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STABILITY OF KDV EQUATION ON A NETWORK WITH BOUNDED AND UNBOUNDED BRANCHES

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Abstract. In this work, we studied the exponential stability of the nonlinear KdV equation posed on a star shaped network with a finite number of branches. On each branch of the network we define a KdV equation posed on a finite domain $(0, \ell_j)$ or the half-line $(0, \infty)$. We start by proving well-posedness and some regularity results. Then, we state the exponential stability of the linear KdV equation by acting with a damping term on some branches. The main idea is to prove a suitable observability inequality. In the nonlinear case, we obtain two kinds of results: The first result holds for small amplitude solutions, and is proved using a perturbation argument from the linear case but without acting on all edges. The second result is a semiglobal stability result, and it is obtained by proving an observability inequality directly for the nonlinear system, but we need to act with damping terms on all the branches. In this case, we are able to prove the stabilization in weighted spaces.

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1. Introduction

In [1], the Korteweg–de Vries (KdV) equation was first proposed to model the behavior of long water waves in a channel. This famous nonlinear third-order dispersive equation arises in various physical systems, including water waves, tsunamis, the transmission of electrical signals in nerve fibers, plasma, cosmology, *etc.* (see, for example, [2–4]). It is a prototypical example of a soliton equation, which admits solutions in the form of solitary waves that preserve their shape and speed during propagation. If we study the KdV equation in a bounded domain, the following model was suggested in [5]

$$\partial_t u + \partial_x u + \partial_x^3 u + u \partial_x u = 0.$$

The KdV equation has been the subject of extensive research in recent years, with a particular focus on its controllability and stabilization properties, which are detailed for instance in [6] and [7]. When it is defined on a network, the KdV equation was proposed to model the pressure of an arterial tree [8]. We also mention [9, 10] where controllability properties were studied and [11, 12] where the exponential stability was achieved by acting with damping terms with time-delay and saturation, respectively (see [13] for more problems related to KdV in

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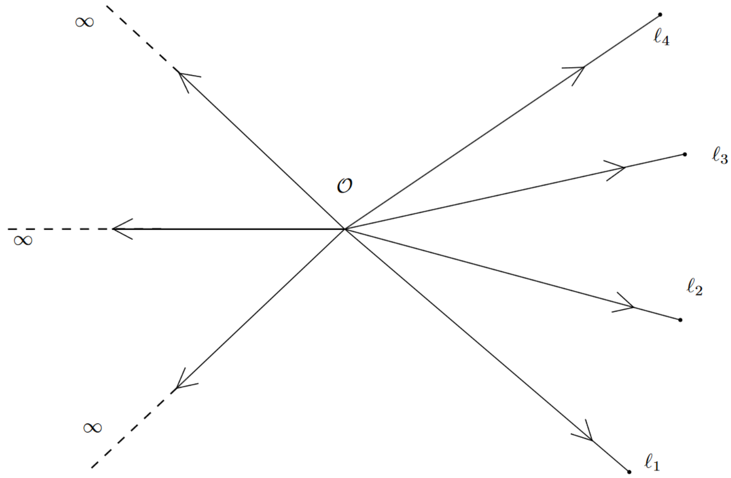

FIGURE 1. Star shaped network for $N_F = 4$ and $N_\infty = 3$.

networks). The main difference of this work with the previously cited is that we consider a star-shaped network mixing bounded and unbounded lengths as, for example, [14, 15] in the case of wave equation.
With respect to the KdV equation defined on the half-line, we can mention, for instance, [5, 16] which focus on the well-posedness properties. In [17], the exact controllability of the linear KdV equation defined on the half-line was obtained by using Carleman estimates. A first result of exponential stability of the KdV equation in the half-line considering a localized damping was derived in [18] under the assumption that the damping term $a(x) \geq c > 0$ in $(0, \delta) \cup (\beta, \infty)$ with $\beta > \delta$ (see [19] for a similar problem in the context of KdV-Burger equation in the whole-line and half-line). Then, in [20] exponential decay of the energy in weighted spaces was derived, and it was noticed that the interval $(0, \delta)$ can be dropped. We can mention also [21] where similar ideas of [20] were applied in the case of a Gear-Grimshaw system modeling long waves. This work is the continuation of [22] (see also [23]) where the linear case was studied. Here we expose both linear and nonlinear problems in a sake of completeness.

In this work inspired by [8, 18] we study the exponential stabilization problem of the KdV equation posed on a star shaped network where the branches mix finite intervals and half-lines.

Let $a < b$, two real numbers, we set

$$[\![a, b]\!] = \mathbb{N} \cap [a, b], \;\;]\!]a, b]\!] = \mathbb{N} \cap (a, b].$$

Let $K = \{k_j : j \in [\![1, N]\!]\}$ be the set of the $N = N_F + N_\infty$ edges of a network $\mathcal{T}$ described as the intervals I_j for $j \in [\![1, N]\!]$, where

$$\begin{cases} I_j = (0, \ell_j) \text{ with } \ell_j > 0 & j \in [\![1, N_F]\!], \\ I_j = (0, \infty) & j \in \,]\!]N_F, N]\!]. \end{cases}$$

The network $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{j=1}^{N} k_j$. We consider a network of $N = N_F + N_\infty$ damped nonlinear KdV equations (see Fig. 1), each one of them defined on I_j for $j \in [\![1, N]\!]$, *i.e.*,

$$(\partial_t u_j + \partial_x u_j + \partial_x^3 u_j + u_j \partial_x u_j + a_j u_j)(t, x) = 0, \; \forall x \in I_j, \; t > 0.$$

These equations are connected by transmission conditions at 0 as follows

$$\begin{cases} u_j(t,0) = u_{j'}(t,0), \quad j,j' \in [\![1,N]\!], \qquad t>0, & \Big(\text{continuity condition}\Big), \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2,\ t>0, & \Big(\text{null-flux like condition}\Big), \end{cases}$$

where $\alpha > \dfrac{N}{2}$ is chosen to have a non-increasing energy (see the end of this section), while the central node conditions are inspired by [8, 11, 12] and are taken in order to model the blood pressure on the arterial tree. For $j \in [\![1,N_F]\!]$, we complement the system with the classical null boundary conditions at the right end,

$$u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0,\ t>0.$$

In the case of the KdV equation posed in bounded domain, three boundary conditions are needed, *i.e.*, two at the right and one at the left. For the half-line $(0,\infty)$ case due to the sense of propagation and the sign of the dispersive term, just one boundary condition is needed to the left for each equation [5, 16][1] At the central node, we have N boundary conditions $N-1$ coming from the continuity and one coming from the flux condition. Finally, we consider initial condition $u_j(0,x) = u_j^0(x)$, $x \in I_j$ for $j \in [\![1,N]\!]$. According to the previous hypothesis, the system studied in this work reads as:

$$\begin{cases} (\partial_t u_j + \partial_x u_j + u_j\partial_x u_j + \partial_x^3 u_j + a_j u_j)(t,x) = 0, & \forall x \in I_j,\ t>0,\ j\in[\![1,N]\!], \\ u_j(t,0) = u_{j'}(t,0), & \forall j,j' \in [\![1,N]\!], \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2, & t>0, \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t>0,\ j \in [\![1,N_F]\!], \\ u_j(0,x) = u_j^0(x), & x \in I_j, j \in [\![1,N]\!], \end{cases} \tag{KdV}$$

and its linearization around zero:

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j + a_j u_j)(t,x) = 0, & \forall x \in I_j,\ t>0,\ j\in[\![1,N]\!], \\ u_j(t,0) = u_{j'}(t,0), & \forall j,j' \in [\![1,N]\!], \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0), & t>0, \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t>0,\ j =\in [\![1,N_F]\!], \\ u_j(0,x) = u_j^0(x), & x \in I_j, j \in [\![1,N]\!], \end{cases} \tag{LKdV}$$

where $\alpha > \dfrac{N}{2}$. The damping terms $(a_j)_{j\in[\![1,N]\!]} \in \mathbb{L}^\infty(\mathcal{T})$, act locally on the branches.

Remark 1.1. The limit case $\alpha = \dfrac{N}{2}$ was rid off in [9, 12] in the context of bounded branches using a specific multiplier (different on each branch) in the well-posedness result. The adaptability of this strategy for unbounded branches is open. As our focus is on stability results, we just analyze the case $\alpha > \dfrac{N}{2}$.

[1] For instance in [16] was shown that for the equation $\partial_t u + \partial_x u + \partial_x^3 u + u\partial_x u = 0$, for $x<0$, two boundary conditions are needed, *i.e.* $u(t,0) = \partial_x u(t,0)) = 0$.

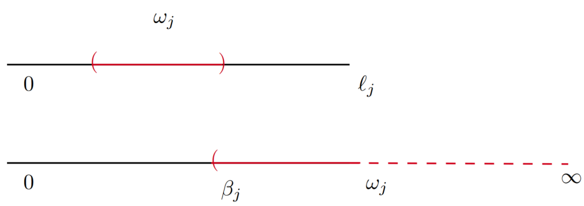

FIGURE 2. Structure of acted region (in red) in the case of bounded and unbounded branch.

Our purpose, is to achieve the exponential stability by acting with the damping terms, not necessarily in all the branches. Let $\mathbb{I}_{act} \subset [\![1, N]\!]$ the set of action index, formally the damping terms, are taken in the following way:

- **No action index:** For $j \in [\![1, N]\!] \setminus \mathbb{I}_{act}$, $a_j \equiv 0$.
- **Local action:** For $j \in \mathbb{I}_{act}$, $a_j(x) \geq c_j > 0$ in a nonempty open set ω_j of I_j.
- **Structure of action set in the half-line case:** For the index $j \in]\!]N_F, N]\!] \cap \mathbb{I}_{act}$, we take a specific structure of the set $\omega_j = (\beta_j, \infty)$, for $\beta_j > 0$ given.

These properties are summarized in

$$\begin{cases} a_j \equiv 0 \quad \text{for} \quad j \in [\![1, N]\!] \setminus \mathbb{I}_{act}, \\ a_j(x) \geq c_j > 0 \quad \text{in } \omega_j \subset I_j, \text{ for } \quad j \in \mathbb{I}_{act}, \\ \omega_j = (\beta_j, \infty), \text{ for } j \in]\!]N_F, N]\!] \ \cap \mathbb{I}_{act}. \end{cases} \tag{1.1}$$

To study the well-posedness properties of (KdV), we need to introduce some specific spaces. Let $s = 1$,2 and for $j \in [\![1, N_F]\!]$ consider the space

$$H_r^s(I_j) = \left\{ v \in H^s(I_j), \ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{i-1} v(\ell_j) = 0, 1 \leq i \leq s \right\},$$

where the index r is related to the null right boundary conditions, and the space $\mathbb{H}_e^s(\mathcal{T})$ defined by

$$\mathbb{H}_e^s(\mathcal{T}) = \Bigg\{ \underline{u} = (u_1, \ldots, u_N)^T \in \prod_{j=1}^{N_F} H_r^s(I_j) \times (H^s(0,\infty))^{N_\infty}, u_j(0) = u_{j'}(0),$$
$$\forall j, j' = 1, \ldots, N \Big\}, \text{with } s = 1, 2,$$

with its associated norm:

$$\|\underline{u}\|^2_{\mathbb{H}_e^1(\mathcal{T})} = \sum_{j=1}^{N} \|u_j\|^2_{H^1(I_j)}, \text{ for } s = 1.$$

We introduce also the product spaces: $\mathbb{H}^3(\mathcal{T}) = \prod_{j=1}^{N} H^3(I_j)$, $\mathbb{L}^\infty(\mathcal{T}) = \prod_{j=1}^{N} L^\infty(I_j)$, and $\mathbb{L}^2(\mathcal{T}) = \prod_{j=1}^{N} L^2(I_j)$, with

$$(\underline{u}, \underline{v})_{\mathbb{L}^2(\mathcal{T})} = \sum_{j=1}^{N_F} \int_0^{\ell_j} u_j v_j \mathrm{d}x + \sum_{j=N_F+1}^{N} \int_0^{\infty} u_j v_j \mathrm{d}x, \quad \forall \underline{u}, \underline{v} \in \mathbb{L}^2(\mathcal{T}),$$

and the weighted spaces

$$L^2_{(1+x)}(0,\infty) := \{f \in L^2(0,\infty) : \int_0^\infty (1+x) f^2 \mathrm{d}x < \infty\},$$
$$L^2_{(1+x^2)}(0,\infty) := \{f \in L^2(0,\infty) : \int_0^\infty (1+x^2) f^2 \mathrm{d}x < \infty\},$$

endowed with the norms

$$\|f\|_{L^2_{(1+x)}(0,\infty)} = \left(\int_0^\infty (1+x) f^2 \mathrm{d}x\right)^{1/2}, \quad \|f\|_{L^2_{(1+x^2)}(0,\infty)} = \left(\int_0^\infty (1+x^2) f^2 \mathrm{d}x\right)^{1/2}.$$

We also define the spaces $B_j = C([0,T], L^2(I_j)) \cap L^2(0,T;H^1(I_j))$ for $j \in [\![1, N_F]\!]$, $B_\infty = \{f \in C([0,T]; L^2_{(1+x^2)}(0,\infty))$; such that $\partial_x f \in L^2(0,T; L^2_{(1+x)}(0,\infty))\}$, and $\mathbb{B} = \prod_{j=1}^{N_F} B_j \times (B_\infty)^{N_\infty}$, endowed with the norms

$$\|u\|_{B_j} = \|u\|_{C([0,T],L^2(I_j))} + \|u\|_{L^2(0,T;H^1(I_j))}, \quad j \in [\![1, N_F]\!],$$
$$\|u\|_{B_\infty} = \|u\|_{C([0,T];L^2_{(1+x^2)}(0,\infty))} + \|\partial_x u\|_{L^2(0,T;L^2_{(1+x)}(0,\infty))}.$$

Finally, we define the spaces $\mathbb{Y} = \prod_{j=1}^{N_F} L^2(I_j) \times (L^2_{(1+x^2)}(0,\infty))^{N_\infty}$ and

$$\mathbb{X}^s = \prod_{j=1}^{N_F} L^2(0,T; H^s(0,\ell_j)) \times (L^2(0,T; H^s_{loc}(0,\infty)))^{N_\infty}, \text{ for } s = -2,-1,0,1.$$

For the systems (KdV) and (LKdV) we define the natural $\mathbb{L}^2(\mathcal{T})$ energy of a solution by

$$E(t) = \frac{1}{2} \sum_{j=1}^{N} \int_{I_j} (u_j(t,x))^2 \mathrm{d}x. \tag{1.2}$$

We can check that for every sufficiently smooth solution of (KdV) or (LKdV) the energy satisfies

$$\dot{E}(t) = -\left(\alpha - \frac{N}{2}\right)(u_1(t,0))^2 - \frac{1}{2}\sum_{j=1}^{N} (\partial_x u_j(t,0))^2 - \sum_{j=1}^{N} \int_{I_j} a_j(x)(u_j(t,x))^2 \mathrm{d}x. \tag{1.3}$$

Observe that, as $a_j \geq 0$, the term $a_j u_j$ provides dissipation to the energy, then $\dot{E}(t) \leq 0$. This work is devoted to prove that indeed the terms $a_j u_j$ provides exponential stability of (LKdV) and (KdV). The article is organized as follows. In Section 2, the well-posedness of (LKdV) and (KdV) is proven using semigroup theory and a fixed point approach. In Section 3 some extra regularity results are obtained for (LKdV) and (KdV). In Section 4, the stabilization problem is studied, and an observability inequality is used to prove exponential stability. Secondly, we deduce a semiglobal exponential stability result for (KdV) by acting with the damping terms on all the branches. In Section 5 also using damping terms actives in all the branches, we show the exponential stability of (KdV) in $\mathbb{Y}$, that is the same spaces for the well-posedness and stability. Finally, we present some conclusions and final remarks.

2. Well-posedness and regularity results for LKdV and KdV system

Our idea is the following one, first we prove a well-posedness result for (LKdV) then we add a boundary source term $g(t)$ at the central node and internal source terms $f_j(t,x)$ to play the role of the nonlinear boundary condition $-\dfrac{N}{3}u_1^2(t,0)$ and internal terms $u_j\partial_x u_j$ respectively. Finally, to pass to the nonlinear case (KdV), we use a fixed point argument. In what follows, we use the well known definitions of *classical* and *mild* solutions [24], Chapter 4.

