (4.7) leads to (4.4). Hence, we complete the proof. □

Remark 4.2. Using the semigroup method (see Sect. 4.3 of [22]) and (4.2), we can also obtain that the solution $(\phi,\varphi)\in C([0,T];L^2(\Omega)^2)$ in this case.

Second, we introduce a localization lemma for the solution (ϕ,φ) of (4.1). Then, we have

Proposition 4.3. *Let $x_0\in\Omega$, $R>0$, and $\delta\in(0,1]$. If $\int_{\Omega\cap B_R}\big(|\phi|^2+|\varphi|^2\big)(x,T)\mathrm{d}x\neq 0$, where $B_R:=B(x_0,R)$, then there are positive constants L_2,L_3,L_4, and L_5 (depending only on R, and δ) such that*

$$0<\frac{\mathbb{E}}{\int_{\Omega\cap B_{(1+\delta)R}}\big(|\phi|^2+|\varphi|^2\big)(x,t)\mathrm{d}x}\le \mathrm{e}^{\frac{L_2}{\theta}},\quad \textit{when } T-\theta\le t\le T, \tag{4.9}$$

where $\mathbb{E}$ is given in (4.5), $B_{(1+\delta)R}:=B(x_0,(1+\delta)R)$, and

$$\frac{1}{\theta}:=L_3\ln\left(\mathrm{e}^{L_4L_MT+L_5(1+\frac{1}{T})}\frac{\mathbb{E}}{\int_{\Omega\cap B_R}\big(|\phi|^2+|\varphi|^2\big)(x,T)\mathrm{d}x}\right), \tag{4.10}$$

with $0<\theta<\min\{1,T/2\}$.

Proof. Write $R_1:=(1+\delta)R$. Let $\sigma_1\in C_0^\infty(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\sigma_1\subset B_{R_1},\ 0\le\sigma_1\le 1,\ \text{and } \sigma_1=1 \text{ on } B_{(1+\delta/2)R}. \tag{4.11}$$

Then, there is a positive number $C=C(R,\delta)>0$ such that

$$\begin{cases} |\nabla\sigma_1(x)|\le C(R,\delta),\ x\in\mathbb{R}^n; \\ \nabla\sigma_1(x)=0,\ \ x\in B_{(1+\delta/2)R}. \end{cases} \tag{4.12}$$

Multiplying the first equation of (4.1) by $\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2\phi$, where $h>0$ will be determined later, integrating it over $\Omega\cap B_{R_1}$, and then using the integration by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_1}}|\sigma_1\phi|^2\cdot\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\mathrm{d}x+\int_{\Omega\cap B_{R_1}}\nabla\phi\cdot\nabla(\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2\phi)\mathrm{d}x \\ &+\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2\phi\cdot F_1\mathrm{d}x=0,\quad t\in[0,T]. \end{aligned}$$

Meanwhile, we can directly check

$$\nabla(\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2\phi)=\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2\nabla\phi+2\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\phi\sigma_1\nabla\sigma_1-2\sigma_1^2\phi\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\cdot\frac{x-x_0}{h}.$$

These lead to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\cdot|\sigma_1\phi|^2\mathrm{d}x+\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2|\nabla\phi|^2\mathrm{d}x\\&=-\int_{\Omega\cap B_{R_1}}2\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\phi\sigma_1\nabla\sigma_1\cdot\nabla\phi\mathrm{d}x+\int_{\Omega\cap B_{R_1}}2\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2\phi\frac{x-x_0}{h}\cdot\nabla\phi\mathrm{d}x\\&-\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2\phi\cdot F_1\mathrm{d}x,\ \ t\in[0,T].\end{aligned}$$

It, along with the Cauchy-Schwarz inequality and (4.2), yields when $t\in[0,T]$,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\cdot|\sigma_1\phi|^2\mathrm{d}x+\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2|\nabla\phi|^2\mathrm{d}x\\&\le C_1(L_M+\frac{R_1^2}{h^2})\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2(|\phi|^2+|\varphi|^2)\mathrm{d}x+\frac{1}{2}\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2|\nabla\phi|^2\mathrm{d}x\\&+4\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}|\nabla\sigma_1|^2|\phi|^2\mathrm{d}x,\end{aligned}$$

where C_1 is a positive number. Moving the term $\frac{1}{2}\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2|\nabla\phi|^2\mathrm{d}x$ to the left-hand side in the above inequality, using (4.12), we deduce when $t\in[0,T]$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\cdot|\sigma_1\phi|^2\mathrm{d}x\\&\le C_1(L_M+\frac{R_1^2}{h^2})\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2(|\phi|^2+|\varphi|^2)\mathrm{d}x+8\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}|\nabla\sigma_1|^2|\phi|^2\mathrm{d}x\\&\le C_1(L_M+\frac{R_1^2}{h^2})\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2(|\phi|^2+|\varphi|^2)\mathrm{d}x+C_2\mathrm{e}^{-\frac{(1+\delta/2)^2R^2}{h}}\int_{\Omega\cap B_{R_1}}|\phi|^2\mathrm{d}x,\end{aligned}$$

where $C_2>1$ is a constant depending only on R and δ. Similarly, it follows from the second equation of (4.1) that,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\cdot|\sigma_1\varphi|^2\mathrm{d}x\le C_1(L_M+\frac{R_1^2}{h^2})\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2(|\phi|^2+|\varphi|^2)\mathrm{d}x\\&+C_2\mathrm{e}^{-\frac{(1+\delta/2)^2R^2}{h}}\int_{\Omega\cap B_{R_1}}|\varphi|^2\mathrm{d}x,\ \ t\in[0,T].\end{aligned}$$

In summary, we have when $t\in[0,T]$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\cdot\sigma_1^2(|\phi|^2+|\varphi|^2)\mathrm{d}x\le C_1(L_M+\frac{R_1^2}{h^2})\int_{\Omega\cap B_{R_1}}\mathrm{e}^{-\frac{|x-x_0|^2}{h}}\sigma_1^2(|\phi|^2+|\varphi|^2)\mathrm{d}x\\&+C_2\mathrm{e}^{-\frac{(1+\delta/2)^2R^2}{h}}\int_{\Omega\cap B_{R_1}}(|\phi|^2+|\varphi|^2)\mathrm{d}x.\end{aligned}$$

Solving this inequality, we obtain when $t \in [0,T]$,

$$\begin{aligned}
&\int_{\Omega\cap B_{R_1}} \mathrm{e}^{-\frac{|x-x_0|^2}{h}} \cdot \sigma_1^2[|\phi|^2+|\varphi|^2](x,T)\mathrm{d}x \\
&\le \exp\big(C_1(L_M+\frac{R_1^2}{h^2})(T-t)\big)\int_{\Omega\cap B_{R_1}} \mathrm{e}^{-\frac{|x-x_0|^2}{h}} \cdot \sigma_1^2(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x \\
&+ C_2\mathrm{e}^{-\frac{(1+\delta/2)^2R^2}{h}} \exp\big(C_1(L_M+\frac{R_1^2}{h^2})(T-t)\big)\int_t^T\int_{\Omega\cap B_{R_1}} (|\phi|^2+|\varphi|^2)(x,s)\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

from which and (4.11), it deduces when $t \in [0,T]$,

$$\begin{aligned}
&\int_{\Omega\cap B_{R}} [|\phi|^2+|\varphi|^2](x,T)\mathrm{d}x \\
&\le \mathrm{e}^{\frac{R^2}{h}} \exp\big(C_1(L_M+\frac{R_1^2}{h^2})(T-t)\big)\int_{\Omega\cap B_{R_1}} \mathrm{e}^{-\frac{|x-x_0|^2}{h}} \cdot \sigma_1^2(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x \\
&+ C_2\mathrm{e}^{\frac{-(1+\delta/2)^2R^2+R^2}{h}} \exp\big(C_1(L_M+\frac{R_1^2}{h^2})(T-t)\big)\int_t^T\int_{\Omega\cap B_{R_1}} (|\phi|^2+|\varphi|^2)(x,s)\mathrm{d}x\mathrm{d}s. \qquad (4.13)
\end{aligned}$$

Now, we let

$$\ell := \frac{\delta+\delta^2/4}{2C_1(1+\delta)^2}. \tag{4.14}$$

Then, we have

$$\frac{C_1\ell R_1^2}{h} = \frac{(1+\delta/2)^2R^2-R^2}{2h}. \tag{4.15}$$

Since $\int_{\Omega\cap B_R}[|\phi|^2+|\varphi|^2](x,T)\mathrm{d}x \neq 0$, we have $\big(\phi(\cdot,T),\varphi(\cdot,T)\big) \neq (0,0)$. This, along with the continuity of (ϕ,φ), shows that

$$\mathbb{E} > 0. \tag{4.16}$$

Next, we take

$$h := \frac{R^2(\delta+\delta^2/4)/2}{\ln\left(Q\frac{C_2\exp\big((C_1+4)L_MT\big)\mathbb{E}}{\frac{1}{e}\int_{\Omega\cap B_R}[|\phi|^2+|\varphi|^2](x,T)\mathrm{d}x}\right)}, \tag{4.17}$$

where $Q = \mathrm{e}^{(R^2(\delta+\delta^2/4)/2)(\frac{2}{T}+1)\ell}$. One can directly check from (4.14), (4.17), and $C_2 > 1$ that $0 < \ell h < \min\{1, \frac{T}{2}\}$. Using (4.13) and (4.15), we find when $\frac{T}{2} < T-\ell h \le t \le T$,

$$\begin{aligned}
\int_{\Omega\cap B_{R}} (|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x &\le \mathrm{e}^{\frac{(1+\delta/2)^2R^2+R^2}{2h}} \exp\big(C_1L_M\ell h\big)\int_{\Omega\cap B_{R_1}} \sigma_1^2(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x \\
&+ C_2\mathrm{e}^{\frac{-(1+\delta/2)^2R^2+R^2}{2h}} \exp\big(C_1L_M\ell h\big)\mathbb{E}. \qquad (4.18)
\end{aligned}$$

