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EXTREMAL FUNCTIONS FOR THE CRITICAL TRUDINGER–MOSER INEQUALITY WITH LOGARITHMIC KERNELS

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Abstract. In this paper we study Moser–Trudinger type inequalities for some nonlocal energy functionals in presence of a logarithmic convolution potential, when the domain is a ball. In particular, we perform a blow-up analysis to prove existence of extremal functions in the borderline case of critical growth. Using this, we sharpen the results in [S. Cingolani and T. Weth *J. London Math. Soc.* **105** (2022) 1897–1935] under critical growth assumptions and give answers to some questions left open in [S. Cingolani and T. Weth *J. London Math. Soc.* **105** (2022) 1897–1935].

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1. Introduction

We consider some nonlocal interaction functionals, which arise in some 2D mathematical models for chemotaxis [1], in the statistical mechanics of selfgravitating clouds [2, 3] and in the description of vortices in turbulent Euler flows [4]. See also [5–9].

In the present paper we are interested to derive Moser–Trudinger type inequalities for energy functionals in presence of a logarithmic convolution potential. Inequalities of this type have been introduced recently in [10], and the purpose of the present paper is to sharpen the results of [10] under critical growth assumptions and to answer some questions left open in [10].

We begin to recall that the classical Trudinger-Moser inequality states that for any bounded domain $\Omega \subset \mathbb{R}^2$ we have

$$\sup_{u \in H_0^1(\Omega), |\nabla u|_2 \leq 1} \int_\Omega \mathrm{e}^{4\pi u^2} \mathrm{d}x = c(\Omega) < +\infty \tag{1.1}$$

where, here and in the following, $|\cdot|_p$ denotes the usual L^p-norm for $1 \leq p \leq \infty$, so $u \mapsto |\nabla u|_2^2$ is the classical Dirichlet integral over Ω (see [11–13]). Carleson and Chang [14] proved that the supremum in (1.1) is attained if $\Omega = B_1 := B_1(0)$ is the unit disc in $\mathbb{R}^2$, and successively Flucher [15] extended this result to arbitrary bounded

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domains in $\mathbb{R}^2$. Related inequalities for unbounded domains have been found by Cao [16], Adachi-Tanaka [17], Ruf [18].

In this paper, as in [10], we consider the nonlocal interaction functional of the form

$$u \mapsto \Phi(u) := \int_{B_1}\int_{B_1} \ln \frac{1}{|x-y|} F(u(x))F(u(y))\,\mathrm{d}x\mathrm{d}y, \tag{1.2}$$

and we wish to analyze the problem of maximizing the quantity (1.2) among functions in the set

$$\mathcal{B}_1 := \{u \in H_0^1(B_1) \ : \ |\nabla u|_2 \le 1\}.$$

Here $F : \mathbb{R} \to [0,\infty)$ is even and continuous on $[0,\infty)$, and it satisfies the weak growth condition

$$F(t) \le c\mathrm{e}^{\alpha t^2} \qquad \text{for } t \in \mathbb{R} \text{ with constants } \alpha, c > 0, \tag{1.3}$$

which ensures that the double integral in (1.2) is well defined for functions $u \in H_0^1(B_1)$. More precisely, splitting the kernel $\ln \frac{1}{|\cdot|}$ into its positive and negative part and defining the functionals $\Phi_\pm : \mathcal{M}(B_1) \to [0,\infty]$ by

$$\Phi_\pm(u) = \int_{B_1}\int_{B_1} \ln^\pm \frac{1}{|x-y|}\, F(u(x))F(u(y))\,\mathrm{d}x\mathrm{d}y,$$

where $\ln^\pm = \max\{\pm\ln, 0\}$ and $\mathcal{M}(B_1)$ denotes the space of the real Lebesgue-measurable functions on B_1, it follows from [10], Lemma 2.5 and Corollary 2.6 that $\Phi_\pm(u) < \infty$ for every $u \in H_0^1(B_1)$, and therefore the quantity in (1.2) has a well-defined finite value

$$\Phi(u) := \int_{B_1}\int_{B_1} \ln\frac{1}{|x-y|}\, F(u(x))F(u(y))\,\mathrm{d}x\mathrm{d}y = \Phi_+(u) - \Phi_-(u) \quad \text{for every } u \in H_0^1(B_1). \tag{1.4}$$

To analyze the maximization problem for Φ within $\mathcal{B}_1$, we put

$$m_1(F) := \sup_{u\in\mathcal{B}_1} \Phi(u). \tag{1.5}$$

If F also satisfies

$$F(s) \le c_1\mathrm{e}^{\alpha s^2} \qquad \text{for } s \in \mathbb{R} \text{ with constants } \alpha < 4\pi,\ c_1 > 0, \tag{1.6}$$

then the classical Trudinger-Moser inequality and the logarithmic Hardy-Littlewood-Sobolev inequality [19], Theorem 2 imply that $m_1(F) < \infty$.

In [10], Theorem 1.2, the following has been proved under the additional assumption that F is increasing on $[0,\infty)$:

(I) If F has at most β-critical growth for some $\beta \le -1$, then $m_1(F) < \infty$, and $m_1(F)$ is attained by Φ in $\mathcal{B}_1$ if $\beta < -1$.
(II) If F has at least β-critical growth for some $\beta > -1$, then $m_1(F) = \infty$.

Here, *at most β-critical growth* means that $F(t) \le c\,\mathrm{e}^{4\pi t^2}(1+|t|)^\beta$ for $t \in \mathbb{R}$ with some constant $c > 0$, while *at least β-critical growth* means that there exist $t_0, c > 0$ with the property that

$$F(t) \ge c\,\mathrm{e}^{4\pi t^2}|t|^\beta \qquad \text{for } |t| \ge t_0.$$

The assumption that F is increasing was used in [10] to reduce the maximization problem, via Schwarz symmetrization, to a problem for radial functions. This extra assumption and the reduction to the radial setting will also play a key role in most of the present paper. We emphasize that [10], Theorem 1.2 shows that $\beta = -1$ is a critical borderline exponent, and it indicates a transition behavior for critically growing nonlinearities F in this sense. To analyze this behavior in detail, we assume in the following that $F : \mathbb{R} \to [0, +\infty)$ has the form

$$F(t) = \frac{\mathrm{e}^{4\pi t^2}}{1+|t|} g(|t|), \tag{1.7}$$

where

(g_0) $g : [0, \infty) \to [0, \infty)$ satisfies

$$g(t) \le \gamma \mathrm{e}^{\gamma t^2} \qquad \text{for } t \ge 0 \text{ with some } \gamma \ge 1. \tag{1.8}$$

For our main results, we also need the following monotonicity condition.

(g_1) The function g is of class C^1 and satisfies $F' \ge 0$ on $(0, \infty)$, *i.e.*,

$$(8\pi t^2 + 8\pi t - 1) g(t) + (1+t) g'(t) \ge 0 \quad \forall t \in (0, \infty). \tag{1.9}$$

Remark 1.1. (i) We note that the weak growth bound (1.8) ensures (1.3).
(ii) We point out that (1.9) holds in particular if

$$g'(t) > \frac{g(t)}{1+t} \quad \text{for } t \in (0,1) \qquad \text{and} \qquad g'(t) \ge 0 \quad \text{for } t \ge 1. \tag{1.10}$$

We also note for later use that, if (1.9) holds, we have $g'(t) \ge \frac{1-8\pi t^2 - 8\pi t}{1+t} g(t) \ge 0$ and therefore $g(t) \ge g(0)$ for $t \in [0, \sqrt{\frac{2\pi+1}{8\pi}} - \frac{1}{2}]$.

In our first theorem, we provide sharp borderline conditions for the problem of maximizing Φ in $\mathcal{B}_1$ depending on the asymptotic behaviour of the function g at infinity. To state our main results, we define

$$C_g := \limsup_{t \to +\infty} g(t) \in [0, \infty]. \tag{1.11}$$

Theorem 1.2. *Suppose that g satisfies (g_0).*

(i) If $C_g < \infty$, then $m_1(F) < \infty$.
(ii) If $\lim_{t\to+\infty} g(t) = +\infty$, then $m_1(F) = +\infty$.
(iii) If g also satisfies (g_1) and $C_g = 0$, then $m_1(F)$ is attained, and every maximizer for Φ in $\mathcal{B}_1$ is, up to sign, a radial and radially decreasing function in $\mathcal{B}_1$.

Remark 1.3. (i) Theorem 1.2(i) is essentially contained in [10], Theorem 1.2, since F has at most (-1)-critical growth if $C_g < \infty$. However, as has been mentioned already, it was assumed in addition in [10], Theorem 1.2 that F is increasing on $[0, \infty)$, and we shall note in Section 2 below that this restriction is unnecessary.
(ii) We notice that Theorem 1.2(ii) and (iii) both improve [10], Theorem 1.2. Indeed, consider first $\sigma \ge 2\log 3$ and the function

$$t \mapsto g(t) = \log^{\sigma}(2+t), \quad t \ge 0$$

Then g satisfies (g_0), (g_1) and $\lim_{t\to+\infty} g(t) = +\infty$, so $m_1(F) = +\infty$ by Theorem 1.2(ii). However, F has not at least β-critical growth if $\beta > -1$, so [10], Theorem 1.2 does not apply. Similarly, we may consider $0 < \sigma < \frac{\log 2}{2}$ and the function

$$t \mapsto g(t) = \frac{t}{2+t} \log^{-\sigma}(2+t), \quad t \geq 0$$

Then g satisfies (g_0), (g_1) and $C_g = 0$, so $m_1(F)$ is attained by Theorem 1.2(iii). On the other hand, F does not have at most β-critical growth if $\beta < -1$, so [10], Theorem 1.2 does not apply.

Theorem 1.2 leaves open the question whether $m_1(F)$ is attained in the purely critical case where $C_g \in (0, \infty)$. The study of this question is the context of the main results of this paper. Our first answer to this question is the following conditional result.

Theorem 1.4. *Suppose that g satisfies (g_0), (g_1) and $C_g \in (0, \infty)$. If there exists $u \in \mathcal{B}_1$ with*

$$\Phi(u) > \pi^2\Big(\frac{g^2(0)}{4} + 2C_g^2\pi \mathrm{e}^2\Big), \tag{1.12}$$

then $m_1(F)$ is attained in $\mathcal{B}_1$.

The sufficient condition (1.12) is a consequence of a detailed analysis of $\Phi(u_n)$ for concentrating sequences. For this we need the following definition.

Definition 1.5. We call a sequence of functions $u_n \in \mathcal{B}_1$ a *Schwarz symmetric concentrating sequence* (*SCS*-sequence in short) if u_n is radial, nonnegative and nonincreasing in the radial variable for every n and satisfies $u_n \rightharpoonup 0$ weakly in $H_0^1(B_1)$ but not strongly.

We then have the following upper bound.

