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ESCAPE FROM COMPACT SETS OF NORMAL CURVES IN SUBFINSLER CARNOT GROUPS

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Abstract. In the setting of subFinsler Carnot groups, we consider curves that satisfy the normal equation coming from the Pontryagin Maximum Principle. We show that, unless it is constant, each such a curve leaves every compact set, quantitatively. Namely, the distance between the points at time 0 and time t grows at least of the order of $t^{1/s}$, where s denotes the step of the Carnot group. In particular, in subFinsler Carnot groups there are no periodic normal geodesics.

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1. Introduction

This paper originates from the following question: can subRiemannian manifolds have arbitrarily short geodesic loops? Important subRiemannian manifolds are Carnot groups, which have dilation structures and therefore are self-similar. Hence, for these spaces, the question rephrases as: do geodesic loops exist in a Carnot group? We remark that not only geodesic loops do not exist in the Euclidean spaces, but also do not exist in normed vector spaces with strictly convex norm. Finite-dimensional normed vector spaces are exactly the Carnot groups of nilpotency step 1, equipped with geodesic distances. We refer to [1] and [2] for an introduction to Carnot groups and the fact that when equipped with their Carnot–Carathéodory distances they are exactly the metric spaces that are homogeneous, locally compact, geodesic, and admit dilations.
Some difficulties in Carnot groups are that the distance function may not be convex, that at every scale there exist pairs of points joined by more than one geodesic, and geodesics may not be globally length-minimizing. However, the biggest difficulty, as in other subRiemannian problems, is the presence of geodesics that are singular points of the end-point map, which we call the *abnormal geodesics*. By the Pontryagin Maximum Principle we know that if a geodesic is not abnormal, then it satisfies a geodesic equation called the *normal equation*. The curves satisfying the normal equation, called *normal curves* (or *normal trajectories*), are more manageable. In this paper we focus on normal curves and we prove that, except for the constant ones, they cannot form loops and actually they leave every compact set in a quantitative way. The main result of this paper is the following statement.

Theorem 1.1 (Normal curves leave compact sets)**.** *Let G be a subFinsler Carnot group of step s. Fix a norm N on $T^*_{\gamma(0)}G$. Then, there exists a constant $\epsilon > 0$ such that for every normal curve $\gamma : \mathbb{R} \to G$ parametrized by*

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arc-length there holds

$$d(\gamma(t),\gamma(t')) \geq \frac{\epsilon}{N(\lambda)^{\frac{1}{s}}}|t-t'|^{\frac{1}{s}} - 1, \quad \forall t,t' \in \mathbb{R}, \tag{1.1}$$

for every covector λ associated to γ, see Definition 2.25.

We remark that if a normal geodesic γ is abnormal then there are several covectors associated to γ. Clearly, considering the covector with minimal norm, one gets the strongest conclusion. As a consequence of Theorem 1.1 we rule out the presence of some geodesic loops.

Corollary 1.2. *In subFinsler Carnot groups normal loops are constant.*

We stress that the above corollary has been know to be true in step 2 Carnot groups (by the complete integration of geodesics) and in jet spaces by the work of Bravo–Doddoli [3]. Moreover, a similar question arose in geometric group theory of the large-scale geometry of nilpotent groups. In fact, because of the work of Hoda [4], it would be important to prove that Carnot groups cannot have isometric copies of the standard unit circle. The latter result is known for finite-dimensional normed spaces and was proven by Creutz, see [5], Lemma 1.7.

1.1. The basic case of loops in subRiemannian Carnot groups

We give here a simpler proof of the non-existence of normal loops in subRiemannian Carnot groups. The same idea will be pushed to prove Corollary 1.2 and to show the quantitative result of Theorem 1.1.

Proof of Corollary 1.2 for subRiemannian Carnot groups. Let G be a subRiemannian Carnot group with left-invariant scalar product $(\cdot,\cdot)$ on the first stratum V_1. Let $\gamma : [0,1] \to G$ be a normal geodesic that makes a loop. Without loss of generality we assume $\gamma(0) = \gamma(1) = 1_G$.

Let $u \in L^2([0,1];V_1)$ be the control of γ, see Section 2.3 for this standard terminology. Since γ is normal, by definition u satisfies the *normal equation*: for some $\lambda \in (T_{\gamma(1)}G)^*$ and all controls $v \in L^2([0,1];V_1)$ we have

$$\lambda(\mathrm{d}\,\mathrm{End}_u\, v) = (u,v)_{L^2}, \tag{1.2}$$

where End is the end-point map from 1_G, see for example [6], Corollary 8.8. When $v = u$ we get

$$\lambda(\mathrm{d}\,\mathrm{End}_u\, u) = ||u||^2_{L^2}. \tag{1.2bis}$$

Now the idea is to consider the curve dilated by the Carnot dilations: for $\tau > 0$ let $\delta_\tau : G \to G$ be the dilation of factor τ. On the one hand, we obviously have $(\delta_\tau \circ \gamma)(1) = 1_G$. On the other hand, the control of the curve $\delta_\tau \circ \gamma$ is τu. Hence we have that

$$\mathrm{End}(\tau u) = \delta_\tau(\mathrm{End}(u)) = 1_G, \quad \forall \tau > 0. \tag{1.3}$$

Equation (1.3) implies that

$$\mathrm{d}\,\mathrm{End}_u\, u = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\mathrm{End}((1+\epsilon)u)\bigg|_{\epsilon=0} = 0. \tag{1.4}$$

Consequently, equations (1.4) and (1.2bis) imply that $||u||_{L^2} = 0$ and therefore γ is the constant curve. □

1.2. The strategy of the proof

We present here the key ideas that we shall use in the proof of Theorem 1.1. We work in the context of subFinsler spaces, smooth manifolds equipped with a distribution, in which the distance between two points is given by the infimum of the lengths of paths tangent to the distribution joining the two points. The length is measured with respect to a continuously varying norm.
We start by writing the normal equation coming from the Pontryagin Maximum Principle in terms of the subdifferentials of the energy, see Proposition 2.14. We use this formulation of the normal equation to get the analog of (1.2bis), which expresses the energy of a normal control u in terms of some covector λ applied to $\mathrm{d}\,\mathrm{End}_u\, u$. We then use the properties of the one parameter subgroup of dilations δ to define the vector field

$$\vec{\delta}(g) := \frac{\mathrm{d}}{\mathrm{d}t}\delta_t(g)\bigg|_{t=1}, \forall g \in G.$$

The first key point is that we have $\vec{\delta}(\mathrm{End}(u)) = \mathrm{d}\,\mathrm{End}_u\, u$, see (3.11). In particular we shall get

$$||u||^2_{L^2} = \lambda(\vec{\delta}(\mathrm{End}(u)). \tag{1.5}$$

We show that $g \mapsto \lambda(\vec{\delta}(g))$ is a finite sum of homogeneous functions of degree of homogeneity smaller than the step s, and therefore it can be bounded by $Cd(1,g)^s$ when $d(1,g) > 1$, for some constant $C > 0$. This bound, together with (1.5), will give (1.1).

1.3. Some technicalities arising from the non-differentiability of the norm

In this article we consider Carnot groups that are arbitrarily equipped with homogeneous geodesic distances. Namely, on the first layer we have symmetric norms that may not come from scalar products, nor are smooth away from 0. Working in such a general setting produces some technical difficulties that we briefly point out in this short section.