After some computations,

$$C_2 \mathrm{e}^{\frac{-(1+\delta/2)^2R^2+R^2}{2h}} \exp\left(C_1 L_M \ell h\right)\mathbb{E} \leq \frac{1}{e}\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x. \tag{4.19}$$

Then by (4.18), and (4.19), we obtain when $\frac{T}{2} < T-\ell h \leq t \leq T$,

$$(1-\frac{1}{e})\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x \leq \mathrm{e}^{\frac{(1+\delta/2)^2R^2+R^2}{2h}} \exp\left(C_1 L_M \ell h\right)\int_{\Omega\cap B_{R_1}}(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x,$$

which, along with (4.16), (4.19), and $C_2>1$, leads to

$$0<\mathbb{E}\leq \mathrm{e}^{\frac{(1+\delta/2)^2R^2}{h}}\int_{\Omega\cap B_{R_1}}(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x,\ \ \frac{T}{2}<T-\ell h\leq t\leq T. \tag{4.20}$$

Let $\theta := \ell h$. Then it follows from (4.14), (4.17), and (4.20),

$$0<\frac{\mathbb{E}}{\int_{\Omega\cap B_{R_1}}(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x}\leq \mathrm{e}^{\frac{(1+\delta/2)^2R^2}{h}} = \mathrm{e}^{\frac{(\delta+\delta^2/4)(1+\delta/2)^2R^2}{2C_1(1+\delta)^2}\frac{1}{\theta}};$$

$$\frac{1}{\theta}=\frac{4C_1(1+\delta)^2}{R^2(\delta+\delta^2/4)^2}\ln\left(eC_2\mathrm{e}^{(C_1+4)L_MT}\mathrm{e}^{(\frac{2}{T}+1)\frac{R^2(\delta+\delta^2/2)^2}{4C_1(1+\delta)^2}}\frac{\mathbb{E}}{\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x}\right),$$

where $\frac{T}{2}<T-\theta\leq t\leq T$. These lead to (4.9) and (4.10). Thus, we complete the proof. □

Remark 4.4. Proposition 4.3 shows that if $\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\neq 0$, then $\int_{\Omega\cap B_{(1+\delta)R}}(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x\neq 0$, for $T-\theta\leq t\leq T$, where θ is the number given in (4.10).

4.2. A local interpolation inequality

In this subsection, we will establish a local interpolation inequality for the solution of (4.1).

Theorem 4.5. *Let $x_0\in\Omega$, $R>0$, and $\delta\in(0,1]$. If $\Omega\cap B_{(1+2\delta)R}$ is star-shaped with x_0, where $B_{(1+2\delta)R}:=B(x_0,(1+2\delta)R)$, then for each $r\in(0,R)$ with $B_r:=B(x_0,r)\subset\Omega$, there are two constants $C=C(R,r,\delta)>0$ and $\alpha=\alpha(R,r,\delta)\in(0,1)$ such that*

$$\begin{aligned}\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\leq \mathrm{e}^{C(1+L_M^2)(\frac{1}{T}+T)}\Big(\int_\Omega(|\phi|^2+|\varphi|^2)(x,0)\mathrm{d}x\Big)^{1-\alpha}&\\ \times\Big(\int_{B_r}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\Big)^{\alpha},&\end{aligned}\tag{4.21}$$

where (ϕ,φ) is the solution of (4.1).

Proof. Without loss of generality, we assume that $\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\neq 0$. The proof of (4.21) will be split into three steps.

Step 1. We first make a localization process.

Write $R_0=(1+2\delta)R$. Let $\sigma_0\in C_0^\infty(\mathbb{R}^n)$ satisfy that

$$\text{supp}\ \sigma_0\subset B_{R_0},\ 0\le\sigma_0\le 1,\ \text{ and }\ \sigma_0=1\ \text{ on }\ B_{(1+3\delta/2)R}. \tag{4.22}$$

Then, there exists a positive constant $C=C(R,\delta)$ such that

$$|\triangle\sigma_0(x)|\le C(R,\delta),\ \text{and}\ |\nabla\sigma_0(x)|\le C(R,\delta),\ \ \text{for}\ x\in\mathbb{R}^n. \tag{4.23}$$

Write $(\chi,\psi)=(\sigma_0\phi,\sigma_0\varphi)$. By simple computations,

$$\begin{cases}\partial_t\chi-\triangle\chi=H_1-\sigma_0F_1, & \text{in }\Omega\times(0,T];\\ \partial_t\psi-\triangle\psi=H_2-\sigma_0F_2, & \text{in }\Omega\times(0,T],\end{cases} \tag{4.24}$$

where $H_1:=-2\nabla\phi\cdot\nabla\sigma_0-\phi\cdot\triangle\sigma_0$ and $H_2:=-2\nabla\varphi\cdot\nabla\sigma_0-\varphi\cdot\triangle\sigma_0$. It is obvious that

$$H_i(x,t)=0,\ \ \text{for}\ x\in B_{(1+3\delta/2)R},\ t\in[0,T]\ (i=1,2). \tag{4.25}$$

Using the Cauchy-Schwarz inequality and (4.23), we obtain

$$\begin{aligned}\int_{\Omega\cap B_{R_0}}|H_1(x,t)|^2\mathrm{d}x&\le\int_{\Omega\cap B_{R_0}}(|\nabla\phi|^2+|\phi|^2)(x,t)\cdot(4|\nabla\sigma_0|^2+|\triangle\sigma_0|^2)(x,t)\mathrm{d}x\\ &\le C_1\int_{\Omega\cap B_{R_0}}(|\nabla\phi|^2+|\phi|^2)(x,t)\mathrm{d}x\end{aligned} \tag{4.26}$$

where C_1 is a positive number only depending on (R,δ). Similarly, we also have

$$\int_{\Omega\cap B_{R_0}}|H_2(x,t)|^2\mathrm{d}x\le C_1\int_{\Omega\cap B_{R_0}}(|\nabla\varphi|^2+|\varphi|^2)(x,t)\mathrm{d}x.$$

This, along with (4.26) and (4.4), implies that

$$\int_{\Omega\cap B_{R_0}}(|H_1(x,t)|^2+|H_2(x,t)|^2)\mathrm{d}x\le C_1L_1(1+t^{-1})\mathbb{E}, \tag{4.27}$$

where the constant L_1 and $\mathbb{E}$ are given in Lemma 4.1.

Taking $(y,z)=(\chi,\psi)$, and $\bar{R}=R_0$ in (2.6), we claim that $\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)(x,t)\cdot G_\lambda(x,t)\mathrm{d}x\neq 0$, when $t\in[T-\theta,T]$, (where θ is the positive number given in Prop. 4.3). Indeed, by (4.22), we can easily find

$$\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)(x,t)\cdot G_\lambda(x,t)\mathrm{d}x\ge\int_{\Omega\cap B_{(1+\delta)R}}(|\phi|^2+|\varphi|^2)(x,t)\cdot G_\lambda(x,t)\mathrm{d}x \tag{4.28}$$

Then, by Remark 4.4 and $\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\neq 0$, we have when $t\in[T-\theta,T]$,

$$\begin{aligned}&\int_{\Omega\cap B_{(1+\delta)R}}(|\phi|^2+|\varphi|^2)(x,t)\cdot G_\lambda(x,t)\mathrm{d}x\\ \ge&\frac{1}{(T-t+\lambda)^{n/2}}\mathrm{e}^{-\frac{(1+\delta)^2R^2}{4(T-t+\lambda)}}\int_{\Omega\cap B_{(1+\delta)R}}(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x>0.\end{aligned}$$

It, together with (4.28), shows that our claim is valid. Thus, we can define the frequency function $N_\lambda(t)$ when $t\in[T-\theta,T]$. Using Lemma 2.4, we obtain when $t\in[T-\theta,T]$,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x+N_\lambda(t)\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x\\ =&\int_{\Omega\cap B_{R_0}}\big[\chi(\partial_t\chi-\triangle\chi)+\psi(\partial_t\psi-\triangle\psi)\big]\cdot G_\lambda\mathrm{d}x,\end{aligned}\tag{4.29}$$

and

$$\frac{\mathrm{d}N_\lambda}{\mathrm{d}t}(t)\le\frac{1}{T-t+\lambda}N_\lambda(t)+\frac{\int_{\Omega\cap B_{R_0}}\big[(\partial_t\chi-\triangle\chi)^2+(\partial_t\psi-\triangle\psi)^2\big]G_\lambda\mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)G_\lambda\mathrm{d}x}.\tag{4.30}$$

It follows from (4.24), and (4.29) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x+N_\lambda(t)\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x\\ =&\int_{\Omega\cap B_{R_0}}\big[\chi(H_1-\sigma_0F_1)+\psi(H_2-\sigma_0F_2)\big]\cdot G_\lambda\mathrm{d}x,\end{aligned}$$

