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A VARIATIONAL APPROACH TO STABILITY RELATIVE TO A SET OF SINGLE-VALUED AND SET-VALUED MAPPINGS

Vo Duc Thinh[1,2], Xiaolong Qin[1,*] and Jen-Chih Yao[3]

Abstract. Based on the limiting normal cone relative to a set, we present in this paper the novel versions of the limiting coderivative relative to a set and subdifferentials relative to a set of multifunctions and singleton mappings, respectively. In addition to giving the necessary and sufficient conditions for the Aubin property relative to a set of multifunctions, the limiting coderivative relative to a set also provides a coderivative criterion for the metric regularity relative to a set of multifunctions. Besides, our study establishes sudifferential characteristics of the metric regularity and the locally Lipschitz continuity relative to a set for single-valued mappings. In finite dimensional spaces, our results are more general than the previous results. Furthermore, we also give examples to illustrate our results.

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1. Introduction

The stability analysis, including the Aubin property (also known as the local Lipschitz-like property or pseudo-Lipschitz), and metric (sub)regularity of multifunctions or/and solution mappings to parameter systems is an essential topic to variational analysis because of their applications; see, *e.g.*, [1–13]. These properties can been characterized through generalized differentials. More precisely, it was demonstrated in [[9], Theorem 5.7] that the Aubin property around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ of a closed-graph multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ can be equivalently characterized by the equality $D^*F(\bar{x}, \bar{y})(0) = \{0\}$. Moreover, the exact Lipschitz-like constant of F around $(\bar{x}, \bar{y})$, $\mathfrak{a}F(\bar{x}, \bar{y})$, can be computed by

$$\mathfrak{a}F(\bar{x}, \bar{y}) = \|D^*F(\bar{x}, \bar{y})\| := \sup \{\|x^*\| \mid (y^*, x^*) \in \operatorname{gph} D^*F(\bar{x}, \bar{y}), y^* \in \mathbb{B}\}.$$

Meanwhile, the metric regularity of a multifunction is also completely characterized, in [[10], Theorem 4.18], through its limiting coderivative as follows: the multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is metrically regular around $(\bar{x}, \bar{y})$ if and only if F^{-1} satisfies the Aubin property at $(\bar{y}, \bar{x})$, which is also equivalent to the fact that the kernel of $D^*F(\bar{x}, \bar{y})$ is only $\{0\}$. This characteristic is known as the Mordukhovich criterion for the metric regularity of multifunctions. By essential applications in variational and optimization theories, the stability properties of

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[1] School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China.
[2] Dong Thap University, Cao Lanh City 870000, Dong Thap, Vietnam.
[3] Research Center for Interneural Computing, China Medical University, Taichung 40402, Taiwan.

[*] Corresponding author: qxlxajh@163.com

multifunctions were generalized in various ways, such as the metric regularity relative to a set (or cone) [8, 14], the directional metric (sub)regularity [6, 14–17], the Aubin property (local Lipschitz-like continuous) relative to a set [8, 18–21], and so on. However, whichever way it is extended, it is of great interest to provide the coderivative characteristic for those stability properties as the Mordukhovich criterions. One such result was mentioned as the directional coderivative characteristic of the directional metric regularity presented in [6], Theorem 5. For the Aubin property relative to a set, by using the projectional coderivative, the authors in [18] established necessary and sufficient characteristics for the Aubin property relative to a set of set-valued mappings. Besides, testing the locally Lipschitz continuous property relative to a set of single-valued mappings due to the projectional subdifferential was presented in that paper. A similar result for the metric regularity relative to a set has not been established through projectional coderivative, however. Recently, the authors in [22] introduced the new relative to a set versions of coderivatives and used them to provide Mordukhovich criterions for the Aubin property and metric regularity relative to a set of multifunctions. Besides, establishing complete characterizations of the relative well-posedness properties with further applications to the stability of affine variational inequalities was presented in [22]. However, the authors in [22] used two different versions of coderivative, which are the *normal coderivative relative to a set* and the *mirror contingent coderivative relative to a set*, to obtain the Mordukhovich criterions for the Aubin property relative to a set and for metric regularity relative to a set of multifunctions. However, the characterizations of the locally Lipschitz continuity and the metric regularity relative to a set of single-valued due to the relative to a set subdifferentials were not established. Almost the same time, inspired by [18, 21], the authors in [19] stated a new Mordukhovich criterion for the Aubin property relative to a set of multifunctions due to the limiting coderivative relative to that set by using another approach. Formulas for calculating and some applications in studying optimality conditions of the limiting coderivative were presented in [23]. Moreover, the necessary optimality conditions for the optimization problems with geometric and generalized equation constraints through the limiting subdifferential and the limiting coderivative relative to a set were established in [23] also. Furthermore, the authors in [24] stated explicit formulas of coderivatives relative to a set of the normal cone mapping to a polyhedron. They also provided a characteristic of the Aubin property relative to a set of that normal cone mapping and thereby give necessary optimality conditions for simple bilevel optimization problems under certain qualification conditions. It is important to know that, although there are some similarities between coderivatives relative to a set in [22] and [19, 23, 24], those coderivatives are different in general.

We, in this paper, first present a new version of the limiting coderivative relative to a set, an improvement of that one in [19], of set-valued mappings due to the limiting normal cone relative to a set $\mathscr{C}$ to their graphic. This novel limiting coderivative relative to $\mathscr{C}$ is also called the $\mathscr{C}$*-coderivative*. By using the $\mathscr{C}$-coderivative, we provide the Mordukhovich criterion versions for the Aubin property and metric regularity relative to $\mathscr{C}$ of set-valued mappings. We only use the $\mathscr{C}$-coderivative to establish both of the Mordukhovich criterions for the Aubin property relative to a set in form of $\mathscr{C} \times \mathscr{D}$ and for metric regularity relative to $\mathscr{C} \times \mathscr{D}$ of a multifunction F. It is important to know that these results are established without the assumption $\mathscr{C} = F^{-1}(\mathscr{D})$, which was used in [22], Theorem 3.9. In addition, we also establish the characteristics for the local Lipschitz continuous and the metric regularity relative to $\mathscr{C}$ of single-valued mappings through their limiting subdifferential relative to $\mathscr{C}$, which is also said to be the $\mathscr{C}$-subdifferential.