In subRiemannian geometry, one says that a curve defined on an arbitrary interval is normal if it admits a lift to the cotangent bundle that solves a particular ODE (the *normal equation*, see Rem. 2.16). In the setting of subFinsler geometry, such an ODE is no longer available. Still, in this setting, one starts by defining normal curves whose domain is $[0, 1]$, or, by affine rescaling, any compact interval. Subsequently, curves whose domain are arbitrary intervals are said to be normal if they are normal when restricted to each compact interval of the domain. We therefore need to pay particular attention to the definition of the covector associated to a normal curve. To do so, we make use of the Lie group structure in order to avoid some technicalities: the key point is that the flow along the curve of any such covector extends to some right-invariant 1-form. In the context of subRiemannian geometry, the two definitions of normal curves coincide, and the covector associated to a normal curve is the initial point of the lift of the curve to the cotangent bundle.

Another problem arising from the non-differentiability of the norm is the lack of the Langrange multipliers rule that is available for smooth norms (see for example [6], Thm. 8.7). We solve this problem re-writing the Pontryagin maximum principle in terms of subdifferentials (Prop. 2.14) and then proving a weaker version of the Lagrange multipliers rule (Cor. 2.21).

1.4. Organization of the paper

The paper is organized as follows: we dedicate Section 2 to a brief presentation of the notions of subFinsler geometry that we need in the paper. We recall the definitions of subFinsler manifolds, Carnot groups, and self-similar spaces; we define normal curves and we characterize them in terms of subdifferentials of the energy. In Section 3, we prove the main results of the paper. We also prove a slight generalization of Corollary 1.2 to self-similar spaces, see Theorem 3.5. Section 4 contains some examples: we show that length-1 normal geodesics can stay arbitrarily close to the starting point and that the exponent $\frac{1}{s}$ in Theorem 1.1 is the biggest possible.

2. Preliminaries

2.1. SubFinsler geometry and Carnot groups

We start recalling some basic definitions and facts from subFinsler geometry following [7] and [8], Section 3.

Definition 2.1. Let M be a smooth manifold. A *distribution* on M is a subbundle of the tangent bundle. We call a distribution *bracket-generating* if, in some neighborhood of every point, the Lie algebra generated by vector fields tangent to the distribution contains a frame for the tangent bundle on that neighborhood. A *subFinsler manifold* is a smooth manifold M, with a bracket-generating distribution Δ and a continuously varying norm $||\cdot||$, equipped with the Carnot–Carathéodory distance

$$\begin{aligned} d_{cc}(x,y) := \inf\{\int_0^1 ||\dot\gamma(t)||\mathrm{d}t \,\big|\gamma : [0,1]\to M \text{ absolutely continuous;} \\ \dot\gamma(t)\in\Delta_{\gamma(t)},\ \text{for a.e. } t\in[0,1]; \gamma(0)=x; \gamma(1)=y\}. \end{aligned} \tag{2.1}$$

The fact that d_{cc} is a distance is guaranteed by Chow–Rashevskii Theorem (see for example [9], Sect. 1). In this paper we focus mainly on Carnot groups: particular Lie groups with left-invariant distributions and norms.

Definition 2.2. A *stratification* of a Lie algebra $\mathfrak{g}$ is a decomposition of $\mathfrak{g}$ as a direct sum $\mathfrak{g} = V_1 \oplus ... \oplus V_s$ with

$$[V_1, V_j] = V_{j+1}, \quad \forall j \in \{1, ..., s\}, \tag{2.2}$$

where $V_{s+1} := \{0\}$. The subspaces V_i are called *strata* of the stratification. A Lie algebra equipped with a stratification is called *stratified*. A *subFinsler Carnot group* is a simply connected Lie group with stratified Lie algebra, with the left-invariant extension of the first stratum as distribution and a left-invariant norm, equipped with the Carnot–Carathéodory distance.

Carnot groups are self-similar, in the sense that there are natural automorphisms that act as homotheties.

Definition 2.3. Let G be Carnot group with stratified Lie algebra $\mathfrak{g} = V_1 \oplus ... \oplus V_s$. The *dilation* $\delta_\tau : G \to G$ of factor τ, with $\tau \in \mathbb{R}$, is the Lie group automorphism defined by setting

$$(\delta_\tau)_*(v) = \tau^i v, \quad \forall i \in \{1, ..., s\},\ \forall v \in V_i. \tag{2.3}$$

The presence of dilations will be crucial in our proof of the non-existence of non-constant periodic normal geodesics, see (3.11).

2.2. Homogeneous spaces and self-similar distances

In differential geometry, the term *homogeneous space* is referred to the quotient space of a Lie group modulo a closed subgroup, in order to still have a transitive action of the Lie group. However, in metric geometry and, more generally in analysis on metric spaces, the term homogeneous is referred to functions that gets multiplied by some dilations of the space. Every metric space admitting a dilation, also called homothety, is said to be self-similar.

In subFinsler geometry, self-similar spaces are well characterized. As differentiable manifold they have a homogeneous structure of a quotient space of a Carnot group modulo the action of a dilation-invariant subgroup via left-multiplication. However, the well-defined action on the right is not by isometries. Hence they are not isometrically homogeneous spaces. They are still called homogeneous because they admit dilations. To avoid this double use of the word homogeneity, we shall call them self-similar (subFinsler) spaces. By the work of Bellaïche ([8, 10]) we know that the metric tangents of subFinsler manifolds are self-similar spaces.

Definition 2.4. A *self-similar subFinsler space* is a subFinsler manifold obtained as the quotient space of a (left-invariant) subFinsler Carnot group with respect to the left-action of a dilation-invariant subgroup, and it is equipped with the quotient distribution. Namely, assume that G is a Carnot group with distribution Δ and left-invariant norm $||\cdot||$ and that $H < G$ is a closed dilation-invariant subgroup. On the quotient manifold ${}_H\backslash^G := \{Hg : g \in G\}$ we define a subFinsler structure that makes the projection $\pi : G \to {}_H\backslash^G$ a submetry: we take $\Delta_{{}_H\backslash^G} := \pi_*\Delta$ as distribution and we define the continuously varing norm on ${}_H\backslash^G$ setting for all $p \in {}_H\backslash^G$ and for all $v \in (\pi_*\Delta)_p \subseteq T_p({}_H\backslash^G)$

$$(||v||_{{}_H\backslash^G})_p := \inf\{||w||_q : q \in \pi^{-1}(p), w \in T_q\Delta, \mathrm{d}\pi_q(w) = v\}. \tag{2.4}$$

Being H dilation invariant, the dilations of G pass to the quotient and define dilations on ${}_H\backslash^G$. Thus, on a self-similar space ${}_H\backslash^G$, we call H the *origin*, since it is the only point fixed by dilations. We remand to [10], Section 7 for a presentation of self-similar spaces.

2.3. Pontryagin Maximum Principle and normal curves

In the remaining part of this section we define the end-point map and we state the Pontryagin Maximum Principle. Afterwards, we present normal curves and we characterize them using subdifferentials of the energy.

Definition 2.5. *(Controls)* Fix a finite-dimensional normed vector space V. We denote by $\Omega(V)$ the Hilbert space $L^2([0,1];V)$ and we call its elements *controls*.