This, together with the Cauchy-Schwarz inequality, indicates when $t\in[T-\theta,T]$

$$\begin{aligned}&|\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x+N_\lambda(t)\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x|\\ \le&\frac{1}{2}\int_{\Omega\cap B_{R_0}}\big[(|H_1|^2+|H_2|^2)+\sigma_0^2(|F_1|^2+|F_2|^2)\big]G_\lambda\mathrm{d}x+\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)G_\lambda\mathrm{d}x.\end{aligned}\tag{4.31}$$

It follows from (4.24), (4.30), and the Cauchy-Schwarz inequality when $t\in[T-\theta,T]$

$$\frac{\mathrm{d}N_\lambda}{\mathrm{d}t}(t)\le\frac{1}{T-t+\lambda}N_\lambda(t)+\frac{2\int_{\Omega\cap B_{R_0}}\big[(|H_1|^2+|H_2|^2)+\sigma_0^2(|F_1|^2+|F_2|^2)\big]G_\lambda\mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)G_\lambda(x,t)\mathrm{d}x}.\tag{4.32}$$

Step 2. Let θ be the positive number given in (4.10) and let parameter $\varepsilon\in(0,\theta)$, which will be determined later. We will estimate the terms $\frac{\int_{\Omega\cap B_{R_0}}\sigma_0^2(|F_1|^2+|F_2|^2)G_\lambda\mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)G_\lambda(x,t)\mathrm{d}x}$, and $\frac{\int_{\Omega\cap B_{R_0}}(|H_1|^2+|H_2|^2)\cdot G_\lambda(x,t)\mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda\mathrm{d}x}$, when $t\in[T-\varepsilon,T](\subseteq[T-\theta,T])$.

We first note that

$$t^{-1}\le\frac{2}{T},\ \text{ when } t\in[T-\varepsilon,T].\tag{4.33}$$

Using (4.25) and (4.28), we obtain when $t\in[T-\varepsilon,T]$,

$$\frac{\int_{\Omega\cap B_{R_0}}(|H_1|^2+|H_2|^2)\cdot G_\lambda\mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda\mathrm{d}x}\le\frac{\int_{\Omega\cap(B_{R_0}\setminus B_{(1+3\delta/2)R})}(|H_1|^2+|H_2|^2)\cdot G_\lambda\mathrm{d}x}{\int_{\Omega\cap B_{(1+\delta)R}}(|\chi|^2+|\psi|^2)\cdot G_\lambda\mathrm{d}x},$$

from which it follows that

$$\frac{\int_{\Omega\cap B_{R_0}}(|H_1|^2+|H_2|^2)\cdot G_\lambda \mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x} \le \frac{\int_{\Omega\cap(B_{R_0}\setminus B_{(1+3\delta/2)R})}(|H_1|^2+|H_2|^2)\mathrm{d}x}{\int_{\Omega\cap B_{(1+\delta)R}}(|\chi|^2+|\psi|^2)\mathrm{d}x}\mathrm{e}^{-\frac{L_6}{T-t+\lambda}},$$

where $L_6 := -\frac{(1+\delta)^2R^2}{4}+\frac{(1+3\delta/2)^2R^2}{4}$. Then, by (4.27), we obtain

$$\frac{\int_{\Omega\cap B_{R_0}}(|H_1|^2+|H_2|^2)\cdot G_\lambda \mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x} \le \frac{C_1L_1(1+t^{-1})\mathbb{E}}{\int_{\Omega\cap B_{(1+\delta)R}}(|\phi|^2+|\varphi|^2)(x,t)\mathrm{d}x}\mathrm{e}^{-\frac{L_6}{T-t+\lambda}}.$$

This, along with Proposition 4.3 and (4.33), implies when $t\in[T-\varepsilon,T]$,

$$\frac{\int_{\Omega\cap B_{R_0}}(|H_1|^2+|H_2|^2)\cdot G_\lambda \mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x} \le 2C_1L_1(1+\frac{1}{T})\mathrm{e}^{\frac{L_2}{\theta}}\mathrm{e}^{-\frac{L_6}{\varepsilon+\lambda}}. \tag{4.34}$$

Let

$$\varepsilon = k\theta, \text{ and } \lambda = \mu\varepsilon, \tag{4.35}$$

where $\mu\in(0,1)$ will be determined later, and

$$k := \min\{\frac{L_6}{2L_2}, \frac{1}{2}\}. \tag{4.36}$$

After some computations, we obtain $L_2 - \frac{L_6}{k(1+\mu)} < 0$. Thus,

$$2C_1L_1(1+\frac{1}{T})\mathrm{e}^{\frac{L_2}{\theta}}\mathrm{e}^{-\frac{L_6}{\varepsilon+\lambda}} \le 2C_1L_1(1+\frac{1}{T}).$$

This, together with (4.34) and $0<\varepsilon<\theta<\min\{1,\frac{T}{2}\}$, indicates

$$\frac{\int_{\Omega\cap B_{R_0}}(|H_1|^2+|H_2|^2)\cdot G_\lambda \mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x} \le 2C_1L_1(1+\frac{1}{T}). \tag{4.37}$$

Next, by (4.2), we can get that

$$\frac{\int_{\Omega\cap B_{R_0}}\sigma_0^2(|F_1|^2+|F_2|^2)G_\lambda \mathrm{d}x}{\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)G_\lambda \mathrm{d}x} \le 4L_M^2. \tag{4.38}$$

Step 3. We prove (4.21).

It, together with (4.31), (4.37), and (4.38), indicates when $t\in[T-\varepsilon,T]$

$$\begin{aligned}&|\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x + N_\lambda(t)\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)\cdot G_\lambda \mathrm{d}x|\\ &\le \big(C_1L_1(1+\frac{1}{T})+2L_M^2+1\big)\int_{\Omega\cap B_{R_0}}(|\chi|^2+|\psi|^2)G_\lambda \mathrm{d}x.\end{aligned}$$

Using (4.32), (4.37), and (4.38), we obtain when $t \in [T-\varepsilon, T]$

$$\frac{\mathrm{d}N_\lambda}{\mathrm{d}t}(t) \le \frac{1}{T-t+\lambda} N_\lambda(t) + 4C_1L_1(1+\frac{1}{T}) + 8L_M^2.$$

Therefore, when $t \in [T-\varepsilon, T]$, the following differential inequalities hold:

$$\begin{cases} |\frac{1}{2}f'(t) + N_\lambda(t)f(t)| \le \big(\hat{C}+1\big)f(t); \\ N_\lambda'(t) \le \frac{1}{T-t+\lambda}N_\lambda(t) + 4\hat{C}, \end{cases} \tag{4.39}$$

where $f(t) = \int_{\Omega\cap B_{R_0}} (|\chi|^2 + |\psi|^2)(x,t)\cdot G_\lambda(x,t)\mathrm{d}x$ and $\hat{C} = C_1L_1(1+\frac{1}{T}) + 2L_M^2$. Using the fact $L_1 = 1 + 4L_M + 2L_M^2 T$ and the Cauchy-Schwarz inequality, we obtain

$$\hat{C} \le C_2(1+L_M^2)(\frac{1}{T} + T), \tag{4.40}$$

where C_2 is a positive number only depending on (R, δ, M).

Take a positive number l satisfying that

$$r^2(1+l) = 2R_0^2(1+D_l), \tag{4.41}$$

where

$$D_l = \frac{\ln(l+1)}{\ln(\frac{2l+1}{l+1})}. \tag{4.42}$$

It is easy to check from (4.41) that the number l only depends on (R, r, δ) and $l > 1$. Then, *via* tuning parameters $\mu \in (0,1)$ in (4.35), we can choose $\lambda > 0$ such that

$$\begin{aligned} \frac{r^2}{8\lambda} &= \ln 2 + K_l + 8(1+D_l)L_M T + \frac{(1+D_l)}{\theta}(L_2 + \frac{2R_0^2}{k}) \\ &\quad + (1+D_l)\ln\Big(\mathrm{e}^{L_4L_MT + L_5(1+\frac{1}{T})}\frac{\mathbb{E}}{\int_{\Omega\cap B_R}(|\phi|^2 + |\varphi|^2)(x,T)\mathrm{d}x}\Big), \end{aligned} \tag{4.43}$$

where $\mathbb{E}$ is given in (4.5), positive numbers L_2, L_4, L_5 are given in Proposition 4.3, and

$$K_l = 2D_l\big[(\hat{C}+1)l\lambda + 4\hat{C}(l\lambda)^2\big] + 2(\hat{C}+1)l\lambda + 8\hat{C}l(l+1)\lambda^2\cdot\ln(l+1). \tag{4.44}$$

After some computations, we have $2l\lambda < \varepsilon < \theta$. These show

$$\mathrm{e}^{-\frac{r^2}{8\lambda}}\cdot\mathrm{e}^{K_l}\cdot\mathrm{e}^{8(1+D_l)L_MT}\cdot\mathrm{e}^{\frac{(1+D_l)}{\theta}(L_2+\frac{2R_0^2}{k})}\cdot\mathbb{E}^{1+D_l} \le \frac{1}{2}\Big(\int_{\Omega\cap B_R}(|\phi|^2 + |\varphi|^2)(x,T)\mathrm{d}x\Big)^{1+D_l}. \tag{4.45}$$

It follows from (4.40), (4.42), (4.44), $l > 1$, and $2l\lambda < \theta < \min\{1, T/2\}$ that

$$\mathrm{e}^{K_l} \le \mathrm{e}^{C_3(1+L_M^2)(\frac{1}{T}+T)}, \tag{4.46}$$

where C_3 is a positive number only depending on (R, r, δ).