Theorem 1.6. *Suppose that g satisfies (g_0), (g_1) and $C_g \in (0, \infty)$.*

(i) For any SCS-sequence $(u_n)_n \subset \mathcal{B}_1$ we have

$$\limsup_{n\to\infty} \Phi(u_n) \leq \pi^2\Big(\frac{g^2(0)}{4} + 2C_g^2\pi \mathrm{e}^2\Big). \tag{1.13}$$

(ii) If $\lim_{t\to\infty} g(t) = C_g$, then there exists a SCS-sequence (u_n) with

$$\lim_{n\to\infty} \Phi(u_n) = \pi^2\Big(\frac{g^2(0)}{4} + 2C_g^2\pi \mathrm{e}^2\Big), \tag{1.14}$$

so the upper bound in (1.13) is sharp. If, in addition,

$$\liminf_{\tau\to\infty}\Big(g(\tau) - C_g\Big)\tau^\rho > 0 \qquad \text{for some } \rho < 1, \tag{1.15}$$

then this sequence satisfies

$$\Phi(u_n) > \pi^2\Big(\frac{g^2(0)}{4} + 2C_g^2\pi \mathrm{e}^2\Big) \qquad \text{for large } n. \tag{1.16}$$

Theorem 1.4 and Theorem 1.6(ii) immediately give rise to the following Theorem on the existence of maximizers.

Theorem 1.7. *Suppose that g satisfies (g_0), (g_1), $\lim_{t\to\infty} g(t) = C_g \in (0,\infty)$ and (1.15). Then $m_1(F)$ is attained in $\mathcal{B}_1$.*

An example for a function g satisfying (g_0), (g_1), $\lim_{t\to\infty} g(t) = C_g \in (0,\infty)$ and (1.15) is given by

$$t \mapsto g(t) = \frac{t}{2+t} + (\kappa + t)^{-\rho} \qquad \text{with } \rho \in (0,1) \text{ and } \kappa \geq 12^{\frac{1}{\rho}}.$$

We finally provide an explicit non-asymptotic condition on g under which $m_1(F)$ is attained in $\mathcal{B}_1$.

Theorem 1.8. *Suppose that g satisfies (g_0), (g_1) and $C_g \in (0,\infty)$. Suppose moreover that g attains its maximum m_g at a point $t_g \in (0,\infty)$, and that*

$$\frac{m_g^2(8\pi t_g^2+1)}{4(t_g+1)^2} > \frac{g^2(0)}{4} + 2C_g^2 \pi \mathrm{e}^2. \tag{1.17}$$

Then $m_1(F)$ is attained in $\mathcal{B}_1$.

To provide an example for a function g satisfying (g_0), (g_1), $C_g \in (0,\infty)$ and (1.17), we consider the function

$$g(t) = C + t\mathrm{e}^{-\frac{t}{\kappa}}$$

which depends on the parameters $C, \kappa > 0$. Clearly we have (g_0) and $C_g = g(0) = C > 0$ in this case, and a direct computation shows that g satisfies (g_1) if $\kappa > 2$ and $C \leq \frac{\kappa-2}{\kappa}\mathrm{e}^{-\frac{1}{\kappa}}$. Moreover, we have $t_g = \kappa$ and $m_g = C + \frac{\kappa}{e} \geq \frac{\kappa}{e}$ in this case, so (1.17) holds if

$$\frac{\kappa^2(8\pi\kappa^2+1)}{4(\kappa+1)^2\mathrm{e}^2} > C^2\Big(\frac{1}{4} + 2\pi\mathrm{e}^2\Big) \tag{1.18}$$

Hence Theorem 1.8 applies to g if

$$\kappa > 2 \qquad \text{and} \qquad 0 < C < \min\Big\{\frac{\kappa-2}{\kappa}\mathrm{e}^{-\frac{1}{\kappa}}, \Big(\frac{\kappa^2(8\pi\kappa^2+1)}{4(\kappa+1)^2\mathrm{e}^2\big(\frac{1}{4}+2\pi\mathrm{e}^2\big)}\Big)^{\frac{1}{2}}\Big\}$$

Note also that Theorem 1.7 does not apply in this case since (1.15) is not satisfied for g.

The paper is organized as follows. In Section 2, we recall some preliminaries related to nonlocal interaction energies and prove the finiteness of the supremum $m_1(F)$ contained in Theorem 1.2(i). Section 3 deals with the unbounded case and the proof of Theorem 1.2(ii). Section 4 is concerned with the existence of extremal functions in the subcritical case, which completes the proof of Theorem 1.2. In Section 5 we perform a change of variable motivated by Carleson and Chang [14], and we find an upper bound for the Schwartz symmetric concentrating sequences. In Section 6 we give an answer to the open problem in [10] and prove the second part of Theorem 1.6. In this section, the construction of a SCS-sequence satisfying (1.14) is inspired by Figuereido *et al.* [20], but we have to work in a complementary parameter regime and the estimates are vastly different from [20]. Finally Section 7 is dedicated to the proof of Theorem 1.8.

Notation. Throughout this paper, if $v : \mathbb{R}^N \to \mathbb{R}$ is a radially symmetric function, we let v also denote the associated function $[0,\infty) \to \mathbb{R}$ of the radial variable $r = |x|$.

2. Preliminaries and finiteness of $m_1(F)$

In this section, we introduce some notation and recall some preliminary results. Moreover, we will complete the proof of Theorem 1.2(i). Throughout the remainder of the paper, we assume that the function g satisfies (g_0), which implies that F is an even continuous function satisfying (1.3). We let $\mathcal{M}_+(\mathbb{R}^2)$ denotes the cone of nonnegative real-valued measurable functions on $\mathbb{R}^2$. If $\Omega \subset \mathbb{R}^2$ is a measurable subset and u is a nonnegative real-valued measurable function on Ω, we also regard u as a function in $\mathcal{M}(\mathbb{R}^2)$ by trivial extension. We then define the quadratic forms $b_\pm : \mathcal{M}_+(\mathbb{R}^2) \to [0,\infty]$ by

$$(v,w) \mapsto b_\pm(v,w) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln^\pm \frac{1}{|x-y|}\, v(x)w(y)\,\mathrm{d}x\mathrm{d}y.$$

Moreover, we define

$$b_0(v,w) := b_+(v,w) - b_-(v,w)$$

for all functions $v, w \in \mathcal{M}_+(\mathbb{R}^2)$ for which $b_\pm(v,w) < \infty$. For the sake of brevity, we also set

$$b_\pm(v) := b_\pm(v,v) \qquad \text{and} \qquad b_0(v) := b_0(v,v) \quad \text{if } b_+(v) < \infty.$$

By definition, we then have

$$\Phi_\pm(u) = b_\pm(1_{B_1}F(u)) \qquad \text{and} \qquad \Phi(u) = b_0(1_{B_1}F(u)) \qquad \text{for } u \in H_0^1(B_1).$$

Moreover, as noted in the introduction, all of these values are finite if $u \in H_0^1(B_1)$. We also recall from the introduction that

$$\mathcal{B}_1 := \{u \in H_0^1(B_1) \;:\; |\nabla u|_2 \le 1\}.$$

In order to study the maximization problem for Φ in the set $\mathcal{B}_1$, it is important to note that the functional Φ increases under Schwarz symmetrization if F is an even, nonnegative and increasing on $[0,\infty)$. This is the consequence of the following Riesz rearrangement type inequalities noted in [10], Lemma 2.3: For $v \in \mathcal{M}_+(\mathbb{R}^2)$ we have

$$b_+(v^*) \ge b_+(v) \qquad \text{and} \qquad b_-(v^*) \le b_-(v), \tag{2.1}$$

where, here and in the following, v^* denotes the Schwarz symmetrization of v. We then let

$$\mathcal{B}_1^* := \{u^* \;:\; u \in \mathcal{B}_1\}$$

denote the corresponding Schwarz symmetrized set of $\mathcal{B}_1$. By the Polya-Szego inequality, we have $\mathcal{B}_1^* \subset \mathcal{B}_1$. Hence the following key corollary readily follows from (2.1).

Corollary 2.1. *Let g satisfy (g_0) and (g_1), and let F be given by (1.7). Then we have*

$$m_1(F) = \sup_{u \in \mathcal{B}_1^*} \Phi(u) \qquad \textit{and} \qquad m_1^+(F) = \sup_{u \in \mathcal{B}_1^*} \Phi^+(u), \tag{2.2}$$

where $m_1^+(F) = \sup_{\mathcal{B}_1} \Phi^+$.

Proof. Since F is nonnegative, even and increasing on $[0,\infty)$ by assumptions (g_0) and (g_1), we have $[1_{B_1}F(u)]^* = 1_{B_1}F(u^*)$. Therefore

$$\Phi(u^*) = b_0(1_{B_1}F(u^*)) = b_0([1_{B_1}F(u)]^*) \geq b_0(1_{B_1}F(u)) = \Phi(u)$$

and

$$\Phi^+(u^*) = b_+(1_{B_1}F(u^*)) = b_+([1_{B_1}F(u)]^*) \geq b_+(1_{B_1}F(u)) = \Phi^+(u) \qquad \text{for } u \in H_0^1(B_1).$$

Since moreover $u^* \in \mathcal{B}_1^* \subset \mathcal{B}_1$ for $u \in \mathcal{B}_1$, the claim follows. □

As a consequence of Corollary 2.1, it is important to study the restrictions of the maps $b_\pm$ to radial functions. As noted in [10], Corollary 2.8, for *radial* functions $v, w \in \mathcal{M}_+(\mathbb{R}^2)$ with $b_\pm(v) < \infty$ we have, by Newton's theorem,

$$\frac{b_0(v,w)}{(2\pi)^2} = \int_0^1 rw(r)\Big(\ln\frac{1}{r}\int_0^r \rho v(\rho)\mathrm{d}\rho + \int_r^1 \rho(\ln\frac{1}{\rho})v(\rho)\mathrm{d}\rho\Big)\mathrm{d}r \tag{2.3}$$

and

$$\frac{b_0(v)}{(2\pi)^2} = 2\int_0^\infty rv(r)\ln\frac{1}{r}\int_0^r \rho v(\rho)\mathrm{d}\rho\mathrm{d}r. \tag{2.4}$$

We also note the following lemmas.

Lemma 2.2. *Let, for $i = 1, 2$, g_i be C^1 nonnegative bounded even functions, and let $\mathcal{B}_{1,rad} := \{u \in \mathcal{B}_1 : u \text{ radial}\}$. Then*

$$\Phi_{g_1,g_2}(u_1,u_2) := \int_0^1 r\frac{\mathrm{e}^{4\pi u_1^2}g_1(u_1(r))}{1+|u_1(r)|}\ln\frac{1}{r}\int_0^r \rho\frac{\mathrm{e}^{4\pi u_2^2(\rho)}g_2(u_2(\rho))}{1+|u_2(\rho)|}\mathrm{d}\rho\,\mathrm{d}r \tag{2.5}$$

defines a bounded functional $\Phi_{g_1,g_2} : \mathcal{B}_{1,rad} \times \mathcal{B}_{1,rad} \to [0,\infty)$.