2.1. Linear case

Note that (LKdV) can be written as

$$\begin{cases} \underline{u}_t(t) = \mathcal{A}\underline{u}(t), \quad t>0, \\ \underline{u}(0) = \underline{u}^0, \end{cases} \tag{2.1}$$

where the operator $\mathcal{A}$ is defined by,

$$\begin{aligned} &\mathcal{A}\underline{u} = -(\partial_x + \partial_x^3 + \underline{a})\underline{u}, \\ &\mathcal{D}(\mathcal{A}) = \left\{ \underline{u} \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^2(\mathcal{T}), \sum_{j=1}^N \frac{\mathrm{d}^2 u_j}{\mathrm{d}x^2}(0) = -\alpha u_1(0) \right\}. \end{aligned}$$

Let $\underline{u} \in \mathcal{D}(\mathcal{A})$, then, after some integrations by parts,

$$(\underline{u}, \mathcal{A}\underline{u})_{\mathbb{L}^2(\mathcal{T})} = -\left(\alpha - \frac{N}{2}\right)(u_1(0))^2 - \frac{1}{2}\sum_{j=1}^N (\partial_x u_j(0))^2 - \sum_{j=1}^N \int_{I_j} a_j (u_j)^2 \mathrm{d}x \le 0.$$

Easy calculations show that $\mathcal{A}^*$ is defined by

$$\begin{aligned} &\mathcal{A}^*\underline{v} = (\partial_x + \partial_x^3 + \underline{a})\underline{v}, \\ &\mathcal{D}(\mathcal{A}^*) = \left\{ \underline{v} \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^1(\mathcal{T}), \sum_{j=1}^N \frac{\mathrm{d}^2 v_j}{\mathrm{d}x^2}(0) = (\alpha - N)v_1(0), \frac{\mathrm{d}v_j}{\mathrm{d}x}(0) = 0, \forall j \in [\![1, N]\!] \right\}. \end{aligned}$$

Similarly, we get that for all $\underline{v} \in \mathcal{D}(\mathcal{A}^*)$

$$(\underline{v}, \mathcal{A}^*\underline{v})_{\mathbb{L}^2(\mathcal{T})} = -\left(\alpha - \frac{N}{2}\right)(v_1(0))^2 - \frac{1}{2}\sum_{j=1}^{N_F} (\partial_x v_j(\ell_j))^2 - \sum_{j=1}^N \int_{I_j} a_j (v_j)^2 \mathrm{d}x \le 0.$$

Finally, $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative, and $\mathcal{A}$ is a densely defined closed operator, thus by [24], Corollary 4.4, Chapter 1 $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions on $\mathbb{L}^2(\mathcal{T})$. Systems (LKdV) and (2.1) are equivalent, thus we deduce the following result.

Theorem 2.1. *Let $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, then, there exists a unique mild solution $\underline{u} \in C([0,\infty); \mathbb{L}^2(\mathcal{T}))$ of (LKdV). Moreover, if $\underline{u}^0 \in \mathcal{D}(\mathcal{A})$, then $\underline{u}$ is a classical solution and $\underline{u} \in C([0,\infty); \mathcal{D}(\mathcal{A})) \cap C^1([0,\infty); \mathbb{L}^2(\mathcal{T}))$.*

2.2. Extra boundary condition and source term

Following [8], we prove now some regularity results for the linear KdV equation with extra boundary source term $g(t)$ at the central node and extra internal term $f_j(t,x)$

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j + a_j u_j)(t,x) = f_j(t,x), & \forall x \in I_j,\ t>0,\ j \in [\![1,N]\!],\\ u_j(t,0) = u_{j'}(t,0), & \forall j,j' \in [\![1,N]\!],\\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) + g(t), & t>0,\\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t>0,\ j\in[\![1,N_F]\!],\\ u_j(0,x) = u_j^0(x), & x\in I_j, j\in[\![1,N]\!]. \end{cases} \tag{2.2}$$

Proposition 2.2. *Let $T>0$, $(\underline{u}^0, g, \underline{f}) \in \mathbb{L}^2(\mathcal{T}) \times L^2(0,T) \times L^1(0,T;\mathbb{L}^2(\mathcal{T}))$, then there exists a unique mild solution $\underline{u} \in C([0,T],\mathbb{L}^2(\mathcal{T}))$ of (2.2). If $(\underline{u}^0, g, \underline{f}) \in D(\mathcal{A}) \times C_0^2([0,T]) \times C^1([0,T];\mathbb{L}^2(\mathcal{T}))$ where $C_0^2([0,T]) := \{\varphi \in C^2([0,T]) : \varphi(0)=0\}$, then, the solution is classical and $\underline{u} \in C([0,\infty);D(\mathcal{A})) \cap C^1([0,\infty);\mathbb{L}^2(\mathcal{T}))$. Moreover, if $\underline{f} = \underline{0}$ and $g=0$, the following estimate holds*

$$\begin{aligned} \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C\Bigg(& \frac{1}{T}\|\underline{u}\|^2_{L^2([0,T];\mathbb{L}^2(\mathcal{T}))} + \|u_1(\cdot,0)\|^2_{L^2(0,T)} + \|\partial_x \underline{u}(\cdot,0)\|^2_{L^2(0,T)} \\ & + \sum_{j=1}^{N} \int_0^T\!\!\int_{I_j} a_j (u_j)^2 \mathrm{d}x\mathrm{d}t \Bigg). \end{aligned} \tag{2.3}$$

Proof. Let $(\underline{u}^0, g, \underline{f}) \in D(\mathcal{A}) \times C_0^2([0,T]) \times C^1([0,T];\mathbb{L}^2(\mathcal{T}))$ and the lifting function $\underline{\phi}$ defined as

$$\phi_j(x) = \begin{cases} \dfrac{(x-\ell_j)^2}{\ell_j^2\left(2\sum_{k=1}^{N_F}\ell_k^{-2} + 2N_\infty + \alpha\right)}, & j\in[\![1,N_F]\!],\\ \dfrac{(x-1)^2}{2\sum_{k=1}^{N_F}\ell_k^{-2} + 2N_\infty + \alpha}\eta(x), & j \in]\!]N_F, N]\!], \end{cases}$$

where $\eta \in C^\infty(0,\infty)$ is a smooth function such that $\eta(x)=1$, for $x\in(0,\delta_0)$ and $\eta(x)=0$ for $x\in(\delta_0+1,\infty)$ for $\delta_0>0$ given. We can easily check that

$$\begin{cases} \phi_j(\ell_j) = \phi_j'(\ell_j) = 0, & \forall j \in [\![1,N_F]\!],\\ \phi_j(0) = \dfrac{1}{2\sum_{k=1}^{N_F}\ell_k^{-2} + 2N_\infty + \alpha} = \phi_{j'}(0), & \forall j,j' \in [\![1,N]\!],\\ \sum_{j=1}^{N} \phi_j''(0) = 1 - \alpha\phi_1(0). \end{cases} \tag{2.4}$$

Define $\underline{v} := \underline{u} - g\underline{\phi}$, then $\underline{u}$ is solution of (2.2) if and only if $\underline{v}$ is solution of (2.5)

$$\begin{cases} (\partial_t v_j + \partial_x v_j + \partial_x^3 v_j + a_j v_j)(t,x) = \tilde{f}_j(t,x), & \forall x \in I_j,\ t>0,\ j \in [\![1,N]\!], \\ v_j(t,0) = v_{j'}(t,0), & \forall j,j' \in [\![1,N]\!], \\ \sum_{j=1}^N \partial_x^2 v_j(t,0) = -\alpha v_1(t,0), & t>0, \\ v_j(t,\ell_j) = \partial_x v_j(t,\ell_j) = 0, & t>0,\ j\in [\![1,N_F]\!], \\ v_j(0,x) = u_j^0(x), & x \in I_j,\ j \in [\![1,N]\!], \end{cases} \tag{2.5}$$

where $\tilde{f}_j(t,x) = f_j(t,x) - \phi_j(x)g'(t) - (\phi_j' + \phi_j''' + a_j\phi_j)(x)g(t)$.

Thus, as $\underline{f} - \underline{\phi}g' - (\underline{\phi}' + \underline{\phi}''' + \underline{a}\underline{\phi})g \in C^1([0,T],\mathbb{L}^2(\mathcal{T}))$, by the classical semigroup theory and Theorem 2.1, we deduce the existence of a unique solution $\underline{v}$ of (2.5). Moreover, $\underline{v} \in C([0,T],D(\mathcal{A})) \cap C^1([0,T];\mathbb{L}^2(\mathcal{T}))$ and hence (2.2) admits a unique classical solution $\underline{u} \in C([0,T],D(\mathcal{A})) \cap C^1([0,T];\mathbb{L}^2(\mathcal{T}))$.

Now, let $\underline{u}$ be a classical solution of (2.2). Multiplying the first line of (2.2) by u_j and integrating on $[0,s]\times I_j$, after some integrations by parts we get for $s \in (0,T)$

$$\begin{aligned} \sum_{j=1}^N \int_{I_j} (u_j(s,x))^2 \mathrm{d}x + \sum_{j=1}^N \int_0^s (\partial_x u_j(t,0))^2 \mathrm{d}t + (2\alpha - N)\int_0^s (u_1(t,0))^2\mathrm{d}t + 2\sum_{j=1}^N \int_0^s\!\!\int_{I_j} a_j (u_j)^2 \mathrm{d}x\mathrm{d}t \\ = \sum_{j=1}^N \int_{I_j} (u_j(0,x))^2\mathrm{d}x + 2\sum_{j=1}^N \int_0^s\!\!\int_{I_j} f_j u_j \mathrm{d}x\mathrm{d}t + 2\int_0^s u_1(t,0)g(t)\mathrm{d}t. \end{aligned} \tag{2.6}$$

Note that

$$\begin{aligned} 2\sum_{j=1}^N \int_0^s\!\!\int_{I_j} f_j u_j \mathrm{d}x\mathrm{d}t + 2\int_0^s u_1(t,0)g(t)\mathrm{d}t \le & 2\sum_{j=1}^N \int_0^T \|f_j\|_{L^2(I_j)} \|u_j\|_{L^2(I_j)}\mathrm{d}t + 2\int_0^T |u_1(t,0)g(t)|\mathrm{d}t \\ \le & 2\sum_{j=1}^N \|u_j\|_{C([0,T],L^2(I_j))} \int_0^T \|f_j\|_{L^2(I_j)}\mathrm{d}t \\ & + 2\int_0^T |u_1(t,0)||g(t)|\mathrm{d}t. \end{aligned} \tag{2.7}$$

Using Young's inequality, we get for all $\varepsilon > 0$

$$\begin{aligned} 2\sum_{j=1}^N \int_0^s\!\!\int_{I_j} f_j u_j \mathrm{d}x\mathrm{d}t + 2\int_0^s u_1(t,0)g(t)\mathrm{d}t \le & \varepsilon \|\underline{u}\|^2_{C([0,T];\mathbb{L}^2(\mathcal{T}))} + \frac{1}{\varepsilon}\|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} \\ & + \varepsilon\|u_1(\cdot,0)\|^2_{L^2(0,T)} + \frac{1}{\varepsilon}\|g\|^2_{L^2(0,T)}. \end{aligned}$$

Next, taking the supremum for $s \in (0,T)$ in each term of the left-hand-side of (2.6), recalling $\alpha > \frac{N}{2}$ and choosing ε small enough we deduce

$$\begin{aligned}\|\underline{u}\|^2_{C([0,T];\mathbb{L}^2(\mathcal{T}))} + \|u_1(\cdot,0)\|^2_{L^2(0,T)} + \|\partial_x \underline{u}(\cdot,0)\|^2_{L^2(0,T)} + \sum_{j=1}^N \int_0^T\int_{I_j} a_j(u_j)^2 \mathrm{d}x\mathrm{d}t \\ \leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|g\|^2_{L^2(0,T)}\right),\end{aligned} \tag{2.8}$$

for a suitable $C>0$, that does not depend on $\underline{u}^0$, $\underline{f}$ and g. Thus, by density of $D(\mathcal{A}) \times C^2_0([0,T]) \times C^1([0,T];\mathbb{L}^2(\mathcal{T}))$ in $\mathbb{L}^2(\mathcal{T}) \times L^2(0,T) \times L^1(0,T;\mathbb{L}^2(\mathcal{T}))$ we extend our result to arbitrary data $(\underline{u}^0, g, \underline{f}) \in \mathbb{L}^2(\mathcal{T}) \times L^2(0,T) \times L^1(0,T;\mathbb{L}^2(\mathcal{T}))$.

Finally, to prove (2.3), consider $\underline{f} = \underline{0}$ and $g=0$. Then, multiplying the first line of (2.2) by $(T-t)u_j$ and integrating on $[0,T]\times I_j$, after some integrations by parts we get

$$\begin{aligned} T\sum_{j=1}^N \int_0^{\ell_j} (u_j(0,x))^2 \mathrm{d}x =& \sum_{j=1}^N \int_0^T\int_{I_j} (u_j)^2 \mathrm{d}x\mathrm{d}t + (2\alpha - N)\int_0^T (T-t)(u_1(t,0))^2\mathrm{d}t \\ &+ \sum_{j=1}^N \int_0^T (T-t)(\partial_x u_j(t,0))^2 \mathrm{d}t + 2\sum_{j=1}^N \int_0^T\int_{I_j} (T-t)a_j(u_j)^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.9}$$

we deduce (2.3) easily from (2.9). □

Remark 2.3. For a single nonlinear KdV equation posed on the half-line, from [25], Theorem 2.1 we know that for any initial data in $L^2(0,\infty)$ we have a unique solution in $C([0,T];L^2(0,\infty))$. In the network case, due to the semigroup approach, an analogous result is quite difficult to achieve. In fact, in the semigroup framework we write our system as $\underline{u}_t = \mathcal{A}\underline{u} + \underline{f}$, where $\underline{f}$ plays the role of the nonlinearity. As it is classical, we ask for $\underline{f} \in L^1(0,T;\mathbb{L}^2(\mathcal{T}))$. For the nonlinear system, this requires that $u_j\partial_x u_j \in L^1(0,T;L^2(0,\infty))$. But we are not able to prove this with initial data in $\mathbb{L}^2(\mathcal{T})$ (see also Rem. 2.5).

Motivated by this remark, we introduce the following proposition to obtain the classical $L^2(0,T;H^1(I_j))$ regularity for solutions of the KdV equation, this will help us to deal with the nonlinearities, but the price to pay is to consider more regular initial conditions on infinite edges.

Proposition 2.4. *Let $(\underline{u}^0, g, \underline{f}) \in \mathbb{Y} \times L^2(0,T) \times L^1(0,T;\mathbb{Y})$, then the mild solution $\underline{u}$ of (2.2) (given by Prop. 2.2) satisfies $\underline{u} \in \mathbb{B}$. Moreover,*

$$\|u_j\|_{B_j} \leq C\left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} + \|\underline{f}\|_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|g\|_{L^2(0,T)}\right),\ j \in [\![1, N_F]\!],$$

$$\begin{aligned}\|u_j\|_{B_\infty} \leq C\Big(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} + \|u_j^0\|_{L^2_{(1+x^2)}(0,\infty)} + \|\underline{f}\|_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|f_j\|_{L^1(0,T;L^2_{(1+x^2)}(0,\infty))} \\ + \|g\|_{L^2(0,T)}\Big),\ j \in]\!]N_F, N]\!].\end{aligned}$$

Proof. Let $\underline{u}$ be a classical solution to (2.2), multiplying the first line of (2.2) by xu_j, integrating on $[0,T]\times I_j$ after some integrations by parts we get

$$\begin{aligned}\frac{1}{2}\int_{I_j} x(u_j(T,x))^2\mathrm{d}x + \frac{3}{2}\int_0^T\!\!\int_{I_j}(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_{I_j} a_j x(u_j)^2\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_{I_j} x(u_j^0)^2\mathrm{d}x \\ +\frac{1}{2}\int_0^T\!\!\int_{I_j}(u_j)^2\mathrm{d}x\mathrm{d}t - \int_0^T u_1(t,0)\partial_x u_j(t,0)\mathrm{d}t + \int_0^T\!\!\int_{I_j} xf_ju_j\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.10}$$

Now, using that for all $x\geq 0$, $x\leq 1+x^2$, we deduce

$$\frac{1}{2}\int_{I_j} x(u_j^0)^2\mathrm{d}x \leq \begin{cases}\dfrac{\ell_j}{2}\|u_j^0\|^2_{L^2(I_j)}, & j\in[\![1,N_F]\!],\\[2ex] \dfrac{1}{2}\|u_j^0\|^2_{L^2_{(1+x^2)}(0,\infty)}, & j\in]\!]N_F,N]\!].\end{cases}$$

Similarly

$$\int_0^T\!\!\int_{I_j} xu_jf_j\mathrm{d}x\mathrm{d}t \leq \begin{cases}\ell_j\|u_j\|_{C([0,T];L^2(I_j))}\|f_j\|_{L^1(0,T;L^2(I_j))}, & j\in[\![1,N_F]\!],\\[2ex] \|u_j\|_{C([0,T];L^2(0,\infty))}\|f_j\|_{L^1(0,T;L^2_{(1+x^2)}(0,\infty))}, & j\in]\!]N_F,N]\!].\end{cases}$$

Thus, using the above inequalities, (2.10) and (2.8), we can obtain

$$\begin{aligned}\|\partial_x u_j\|^2_{L^2(0,T;L^2(I_j))} &\leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|g\|^2_{L^2(0,T)}\right),\ j\in[\![1,N_F]\!],\\ \|\partial_x u_j\|^2_{L^2(0,T;L^2(0,\infty))} &\leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|u_j^0\|^2_{L^2_{(1+x^2)}(0,\infty)} + \|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))}\right.\\ &\qquad \left.+\|f_j\|^2_{L^1(0,T;L^2_{(1+x^2)}(0,\infty))} + \|g\|^2_{L^2(0,T)}\right),\ j\in]\!]N_F,N]\!].\end{aligned} \tag{2.11}$$

From the last inequalities we get $\partial_x\underline{u}\in L^2(0,T;\mathbb{L}^2(\mathcal{T}))$, in particular for $j\in[\![1,N_F]\!]$, $\partial_x u_j\in B_j$. Now, for $j\in]\!]N_F,N]\!]$ multiplying by x^2u_j, integrating on $[0,s]\times(0,\infty)$ after some integrations by parts we get

$$\begin{aligned}\frac{1}{2}\int_0^\infty x^2(u_j(s,x))^2\mathrm{d}x + 3\int_0^s\!\!\int_0^\infty x(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^s\!\!\int_0^\infty a_jx^2(u_j)^2\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_0^\infty x^2(u_j^0)^2\mathrm{d}x\\ + \int_0^s\!\!\int_0^\infty x(u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^s(u_1(t,0))^2\mathrm{d}t + \int_0^s\!\!\int_0^\infty x^2f_ju_j\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.12}$$