Let M be a subFinsler manifold with distribution Δ, choose $X_1, ..., X_m \in \mathrm{Vec}(M)$ such that $\mathrm{Span}(\{X_1(p), ..., X_m(p)\}) = \Delta_p$ for every $p \in M$. Notice that such a set of vector fields always exists by Whitney's Theorem, see [6], Section 3.1.4. To each control $u \in \Omega(\mathbb{R}^m)$ and each point $p \in M$ we associate the unique curve $\gamma_u : [0,1] \to M$ solving

$$\begin{cases} \gamma_u'(t) = \sum_{i=1}^m u_i(t) X_i(\gamma_u(t)), \text{ for a.e. } t \in [0,1]; \\ \gamma_u(0) = p. \end{cases} \tag{2.5}$$

Remark 2.6. The existence and uniqueness of a solution of (2.5) is guaranteed by the Carathéodory Existence and Uniqueness Theorem (see [11], Thm. 3.4). *A priori*, for arbitrary manifolds, the maximal interval of definition of this solution γ_u may be strictly contained in $[0,1]$. When the manifold is complete, like in the case of Lie groups, short-time solutions stay in compact sets, hence we have global solutions, see [12], Chapter 2, Theorem 1.3. In our article, to simplify the notation, we will assume the solution γ_u of (2.5) to be defined on the whole interval $[0,1]$ for every $u \in L^2([0,1];\mathbb{R}^m)$. This is not restrictive for our purposes, since, as we said, on Carnot groups the latter statement is always true.

We shall say that γ_u is the *curve with control* u *starting at* p, or that u is a *control* of γ_u. Two different controls could lead to the same curve, because the vector fields X_j's may not be linearly independent at some point. Moreover, every curve of finite length admits a reparametrization with control in Ω. We use equation (2.5) to define the flow along a control. This will be useful for the computation of the differential of the end-point map.

Definition 2.7. *(Flow along a control)* Let M be a subFinsler manifold with distribution Δ, let $X_1, ..., X_m \in \mathrm{Vec}(M)$ be such that $\mathrm{Span}(\{X_1(p), ..., X_m(p)\}) = \Delta_p$ for every $p \in M$. The *flow along a control* $u \in \Omega(\mathbb{R}^m)$ is the map $\phi : [0,1] \times M \to M$ defined as

$$\phi^t(p) := \gamma_u(t), \quad \forall t \in [0,1], \quad \forall p \in M, \tag{2.6}$$

where γ_u is the curve solving (2.5). For fixed $s, t \in [0,1]$, with $s < t$, we will write $\phi_s^t : M \to M$ for the flow from s to t:

$$\phi_s^t := \phi^t \circ (\phi^s)^{-1}. \tag{2.7}$$

Example 2.8. Let G be a subFinsler Carnot group with first stratum V_1. Let $X_1, ..., X_m$ be a left-invariant frame of the first stratum. Using the frame $X_1, ..., X_m$ identify $L^2([0,1];\mathbb{R}^m)$ with $L^2([0,1];V_1)$. For all $p \in G$ and $u \in L^2([0,1];V_1)$, equation (2.5) rewrites as

$$\begin{cases} \gamma_u'(t) = \mathrm{d}L_{\gamma_u(t)} u(t), \text{ for a.e. } t \in [0,1]; \\ \gamma_u(0) = p. \end{cases} \tag{2.8}$$

In particular, if $\phi : [0,1] \times G \to G$ is the flow along a control $u \in L^2([0,1];V_1)$, see (2.6), then it is trivial to check that for all $p \in G$ the curve $t \mapsto L_p(\phi^t(1_G))$ solves (2.8). Consequently,

$$\phi^t(p) = R_{\phi^t(1_G)}(p), \ \forall p \in G, \forall t \in [0,1]. \tag{2.9}$$

Definition 2.9. *(End-point map)* Let M be a subFinsler manifold with distribution Δ. Choose $X_1, ..., X_m \in \mathrm{Vec}(M)$ such that $\mathrm{Span}(\{X_1(p), ..., X_m(p)\}) = \Delta_p$ for all $p \in M$. Fix a point $p \in M$. The *end-point map* $\mathrm{End} : \Omega(\mathbb{R}^m) \to M$ associated to $X_1, ..., X_m$ and p is defined as

$$\mathrm{End}(u) := \gamma_u(1), \tag{2.10}$$

where γ_u solves (2.5). In a Carnot group we usually assume the point p to be the identity element and $X_1, ..., X_m$ to be a left-invariant frame of the first stratum.

Proposition 2.10. (Differential of the end-point map, [6], Prop. 8.5). *Let M be a subFinsler manifold with distribution Δ, let $X_1, ..., X_m \in \mathrm{Vec}(M)$ such that $\Delta_p = \mathrm{Span}(\{X_1(p), ..., X_m(p)\})$ for every $p \in M$. Fix $q \in M$. Denote with* $\mathrm{End} : \Omega(\mathbb{R}^m) \to M$ *the end-point map associated to $X_1, ..., X_m$ and q. Then* End *is smooth on* $\Omega(\mathbb{R}^m)$ *and*

$$\mathrm{d}\,\mathrm{End}_u\, v = \int_0^1 (\phi_t^1)_* \sum_{i=1}^m v_i(t) X_i(\gamma(t)) \mathrm{d}t, \qquad \forall u, v \in \Omega(\mathbb{R}^m), \tag{2.11}$$

where ϕ is the flow associated to the control u and $\gamma(t) := \phi^t(q)$, see (2.6).

Definition 2.11. *(Energy)* Let M be a subFinsler manifold with a continuously varying norm $||\cdot|| : \Delta \to \mathbb{R}$. The *energy* at a point $p \in M$ is the function $\mathrm{E}_p : \Delta_p \to \mathbb{R}$ defined as

$$\mathrm{E}_p := \frac{1}{2} ||\cdot||_p^2.$$

For each absolutely continuous curve $\gamma : [0,1] \to M$, with $\dot\gamma(t) \in \Delta$ for a.e. $t \in [0,1]$, we define its *energy* as $\frac{1}{2}\int_0^1 ||\dot\gamma(t)||^2 \mathrm{d}t$.

The curves realizing the infimum in (2.1) are called *length-minimizing*. It is well-known that the same infimum is realized by minimizers of the energy, considering length-minimizing curves re-parametrized by constant speed.

We are now ready to state the well known Pontryagin Maximum Principle, which gives first-order necessary conditions for curves to be energy-minimizing.

Theorem 2.12. (PMP, [13], Thm. 12.10). *Let M be a subFinsler manifold with distribution Δ, let $X_1, ..., X_m \in \mathrm{Vec}(M)$ such that $\Delta_p = \mathrm{Span}\{X_1(p), ..., X_m(p)\}$ for every $p \in M$. For every $\nu \in \mathbb{R}$ and $v = (v_1, ..., v_m) \in \mathbb{R}^m$; define $h_{v,\nu} : T^*M \to \mathbb{R}$ as*

$$h_{v,\nu}(\eta) := \left\langle \eta, \sum_{i=1}^{m} v_i X_i(\pi(\eta)) \right\rangle + \nu \, \mathrm{E}_{\pi(\eta)} \left(\sum_{i=1}^{m} v_i X_i(\pi(\eta)) \right), \quad \forall \eta \in T^*M, \tag{2.12}$$

where $\pi : T^*M \to M$ is the canonical projection. If a curve $\gamma : [0,1] \to M$ with control $u \in \Omega(\mathbb{R}^m)$ is energy-minimizing, then there exists $\nu \in \{-1, 0\}$ and an absolutely continuous curve $\eta : [0,1] \to T^*M$ such that $(\eta(t), \nu) \neq 0$ for all $t \in [0,1]$ and

$$\dot{\eta}(t) = \vec{h}_{u(t),\nu}(\eta(t)), \ \textit{for a.e. } t \in [0,1]; \tag{2.13}$$

$$h_{u(t),\nu}(\eta(t)) \geq h_{v,\nu}(\eta(t)), \quad \forall v \in \mathbb{R}^m, \ \textit{for a.e. } t \in [0,1], \tag{2.14}$$

where $\vec{h}_{u(t),\nu}(\eta(t))$ is the Hamiltonian vector field associated to $h_{u(t),\nu}$, see [13], Section 11.5.2 or (2.20).

In this paper we will use a reformulation of Proposition 2.12. We first need to recall the definition of subdifferential.