This paper is organized as follows. In Section 2, we first recall the notions and some basic properties of the proximal and limiting normal cones relative to a set. In Section 3, we introduce some concepts of the stability property relative to a set including the Aubin property and metric regularity relative to a set of set-valued mappings and the locally Lipschitz continuity and metric regularity relative to a set of single-valued. Besides, the relations between these stability properties are also established in this section. On the basis of the limiting normal cone relative to a set, we propose the new concept of the limiting coderivative relative to a set of multifunctions in Section 4. We then apply this new limiting coderivative relative to a set to characterize the necessary and sufficient conditions for the Aubin property and metric regularity relative to a set of multifunctions. In Section 5, we introduce the notions of the new limiting subdifferential relative to a set of single-valued mappings, and we provide the characteristics for the local Lipschitz continuous and the metric regularity relative to a set of single-valued mappings due to these subdifferentials. In addition, we present some examples to illustrate obtained results. The last section, Section 6, ends this paper with concluding remarks.

2. Preliminaries and auxiliaries

We assume, through out of this paper, that all of spaces are finite dimension spaces with Euclidean norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$. Their Cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ is also equipped an Euclidean norm defined by $\|(x,y)\| := \sqrt{\|x\|_n^2 + \|y\|_m^2}$. We use $\|\cdot\|$ for any Euclidean space if there is no confusion. $\mathbb{B}(x,r)$ is used to denote the closed ball centered at x with radius $r > 0$ and $\mathbb{B} := \mathbb{B}(0,1)$, while $\mathbb{R}^s_+ := \{x = (x_1,\ldots,x_s) \in \mathbb{R}^s \mid x_i \geq 0,\ \forall i = 1,\ldots,s\}$. Set $\operatorname{cone}(\Omega) := \{\lambda\omega \mid \lambda \geq 0, \omega \in \Omega\}$ and $\mathscr{R}(\bar{x},\Omega) := \{x^* \in \mathbb{R}^s \mid \exists t_0 > 0 : \bar{x} + tx^* \in \Omega\ \forall t \in (0,t_0]\}$. Given a sequence of real numbers (t_k), we write $t_k \to 0^+$ if $t_k \to 0$ and $t_k > 0$ for all k. Let $\bar{x} \in \Omega$. We borrow the notation $x_k \xrightarrow{\Omega} \bar{x}$ to present that $x_k \to \bar{x}$ and $x_k \in \Omega$ for all $k \in \mathbb{N}$. The *distance function* to Ω, $d(\cdot,\Omega) : \mathbb{R}^s \to \mathbb{R}$, is defined by $d(x,\Omega) := \inf_{u\in\Omega} \|u - x\|$ for all $x \in \mathbb{R}^s$. The set-valued mapping $\Pi(\cdot,\Omega) : \mathbb{R}^s \rightrightarrows \mathbb{R}^s$, with $\Pi(x,\Omega) := \{u \in \mathbb{R}^s \mid \|u - x\| = d(x,\Omega)\}$ is called the *set-valued mapping projection* onto Ω, and the set $\Pi(x,\Omega)$, which can be empty, is said to be the *Euclidean projector set* of x onto Ω. It is known that if Ω is closed, then $\Pi(x,\Omega) \neq \emptyset$ for any $x \in \mathbb{R}^s$. Given $u \in \Omega$, we define $\Pi^{-1}(u,\Omega) := \{x \in \mathbb{R}^s \mid u \in \Pi(x,\Omega)\}$, which is always nonempty (in fact, $u \in \Pi^{-1}(u,\Omega)$ for all $u \in \Omega$). The *proximal* and *Fréchet/regular normal cones* to Ω at $\bar{x}$ are respectively given (see [[10], Def. 1.1 and p. 240,] or [[25], p. 22]) by

$$N^p(\bar{x},\Omega) := \operatorname{cone}[\Pi^{-1}(\bar{x},\Omega) - \bar{x}]$$

and

$$\hat{N}(\bar{x},\Omega) := \left\{ x^* \in \mathbb{R}^s \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

Let Ω be locally closed around $\bar{x}$. The *limiting normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x},\Omega) := \operatorname*{Lim\,sup}_{x\to\bar{x}} \Big(\operatorname{cone}[x - \Pi(x,\Omega)]\Big),$$

where $\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x)$ signifies the *sequential Painlevé-Kuratowski upper/outer limit* of F at $\bar{x} \in \operatorname{dom} F$ given by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \{y \in \mathbb{R}^m \mid \exists x_k \to \bar{x}, y_k \to y \text{ with } y_k \in F(x_k) \text{ for all } k \in \mathbb{N}\}.$$

The *tangent/contingent cone* to Ω at $\bar{x}$ is given by

$$T(\bar{x},\Omega) := \left\{v \in \mathbb{R}^s \mid \exists t_k \to 0^+, v_k \to \bar{v} \text{ such that } \bar{x} + t_k v_k \in \Omega, \forall k \in \mathbb{N}\right\}.$$