Note that, for all $\varepsilon>0$

$$\begin{aligned}\int_0^s\!\!\int_0^\infty x(u_j)^2\mathrm{d}x\mathrm{d}t &\leq \int_0^s\|xu_j\|_{L^2(0,\infty)}\|u_j\|_{L^2(0,\infty)}\mathrm{d}t \leq \|xu_j\|_{C([0,T];L^2(0,\infty))}\int_0^T\|u_j\|_{L^2(0,\infty)}\mathrm{d}t\\ &\leq \frac{\varepsilon}{2}\|xu_j\|^2_{C([0,T];L^2(0,\infty))} + \frac{T^2}{2\varepsilon}\|u_j\|^2_{C([0,T];L^2(0,\infty))}.\end{aligned} \tag{2.13}$$

$$\int_0^s\int_0^\infty x^2 f_j u_j \mathrm{d}x\mathrm{d}t \leq \int_0^s \|xu_j\|_{L^2(0,\infty)}\|xf_j\|_{L^2(0,\infty)}\mathrm{d}t \leq \|xu_j\|_{C([0,T];L^2(0,\infty))}\int_0^T \|xf_j\|_{L^2(0,\infty)}\mathrm{d}t$$
$$\leq \frac{\varepsilon}{2}\|xu_j\|^2_{C([0,T];L^2(0,\infty))} + \frac{1}{2\varepsilon}\|f_j\|^2_{L^1(0,T;L^2_{(1+x^2)}(0,\infty))}.$$

Then, taking the supremum for $s \in (0,T)$ in (2.12), using (2.8) and choosing ε small enough we deduce

$$\|xu_j\|^2_{C([0,T];L^2(0,\infty))} + \|\sqrt{x}\partial_x u_j\|^2_{L^2(0,T;L^2(0,\infty))} \leq C\bigg(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|u_j^0\|^2_{L^2_{(1+x^2)}(0,\infty)}$$
$$+\|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|f_j\|^2_{L^1(0,T;L^2_{(1+x^2)}(0,\infty))} + \|g\|^2_{L^2(0,T)}\bigg),\ j \in]\!]N_F, N]\!]. \tag{2.14}$$

We observe from (2.14), $u_j \in B_\infty$ for $j \in]\!]N_F, N]\!]$ and by (2.11), $\partial_x\underline{u} \in L^2(0,T;\mathbb{L}^2(\mathcal{T}))$, therefore $\underline{u} \in \mathbb{B}$. □

Remark 2.5. Note that in the proof of Proposition 2.4 we consider for the unbounded branches ($j \in [\![N_F, N]\!]$) initial data such that $u^0 \in L^2_{(1+x^2)}(0,\infty)$. One could ask why not only consider $u^0 \in L^2_{(1+x)}(0,\infty)$. First, note that an estimate as (2.11) it is possible to obtain too

$$\|\partial_x u_j\|^2_{L^2(0,T;L^2(0,\infty))} \leq C\Big(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|u_j^0\|^2_{L^2_{(1+x)}(0,\infty)} + \|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|f_j\|^2_{L^1(0,T;L^2_{(1+x)}(0,\infty))}$$
$$+\|g\|^2_{L^2(0,T)}\Big),\ j \in]\!]N_F, N]\!],$$

this means that if $u_j^0 \in L^2_{(1+x)}(0,\infty)$ for $j \in]\!]N_F, N]\!]$, we still have $\partial_x\underline{u} \in L^2(0,T;\mathbb{L}^2(\mathcal{T}))$. But we can not prove that $u_j \in B_\infty$ for $j \in]\!]N_F, N]\!]$, because the multiplier x^2 is needed. The space B_∞ it is strongly used to deal with the nonlinearity, see Lemma 2.7.

2.3. Nonlinear case

The aim of this subsection is to use the well-posedness result for (LKdV) and a fixed point approach to obtain the well-posedness of (KdV). In this spirit, the next three lemmas are needed to deal with the nonlinearities.

Lemma 2.6. *Let $y,\ z \in L^2(0,T;H^1(0,L))$. Then $y\partial_x y \in L^1(0,T;L^2(0,L))$ and the map*

$$y \in L^2(0,T;H^1(0,L)) \mapsto y\partial_x y \in L^1(0,T;L^2(0,L))$$

is continuous. Moreover, we have

$$\|y\partial_x y - z\partial_x z\|_{L^1(0,T;L^2(0,L))} \leq C\left(\|y\|_{L^2(0,T;H^1(0,L))} + \|z\|_{L^2(0,T;H^1(0,L))}\right)\|y-z\|_{L^2(0,T;H^1(0,L))}. \tag{2.15}$$

Lemma 2.7. *Let $y,\ z \in B_\infty$. Then $y\partial_x y \in L^1(0,T;L^2_{(1+x^2)}(0,\infty))$ and the map*

$$y \in B_\infty \mapsto y\partial_x y \in L^1(0,T;L^2_{(1+x^2)}(0,\infty))$$

is continuous. Moreover, we have

$$\|y\partial_x y - z\partial_x z\|_{L^1(0,T;L^2_{(1+x^2)}(0,\infty))} \leq C(T^{1/4}+T^{1/2})\left(\|y\|_{B_\infty} + \|z\|_{B_\infty}\right)\|y-z\|_{B_\infty}. \tag{2.16}$$

Finally, for the nonlinearity in the central node condition we have the following result

Lemma 2.8. *Let $\underline{u} \in \mathbb{B}$, then, $(u_1(t,0))^2 \in L^2(0,T)$ and the map*

$$\underline{u} \in \mathbb{B} \mapsto (u_1(t,0))^2 \in L^2(0,T)$$

is continuous. Moreover, we have the estimate,

$$\|u_1^2(\cdot,0)\|_{L^2(0,T)} \leq \frac{1}{\sqrt{2}}\|\underline{u}\|_{\mathbb{B}}^2. \tag{2.17}$$

The proofs of Lemma 2.6 and Lemma 2.8 can be found in [26], Proposition 4.1 and [8], Proposition 2.6 respectively. Concerning Lemma 2.7 the proof is done in Appendix A.

Now, we are ready to prove the well-posedness result for (KdV) using the tools developed in the past sections. We call *mild* solution of (KdV) any mild solution of (2.2) with $f_j = -u_j\partial_x u_j$ and $g = -\frac{N}{3}(u_1(t,0))^2$, *i.e.* For given $\underline{u}^0 \in \mathbb{Y}$ we search for a fix point of the map that for any $(g, \underline{f}) \in \times L^2(0,T) \times L^1(0,T;\mathbb{Y})$ associate the respective mild solution of (2.2).

Theorem 2.9. *Let $(\ell_j)_{j=1}^{N_F} \subset (0,+\infty)^{N_F}$, $T > 0$, there exist $C,\epsilon > 0$ such that for all $\underline{u}^0 \in \mathbb{Y}$, with $\|\underline{u}^0\|_{\mathbb{Y}} \leq \epsilon$, then (KdV) has a unique mild solution $\underline{u} \in \mathbb{B}$. Moreover, it satisfies*

$$\|\underline{u}\|_{\mathbb{B}} \leq C\|\underline{u}^0\|_{\mathbb{Y}}.$$

Proof. Let $T^* > 0$ arbitrary and $\underline{u}^0 \in \mathbb{Y}$, with $\|\underline{u}^0\|_{\mathbb{Y}} \leq \epsilon$ where $\epsilon > 0$ will be chosen later and $\underline{u} \in \mathbb{B}$. Thanks to Lemmas 2.6, 2.7 and 2.8 we get that $\left(\underline{u}^0, -\frac{N}{3}(u_1(\cdot,0))^2, -\underline{u}\partial_x\underline{u}\right) \in \mathbb{Y} \times L^2(0,T^*) \times L^1(0,T^*;\mathbb{Y})$ and by Proposition 2.4 we can consider the map $\Phi : \mathbb{B} \to \mathbb{B}$ defined by $\Phi(\underline{u}) = \underline{v}$ where $\underline{v}$ is the mild solution of

$$\begin{cases} (\partial_t v_j + \partial_x v_j + \partial_x^3 v_j + a_j v_j)(t,x) = -u_j\partial_x u_j, & \forall x \in I_j,\ t > 0,\ j \in [\![1,N]\!],\\ v_j(t,0) = v_{j'}(t,0), & \forall j,j' \in [\![1,N]\!],\\ \displaystyle\sum_{j=1}^{N} \partial_x^2 v_j(t,0) = -\alpha v_1(t,0) - \frac{N}{3}(u_1(t,0))^2, & t > 0,\\ v_j(t,\ell_j) = \partial_x v_j(t,\ell_j) = 0, & t > 0,\ j \in [\![1,N_F]\!],\\ v_j(0,x) = u_j^0(x), & x \in I_j, j \in [\![1,N]\!]. \end{cases} \tag{2.18}$$

Then, $\underline{u} \in \mathbb{B}$ is a solution of (KdV) if $\underline{u}$ is a fixed point of Φ. From Proposition 2.4, Lemma 2.6 and Lemma 2.8, we get for all $\underline{u}$, $\underline{\tilde{u}} \in \mathbb{B}$

$$\|\Phi(\underline{u})\|_{\mathbb{B}} = \|\underline{v}\|_{\mathbb{B}} \leq C_{T^*}\left(\|\underline{u}^0\|_{\mathbb{Y}} + \|\underline{u}\|_{\mathbb{B}}^2\right),$$

$$\|\Phi(\underline{u}) - \Phi(\underline{\tilde{u}})\|_{\mathbb{B}} \leq C_{T^*}\left(\|\underline{u}\|_{\mathbb{B}} + \|\underline{\tilde{u}}\|_{\mathbb{B}}\right)\|\underline{u} - \underline{\tilde{u}}\|_{\mathbb{B}}.$$

We take $R > 0$ to be defined later, and restrict Φ to $B_{\mathbb{B}}(0,R) := \{\underline{u} \in \mathbb{B} : \|\underline{u}\|_{\mathbb{B}} \leq R\}$, then, for all $\underline{u}$, $\underline{\tilde{u}} \in B_{\mathbb{B}}(0,R)$, we have

$$\begin{aligned} &\|\Phi(\underline{u})\|_{\mathbb{B}} \leq C_{T^*}(\epsilon + R^2),\\ &\|\Phi(\underline{u}) - \Phi(\underline{\tilde{u}})\|_{\mathbb{B}} \leq 2C_{T^*}R\|\underline{u} - \underline{\tilde{u}}\|_{\mathbb{B}}. \end{aligned}$$

Thus, if $R < \dfrac{1}{2C_{T^*}}$ and $\epsilon > 0$ such that, $C_{T^*}(\epsilon + R^2) < R$ we obtain the local well-posedness result applying the Banach fixed point Theorem. If $T \leq T^*$ then the solution up to T corresponds to the restriction to $[0,T]$ of the above solution. For $T > T^*$, we can extend our solution by using some energy estimates. In fact, let $j \in [\![1, N]\!]$, multiplying (KdV) by u_j and integrating on $[0, s] \times I_j$, after some integrations by parts and summing over $j \in [\![1, N]\!]$ we get

$$\begin{aligned}\sum_{j=1}^{N}\int_{I_j}(u_j(s,x))^2\mathrm{d}x + \sum_{j=1}^{N}\int_0^s(\partial_x u_j(t,0))^2\mathrm{d}t + (2\alpha - N)\int_0^s(u_1(t,0))^2\mathrm{d}t + 2\sum_{j=1}^{N}\int_0^s\int_{I_j}a_j(u_j)^2\mathrm{d}x\mathrm{d}t \\ = \sum_{j=1}^{N}\int_{I_j}(u_j(0,x))^2\mathrm{d}x.\end{aligned} \tag{2.19}$$

As in (2.8) we see that

$$\|\underline{u}\|^2_{C([0,T];\mathbb{L}^2(\mathcal{T}))} + (2\alpha - N)\|u_1(\cdot,0)\|^2_{L^2(0,T)} + \|\partial_x\underline{u}(\cdot,0)\|^2_{L^2(0,T)} + 2\sum_{j=1}^{N}\int_0^T\int_{I_j}a_j(u_j)^2\mathrm{d}x\mathrm{d}t \le \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}. \tag{2.20}$$

Similarly, multiplying the first line of (KdV) by xu_j, integrating on $[0, s] \times I_j$ after some integrations by parts we get

$$\begin{aligned}\frac{1}{2}\int_{I_j}x(u_j(s,x))^2\mathrm{d}x + \frac{3}{2}\int_0^s\int_{I_j}(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^s\int_{I_j}a_j x(u_j)^2\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_{I_j}x(u_j^0)^2\mathrm{d}x \\ + \frac{1}{2}\int_0^s\int_{I_j}(u_j)^2\mathrm{d}x\mathrm{d}t - \int_0^s u_1(t,0)\partial_x u_j(t,0)\mathrm{d}t + \frac{1}{3}\int_0^s\int_{I_j}u_j^3\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.21}$$

The problematic term to estimate is $\int_0^s\int_{I_j}u_j^3\mathrm{d}x\mathrm{d}t$. Note that

$$\int_0^s\int_{I_j}u_j^3\mathrm{d}x\mathrm{d}t \le \int_0^T\int_{I_j}|u_j|^3\mathrm{d}x\mathrm{d}t \le \int_0^T\|u_j\|_{L^\infty(I_j)}\|u_j\|^2_{L^2(I_j)}\mathrm{d}t \le C\|u_j\|^2_{C([0,T];L^2(I_j))}\int_0^T\|u_j\|_{H^1(I_j)}\mathrm{d}t,$$

where $C > 0$ is the constant of the Sobolev embedding of $H^1(I_j)$ in $L^\infty(I_j)$. Thus, by Cauchy–Schwarz inequality and (2.20)

$$\int_0^T\int_{I_j}u_j^3\mathrm{d}x\mathrm{d}t \le C\sqrt{T}\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}\|u_j\|_{L^2(0,T;H^1(I_j))}\mathrm{d}t. \tag{2.22}$$

Consider now, $j \in [\![1, N_F]\!]$, we get from (2.21)

$$\begin{aligned}\int_0^T\int_{I_j}(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t \le & C\Big(T\|u_j^0\|^2_{L^2(I_j)} + \|\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} + \|u_1(\cdot,0)\|^2_{L^2(0,T)} + \|\partial_x u_j(\cdot,0)\|^2_{L^2(0,T)} \\ & + \int_0^T\int_{I_j}u_j^3\mathrm{d}x\mathrm{d}t\Big).\end{aligned}$$

We deduce from (2.21), (2.20) and (2.22) that

$$\int_0^T\int_{I_j}(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t \le C(1+T)\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}\|\partial_x\underline{u}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}\right). \tag{2.23}$$

Therefore, by Young inequality, we conclude that $u_j \in B_j$ for $j \in [\![1, N_F]\!]$. Similarly, for $j \in]\!]N_F, N]\!]$, we deduce from (2.21)

$$\int_{I_j} x(u_j(s,x))^2 \mathrm{d}x + \int_0^s\int_{I_j} (\partial_x u_j)^2 \mathrm{d}x\mathrm{d}t \leq C(1+T)\left(\|u_j^0\|^2_{L^2_{(1+x)}(0,\infty)} + \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}\|\partial_x \underline{u}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}\right).$$

Therefore, taking the supremum for $s \in (0,T)$

$$\|\sqrt{x}u_j\|_{C([0,T];L^2(0,\infty))} + \|\partial_x u_j\|_{L^2(0,T;L^2(0,\infty))} \leq C(1+T)\left(\|u_j^0\|^2_{L^2_{(1+x)}(0,\infty)} + \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{u}^0\|^4_{\mathbb{L}^2}\right). \tag{2.24}$$

Now, multiplying by $x^2 u_j$ the first line of (KdV), integrating on $[0,s]\times(0,\infty)$ for $j \in]\!]N_F, N]\!]$, following (2.12) we get

$$\begin{aligned}\frac{1}{2}\int_0^\infty x^2(u_j(s,x))^2\mathrm{d}x + 3\int_0^s\int_0^\infty x(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^s\int_0^\infty a_j x^2(u_j)^2\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_0^\infty x^2(u_j^0)^2\mathrm{d}x \\ + \int_0^s\int_0^\infty x(u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^s (u_1(t,0))^2\mathrm{d}t + \frac{2}{3}\int_0^s\int_0^\infty x u_j^3\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{1}{2}\int_0^\infty x^2(u_j(s,x))^2\mathrm{d}x + 3\int_0^s\int_0^\infty x(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^s\int_0^\infty a_j x^2(u_j)^2\mathrm{d}x\mathrm{d}t \\ &\leq \frac{1}{2}\int_0^\infty x^2(u_j^0)^2\mathrm{d}x + \int_0^s\int_0^\infty x(u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^s (u_1(t,0))^2\mathrm{d}t + \frac{2}{3}\int_0^s\int_0^\infty x|u_j|^3\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.25}$$

Again, the problematic term to estimate is $\int_0^s\int_0^\infty x|u_j|^3\mathrm{d}x\mathrm{d}t$. Note that similar to the previous case, we can obtain

$$\int_0^s\int_0^\infty x|u_j|^3\mathrm{d}x\mathrm{d}t \leq \int_0^T\int_0^\infty x u_j^3\mathrm{d}x\mathrm{d}t \leq C\|xu_j\|^2_{C([0,T];L^2(0,\infty))}\|u_j\|_{L^2(0,T;H^1(0,\infty))}.$$