Definition 2.13. *(Subdifferentials)* Let V be a vector space and $f : V \to \mathbb{R}$. A *subdifferential* of f at $v \in V$ is a linear function $a : V \to \mathbb{R}$ such that

$$a(u - v) \leq f(u) - f(v), \quad \forall u \in V. \tag{2.15}$$

We will use $\partial_v f$ to denote the *set of subdifferentials* of f at v.

It is an easy exercise to check that the set of subdifferentials of a convex function is always a non-empty closed convex set. Let V be a vector space endowed with a norm $\| \cdot \|$. In this article, we are interested in subdifferentials of the energy function $\mathrm{E} := \frac{1}{2} \| \cdot \|^2$. We briefly recall here two properties that subdifferentials of E possess. Let $u \in V$ and $a \in \partial_u \mathrm{E}$. Then

$$a(u) = \|u\|^2; \tag{2.16}$$

$$a(v) \leq \|v\| \|u\|, \quad \forall v \in V. \tag{2.17}$$

For a proof of (2.16) and (2.17) we refer to [14], Lemma 2.19. In sub-Finsler geometry, we can state the PMP in terms of subdifferentials of the energy function.

Proposition 2.14. (subFinsler PMP revised) *Let M be a subFinsler manifold with distribution Δ, let $X_1, ..., X_m \in \mathrm{Vec}(M)$ be such that $\Delta_p = \mathrm{Span}(\{X_1(p), ..., X_m(p)\})$ for every $p \in M$. If a curve $\gamma : [0,1] \to M$ with control u is energy-minimizing, then there exists $\lambda \in T^*_{\gamma(1)}M$ such that either*

$$\left(\phi_t^1\right)^* \lambda \in \partial_{\dot{\gamma}(t)} \mathrm{E}_{\gamma(t)}, \quad \textit{for a.e. } t \in [0,1], \tag{2.18}$$

or $\lambda \neq 0$ and

$$\lambda \left(\left(\phi_t^1\right)_* X \right) = 0, \quad \forall t \in [0,1], \ \forall X \in \Delta_{\gamma(t)}, \tag{2.19}$$

the function ϕ denoting the flow along the control u, see (2.6).

Before the proof of Proposition 2.14, we make two small remarks.

Remark 2.15. The above λ may not be unique, and some such curve γ may admit a λ with property (2.18) and another λ with property (2.19).

Remark 2.16. If the energy $E_{\gamma(t)}$ is differentiable at $\dot\gamma(t)$, then $\partial_{\dot\gamma(t)} \mathrm{E}_{\gamma(t)} = \{\mathrm{d}_{\dot\gamma(t)} E_{\gamma(t)}\}$. Thus, equation (2.18) rewrites as

$$\left(\phi_t^1\right)^* \lambda = \mathrm{d}_{\dot\gamma(t)} E_{\gamma(t)}.$$

However, in this paper we do not assume the differentiability of the energy, since it is not needed for our purposes.

Proof of Proposition 2.14. By Proposition 2.12 if a curve $\gamma : [0,1] \to M$ with control u is energy-minimizing then there exist $\nu \in \{-1, 0\}$ and an absolutely continuous curve $\eta : [0,1] \to T^*M$ such that $\eta(0) \in T^*_{\gamma(0)}M$, $(\eta(t), \nu) \neq 0$ for all $t \in [0,1]$, and equations (2.13) and (2.14) hold. It is well known (see for example [13], Sect. 12.2) that equation (2.13) rewrites for all $t \in [0,1]$ as

$$\begin{cases} \pi(\eta(t)) = \gamma(t); \\ \eta(t) = \left(\phi_t^1\right)^* \eta(1). \end{cases} \tag{2.20}$$

Setting $\lambda := \eta(1)$, by (2.20) and the definition of $h_{v,\nu}$ in (2.12) we have that

$$h_{v,\nu}(\eta(t)) = \left\langle \left(\phi_t^1\right)^* \lambda, \sum_{i=1}^m v_i X_i(\gamma(t)) \right\rangle + \nu\, \mathrm{E}_{\gamma(t)} \left(\sum_{i=1}^m v_i X_i(\gamma(t)) \right). \tag{2.21}$$

In the first case, when $\nu = -1$, equation (2.14) is equivalent to, for all $v \in \mathbb{R}^m$,

$$\left\langle \left(\phi_t^1\right)^* \lambda, \sum_{i=1}^m v_i X_i(\gamma(t)) - \dot\gamma(t) \right\rangle \leq \mathrm{E}_{\gamma(t)} \left(\sum_{i=1}^m v_i X_i(\gamma(t)) \right) - \mathrm{E}_{\gamma(t)} \left(\dot\gamma(t)\right).$$

When v varies, the vector $\sum_{i=1}^m v_i X_i(\gamma(t))$ gives an arbitrary element of the domain of $\mathrm{E}_{\gamma(t)}$. Hence, we have that equation (2.14) is equivalent to (2.18). In the second case, when $\nu = 0$, then necessarely $\lambda \neq 0$. By (2.21), equation (2.14) rewrites as

$$\left\langle \left(\phi_t^1\right)^* \lambda, \sum_{i=1}^m v_i X_i(\gamma(t)) - \dot\gamma(t) \right\rangle \leq 0.$$

Noticing that we are applying $\left(\phi_t^1\right)^* \lambda$ to an arbitrary element of $\Delta_{\gamma(t)}$, we finally deduce that in this second case equation (2.14) is equivalent to (2.19). □

Definition 2.17. *(Normal and abnormal curves)* In the setting of Proposition 2.14, let $\gamma : [0,1] \to M$ be an absolutely continuous curve with control u. We say that γ is a *normal curve* if there exists a covector $\lambda \in T^*_{\gamma(1)}M$ such that (2.18) holds. While we say that γ is an *abnormal curve* if there exist a covector $\lambda \in T^*_{\gamma(1)}M \setminus \{0\}$ such that (2.19) holds.

Note that a curve could be normal and abnormal, simultaneously. We next define normal curves also with domain different than $[0,1]$.

Definition 2.18. Let M be a subFinsler manifold with distribution Δ and let I be an interval. An absolutely continuous curve $\gamma : I \to M$ is a *normal curve* if the restriction of γ to every compact subinterval of I, when affinely re-parametrized in $[0,1]$, is a normal curve in the sense of Definition 2.17.

The following remark ensures that the restriction of a normal curve to every subinterval of its domain is a normal curve and therefore that Definition 2.18 is well posed.

Remark 2.19. Let M be a subFinsler manifold with distribution Δ. Let $\gamma : [0,1] \to M$ be an absolutely continuous curve with control u and $\lambda \in T^*_{\gamma(1)}M$ be a covector for which (2.18) holds. Fix $a, b \in [0,1]$, with $a < b$, and consider the curve $\gamma_{ab} : [0,1] \to M$ defined by

$$\gamma_{ab}(t) := \gamma(a + (b-a)t), \quad \forall t \in [0,1]. \tag{2.22}$$

We claim that γ_{ab} satisfies (2.18) with covector $\lambda_{ab} := \left(\phi^1_b\right)^* (b-a)\lambda$, where ϕ denotes the flow along u. Indeed, we have for all $t \in [0,1]$ that

$$\dot\gamma_{ab}(t) = (b-a)\dot\gamma(a + (b-a)t), \tag{2.23}$$
$$(\phi_{ab})^1_t = \phi^b_{a+(b-a)t}, \tag{2.24}$$

where ϕ_{ab} denotes the flow along the control $t \mapsto (b-a)u(a + (b-a)t)$ of γ_{ab}. Thus, for all $t \in [0,1]$, we have

$$\begin{aligned} \left((\phi_{ab})^1_t\right)^* \lambda_{ab} &= \left((\phi_{ab})^1_t\right)^* \left(\phi^1_b\right)^* (b-a)\lambda \\ &\overset{(2.24)}{=} (\phi^b_{a+(b-a)t})^* \left(\phi^1_b\right)^* (b-a)\lambda \\ &= (b-a)(\phi^1_{a+(b-a)t})^*\lambda. \end{aligned} \tag{2.25}$$