Remark 2.1. (i) We have from the definition (see [[20], Ex. 6.16]) that $x^* \in N^p(\bar{x},\Omega)$ if and only if there exists $t_0 > 0$ such that $\bar{x} \in \Pi(\bar{x} + t_0x^*,\Omega)$ and $\{\bar{x}\} = \Pi(\bar{x} + tx^*,\Omega)$ for all $t \in (0,t_0)$. Moreover, it is also known (see [[10], p. 240] or [[25], Prop. 1.5]) that

$$\begin{aligned} N^p(\bar{x},\Omega) &= \left\{x^* \mid \exists\delta > 0 \text{ such that } \langle x^*, x - \bar{x}\rangle \leq \delta\|x - \bar{x}\|^2\ \forall x \in \Omega\right\} \\ &= \left\{x^* \mid \exists\delta,\theta > 0 \text{ such that } \langle x^*, x - \bar{x}\rangle \leq \delta\|x - \bar{x}\|^2\ \forall x \in \Omega \cap \mathbb{B}(\bar{x},\theta)\right\}. \end{aligned}$$

(ii) It is known from [[10], p. 240] that $N^p(\bar{x}, \Omega) \subset \hat{N}(\bar{x}, \Omega)$ and

$$N(\bar{x}, \Omega) = \operatorname*{Lim\,sup}_{x \to \bar{x}} \hat{N}(x, \Omega) = \operatorname*{Lim\,sup}_{x \to \bar{x}} N^p(x, \Omega).$$

(iii) In general, $N^p(\bar{x}, \Omega)$ is a convex cone, but it is not closed (see [[10], p. 240]).

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ be an extended real-valued mapping. Given $\mathscr{C} \subset \mathbb{R}^n$, we define $f_{\mathscr{C}} : \mathbb{R}^n \to \bar{\mathbb{R}}$ by

$$f_{\mathscr{C}}(x) := \begin{cases} f(x) & \text{if } x \in \mathscr{C}, \\ \infty & \text{otherwise.} \end{cases}$$

We respectively denote the *domain*, the *range*, the *graph*, and the *epi-graph* of f by

$$\operatorname{dom} f := \{x \in \mathbb{R}^n \mid f(x) < \infty\}, \ \operatorname{rge} f := \{y \in \mathbb{R} \mid \exists x \in \operatorname{dom} f, f(x) = y\},$$

and

$$\operatorname{gph} f := \{(x, f(x)) \mid x \in \operatorname{dom} f\}, \ \operatorname{epi} f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \operatorname{dom} f, \alpha \geq f(x)\}.$$

Mapping f is called *lower semi-continuous relative to* $\mathscr{C}$ *around* $\bar{x}$ if, for any sequence $x_k \xrightarrow{\mathscr{C}} \bar{x}$, $\liminf_{k\to\infty} f(x_k) \geq f(\bar{x})$. f is called *continuous relative to* $\mathscr{C}$ *at* $\bar{x}$ if $\lim_{k\to\infty} f(x_k) = f(\bar{x})$ for any sequence $x_k \xrightarrow{\mathscr{C}} \bar{x}$. The set of all lower semi-continuous mapping relative to $\mathscr{C}$ around $\bar{x}$ is denoted by $\mathscr{F}_{\mathscr{C}}(\bar{x})$, while $C_{\mathscr{C}}(\bar{x})$ stands for the set of all continuous mapping relative to $\mathscr{C}$ at $\bar{x}$. If $\mathscr{C}$ is a neighborhood of $\bar{x}$, then we write $\mathscr{F}(\bar{x})$ and $C(\bar{x})$ instead of $\mathscr{F}_{\mathscr{C}}(\bar{x})$ and $C_{\mathscr{C}}(\bar{x})$. We say that f is *level-bounded* if the level set $\{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ is bounded (possibly empty) for any $\alpha \in \mathbb{R}$.

Given the multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we denote

$$\operatorname{dom} F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}, \ \operatorname{rge} F := \{y \in \mathbb{R}^m \mid \exists x \in \operatorname{dom} F, y \in F(x)\}$$

and

$$\operatorname{gph} F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\}$$

and call the *domain*, the *range*, and the *graph* of F respectively.

Given $\mathscr{C} \subset \mathbb{R}^n$ and $\mathscr{D} \subset \mathbb{R}^m$, we respectively define the multifunctions $F_{\mathscr{C}}, F^{\mathscr{D}}, F^{\mathscr{D}}_{\mathscr{C}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ as follows:

$$F_{\mathscr{C}}(x) := \begin{cases} F(x) & \text{if } x \in \mathscr{C}, \\ \emptyset & \text{otherwise,} \end{cases}, \quad F^{\mathscr{D}}(x) := F(x) \cap \mathscr{D}$$

and

$$F^{\mathscr{D}}_{\mathscr{C}}(x) := \begin{cases} F(x) \cap \mathscr{D} & \text{if } x \in \mathscr{C}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence,

$$\operatorname{dom} F_{\mathscr{C}} = \operatorname{dom} F \cap \mathscr{C} \ \text{ and } \ \operatorname{gph} F_{\mathscr{C}} = \operatorname{gph} F \cap (\mathscr{C} \times \mathbb{R}^m),$$

and

$$\operatorname{dom} F^{\mathscr{D}} = \operatorname{dom} F \cap F^{-1}(\mathscr{D}) \ \text{ and } \operatorname{gph} F^{\mathscr{D}} = \operatorname{gph} F \cap (\mathbb{R}^n \times \mathscr{D}),$$

while

$$\operatorname{dom} F^{\mathscr{D}}_{\mathscr{C}} := \operatorname{dom} F \cap \mathscr{C} \cap F^{-1}(\mathscr{D}), \ \text{ and } \operatorname{gph} F^{\mathscr{D}}_{\mathscr{C}} = \operatorname{gph} F \cap (\mathscr{C} \times \mathscr{D}).$$

The *epigraphical/profile mapping* of F, denoted by $\mathscr{E}F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, is defined by

$$\mathscr{E}F(x) := F(x) + \mathbb{R}^m_+ \text{ for all } x \in \mathbb{R}^n.$$

If $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a single-valued mapping, then $\operatorname{gph} \mathscr{E}f = \operatorname{epi} f$.

We now recall the concepts of the limiting normal cone to sets, the limiting corderivative of multifunctions, and the limiting subdifferential of single-valued mappings.