Now, for all $\varepsilon > 0$ using (2.13) and

$$\int_0^T\int_0^\infty x|u_j|^3\mathrm{d}x\mathrm{d}t \leq \frac{C\varepsilon}{2}\|xu_j\|^2_{C([0,T];L^2(0,\infty))} + \frac{C}{2\varepsilon}\|u_j\|_{L^2(0,T;H^1(0,\infty))}.$$

Similarly

$$\begin{aligned}\int_0^T\int_0^\infty x(u_j)^2\mathrm{d}x\mathrm{d}t &\leq \int_0^T\left(\int_0^\infty x^2(u_j)^2\mathrm{d}x\right)^{1/2}\left(\int_0^\infty (u_j)^2\mathrm{d}x\right)^{1/2}\mathrm{d}t \\ &\leq \frac{C\varepsilon}{2}\|xu_j\|^2_{C([0,T];L^2(0,\infty))} + \frac{C}{2\varepsilon}\|u_j\|_{C([0,T];L^2(0,\infty))}.\end{aligned}$$

Then, from (2.25) using the previous inequalities we get that for all $\varepsilon > 0$ and for all $s \in (0,T)$

$$\begin{aligned}
&\frac{1}{2}\|xu_j(s,\cdot)\|^2_{L^2(0,\infty)} + 3\int_0^s \|\sqrt{x}\partial_x u_j(t,\cdot)\|^2_{L^2(0,\infty)}\mathrm{d}t \\
&\leq \frac{1}{2}\|xu_j^0\|^2_{L^2(0,\infty)} + \int_0^T (u_1(t,0))^2\mathrm{d}t + \frac{C\varepsilon}{2}\|xu_j\|^2_{C([0,T];L^2(0,\infty))} + \frac{C}{2\varepsilon}\|u_j\|_{L^2(0,T;H^1(0,\infty))} \\
&\quad + \frac{C}{2\varepsilon}\|u_j\|_{C([0,T];L^2(0,\infty))}.
\end{aligned}$$

Taking the supremum for $s \in (0,T)$ and using and choosing ε small enough we deduce

$$\begin{aligned}
\|xu_j\|^2_{C([0,T];L^2(0,\infty))} + \|\sqrt{x}\partial_x u_j\|^2_{L^2(0,T;L^2(0,\infty))} \leq & C\left(\|xu_j^0\|^2_{L^2(0,\infty)} + \int_0^T (u_1(t,0))^2\mathrm{d}t\right. \\
&\left. + \|u_j\|^2_{C([0,T];L^2(0,\infty)} + \|u_j\|^2_{L^2(0,T;H^1(0,\infty)}\right).
\end{aligned}$$

By using (2.20), (2.23) and (2.24) we observe

$$\begin{aligned}
\|xu_j\|^2_{C([0,T];L^2(0,\infty))} + \|\sqrt{x}\partial_x u_j\|^2_{L^2(0,T;L^2(0,\infty))} \leq & C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \sum_{j=N_F+1}^{N}\|u_j^0\|^2_{L^2_{(1+x^2)}(0,\infty)}\right. \\
&\left. + \|\underline{u}^0\|^4_{\mathbb{L}^2(\mathcal{T})}\right).
\end{aligned} \tag{2.26}$$

Thus, we conclude that $u_j \in B_\infty$ for $j \in]\!] N_F, N]\!]$. □

3. Hidden regularity

As explained before Proposition 2.4, more regular initial conditions ($u_j^0 \in L^2_{(1+x^2)}(0,\infty)$, $j \in]\!] N_F, N]\!]$) were considered in order to demonstrate our well-posedness result for (KdV). The same assumption was used in [18], Theorem 2.2 and [27], Theorem 4 to obtain $L^2(0,T;H^1(0,\infty))$ and B_∞ regularity, respectively. Nevertheless, we can still prove a similar regularity result, depending only on the $\mathbb{L}^2(\mathcal{T})$ norm of the initial data.

Definition 3.1. A function $\gamma : [0,\infty) \to (0,\infty)$ is said to be a class $\mathcal{K}_\infty$ function if γ is continuous, nonnegative, increasing, vanishing at 0, and such that $\lim_{x\to\infty}\gamma(x) = \infty$.

Proposition 3.2. *Let $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ (resp. $\underline{u}^0 \in \mathbb{Y}$ with $\|\underline{u}^0\|_{\mathbb{Y}} \leq \varepsilon$, for $\varepsilon > 0$ small enough). Consider $\underline{u}$ the associate mild solution of (LKdV) (resp. (KdV)). Then, $\underline{u} \in \mathbb{X}^1$, moreover, the following estimates hold*

- *There exists $C > 0$ such that for all $j \in [\![1, N_F]\!]$:*

$$\int_0^T\int_{I_j} (\partial_x u_j)^2\mathrm{d}x\mathrm{d}t \leq \begin{cases} C\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}, & \text{for (LKdV)}, \\ C(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{u}^0\|^4_{L^2(\mathcal{T})}), & \text{for (KdV)}. \end{cases} \tag{3.1}$$

- *For any $x_0 > 0$, there exist γ, a function of class $\mathcal{K}_\infty$ and $C_{x_0} > 0$ such that for all $j \in]\!] N_F, N]\!]$:*

$$\int_0^T\int_{x_0}^{x_0+1} (\partial_x u_j)^2\mathrm{d}x\mathrm{d}t \leq \begin{cases} C_{x_0}\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}, & \text{for (LKdV)}, \\ C_{x_0}\gamma(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}), & \text{for (KdV)}. \end{cases} \tag{3.2}$$

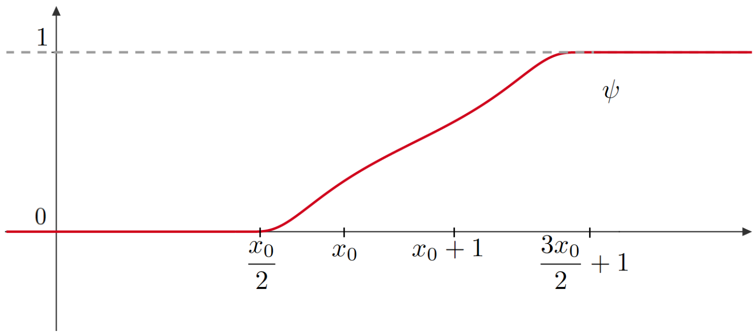

FIGURE 3. Graph of the function $\psi(x)$.

Proof. Let $\lambda = 0$ or $\lambda = 1$ and consider the system

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \lambda u_j \partial_x u_j + \partial_x^3 u_j + a_j u_j)(t,x) = 0, & \forall x \in I_j,\ t>0,\ j \in [\![1,N]\!], \\ u_j(t,0) = u_{j'}(t,0), & \forall j,j' \in [\![1,N]\!], \\ \displaystyle\sum_{j=1}^N \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \lambda \frac{N}{3}(u_1(t,0))^2, & t>0, \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t>0,\ j \in [\![1,N_F]\!], \\ u_j(0,x) = u_j^0(x), & x \in I_j, j \in [\![1,N]\!], \end{cases} \tag{3.3}$$

which represents (LKdV) ($\lambda = 0$) or (KdV) ($\lambda = 1$).

First, (3.1) can be easily deduced from (2.11) and (2.23). We focus now in the case, $j \in]\!]N_F, N]\!]$. Inspired by [25], Theorem 2.1, let $x_0 > 0$ and $K_{1,x_0}, K_{2,x_0} > 0$ depending on x_0. Consider $\psi \in C^\infty(\mathbb{R})$ an increasing function satisfying the following properties (see Fig. 3)

$$\begin{cases} \psi(x) = 0, & \text{for } x \le \frac{x_0}{2}, \\ \psi(x) = 1, & \text{for } x \ge \frac{3x_0}{2} + 1, \\ \psi'(x) \ge K_{1,x_0}, & \text{for } x \in [x_0, x_0+1], \\ \psi'(x) \ge 0, & \text{for } x \in \mathbb{R}, \\ |\psi^{(k)}(x)| \le K_{2,x_0}, & \text{for } x \in \mathbb{R}, k = 0,1,2,3, \\ \sqrt{\psi'} \in H^1\left(\frac{x_0}{2}, \frac{3x_0}{2}+1\right). \end{cases} \tag{3.4}$$

Multiplying the $j-th$ equation of (3.3) by $u_j(t,x)\psi(x)$, and integrating over $(0,\infty)$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (u_j)^2\psi(x)\mathrm{d}x + \frac{1}{2}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}(u_j)^2\psi(x)\mathrm{d}x + \int_0^\infty \partial_x^3 u_j u_j \psi(x)\mathrm{d}x + \frac{\lambda}{3}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}u_j^3\psi(x)\mathrm{d}x \\ + \int_0^\infty a_j (u_j)^2 \psi(x)\mathrm{d}x = 0,$$

then, after some integrations by parts get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (u_j)^2\psi(x)\mathrm{d}x + \frac{3}{2}\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x + \int_0^\infty a_j(u_j)^2\psi(x)\mathrm{d}x = \frac{\lambda}{3}\int_0^\infty u_j^3\psi'(x)\mathrm{d}x + \frac{1}{2}\int_0^\infty (u_j)^2(\psi'(x)+\psi'''(x))\mathrm{d}x. \tag{3.5}$$

If $\lambda = 0$ (LKdV), recalling the definition of ψ (3.4), we observe that

$$K_{1,x_0}\int_{x_0}^{x_0+1}(\partial_x u_j)^2\mathrm{d}x \le \int_{x_0}^{x_0+1}(\partial_x u_j)^2\psi'(x)\mathrm{d}x \le \int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x. \tag{3.6}$$

Thus using, (3.6) and (3.4) in (3.5) we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty |u_j|^2\psi(x)\mathrm{d}x + \frac{3}{2}K_{1,x_0}\int_{x_0}^{x_0+1}|\partial_x u_j|^2\mathrm{d}x \le K_{2,x_0}\int_{x_0}^{x_0+1}|u_j|^2\mathrm{d}x,$$

We conclude the proof of (3.2) in the case $\lambda = 0$ integrating t between $[0,T]$ and using (2.20).

Now we focus on the case $\lambda = 1$ (KdV), again the tricky term is $\int_0^\infty u_j^3\psi'(x)\mathrm{d}x$. This nonlinear term can be estimated in the following manner:

Lemma 3.3. *Let $\underline{u}$ the unique mild solution of (KdV), then for $j \in]\!]N_F, N]\!]$ the following estimate holds:*

$$\frac{1}{3}\int_0^\infty u_j^3\psi'(x)\mathrm{d}x \le \frac{1}{6}\left(\int_0^\infty (u_j)^2|\psi''(x)|\mathrm{d}x\right)^{1/2}\int_0^\infty (u_j)^2\sqrt{\psi'(x)}\mathrm{d}x + \frac{1}{3\sqrt{2}}\left(\int_0^\infty (u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4} \times\left(\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4}\int_0^\infty (u_j)^2\sqrt{\psi'(x)}\mathrm{d}x.$$

Proof. The proof of Lemma 3.3 is given in Appendix B. □

Using this Lemma in (3.5) we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (u_j)^2\psi(x)\mathrm{d}x + \frac{3}{2}\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x \le \frac{1}{6}\left(\int_0^\infty (u_j)^2|\psi''(x)|\mathrm{d}x\right)^{1/2}\int_0^\infty (u_j)^2\sqrt{\psi'(x)}\mathrm{d}x + \frac{1}{3\sqrt{2}}\left(\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4}\left(\int_0^\infty (u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4}\int_0^\infty (u_j)^2\sqrt{\psi'(x)}\mathrm{d}x + \frac{1}{2}\int_0^\infty (u_j)^2(\psi'(x)+\psi'''(x))\mathrm{d}x.$$

Now by (3.4) we get for some $M_{x_0} > 0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (u_j)^2\psi(x)\mathrm{d}x + \frac{3}{2}\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x \le M_{x_0}\left(\left(\int_0^\infty (u_j)^2\mathrm{d}x\right)^{1/2}\int_0^\infty (u_j)^2\mathrm{d}x + \left(\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4}\left(\int_0^\infty (u_j)^2\mathrm{d}x\right)^{1/4}\int_0^\infty (u_j)^2\mathrm{d}x + \int_0^\infty (u_j)^2\mathrm{d}x\right). \tag{3.7}$$

Now, by (2.20) in (3.7), there exists $\gamma_1(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}) = \max\left(\|\underline{u}^0\|^3_{\mathbb{L}^2(\mathcal{T})}, \|\underline{u}^0\|^{5/2}_{\mathbb{L}^2(\mathcal{T})}, \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}\right) > 0$, such that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (u_j)^2\psi(x)\mathrm{d}x + \frac{3}{2}\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x \leq & 2M_{x_0}\gamma_1(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}) \\ & + M_{x_0}\gamma_1(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})})\left(\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4}\end{aligned}$$

Using $ab^{1/4} \leq 3\cdot 4^{-4/3}a^{4/3} + b$, with $a = M_{x_0}\gamma_1(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})})$ and $b = \displaystyle\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x$ we derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (u_j)^2\psi(x)\mathrm{d}x + \int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x &\leq 2\left(2M_{x_0}\gamma_1(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}) + 3\cdot 4^{-4/3}M_{x_0}^{4/3}\gamma_1(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})})^{4/3}\right) \\ &\leq C_{x_0}\gamma(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}),\end{aligned}$$

where $C_{x_0} = 2\max\left(2M_{x_0}, 3\cdot 4^{-4/3}M_{x_0}^{4/3}\right)$ and $\gamma(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}) = \gamma_1(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}) + \gamma_1(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})})^{4/3}$. We conclude, as in the case $\lambda = 0$, using (3.6) and integrating t between $[0, T]$ and obtaining (3.2). ☐

Several remarks are in order,

Remark 3.4. Note that the smallness condition on the initial data $\|\underline{u}^0\|_{\mathbb{Y}} \leq \varepsilon$ is just used to ensure the existence of solutions. In particular, if we are able to prove the existence of solutions without this assumption, Proposition 3.2 is still valid.

Remark 3.5. An important fact about the last proposition is that in the right-hand-side of estimates (3.1) and (3.2) we only have the $\mathbb{L}^2(\mathcal{T})$ norm of the initial data. As we see in the proof of the well-posedness of (KdV) the introduction of the weighted spaces is necessary in our proof due to the perturbation approach. In our best knowledge, a well-posedness result for (KdV) with initial data in $\mathbb{L}^2(\mathcal{T})$ is an open problem.

Remark 3.6. We can build a function ψ satisfying (3.4) in the following way: consider the bump function $\kappa \in C^\infty(\mathbb{R})$ defined by $\kappa(x) = \mathrm{e}^{-\frac{1}{x}}$, for $x > 0$ and $\kappa(x) = 0$ for $x \leq 0$. Then, we can take

$$\psi(x) = \frac{\kappa(x - \frac{x_0}{2})}{\kappa(x - \frac{x_0}{2}) + \kappa(\frac{3x_0}{2} + 1 - x)},$$

it is not difficult to check that the above function satisfies all the hypotheses of (3.4).

4. Exponential stability in $\mathbb{L}^2(\mathcal{T})$

In this section, we prove our results related with the exponential stability in $\mathbb{L}^2(\mathcal{T})$. First, we study (LKdV), in this case we are able to prove the stability without acting in all the branches of the network. Then, using a perturbation argument, we obtain a stability result for (KdV) but for small and more regular initial data. Finally, we present a semiglobal stability result for (KdV) but we need to act in all the branches.

First, note that to prove the exponential stability, it is enough to prove the following observability inequality, with E defined in (1.2)

$$E(0) \leq C_{obs}\int_0^T \left((u_1(t,0))^2 + \sum_{j=1}^N (\partial_x u_j(t,0))^2 + \sum_{j=1}^N \int_{I_j} a_j(u_j)^2\mathrm{d}x\right)\mathrm{d}t. \tag{Obs1}$$

Indeed, using (Obs1) and dissipation law (1.3) we can show $E(t) \leq \gamma E(0)$ for $0 < \gamma < 1$, finally as (KdV) (or (LKdV)) is invariant by translation in time, we derive the exponential decay. This idea was used in several works as [11, 12, 18, 28].
We recall the set $\mathcal{N}$ of critical lengths for the KdV equation introduced by Rosier in [26] defined by

$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, k, l \in \mathbb{N}^* \right\},$$

and we define $I_c = \{j \in [\![1, N_F]\!];\ \ell_j \in \mathcal{N}\}$ the set of critical index, $I_\infty =]\!]N_F, N]\!]$ and I_c^* (resp I_∞^*) be the subset of I_c (resp I_∞) where we remove one index.

4.1. Linear case

In this part, we will prove the first stabilization result.