Proof. The result follows directly from [10], Lemma 2.10, applied with $\beta_1 = \beta_2 = -1$. □

Lemma 2.3. *Suppose that g satisfies (g_0) and $C_g \in [0,\infty)$. Then the functional Φ_- is uniformly bounded on $\mathcal{B}_1$, i.e., we have $m_1^-(F) = \sup_{\mathcal{B}_1}\Phi^- < \infty$.*

Proof. Let $u \in \mathcal{B}_1$. Then we have

$$\Phi_-(u) = b_-(F(u),F(u)) = \int_{B_1}\int_{B_1}\ln^-\frac{1}{|x-y|}\,F(u(x))F(u(y))\,\mathrm{d}x\mathrm{d}y \leq (\ln 2)\|F(u)\|^2_{L^1(B_1)}$$

where $\|F(u)\|_{L^1(B_1)} \leq c_1\int_{B_1}\mathrm{e}^{4\pi u^2}\,\mathrm{d}x \leq c_2$ with constants $c_1, c_2 > 0$ independent of u by assumption and by the Trudinger-Moser inequality (1.1). □

Now Theorem 1.2(i) is a direct consequence of the following Proposition.

Proposition 2.4. *Suppose that g satisfies (g_0) and $C_g \in [0,\infty)$. Then we have*

$$m_1(F) \leq m_1^+(F) < \infty, \qquad \textit{where} \qquad m_1^+(F) = \sup_{\mathcal{B}_1}\Phi^+.$$

Proof. We note that F has at most (-1)-critical growth if $C_g < \infty$, and therefore the assertion is already proved in [10], Proposition 3.2 under the additional assumption that F is increasing. So it remains to prove $m_1^+(F) < \infty$ without this extra assumption. Since g satisfies (g_0) and $C_g < \infty$, we can choose $\kappa > 0$ large enough such that

$$F(t) \le F_\kappa(t) := \kappa \frac{1 + \mathrm{e}^{1-t} t}{1+t} \mathrm{e}^{4\pi t^2}$$

Moreover, the function F_κ is increasing and also has at most (-1)-critical growth, so we have $m_1^+(F_\kappa) < \infty$ by [10], Proposition 3.2. Moreover,

$$\begin{aligned}\Phi^+(u) = \int_{B_1} \int_{B_1} \ln^+ \frac{1}{|x-y|} F(u(x))F(u(y))\,\mathrm{d}x\mathrm{d}y &\le \int_{B_1} \int_{B_1} \ln^+ \frac{1}{|x-y|} F_\kappa(u(x))F_\kappa(u(y))\,\mathrm{d}x\mathrm{d}y \\ &\le m_1^+(F_\kappa) \qquad \text{for } u \in \mathcal{B}_1,\end{aligned}$$

which shows the required finiteness of $m_1^+(F)$. □

3. The unbounded case

In this section we shall complete the proof of Theorem 1.2(ii), which we restate in the following Proposition.

Proposition 3.1. *Suppose that g satisfies (g_0) and $\lim\limits_{s \to +\infty} g(s) = \infty$. Then there exists a sequence of functions $u_n \in \mathcal{B}_1 \cap L^\infty(B_1)$ with $\Phi(u_n) \to \infty$ as $n \to \infty$.*

Proof. Set $v_n = F(u_n)$. For $n \in \mathbb{N}$, $n \ge 2$, we now define $u_n = m_n \in H_0^1(B_1) \cap L^\infty(B_1)$ as in [18], equation (2.12), namely

$$m_n(x) = \frac{1}{\sqrt{2\pi}} \frac{\ln(|x|)}{(\ln n)^{1/2}} \big(1 - \frac{1}{4 \ln n}\big)^{1/2}, \qquad \frac{1}{n} \le |x| \le 1$$

and

$$m_n(x) = \frac{1}{\sqrt{2\pi}} (\ln n)^{1/2} \big(1 - \frac{1}{4 \ln n}\big)^{1/2}, \qquad 0 \le |x| \le \frac{1}{n}.$$

As noted in [18], p. 346, we then have $|\nabla u_n|_2 \le 1$ for n large and thus $u_n \in \mathcal{B}_1^*$. Moreover, $v_n := F(u_n) \in L^\infty(B_1)$ and therefore

$$\Phi_\pm(u_n) = b_\pm(v_n, v_n) < \infty \qquad \text{for } n \in \mathbb{N}.$$

We have, for n large,

$$v_n = \frac{g\big(\sqrt{\frac{\ln n}{2\pi} - \frac{1}{8\pi}}\big) \mathrm{e}^{4\pi\left(\frac{\ln n}{2\pi} - \frac{1}{8\pi}\right)}}{\big(1 + |\sqrt{\frac{\ln n}{2\pi} - \frac{1}{8\pi}}|\big)} \;\ge\; c_1 \frac{n^2 g\big(\sqrt{\frac{\ln n}{2\pi} - \frac{1}{8\pi}}\big)}{\sqrt{\ln n}} \qquad \text{on } B_{\frac{1}{n}}(0)$$

with some constant $c_1 > 0$. We derive that

$$
\begin{aligned}
\frac{b_0(v_n,v_n)}{(2\pi)^2} &\geq 2\int_0^{\frac{1}{n}} r v_n(r)\ln\frac{1}{r}\int_0^r \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r \\
&\geq 2c_1^2\frac{n^4[g(\sqrt{\frac{\ln n}{2\pi}-\frac{1}{8\pi}})]^2}{\ln n}\int_0^{\frac{1}{n}} r\ln\frac{1}{r}\int_0^r \rho\mathrm{d}\rho\mathrm{d}r \\
&= c_1^2\frac{n^4[g(\sqrt{\frac{\ln n}{2\pi}-\frac{1}{8\pi}})]^2}{\ln n}\int_0^{\frac{1}{n}} r^3\ln\frac{1}{r}\mathrm{d}r = -c_1^2\frac{n^4[g(\sqrt{\frac{\ln n}{2\pi}-\frac{1}{8\pi}})]^2}{\ln n}\Big(\frac{r^4}{4}\ln r\Big|_0^{\frac{1}{n}}-\int_0^{\frac{1}{n}}\frac{r^3}{4}\,\mathrm{d}r\Big) \\
&= -c_1^2\frac{n^4[g(\sqrt{\frac{\ln n}{2\pi}-\frac{1}{8\pi}})]^2}{\ln n}\Big(\frac{r^4}{4}\ln r-\frac{r^4}{16}\Big)\Big|_0^{\frac{1}{n}} \geq \frac{1}{4}c_1^2 g^2\Big(\sqrt{\frac{\ln n}{2\pi}-\frac{1}{8\pi}}\Big)
\end{aligned}
$$

so that $b_0(v_n,v_n)\to\infty$ as $n\to\infty$. This shows that $\Phi(u_n)=b_0(v_n,v_n)\to\infty$ as $n\to\infty$, as required. □

4. Continuity properties of Φ and existence of maximizers in the subcritical case

In this section we provide an abstract strong continuity result for the functional Φ, which is partly based on [21], Theorem 1.6. Moreover, we will complete the proof of Theorem 1.2.

Proposition 4.1. *Suppose that g satisfies (g_0) and (g_1), and let $(u_n)_n$ be a sequence in $\mathcal{B}_1^*$ with $u_n \rightharpoonup u$ in $H_0^1(B_1)$. Suppose moreover that one of the following conditions is satisfied:*

(i) $u\neq 0$.
(ii) $C_g=0$, *i.e.,* $\lim_{t\to\infty} g(t)=0$.

Then we have

$$\lim_{n\to\infty}\Phi(u_n)=\Phi(u).$$

Proof. Since $u_n\in\mathcal{B}_1^*$ for all n, it is easy to deduce from the weak convergence $u_n\rightharpoonup u$ in $H_0^1(B_1)$ that $u\in\mathcal{B}_1^*$.
We now assume (i) first, so we assume that $u\neq 0$. Then [21], Theorem 1.6 implies that

$$\int_{B_1}\mathrm{e}^{(4\pi+t)u_n^2}\,\mathrm{d}x \qquad \text{is bounded for some } t>0, \tag{4.1}$$

and thus

$$\mathrm{e}^{4\pi u_n^2}\to\mathrm{e}^{4\pi u^2} \qquad \text{in } L^1(B_1). \tag{4.2}$$

Set $v_n:=1_{B_1}F(u_n)$ for $n\in\mathbb{N}$ and $v:=1_{B_1}F(u)$. By (4.1), v_n is bounded in $L^{s_0}(\mathbb{R}^2)$ with $s_0=1+\frac{t}{4\pi}>1$. Moreover, since

$$v_n\to v \qquad \text{in } L^1(B_1),$$

interpolation yields that

$$v_n\to v \qquad \text{in } L^s(\mathbb{R}^2) \qquad \text{for } 1\leq s<s_0.$$

Moreover, by (2.4),

$$\frac{\Phi(u_n)-\Phi(u)}{2(2\pi)^2} = \int_0^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r - \int_0^1 r v(r)\ln\frac{1}{r}\int_0^r \rho v(\rho)\mathrm{d}\rho\mathrm{d}r$$
$$= \int_0^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho[v_n(\rho)-v(\rho)]\mathrm{d}\rho\mathrm{d}r + \int_0^1 r[v_n(r)-v(r)]\ln\frac{1}{r}\int_0^r \rho v(\rho)\mathrm{d}\rho\mathrm{d}r$$

where, for fixed $s\in(1,s_0)$,

$$\Big|\int_0^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho[v_n(\rho)-v(\rho)]\mathrm{d}\rho\mathrm{d}r\Big| \le \frac{|v_n-v|_s}{2\pi}\int_0^1 r|B_r|^{\frac{1}{s'}}v_n(r)\ln\frac{1}{r}\mathrm{d}r$$
$$\le \frac{|v_n-v|_s}{2\pi^{1-\frac{1}{s'}}}\int_0^1 r^{1+\frac{2}{s'}}v_n(r)\ln\frac{1}{r}\mathrm{d}r \le C|v_n-v|_s|v_n|_1 \to 0 \qquad \text{as } n\to\infty$$

with $C := \frac{\pi^{\frac{1}{s'}-2}}{4}\sup\limits_{r\in(0,1]} r^{\frac{2}{s'}}\ln\frac{1}{r}$ and also

$$\Big|\int_0^1 r[v_n(r)-v(r)]\ln\frac{1}{r}\int_0^r \rho v(\rho)\mathrm{d}\rho\mathrm{d}r\Big| \le \frac{|v|_s}{2\pi}\int_0^1 r|B_r|^{\frac{1}{s'}}[v_n(r)-v(r)]\ln\frac{1}{r}\mathrm{d}r$$
$$\le \frac{|v|_s}{2\pi^{1-\frac{1}{s'}}}\int_0^1 r^{1+\frac{2}{s'}}[v_n(r)-v(r)]\ln\frac{1}{r}\mathrm{d}r \le C|v|_s|v_n-v|_1 \to 0 \qquad \text{as } n\to\infty.$$

We thus conclude that $\Phi(u_n)\to\Phi(u)$ as $n\to\infty$, as claimed.