Next, we choose $t_3 = T$, $t_2 = T - l\lambda$, and $t_1 = T - 2l\lambda$. Clearly, $T - \varepsilon < t_1 < t_2 < t_3 = T$. Using (4.39) and Lemma 2.5, we obtain that

$$f(t_2)^{1+D_l} \leq f(t_3) \cdot f(t_1)^{D_l} \cdot \mathrm{e}^{K_l}. \tag{4.47}$$

Meanwhile, by (4.3) and (4.22), we observe

$$f(t_1) = \int_{\Omega \cap B_{R_0}} (|\chi|^2 + |\psi|^2)(x, t_1) \cdot \mathrm{e}^{-\frac{|x-x_0|^2}{4\lambda(2l+1)}} \mathrm{d}x \leq \mathrm{e}^{4L_M T} \mathbb{E};$$

$$f(t_3) = \int_{\Omega \cap B_{R_0}} (|\chi|^2 + |\psi|^2)(x, T) \cdot \mathrm{e}^{-\frac{|x-x_0|^2}{4\lambda}} \mathrm{d}x \leq \int_{B_r} (|\phi|^2 + |\varphi|^2)(x, T) \mathrm{d}x + \mathrm{e}^{-\frac{r^2}{4\lambda}} \mathrm{e}^{4L_M T} \mathbb{E}.$$

These, together with (4.47), indicate that

$$\begin{aligned} \Big[\int_{\Omega \cap B_{R_0}} (|\chi|^2 + |\psi|^2)(x, t_2) \mathrm{d}x \Big]^{1+D_l} &\leq \mathrm{e}^{\frac{R_0^2(1+D_l)}{4\lambda(l+1)}} \cdot \mathrm{e}^{K_l} \cdot \mathrm{e}^{4L_M T} \\ &\quad \times \Big(\int_{B_r} (|\phi|^2 + |\varphi|^2)(x, T) \mathrm{d}x + \mathrm{e}^{-\frac{r^2}{4\lambda}} \mathbb{E} \Big) \cdot \mathrm{e}^{4D_l L_M T} \cdot \mathbb{E}^{D_l}. \end{aligned}$$

Applying (4.41), we obtain

$$\begin{aligned} \Big[\int_{\Omega \cap B_{R_0}} (|\chi|^2 + |\psi|^2)(x, t_2) \mathrm{d}x \Big]^{1+D_l} &\leq \mathrm{e}^{K_l} \cdot \mathrm{e}^{4(1+D_l) L_M T} \\ &\quad \times \Big(\mathrm{e}^{\frac{r^2}{8\lambda}} \int_{B_r} (|\phi|^2 + |\varphi|^2)(x, T) \mathrm{d}x + \mathrm{e}^{-\frac{r^2}{8\lambda}} \mathbb{E} \Big) \cdot \mathbb{E}^{D_l}. \end{aligned} \tag{4.48}$$

It follows from (4.3) and Proposition 4.3 that

$$\begin{aligned} \int_{\Omega \cap B_R} (|\phi|^2 + |\varphi|^2)(x, T) \mathrm{d}x &\leq \mathrm{e}^{4L_M T} \cdot \mathbb{E} \\ &\leq \mathrm{e}^{4L_M T} \cdot \mathrm{e}^{\frac{1}{\theta}(L_2 + \frac{2R_0^2}{k})} \cdot \int_{\Omega \cap B_{(1+\delta)R}} \big(|\phi|^2 + |\varphi|^2 \big)(x, t_2) \mathrm{d}x. \end{aligned}$$

This, along with (4.48) and (4.22), yields

$$\begin{aligned} \Big[\int_{\Omega \cap B_R} (|\phi|^2 + |\varphi|^2)(x, T) \mathrm{d}x \Big]^{1+D_l} &\leq \mathrm{e}^{K_l} \cdot \mathrm{e}^{8(1+D_l) L_M T} \cdot \mathrm{e}^{\frac{(1+D_l)}{\theta}(L_2 + \frac{2R_0^2}{k})} \\ &\quad \times \Big(\mathrm{e}^{\frac{r^2}{8\lambda}} \int_{B_r} (|\phi|^2 + |\varphi|^2)(x, T) \mathrm{d}x + \mathrm{e}^{-\frac{r^2}{8\lambda}} \mathbb{E} \Big) \cdot \mathbb{E}^{D_l}. \end{aligned} \tag{4.49}$$

By (4.45), we obtain

$$\begin{aligned} \Big[\int_{\Omega \cap B_R} (|\phi|^2 + |\varphi|^2)(x, T) \mathrm{d}x \Big]^{1+D_l} &\leq 2\mathrm{e}^{K_l} \cdot \mathrm{e}^{8(1+D_l) L_M T} \cdot \mathrm{e}^{\frac{(1+D_l)}{\theta}(L_2 + \frac{2R_0^2}{k})} \\ &\quad \times \Big(\mathrm{e}^{\frac{r^2}{8\lambda}} \int_{B_r} (|\phi|^2 + |\varphi|^2)(x, T) \mathrm{d}x \Big) \cdot \mathbb{E}^{D_l}. \end{aligned} \tag{4.50}$$

It follows from (4.10) that

$$e^{\frac{(1+D_l)}{\theta}(L_2+\frac{2R_0^2}{k})}=\Big(e^{L_4L_MT+L_5(1+\frac{1}{T})}\frac{\mathbb{E}}{\int_{\Omega\cap B_R}\big(|\phi|^2+|\varphi|^2\big)(x,T)\mathrm{d}x}\Big)^{(1+D_l)L_3(L_2+\frac{2R_0^2}{k})}.$$

This, together with (4.43) and (4.50), indicates

$$\begin{aligned}&\big[\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\big]^{1+D_l}\\ &\leq L\Big(\frac{\mathbb{E}}{\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x}\Big)^{(1+D_l)[2L_3(L_2+\frac{2R_0^2}{k})+1]}\int_{B_r}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\cdot\mathbb{E}^{D_l},\end{aligned}$$

where

$$L=4e^{2K_l}\cdot e^{16(1+D_l)L_MT}\cdot e^{[L_4L_MT+L_5(1+\frac{1}{T})]\cdot(1+D_l)\cdot[2L_3(L_2+\frac{2R_0^2}{k})+1]}. \tag{4.51}$$

Thus,

$$\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\leq L^\alpha\Big(\int_{B_r}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\Big)^\alpha\cdot\mathbb{E}^{1-\alpha}, \tag{4.52}$$

where

$$\alpha=\frac{1}{2(1+D_l)\cdot(1+L_3(L_2+\frac{2R_0^2}{k}))}. \tag{4.53}$$

It is easy to check from (4.36), (4.42), and Proposition 4.3 that α only depends on (R,r,δ). Further, by a standard energy estimate, we have

$$\mathbb{E}\leq(1+T\cdot e^{4L_MT})\int_\Omega(|\phi|^2+|\varphi|^2)(x,0)\mathrm{d}x.$$

Since $1+T\cdot e^{4L_MT}\leq(1+T)e^{4L_MT}\leq e^T\cdot e^{4L_MT}$, we obtain

$$\mathbb{E}\leq e^{(1+4L_M)T}\int_\Omega(|\phi|^2+|\varphi|^2)(x,0)\mathrm{d}x.$$

Combining the above with (4.52) yields

$$\begin{aligned}\int_{\Omega\cap B_R}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\leq L^\alpha\cdot e^{(1-\alpha)(1+4L_M)T}&\Big(\int_{B_r}(|\phi|^2+|\varphi|^2)(x,T)\mathrm{d}x\Big)^\alpha\\ &\times\Big(\int_\Omega(|\phi|^2+|\varphi|^2)(x,0)\mathrm{d}x\Big)^{1-\alpha}.\end{aligned}$$

This, along with (4.51), (4.53), (4.36), and (4.46), indicates (4.21). We complete the proof. □

Remark 4.6. Indeed, (4.21) is a local interpolation inequality of the boundary case for the system (4.1).

When x_0 is a interior point of Ω, there is a positive number R satisfying $B(x_0,3R)\subseteq\Omega$. Taking $\delta=1$ in (4.22), we observe that $B(x_0,(1+2\delta)R)=B(x_0,3R)$ and $\Omega\cap B(x_0,(1+2\delta)R)=B(x_0,(1+2\delta)R)$. Then, by the same argument used in Theorem 4.5, we have for each $r \in (0, R)$, there are two constants $C = C(R, r) > 0$ and $\alpha = \alpha(R, r) \in (0, 1)$ such that

$$\int_{B(x_0,R)} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x \le \mathrm{e}^{C(1+L_M^2)(\frac{1}{T}+T)} \Big(\int_\Omega (|\phi|^2 + |\varphi|^2)(x, 0)\mathrm{d}x \Big)^{1-\alpha} \times \Big(\int_{B(x_0,r)} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x \Big)^{\alpha}, \tag{4.54}$$

where (ϕ, φ) is the solution of (4.1). It can be regarded the local interpolation inequality of the interior case for the system (4.1). We omit the detailed proof.

5. Proof of the main results

In this section, we will prove Theorem 1.1 and Theorem 1.3.

5.1. Proof of Theorem 1.1

Proof. The proof will be organized in two steps.
Step 1. We prove (1.3)*.*

Since ω is a nonempty open subset of Ω, we can find $x_0 \in \omega$ and $r > 0$ such that the open ball $B(x_0, r) \subseteq \omega$.