Moreover, since γ and λ satisfy (2.18), we have

$$(\phi^1_{a+(b-a)t})^*\lambda \in \partial_{\dot\gamma(a+(b-a)t)} \operatorname{E}_{\gamma(a+(b-a)t)}, \quad \forall t \in [0,1], \tag{2.26}$$

By definition of subdifferential and of E, if for some $p \in M$, $v \in T_pM$, and $a \in T^*_pM$ we have that $a \in \partial_v \operatorname{E}_p$, then $\tau a \in \partial_{\tau v} \operatorname{E}_p$ for every $\tau > 0$. Therefore, equations (2.23), (2.25), and (2.26) imply

$$\left((\phi_{ab})^1_t\right)^* \lambda_{ab} \in \partial_{\dot\gamma_{ab}(t)} \operatorname{E}_{\gamma_{ab}(t)}, \quad \forall t \in [0,1].$$

With the very same strategy used to prove Remark 2.19, given a normal curve in the sense of Definition 2.18, we obtain the following formula for the covector of the affinely reparametrized curve.

Remark 2.20. Let M be a subFinsler manifold with distribution Δ and $T \in \mathbb{R}$. Let $\gamma : [0,T] \to M$ be a normal curve and define $\bar\gamma : [0,1] \to M$ by setting $\bar\gamma(t) := \gamma(Tt)$. Let u be a control of $\bar\gamma$ and let ϕ be its flow. Let $\bar\lambda \in T^*_{\bar\gamma}M$ be a covector for which $\left(\phi^1_t\right)^* \bar\lambda \in \partial_{\dot{\bar\gamma}(t)} \operatorname{E}_{\bar\gamma(t)}$ holds. Then for all $t \in [0,T]$ we have

$$\left(\phi^1_{\frac{t}{T}}\right)^* \frac{1}{T}\bar\lambda \in \partial_{\dot\gamma(t)} \operatorname{E}_{\gamma(t)}.$$

From Proposition 2.14 and the formula of the differential of End in Proposition 2.10 we can get an analog of (1.2bis) for subFinsler manifolds.

Corollary 2.21. (Normal equation and end-point map) *Let M be a subFinsler manifold with distribution Δ. Choose $X_1, ..., X_m \in \operatorname{Vec}(M)$ such that $\operatorname{Span}(\{X_1(p), ..., X_m(p)\}) = \Delta_p$ for every $p \in M$ and fix $q \in M$. Denote* *with* $\mathrm{End}:\Omega(\mathbb{R}^m)\to M$ *the end-point map associated to* $X_1,...,X_m$ *and* q*. Let* $\gamma:[0,1]\to M$ *be a normal curve with control* u *such that* $\gamma(0)=q$*. Let* $\lambda\in T^*_{\gamma(1)}M$ *be a covector for which* (2.18) *holds. Then*

$$\lambda(\mathrm{d}\,\mathrm{End}_u\,u)=\int_0^1||\dot\gamma(t)||^2_{\gamma(t)}\mathrm{d}t. \tag{2.27}$$

Proof. By Proposition 2.10 taking $v=u$, we have

$$\begin{aligned}\lambda(\mathrm{d}\,\mathrm{End}_u\,u)&=\int_0^1\lambda\left(\left(\phi_t^1\right)_*\sum_{i=1}^m u_i(t)X_i(\gamma(t))\right)\mathrm{d}t,\\ &=\int_0^1\lambda\left(\left(\phi_t^1\right)_*\ \dot\gamma(t)\right)\mathrm{d}t.\end{aligned}\tag{2.28}$$

By (2.16) and (2.18) we have

$$\left(\left(\phi_t^1\right)^*\lambda\right)(\dot\gamma(t))=||\dot\gamma(t)||^2,\ \text{for a.e. } t\in[0,1], \tag{2.29}$$

Equations (2.28) and (2.29) imply (2.27).

□

In the case of Carnot groups we always choose a left-invariant frame as the set of vector fields spanning the distribution Δ at each point. Therefore, we interpret controls as functions $u\in L^2([0,1];\Delta_1)$: the left translation to the origin of the derivative of the associated curve. The flow along a control coincides with the right-translation along the associated curve, hence we rewrite Proposition 2.14 and Corollary 2.21 as follows:

Proposition 2.22. *Let* G *be a subFinsler Carnot group with first stratum* V_1 *equipped with a left-invariant norm. Let* $\{X_1,...,X_m\}$ *be a left-invariant frame of the first stratum. Let* End *be the end-point map associated to* $X_1,...,X_m$ *and* 1_G*. Let* $\gamma:[0,1]\to G$ *be an absolutely continuous curve with control* $u\in L^2([0,1];V_1)$ *and such that* $\gamma(0)=1_G$*. Then, the curve* γ *is normal if and only if there exist* $\lambda\in(\mathrm{Lie}(G))^*$ *such that*

$$\lambda\circ\mathrm{Ad}_{\gamma(t)}\in\partial_{u(t)}\,\mathrm{E}_1,\ \textit{for a.e. } t\in[0,1]. \tag{2.30}$$

Moreover, if λ *is a right-invariant 1-form on* G *satisfying* (2.30)*, then*

$$\lambda(\mathrm{d}\,\mathrm{End}_u\,u)=||u||^2_{L^2}. \tag{2.31}$$

Proof. Given a right-invariant 1-form λ on G we denote by $\bar\lambda$ its restriction at $T_{\gamma(1)}G$, so $\bar\lambda\in T^*_{\gamma(1)}G$. Since the flow along u is $\phi^t=R_{\gamma(t)}$, see (2.9), there holds

$$\begin{aligned}\left(\phi_t^1\right)^*\bar\lambda&=\bar\lambda\circ\mathrm{d}R_{\gamma(1)}\circ\mathrm{d}R^{-1}_{\gamma(t)}\\ &=\lambda\circ\mathrm{d}R^{-1}_{\gamma(t)},\qquad\forall t\in[0,1].\end{aligned}$$

Consequently, the curve γ is normal, which by definition means that some covector $\bar\lambda\in T^*_{\gamma(1)}G$ satisfies (2.18), if and only if

$$\lambda\circ\mathrm{d}R^{-1}_{\gamma(t)}\in\partial_{\dot\gamma(t)}\,\mathrm{E}_{\gamma(t)},\ \text{for a.e. } t\in[0,1].$$

Being $\dot\gamma = \mathrm{d}L_\gamma u$ (see (2.8)) and being E left-invariant, the last equation is equivalent to

$$\lambda \circ \mathrm{d}R^{-1}_{\gamma(t)} \circ \mathrm{d}L_{\gamma(t)} \in \partial_{u(t)} \mathrm{E}_1, \text{ for a.e. } t \in [0,1],$$

which is exactly (2.30). Thus the first part of the proposition is proved. Finally, Corollary 2.21 rephrases as equation (2.31). □

We next prove a property of normal curves that, in the case the norm is smooth, is a well known fact: Normal curves have constant speed.