Next we assume (ii), and by (i) we may assume that $u=0$, so $u_n \rightharpoonup 0$ in $H_0^1(B_1)$. Since $H_0^1(B_1)$ is compactly embedded into $L^p(B_1)$ for $2<p<\infty$, we have

$$u_n \to 0 \qquad \text{in } L^p(B_1) \text{ for } 2<p<\infty. \tag{4.3}$$

Since $u_n\in\mathcal{B}_1^*$ for every $n\in\mathbb{N}$, (4.3) implies that

$$u_n\to 0 \qquad \text{uniformly in } [\delta,1] \text{ for every } \delta\in(0,1). \tag{4.4}$$

We now write $F=\kappa_0+\tilde{F}$ with $\kappa_0=F(0)$, where the function $\tilde{F}=F-\kappa_0$ is also even, nonnegative and increasing on $[0,\infty)$. Moreover, it satisfies $\tilde{F}(0)=0$ and

$$\tilde{F}(t)\le c_1\mathrm{e}^{4\pi t^2} \qquad \text{for } t\in\mathbb{R} \text{ with a constant } c_1>0. \tag{4.5}$$

With

$$v_n := \tilde{F}(u_n) \qquad \text{for } n\in\mathbb{N},$$

we then have

$$\Phi(u_n) = b_0(v_n) + 2b_0\big(1_{B_1}\kappa_0, v_n\big) + b_0(\kappa_0 1_{B_1}) = b_0(v_n) + 2b_0\big(1_{B_1}\kappa_0, v_n\big) + \Phi(0). \tag{4.6}$$

By (2.3) we have

$$b_0\big(1_{B_1}\kappa_0, v_n\big) = (2\pi)^2\kappa_0 \int_0^1 r v_n(r)h(r)\mathrm{d}r \qquad \text{with} \quad h(r) = \ln\frac{1}{r}\int_0^r \rho\mathrm{d}\rho + \int_r^1 \rho(\ln\frac{1}{\rho})\mathrm{d}\rho.$$

Moreover, for any $\delta \in (0,1)$ we have, by (1.1) and (4.5),

$$\Big|\int_0^\delta r v_n h(r)\mathrm{d}r\Big| \le \frac{c_1}{2\pi}\|h\|_{L^\infty(0,\delta)}\int_{B_1} \mathrm{e}^{4\pi u_n^2}\mathrm{d}x \le \frac{c_1 c(B_1)}{2\pi}\|h\|_{L^\infty(0,\delta)}. \tag{4.7}$$

By (4.4) and since $\tilde{F}(0) = 0$, we also have

$$v_n \to 0 \qquad \text{uniformly in } [\delta, 1] \text{ for every } \delta \in (0,1). \tag{4.8}$$

Combining (4.7), (4.8) and the fact that $h(r) \to 0$ as $r \to 0$, we see that

$$b_0\big(1_{B_1}\kappa_0, v_n\big) \to 0 \qquad \text{as } n \to \infty.$$

To prove that $\Phi(u_n) \to \Phi(0)$ as $n \to \infty$, it thus remains, by (4.6), to show that

$$b_0(v_n) \to 0 \qquad \text{as } n \to \infty. \tag{4.9}$$

To see (4.9), we note that for every $\delta \in (0,1)$ we have

$$\frac{b_0(v_n)}{2(2\pi)^2} = \int_0^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r = M_n^\delta + N_n^\delta,$$

where, by (1.1), (4.5) and (4.8)

$$\begin{aligned} M_n^\delta &:= \int_\delta^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r \le c_1\int_\delta^1 r v_n(r)\ln\frac{1}{r}\int_0^1 \rho\mathrm{e}^{4\pi u_n^2(\rho)}\mathrm{d}\rho\mathrm{d}r \\ &\le \frac{c_1 c(B_1)}{2\pi}\int_\delta^1 r v_n(r)\ln\frac{1}{r}\mathrm{d}r \to 0 \qquad \text{as } n \to \infty. \end{aligned} \tag{4.10}$$

To estimate

$$N_n^\delta := \int_0^\delta r v_n(r)\ln\frac{1}{r}\int_0^r \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r$$

we fix $\varepsilon \in (0, \frac{1}{4\pi})$ and define, for any $n \in \mathbb{N}$,

$$A_n^+ = \{r \in (0,1] \;:\; u_n(r) \ge \sqrt{\varepsilon(-\ln r)}\}, \qquad A_n^- := \{r \in (0,1] \;:\; u_n(r) < \sqrt{\varepsilon(-\ln r)}\}.$$

Since $\tilde{F}$ is an increasing function and g is bounded on $[0,\infty)$ as a consequence of the assumptions $C_g = 0$, we then have

$$v_n(r) \le \tilde{F}(\sqrt{\varepsilon(-\ln r)}) \le F(\sqrt{\varepsilon(-\ln r)}) = \frac{g(\sqrt{\varepsilon(-\ln r)})\mathrm{e}^{-4\pi\varepsilon\ln r}}{1+\sqrt{\varepsilon(-\ln r)}} \le M r^{-4\pi\varepsilon} \qquad \text{for } r \in A_n^-, \tag{4.11}$$

with some constant $M > 0$ and

$$v_n(r) \le \frac{g(u_n(r))\mathrm{e}^{4\pi u_n^2(r)}}{1+\sqrt{\varepsilon(-\ln r)}} \le \frac{C_\varepsilon(r)\mathrm{e}^{4\pi u_n^2(r)}}{1+\sqrt{\varepsilon(-\ln r)}} \qquad \text{for } r \in A_n^+ \tag{4.12}$$

with the increasing function

$$r \mapsto C_\varepsilon(r) := \sup\{g(t) \ : \ t \ge \sqrt{\varepsilon(-\ln r)}\}.$$

In particular, we thus have

$$v_n(r) \le C_\varepsilon(1)\mathrm{e}^{4\pi u_n^2(r)} \qquad \text{for } r \in A_n^+. \tag{4.13}$$

We now write

$$N_n^\delta = \int_{A_n^- \cap(0,\delta)} r v_n(r) \ln\frac{1}{r}\int_0^r \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r + \int_{A_n^+ \cap(0,\delta)} r v_n(r) \ln\frac{1}{r}\int_0^r \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r,$$

where, by (4.11) and since $\varepsilon \in (0, \frac{1}{4\pi})$,

$$\begin{aligned}
&\int_{A_n^- \cap(0,\delta)} r v_n(r) \ln\frac{1}{r}\int_0^r \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r \le M^2 \int_0^\delta r^{1-4\pi\varepsilon}\ln\frac{1}{r}\int_0^1 \rho^{1-4\pi\varepsilon}\mathrm{d}\rho\mathrm{d}r \\
&\le M^2\int_0^\delta r^{1-4\pi\varepsilon}\ln\frac{1}{r}\mathrm{d}r = M^2\Big(\frac{\delta^{1-4\pi\varepsilon}}{1-4\pi\varepsilon} - \frac{\delta^{2-4\pi\varepsilon}\ln\delta}{2-4\pi\varepsilon}\Big)
\end{aligned} \tag{4.14}$$

for all $n \in \mathbb{N}$. Moreover, we have

$$\begin{aligned}
&\int_{A_n^+ \cap(0,\delta)} r v_n(r) \ln\frac{1}{r}\int_0^r \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r \\
&= \int_{A_n^+ \cap(0,\delta)} r v_n(r) \ln\frac{1}{r}\int_{A_n^+\cap(0,r)} \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r + \int_{A_n^+ \cap(0,\delta)} r v_n(r) \ln\frac{1}{r}\int_{A_n^-\cap(0,r)} \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r,
\end{aligned}$$

where, by (4.12)

$$\begin{aligned}
&\int_{A_n^+ \cap(0,\delta]} r v_n(r) \ln\frac{1}{r}\int_{A_n^+\cap[0,r]} \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r \\
&\le \int_{A_n^+ \cap(0,\delta]} r\frac{C_\varepsilon(r)\mathrm{e}^{4\pi u_n^2(r)}}{1+\sqrt{\varepsilon(-\ln r)}}\ln\frac{1}{r}\int_{A_n^+\cap[0,r]} \rho\frac{C_\varepsilon(\rho)\mathrm{e}^{4\pi u_n^2(\rho)}}{1+\sqrt{\varepsilon(-\ln\rho)}}\mathrm{d}\rho\mathrm{d}r \\
&\le C_\varepsilon(\delta)^2\int_{A_n^+ \cap(0,\delta]}\frac{(-\ln r)r\mathrm{e}^{4\pi u_n^2(r)}}{[1+\sqrt{\varepsilon(-\ln r)}]^2}\int_0^r \rho\mathrm{e}^{4\pi u_n^2(\rho)}\mathrm{d}\rho\mathrm{d}r \\
&\le \frac{C_\varepsilon(\delta)^2}{\varepsilon}\Big(\int_0^1 r\mathrm{e}^{4\pi u_n^2(r)}\mathrm{d}r\Big)^2 \le \frac{\big(c(B_1)\big)^2}{(2\pi)^2\varepsilon}C_\varepsilon(\delta)^2
\end{aligned} \tag{4.15}$$

by (1.1). Furthermore, by (4.11) and (4.13),

$$\int_{A_n^+ \cap[0,\delta]} r v_n(r) \ln\frac{1}{r}\int_{A_n^-\cap[0,r]} \rho v_n(\rho)\mathrm{d}\rho\mathrm{d}r \le C_\varepsilon(1)M\int_{A_n^+\cap[0,\delta]} r\mathrm{e}^{4\pi u_n^2(r)}\ln\frac{1}{r}\int_0^r \rho^{1-4\pi\varepsilon}\mathrm{d}\rho\mathrm{d}r$$

$$\begin{aligned}&\leq \frac{C_\varepsilon(1)M}{2-4\pi\varepsilon}\int_0^\delta r^{3-4\pi\varepsilon}\mathrm{e}^{4\pi u_n^2(r)}\ln\frac{1}{r}\,\mathrm{d}r \leq C_\varepsilon(1)M\sup_{s\in[0,\delta]}\left(s^{2-4\pi\varepsilon}\ln\frac{1}{s}\right)\int_0^1 r\mathrm{e}^{4\pi u_n^2(r)}\,\mathrm{d}r\\&\leq \frac{C_\varepsilon(1)Mc(B_1)}{2\pi}\sup_{s\in[0,\delta]}\left(s^{2-4\pi\varepsilon}\ln\frac{1}{s}\right)\end{aligned}\tag{4.16}$$

again by (1.1). We now observe that $\lim_{\delta\to 0^+} C_\varepsilon(\delta) = \limsup_{t\to+\infty} g(t) = 0$ by assumption (ii). So, as $\varepsilon \in (0, \frac{1}{4\pi})$ the RHS of (4.14), (4.15) and (4.16) tend to zero as $\delta \to 0^+$. Hence we infer that

$$\lim_{\delta\to 0^+}\sup_{n\in\mathbb{N}} N_n^\delta = 0.$$

Combining this with (4.10), we infer (4.9), as claimed.