We first deal with the boundary of Ω. Since $\Omega \subset \mathbb{R}^n$ is a bounded domain with a C^2 boundary $\partial\Omega$, it is locally star-shaped, *i.e.* for each $d \in \partial\Omega$, there is a triplet $(x_d, R_d, \delta_d) \in \Omega \times \mathbb{R}^+ \times (0, 1]$ such that

$$|d - x_d| < R_d \ \text{ and } \ \Omega \cap B(x_d, (1 + 2\delta_d)R_d) \text{ is star-shaped with } x_d. \tag{5.1}$$

(See Thm. 8 of [23].) It follows from (5.1) that

$$\partial\Omega \subset \cup_{d\in\partial\Omega} B(x_d, R_d),$$

where the triplet $(x_d, R_d, \delta_d) \in \Omega \times \mathbb{R}^+ \times (0, 1]$ corresponding to $d \in \partial\Omega$ is given in (5.1). Then by the compactness of $\partial\Omega$, we can find a finite set of triplets $(x_i, R_i, \delta_i) \in \Omega \times \mathbb{R}^+ \times (0, 1]$ $(i = 1, 2, \dots, m_1)$ such that $\partial\Omega \subset \cup_{i=1,2,\dots,m_1} B(x_i, R_i)$ and such that each $\Omega \cap B(x_i, (1 + 2\delta_i)R_i)$ is star-shaped with respect to x_i. For each $i \in \{1, 2, \dots, m_1\}$, we choose $\rho_i \in (0, R_i)$ and finitely many points $d_{i,1}, d_{i,2}, \dots, d_{i,q_i} \in \Omega$ so that

$$\begin{cases} x_i = d_{i,1}; \\ B(d_{i,j}, \rho_i/2) \subset B(d_{i,j+1}, \rho_i),\ \forall\, j = 1, 2, \dots, q_i - 1; \\ B(d_{i,q_i}, \rho_i) \subset B(x_0, r); \\ B(d_{i,j}, 3\rho_i) \subset \Omega,\ \forall\, j = 1, 2, \dots, q_i, \end{cases} \tag{5.2}$$

which forms a chain of balls along a curve connecting $d_{i,1}$ with d_{i,q_i} in Ω. Using (4.21), and propagating this interpolation inequality along the chain of balls (5.2), we can find constants $D_i = D_i(R_i, \delta_i, \Omega, \omega) > 0$ and $\alpha_i = \alpha_i(R_i, \delta_i, \Omega, \omega) \in (0, 1)$ such that

$$\int_{\Omega\cap B(x_i,R_i)} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x \le \mathrm{e}^{D_i(1+L_M^2)(\frac{1}{T}+T)} \Big(\int_\Omega \big(|\phi|^2 + |\varphi|^2\big)(x, 0)\mathrm{d}x \Big)^{1-\alpha_i} \times \Big(\int_{B(x_0,\, r)} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x \Big)^{\alpha_i},$$

where (ϕ, φ) is the solution of (4.1). (See Thm. 1.1 of [24].) Write

$$\Theta_1 = \cup_{i=1,2,\ldots,m_1} \big[\Omega \cap B(x_i, R_i)\big].$$

In summary, there exist two constants $D_1 = D_1(\Omega, \omega) > 0$ and $\beta_1 = \beta_1(\Omega, \omega) \in (0, 1)$ such that

$$\begin{aligned}\int_{\Theta_1} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x \le \mathrm{e}^{D_1(1+L_M^2)(\frac{1}{T}+T)} \Big(\int_\Omega \big(|\phi|^2 + |\varphi|^2\big)(x, 0)\mathrm{d}x\Big)^{1-\beta_1} \\ \times \Big(\int_{B(x_0,r)} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x\Big)^{\beta_1}. \end{aligned} \tag{5.3}$$

We second deal with the interior of Ω. It is obvious that there exists a compact subset $\Theta_2 \subset \Omega$ such that $\Omega \subseteq \Theta_1 \cup \Theta_2$. By the compactness of Θ_2, there is a constant $R > 0$ and finitely many points $y_1, y_2 \ldots, y_{m_2} \in \Omega$ such that $\Theta_2 \subset \cup_{i=1,2,\ldots,m_2} B(y_i, R)$ and $B(y_i, 3R) \subset \Omega$ for each $i \in \{1, 2, \ldots, m_2\}$. Then, by (4.54) and propagating this interpolation inequality along a chain of balls, we can find constants $C_i = C_i(R, \Omega, \omega) > 0$ and $\gamma_i = \gamma_i(R, \Omega, \omega) \in (0, 1)$ such that

$$\begin{aligned}\int_{B(y_i,R)} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x \le \mathrm{e}^{C_i(1+L_M^2)(\frac{1}{T}+T)} \Big(\int_\Omega \big(|\phi|^2 + |\varphi|^2\big)(x, 0)\mathrm{d}x\Big)^{1-\gamma_i} \\ \times \Big(\int_{B(x_0,\, r)} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x\Big)^{\gamma_i}, \end{aligned}$$

where (ϕ, φ) is the solution of (4.1). It follows from $\Theta_2 \subset \cup_{i=1,2,\ldots,m_2} B(y_i, R)$ that there exist two constants $D_2 = D_2(\Omega, \omega) > 0$ and $\beta_2 = \beta_2(\Omega, \omega) \in (0, 1)$ such that

$$\begin{aligned}\int_{\Theta_2} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x \le \mathrm{e}^{D_2(1+L_M^2)(\frac{1}{T}+T)} \Big(\int_\Omega \big(|\phi|^2 + |\varphi|^2\big)(x, 0)\mathrm{d}x\Big)^{1-\beta_2} \\ \times \Big(\int_{B(x_0,r)} (|\phi|^2 + |\varphi|^2)(x, T)\mathrm{d}x\Big)^{\beta_2}. \end{aligned} \tag{5.4}$$

Finally, by (5.3), (5.4), we obtain (1.3).

Step 2. We prove (1.4).

The proof will be split into two sub-steps.

Sub-step 2.1. We prove that $(\phi(\cdot, t), \varphi(\cdot, t)) \neq (0, 0)$ *when* $t \in [0, T]$.

By contradiction, we suppose that $(\phi(\cdot, t), \varphi(\cdot, t)) = (0, 0)$ for some $t \in (0, T]$. We define that

$$T_0 := \inf\{t \in (0, T] \;\big|\; (\phi(\cdot, t), \varphi(\cdot, t)) = (0, 0)\}.$$

Using $(u_0^1, v_0^1) \neq (u_0^2, v_0^2)$, *i.e.* $(\phi(\cdot, 0), \varphi(\cdot, 0)) \neq (0, 0)$, we have that $0 < T_0 \le T$. This, along with the continuity of $(\phi(\cdot, t), \varphi(\cdot, t))$, yields

$$(\phi(\cdot, T_0), \varphi(\cdot, T_0)) = 0 \text{ and } (\phi(\cdot, t), \varphi(\cdot, t)) \neq (0, 0), \text{ for } t \in [0, T_0). \tag{5.5}$$

Define a function

$$\zeta(t) := \frac{\|\phi(\cdot, t)\|_2^2 + \|\varphi(\cdot, t)\|_2^2}{\|\phi(\cdot, t)\|_{H^{-1}}^2 + \|\varphi(\cdot, t)\|_{H^{-1}}^2},\ t \in [0, T_0), \tag{5.6}$$

where $\|\cdot\|_{H^{-1}}$ is the norm of $H^{-1}(\Omega)$. We claim the following backward uniqueness estimate for $(\phi(\cdot,t),\varphi(\cdot,t))$:

$$\begin{aligned}&\|\phi(\cdot,0)\|_{H^{-1}}^2+\|\varphi(\cdot,0)\|_{H^{-1}}^2\\ \le\ &\exp\big(2\big(\mathrm{e}^{2L_M^2T}\zeta(0)+2L_M\mathrm{e}^{L_M^2T}\sqrt{\zeta(0)}\big)T\big)\big(\|\phi(\cdot,t)\|_{H^{-1}}^2+\|\varphi(\cdot,t)\|_{H^{-1}}^2\big),\end{aligned}\tag{5.7}$$

where $t\in[0,T_0)$. By the standard energy method, we have when $t\in[0,T]$,

$$\begin{cases}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\phi\|_2^2+\|\varphi\|_2^2\big)+\big(\|\phi\|_{H_0^1}^2+\|\varphi\|_{H_0^1}^2\big)=-\langle F_1,\phi\rangle-\langle F_2,\varphi\rangle;\\ \frac12\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2\big)+\big(\|\phi\|_2^2+\|\varphi\|_2^2\big)=-\langle F_1,(-\triangle)^{-1}\phi\rangle_{H^{-1},H_0^1}-\langle F_2,(-\triangle)^{-1}\varphi\rangle_{H^{-1},H_0^1}.\end{cases}\tag{5.8}$$

(Here, $\|\cdot\|_{H_0^1}$ is the norm of $H_0^1(\Omega)$, $\langle\cdot,\cdot\rangle$ is the inner product of $L^2(\Omega)$, and $\langle\cdot,\cdot\rangle_{H^{-1},H_0^1}$ stands for the pair between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.) By (5.8), we obtain when $t\in[0,T_0)$,

$$\begin{aligned}\zeta'(t)=\ &\frac{2}{(\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2)^2}\big((-\langle F_1,\phi\rangle-\langle F_2,\varphi\rangle-\|\phi\|_{H_0^1}^2-\|\varphi\|_{H_0^1}^2)(\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2)\\ &-(\langle -F_1,(-\triangle)^{-1}\phi\rangle_{H^{-1},H_0^1}+\langle -F_2,(-\triangle)^{-1}\varphi\rangle_{H^{-1},H_0^1}-\|\phi\|_2^2-\|\varphi\|_2^2)(\|\phi\|_2^2+\|\varphi\|_2^2)\big).\end{aligned}\tag{5.9}$$