Theorem 4.1. *Let $\mathbb{I}_{act} \supseteq I_c^* \cup I_\infty$ or $\mathbb{I}_{act} \supseteq I_c \cup I_\infty^*$, assume that the damping terms $(a_j)_{j\in[\![1,N]\!]}$ satisfy (1.1). Then, there exist C, $\mu > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, the energy the unique solution of (LKdV) satisfies $E(t) \leq CE(0)\mathrm{e}^{-\mu t}$ for all $t > 0$.*

Proof. As we said at the beginning of the section, it is enough to prove (Obs1). To prove it, we follow a contradiction argument as in [26]. Suppose that (Obs1) is false, then there exists $(\underline{u}^{0,n})_{n\in\mathbb{N}} \subset \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^{0,n}\|_{\mathbb{L}^2(\mathcal{T})} = 1$ and such that

$$\begin{aligned}
&\|u_1^n(t,0)\|_{L^2(0,T)} \to 0,\\
&\|\partial_x \underline{u}^n(t,0)\|_{L^2(0,T)} \to 0,\\
&\sum_{j=1}^N \int_0^T\int_{I_j} a_j (u_j^n)^2 \mathrm{d}x\mathrm{d}t \to 0, \text{ as } n \to \infty,
\end{aligned}$$

where $\underline{u}^n$, is the associated solution of (LKdV) with initial data $\underline{u}^{0,n}$ given by Theorem 2.1.

By Proposition 3.2 we get that $(\underline{u}^n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}^1$, as $\partial_t u_j^n = -\partial_x u_j^n - \partial_x^3 u_j^n - a_j u_j^n$, we get that $(\partial_t u_j^n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}^{-2}$. Using [29], Corollary 4 we can extract a subsequence denoted by $(\underline{u}^n)_{n\in\mathbb{N}}$ which is convergent in $\mathbb{X}^0$, by (2.3), we get that $(\underline{u}^{0,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}^2(\mathcal{T})$. Let $\underline{u}^0 = \lim_{n\to\infty} \underline{u}^{0,n}$ and $\underline{u}$ the unique mild solution of (LKdV) associated to $\underline{u}^0$. Then, we have that $\underline{u}$ solves the following problem

$$\begin{cases}
\partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in I_j,\ t \in (0,T),\ j \in [\![1,N]\!],\\
u_j(t,0) = \partial_x u_j(t,0) = 0, & \forall j \in [\![1,N]\!],\\
\sum_{j=1}^N \partial_x^2 u_j(t,0) = 0, & t \in (0,T),\\
u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0,\ j \in [\![1,N_F]\!],\\
u_j \equiv 0 & \text{in } (0,T)\times\omega_j,\ j \in \mathbb{I}_{act},\\
u_j(0,x) = u_j^0(x), & x \in I_j, j \in [\![1,N]\!],\\
\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1.
\end{cases} \tag{4.1}$$

Here we have two cases:

- $\mathbb{I}_{act} \supseteq I_c^* \cup I_\infty$. In this case, for $j \in I_\infty$, $w = u_j$ solves

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w = 0, & \forall x \in (0,\infty),\ t \in (0,T), \\ w(t,0) = \partial_x w(t,0) = 0, & t \in (0,T), \\ w \equiv 0 & \text{in } (0,T) \times \omega_j. \end{cases}$$

Then, by Holmgren's Theorem (see also [18], Thm. 1.1), $w \equiv 0$ in $(0,\infty) \times (0,T)$. Therefore, we have the following problem

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in (0,\ell_j),\ t \in (0,T),\ j \in [\![1, N_F]\!], \\ u_j(t,0) = \partial_x u_j(t,0) = 0, & \forall j \in [\![1, N_F]\!], \\ \displaystyle\sum_{j=1}^{N_F} \partial_x^2 u_j(t,0) = 0, & t \in (0,T), \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0,\ j \in [\![1, N_F]\!], \\ u_j \equiv 0 & \text{in } (0,T) \times \omega_j,\ j \in I_c^*, \\ \left(\displaystyle\sum_{j=1}^{N_F} \|u_j^0\|^2_{L^2(0,\ell_j)} \right)^{1/2} = 1. \end{cases}$$

The above system is exactly the same studied in [8]. Thus, by [8], Theorem 3.1 as we are acting in I_c^* we get $u_j \equiv 0$ for $j \in [\![1, N_F]\!]$ and finally, $\underline{u} \equiv 0$ which is a contradiction with the fact $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1$.

- $\mathbb{I}_{act} \supseteq I_c \cup I_\infty^*$. Let $j \in I_\infty^*$, the same argument used in the previous case shows that $u_j \equiv 0$ on $(0,\infty) \times (0,T)$. Similarly, by Holmgren's theorem for all $j \in I_c$ $u_j \equiv 0$ in $(0,\ell_j) \times (0,T)$. Now, for $j \in [\![1, N_F]\!] \cap (I_c)^c$, u_j solves

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in (0,\ell_j),\ t \in (0,T), \\ u_j(t,0) = \partial_x u_j(t,0) = 0, & t > 0, \\ u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0, & t > 0. \end{cases}$$

Then, as $\ell_j \notin \mathcal{N}$ by [26], Lemma 3.5, $u_j \equiv 0$ in $(0,\ell_j) \times (0,T)$. Finally, let $j \in]\!] N_F, N]\!] \cap (I_\infty^*)^c$, then $w = u_j$ solves

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w = 0, & \forall x \in (0,\infty),\ t \in (0,T), \\ w(t,0) = \partial_x w(t,0) = \partial_x^2 w(t,0) = 0, & t \in (0,T). \end{cases} \tag{4.2}$$

It is enough to see that, due to the three null boundary conditions in 0[2], the unique solution is $w \equiv 0$.

□

Theorem 4.1 is optimal in the sense of acted branches. For instance, if we take a smaller set of acted index, we can not derive the result as shows the next proposition.

Proposition 4.2. *Let $\mathbb{I}_{act} \subsetneq I_c^* \cup I_\infty$ or $\mathbb{I}_{act} \subsetneq I_c \cup I_\infty^*$, assume that the damping terms $(a_j)_{j \in [\![1,N]\!]}$ satisfy (1.1). Then, there exists a nontrivial solution of (4.1).*

[2]See for instance [17] where an implicit controllability result is obtained by imposing three boundary conditions at 0.

Proof. It is enough to consider the case where we remove one index more. Let I_c^{**} (resp I_∞^{**}) be the subset of I_c (resp I_∞) where we remove two indexes. Let us show that, under these conditions, there exists nontrivial solutions of (4.1).

- $\underline{\mathbb{I}_{act} = I_c^{**} \cup I_\infty}$. We get for all $j \in I_\infty$ $u_j \equiv 0$. Thus, we get the optimality from [8], Lemma 3.2.
- $\underline{\mathbb{I}_{act} = I_c^{*} \cup I_\infty^{*}}$. Consider $N_F = N_\infty = 1$, $\ell_1 = 2\pi$. Following the computations of the proof of Theorem 4.1 we obtain the system.

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in I_j,\ t \in (0,T),\ j = 1,2 \\ u_1(t,0) = \partial_x u_1(t,0) = 0, & t > 0, \\ u_2(t,0) = \partial_x u_2(t,0) = 0, & t > 0, \\ u_1(t,2\pi) = \partial_x u_1(t,2\pi) = 0, & t > 0, \\ \partial_x^2 u_1(t,0) + \partial_x^2 u_2(t,0) = 0, & t \in (0,T), \\ u_j(0,x) = u_j^0(x), & x \in I_j, j = 1,2, \\ \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1. \end{cases}$$

Consider the stationary functions $u_1 = K_1(1-\cos(x))$ and $u_2 = K_2(1-\cos(x))$ for $x \in (0,2\pi)$ and $u_2 = 0$ for $x \geq 2\pi$. We observe that u_1 and u_2 satisfies a linear KdV equation. Moreover, $u_1(0) = \partial_x u_1(0) = u_1(2\pi) = \partial_x u_1(2\pi) = 0$, $u_2(0) = \partial_x u_2(0) = 0$ and $\partial_x^2 u_1(0) + \partial_x^2 u_1(0) = K_1 + K_2$. Therefore, if $K_1 = -K_2$ we found nontrivial solutions of (4.1) and $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1$.

- $\underline{\mathbb{I}_{act} = I_c \cup I_\infty^{**}}$. Consider the case $N_F = 0$. Then we obtain the system

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in (0,\infty),\ t \in (0,T),\ j = 1,2 \\ u_1(t,0) = \partial_x u_1(t,0) = 0, & t > 0, \\ u_2(t,0) = \partial_x u_2(t,0) = 0, & t > 0, \\ \partial_x^2 u_1(t,0) + \partial_x^2 u_2(t,0) = 0, & t \in (0,T), \\ u_j(0,x) = u_j^0(x), & x \in (0,\infty), \\ \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1. \end{cases}$$

Similarly to the past case, we can consider, $u_1 = K_1(1-\cos(x))$ for $x \in (0,2\pi)$ and $u_1 = 0$ for $x \geq 2\pi$, $u_2 = K_2(1-\cos(x))$ for $x \in (0,2\pi)$ and $u_2 = 0$ for $x \geq 2\pi$, such that $K_1 = -K_2$ and $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} = 1$. This concludes the proof. □

4.2. Nonlinear case: small amplitude solutions

In this subsection, we will prove a stabilization result for (KdV) by imposing small amplitude solutions.

Theorem 4.3. *Let $\mathbb{I}_{act} \supseteq I_c^* \cup I_\infty$ or $\mathbb{I}_{act} \supseteq I_c \cup I_\infty^*$, assume that the damping terms $(a_j)_{j \in [\![1,N]\!]}$ satisfy (1.1). Then there exist C, $\tilde{C}$, μ, $\epsilon > 0$ such that for all $\underline{u}^0 \in \mathbb{Y}$, with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \epsilon$ and $\sum_{j=N_F+1}^{N} \|u_j^0\|^2_{L^2_{(1+x^2)}(0,\infty)} \leq \tilde{C}\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}$, the energy of any solution of (KdV) satisfies $E(t) \leq CE(0)\mathrm{e}^{-\mu t}$ for all $t > 0$.*

Proof. Following the same idea as the linear case, it is enough to show that $\|\underline{u}\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))} \leq \gamma \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$ for $0 < \gamma < 1$ and $\underline{u}$ solution of (KdV). Let $\underline{u} = \underline{u}^1 + \underline{u}^2$ where $\underline{u}^1$ and $\underline{u}^2$ are respectively solutions of

$$\begin{cases} (\partial_t u_j^1 + \partial_x u_j^1 + \partial_x^3 u_j^1 + a_j u_j^1)(t,x) = 0, & \forall x \in I_j,\ t>0,\ j \in [\![1,N]\!], \\ u_j^1(t,0) = u_{j'}^1(t,0), & \forall j,j' \in [\![1,N]\!], \\ \displaystyle\sum_{j=1}^N \partial_x^2 u_j^1(t,0) = -\alpha u_1^1(t,0), & t>0, \\ u_j^1(t,\ell_j) = \partial_x u_j^1(t,\ell_j) = 0, & t>0,\ j \in [\![1,N_F]\!], \\ u_j^1(0,x) = u_j^0(x), & x \in I_j, j \in [\![1,N]\!], \end{cases} \tag{4.3}$$

and

$$\begin{cases} (\partial_t u_j^2 + \partial_x u_j^2 + \partial_x^3 u_j^2 + a_j u_j^2)(t,x) = -u_j \partial_x u_j, & \forall x \in I_j,\ t>0,\ j \in [\![1,N]\!], \\ u_j^2(t,0) = u_{j'}^2(t,0), & \forall j,j' \in [\![1,N]\!], \\ \displaystyle\sum_{j=1}^N \partial_x^2 u_j^2(t,0) = -\alpha u_1^2(t,0) - \frac{N}{3}(u_1(t,0))^2, & t>0, \\ u_j^2(t,\ell_j) = \partial_x u_j^2(t,\ell_j) = 0, & t>0,\ j \in [\![1,N_F]\!], \\ u_j^2(0,x) = 0, & x \in I_j, j \in [\![1,N]\!]. \end{cases} \tag{4.4}$$

Then, using Theorem 4.1 for $\underline{u}^1$, Proposition 2.2 and (2.8) for $\underline{u}^2$ we get

$$\begin{aligned} \|\underline{u}\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))} &\leq \|\underline{u}^1\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))} + \|\underline{u}^2\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))} \\ &\leq \gamma \|\underline{u}^0\|_{\mathbb{L}(\mathcal{T})} + C\left(\|\underline{u}\partial_x \underline{u}\|_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|u_1(\cdot,0)^2\|_{L^2(0,T)} \right), \end{aligned} \tag{4.5}$$

where $\gamma < 1$. By Lemma 2.6, Lemma 2.7 and Lemma 2.8 we get $\|\underline{u}\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))} \leq \gamma \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} + C\|\underline{u}\|_{\mathbb{B}}^2$. In the sequel, a more explicit estimate of $\underline{u}$ solution of (KdV) in the $\mathbb{B}$ norm is needed. We deduce from (2.23) and (2.24)

$$\|\partial_x \underline{u}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}^2 \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \sum_{j=N_F+1}^N \|u_j^0\|_{L^2_{(1+x^2)}(0,\infty)}^2 + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^4 \right). \tag{4.6}$$

Thus, by (2.20) and (4.6)

$$\|\underline{u}\|_{\mathbb{B}}^2 \leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \sum_{j=N_F+1}^N \|u_j^0\|_{L^2_{(1+x^2)}(0,\infty)}^2 + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^4 \right). \tag{4.7}$$

Using this estimation and the smallness assumption on the initial data and (4.5), we get $\|\underline{u}\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))} \leq (\gamma + C(1+\tilde{C})\epsilon + C\epsilon^3)\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$. We conclude by choosing $\epsilon > 0$ small enough such that $\tilde{\gamma} = \gamma + C(1+\tilde{C})\epsilon + C\epsilon^3 < 1$, which is possible because $\gamma < 1$. □

Theorem 4.3 is interesting in the sense that we derive an exponential stability result for the nonlinear system without all the damping actives (see Fig. 4).

In the sequel results, we need to all the damping terms to be active, but as will be shown in Section 5 we are able to show the exponential stability in the same spaces as our well-posedness result.

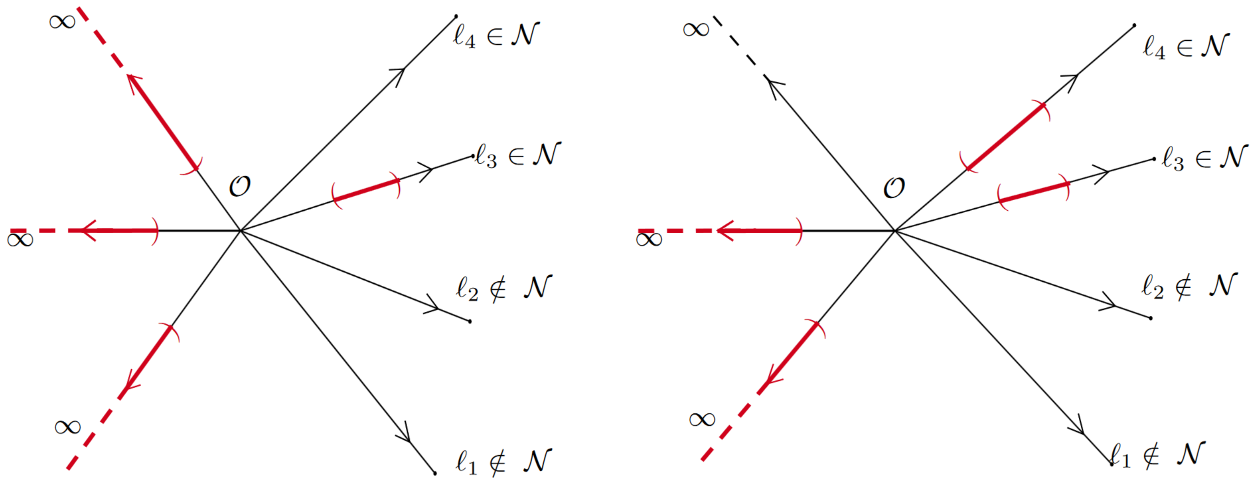

FIGURE 4. Possible acted branches of Theorem 4.1 and Theorem 4.3.

4.3. Nonlinear case: semiglobal result

In this subsection we prove an exponential stability result in $\mathbb{L}^2(\mathcal{T})$ working directly with the nonlinear system but acting in all branches.