Definition 2.2. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $f : \mathbb{R}^n \to \bar{\mathbb{R}}$.

(i) [[10], Definition 3.32]. The multifunction $D^*F(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}), \operatorname{gph} F)\} \text{ for all } y^* \in \mathbb{R}^m$$

is called the *limiting coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$.

(ii) [[10], Definition 1.77]. The *limiting subdifferential* of f at $\bar{x} \in \operatorname{dom} f$ is the following set

$$\partial f(\bar{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, -1) \in N((\bar{x}, f(\bar{x})), \operatorname{epi} f)\}.$$

We now introduce a version relative to a set of the proximal and limiting normal cones as follows.

Definition 2.3. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, and let $\mathscr{C}$ be locally closed around $\bar{x} \in \Omega \cap \mathscr{C}$.

(i) The *proximal normal cone relative to* $\mathscr{C}$ to Ω at $\bar{x}$ is defined by

$$N^p_{\mathscr{C}}(\bar{x}, \Omega) := \left\{x^* \in \mathbb{R}^n \mid \exists t_0 > 0 \text{ such that } \bar{x} + tx^* \in \Pi^{-1}(\bar{x}, \Omega \cap \mathscr{C}) \cap \mathscr{C} \text{ for all } t \in (0, t_0]\right\}.$$

(ii) The *limiting normal cone relative to* $\mathscr{C}$ to Ω at $\bar{x}$ is defined by

$$N_{\mathscr{C}}(\bar{x}, \Omega) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega \cap \mathscr{C}} \bar{x}} N^p_{\mathscr{C}}(x, \Omega).$$

Remark 2.4. In the case that $\mathscr{C}$ is convex, the normal cones relative to $\mathscr{C}$ in Definition 2.3 reduce to those ones in [[19], Definition 5] respectively.

Some properties of the proximal and limiting normal cones relative to a set are presented in the following proposition.

Proposition 2.5. *Let $\Omega, \mathscr{C} \subset \mathbb{R}^s$ be nonempty locally closed around $\bar{x} \in \Omega \cap \mathscr{C}$. The following assertions hold:*

(i)

$$N^p_{\mathscr{C}}(\bar{x}, \Omega) = \Bigg\{ x^* \mid \exists t_0 > 0 \text{ such that } \bar{x} + tx^* \in \mathscr{C}, \text{ and } \langle x^*, x - \bar{x} \rangle \le \frac{1}{2t}\|x - \bar{x}\|^2 \text{ for all } x \in \Omega \cap \mathscr{C} \text{ and } t \in (0, t_0] \Bigg\}. \tag{2.1}$$

(ii) *The proximal normal cone with respect to a set can be computed by*

$$N^p_{\mathscr{C}}(\bar{x}, \Omega) = N^p(\bar{x}, \Omega \cap \mathscr{C}) \cap \mathscr{R}(\bar{x}, \mathscr{C}). \tag{2.2}$$

Consequently,

$$N^p_{\mathscr{C}}(\bar{x}, \Omega) = \Big\{ x^* \mid \exists \theta, \delta, p > 0 \text{ such that } \bar{x} + px^* \in \mathscr{C} \text{ and } \langle x^*, x - \bar{x} \rangle \le \delta \|x - \bar{x}\|^2 \ \forall x \in \Omega \cap \mathscr{C} \cap \mathbb{B}(\bar{x}, \theta) \Big\}. \tag{2.3}$$

If, in addition, $\mathscr{C}$ is convex, then

$$N^p_{\mathscr{C}}(\bar{x},\Omega) := \left\{x^* \in \mathbb{R}^n \mid \exists p > 0 \text{ such that } \bar{x} + px^* \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{C}\right\}. \tag{2.4}$$

(iii) *$N^p_{\mathscr{C}}(\bar{x},\Omega)$ is a cone. In addition, if $\mathscr{C}$ is convex, then $N^p_{\mathscr{C}}(\bar{x},\Omega)$ is convex. In this case, we have*

$$N^p_{\mathscr{C}}(\bar{x},\Omega) = \left\{x^* \in \mathbb{R}^n \mid \exists t > 0 : \bar{x} + tx^* \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{C}\right\} \tag{2.5}$$

$$= \left\{x^* \mid \exists t > 0 : \bar{x} + tx^* \in \mathscr{C} \text{ and } \langle x^*, x-\bar{x}\rangle \le \frac{1}{2t}\|x-\bar{x}\|^2 \text{ for all } x \in \Omega\cap\mathscr{C}\right\}. \tag{2.6}$$

(iv) *$N_{\mathscr{C}}(\bar{x},\Omega)$ is a closed cone.*

(v) $N_{\mathscr{C}}(\bar{x},\Omega) \subset \operatorname*{Lim\,sup}_{x\xrightarrow{\mathscr{C}}\bar{x}} (\text{cone}\,[x-\Pi(x,\Omega\cap\mathscr{C})])$ *. If, in addition, $\mathscr{C}$ is convex, then*

$$N_{\mathscr{C}}(\bar{x},\Omega) = \operatorname*{Lim\,sup}_{x\xrightarrow{\mathscr{C}}\bar{x}} (\text{cone}\,[x-\Pi(x,\Omega\cap\mathscr{C})]) \tag{2.7}$$

$$= \operatorname*{Lim\,sup}_{\substack{\epsilon\to 0^+\\ x\xrightarrow{\mathscr{C}\cap\Omega}\bar{x}}} \hat{N}^\epsilon_{\mathscr{C}}(x,\Omega), \tag{2.8}$$

where $\hat{N}^\epsilon_{\mathscr{C}}(x,\Omega) := \hat{N}_\epsilon(x,\Omega\cap\mathscr{C})\cap\mathscr{R}(x,\mathscr{C})$.