After some computations, we obtain

$$\begin{aligned}&(\|\phi\|_2^2+\|\varphi\|_2^2)^2+(\langle F_1,(-\triangle)^{-1}\phi\rangle_{H^{-1},H_0^1}+\langle F_2,(-\triangle)^{-1}\varphi\rangle_{H^{-1},H_0^1})(\|\phi\|_2^2+\|\varphi\|_2^2)\\ \le\ &\big(\|-\triangle\phi+\frac{F_1}{2}\|_{H^{-1}}\cdot\|(-\triangle)^{-1}\phi\|_{H_0^1}+\|-\triangle\varphi+\frac{F_2}{2}\|_{H^{-1}}\cdot\|(-\triangle)^{-1}\varphi\|_{H_0^1}\big)^2.\end{aligned}$$

Then, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}&(\|\phi\|_2^2+\|\varphi\|_2^2)^2+(\langle F_1,(-\triangle)^{-1}\phi\rangle_{H^{-1},H_0^1}+\langle F_2,(-\triangle)^{-1}\varphi\rangle_{H^{-1},H_0^1})(\|\phi\|_2^2+\|\varphi\|_2^2)\\ \le\ &\big(\|\phi\|_{H_0^1}^2+\|\varphi\|_{H_0^1}^2+\|\frac{F_1}{2}\|_{H^{-1}}^2+\|\frac{F_2}{2}\|_{H^{-1}}^2+\langle\phi,F_1\rangle+\langle\varphi,F_2\rangle\big)\cdot\big(\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2\big)\end{aligned}$$

Together with (5.9), we obtain when $t\in[0,T_0)$,

$$\zeta'(t)\le\frac{2}{\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2}(\|\frac{F_1}{2}\|_{H^{-1}}^2+\|\frac{F_2}{2}\|_{H^{-1}}^2).$$

Applying (4.2), then solving this inequality, we have when $t\in[0,T_0)$,

$$\zeta(t)\le\mathrm{e}^{2L_M^2t}\zeta(0).\tag{5.10}$$

Then, by the second equation of (5.8), and (5.10), we obtain when $t\in[0,T_0)$,

$$\begin{aligned}0=\ &\frac12\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2\big)+\big(\|\phi\|_2^2+\|\varphi\|_2^2\big)+\langle F_1,(-\triangle)^{-1}\phi\rangle_{H^{-1},H_0^1}+\langle F_2,(-\triangle)^{-1}\varphi\rangle_{H^{-1},H_0^1}\\ \le\ &\frac12\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2\big)+\zeta(t)\big(\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2\big)+\|F_1\|_{H^{-1}}\|\phi\|_{H^{-1}}+\|F_2\|_{H^{-1}}\|\varphi\|_{H^{-1}}\end{aligned}$$

which, along with the Cauchy-Schwarz inequality, and (5.10), indicates when $t\in[0,T_0)$,

$$0\le\frac12\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2\big)+\big(\mathrm{e}^{2L_M^2T}\zeta(0)+2L_M\mathrm{e}^{L_M^2T}\sqrt{\zeta(0)}\big)\big(\|\phi\|_{H^{-1}}^2+\|\varphi\|_{H^{-1}}^2\big).$$

Solving this inequality, we obtain (5.7).

Next, it follows from (5.7) when $t \in [0, T_0)$,

$$\frac{\|\phi(\cdot,0)\|_2^2 + \|\varphi(\cdot,0)\|_2^2}{\|\phi(\cdot,t)\|_2^2 + \|\varphi(\cdot,t)\|_2^2} \le \frac{\|\phi(\cdot,0)\|_{H^{-1}}^2 + \|\varphi(\cdot,0)\|_{H^{-1}}^2}{\|\phi(\cdot,t)\|_{H^{-1}}^2 + \|\varphi(\cdot,t)\|_{H^{-1}}^2}\zeta(0)$$
$$\le \zeta(0)\exp\big(2\big(e^{2L_M^2T}\zeta(0) + 2L_M e^{L_M^2T}\sqrt{\zeta(0)}\big)T\big). \tag{5.11}$$

By Remark 4.2, we have $(\phi,\varphi) \in C([0,T]; L^2(\Omega)^2)$. This, along with (5.5), implies that

$$\lim_{t\to T_0^-}\big(\|\phi(\cdot,t)\|_2^2 + \|\varphi(\cdot,t)\|_2^2\big) = \|\phi(\cdot,T_0)\|_2^2 + \|\varphi(\cdot,T_0)\|_2^2 = 0,$$

which contradicts (5.11). We complete the proof of *Sub-step 2.1.*
Sub-step 2.2. We prove (1.4).

By *Sub-step 2.1*, the function $t \to \frac{\|\phi(\cdot,t)\|_2^2+\|\varphi(\cdot,t)\|_2^2}{\|\phi(\cdot,t)\|_{H^{-1}}^2+\|\varphi(\cdot,t)\|_{H^{-1}}^2}$ is well defined when $t \in [0,T]$. We still use $\zeta(\cdot)$ to denote this function on $[0,T]$. Then by the same method in the proof of (5.11), we obtain that

$$\|\phi(\cdot,0)\|_2^2 + \|\varphi(\cdot,0)\|_2^2 \le \zeta(0)\exp\big(2\big(e^{2L_M^2T}\zeta(0) + 2L_M e^{L_M^2T}\sqrt{\zeta(0)}\big)T\big)\big(\|\phi(\cdot,T)\|_2^2 + \|\varphi(\cdot,T)\|_2^2\big).$$

Since $\zeta(0) \ge 1$, we have that $\sqrt{\zeta(0)} \le \zeta(0)$ and $\zeta(0) < \exp\big(\zeta(0)\big)$. Together with these, we obtain

$$\|\phi(\cdot,0)\|_2^2 + \|\varphi(\cdot,0)\|_2^2 \le \exp\big(2(1 + e^{2L_M^2T}T + 2L_M e^{L_M^2T}T)\cdot\zeta(0)\big)\big(\|\phi(\cdot,T)\|_2^2 + \|\varphi(\cdot,T)\|_2^2\big)$$
$$\le \exp\big(2(1 + 2Te^{L_M + 2L_M^2T})\cdot\zeta(0)\big)\big(\|\phi(\cdot,T)\|_2^2 + \|\varphi(\cdot,T)\|_2^2\big),$$

which, along with (1.3), yields (1.4). Hence, we complete the proof of Theorem 1.1. □

5.2. Proof of Theorem 1.3

Proof. Let (u_i, v_i) $(i = 1,2)$ be the solution to (1.1) with initial value (u_i^0, v_i^0). Arbitrarily fix $\epsilon \in (0,T)$. Write $T_\epsilon = T - \epsilon$, $\phi = u_1 - u_2$, and $\varphi = v_1 - v_2$. We define two functions ϕ_ϵ and φ_ϵ on $\Omega \times [0, T_\epsilon]$ by setting $\phi_\epsilon(x,t) := \phi(x, t+\epsilon)$ and $\varphi_\epsilon(x,t) := \varphi(x,t+\epsilon)$.

Since $(u_i^0, v_i^0) \in L^q(\Omega)^2$ (where $i = 1,2$, and $\max\{\frac{n(p-1)}{2}, p\} < q < +\infty$), it follows from Theorem 3.2 that $\big(\phi_\epsilon(\cdot,0), \varphi_\epsilon(\cdot,0)\big) = \big((u_1-u_2)(\cdot,\epsilon), (v_1-v_2)(\cdot,\epsilon)\big) \in L^\infty(\Omega)^2$, and $(\phi_\epsilon, \varphi_\epsilon) \in L^\infty((0,T_\epsilon); L^\infty(\Omega)^2)$. Using Theorem 1.1, we obtain
(i) There are constants $\beta = \beta(\Omega,\omega) \in (0,1)$ and $C = C(\Omega,\omega) > 0$ such that

$$\int_\Omega (|\phi_\epsilon|^2 + |\varphi_\epsilon|^2)(x,T_\epsilon)\mathrm{d}x \le e^{C(1+L_{M_\epsilon}^2)(\frac{1}{T_\epsilon}+T_\epsilon)}\Big(\int_\Omega (|\phi_\epsilon|^2 + |\varphi_\epsilon|^2)(\cdot,0)\mathrm{d}x\Big)^{1-\beta}$$
$$\times\Big(\int_\omega (|\phi_\epsilon|^2 + |\varphi_\epsilon|^2)(x,T_\epsilon)\mathrm{d}x\Big)^{\beta}, \tag{5.12}$$

(ii) If $(\phi_\epsilon(\cdot,0), \varphi_\epsilon(\cdot,0)) \ne (0,0)$, then there exist a positive number $C = C(\Omega,\omega)$ such that

$$\int_\Omega (|\phi_\epsilon|^2 + |\varphi_\epsilon|^2)(\cdot,0)\mathrm{d}x \le e^{C\Big((1+L_{M_\epsilon}^2)(\frac{1}{T_\epsilon}+T_\epsilon) + [1+T_\epsilon e^{(L_{M_\epsilon}+2L_{M_\epsilon}T_\epsilon)}]\frac{\|\phi_\epsilon(\cdot,0)\|_2^2+\|\varphi_\epsilon(\cdot,0)\|_2^2}{\|\phi_\epsilon(\cdot,0)\|_{H^{-1}}^2+\|\varphi_\epsilon(\cdot,0)\|_{H^{-1}}^2}\Big)}$$
$$\times\int_\omega (|\phi_\epsilon|^2 + |\varphi_\epsilon|^2)(x,T_\epsilon)\mathrm{d}x. \tag{5.13}$$

where $M_\epsilon = \max\{\|(u_i, v_i)\|_{L^\infty(\epsilon,T;L^\infty(\Omega)^2)} \mid i = 1, 2\}$ and $L_{M_\epsilon} > 0$ is the Lipschitz constant of f_i $(i = 1, 2)$ on the domain $\mathcal{D}_M = \{(s_1, s_2) \in \mathbb{R}^2 \mid |s_1| + |s_2| \leq M_\epsilon\}$,. Taking $\epsilon = \frac{T}{2}$ in (5.12), then using (3.4), we obtain (1.6).