Theorem 4.4. *Assume that the damping terms $(a_j)_{j\in[\![1,N]\!]}$ satisfy (1.1) and let $R > 0$. If $\mathbb{I}_{act} = [\![1, N]\!]$, then, there exist $C(R) > 0$ and $\mu(R) > 0$ such that for all $\underline{u}^0 \in \mathbb{Y}$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, the energy of any solution of (KdV) satisfies $E(t) \leq CE(0)\mathrm{e}^{-\mu t}$ for all $t > 0$.*

Proof. Our idea will be to prove the observability inequality (Obs1) directly for (KdV). Similar to the linear case, multiplying (KdV) by $(T - t)u_j$ and integrating on $[0, T] \times I_j$, after some integrations by parts we get

$$
\begin{aligned}
T\sum_{j=1}^{N}\int_{I_j}(u_j(0,x))^2\mathrm{d}x =& \sum_{j=1}^{N}\int_0^T\int_{I_j}(u_j)^2\mathrm{d}x\mathrm{d}t + (2\alpha - N)\int_0^T (T-t)(u_1(t,0))^2\mathrm{d}t \\
&+ \sum_{j=1}^{N}\int_0^T (T-t)(\partial_x u_j(t,0))^2\mathrm{d}t + 2\sum_{j=1}^{N}\int_0^T\int_{I_j}(T-t)a_j(u_j)^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \leq& \frac{1}{T}\|\underline{u}\|^2_{L^2([0,T];\mathbb{L}^2(\mathcal{T}))} + (2\alpha - N)\|u_1(\cdot,0)\|^2_{L^2(0,T)} + \|\partial_x\underline{u}(\cdot,0)\|^2_{L^2(0,T)} \\
&+ 2\sum_{j=1}^{N}\int_0^T\int_{I_j} a_j(u_j)^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.8}
$$

Thus, in order to show the observability inequality (Obs1), we show

$$
\|\underline{u}\|^2_{L^2([0,T];\mathbb{L}^2(\mathcal{T}))} \leq C_{obs2}\int_0^T\left((u_1(t,0))^2 + \sum_{j=1}^{N}(\partial_x u_j(t,0))^2 + \sum_{j=1}^{N}\int_{I_j} a_j(u_j)^2\mathrm{d}x\right)\mathrm{d}t. \tag{Obs2}
$$

To prove it, we follow a contradiction argument as in [28]. Suppose that (Obs2) is false, then there exists $(\underline{u}^n)_{n\in\mathbb{N}} \subset C([0,T];\mathbb{L}^2(\mathcal{T}))$ such that $\underline{u}^n$ is solution of (KdV) and

$$\lim_{n\to\infty} \frac{\|\underline{u}^n\|^2_{L^2([0,T];\mathbb{L}^2(\mathcal{T}))}}{\displaystyle\int_0^T \left((u_1^n(t,0))^2 + \sum_{j=1}^N (\partial_x u_j^n(t,0))^2 + \sum_{j=1}^N \int_{I_j} a_j (u_j^n)^2 \mathrm{d}x \right) \mathrm{d}t} = \infty. \tag{4.9}$$

Let $\lambda_n = \|\underline{u}^n\|_{L^2([0,T];\mathbb{L}^2(\mathcal{T}))}$ and $\underline{v}^n = \frac{\underline{u}^n}{\lambda_n}$, then $\underline{v}^n$ satisfies

$$\begin{cases} (\partial_t v_j^n + \partial_x v_j^n + \lambda_n v_j^n \partial_x v_j^n + \partial_x^3 v_j^n + a_j v_j^n)(t,x) = 0, & \forall x \in I_j,\ t>0,\ j \in [\![1,N]\!], \\ v_j^n(t,0) = v_{j'}^n(t,0), & \forall j,j' \in [\![1,N]\!], \\ \displaystyle\sum_{j=1}^N \partial_x^2 v_j^n(t,0) = -\alpha v_1^n(t,0) - \lambda_n \frac{N}{3}(v_1^n(t,0))^2 & t>0, \\ v_j^n(t,\ell_j) = \partial_x v_j^n(t,\ell_j) = 0, & t>0,\ j \in [\![1,N_F]\!], \end{cases} \tag{4.10}$$

and

$$\begin{aligned} &\|\underline{v}^n\|_{L^2([0,T];\mathbb{L}^2(\mathcal{T}))} = 1, \\ &\int_0^T \left((v_1^n(t,0))^2 + \sum_{j=1}^N (\partial_x v_j^n(t,0))^2 + \sum_{j=1}^N \int_{I_j} a_j (v_j^n)^2 \mathrm{d}x \right) \mathrm{d}t \longrightarrow 0. \end{aligned} \tag{4.11}$$

Using (4.8), (4.9) and (4.11), we get that $(\underline{v}(0,\cdot))_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^2(\mathcal{T})$ and then by Proposition 3.2 (and Rem. 3.4) we get $(\underline{v}^n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}^1$. Note that

$$\begin{aligned} &\|v_j^n \partial_x v_j^n\|_{L^1(0,T;L^2(I_j))} \le C \|v_j^n\|_{L^\infty(0,T;L^2(I_j))} \|v_j^n\|_{L^2(0,T;H^1(I_j))}, \quad j \in [\![1,N_F]\!], \\ &\|v_j^n \partial_x v_j^n\|_{L^1(0,T;L^2_{loc}(0,\infty))} \le C \|v_j^n\|_{L^\infty(0,T;L^2_{loc}(0,\infty))} \|v_j^n\|_{L^2(0,T;H^1_{loc}(0,\infty))}, \quad j \in]\!]N_F,N]\!], \end{aligned}$$

which implies that $(v_j^n \partial_x v_j^n)_{n\in\mathbb{N}}$ is a subset of $L^1(0,T;L^2(I_j))$ (resp. $L^1(0,T;L^2_{loc}(0,\infty))$) for $j \in [\![1,N_F]\!]$ (resp. $j \in]\!]N_F,N]\!]$). Moreover, as $\|\underline{u}^n(0,\cdot)\|_{\mathbb{L}^2(\mathcal{T})} \le R$ by (2.20) we deduce $\lambda_n \le R$. Then, similarly to the linear case, observing that $\partial_t v_j^n = -\partial_x v_j^n - \partial_x^3 v_j^n - \lambda_n v_j^n \partial_x v_j^n - a_j v_j^n$, we get that $(\partial_t v_j^n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}^{-2}$, using [29], Corollary 4 we can extract a subsequence denoted by $(\underline{v}^n)_{n\in\mathbb{N}}$ which is convergent in $\mathbb{X}^0$ to $\underline{v}$. By the structure of ω_j in (1.1) and following [18] we get

$$\|\underline{v}\|^2_{L^2([0,T];\mathbb{L}^2(\mathcal{T}))} = \sum_{j=1}^N \left(\int_0^T \int_{\overline{\omega_j}} (v_j)^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\overline{\omega_j}^c} (v_j)^2 \mathrm{d}x\mathrm{d}t \right) = 1.$$

Thus, for all $K_j \subset (0,\infty)$ for $j \in]\!] N_F, N]\!]$ we get

$$\begin{aligned}
&\|\partial_x \underline{v}(t,0)\|^2_{L^2(0,T)} + \|v_1(t,0)\|^2_{L^2(0,T)} + \sum_{j=1}^{N_F}\int_0^T\!\!\int_{\omega_j} a_j(v_j)^2 \mathrm{d}x\mathrm{d}t + \sum_{j=N_F+1}^{N}\int_0^T\!\!\int_{K_j} a_j(v_j)^2 \mathrm{d}x\mathrm{d}t \\
&\leq \liminf \Bigg(\|\partial_x \underline{v}^n(t,0)\|^2_{L^2(0,T)} + \|v_1^n(t,0)\|^2_{L^2(0,T)} + \sum_{j=1}^{N_F}\int_0^T\!\!\int_{\omega_j} a_j(v_j^n)^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \sum_{j=N_F+1}^{N}\int_0^T\!\!\int_{K_j} a_j(v_j^n)^2 \mathrm{d}x\mathrm{d}t \Bigg) = 0.
\end{aligned}$$

In addition, as $(\lambda_n)_{n\in\mathbb{N}}$ is bounded, we can extract a convergent subsequence, such that $\lambda_n \to \lambda \geq 0$. We have now two situations. If $\lambda = 0$, the system solved by $\underline{v}$ is linear. Therefore, as we are acting in all the branches by Holmgrem's Theorem $\underline{v} \equiv 0$, that contradicts the fact that $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$. In the case $\lambda > 0$ we have for $j \in [\![1, N_F]\!]$ that v_j solves

$$\begin{cases}
\partial_t v_j + \partial_x v_j + \lambda v_j \partial_x v_j + \partial_x^3 v_j = 0, & \forall x \in (0,\ell_j),\ t \in (0,T), \\
v_j(t,0) = \partial_x v_j(t,0) = 0, & \\
v_j(t,\ell_j) = \partial_x v_j(t,\ell_j) = 0, & t > 0, \\
v_j \equiv 0, & \text{in } (0,T)\times\omega_j.
\end{cases}$$

Then, the idea is to apply the following unique continuation property.

Lemma 4.5. *[[30], Lemma 3.5] Let $L > 0$ and $T > 0$ be two real numbers, and let $\omega \subset (0,L)$ be a nonempty open set. If $v \in L^\infty(0,T;H^1(0,L))$ solves*

$$\begin{cases}
\partial_t v + \partial_x v + a(v)\partial_x v + \partial_x^3 v_j = 0, & \forall x \in (0,\ell_j),\ t \in (0,T), \\
v(t,0) = v(t,L) = 0, & t > 0, \\
v \equiv 0, & \textit{in } (0,T)\times\omega_j
\end{cases}$$

with $a \in C^0(\mathbb{R},\mathbb{R})$, then $v \equiv 0$.

Note that we can not apply directly to v_j, $j \in [\![1, N_F]\!]$ Lemma 4.5, but by [30], Theorem 1.2, Remark (ii), for any $\varepsilon > 0$, $v_j \in C([\varepsilon,T];H^3(I_j)) \cap L^2(\varepsilon,T;H^4(I_j))$ and thus, we have enough regularity. Then, we get $v_j \equiv 0$ for $j \in [\![1, N_F]\!]$.

For $j \in]\!] N_F, N]\!]$,

$$\begin{cases}
\partial_t v_j + \partial_x v_j + \lambda v_j \partial_x v_j + \partial_x^3 v_j = 0, & \forall x \in (0,\infty),\ t \in (0,T), \\
v_j(t,0) = \partial_x v_j(t,0) = 0, & t > 0, \\
v_j \equiv 0, & \text{in } (0,T)\times\omega_j.
\end{cases}$$

Let $L > \beta_j$, as $v_j(0,x) \in L^2(0,\infty)$, then $v_j \in C([0,T];L^2(0,\infty)) \cap L^2(0,T;H^1_{loc}(0,\infty))$. By using $v_j \equiv 0$, in $(0,T)\times\omega_j$, we get $v_j(\cdot,L) = 0$. Then $w_j = v_j\Big|_{(0,L)}$ solves

$$\begin{cases} \partial_t w_j + \partial_x w_j + \lambda w_j \partial_x w_j + \partial_x^3 w_j = 0, & \forall x \in (0,L),\ t \in (0,T), \\ w_j(t,0) = w_j(t,L) = \partial_x w_j(t,0) = 0, & t > 0, \\ w_j \equiv 0, & \text{in } (0,T)\times(\beta_j, L), \\ w_j(0,x) \in L^2(0,L). \end{cases}$$

Thus, as in the past case, we get $w_j = v_j \equiv 0$ in $(0,T)\times(0,L)$ and as $L > \beta_j$ it is arbitrary we deduce that $v_j \equiv 0$ for $j \in]\!] N_F, N]\!]$. Finally $\underline{v} \equiv 0$, that contradicts $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$. □

Remark 4.6. Note that if we are able to prove a well-posedness result for (KdV) with initial data only in $\mathbb{L}^2(\mathcal{T})$, Theorem 4.4 still holds in this case, but we can not guarantee the same for Theorem 4.3, because we strongly use that $\underline{u}^0 \in \mathbb{Y}$.

5. Exponential stability in weighted Sobolev spaces

In this section, we present our main result about the exponential stability of (KdV) in the space $\mathbb{Y}$, which is the space of initial data for the well-posedness result. This section is inspired by [20], Section 3.1 where the exponential stability of a single nonlinear KdV equation in weighted spaces was deduced. Note first, that by Theorem 4.4 we already have the exponential stability in $L^2(I_j)$ for $j \in [\![1,N]\!]$, thus we only have to prove the exponential stability in $L^2_{(1+x^2)}(0,\infty)$ for $j \in]\!] N_F, N]\!]$.

Take $V_0(\underline{u}(t,\cdot)) = E(t)$, where $E(t)$ is defined by (1.2) and for $m = 1$,2 we define

$$V_m(\underline{u}) = \frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(1+x^m)(u_j)^2\mathrm{d}x + d_{m-1}V_{m-1}(\underline{u}), \tag{5.1}$$

where d_0, $d_1 > 0$ are large enough.

5.1. Exponential stability in the case $m = 1$

The idea of the following lines is to deduce first the exponential stability with energy V_1. To show that, we use the exponential stability in $\mathbb{L}^2(\mathcal{T})$ and the observability inequality (Obs2). More specifically, the aim of this subsection is to prove the following result

Proposition 5.1. *Assume that the damping terms $(a_j)_{j\in[\![1,N]\!]}$ satisfy* (1.1) *and let $R > 0$. If $\mathbb{I}_{act} = [\![1,N]\!]$, then, there exists $C(R) > 0$ such that for all $\underline{u}^0 \in \mathbb{Y}$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \le R$, we have for $\underline{u}$ solution of* (KdV)

$$V_1(\underline{u}(T,\cdot)) - V_1(\underline{u}^0) \le -C(R)V_1(\underline{u}^0). \tag{5.2}$$

Clearly, (5.2) gives the desired decay, $V_1(\underline{u}) \le C\mathrm{e}^{-\mu t}V_1(\underline{u}^0)$.

Proof. We start by noticing that, using (2.19) and (2.21) it is not difficult to see that

$$\begin{aligned} V_1(\underline{u}) - V_1(\underline{u}^0) + \frac{1+d_0}{2}\Bigg((2\alpha - N)\int_0^T (u_1(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2 \mathrm{d}t \\ +2\sum_{j=1}^N \int_0^T\int_{I_j} a_j (u_j)^2 \mathrm{d}x\mathrm{d}t\Bigg) + \frac{3}{2}\sum_{j=1}^N \int_0^T\int_{I_j} (\partial_x u_j)^2 \mathrm{d}x\mathrm{d}t - \frac{1}{2}\sum_{j=1}^N \int_0^T\int_{I_j} (u_j)^2 \mathrm{d}x\mathrm{d}t \\ -\frac{1}{3}\sum_{j=1}^N \int_0^T\int_{I_j} u_j^3 \mathrm{d}x\mathrm{d}t + \sum_{j=1}^N \int_0^T u_1(t,0)\partial_x u_j(t,0)\mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} x a_j (u_j)^2 \mathrm{d}x\mathrm{d}t = 0. \end{aligned} \tag{5.3}$$

Note now that, $\int_{I_j} u_j^3 \mathrm{d}x \leq \|u_j\|_{L^\infty(I_j)}\|u_j\|^2_{L^2(I_j)}$, and by [31], Corollary 1.2, we obtain $\int_{I_j} u_j^3 \mathrm{d}x \leq \|\partial_x u_j\|^{1/2}_{L^2(I_j)}\|u_j\|^{5/2}_{L^2(I_j)}$. Then for all $\varepsilon > 0$, using Young inequality $ab = (4\varepsilon)^{1/4} a \frac{1}{(4\varepsilon)^{1/4}} b \leq \varepsilon |a|^4 + C_\varepsilon |b|^{4/3}$ we get

$$\int_{I_j} u_j^3 \mathrm{d}x \leq \varepsilon \|\partial_x u_j\|^2_{L^2(I_j)} + C_\varepsilon \|u_j\|^{10/3}_{L^2(I_j)}.$$

If we choose $\underline{u}^0$ such that, $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq 1$, we observe that

$$\|u_j\|^{10/3}_{L^2(I_j)} \leq \|u_j\|^{4/3}_{L^2(I_j)}\|u_j\|^2_{L^2(I_j)} \leq C\|\underline{u}^0\|^{4/3}_{\mathbb{L}^2(\mathcal{T})}\|u_j\|^2_{L^2(I_j)} \leq C\|u_j\|^2_{L^2(I_j)}.$$

Therefore

$$\sum_{j=1}^N \int_0^T\int_{I_j} u_j^3 \mathrm{d}x\mathrm{d}t \leq \varepsilon \sum_{j=1}^N \int_0^T\int_{I_j} (\partial_x u_j)^2 \mathrm{d}x\mathrm{d}t + C_\varepsilon \sum_{j=1}^N \int_0^T\int_{I_j} (u_j)^2 \mathrm{d}x\mathrm{d}t. \tag{5.4}$$

Moreover, using (Obs2) we derive

$$\begin{aligned} \sum_{j=1}^N \int_0^T\int_{I_j} u_j^3 \mathrm{d}x\mathrm{d}t \leq & \varepsilon \sum_{j=1}^N \int_0^T\int_{I_j} (\partial_x u_j)^2 \mathrm{d}x\mathrm{d}t + C_\varepsilon C_{obs2}\Bigg(\int_0^T (u_1(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2 \mathrm{d}t \\ & + \sum_{j=1}^N \int_0^T\int_{I_j} a_j (u_j)^2 \mathrm{d}x\mathrm{d}t\Bigg). \end{aligned}$$

Using this inequality, (Obs2), with ε small enough and d_0 big enough in (5.3) we conclude the existence of $\tilde{C} > 0$ such that

$$\begin{aligned} V_1(\underline{u}) - V_1(\underline{u}^0) \leq - \widetilde{C}\Bigg(\int_0^T (u_1(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} (1+x) a_j (u_j)^2 \mathrm{d}x\mathrm{d}t \\ + \sum_{j=1}^N \int_0^T\int_{I_j} (\partial_x u_j)^2 \mathrm{d}x\mathrm{d}t\Bigg). \end{aligned} \tag{5.5}$$

Now, we state the following Lemma,

Lemma 5.2. *Let* $\underline{u}^0$ *and* $(a_j)_{j\in[\![1,N]\!]}$ *as in the statement of Proposition 5.1. Then, there exists* $C>0$ *such that*

$$\int_0^T V_1(\underline{u})\mathrm{d}t \leq C\left(\int_0^T (u_1(t,0))^2\mathrm{d}t + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2\mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} (1+x)a_j(u_j)^2\mathrm{d}x\mathrm{d}t\right). \tag{5.6}$$

Proof. The proof of this Lemma is postponed to Appendix C. □

Now, we estimate the term $V_1(\underline{u}^0)$. First, using (4.8) and (Obs2) we observe

$$\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C\left(\int_0^T (u_1(t,0))^2\mathrm{d}t + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2\mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} a_j(u_j)^2\mathrm{d}x\mathrm{d}t\right). \tag{5.7}$$