Proposition 2.23. *Let G be a Carnot group and $\gamma : [0,1] \to G$ be a normal curve. Then γ is parametrized by multiple of arc-length.*

Proof. Let u be the control for γ and $\lambda \in \mathrm{Lie}(G)^*$ be a covector such that (2.30) holds. For $t \in [0,1]$, set $F(t) := \|u(t)\|$. Denote with $\|\cdot\|_* : \mathrm{Lie}(G)^* \to \mathbb{R}$ the Lipschitz function

$$\|\eta\|_* := \max\{\eta(v) \mid v \in V_1, \|v\| = 1\}, \quad \forall \eta \in T^*_1 G,$$

We first remark that by equations (2.16), (2.17), and (2.30), we have for a.e. $t \in [0,1]$ that

$$F(t) = \left\|\lambda \circ \mathrm{Ad}_{\gamma(t)}\right\|_* . \tag{2.32}$$

Since $\lambda \circ \mathrm{Ad}_\gamma$ is a continuous function defined on a compact set, it is bounded, thus by (2.32) we have $u \in L^\infty([0,1], V_1)$ and therefore γ is a Lipschitz function. Using again (2.32), we have that F is Lipschitz and differentiable almost everywhere.

We claim that F is non decreasing. Indeed, we show that $F'(t) \geq 0$ for all $t \in A := \{t' \in [0,1] \mid F \text{ and } \gamma \text{ are differentiable at } t'\}$. Fix $t \in A$. For $\epsilon > 0$ we have

$$\begin{aligned} F(t+\epsilon)\|u(t)\| &\overset{(2.32)}{=} \left\|\lambda \circ \mathrm{Ad}_{\gamma(t+\epsilon)}\right\|_* \|u(t)\| \\ &\geq \lambda \circ \mathrm{Ad}_{\gamma(t+\epsilon)}(u(t)) \\ &= \lambda \circ \mathrm{Ad}_{\gamma(t)}(u(t)) + \epsilon\lambda \circ \mathrm{Ad}_{\gamma(t)}([u(t), u(t)]) + \mathrm{o}(\epsilon) \\ &\overset{(2.30),(2.16)}{=} F(t)\|u(t)\| + \mathrm{o}(\epsilon). \end{aligned}$$

If $\|u(t)\| \neq 0$ we get from the above equation that $F'(t) \geq 0$. If $F(t) = \|u(t)\| = 0$, then necessarily $F'(t) \geq 0$ since $F \geq 0$. Thus the claim is proved.

We are left to show that F is non-increasing. Consider the curve $\sigma : [0,1] \to G$ defined for all $t \in [0,1]$ by $\sigma(t) := \gamma(1-t)$. It is easy to check that σ and $-\lambda$ solve (2.30), thus σ is a normal curve. In particular, by the reasoning made above, the norm of its control $[0,1] \ni t \mapsto u(1-t) \in V_1$ is non-decreasing. As a consequence, we have that F is a non-increasing function. □

Remark 2.24. Let G be a Carnot group, $I \subseteq \mathbb{R}$ is an interval, $\gamma : I \to G$ a normal geodesic, and $u := \mathrm{d}L^{-1}_\gamma \dot\gamma$. From Proposition 2.22 and Remark 2.20 we have that for every compact interval $J \subseteq I$ there exists a covector $\lambda \in T^*_1 G$ such that for every $t \in J$ we have

$$\lambda \circ \mathrm{Ad}_{\gamma(t)} \in \partial_{u(t)} \mathrm{E}_1 . \tag{2.33}$$

A priori the covector λ depends on the interval J. However, similarly as in [14], Section 3.1, one can actually prove that λ can be chosen independently of the interval J. Consequently, one can choose the covector λ in a way that equation (2.33) holds for every $t \in I$.

Definition 2.25. Let G be a subFinsler Carnot group, I be an interval, with $0 \in I$, and $\gamma : I \to G$ be a normal curve. Set $u := \mathrm{d}L_\gamma^{-1}\dot\gamma$. If $\gamma(0) = 1_G$, we say that $\lambda \in T^*_{\gamma(0)}G$ is a *covector associated to* γ if

$$\lambda \circ \mathrm{Ad}_{\gamma(t)} \in \partial_{u(t)} \mathrm{E}_1, \text{ for a.e. } t \in I. \tag{2.34}$$

If $\gamma(0) \neq 1_G$, the *covector associated to* γ is by definition $\lambda := \tilde\lambda \circ \mathrm{d}R^{-1}_{\gamma(0)}$, where $\tilde\lambda$ is the covector associated to $L^{-1}_{\gamma(0)} \circ \gamma$.

3. Proof of main results

3.1. The differential of the dilations

This subsection is devoted to a computation of the differential of the one-parameter family δ of dilations in Carnot groups. The aim is to write the vector field

$$\vec\delta(g) := \frac{\mathrm{d}}{\mathrm{d}\tau}\delta_\tau(g)\Big|_{\tau=1} \tag{3.1}$$

as a linear combination of right-invariant vector-fields. Then, using the particular form of the coefficients of this combination, we will get an estimate of $\lambda(\vec\delta(g))$ in terms of the distance $d(1, g)$ for every right-invariant 1-form λ.

Definition 3.1. Let G be a Carnot group with stratified Lie algebra $\mathfrak{g} = V_1 \oplus ... \oplus V_s$. An ordered basis $X_1, ..., X_m$ of $\mathfrak{g}$ is *adapted to the stratification* if

$$X_i \in V_j, \quad \forall j \in 1, ..., s, \forall i \in \{1, ..., m\} \text{ s.t. } \sum_{1\le k\le j-1} \dim(V_k) < i \le \sum_{1\le k\le j} \dim(V_k),$$

where we are setting $V_0 := \{0\}$.
We say that a vector $X \in \mathfrak{g}$ has *degree of homogeneity* $j \in \{1, ..., s\}$ if $X \in V_j$. We say that a function $f : G \to \mathbb{R}$ is *homogeneous* of degree $\alpha \in \mathbb{R}$ if $f \circ \delta_\tau = \tau^\alpha f$ for all $\tau \geq 0$.

Lemma 3.2. *Let G be a Carnot group, fix a basis $X_1, ..., X_m$ adapted to the stratification and let $\vec\delta$ be as in (3.1). Then*

$$\vec\delta = \sum_{i=1}^m P_i X_i^\dagger, \tag{3.2}$$

where, for each $i \in \{1, ..., m\}$, we denote by $X_i^\dagger$ the right-invariant extension of X_i and $P_i : G \to \mathbb{R}$ are homogeneous functions of degree d_i, the integer $d_i \in \mathbb{N}$ being the degree of homogeneity of X_i.

Proof. Fix $\tau > 0$. For all $g \in G$ we have

$$\left((\delta_\tau)_* \vec\delta\right)(g) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\delta_\tau \circ \delta_{1+\epsilon}(\delta_\tau^{-1}(g))\Big|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\delta_{1+\epsilon}(g)\Big|_{\epsilon=0} = \vec\delta(g). \tag{3.3}$$

Moreover, for all $i \in \{1, ..., m\}$ there holds

$$(\delta_\tau)_* X_i^\dagger = \tau^{d_i} X_i^\dagger, \tag{3.4}$$

since $(\delta_\tau)_* X_i^\dagger(1_G) = \tau^{d_i} X_i^\dagger(1_G)$ and $(\delta_\tau)_* X_i^\dagger$ is a right-invariant vector field, being δ_τ a group homomorphism and $X_i^\dagger$ right-invariant.