If $\big(\phi(\cdot, T), \varphi(\cdot, T)\big) = (0, 0)$ over ω, *i.e.*, $\big(\phi_\epsilon(\cdot, T_\epsilon), \varphi_\epsilon(\cdot, T_\epsilon)\big) = (0, 0)$, then by (5.13), we have $\big(\phi_\epsilon(\cdot, 0), \varphi_\epsilon(\cdot, 0)\big) = (0, 0)$ over Ω, *i.e.*, $\big(\phi(\cdot, \epsilon), \varphi(\cdot, \epsilon)\big) = (0, 0)$ over Ω. Using Theorem 3.2, $(u_i, v_i) \in C([0, T]; L^q(\Omega)^2)$ $(i = 1, 2)$. Letting $\epsilon \to 0+$, we obtain $\big(\phi(\cdot, 0), \varphi(\cdot, 0)\big) = (0, 0)$, *i.e.* $(u_1^0, v_1^0) = (u_2^0, v_2^0)$. We complete the proof. □

Remark 5.1. We obtain inequality (1.6) by (1.3). Therefore, the index β in (1.6) can not be taken the value 1.

Appendix A.

This appendix is devoted to the proof of Lemma 2.2. We borrow the idea from the proof of Theorem A1 in [21]. Now, we first introduce two inequalities.

Lemma A.1. *Let $T > 0$. Suppose that $g(\cdot) \in C^1[0, T]$ with $g(t) \geq 0$ for all $t \in [0, T]$. If there exist three positive numbers α, A, B such that*

$$g'(t) + Ag(t)^{1+\alpha} \leq Bg(t) \ \text{ for each } \ t \in [0, T],$$

then

$$g(t) \leq \big(\frac{1}{\alpha A t}\big)^{\frac{1}{\alpha}} \mathrm{e}^{Bt} \ \text{ for each } \ t \in (0, T].$$

This inequality can be found in [21].

Lemma A.2. *Let $u, v \in H_0^1(\Omega)$. Then, there exists a positive number C only depending on n such that*

$$\big(\int_\Omega (|u|^2 + |v|^2)\mathrm{d}x\big)^{\frac{n+2}{n}} \leq C\big(\int_\Omega (|\nabla u|^2 + |\nabla v|^2)\mathrm{d}x\big)\big(\int_\Omega (|u| + |v|)\mathrm{d}x\big)^{\frac{4}{n}}. \tag{A.1}$$

Proof. It is clear that (A.1) holds, when $(u, v) = (0, 0)$. Assume that $(u, v) \neq (0, 0)$.

Now, we define $w = \sqrt{u^2 + v^2}$. Clearly, $w \in H_0^1(\Omega)$. By Gagliardo-Nirenberg's inequality, there exists a positive number C only depending on n such that

$$\big(\int_\Omega |w|^2 \mathrm{d}x\big)^{\frac{n+2}{n}} \leq C\big(\int_\Omega |\nabla w|^2 \mathrm{d}x\big)\big(\int_\Omega |w| \mathrm{d}x\big)^{\frac{4}{n}}.$$

By simple computation, we deduce that

$$\big(\int_\Omega (|u|^2 + |v|^2)\mathrm{d}x\big)^{\frac{n+2}{n}} \leq C\big(\int_\Omega \frac{|u\nabla u + v\nabla v|^2}{|u|^2 + |v|^2}\mathrm{d}x\big)\big(\int_\Omega \sqrt{|u|^2 + |v|^2}\mathrm{d}x\big)^{\frac{4}{n}}.$$

This, together with the Cauchy-Schwarz inequality, leads to (A.1). Hence, we complete the proof. □

Proof of Lemma 2.2.

Proof. The proof will be split into the following two steps.

Step 1: Given $(u_0, v_0) \in L^\infty(\Omega)^2$, we will prove that (2.2) have a unique solution $(u, v) \in L^\infty((0, T; L^\infty(\Omega)^2)$. Moreover, the solution (u, v) satisfies that for $t \in [0, T]$,

$$\|\big(u(t), v(t)\big)\|_\infty \leq C\mathrm{e}^{CL^\vartheta t}\|(u_0, v_0)\|_\infty, \tag{A.2}$$

where the numbers L, ϑ are given in (2.4), and the positive number C only depends on $(n, \sigma, \gamma, \Omega)$.

We first write $\mathcal{A}_1 := L^\infty(0,T;L^\infty(\Omega)^2)$. For each $(\xi,\eta) \in \mathcal{A}_1$, we define function $(u(\cdot;\xi,\eta), v(\cdot;\xi,\eta))$ as follows:

$$\begin{cases} u(t;\xi,\eta) := \mathrm{e}^{t\triangle}u_0 + \int_0^t \mathrm{e}^{(t-s)\triangle}(a_1\xi + b_1\eta)\mathrm{d}s, \\ v(t;\xi,\eta) := \mathrm{e}^{t\triangle}v_0 + \int_0^t \mathrm{e}^{(t-s)\triangle}(a_2\xi + b_2\eta)\mathrm{d}s, \end{cases}$$

where $t \in [0,T]$. Using Lemma 2.1, we observe that $(u(\cdot;\xi,\eta), v(\cdot;\xi,\eta)) \in \mathcal{A}_1$. Now, we define a map $\Psi_1 : \mathcal{A}_1 \to \mathcal{A}_1$ by setting $\Psi_1(\xi,\eta) := (u(\cdot;\xi,\eta), v(\cdot;\xi,\eta))$. For $(\xi_i,\eta_i) \in \mathcal{A}_1$ $(i = 1,2)$, we have

$$\begin{aligned} \|\Psi_1(\xi_2,\eta_2)(t) - \Psi_1(\xi_1,\eta_1)(t)\|_\infty = \| &\int_0^t \mathrm{e}^{(t-s)\triangle}(a_1(\xi_2-\xi_1) + b_1(\eta_2-\eta_1))\mathrm{d}s\|_\infty \\ + \| &\int_0^t \mathrm{e}^{(t-s)\triangle}(a_2(\xi_2-\xi_1) + b_2(\eta_2-\eta_1))\mathrm{d}s\|_\infty. \end{aligned}$$

Using Lemma 2.1 again, we obtain

$$\|\Psi_1(\xi_2,\eta_2) - \Psi_1(\xi_1,\eta_1)\|_{\mathcal{A}_1} \le \frac{T^{1-\kappa}}{1-\kappa}L\|(\xi_2-\xi_1, \eta_2-\eta_1)\|_{\mathcal{A}_1}, \tag{A.3}$$

where L is given in (2.4), and $\kappa = \frac{n}{2\sigma}$. We first consider the case that

$$\frac{T^{1-\kappa}}{1-\kappa}L < \frac{1}{2}. \tag{A.4}$$

Applying (A.3) and (A.4), Ψ_1 is a strict contraction map on $\mathcal{A}_1$, and there exists a unique solution $(u,v) \in \mathcal{A}_1$ to the system (2.2). Clearly, $\Psi_1(0,0) = (\mathrm{e}^{t\triangle}u_0, \mathrm{e}^{t\triangle}v_0) \in \mathcal{A}_1$. Then by taking $(\xi_1,\eta_1) = (u,v)$ and $(\xi_2,\eta_2) = (0,0)$ in (A.3), we obtain

$$\|(u,v)\|_{\mathcal{A}_1} = \|\Psi_1(u,v)\|_{\mathcal{A}_1} \le \|\Psi_1(u,v) - \Psi_1(0,0)\|_{\mathcal{A}_1} + \|\Psi_1(0,0)\|_{\mathcal{A}_1}.$$

which, together with Lemma 2.1 and (A.3), leads to

$$\|(u,v)\|_{\mathcal{A}_1} \le 2\|(u_0,v_0)\|_\infty.$$

Thus, (A.2) holds in this case.

Secondly, if (A.4) does not hold, we choose another $T_0 \in [0,T]$ such that $\frac{T_0^{1-\kappa}}{1-\kappa}L < \frac{1}{2}$, and work on this problem in $[0,T_0]$. Then, by a standard iteration argument, we can get the desired results in the second case.

Step 2: We will prove (2.3).

By the duality technique (see Sect. 2 of [25]), we can obtain that there exists $C = C(n,\sigma,\gamma,\Omega)$ such that

$$\|\big(u(t),v(t)\big)\|_1 \le C\mathrm{e}^{CL^\vartheta t}\|(u_0,v_0)\|_1, \tag{A.5}$$

where L, ϑ are given in (2.4). This, along with (A.2), and Riesz-Thorin's interpolation theorem, shows

$$\|\big(u(t),v(t)\big)\|_p \le C\mathrm{e}^{CL^\vartheta t}\|(u_0,v_0)\|_p, \tag{A.6}$$

where $1 \le p \le \infty$.