□

The following Proposition completes the proof of Theorem 1.2.

Proposition 4.2. *Suppose that g satisfies (g_0) and (g_1) with $C_g = 0$. Then the value $m_1(F) < \infty$ is attained by a function $u \in \mathcal{B}_1^*$.*

Proof. Let $(u_n)_n$ be a maximizing sequence in $\mathcal{B}_1$ for m_F. By Polya-Szego inequality, we may assume that $u_n \in \mathcal{B}_1^*$ for $n \in \mathbb{N}$. Since $\mathcal{B}_1$ is bounded in $H_0^1(B_1)$, we may also assume that $u_n \rightharpoonup u \in H_0^1(B_1)$ with $u \in \mathcal{B}_1^*$. By Proposition 4.1, we then have

$$m_1(F) = \lim_{n\to\infty}\Phi(u_n) = \Phi(u),$$

so $m_1(F)$ is attained at $u \in \mathcal{B}_1^*$. □

5. An upper bound for Schwarz symmetric concentrating sequences

In this section, we shall prove the first part of Theorem 1.6, namely the asymptotic upper bound for Schwarz symmetric concentrating sequences given in (1.13). As a consequence of this upper bound and the continuity criterion given in Proposition 4.1, we will then readily complete the proof of Theorem 1.4 in the end of this section.

For the proof of Theorem 1.6, it will be useful to apply a change of variables motivated by [14]. More precisely, we have the following.

Lemma 5.1. *Let $u \in H_0^1(B_1)$ be a radial nonnegative and radially decreasing function, and let $w : [0,\infty) \to \mathbb{R}$ be defined by $w(t) = (4\pi)^{1/2}u(\mathrm{e}^{-t/2})$ (where we identify u with its profile function in the radial variable). Then $w \in H^1_{loc}(\mathbb{R}^+)$ is an increasing function with $w(0) = 0$. Moreover, we have*

$$\int_0^\infty (w'(t))^2\mathrm{d}t = 2\pi\int_0^1 (u'(r))^2 r\mathrm{d}r = \int_{B_1}|\nabla u|^2\mathrm{d}x,\tag{5.1}$$

and

$$\Phi(u) = \pi^2\int_0^\infty \frac{y\mathrm{e}^{w^2(y)-y}g((4\pi)^{-1/2}w(y))}{1+(4\pi)^{-1/2}w(y)}\int_y^\infty \frac{\mathrm{e}^{w^2(x)-x}g((4\pi)^{-1/2}w(x))}{1+(4\pi)^{-1/2}w(x)}\mathrm{d}x\mathrm{d}y.\tag{5.2}$$

Proof. By definition, w is an increasing function satisfying $w(0) = 0$, and (5.1) follows by a straightforward computation. Moreover, since

$$\Phi(u) = 2(2\pi)^2 \int_0^1 rF(u(r)) \ln \frac{1}{r} \int_0^r \rho F(u(\rho)) \mathrm{d}\rho \mathrm{d}r$$

by (2.4), we have, by the change of variables $r = \mathrm{e}^{-t/2}$,

$$\begin{aligned}
\Phi(u) &= \pi^2 \int_0^\infty t\mathrm{e}^{-t} F((4\pi)^{-1/2} w(t)) \int_0^\infty \mathrm{e}^{-(s+t)} F((4\pi)^{-1/2} w(s+t)) \mathrm{d}s\mathrm{d}t \\
&= \pi^2 \int_0^\infty \frac{t\mathrm{e}^{w^2(t)-t} g((4\pi)^{-1/2} w(t))}{1+(4\pi)^{-1/2} w(t)} \int_0^\infty \frac{\mathrm{e}^{w^2(s+t)-(s+t)} g((4\pi)^{-1/2} w(s+t))}{1+(4\pi)^{-1/2} w(s+t)} \mathrm{d}s\mathrm{d}t \\
&= \pi^2 \int_0^\infty \frac{y\mathrm{e}^{w^2(y)-y} g((4\pi)^{-1/2} w(y))}{1+(4\pi)^{-1/2} w(y)} \int_y^\infty \frac{\mathrm{e}^{w^2(x)-x} g((4\pi)^{-1/2} w(x))}{1+(4\pi)^{-1/2} w(x)} \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

□

We may now complete the

Proof of Theorem 1.6(i). Let $(u_n)_n \subset \mathcal{B}_1$ be any SCS-sequence, and let $w_n : [0,\infty) \to \mathbb{R}$ be defined by $w_n(t) = (4\pi)^{1/2} u_n(\mathrm{e}^{-t/2})$ for $n \in \mathbb{N}$. By Lemma 5.1, we then have $w_n \in H^1_{loc}(\mathbb{R}_+)$ with $w_n(0) = 0$,

$$\int_0^\infty (w_n')^2 dt \le 1 \qquad \text{for all } n \in \mathbb{N} \tag{5.3}$$

and

$$\int_0^A w_n' \mathrm{d}t \to 0 \qquad \text{as } n \to \infty \text{ for every } A > 0. \tag{5.4}$$

Moreover, w_n is nondecreasing for every n, and by (5.2) it suffices to show that

$$\begin{aligned}
\limsup_{n\to\infty} \int_0^{+\infty} \int_y^{+\infty} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1+(4\pi)^{-1/2} w_n(y)} \frac{\mathrm{e}^{w_n^2(x)-x} g((4\pi)^{-1/2} w_n(x))}{1+(4\pi)^{-1/2} w_n(x)} \mathrm{d}x\mathrm{d}y \\
\le \frac{g^2(0)}{4} + 2C_g^2 \pi \mathrm{e}^2.
\end{aligned} \tag{5.5}$$

To see this, we first note that (5.3) yields

$$w_n^2(t) = \Big(\int_0^t w_n'(s)\,\mathrm{d}s\Big)^2 \le t \int_0^\infty (w_n')^2 \le t \qquad \text{for every } t \ge 0, \tag{5.6}$$

while (5.4) gives

$$w_n \to 0 \qquad \text{locally uniformly on } [0,\infty]. \tag{5.7}$$

We now define, for $n \in \mathbb{N}$,

$$a_n := \inf\{t \in [3,\infty) \;:\; w_n^2(t) \ge t - 3\log t\} \quad \text{in } [3,\infty].$$

So if $a_n = \infty$, then $w_n^2(t) \le t - 3\log t$ for all $t \ge 3$, while a_n is the first point $a_n \in [1, +\infty)$ with $w_n^2(a_n) = a_n - 3\log a_n$ if $a_n < \infty$. We also note that

$$a_n \to \infty \qquad \text{as } n \to \infty \tag{5.8}$$

by (5.7). Moreover, we have the estimate

$$\limsup_{n\to\infty} \int_{a_n}^{\infty} \mathrm{e}^{w_n^2(x)-x}\,\mathrm{d}x \le e. \tag{5.9}$$

Indeed, this is true by [14], p. 117 if a_n is replaced by $\tilde{a}_n = \inf\{t \in [1,\infty) : w_n^2(t) \ge t - 2\log t\}$. Note that $a_n \le \tilde{a}_n$, and, moreover

$$\limsup_{n\to\infty} \int_{a_n}^{\tilde{a}_n} \mathrm{e}^{w_n^2(x)-x}\,\mathrm{d}x \le \limsup_{n\to\infty} \int_{a_n}^{\tilde{a}_n} \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{a_n} - \frac{1}{\tilde{a}_n} \to 0 \quad \text{as } n \to \infty,$$

where we have used the convention $\frac{1}{\infty} = 0$. Hence (5.9) holds.

Next we show that

$$\limsup_{n\to\infty} \int_0^{a_n} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1 + (4\pi)^{-1/2} w_n(y)} \mathrm{d}y \le g(0). \tag{5.10}$$

On the one hand, we have, for fixed $A > 0$,

$$\begin{aligned}\int_0^{a_n} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1 + (4\pi)^{-1/2} w_n(y)} \mathrm{d}y &= \int_0^{A} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1 + (4\pi)^{-1/2} w_n(y)} \mathrm{d}y \\ &\quad + \int_A^{a_n} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1 + (4\pi)^{-1/2} w_n(y)} \mathrm{d}y\end{aligned}$$

where, since $w_n \to 0$ uniformly on $[0, A]$,

$$\int_0^{A} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1 + (4\pi)^{-1/2} w_n(y)} \mathrm{d}y \to g(0) \int_0^A y\mathrm{e}^{-y}\,\mathrm{d}y = g(0)(1 - (1+A)\mathrm{e}^{-A}) \quad \text{as } n \to \infty$$

and

$$\begin{aligned}\int_A^{a_n} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1 + (4\pi)^{-1/2} w_n(y)} \mathrm{d}y &\le \|g\|_{L^\infty} \int_A^{a_n} y\mathrm{e}^{-3\log y}\,\mathrm{d}y = \|g\|_{L^\infty} \int_A^{a_n} \frac{1}{y^2}\,\mathrm{d}y \\ &\to \|g\|_{L^\infty} \int_A^{\infty} \frac{1}{y^2}\,\mathrm{d}y = \frac{\|g\|_{L^\infty}}{A} \qquad \text{as } n \to \infty.\end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} \int_0^{a_n} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1 + (4\pi)^{-1/2} w_n(y)} \mathrm{d}y \le g(0)(1 - (1+A)\mathrm{e}^{-A}) + \frac{\|g\|_{L^\infty}}{A},$$

for all $A > 0$, which gives (5.10).