Since $\vec{\delta}$ is a smooth vector field there exist smooth functions $P_1, ..., P_m : G \to \mathbb{R}$ such that (3.2) holds. To conclude the proof of the lemma we have to show that

$$P_i \circ \delta_\tau = \tau^{d_i} P_i, \quad \forall i \in \{1, ..., m\}. \tag{3.5}$$

For all $i \in \{1, ..., m\}$ we have

$$\begin{aligned} \sum_{i=1}^{m} P_i X_i^\dagger &\overset{(3.2)}{=} \vec{\delta} \\ &\overset{(3.3)}{=} (\delta_\tau)_* \vec{\delta} \\ &\overset{(3.2)}{=} (\delta_\tau)_* \left(\sum_{i=1}^{m} P_i X_i^\dagger \right) \\ &\overset{(3.4)}{=} \sum_{i=1}^{m} \tau^{d_i} (P_i \circ \delta_\tau^{-1}) X_i^\dagger, \end{aligned}$$

and consequently (3.5) holds. □

The identity found in Lemma 3.2 allows us to give an estimate of the value that we get applying a right-invariant 1-form to the vector $\vec{\delta}(g)$ defined in (3.1). Up to constants, the bound we find depends only on the covector and on the distance of g from the origin. To get a quantitative estimate we fix a norm on the Lie algebra.

Definition 3.3. Let G be a Lie group and $\lambda \in T_1^*G$. Choose a basis $\{X_1, ..., X_n\}$ of T_1G and define

$$N(\lambda) := \sum_{i=1}^{n} |\lambda(X_i)|. \tag{3.6}$$

Up to a multiplicative constant the norm N is equivalent to every other norm on T_1^*G.

Lemma 3.4. *Let G be a Carnot group. Define $\vec{\delta}$ as in (3.1). There exists a constant $C > 0$ such that for every right-invariant 1-form λ we have*

$$\lambda(\vec{\delta}(g)) \le C N(\lambda_1) \max(d(1, g), d(1, g)^s), \qquad \forall g \in G, \tag{3.7}$$

where $N(\lambda_1)$ is defined as in Definition 3.3.

Proof. Choose a basis $X_1, ..., X_n$ adapted to the stratification; call $\{X_i^\dagger\}_{i \in \{1,...,n\}}$ the right-invariant extension of this basis. Let $d_i \in \mathbb{N}$ be the degree of homogeneity of X_i. From Lemma 3.2 we have that (3.2) holds. For all $i \in \{1, ..., n\}$, being the function P_i homogeneous of degree d_i, there exist a constant $C_i > 0$ such that

$$P_i(g) \le C_i d(1, g)^{d_i}, \quad \forall g \in G. \tag{3.8}$$

Setting $C = \max_{1,...,n} C_i$, for every right-invariant 1-form λ, by definition of $N(\lambda_1)$, we have

$$\lambda(\vec{\delta}(g)) \overset{(3.2)}{=} \sum_{i=1}^{n} P_i(g) \lambda(X_i) \overset{(3.6),(3.8)}{\le} C N(\lambda_1) \max(d(1, g), d(1, g)^s), \quad \forall g \in G. \tag{3.9}$$

This concludes the proof of the lemma. □

3.2. Proof of Theorem 1.1

We are now ready to prove the main result of this paper.

Proof of Theorem 1.1. Let $\gamma : \mathbb{R} \to G$ be a normal curve parametrized by arc-length. Without loss of generality we assume $t' = 0$, $t > 0$ and $\gamma(0) = 1_G$. Since all norms on $T^*_{1_G}G$ are equivalent, we don't lose generality assuming that N is the norm defined by (3.6). Choose a covector λ associated to γ (see Def. 2.25). Denote with λ also the right-invariant extension of λ. Denote with V_1 the first stratum of the stratification of Lie(G). For $t > 0$, denote with $u_t \in L^2([0,1]; V_1)$ the control of the curve $\gamma_t : [0,1] \to G$, $\gamma_t(\tau) := \gamma|_{[0,t]}(t\tau)$ for all $\tau \in [0,1]$. For every $t > 0$, we have

$$\mathrm{d}\,\mathrm{End}_{u_t} u_t = \frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{End}(\tau u_t)\bigg|_{\tau=1}. \tag{3.10}$$

Being τu_t the control of $\delta_\tau \gamma_t$, we can rewrite (3.10) as

$$\mathrm{d}\,\mathrm{End}_{u_t} u_t = \frac{\mathrm{d}}{\mathrm{d}\tau} \delta_\tau(\mathrm{End}(u_t))\bigg|_{\tau=1} = \vec{\delta}(\mathrm{End}(u_t)), \quad \forall t > 0, \tag{3.11}$$

where $\vec{\delta}$ is defined by (3.1). Applying Lemma 3.4, we get that there exists a constant $C > 0$ such that

$$\lambda(\vec{\delta}(\mathrm{End}(u_t))) \le CN(\lambda) \max(d(1, \mathrm{End}(u_t)), d(1, \mathrm{End}(u_t))^s), \quad \forall t > 0. \tag{3.12}$$

By (3.11) and (3.12), and being $\mathrm{End}(u_t) = \gamma(t)$ for all $t > 0$, we have

$$\lambda(\mathrm{d}\,\mathrm{End}_{u_t} u_t) \le CN(\lambda) \max(d(1, \gamma(t)), d(1, \gamma(t))^s), \quad \forall t > 0. \tag{3.13}$$

Moreover, we have from Remark 2.20 and Proposition 2.22 that

$$t\lambda(\mathrm{d}\,\mathrm{End}_{u_t} u_t) = (\mathrm{length}(\gamma|_{[0,t]}))^2 = t^2, \quad \forall t > 0. \tag{3.14}$$

From equations (3.13) and (3.14) we get for every $t > 0$ that

$$t \le CN(\lambda) \max(d(1, \gamma(t)), d(1, \gamma(t))^s). \tag{3.15}$$

In particular, for $t > CN(\lambda)$ we must have

$$d(1, \gamma(t)) > 1,$$

and therefore

$$\max(d(1, \gamma(t)), d(1, \gamma(t))^s) = d(1, \gamma(t))^s.$$

Thus, for all $t > CN(\lambda)$, equation (3.15) becomes

$$t \le CN(\lambda) d(1, \gamma(t))^s. \tag{3.16}$$

Set $\epsilon := C^{-\frac{1}{s}}$. If $t > CN(\lambda)$ equation (1.1) is a conseqeuence of (3.16). If $t \le CN(\lambda)$ equation (1.1) is trivially true being the right-hand side less or equal than 0. We showed that (1.1) holds for every $t > 0$, thus we concluded the proof of the theorem. □

3.3. Proof of Corollary 1.2

This section is devoted to the proof of Corollary 1.2. We will actually prove a stronger statement:

Theorem 3.5. *In self-similar spaces, normal loops starting from the origin are constant.*

Proof. Let G be a Carnot group and $H < G$ be a subgroup invariant under the action of dilations. Denote with $\pi : G \to {}_{H}\backslash^{G}$ the canonical projection. Fix a left-invariant frame $X_1, ..., X_m$ of the first stratum of G and write $\bar{X}_i := \pi_* X_i$ for $i \in \{1, ..., m\}$. Let $\gamma : [0,1] \to {}_{H}\backslash^{G}$ be a normal curve, with control $u \in L^2([0,1]; \mathbb{R}^m)$, such that $\gamma(0) = \gamma(1) = \pi(1_G)$. Being γ normal, by Corollary 2.21 there exists a covector $\lambda \in T^*_{\gamma(1)}\left({}_{H}\backslash^{G}\right)$ such that

$$\lambda(\mathrm{d}\,\mathrm{End}_u\, u) = \int_0^1 ||\dot{\gamma}(t)||^2_{\gamma(t)} \mathrm{d}t. \tag{3.17}$$

If we denote with $\delta_\tau : {}_{H}\backslash^{G} \to {}_{H}\backslash^{G}$ the dilation of a factor $\tau \in \mathbb{R}$, we have $(\delta_\tau)_* \bar{X}_i = \tau \bar{X}_i$. Thus, the control of the curve $\delta_\tau \circ \gamma$ is τu. As a consequence,

$$\mathrm{d}\,\mathrm{End}_u\, u = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \mathrm{End}((1+\epsilon)u)\Big|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \delta_{(1+\epsilon)}(\mathrm{End}(u))\Big|_{\epsilon=0} = 0, \tag{3.18}$$

where in the last equality we used that $\delta_{(1+\epsilon)}(\mathrm{End}(u))$ is constantly equal to $\pi(1_G)$. Using equations (3.17) and (3.18) we deduce that the curve γ is constant. ☐

4. Some examples

4.1. Curves with end-point arbitrarely close to the origin

Even if by Theorem 3.5 we know that in Carnot groups non-constant normal loops do not exists, we can find normal curves of length 1 whose end-points are arbitrarily close. We present a simple example in the Heisenberg group.