Proof. (i) Fixing $x^* \in N^p_{\mathscr{C}}(\bar{x},\Omega)$, one sees that there exists $t_0 > 0$ such that $\bar{x} + tx^* \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{C}$ for all $t \in (0,t_0]$. Thus, for any $x \in \Omega\cap\mathscr{C}$ and $t \in (0,t_0)$, we have $\|\bar{x}-\bar{x}-tx^*\|^2 \le \|x-\bar{x}-tx^*\|^2$, which implies that

$$t^2\|x^*\|^2 \le \|x-\bar{x}\|^2 - 2t\langle x^*, x-\bar{x}\rangle + t^2\|x^*\|^2.$$

This means that

$$\langle x^*, x-\bar{x}\rangle \le \frac{1}{2t}\|x-\bar{x}\|^2 \text{ for all } x \in \mathscr{C}\cap\Omega \text{ and } t \in (0,t_0].$$

Otherwise, let $x^* \in \mathbb{R}^s$ and $t_0 > 0$ satisfy $\bar{x} + tx^* \in \mathscr{C}$ and $\langle x^*, x-\bar{x}\rangle \le \frac{1}{2t}\|x-\bar{x}\|^2$ for all $x \in \Omega\cap\mathscr{C}$ and $t \in (0,t_0]$. Then

$$t^2\|x^*\|^2 \le \|x-\bar{x}\|^2 - 2t\langle x^*, x-\bar{x}\rangle + t^2\|x^*\|^2,$$

which follows that $\|\bar{x}-\bar{x}-tx^*\|^2 \le \|x-\bar{x}-tx^*\|^2$ for all $x \in \mathscr{C}\cap\Omega$. This implies that $\bar{x}+tx^* \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})$ for all $t \in (0,t_0]$. Combining this with $\bar{x}+tx^* \in \mathscr{C}$ for all $t \in (0,t_0]$, we have $x^* \in N^p_{\mathscr{C}}(\bar{x},\Omega)$.

(ii) Let $y^* \in N^p_{\mathscr{C}}(\bar{x},\Omega)$. By the definition, we find $t_0 > 0$ such that $\bar{x}+ty^* \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{C}$ for all $t \in (0,t_0]$, which implies from the definition of proximal normal cones that $y^* \in N^p(\bar{x},\Omega\cap\mathscr{C})$. Hence, $y^* \in N^p(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{R}(\bar{x},\mathscr{C})$, which means that

$$N^p_{\mathscr{C}}(\bar{x},\Omega) \subset N^p(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{R}(\bar{x},\mathscr{C}).$$

To prove the opposite inclusion, we take $y^* \in N^p(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{R}(\bar{x},\mathscr{C})$. There exist by the definition $t_0 > 0, t_1 > 0$ such that $\bar{x}+ty^* \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})$ for any $t \in (0,t_0]$ and $\bar{x}+\tilde{t}y^* \in \mathscr{C}$ for any $\tilde{t} \in (0,t_1]$. Picking $\bar{t} := \min\{t_0,t_1\}$, we see that $\bar{x}+ty^* \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{C}$ for all $t \in (0,\bar{t}]$, which derives that $y^* \in N^p_{\mathscr{C}}(\bar{x},\Omega)$. Therefore,

$$N^p(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{R}(\bar{x},\mathscr{C}) \subset N^p_{\mathscr{C}}(\bar{x},\Omega).$$

If $\mathscr{C}$ is convex, then $\mathscr{R}(\bar{x},\mathscr{C}) = \{u \in \mathbb{R}^s \mid \exists p > 0 \text{ such that } \bar{x} + pu \in \mathscr{C}\}$. This combines with (2.2) to give us (2.4).

(iii) It is easy to see from the definition that $N^p_{\mathscr{C}}(\bar{x},\Omega)$ is a cone and the inclusion "$\subset$" in (2.5) holds. We next show the inclusion "$\supset$" in the case that $\mathscr{C}$ is convex. Take $u \in \{x^* \in \mathbb{R}^n \mid \exists t > 0 : \bar{x} + tx^* \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{C}\}$. If $u = 0$, then $u \in N^p_{\mathscr{C}}(\bar{x},\Omega)$. If $u \neq 0$, then there exists $t_0 > 0$ such that $\bar{x} + t_0 u \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{C}$. For any $t \in (0,t_0]$, we have $\bar{x} + tu = (1 - \frac{t}{t_0})\bar{x} + \frac{t}{t_0}(\bar{x} + t_0 u) \in \mathscr{C}$. Moreover, it implies from $\bar{x} + t_0 u \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})$ that $\bar{x} + tu \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})$ for all $t \in (0,t_0]$, which follows that $\bar{x} + tu \in \Pi^{-1}(\bar{x},\Omega\cap\mathscr{C})\cap\mathscr{C}$ for all $t \in (0,t_0]$. Thus we have $u \in N^p_{\mathscr{C}}(\bar{x},\Omega)$ and hence the inclusion "$\supset$" holds.