Next, we split the integral in (5.5) into three parts, *i.e.*,

$$\begin{aligned}
&\int_0^{+\infty}\int_y^{+\infty}\frac{y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\frac{\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)}\mathrm{d}x\mathrm{d}y\\
&\qquad=\int_0^{a_n}\int_{a_n}^{+\infty}\dots\mathrm{d}x\mathrm{d}y+\int_0^{a_n}\int_y^{a_n}\dots\mathrm{d}x\mathrm{d}y+\int_{a_n}^{+\infty}\int_y^{+\infty}\dots\mathrm{d}x\mathrm{d}y\\
&\qquad=: I_n+J_n+K_n. \qquad (5.11)
\end{aligned}$$

By (5.9) and (5.10) we have,

$$\begin{aligned}
I_n&=\left(\int_0^{a_n}\frac{y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\mathrm{d}y\right)\left(\int_{a_n}^{+\infty}\frac{\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)}\mathrm{d}x\right)\\
&\le\frac{\|g\|_{L^\infty}}{1+(4\pi)^{-1/2}w_n(a_n)}\left(\int_0^{a_n}\frac{y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\mathrm{d}y\right)\left(\int_{a_n}^{+\infty}\mathrm{e}^{w_n^2(x)-x}\mathrm{d}x\right)\\
&\to 0\quad\text{as } n\to+\infty. \qquad (5.12)
\end{aligned}$$

To estimate J_n, we follow the idea of proving (5.10). We have

$$\begin{aligned}
J_n&\le\int_0^{a_n}\int_y^{a_n}y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))\mathrm{d}x\mathrm{d}y\\
&\le\int_0^A\int_y^A\dots\mathrm{d}x\mathrm{d}y\;+\;\int_0^A\int_A^{a_n}\dots\mathrm{d}x\mathrm{d}y\;+\;\int_A^{a_n}\int_y^{a_n}\dots\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where, as $n\to\infty$, by (5.7),

$$\begin{aligned}
&\int_0^A y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))\int_y^A\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))\mathrm{d}x\mathrm{d}y\\
&\to\quad g^2(0)\int_0^A y\mathrm{e}^{-y}\int_y^A\mathrm{e}^{-x}\mathrm{d}x\mathrm{d}y\le g^2(0)\int_0^A y\mathrm{e}^{-y}\int_y^\infty\mathrm{e}^{-x}\mathrm{d}x\mathrm{d}y=g^2(0)\int_0^A y\mathrm{e}^{-2y}\mathrm{d}y\\
&\qquad=\frac{g^2(0)}{4}\int_0^{2A}y\mathrm{e}^{-y}\mathrm{d}y=\frac{g^2(0)}{4}\Big(1-(1+2A)\mathrm{e}^{-2A}\Big), \qquad (5.13)\\
&\int_0^A y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))\int_A^{a_n}\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))\mathrm{d}x\mathrm{d}y\\
&\le\|g\|_{L^\infty}^2\int_0^A y\mathrm{e}^{w_n^2(y)-y}\int_A^\infty\frac{1}{x^3}\mathrm{d}x\mathrm{d}y=\frac{\|g\|_{L^\infty}^2}{2A^2}\int_0^A y\mathrm{e}^{w_n^2(y)-y}\mathrm{d}y\\
&\to\quad\frac{\|g\|_{L^\infty}^2}{2A^2}\int_0^A y\mathrm{e}^{-y}\mathrm{d}y=\frac{\|g\|_{L^\infty}^2}{2A^2}\Big(1-(1+2A)\mathrm{e}^{-2A}\Big), \qquad (5.14)
\end{aligned}$$

and

$$\begin{aligned}
&\int_A^{a_n}y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))\int_y^{a_n}\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))\mathrm{d}x\mathrm{d}y\\
&\le\|g\|_{L^\infty}^2\int_A^{a_n}\frac{1}{y^2}\int_y^{a_n}\frac{1}{x^3}\mathrm{d}x\mathrm{d}y\le\|g\|_{L^\infty}^2\int_A^\infty\frac{1}{y^2}\int_A^\infty\frac{1}{x^3}\mathrm{d}x\mathrm{d}y=\frac{\|g\|_{L^\infty}^2}{2A^3}\qquad\text{for all } n\in\mathbb{N}. \qquad (5.15)
\end{aligned}$$

Consequently, by sending $A\to\infty$ in (5.13), (5.14) and (5.15), we deduce that

$$\limsup_{n\to\infty} J_n \le \frac{g^2(0)}{4}. \tag{5.16}$$

To estimate K_n, we fix $\varepsilon\in(0,1)$ and define, for any $n\in\mathbb{N}$,

$$H_n^+ = \{x\in(a_n,+\infty) : w_n(x)\ge \varepsilon\sqrt{x}\} \qquad \text{and} \qquad H_n^- = \{x\in(a_n,+\infty) : w_n(x) < \varepsilon\sqrt{x}\}.$$

Then, for n sufficiently large we have

$$\begin{aligned}
&\int_{[a_n,+\infty)\cap H_n^+}\int_{[y,+\infty)\cap H_n^+} \frac{y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\frac{\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)}\mathrm{d}x\mathrm{d}y\\
&\le \int_{[a_n,+\infty)\cap H_n^+}\int_{[y,+\infty)\cap H_n^+} \frac{yg((4\pi)^{-1/2}w_n(y))g((4\pi)^{-1/2}w_n(x))}{\left(1+\varepsilon(4\pi)^{-1/2}\sqrt{y}\right)^2}\mathrm{e}^{w_n^2(x)+w_n^2(y)-(x+y)}dxdy\\
&\le \frac{4\pi}{\varepsilon^2}\int_{a_n}^{\infty}\int_y^{\infty}\mathrm{e}^{w_n^2(x)+w_n^2(y)-(x+y)}g((4\pi)^{-1/2}w_n(y))g((4\pi)^{-1/2}w_n(x))dxdy\\
&= \frac{2\pi}{\varepsilon^2}\left(\int_{a_n}^{\infty}\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))\mathrm{d}x\right)^2 \le \frac{2\pi\mathrm{e}^2}{\varepsilon^2}\left(C_g+o(1)\right)^2.
\end{aligned} \tag{5.17}$$

The equality is the last line follows from the symmetry of the integrand and Fubini's theorem, and in the last inequality we used (5.9) together with the fact that

$$w_n(x)\ge w(a_n)=\sqrt{a_n-3\log a_n} \qquad \text{on } [a_n,\infty),$$

while $\sqrt{a_n-3\log a_n}\to\infty$ as $a_n\to\infty$. Moreover, for n sufficiently large,

$$\begin{aligned}
&\int_{[a_n,+\infty)\cap H_n^+}\int_{[y,+\infty)\cap H_n^-} \frac{y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\frac{\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)}\mathrm{d}x\mathrm{d}y\\
&\le \int_{[a_n,+\infty)\cap H_n^+}\int_{[y,+\infty)\cap H_n^-} \frac{y\mathrm{e}^{w_n^2(y)-y}\mathrm{e}^{(\varepsilon^2-1)x}g((4\pi)^{-1/2}w_n(x))g((4\pi)^{-1/2}w_n(y))}{\left(1+\varepsilon(4\pi)^{-1/2}\sqrt{y}\right)^2}dxdy\\
&\le \left(C_g+o(1)\right)^2\times\frac{\mathrm{e}^{(\varepsilon^2-1)a_n}}{1-\varepsilon^2}\times\sup_{y\in[a_n,+\infty)}\frac{y}{\left(1+\varepsilon(4\pi)^{-1/2}\sqrt{y}\right)^2}\times\int_{a_n}^{\infty}\mathrm{e}^{w_n^2(y)-y}\mathrm{d}y\\
&\le \left(C_g+o(1)\right)^2\times\frac{4\pi\mathrm{e}^{(\varepsilon^2-1)a_n}}{\varepsilon^2(1-\varepsilon^2)}\times\int_{a_n}^{\infty}\mathrm{e}^{w_n^2(y)-y}\mathrm{d}y\to 0 \quad \text{as } n\to\infty,
\end{aligned} \tag{5.18}$$

while similarly

$$\limsup_{n\to\infty}\int_{[a_n,+\infty)\cap H_n^-}\int_{[y,+\infty)\cap H_n^+} \frac{y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\frac{\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)}\mathrm{d}x\mathrm{d}y\le 0,$$

and

$$\limsup_{n\to\infty}\int_{[a_n,+\infty)\cap H_n^-}\int_{[y,+\infty)\cap H_n^-} \frac{y\mathrm{e}^{w_n^2(y)-y}g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\frac{\mathrm{e}^{w_n^2(x)-x}g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)}\mathrm{d}x\mathrm{d}y\le 0.$$

Combining these asymptotic estimates with (5.17) and (5.18), we obtain that

$$K_n \le \frac{2\pi e^2}{\varepsilon^2} \left(C_g + o(1)\right)^2 \qquad \text{as } n \to \infty.$$

Since ε can chosen arbitrarily close to 1 in this estimate, we obtain that

$$\limsup_{n\to\infty} K_n \le 2C_g^2 \pi e^2. \tag{5.19}$$

Collecting the asymptotic estimates for I_n, J_n and K_n given in (5.12),(5.16) and (5.19), we get (5.5). This finishes the proof of (1.13). □

As announced, we will now readily complete the

Proof of Theorem 1.4. By assumption, we have

$$m_1(F) > \pi^2 \Big(\frac{g^2(0)}{4} + 2C_g^2 \pi e^2 \Big). \tag{5.20}$$

Let $(u_n)_n$ be a maximizing sequence in $\mathcal{B}_1$ for m_F. By the Polya-Szego inequality, we may assume that $u_n \in \mathcal{B}_1^*$ for $n \in \mathbb{N}$. Moreover, we may pass to a subsequence with $u_n \rightharpoonup u \in H_0^1(B_1)$. If $(u_n)_n$ is an SCS-sequence, then it follows from Theorem 1.6(i) that

$$m_1(F) = \lim_{n\to\infty} \Phi(u_n) \le \pi^2 \Big(\frac{g^2(0)}{4} + 2C_g^2 \pi e^2 \Big),$$

contrary to (5.20). Hence $(u_n)_n$ is no SCS-sequence, which implies that $u \ne 0$ and therefore

$$m_1(F) = \lim_{n\to\infty} \Phi(u_n) = \Phi(u),$$

by Proposition 4.1, showing that $m_1(F)$ is attained at $u \in \mathcal{B}_1^*$. □

6. Sharpness of the upper limit and convergence from above

In this section, we shall prove the sharpness of the upper bound in (1.13). Precisely we recognize the second part of Theorem 1.6, namely the existence of a Schwarz symmetric concentrating sequence satisfying (1.14), assuming that

$$\exists C_g = \lim_{t\to\infty} g(t) \in (0,\infty). \tag{6.1}$$

If in addition (1.15) holds, such a SCS sequence satisfies (1.16).

As in the last section, we will consider the transformation $r = e^{-t/2}$ of the radial variable. By Lemma 5.1, the proof of (1.14) and (1.16) is reduced to the following Proposition.

Proposition 6.1. *Let g satisfy (g_0), (g_1) and (6.1). There exists a sequence of increasing functions $w_n \in H^1_{loc}(\mathbb{R}_+)$ with $w_n(0) = 0$ for all $n \in \mathbb{N}$ and the following properties:*

(i)

$$\int_0^\infty (w_n')^2 dt = 1 \quad \textit{for all } n \in \mathbb{N}, \quad \textit{and} \quad \int_0^A w_n' dt \to 0 \quad \textit{as } n \to \infty \textit{ for every } A > 0. \tag{6.2}$$

(ii)

$$\lim_{n\to\infty}\int_0^\infty \frac{y\mathrm{e}^{{w_n}^2(y)-y}g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\int_y^\infty \frac{\mathrm{e}^{{w_n}^2(x)-x}g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)}\mathrm{d}x\mathrm{d}y = \frac{g^2(0)}{4}+2C_g^2\pi\mathrm{e}^2. \quad (6.3)$$

(iii) If in addition ([1.15]) holds, then

$$\int_0^\infty \frac{y\mathrm{e}^{{w_n}^2(y)-y}g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\int_y^\infty \frac{\mathrm{e}^{{w_n}^2(x)-x}g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)}\mathrm{d}x\mathrm{d}y > \frac{g^2(0)}{4}+2C_g^2\pi\mathrm{e}^2 \quad (6.4)$$

for large n.