Secondly, multiplying (KdV) by $(T-t)xu_j$ and integrating on $[0,T]\times I_j$, after some integrations by parts we get

$$\begin{aligned} T\sum_{j=1}^N \int_{I_j} x(u_j(0,x))^2\mathrm{d}x =& \sum_{j=1}^N \int_0^T\int_{I_j} x(u_j)^2\mathrm{d}x\mathrm{d}t + 3\sum_{j=1}^N \int_0^T\int_{I_j} (T-t)(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t \\ &- \sum_{j=1}^N \int_0^T\int_{I_j} (T-t)(u_j)^2\mathrm{d}x\mathrm{d}t + 2\sum_{j=1}^N \int_0^T (T-t)u_1(t,0)\partial_x u_j(t,0)\mathrm{d}t \\ &+2\sum_{j=1}^N \int_0^T\int_{I_j} (T-t)xa_j(u_j)^2\mathrm{d}x\mathrm{d}t - \frac{2}{3}\sum_{j=1}^N \int_0^T\int_{I_j} (T-t)u_j^3\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Recalling that $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}\leq 1$ we can use (5.4) to obtain

$$\begin{aligned} \sum_{j=1}^N \int_{I_j} x(u_j(0,x))^2\mathrm{d}x \leq& C\left(\sum_{j=1}^N \int_0^T\int_{I_j} (1+x)(u_j)^2\mathrm{d}x\mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} (\partial_x u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^T (u_1(t,0))^2\mathrm{d}t\right. \\ &\left. + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2\mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} xa_j(u_j)^2\mathrm{d}x\mathrm{d}t\right) \\ \leq& C\left(\int_0^T V_1(\underline{u})\mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} (\partial_x u_j)^2\mathrm{d}x\mathrm{d}t + \int_0^T (u_1(t,0))^2\mathrm{d}t\right. \\ &\left. + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2\mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} xa_j(u_j)^2\mathrm{d}x\mathrm{d}t\right). \end{aligned}$$

Using this inequality, Lemma 5.2 and (5.7) we deduce

$$\begin{aligned} V_1(\underline{u}^0) \leq & C\left(\int_0^T (u_1(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} (1+x) a_j (u_j)^2 \mathrm{d}x\mathrm{d}t \right.\\ & \left. + \sum_{j=1}^N \int_0^T\int_{I_j} (\partial_x u_j)^2 \mathrm{d}x\mathrm{d}t \right). \end{aligned} \tag{5.8}$$

Finally, joining the estimates (5.5) and (5.8), we obtain (5.2) in the case $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq 1$. To conclude, note that as $\underline{u}^0 \in \mathbb{Y} \subset \mathbb{L}^2(\mathcal{T})$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ using Theorem 4.4 we know that $\|\underline{u}(t,\cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq \tilde{C}\mathrm{e}^{-\tilde{\mu}t}\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$, for some $\tilde{C} = \tilde{C}(R)$ and $\tilde{\mu} = \tilde{\mu}(R)$. Now, as $\underline{u} \in C([0,T];\mathbb{L}^2(\mathcal{T}))$ taking $T > 0$ such that $\tilde{C}\mathrm{e}^{-\tilde{\mu}T}R < 1$ we deduce our stability result. □

Remark 5.3. As in the proof of (4.4), we use $\underline{u}^0 \in \mathbb{Y}$ only for the existence of solutions $\underline{u} \in C([0,T];\mathbb{L}^2(\mathcal{T}))$. In this case, if we are able to prove a well-posedness result for initial data $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, such that $u_j^0 \in L^2_{(1+x)}(0,\infty)$ for $j \in]\!]N_F, N]\!]$ Proposition 5.1 still holds. Moreover, as was pointed in [20], Remark 1 in the case of a single KdV equation in the half-line adapting the ideas of [20], Theorem 2.5 it is possible to show the existence of solution mild solutions in $C([0,T];L^2_{(1+x)}(0,\infty))$ with initial data in $L^2_{(1+x)}(0,\infty)$ but the uniqueness is an open problem. We expect the same behavior in the network case.

5.2. Exponential stability in the case $m = 2$

In this part, using all the tools developed in the past sections, we show the exponential stability of (KdV) in $\mathbb{Y}$.

Theorem 5.4. *Assume that the damping terms $(a_j)_{j\in[\![1,N]\!]}$ satisfy (1.1) and let $R > 0$. If $\mathbb{I}_{act} = [\![1,N]\!]$, then, there exist $C(R) > 0$ and $\mu(R) > 0$ such that for all $\underline{u}^0 \in \mathbb{Y}$ with $\|\underline{u}^0\|_{\mathbb{Y}} \leq R$, we have for $\underline{u}$ solution of (KdV) $V_2(t) \leq CV_2(0)\mathrm{e}^{-\mu t}$, for all $t > 0$.*

Proof. We start by noting that as in Proposition 5.1 it is enough to prove that for some $C(R) > 0$

$$V_2(\underline{u}(T,\cdot)) - V_2(\underline{u}^0) \leq -C(R)V_2(\underline{u}^0), \tag{5.9}$$

which is equivalent to show the existence of some $C = C(R) > 0$ such that

$$\begin{aligned} V_2(\underline{u}) - V_2(\underline{u}^0) \leq -C & \left(\int_0^T (u_1(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} (1+x^2) a_j (u_j)^2 \mathrm{d}x\mathrm{d}t \right.\\ & \left. + \sum_{j=1}^N \int_0^T\int_{I_j} (1+x)(\partial_x u_j)^2 \mathrm{d}x\mathrm{d}t \right), \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} V_2(\underline{u}^0) \leq C & \left(\int_0^T (u_1(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T (\partial_x u_j(t,0))^2 \mathrm{d}t + \sum_{j=1}^N \int_0^T\int_{I_j} (1+x^2) a_j (u_j)^2 \mathrm{d}x\mathrm{d}t \right.\\ & \left. + \sum_{j=1}^N \int_0^T\int_{I_j} (1+x)(\partial_x u_j)^2 \mathrm{d}x\mathrm{d}t \right). \end{aligned} \tag{5.11}$$

Let $V=V_2-d_1V_1$, using (2.19), (2.25) and deriving a similar computation for a bounded branch is not difficult to see that

$$V(\underline{u})-V(\underline{u}^0)+\frac{1}{2}\left((2\alpha-N)\int_0^T(u_1(t,0))^2\mathrm{d}t+\sum_{j=1}^N\int_0^T(\partial_xu_j(t,0))^2\mathrm{d}t+2\sum_{j=1}^N\int_0^T\int_{I_j}a_j(u_j)^2\mathrm{d}x\mathrm{d}t\right)$$
$$+3\sum_{j=1}^N\int_0^T\int_{I_j}x(\partial_xu_j)^2\mathrm{d}x\mathrm{d}t-\sum_{j=1}^N\int_0^T\int_{I_j}x(u_j)^2\mathrm{d}x\mathrm{d}t-\frac{2}{3}\sum_{j=1}^N\int_0^T\int_{I_j}xu_j^3\mathrm{d}x\mathrm{d}t-N\int_0^T(u_1(t,0))^2\mathrm{d}t \quad (5.12)$$
$$+\sum_{j=1}^N\int_0^T\int_{I_j}x^2a_j(u_j)^2\mathrm{d}x\mathrm{d}t=0.$$

Now, by (5.5) and (5.6) we get

$$\sum_{j=1}^N\int_0^T\int_{I_j}x(u_j)^2\mathrm{d}x\mathrm{d}t\le\int_0^TV_1(\underline{u})\mathrm{d}t\le-C\Big(V_1(\underline{u})-V_1(\underline{u}^0)\Big). \quad (5.13)$$

Secondly, as $\left|\int_{I_j}xu_j^3\mathrm{d}x\right|\le\|u_j\|_{L^\infty(I_j)}\int_{I_j}x(u_j)^2\mathrm{d}x$, we get similarly as in the past section $\left|\int_{I_j}xu_j^3\mathrm{d}x\right|\le\|\partial_xu_j\|_{L^2(I_j)}^{1/2}\|u_j\|_{L^2(I_j)}^{1/2}\int_{I_j}x(u_j)^2\mathrm{d}x$, then for all $\varepsilon>0$ using Young inequality

$$\int_{I_j}xu_j^3\mathrm{d}x\le\varepsilon\|\partial_xu_j\|_{L^2(I_j)}^2+C_\varepsilon\|u_j\|_{L^2(I_j)}^2\left(\int_{I_j}x(u_j)^2\mathrm{d}x\right)^2.$$

Assume now that $\sum_{j=1}^N\int_{I_j}x(u_j(t,x))^2\mathrm{d}x<1$, for all $t>0$, then we get

$$\sum_{j=1}^N\int_0^T\int_{I_j}xu_j^3\mathrm{d}x\mathrm{d}t\le\varepsilon\sum_{j=1}^N\int_0^T\int_{I_j}(\partial_xu_j)^2\mathrm{d}x\mathrm{d}t+C_\varepsilon\sum_{j=1}^N\int_0^T\int_{I_j}(u_j)^2\mathrm{d}x\mathrm{d}t. \quad (5.14)$$

Finally, using this inequality for $\varepsilon>0$ small enough, $d_1>0$ big enough, (5.13) and (Obs2) we see

$$V_2(\underline{u})-V_2(\underline{u}^0)\le-C\left(\int_0^T(u_1(t,0))^2\mathrm{d}t+\sum_{j=1}^N\int_0^T(\partial_xu_j(t,0))^2\mathrm{d}t+\sum_{j=1}^N\int_0^T\int_{I_j}(1+x^2)a_j(u_j)^2\mathrm{d}x\mathrm{d}t\right.$$
$$\left.+\sum_{j=1}^N\int_0^T\int_{I_j}(1+x)(\partial_xu_j)^2\mathrm{d}x\mathrm{d}t\right)+\frac{d_1}{2}\Big(V_1(\underline{u})-V_1(\underline{u}^0)\Big).$$

Finally, (5.5) yields (5.10). Let us check (5.11), multiplying (KdV) by $(T-t)x^2u_j$ and integrating on $[0,T]\times I_j$, after some integrations by parts we get

$$\frac{T}{2}\sum_{j=1}^N\int_{I_j}x^2(u_j^0)^2\mathrm{d}x=3\sum_{j=1}^N\int_0^T\int_{I_j}(T-t)x(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t+\sum_{j=1}^N\int_0^T\int_{I_j}(T-t)a_jx^2(u_j)^2\mathrm{d}x\mathrm{d}t$$
$$+\sum_{j=1}^N\int_0^T\int_{I_j}(T-t)x(u_j)^2\mathrm{d}x\mathrm{d}t+N\int_0^T(T-t)(u_1(t,0))^2\mathrm{d}t-\sum_{j=1}^N\frac{2}{3}\int_0^T\int_{I_j}(T-t)xu_j^3\mathrm{d}x\mathrm{d}t.$$

As we assumed $\displaystyle\sum_{j=1}^N\int_{I_j}x(u_j(t,x))^2\mathrm{d}x<1$ for all $t>0$, we get

$$\sum_{j=1}^N\int_{I_j}x^2(u_j(0,x))^2\mathrm{d}x\leq C\left(\sum_{j=1}^N\int_0^T\int_{I_j}(1+x)(u_j)^2\mathrm{d}x\mathrm{d}t+\sum_{j=1}^N\int_0^T\int_{I_j}x(\partial_x u_j)^2\mathrm{d}x\mathrm{d}t+\int_0^T(u_1(t,0))^2\mathrm{d}t\right.$$
$$\left.+\sum_{j=1}^N\int_0^T\int_{I_j}x^2a_j(u_j)^2\mathrm{d}x\mathrm{d}t\right).$$

From where we obtain (5.11) using (5.6), (5.7) and (5.8). Thus, we obtain the exponential stability in the case $\displaystyle\sum_{j=1}^N\int_{I_j}x(u_j(t,x))^2\mathrm{d}x<1$. To conclude, note that as $\underline{u}^0\in\mathbb{Y}$ with $\|\underline{u}^0\|_{\mathbb{Y}}\leq R$ which in particular implies $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}\leq R$ and $V_1(\underline{u}^0)\leq\tilde{R}$ using Proposition 5.1 we know that $V_1(\underline{u}(t,\cdot))\leq\tilde{C}\mathrm{e}^{-\tilde{\mu}t}V_1(\underline{u}^0)$, for some $\tilde{C}=\tilde{C}(\tilde{R})$ and $\tilde{\mu}=\tilde{\mu}(\tilde{R})$. Now, as $\underline{u}\in\mathbb{B}$ taking $T>0$ such that $\tilde{C}\mathrm{e}^{-\tilde{\mu}T}\tilde{R}<1$ we deduce our stability result. □

Conclusions and final remarks

The well-posedness and exponential stability of the nonlinear KdV equation posed in star shaped network mixing bounded, and unbounded edges were investigated. The well-posedness results were obtained by considering first the linearization around zero and semigroup theory. Then, by the introduction of some weighted Sobolev spaces and a fixed point approach, the well-posedness for the original nonlinear system was deduced. In this sense, as was pointed in Remark 3.5 the introduction of the weighted spaces is due to the perturbation approach. Regarding that, note that a stability result was presented in Theorem 4.4 which in spirit did not use that the initial data is in $\mathbb{Y}$ and thus the open problem about well-posedness with the classical $\mathbb{L}^2(\mathcal{T})$ initial data is interesting. In [20] the strategy to prove the well-posedness was quite different. The idea of [20] was first to derive the well-posedness in weighted spaces with exponential weights using semigroup theory and fixed point results. Then, by compactness argument the well-posedness in the spaces $L^2_{(1+x)^m}(0,\infty)$ was deduced in [20], we expect that similar ideas can be applied in the networks case. Actually in that work, the authors were able to prove the exponential stability in the spaces $L^2_{(1+x)^m}(0,\infty)$, for $m\geq 1$.

An interesting open problem becoming of our contradiction strategy to prove the observability inequalities is the possibility to remove one index in Theorem 4.4 (or Prop. 5.1, Theorem 5.4). For instance, if we remove one index for $j\in[\![1,N_F]\!]$ and following the proof of Theorem 4.4, we are asking to prove that the unique solution of

$$\begin{cases}\partial_t v+\partial_x v+v\partial_x v+\partial_x^3 v=0, & \forall x\in(0,\ell),\ t\in(0,T),\\ v(t,0)=\partial_x v(t,0)=\partial_x^2 v(t,0)=0, & t>0,\\ v(t,\ell)=\partial_x v(t,\ell)=0, & t>0,\end{cases}\tag{5.15}$$

is the null solution. Up to our knowledge, this is an open problem, but it is known that the condition $\partial_x^2 v(t,0)=0$ is really needed. In fact, in [32] the following result was proved regarding the stationary solutions of the KdV equation, considering the system

$$\partial_x \phi + \frac{1}{2}\partial_x(\phi^2) + \partial_x^3 \phi = 0, \quad \text{in} \;\; [0,L], \tag{5.16}$$

Theorem 5.5 (Theorem 1, [32]). *For all $L \in (0,2\pi)$, there exists a stationary solution $\phi \in C^\infty(\mathbb{R})$ of (5.16) with boundary conditions $\phi(0)=\partial_x\phi(0)=0$, satisfying $\phi(x+L)=\phi(x)$, $\forall x \in \mathbb{R}$ and $\partial_x^2\phi(0)\neq 0$.*

Note that by the periodicity we have in particular $\phi(L)=\partial_x\phi(L)=0$. In [33] a more general result was presented,

Lemma 5.6 (Lemma 1, [33]). *If $\phi \in C^3([0,L])$ is a solution of (5.16) with boundary conditions $\phi(0)=\phi(L)=\partial_x\phi(L)=0$, then it is infinitely smooth and periodic with period L.*

Theorem 5.7 (Theorem 1, [33]). *If $L^2 \neq 4\pi^2$, then there exists a unique non-trivial solution of period L of (5.16) with boundary conditions $\phi(0)=\phi(L)=\partial_x\phi(L)=0$. If $L^2=4\pi^2$ such a solution does not exist.*

By Lemma 5.6 and Theorem 5.7, we have the existence of a stationary solution $\phi \in C^\infty([0,L])$ of (5.15). But this solution does not satisfy $\partial_x^2\phi(0)=0$. Indeed, consider the substitution $\eta(x)=\frac{L^2}{4}\phi\Big(\frac{L}{4}(x+1)\Big)$, then η satisfies system

$$b\partial_x\eta + \eta\partial_x\eta + \partial_x^3\eta = 0, \quad \text{in} \;\; [-1,1], \eta(-1)=\partial_x\eta(-1)=0,$$

where $b=\frac{L^2}{4}$. Note that this equation is equivalent to $b\eta + \dfrac{1}{2}\eta^2 + \partial_x^2\eta = c$, for some $c \in \mathbb{R}$. In particular, if $\partial_x^2\eta(-1)=0$ we get $c=0$. But from [33], Theorem 1 and [33], Lemma 2 we have that c must be not null. Similarly, in [34], Section 4.4 the existence of some stationary solutions η of KdV equation in $[-1,1]$ with $\eta(-1)=\eta(1)=\partial_x\eta(-1)=\partial_x\eta(1)=0$ was shown, but $\partial_x^2\eta(-1)\neq 0$.