(iv) We first prove that $N_{\mathscr{C}}(\bar{x},\Omega)$ is a cone. Take $x^* \in N_{\mathscr{C}}(\bar{x},\Omega)$ and $\lambda > 0$. There exist by the definition sequences $x_k \xrightarrow{\mathscr{C}\cap\Omega} \bar{x}$ and $x_k^* \to x^*$ such that $x_k^* \in N^p_{\mathscr{C}}(x_k,\Omega)$ for all $k \in \mathbb{N}$, which implies that $\lambda x_k^* \in N^p_{\mathscr{C}}(x_k,\Omega)$ for all $k \in \mathbb{N}$ due to the conical property of $N^p_{\mathscr{C}}(x_k,\Omega)$ (see (iii)). Since $\lambda x_k^* \to \lambda x^*$, we obtain $\lambda x^* \in N_{\mathscr{C}}(\bar{x},\Omega)$. Hence, $N_{\mathscr{C}}(\bar{x},\Omega)$ is a cone. Now, we prove that $N_{\mathscr{C}}(\bar{x},\Omega)$ is closed. To see this, we take a sequence $(x_k^*) \subset N_{\mathscr{C}}(\bar{x},\Omega)$ satisfying $x_k^* \to x^*$. We demonstrate that $x^* \in N_{\mathscr{C}}(\bar{x},\Omega)$ also. Indeed, if there exists $k \in \mathbb{N}$ satisfying $x_k^* = x^*$, then $x^* \in N(\bar{x},\Omega)$. Thus we only need to prove this assertion in the case that $x_k^* \neq x^*$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, there exist, by the definition, sequences $x_{k_m} \xrightarrow{\Omega\cap\mathscr{C}} \bar{x}, x_{k_m}^* \to x_k^*$ such that $x_{k_m}^* \in N^p_{\mathscr{C}}(x_{k_m},\Omega)$. Since $x_{k_m}^* \to x_k^*$ and $x_k^* \neq x^*$, one sees that there exists, for each $k \in \mathbb{N}$, $k_{m_0} \in \mathbb{N}$ such that $\|x_{k_m}^* - x_k^*\| \leq \|x_k^* - x^*\|$ for all $k_m \geq k_{m_0}$. For each $k \in \mathbb{N}$, we take $\tilde{x}_k := x_{k_{m_0}}$ and $\tilde{x}_k^* := x_{k_{m_0}}^*$. Then $\tilde{x}_k \xrightarrow{\Omega\cap\mathscr{C}} \bar{x}$, $\tilde{x}_k^* \in N^p_{\mathscr{C}}(\tilde{x}_k,\Omega)$, and

$$\|\tilde{x}_k^* - x^*\| \leq \|\tilde{x}_k^* - x_k^*\| + \|x_k^* - x^*\| = \|x_{k_{m_0}}^* - x_k^*\| + \|x_k^* - x^*\| \leq 2\|x_k^* - x^*\| \to 0.$$

This gives us the assertion that $x^* \in N_{\mathscr{C}}(\bar{x},\Omega)$, so $N_{\mathscr{C}}(\bar{x},\Omega)$ is closed.

(v) Let $x^* \in N_{\mathscr{C}}(\bar{x},\Omega)$. There exist, by the definition, sequences $x_k \xrightarrow{\mathscr{C}\cap\Omega} \bar{x}$ and $x_k^* \to x^*$ such that $x_k^* \in N^p_{\mathscr{C}}(x_k,\Omega)$ for all $k \in \mathbb{N}$. This follows that, for each $k \in \mathbb{N}$, there exists $t_k > 0$, which can be assumed to converge to 0, such that $\tilde{x}_k := x_k + t_k x_k^* \in \Pi^{-1}(x_k,\Omega\cap\mathscr{C})\cap\mathscr{C}$, which means that

$$x_k^* = \frac{\tilde{x}_k - x_k}{t_k} \in \text{cone}\,[\tilde{x}_k - \Pi(\tilde{x}_k,\Omega\cap\mathscr{C})].$$

Thus $x^* \in \operatorname*{Lim\,sup}_{x \xrightarrow{\mathscr{C}} \bar{x}} [\text{cone}\,(x - \Pi(x,\Omega\cap\mathscr{C}))]$, which gives us that

$$N_{\mathscr{C}}(\bar{x},\Omega) \subset \operatorname*{Lim\,sup}_{x \xrightarrow{\mathscr{C}} \bar{x}} [\text{cone}\,(x - \Pi(x,\Omega\cap\mathscr{C}))]\,.$$

If, in addition, $\mathscr{C}$ is convex, then, for any $\epsilon > 0$ small enough, $\mathscr{C}\cap\mathbb{B}(\bar{x},\epsilon)$ is a closed and convex set. Taking $x^* \in \operatorname*{Lim\,sup}_{x \xrightarrow{\mathscr{C}} \bar{x}} [\text{cone}\,(x - \Pi(x,\Omega\cap\mathscr{C}))]$, one sees that there exist sequences $x_k \xrightarrow{\mathscr{C}\cap\mathbb{B}(\bar{x},\epsilon)} \bar{x}$, $x_k^* \to x^*$ such that $x_k^* \in \text{cone}\,[x_k - \Pi(x_k,\Omega\cap\mathscr{C})]$. If $x_k^* = 0$, then $x_k^* \in N^p_{\mathscr{C}}(x_k,\Omega)$. If $x_k^* \neq 0$, then one sees from the fact that $x_k^* \in \text{cone}\,[x_k - \Pi(x_k,\Omega\cap\mathscr{C})]$ that there exist $\tilde{x}_k \in \Pi(x_k,\Omega\cap\mathscr{C})$ and $\alpha_k > 0$ such that $x_k^* = \alpha_k(x_k - \tilde{x}_k)$. Hence, $\tilde{x}_k + \frac{1}{\alpha_k}x_k^* = x_k \in \Pi^{-1}(\tilde{x}_k,\Omega\cap\mathscr{C})\cap\mathscr{C}$, which implies that $x_k^* \in N^p_{\mathscr{C}}(\tilde{x}_k,\Omega)$ due to (2.5). On the other hand, since $x_k \in \mathbb{B}(\bar{x},\epsilon)$ and $x_k \in \Pi^{-1}(\tilde{x}_k,\Omega\cap\mathscr{C})$, one has $\|x_k - \tilde{x}_k\| \leq \|x_k - \bar{x}\| \leq \epsilon$. This implies, by the triangular inequality, that $\|\tilde{x}_k - \bar{x}\| \leq 2\|x_k - \bar{x}\| \leq 2\epsilon$, which converges to 0 as $\epsilon \to 0$. Thus $x^* \in N_{\mathscr{C}}(\bar{x},\Omega)$, which means that

$$N_{\mathscr{C}}(\bar{x},\Omega) \supset \operatorname*{Lim\,sup}_{x \xrightarrow{\mathscr{C}} \bar{x}} [\text{cone}\,(x - \Pi(x,\Omega\cap\mathscr{C})]\,.$$