The remainder of this section is devoted to the proof of Proposition 6.1. We fix $s \in (0, \frac{1}{2})$ in the following. Moreover, for $n \in \mathbb{N}$, we set $\delta_n = \frac{s\log n}{n}$ and define $w_n \in H^1_{loc}(\mathbb{R}_+)$ by

$$w_n(t) = \begin{cases} \dfrac{t}{n^{1/2}}\left(1-\delta_n\right)^{1/2}, & 0 \le t \le n, \\ \dfrac{1}{(n(1-\delta_n))^{1/2}}\log\dfrac{A_n+1}{A_n+\mathrm{e}^{-(t-n)}} + (n(1-\delta_n))^{1/2}, & t \ge n. \end{cases} \quad (6.5)$$

Here we choose $A_n > 0$ such that

$$\int_0^\infty |w_n'(t)|^2\,\mathrm{d}t = 1, \qquad \textit{i.e.} \quad \int_n^\infty |w_n'(t)|^2\,\mathrm{d}t = \delta_n. \quad (6.6)$$

To see that such a value A_n exists, we note that (6.6) is equivalent to

$$\frac{1}{n(1-\delta_n)}\Big(\log\frac{A_n+1}{A_n} - \frac{1}{A_n+1}\Big) = \delta_n,$$

i.e.,

$$\frac{A_n+1}{A_n} = \mathrm{e}^{\frac{1}{A_n+1}}\mathrm{e}^{s\log n - s\log n\delta_n} = \mathrm{e}^{\frac{1}{A_n+1}}n^s n^{-s\delta_n}. \quad (6.7)$$

So A_n is chosen as an intersection point of the functions

$$x \mapsto f(x) := \frac{x+1}{x} \quad \text{and} \quad x \mapsto h_n(x) := \mathrm{e}^{\frac{1}{x+1}}n^s n^{-s\delta_n},$$

which exists since

$$\lim_{x\to 0^+} f(x) = +\infty > en^s n^{-s\delta_n} = h_n(0)$$

and

$$\lim_{x\to\infty} f(x) = 1 < n^s n^{-s\delta_n} =: h_{n,\infty} = \lim_{x\to\infty} h_n(x).$$

This choice of w_n is inspired by Figuereido *et al.* [20], who considered the special case $s = 2$ in the definition of w_n. For our purposes, it turns out that the complementary choice of $s \in (0, \frac{1}{2})$ is essential.

Partly following [20], we first note some elementary estimates. Since $h_n(x) \geq h_{n,\infty} \to \infty$ as $n \to \infty$ for $x > 0$, it follows that $A_n \to 0$. More precisely, we have

Lemma 6.2. *As $n \to \infty$, we have*

$$A_n = n^{-s} \frac{1}{e + O(n^{-s})} = \frac{1}{e} n^{-s} + O(n^{-2s}) \tag{6.8}$$

and

$$\int_n^\infty \mathrm{e}^{w_n^2(t)-t}\,\mathrm{d}t \geq e - n^{-s} + o(n^{-s}). \tag{6.9}$$

Proof. As $n \to \infty$, we have

$$n^{-s\delta_n} = \mathrm{e}^{-s\delta_n \log n} = \mathrm{e}^{-s^2 \frac{\log^2 n}{n}} = 1 + O(\frac{\log^2 n}{n}) = 1 + o(1). \tag{6.10}$$

Inserting this expansion in (6.7) gives $A_n = O(n^{-s})$ and

$$\frac{A_n + 1}{A_n n^s} = e\mathrm{e}^{-\frac{A_n}{A_n+1}} n^{-s\delta_n} = e(1 + O(n^{-s}))(1 + O(\frac{\log^2 n}{n})) = e + O(n^{-s}). \tag{6.11}$$

as $n \to \infty$. We point out that here it is essential that $s < 1$. Consequently,

$$1 + \frac{1}{A_n} = \frac{A_n + 1}{A_n} = n^s(e + O(n^{-s}))$$

and hence (6.8) follows.
To see (6.9), we follow the calculations in [20], p. 147, replacing $s = 2$ by $s \in (0, \frac{1}{2})$ to see that

$$\int_n^\infty \mathrm{e}^{w_n^2(t)-t}\,\mathrm{d}t \geq \frac{A_n + 1}{A_n n^s},$$

which by (6.10) and (6.11) gives

$$\begin{aligned}\int_n^\infty \mathrm{e}^{w_n^2(t)-t}\,\mathrm{d}t &\geq e\mathrm{e}^{-A_n} n^{-s\delta_n} \geq e(1 - A_n)(1 + O(\frac{\log^2 n}{n})) \\ &= e(1 - \frac{n^{-s}}{e} + O(n^{-2s}))(1 + O(\frac{\log^2 n}{n})) \geq e - n^{-s} + o(n^{-s}).\end{aligned}$$

□

Next, assuming that (g_0), (g_1) and (6.1) hold, we consider a similar decomposition as in (5.11) for the double integral in (6.3), with a_n replaced by n. First, we have

$$\begin{aligned}\tilde{I}_n :=& \int_0^n \int_n^{+\infty} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1 + (4\pi)^{-1/2} w_n(y)} \frac{\mathrm{e}^{w_n^2(x)-x} g((4\pi)^{-1/2} w_n(x))}{1 + (4\pi)^{-1/2} w_n(x)} \mathrm{d}x\mathrm{d}y \\ =& \int_0^n \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2} w_n(y))}{1 + (4\pi)^{-1/2} w_n(y)}\,\mathrm{d}y \int_n^{+\infty} \frac{\mathrm{e}^{w_n^2(x)-x} g((4\pi)^{-1/2} w_n(x))}{1 + (4\pi)^{-1/2} w_n(x)} \mathrm{d}x \\ =& \tilde{I}_n^1 \times \tilde{I}_n^2,\end{aligned}$$

where, by (6.1),

$$
\begin{aligned}
\tilde{I}_n^1 &\geq \int_{\frac{n}{2}}^{n} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)}\,\mathrm{d}y \\
&\geq \Big(\inf_{\frac{n}{2}\leq y\leq n} \frac{y}{1+(4\pi)^{-1/2}w_n(y)}\Big)\,(C_g+o(1)) \int_{\frac{n}{2}}^{n} \mathrm{e}^{w_n^2(y)-y}\,\mathrm{d}y \\
&= \Big(\inf_{\frac{n}{2}\leq y\leq n} \frac{y}{1+(4\pi)^{-1/2}\frac{y}{n^{1/2}}(1-\delta_n)^{1/2}}\Big)\,(C_g+o(1)) \int_{\frac{n}{2}}^{n} \mathrm{e}^{\frac{1-\delta_n}{n}y^2-y}\,\mathrm{d}y \\
&= \frac{n}{2(1+(4\pi)^{-1/2}\frac{\sqrt{n(1-\delta_n)}}{2})}\,(C_g+o(1)) \int_{\frac{n}{2}}^{n} \mathrm{e}^{\frac{1-\delta_n}{n}y^2-y}\,\mathrm{d}y \\
&= \frac{n}{2+(4\pi)^{-1/2}\sqrt{n(1-\delta_n)}}\,(C_g+o(1)) \int_{\frac{n}{2}}^{n} \mathrm{e}^{\frac{1}{n}y(y-n)}\mathrm{e}^{-\frac{\delta_n}{n}y^2}\,\mathrm{d}y \\
&\geq \frac{n\mathrm{e}^{-n\delta_n}}{2+(4\pi)^{-1/2}\sqrt{n(1-\delta_n)}}\,(C_g+o(1)) \int_{\frac{n}{2}}^{n} \mathrm{e}^{\frac{1}{n}y(y-n)}\,\mathrm{d}y \\
&= \frac{n^{1-s}}{2+(4\pi)^{-1/2}\sqrt{n(1-\delta_n)}}\,(C_g+o(1)) \int_{0}^{\frac{n}{2}} \mathrm{e}^{\frac{1}{n}y(y-n)}\,\mathrm{d}y \\
&\geq \frac{n^{1-s}}{2+(4\pi)^{-1/2}\sqrt{n(1-\delta_n)})}\,(C_g+o(1)) \int_{0}^{\frac{n}{2}} \mathrm{e}^{-y}\,\mathrm{d}y \\
&\geq \frac{n^{1-s}}{2+(4\pi)^{-1/2}\sqrt{n(1-\delta_n)})}\,(C_g+o(1))\Big(1+o(1)\Big) \\
&= \Big(\sqrt{4\pi}n^{1/2-s}+o(n^{1/2-s})\Big)\,(C_g+o(1)) = C_g\sqrt{4\pi}n^{1/2-s}+o(n^{1/2-s})
\end{aligned}
$$

and, by (6.9),

$$
\begin{aligned}
\tilde{I}_n^2 &\geq \frac{C_g+o(1)}{1+(4\pi)^{-1/2}\Big(\frac{1}{\left(n(1-\delta_n)^{1/2}\right)}\log\frac{A_n+1}{A_n}+(n\,(1-\delta_n))^{1/2}\Big)} \int_{n}^{+\infty} \mathrm{e}^{w_n^2(x)-x}\mathrm{d}x \\
&= \Big(\sqrt{4\pi}n^{-\frac{1}{2}}+o(n^{-\frac{1}{2}})\Big)\Big(C_g+o(1)\Big)\Big(e+O(n^{-s})\Big) = C_g\sqrt{4\pi}en^{-\frac{1}{2}}+o(n^{-\frac{1}{2}}).
\end{aligned}
$$

Consequently,

$$
\tilde{I}_n \geq \Big(C_g\sqrt{4\pi}n^{1/2-s}+o(n^{1/2-s})\Big)\Big(C_g\sqrt{4\pi}en^{-\frac{1}{2}}+o(n^{-\frac{1}{2}})\Big) \geq 4C_g^2\pi en^{-s}+o(n^{-s}). \tag{6.12}
$$

Moreover, fixing $\tau\in(0,\frac{1}{2}-s)$, we find that

$$
0\leq(4\pi)^{-1/2}w_n(x)\leq n^{\tau-\frac{1}{2}}\,(1-\delta_n)^{1/2}\leq\frac{\sqrt{3}-1}{2}\quad\text{for } x\in[0,n^{\tau}] \text{ and } n \text{ sufficiently large,}
$$

where δ_0 is chosen as in Remark 1.1 so that $g(t)\geq g(0)$ for $0\leq t\leq\frac{\sqrt{3}-1}{2}$. Consequently,

$$\begin{aligned}
\tilde{J}_n :&= \int_0^n \int_y^n \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)} \frac{\mathrm{e}^{w_n^2(x)-x} g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)} \mathrm{d}x\mathrm{d}y \\
&\geq \int_0^{n^\tau} \int_y^{n^\tau} \frac{y\mathrm{e}^{-y} g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)} \frac{\mathrm{e}^{-x} g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)} dxdy \\
&\geq \left(\frac{g^2(0)}{1+(4\pi)^{-1/2} n^{\tau-\frac{1}{2}} \left(1-\delta_n\right)^{1/2}} \right)^2 \int_0^{n^\tau} \int_y^{n^\tau} y\mathrm{e}^{-x-y} dxdy \\
&= \left(\frac{g^2(0)}{1+(4\pi)^{-1/2} n^{\tau-\frac{1}{2}} \left(1-\delta_n\right)^{1/2}} \right)^2 \left(\frac{1}{4} + O(n^\tau)\mathrm{e}^{-n^\tau} \right) \\
&= g^2(0) \left(1+O(n^{\tau-\frac{1}{2}})\right) \left(\frac{1}{4} + O(n^\tau)\mathrm{e}^{-n^\tau} \right) = \frac{g^2(0)}{4} + O(n^{\tau-\frac{1}{2}}) \qquad \text{as } \ n \to \infty.
\end{aligned} \tag{6.13}$$