Definition 4.1. The *3-dimensional Heisenberg group* is the Carnot group $(\mathbb{R}^3, \cdot)$ with product law

$$(x, y, z) \cdot (x', y', z') := (x + x', y + y', z + z' - \frac{1}{2}(x'y - xy')), \forall x, y, z, x', y', z' \in \mathbb{R}.$$

In the Heisenberg group we choose as orthonormal left-invariant frame of the first stratum the two vector fields $\{X, Y\}$ defined for all $x, y, z \in \mathbb{R}$ by

$$X(x, y, z) := \partial_x - \frac{y}{2}\partial_z;$$
$$Y(x, y, z) := \partial_y + \frac{x}{2}\partial_z.$$

Proposition 4.2. *Let $G := (\mathbb{R}^3, \cdot)$ be the subRiemannian 3-dimensional Heisenberg group. For every $\epsilon > 0$ there exists a normal curve $\gamma : [0,1] \to \mathbb{R}^3$, parametrized by arclength, such that $\gamma(0) = 1_G$ and $d(\gamma(1), 1_G) < \epsilon$.*

Proof. For every $N \in \mathbb{N}$, in the coordinates of Definition 4.1, consider the curve $\gamma_N : [0,1] \to G$ defined as $\gamma_N(t) := (x_N(t), y_N(t), z_N(t))$ with

$$x_N(t) := \frac{\cos(2\pi N t) - 1}{2\pi N};$$

$$y_N(t) := \frac{\sin(2\pi N t)}{2\pi N};$$

$$z_N(t) := \frac{2\pi N t - \sin(2\pi N t)}{8(\pi N)^2}.$$

The curve γ_N is the horizontal lift of a circle of radius $\frac{1}{2\pi N}$ travelled N times and is a normal geodesic parametrized by arc-length (see for example [7], Section 1 for a characterization of geodesics in the Heisenberg group). We have $\gamma_N(0) = 1_G$ and $\gamma_N(1) = (0, 0, \frac{1}{4\pi N})$. Therefore, for every $\epsilon > 0$ there exists $N > 0$ such that $\gamma_N(1) \in B(1_G, \epsilon)$. □

4.2. Optimality of the exponent in Theorem 1.1

In complete generality we cannot obtain an estimate as the one in Theorem 1.1 with an exponent bigger than the reciprocal of the step. Indeed, for every $s \in \mathbb{N}$, we provide an example of a normal curve γ, in a filiform group of step s, for which the distance of $\gamma(t)$ from the origin is bounded above by a constant times $t^{\frac{1}{s}}$.

Definition 4.3. The *subRiemannian filiform group of first type* is the Carnot group with stratified Lie algebra $\mathfrak{g} = V_1 \oplus ... \oplus V_s$ with basis $X_1, Y_1, ..., Y_s$, $\{X_1, Y_1\}$ being an orthonormal basis of V_1, $V_j = \mathrm{Span}(Y_j)$ for $j = 2, ..., s$, and only non-trivial bracket relations $[X_1, Y_i] = Y_{i+1}$, for every $i = 1, ..., s-1$.

The Lie algebra $\mathfrak{g}$ in Definition 4.3 is an example of a Goursat distribution [15], p. 81. We recall that in the subRiemannian filiform groups of first type all length-minimizing curves are normal (see for example [16], Prop. 4.1).

Proposition 4.4. *Let G be the sub-Riemmanian filiform group of first type of step s. Let Y_s be a vector spanning the s-th stratum and $\gamma : \mathbb{R} \to G$ be a normal curve with $\gamma(0) = 1_G$, $\gamma(1) = \exp(Y_s)$ and $\gamma|_{[0,1]}$ energy-minimizing. Then there exists $C > 0$ such that for every $t > 1$ there holds*

$$d(1, \gamma(t)) < C t^{\frac{1}{s}}. \tag{4.1}$$

Proof. Let $\{X_1, X_2\}$ be an orthonormal frame of the first stratum. Let λ be a covector associated to γ, see Definition 2.25. Since the energy at the origin is differentiable with differential

$$\mathrm{d}(\mathrm{E}_1)_w v = (w, v), \quad \forall v, w \in T_1 G,$$

equation (2.34) rewrites as

$$\lambda(\mathrm{Ad}_{\gamma(t)} X_i) = (u(t), X_i), \quad \forall t \in \mathbb{R}, \forall i \in \{1, 2\}. \tag{4.2}$$

We claim that

$$\gamma(m + t) = \exp(m Y_s)\gamma(t), \quad \forall m \in \mathbb{N}, \forall t \in \mathbb{R}. \tag{4.3}$$

Indeed, the curve $t \mapsto \exp(mY_s)\gamma(t)$ solves the normal equation (4.2) with respect to the covector λ: for all $t \in \mathbb{R}$ we have

$$\begin{aligned}(\exp(mY_s)\gamma(t))' &= \mathrm{d}L_{\exp(mY_s)}\gamma(t)' \\ &= \mathrm{d}L_{\exp(mY_s)} \sum_{i=1}^{2} \lambda(\mathrm{Ad}_{\gamma(t)} X_i) X_i(\gamma(t)) \\ &= \sum_{i=1}^{2} \lambda(\mathrm{Ad}_{\exp(mY_s)\gamma(t)} X_i) X_i(\exp(mY_s)\gamma(t)),\end{aligned} \tag{4.4}$$

where in the second equality we used that γ solves (4.2) and the last equality comes from the fact that $\exp(mY_s)$ is in the center.

We have

$$d(1, \exp(mY_s)) = d(1, \delta_{m^{\frac{1}{s}}}(\exp(Y_s))) = m^{\frac{1}{s}} d(1, \exp(Y_s)), \quad \forall m \in \mathbb{N}. \tag{4.5}$$

With equation (4.5) together with (4.3) and the triangle inequality we obtain

$$d(1, \gamma(m+t)) \leq m^{\frac{1}{s}} d(1, \exp(Y_s)) + d(1, \gamma(t)), \quad \forall m \in \mathbb{N}, \forall t \in \mathbb{R}. \tag{4.6}$$

In particular, being $\gamma|_{[0,1]}$ length minimizing and $\gamma(1) = \exp(Y_s)$, we have

$$\begin{aligned} d(1, \gamma(m+t)) &\leq m^{\frac{1}{s}} d(1, \exp(Y_s)) + d(1, \exp(Y_s)) \\ &\leq d(1, \exp(Y_s))(m^{\frac{1}{s}} + 1) \\ &< C m^{\frac{1}{s}} \\ &\leq C(m+t)^{\frac{1}{s}}, \quad \forall m > 1, \forall t \in [0,1], \end{aligned} \tag{4.7}$$

where $C = 2d(1, \exp(Y_s))$. This concludes the proof of the proposition. □

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