Thus we obtain equation (2.7). To prove (2.8), it is sufficient to demonstrate inclusion "$\supset$" because inclusion "$\subset$" holds due to the relation $N^p_{\mathscr{C}}(x,\Omega) \subset \hat{N}^\epsilon_{\mathscr{C}}(x,\Omega)$ for all $x \in \Omega$ and $\epsilon \geq 0$. Let

$$x^* \in \operatorname*{Lim\,sup}_{\substack{\epsilon \to 0^+ \\ x \xrightarrow{\mathscr{C}\cap\Omega} \bar{x}}} \hat{N}^\epsilon_{\mathscr{C}}(x,\Omega).$$

There exist sequences $\epsilon_k \to 0^+$, $x_k \xrightarrow{\mathscr{C}\cap\Omega} \bar{x}$ and $x_k^* \to x^*$ such that $x_k^* \in \hat{N}^{\epsilon_k}_{\mathscr{C}}(x_k,\Omega)$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, one sees that there exists $p_k > 0$ such that $x_k + p_k x_k^* \in \mathscr{C}$. Since $x_k^* \in \hat{N}^{\epsilon_k}_{\mathscr{C}}(x_k,\Omega) \subset \hat{N}_{\epsilon_k}(x_k, \Omega \cap \mathscr{C})$, one has that there exists $\delta_k > 0$ such that

$$\langle x_k^*, x - x_k \rangle \leq 2\epsilon_k \|x - x_k\| \ \forall x \in \Omega \cap \mathscr{C} \cap \mathbb{B}(x_k, \delta_k). \tag{2.9}$$

Pick $\alpha_k \in (0, p_k)$ satisfying $\alpha_k \|x_k^*\| < \frac{1}{2}\delta_k$ and $\alpha_k \to 0$. According to the convexity of $\mathscr{C}$ and $x_k, x_k + p_k x_k^* \in \mathscr{C}$, we have

$$x_k + \alpha_k x_k^* = (1 - \frac{\alpha_k}{p_k})x_k + \frac{\alpha_k}{p_k}(x_k + p_k x_k^*) \in \mathscr{C}$$

Taking $w_k \in \Pi(x_k + \alpha_k x_k^*, \Omega \cap \mathscr{C})$, one has the following assertion

$$\|w_k - x_k\|^2 - 2\alpha_k \langle x_k^*, w_k - x_k \rangle + \alpha_k^2 \|x_k^*\|^2 = \|w_k - x_k - \alpha_k x_k^*\|^2 \leq \alpha_k^2 \|x_k^*\|^2$$

which is equivalent to

$$\|w_k - x_k\|^2 \leq 2\alpha_k \langle x_k^*, w_k - x_k \rangle. \tag{2.10}$$

This implies from $w_k \in \Pi(x_k + \alpha_k x_k^*, \Omega \cap \mathscr{C})$ that $w_k \in \Omega \cap \mathscr{C}$ and

$$\|w_k - x_k - \alpha_k x_k^*\| \leq \|x_k - x_k - \alpha_k x_k^*\| = \alpha_k \|x_k^*\|.$$

This follows that $\|w_k - x_k\| \leq 2\alpha_k \|x_k^*\| < \delta_k$, so $w_k \in \mathbb{B}(x_k, \delta_k)$. Taking (2.9) into account, we have $\langle x_k^*, w_k - x_k \rangle \leq 2\epsilon_k \|w_k - x_k\|$. Combining this and (2.10), we obtain $\|w_k - x_k\| \leq 4\epsilon_k \alpha_k$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, we set $w_k^* := x_k^* + \frac{1}{\alpha_k}(x_k - w_k)$. Then $w_k + \alpha_k w_k^* = x_k + \alpha_k x_k^* \in \mathscr{C}$ and

$$\|w_k^* - x^*\| \leq \|x_k^* - x^*\| + \|w_k^* - x_k^*\| = \|x_k^* - x^*\| + \frac{1}{\alpha_k}\|x_k - w_k\| \leq \|x_k^* - x^*\| + 4\epsilon_k.$$

Hence $w_k^* \to x^*$ as $k \to \infty$. On the other hand, for any $x \in \Omega \cap \mathscr{C}$, we have

$$\begin{aligned} 0 &\leq \|x_k + \alpha_k x_k^* - x\|^2 - \|x_k + \alpha_k x_k^* - w_k\|^2 \\ &= 2\langle x_k + \alpha_k x_k^* - w_k, w_k - x \rangle + \|w_k - x\|^2 \\ &= 2\alpha_k \langle w_k^*, w_k - x \rangle + \|w_k - x\|^2, \end{aligned}$$

which implies that

$$\langle w_k^*, x - w_k \rangle \leq \frac{1}{2\alpha_k}\|w_k - x\|^2 \ \forall x \in \mathscr{C} \cap \Omega.$$

It results that $w_k^* \in N^p(w_k, \Omega \cap \mathscr{C})$ and then $w_k^* \in N^p_{\mathscr{C}}(w_k, \Omega)$ due to $w_k + \alpha_k w_k^* \in \mathscr{C}$. It follows that $x^* \in N_{\mathscr{C}}(\bar{x}, \Omega)$. Therefore,

$$\operatorname*{Lim\,sup}_{\substack{\epsilon \to 0^+ \\ x \xrightarrow{\mathscr{C}\cap\Omega} \bar{x}}} \hat{N}^{\epsilon}_{\mathscr{C}}(x, \Omega) \subset N_{\mathscr{C}}(\bar{x}, \Omega).$$

Hence, the proof is completed. □

3. Metric regularity and locally Lipschitz continuities relative to a set of mappings

We first, in this section, present the notions of the metric regularity and Aubin property relative to a set of set-valued mappings as follows. In what follow, we always assume that sets are nonempty.