In addition, again recalling (5.15) and the regularizing effect of the nonlinear KdV equation, we get using the boundary conditions and the equation that all the spatial derivatives evaluated at $x=0$ are null, *i.e.* $\partial_x^k u(t,0)=0$ for all $k \in \mathbb{N}$. Thus, if we can show that the solution of (5.15) is analytic we obtain that $u \equiv 0$, but this it is also unknown. In the linear case, is known, that the semigroup generated by the linear KdV equation is not analytic but a semigroup of Gevrey class $\delta > 3/2$ for all lengths $L>0$ see [35], Theorem 1.1.

Appendix A. Proof of Lemma 2.7

Let y, $z \in B_\infty$, then

$$\|y\partial_x y - z\partial_x z\|_{L^1(0,T;L^2_{(1+x^2)}(0,\infty))} \leq \int_0^T \|(y-z)\partial_x y\|_{L^2_{(1+x^2)}(0,\infty)}\mathrm{d}t + \int_0^T \|z(\partial_x y - \partial_x z)\|_{L^2_{(1+x^2)}(0,\infty)}\mathrm{d}t. \tag{A.1}$$

Here we follow the strategies used in [27]. Let $w=y-z$, thus, the first term to estimate is $\|w\partial_x y\|_{L^1(0,T;L^2_{(1+x^2)}(0,\infty))}$,

$$\int_0^T \|w\partial_x y\|_{L^2_{(1+x^2)}(0,\infty)}\mathrm{d}t \leq \int_0^T \|w\partial_x y\|_{L^2(0,\infty)}\mathrm{d}t + \int_0^T \|xw\partial_x y\|_{L^2(0,\infty)}\mathrm{d}t,$$

the first term can be estimated as

$$\begin{aligned}\int_0^T \|w\partial_x y\|_{L^2(0,\infty)}\mathrm{d}t &\leq \int_0^T \|w\|_{L^\infty(0,\infty)}\|\partial_x y\|_{L^2(0,\infty)}\mathrm{d}t \\ &\leq C\int_0^T \|w\|_{H^1(0,\infty)}\|\partial_x y\|_{L^2(0,\infty)}\mathrm{d}t \\ &\leq C\|w\|_{L^2(0,T;H^1(0,\infty))}\|y\|_{L^2(0,T;H^1(0,\infty))} \\ &\leq C\|w\|_{B_\infty}\|y\|_{B_\infty},\end{aligned} \tag{A.2}$$

where we have used the embedding of $H^1(0,\infty)$ in $L^\infty(0,\infty)$. For the other term, we can observe that

$$\begin{aligned}\int_0^T \|xw\partial_x y\|_{L^2(0,\infty)}\mathrm{d}t &= \int_0^T \left(\int_0^\infty x^2(w)^2(\partial_x y)^2\mathrm{d}x\right)^{1/2}\mathrm{d}t = \int_0^T \left(\int_0^\infty (\sqrt{x}w)^2(\sqrt{x}\partial_x y)^2\mathrm{d}x\right)^{1/2}\mathrm{d}t \\ &\leq \int_0^T \|\sqrt{x}w\|_{L^\infty(0,\infty)}\|\partial_x y\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t.\end{aligned} \tag{A.3}$$

We cannot apply directly apply the embedding of $H^1(0,\infty)$ in $L^\infty(0,\infty)$ to estimate the $L^\infty(0,\infty)$ norm of $\sqrt{x}w$. In fact, we have, $\partial_x(\sqrt{x}w) = \dfrac{1}{2\sqrt{x}}w + \sqrt{x}\partial_x w$ which is not necessarily in $L^2(0,\infty)$. With this in mind, we study the term $\|\sqrt{x}w\|_{L^\infty(0,\infty)}$ in the following way:

$$\|\sqrt{x}w\|_{L^\infty(0,\infty)} = \sup\left\{\|\sqrt{x}w\|_{L^\infty(0,1)}, \|\sqrt{x}w\|_{L^\infty(1,\infty)}\right\},$$

for the first term, as $x \in (0,1)$

$$\begin{aligned}\|\sqrt{x}w\|_{L^\infty(0,1)} &\leq \|w\|_{L^\infty(0,1)} \leq C\|w\|^{1/2}_{L^2(0,1)}\|\partial_x w\|^{1/2}_{L^2(0,1)} \leq C\|w\|^{1/2}_{L^2(0,\infty)}\|\partial_x w\|^{1/2}_{L^2(0,\infty)} \\ &\leq \|w\|^{1/2}_{L^2_{(1+x^2)}(0,\infty)}\|\partial_x w\|^{1/2}_{L^2_{(1+x)}(0,\infty)}.\end{aligned} \tag{A.4}$$

Similarly,

$$\|\sqrt{x}w\|_{L^\infty(1,\infty)} \leq C\|\sqrt{x}w\|^{1/2}_{L^2(1,\infty)}\|\partial_x(\sqrt{x}w)\|^{1/2}_{L^2(1,\infty)}.$$

Using that $x \geq 1$, we observe that $\|\partial_x(\sqrt{x}w)\|^{1/2}_{L^2(1,\infty)} \leq C(\|w\|_{L^2(1,\infty)} + \|\sqrt{x}\partial_x w\|_{L^2(1,\infty)})^{1/2}$ and thus

$$\begin{aligned}\|\sqrt{x}w\|_{L^\infty(1,\infty)} &\leq C\left(\|\sqrt{x}w\|^{1/2}_{L^2(1,\infty)}\left[\|w\|_{L^2(1,\infty)} + \|\sqrt{x}\partial_x w\|_{L^2(1,\infty)}\right]^{1/2}\right) \\ &\leq C\left(\|w\|^{1/2}_{L^2_{(1+x^2)}(0,\infty)}\left[\|w\|_{L^2_{(1+x^2)}(0,\infty)} + \|\partial_x w\|_{L^2_{(1+x)}(0,\infty)}\right]^{1/2}\right) \\ &\leq C\left(\|w\|^{1/2}_{L^2_{(1+x^2)}(0,\infty)}\|\partial_x w\|^{1/2}_{L^2_{(1+x)}(0,\infty)} + \|w\|_{L^2_{(1+x^2)}(0,\infty)}\right).\end{aligned} \tag{A.5}$$

Then by (A.4) and (A.5)

$$\|\sqrt{x}w\|_{L^\infty(0,\infty)} \leq C\left(\|w\|^{1/2}_{L^2_{(1+x^2)}(0,\infty)}\|\partial_x w\|^{1/2}_{L^2_{(1+x)}(0,\infty)} + \|w\|_{L^2_{(1+x^2)}(0,\infty)}\right).$$

Using this in (A.3) we obtain

$$\begin{aligned}\int_0^T \|xw\partial_x y\|_{L^2(0,\infty)}\mathrm{d}t \leq & C\int_0^T \left(\|w\|^{1/2}_{L^2_{(1+x^2)}(0,\infty)}\|\partial_x w\|^{1/2}_{L^2_{(1+x)}(0,\infty)} + \|w\|_{L^2_{(1+x^2)}(0,\infty)}\right) \\ & \times \|\partial_x y\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t \\ \leq & C\left(\int_0^T \|w\|^{1/2}_{L^2_{(1+x^2)}(0,\infty)}\|\partial_x w\|^{1/2}_{L^2_{(1+x)}(0,\infty)}\|\partial_x y\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t \right. \\ & \left. + \int_0^T \|w\|_{L^2_{(1+x^2)}(0,\infty)}\|\partial_x y\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t\right) \\ \leq & C\left(\|w\|^{1/2}_{C([0,T];L^2_{(1+x^2)}(0,\infty))}\int_0^T \|\partial_x w\|^{1/2}_{L^2_{(1+x)}(0,\infty)}\|\partial_x y\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t \right. \\ & \left. +\|w\|_{C([0,T];L^2_{(1+x^2)}(0,\infty))}\int_0^T \|\partial_x y\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t\right) \\ \leq & C\left[\|w\|^{1/2}_{B_\infty}\left(\int_0^T \|\partial_x w\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t\right)^{1/2}\left(\int_0^T \|\partial_x y\|^2_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t\right)^{1/2} \right. \\ & \left. +\|w\|_{B_\infty}\int_0^T \|\partial_x y\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t\right] \\ \leq & C\left[\|w\|^{1/2}_{B_\infty}\|y\|_{B_\infty}\left(\int_0^T \|\partial_x w\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t\right)^{1/2} \right. \\ & \left. +\|w\|_{B_\infty}\int_0^T \|\partial_x y\|_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t\right].\end{aligned}$$

Using Cauchy–Schwarz in the two remaining time integrals

$$\begin{aligned}\int_0^T \|xw\partial_x y\|_{L^2(0,\infty)}\mathrm{d}t \leq & C\left\{\|w\|^{1/2}_{B_\infty}\|y\|_{B_\infty}\left[\left(\int_0^T \|\partial_x w\|^2_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t\right)^{1/2}\left(\int_0^T \mathrm{d}t\right)^{1/2}\right]^{1/2} \right. \\ & \left. +\|w\|_{B_\infty}\left(\int_0^T \|\partial_x y\|^2_{L^2_{(1+x)}(0,\infty)}\mathrm{d}t\right)^{1/2}\left(\int_0^T \mathrm{d}t\right)^{1/2}\right\} \\ \leq & C(T^{1/4}+T^{1/2})\|w\|_{B_\infty}\|y\|_{B_\infty}.\end{aligned}$$

Therefore

$$\int_0^T \|w\partial_x y\|_{L^2_{(1+x^2)}(0,\infty)}\mathrm{d}t \leq C\|w\|_{B_\infty}\|y\|_{B_\infty}.$$

For the second term of (A.1), it is enough to note that it can be written as $\|y\partial_x w\|_{L^1(0,T;L^2_{(1+x^2)}(0,\infty))}$ for $w = y - z$.

Appendix B. Proof of Lemma 3.3

In this part we prove Lemma 3.3. First note that as $\psi'(x) = 0$ for $x \notin \left(\frac{x_0}{2}, \frac{3x_0}{2}+1\right)$, thus

$$\frac{1}{3}\int_0^\infty u_j^3\psi'(x)\mathrm{d}x = \frac{1}{3}\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}+1} u_j^3\psi'(x)\mathrm{d}x \leq \frac{1}{3}\sup_{x\in(\frac{x_0}{2},\frac{3x_0}{2}+1)} |u_j(t,x)\sqrt{\psi'(x)}| \int_{\frac{x_0}{2}}^{\frac{3x_0}{2}+1} (u_j)^2\sqrt{\psi'(x)}\mathrm{d}x.$$

Define $f = u_j(t,x)\sqrt{\psi'(x)}$, by Theorem 2.9 $\underline{u} \in \mathbb{B}$ and we can consider that $u_j(t,\cdot) \in H^1(\frac{x_0}{2}, \frac{3x_0}{2}+1)$. Moreover, by (3.4) $\sqrt{\psi'} \in H^1(\frac{x_0}{2}, \frac{3x_0}{2}+1)$, then as $H^1(\frac{x_0}{2}, \frac{3x_0}{2}+1)$ is an algebra, we get that $f \in H^1(\frac{x_0}{2}, \frac{3x_0}{2}+1)$. Now, observe that

$$\begin{aligned} |f(x)||f'(x)| &= \left|u_j\sqrt{\psi'(x)}\right| \left|\partial_x u_j\sqrt{\psi'(x)} + \frac{u_j\psi''(x)}{2\sqrt{\psi'(x)}}\right| \\ &\leq |u_j\partial_x u_j\psi'(x)| + \frac{1}{2}\left|u_j^2\psi''(x)\right|. \end{aligned} \tag{B.1}$$

As $f \in H^1(\frac{x_0}{2}, \frac{3x_0}{2}+1)$, using [31], Corollary 1.2 we have

$$\sup_{x\in(\frac{x_0}{2},\frac{3x_0}{2}+1)} |f(x)| \leq \frac{1}{\sqrt{2}}\left(\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}+1} |f(x)||f'(x)|\mathrm{d}x\right)^{1/2},$$

thus

$$\begin{aligned} \sup_{x\in(\frac{x_0}{2},\frac{3x_0}{2}+1)} |u_j(t,x)\sqrt{\psi'(x)}| &\leq \frac{1}{\sqrt{2}}\left(\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}+1} |u_j\partial_x u_j\psi'(x)|\,\mathrm{d}x + \frac{1}{2}\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}+1} \left|u_j^2\psi''(x)\right|\mathrm{d}x\right)^{1/2} \\ &\leq \frac{1}{\sqrt{2}}\left(\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}+1} |u_j\partial_x u_j\psi'(x)|\,\mathrm{d}x\right)^{1/2} + \frac{1}{2}\left(\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}+1} \left|u_j^2\psi''(x)\right|\mathrm{d}x\right)^{1/2}. \end{aligned}$$

Writing $u_j\partial_x u_j\psi'(x) = u_j\sqrt{\psi'(x)}\partial_x u_j\sqrt{\psi'(x)}$, we get

$$\begin{aligned} \sup_{x\in(\frac{x_0}{2},\frac{3x_0}{2}+1)} |u_j(t,x)\sqrt{\psi'(x)}| \leq & \frac{1}{\sqrt{2}}\left(\int_0^\infty (u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4}\left(\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4} \\ &+ \frac{1}{2}\left(\int_0^\infty (u_j)^2|\psi''(x)|\mathrm{d}x\right)^{1/2}. \end{aligned}$$

Finally, we deduce

$$\begin{aligned} \frac{1}{3}\int_0^\infty u_j^3\psi'(x)\mathrm{d}x \leq & \frac{1}{6}\left(\int_0^\infty (u_j)^2|\psi''(x)|\mathrm{d}x\right)^{1/2}\int_0^\infty (u_j)^2\sqrt{\psi'(x)}\mathrm{d}x + \frac{1}{3\sqrt{2}}\left(\int_0^\infty (u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4} \\ &\times\left(\int_0^\infty (\partial_x u_j)^2\psi'(x)\mathrm{d}x\right)^{1/4}\int_0^\infty (u_j)^2\sqrt{\psi'(x)}\mathrm{d}x. \end{aligned}$$

Appendix C. Proof of Lemma 5.2

First, note that

$$\int_0^T V_1(\underline{u})\mathrm{d}t = \frac{1}{2}\sum_{j=1}^N \int_0^T\int_{I_j} (1+x)(u_j)^2\mathrm{d}x\mathrm{d}t + \frac{d_0}{2}\sum_{j=1}^N \int_0^T\int_{I_j} (u_j)^2\mathrm{d}x\mathrm{d}t,$$

clearly as $\underline{u}^0 \in \mathbb{Y} \subset \mathbb{L}^2(\mathcal{T})$, from (Obs2)

$$\begin{aligned}\frac{1+d_0}{2}\sum_{j=1}^N \int_0^T\int_{I_j} (u_j)^2\mathrm{d}x\mathrm{d}t \leq& \frac{C_{obs2}(1+d_0)}{2}\left(\int_0^T (u_1(t,0))^2\mathrm{d}t + \sum_{j=1}^N\int_0^T (\partial_x u_j(t,0))^2\mathrm{d}t\right.\\ &\left.+\sum_{j=1}^N \int_0^T\int_{I_j} a_j(u_j)^2\mathrm{d}x\mathrm{d}t\right).\end{aligned}$$

For the other term, we observe that

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^N \int_0^T\int_{I_j} x(u_j)^2\mathrm{d}x\mathrm{d}t =& \frac{1}{2}\sum_{j=1}^{N_F} \int_0^T\int_{I_j} x(u_j)^2\mathrm{d}x\mathrm{d}t + \frac{1}{2}\sum_{j=N_F+1}^{N} \int_0^T\int_0^{\beta_j} x(u_j)^2\mathrm{d}x\mathrm{d}t\\ &+\frac{1}{2}\sum_{j=N_F+1}^{N} \int_0^T\int_{\beta_j}^{\infty} x(u_j)^2\mathrm{d}x\mathrm{d}t\\ \leq& \frac{\max_{j\in[\![1,N_F]\!]}\ell_j}{2}\sum_{j=1}^{N_F}\int_0^T\int_{I_j}(u_j)^2\mathrm{d}x\mathrm{d}t + \frac{1}{2}\sum_{j=N_F+1}^{N} \int_0^T\int_{\beta_j}^{\infty} x(u_j)^2\mathrm{d}x\mathrm{d}t\\ &+\frac{\max_{j\in[\![1,N_F]\!]}\beta_j}{2}\sum_{j=N_F+1}^{N} \int_0^T\int_0^{\beta_j} x(u_j)^2\mathrm{d}x\mathrm{d}t\\ \leq& C\left(\int_0^T (u_1(t,0))^2\mathrm{d}t + \sum_{j=1}^N\int_0^T (\partial_x u_j(t,0))^2\mathrm{d}t + \sum_{j=1}^N\int_0^T\int_{I_j} a_j(u_j)^2\mathrm{d}x\mathrm{d}t\right)\\ &+\frac{1}{2}\sum_{j=N_F+1}^{N} \int_0^T\int_{\beta_j}^{\infty} x\frac{a_j}{c_j}(u_j)^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^N \int_0^T\int_{I_j} x(u_j)^2\mathrm{d}x\mathrm{d}t \leq& C\left(\int_0^T (u_1(t,0))^2\mathrm{d}t + \sum_{j=1}^N\int_0^T (\partial_x u_j(t,0))^2\mathrm{d}t\right.\\ &\left.+\sum_{j=1}^N \int_0^T\int_{I_j} (1+x)a_j(u_j)^2\mathrm{d}x\mathrm{d}t\right).\end{aligned} \tag{C.1}$$

Joining these two estimates, we deduce (5.6).

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