We assume that $(u_0, v_0) \neq (0, 0)$ (In fact, Lem. 2.2 is valid when $(u_0, v_0) = (0, 0)$). Multiplying the first equation of (2.2) by u, multiplying the second equation of (2.2) by v, and integrating them over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|u|^2+|v|^2)\mathrm{d}x+\int_\Omega(|\nabla u|^2+|\nabla v|^2)\mathrm{d}x\\&=-\int_\Omega(a_1u^2+b_1uv+a_2uv+b_2v^2)\mathrm{d}x.\end{aligned}$$

Let $\sigma' \in [1, +\infty]$ satisfy $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$. Using Hölder and Gagliardo-Nirenberg's inequalities, we have

$$\begin{aligned}\int_\Omega |a_1|u^2\mathrm{d}x \leq \|a_1\|_\sigma\|u\|_{2\sigma'}^2 &\leq C\|a_1\|_\sigma(\int_\Omega|\nabla u|^2\mathrm{d}x)^{\frac{n}{2\sigma}}(\int_\Omega|u|^2\mathrm{d}x)^{\frac{2\sigma-n}{2\sigma}}\\&\leq C(\int_\Omega|\nabla u|^2\mathrm{d}x)^{\frac{n}{2\sigma}}(L^\vartheta\int_\Omega|u|^2\mathrm{d}x)^{\frac{2\sigma-n}{2\sigma}},\end{aligned}$$

where the constant C is only dependent of n, $\vartheta = \frac{2\sigma}{2\sigma-n}$, and

$$L=\sum_{i=1,2}\big(\|a_i\|_{L^\infty((0,T);L^\sigma(\Omega))}+\|b_i\|_{L^\infty((0,T);L^\sigma(\Omega))}\big).$$

Then, by Young's inequality, it yields

$$C(n)\|a_1\|_\sigma(\int_\Omega|\nabla u|\mathrm{d}x)^{\frac{n}{2\sigma}}(\int_\Omega|u|^2\mathrm{d}x)^{\frac{2\sigma-n}{2\sigma}}\leq\frac{1}{4}\int_\Omega|\nabla u|^2\mathrm{d}x+C(n)L^\vartheta\int_\Omega|u|^2\mathrm{d}x.$$

Thus, we obtain

$$|\int_\Omega a_1u^2\mathrm{d}x|\leq\frac{1}{4}\int_\Omega|\nabla u|^2\mathrm{d}x+C(n)L^\vartheta\int_\Omega|u|^2\mathrm{d}x.$$

By the same method, we deduce that

$$\begin{aligned}&|\int_\Omega a_1u^2\mathrm{d}x|+|\int_\Omega b_1uv\mathrm{d}x|+|\int_\Omega a_2uv\mathrm{d}x|+|\int_\Omega b_2v^2\mathrm{d}x|\\&\leq\frac{1}{2}\int_\Omega(|\nabla u|^2+|\nabla v|^2)\mathrm{d}x+C(n)L^\vartheta\int_\Omega(|u|^2+|v|^2)\mathrm{d}x.\end{aligned}$$

In summary, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|u|^2+|v|^2)\mathrm{d}x+\int_\Omega(|\nabla u|^2+|\nabla v|^2)\mathrm{d}x\leq C(n)L^\vartheta\int_\Omega(|u|^2+|v|^2)\mathrm{d}x.$$

It, along with Lemma A.2 and (A.5), indicates that

$$\Big(\int_\Omega(|u|^2+|v|^2)\mathrm{d}x\Big)^{\frac{n+2}{n}}\leq(C\mathrm{e}^{CL^\vartheta t}\|(u_0,v_0)\|_1)^{\frac{4}{n}}\int_\Omega(|\nabla u|^2+|\nabla v|^2)\mathrm{d}x,$$

where $C = C(n, \sigma, \gamma, \Omega)$. Define a function $f(t) := \int_\Omega (|u|^2 + |v|^2)(x,t)\mathrm{d}x,\ t \in [0,T]$. Clearly, $f(t)$ satisfies the following differential inequality:

$$f'(t) + Af(t)^{1+\frac{2}{n}} \leq Bf(t),$$

with $A = (C\mathrm{e}^{CL^\vartheta t}\|(u_0,v_0)\|_1)^{-\frac{4}{n}}$, and $B = CL^\vartheta$. By Lemma A.1, we obtain

$$f(t) \leq \big(\frac{n}{2At}\big)^{\frac{n}{2}}\mathrm{e}^{Bt}.$$

Thus,

$$\|(u,v)\|_2 \leq C\mathrm{e}^{CL^\vartheta t}t^{-\frac{n}{4}}\|(u_0,v_0)\|_1. \tag{A.7}$$

Using the duality technique again, we obtain

$$\|(u,v)\|_\infty \leq C\mathrm{e}^{CL^\vartheta t}t^{-\frac{n}{4}}\|(u_0,v_0)\|_2.$$

This, along with (A.7), shows

$$\|(u,v)\|_\infty \leq C\mathrm{e}^{CL^\vartheta t}t^{-\frac{n}{2}}\|(u_0,v_0)\|_1,$$

where the number C only depends on $(n, \sigma, \gamma, \Omega)$. It, together with (A.2) and Riesz-Thorin's interpolation theorem, indicates (2.3). This completes the proof. □

Acknowledgements. This work was partially supported by Zhejiang Provincial Natural Science Foundation of China under Grant No. LY24A010014, and by Faculty Development Fund granted by College of Arts and Sciences, University of Cincinnati.

Conflict of Interest. The authors declare that they have no competing interests.

References

[1] L. Allen, B. Bolker, Y. Lou and N. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model. *Disc. Cont. Dyn. Syst.* **21** (2008) 1–20.

[2] A. Friedman, Partial Differential Equations of Parabolic Type. Dover Pub., Mineola, New York (2008).

[3] R. Peng, Asymptotic profiles of the positive steady state for an SIS epidemic reaction–diffusion model. Part I. *J. Diff. Equ.* **247** (2009) 1096–1119.

[4] Y. Wu and X. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism. *J. Diff. Equ.* **261** (2016) 4424–4447.

[5] P. Albano and D. Tataru, Unique continuation for second-order parabolic operators at the initial time. *Proc. Am. Math. Soc.* **132** (2004) 1077–1085.

[6] H. Koch and D. Tataru, Carleman estimates and unique continuation for second order parabolic equations with nonsmooth coefficients. *Commun. PDE* **34** (2009) 305–366.

[7] K.D. Phung and G. Wang, Quantitative unique continuation for the semilinear heat equation in a convex domain. *J. Funct. Anal.* **259** (2010) 1230–1247.

[8] C. Poon, Unique continuation for parabolic equations. *Commun. PDE* **21** (1996) 521–539.

[9] G. Wang, M. Wang and Y. Zhang, Observability and unique continuation inequalities for the Schrödinger equation. *J. Eur. Math. Soc.* **21** (2019) 3513–3572.

[10] G. Zheng, D. Xu and T. Wang, A unique continuation property for a class of parabolic differential inequalities in a bounded domain. *Commun. Pure Appl. Anal.* **20** (2021) 547–558.

[11] J. Coron, Control and Nonlinearity. AMS Providence, Rhode Island (2007).

[12] X. Fu, J. Yong and X. Zhang, Controllability and observability of a heat equation with hyperbolic memory kernel. *J. Diff. Equ.* **247** (2009) 2395–2439.

[13] P. Lissy and E. Zuazua, Internal observability for coupled systems of linear partial differential equations. *SIAM J. Control. Optim.* **57** (2019) 832–853.
[14] Q. Lü, Observability estimate and state observation problems for stochastic hyperbolic equations. *Inverse Probl.* **29** (2013) Article 095011.
[15] Q. Lü, Observability estimate for stochastic Schrödinger equations and its applications. *SIAM J. Control Optim.* **51** (2013) 121–144.
[16] K.D. Phung and G. Wang, An observability estimate for parabolic equations from a measurable set in time and its applications. *J. Eur. Math. Soc.* **15** (2013) 681–703.
[17] K.D. Phung, Carleman commutator approach in logarithmic convexity for parabolic equations. *Math. Control Relat. F* **8** (2018) 899–933.
[18] F.J. Almgren Jr., Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents, in Minimal Submanifolds and Geodesics. North-Holland, Amsterdam-New York (1979) 1–6.
[19] N. Garofalo and F. Lin, Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J.* **35** (1986) 245–268.
[20] Q. Lü and Z. Yin, Unique continuation for stochastic heat equations. *ESAIM Control Optim. Calc. Var.* **21** (2015) 378–398.
[21] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data. *J. Anal. Math.* **68** (1996) 277–304.
[22] T. Cazenave and A. Haraux, An Introduction to Semilinear Evolution Equations. Clarendon Press, Oxford (1998).
[23] J. Apraiz, L. Escauriaza, G. Wang and C. Zhang, Observability inequalities and measurable sets. *J. Eur. Math. Soc.* **16** (2014) 2433–2475.
[24] G. Wang and G. Zheng, Unique continuation inequalities for the parabolic-elliptic chemotaxis system. *J. Diff. Equ.* **317** (2022) 524–560.
[25] D. Daners, Heat kernal estimates for operators with boundary conditions. *Math. Nachr.* **217** (2000) 13–41.