Finally, we have

$$\begin{aligned}
\tilde{K}_n :&= \int_n^{+\infty} \frac{y\mathrm{e}^{w_n^2(y)-y} g((4\pi)^{-1/2}w_n(y))}{1+(4\pi)^{-1/2}w_n(y)} \int_y^{+\infty} \frac{\mathrm{e}^{w_n^2(x)-x} g((4\pi)^{-1/2}w_n(x))}{1+(4\pi)^{-1/2}w_n(x)} \mathrm{d}x\mathrm{d}y \\
&\geq \frac{n}{\left(1+(4\pi)^{-1/2}\left(\frac{1}{(n(1-\delta_n))^{1/2}} \log \frac{A_n+1}{A_n} + \left(n\left(1-\delta_n\right)\right)^{1/2}\right)\right)^2} \\
&\quad \times \int_n^{+\infty} \int_y^{+\infty} \mathrm{e}^{w_n^2(x)+w_n^2(y)-x-y} g((4\pi)^{-1/2}w_n(x)) g((4\pi)^{-1/2}w_n(y)) dxdy,
\end{aligned} \tag{6.14}$$

where

$$\begin{aligned}
&\frac{1}{\left(n\left(1-\delta_n\right)\right)^{1/2}} \log \frac{A_n+1}{A_n} + \left(n\left(1-\delta_n\right)\right)^{1/2} = \frac{\log \frac{1}{A_n} + o(1)}{\left(n\left(1-\delta_n\right)\right)^{1/2}} + \left(n\left(1-\delta_n\right)\right)^{1/2} \\
&= \frac{\log(en^s + o(n^s)) + o(1)}{\left(n\left(1-\delta_n\right)\right)^{1/2}} + \left(n\left(1-\delta_n\right)\right)^{1/2} = \frac{1+s\log n + o(1)}{\left(n\left(1-\delta_n\right)\right)^{1/2}} + \left(n\left(1-\delta_n\right)\right)^{1/2} \\
&= \left(n\left(1-\delta_n\right)\right)^{1/2} + o(1),
\end{aligned} \tag{6.15}$$

and hence

$$\begin{aligned}
&\frac{n}{\left(1+(4\pi)^{-1/2}\left(\frac{1}{(n(1-\delta_n))^{1/2}} \log \frac{A_n+1}{A_n} + \left(n\left(1-\delta_n\right)\right)^{1/2}\right)\right)^2} \\
&= \frac{n}{\left(1+(4\pi)^{-1/2}\left(\left(n\left(1-\delta_n\right)\right)^{1/2} + o(1)\right)\right)^2} = \frac{1}{\left(n^{-\frac{1}{2}}+(4\pi)^{-1/2}\left(\left(1-\delta_n\right)^{1/2} + o(n^{-\frac{1}{2}})\right)\right)^2} \\
&= 4\pi + O(n^{-\frac{1}{2}}),
\end{aligned} \tag{6.16}$$

which implies that

$$\begin{aligned}\tilde{K}_n &\geq \left(4\pi + O(n^{-\frac{1}{2}})\right) \times \frac{1}{2} \times \left(\int_n^{+\infty} \mathrm{e}^{w_n^2(x)-x} g((4\pi)^{-1/2} w_n(x))\mathrm{d}x\right)^2 \\ &\geq \left(2\pi + O(n^{-\frac{1}{2}})\right)\left(e - n^{-s} + o(n^{-s})\right)^2 (C_g + o(1))^2 \\ &= \left(2\pi + O(n^{-\frac{1}{2}})\right)\left(\mathrm{e}^2 - 2en^{-s} + o(n^{-s})\right)(C_g + o(1))^2\end{aligned} \tag{6.17}$$

as $n \to \infty$. Thus,

$$\liminf_{n\to+\infty} \tilde{K}_n \geq 2C_g^2 \pi \mathrm{e}^2. \tag{6.18}$$

Now, combining estimates (6.12), (6.13) and (6.18), we conclude that

$$\liminf_{n\to\infty} \int_0^\infty \frac{y\mathrm{e}^{w^2(y)-y} g((4\pi)^{-1/2}w(y))}{1+(4\pi)^{-1/2}w(y)} \int_y^\infty \frac{\mathrm{e}^{w^2(x)-x} g((4\pi)^{-1/2}w(x))}{1+(4\pi)^{-1/2}w(x)} \mathrm{d}x\mathrm{d}y \geq \frac{g^2(0)}{4} + 2C_g^2\pi\mathrm{e}^2.$$

Combining this with (5.5) which also holds for the sequence $(w_n)_n$ defined in (6.5), we obtain (6.3). This finishes the proof of Proposition 6.1(ii).

To prove Part (iii) of Proposition 6.1, we now assume (1.15), which implies that there exists constants $C_2, t_0 > 0$ with

$$g(t) \geq C_g + C_2\tau^{-\rho} \qquad \text{for } t \geq t_0.$$

Since for n sufficiently large we have

$$(4\pi)^{-1/2} w_n(x) \geq t_0 \qquad \text{for } x \geq n,$$

the estimate (6.17) can be improved, for n large, as

$$\begin{aligned}\tilde{K}_n &\geq \left(4\pi + O(n^{-\frac{1}{2}})\right) \times \frac{1}{2} \times \left(\int_n^{+\infty} \mathrm{e}^{w_n^2(x)-x} g((4\pi)^{-1/2} w_n(x))\mathrm{d}x\right)^2 \\ &\geq \left(2\pi + O(n^{-\frac{1}{2}})\right)\left(e - n^{-s} + o(n^{-s})\right)^2 \left(C_g + C_2(4\pi)^{\frac{\rho}{2}} \inf_{x\geq n} w_n^{-\rho}(x)\right)^2 \\ &= \left(2\pi + O(n^{-\frac{1}{2}})\right)\left(\mathrm{e}^2 - 2en^{-s} + o(n^{-s})\right)\left(C_g + C_2(4\pi)^{\frac{\rho}{2}}\left((n(1-\delta_n))^{\frac{1}{2}} + o(1)\right)^{-\rho}\right)^2 \\ &= 2C_g^2\pi\mathrm{e}^2 + 4(4\pi)^{\frac{\rho}{2}} C_g C_2 \pi \mathrm{e}^2 n^{-\rho/2} + O(n^{-s}).\end{aligned} \tag{6.19}$$

Combining (6.12), (6.13) and (6.19), we conclude that

$$\begin{aligned}&\int_0^\infty \frac{y\mathrm{e}^{w^2(y)-y} g((4\pi)^{-1/2}w(y))}{1+(4\pi)^{-1/2}w(y)} \int_y^\infty \frac{\mathrm{e}^{w^2(x)-x} g((4\pi)^{-1/2}w(x))}{1+(4\pi)^{-1/2}w(x)} \mathrm{d}x\mathrm{d}y \\ &\geq \frac{g^2(0)}{4} + O(n^{\tau-\frac{1}{2}}) + 2C_g^2\pi\mathrm{e}^2 + 4(4\pi)^{\frac{\rho}{2}} C_g C_2 \pi\mathrm{e}^2 n^{-\rho/2} + O(n^{-s}) \\ &= \frac{g^2(0)}{4} + 2C_g^2\pi\mathrm{e}^2 + 4(4\pi)^{\frac{\rho}{2}} C_g C_2\pi\mathrm{e}^2 n^{-\rho/2} + O(n^{-s})\end{aligned} \tag{6.20}$$

as $n \to \infty$. Here we used in the last step that we have chosen $\tau < \frac{1}{2} - s$. Since $\rho < 1$, we can assume that $s \in (\frac{\rho}{2}, \frac{1}{2})$ was chosen here. We then conclude that (6.4) holds for n large, as claimed. This finishes the proof of Proposition 6.1(iii).

7. A further sufficient condition

In this section, we give the proof of Theorem 1.8. So let us assume that g satisfies (g_0), (g_1), and that $m_g = \max\limits_{(0,\infty)} g$ is attained at a point $t_g > 0$. We then define the function

$$w : [0,\infty) \to [0,\infty), \qquad w(x) = \begin{cases} \dfrac{x}{\sqrt{4\pi} t_g}, & x \in [0, 4\pi t_g^2]; \\ \sqrt{4\pi} t_g, & x \in (4\pi t_g^2, \infty). \end{cases}$$

Then, $w \in H^1_{loc}(\mathbb{R}^+)$ is an increasing function with $w(0) = 0$ and $\int_0^\infty (w'(t))^2 \mathrm{d}t = 1$.

By Theorem 1.4 and Lemma 5.1, it suffices to show that

$$\int_0^\infty \frac{y\mathrm{e}^{w^2(y)-y} g((4\pi)^{-1/2} w(y))}{1 + (4\pi)^{-1/2} w(y)} \int_y^\infty \frac{\mathrm{e}^{w^2(x)-x} g((4\pi)^{-1/2} w(x))}{1 + (4\pi)^{-1/2} w(x)} \mathrm{d}x\mathrm{d}y > \frac{g^2(0)}{4} + 2C_g^2 \pi \mathrm{e}^2. \tag{7.1}$$

Indeed, using the definition of w, we have

$$\begin{aligned} &\int_0^\infty \frac{y\mathrm{e}^{w^2(y)-y} g((4\pi)^{-1/2} w(y))}{1 + (4\pi)^{-1/2} w(y)} \int_y^\infty \frac{\mathrm{e}^{w^2(x)-x} g((4\pi)^{-1/2} w(x))}{1 + (4\pi)^{-1/2} w(x)} \mathrm{d}x\mathrm{d}y \\ &\geq \int_{4\pi t_g^2}^\infty \frac{y\mathrm{e}^{w^2(y)-y} g((4\pi)^{-1/2} w(y))}{1 + (4\pi)^{-1/2} w(y)} \int_y^\infty \frac{\mathrm{e}^{w^2(x)-x} g((4\pi)^{-1/2} w(x))}{1 + (4\pi)^{-1/2} w(x)} \mathrm{d}x\mathrm{d}y \\ &= \frac{m_g^2}{(t_g+1)^2} \int_{4\pi t_g^2}^\infty \int_y^\infty y\mathrm{e}^{4\pi t_g^2 - y} \mathrm{e}^{4\pi t_g^2 - x} dx dy = \frac{m_g^2 (8\pi t_g^2 + 1)}{4(t_g+1)^2}. \end{aligned}$$

Hence (7.1) follows from assumption (1.17), and this finishes the proof of Theorem 1.8.

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