Definition 3.1. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Let $\mathscr{C} \subset \mathbb{R}^n$ and $\mathscr{D} \subset \mathbb{R}^m$ be locally closed around $\bar{x}$ and $\bar{y}$, respectively, and let $\operatorname{gph} F^{\mathscr{D}}_{\mathscr{C}}$ be a locally closed around $(\bar{x}, \bar{y})$.

(i) F is called to have the *Aubin property (*or *local Lipschitz-like property) relative to* $\mathscr{C} \times \mathscr{D}$ around $(\bar{x}, \bar{y})$ if there exist $\kappa > 0$ and neighborhoods V of $\bar{y}$ and U of $\bar{x}$ such that

$$F^{\mathscr{D}}(u) \cap V \subset F^{\mathscr{D}}(x) + \kappa\|u - x\|\mathbb{B} \quad \forall x, u \in \mathscr{C} \cap U. \tag{3.1}$$

The above κ is called a *Aubin modulus relative to* $\mathscr{C} \times \mathscr{D}$ of F around $(\bar{x}, \bar{y})$. The infimum of all of Aubin modulus relative to $\mathscr{C} \times \mathscr{D}$ of F around $(\bar{x}, \bar{y})$ is called the *exact Aubin modulus relative to* $\mathscr{D}$ of F around $(\bar{x}, \bar{y})$ and denoted by $\operatorname{lip}^{\mathscr{D}}_{\mathscr{C}} F(\bar{x}, \bar{y})$. If $\mathscr{C} = \operatorname{dom} F$ and $\mathscr{D} = \mathbb{R}^m$, then we say that F has the *relative Aubin property* around $(\bar{x}, \bar{y})$.

(ii) F is called to satisfy the *metric regularity relative to* $\mathscr{C} \times \mathscr{D}$ around $(\bar{x}, \bar{y})$ if there exist $\kappa > 0$ and neighborhoods V of $\bar{y}$ and U of $\bar{x}$ such that

$$d(x, F^{-1}(y) \cap \mathscr{C}) \leq \kappa d(y, F(x) \cap \mathscr{D}) \; \forall x \in U \cap \mathscr{C}, y \in V \cap \mathscr{D}.$$

The above κ is called a *regular modulus relative to* $\mathscr{C} \times \mathscr{D}$ of F around $(\bar{x}, \bar{y})$. The infimum of all of regular modulus relative to $\mathscr{C} \times \mathscr{D}$ of F around $(\bar{x}, \bar{y})$ is called the *exact regular modulus relative to* $\mathscr{C} \times \mathscr{D}$ of F around $(\bar{x}, \bar{y})$ and denoted by $\operatorname{reg}^{\mathscr{D}}_{\mathscr{C}} F(\bar{x}, \bar{y})$. If $\mathscr{D} = \operatorname{reg} F$ and $\mathscr{C} = \mathbb{R}^n$, then F is called to be *metrically regular relative to the image* around $(\bar{x}, \bar{y})$.

Remark 3.2. (i) In Definition 3.1 (i), the Aubin property relative to $\mathscr{C} \times \mathbb{R}^m$ coincides with the Aubin property relative to $\mathscr{C}$ in [[20], Definition 9.36]. In addition, F satisfies the Aubin property relative to $\mathscr{C} \times \mathscr{D}$ is equivalent to the fact that $F^{\mathscr{D}}$ satisfies the Aubin property relative to $\mathscr{C}$ in sense of [[20], Definition 9.36].

(ii) The metric regularity relative to a set in Definition 3.1 (ii) is different from that in [8], which satisfies

$$d(x, F^{-1}(y) \cap \mathscr{C}) \leq \kappa d(y, F(x)) \; \forall x \in U \cap \mathscr{C}, y \in V \cap \mathscr{D}.$$

However, in the case that $\mathscr{D} := \operatorname{rge} F$, they are equivalent and reduce to the metric regularity relative to the image around $(\bar{x}, \bar{y})$.

(iii) The metric regularity relative to a set in Definition 3.1 (ii), which does not need the assumption $\mathscr{C} = F^{-1}(\mathscr{D})$, are more general than that one in [[22], Theorem 3.8]. In the case of $\mathscr{C} = F^{-1}(\mathscr{D})$, they are the same.

It is known that the Aubin property of a set-valued mapping is equivalent to the metric regularity of its inverse mapping (*cf.* [10, 20]). The similar result for the Aubin property and metric regularity relative to a set is presented in the following theorem.

Theorem 3.3. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, and let $\mathscr{C}, \mathscr{D}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $(\bar{x}, \bar{y}) \in \text{gph} F_{\mathscr{C}}^{\mathscr{D}}$. Assume that $\mathscr{C} \times \mathscr{D}$ and $\text{gph} F_{\mathscr{C}}^{\mathscr{D}}$ are locally closed sets around $(\bar{x}, \bar{y})$. For the following assertions:*
(i) *F^{-1} is metrically regular relative to $\mathscr{D} \times \mathscr{C}$ around $(\bar{y}, \bar{x})$ with the modulus τ;*
(ii) *F satisfies the Aubin property relative to $\mathscr{C} \times \mathscr{D}$ around $(\bar{x}, \bar{y})$ with the modulus τ;*
(iii) *There exist neighborhoods V of $\bar{y}$ and U of $\bar{x}$ and a positive number $\gamma > 0$ such that*

$$d(y, F(x) \cap \mathscr{D}) \leq \tau d(x, F^{-1}(y) \cap \mathscr{C}) \ \forall x \in U \cap \mathscr{C}, y \in V \cap \mathscr{D} \text{ with } d(x, F^{-1}(y) \cap \mathscr{C}) \leq \gamma,$$

it holds that (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii). *In addition, if $\mathscr{C} \subset F^{-1}(\mathscr{D})$, then* (ii) $\Rightarrow$ (iii).

Proof. "(iii) $\Rightarrow$ (i)". Take $\tilde{U} := U \cap \mathbb{B}(\bar{x}, \nu)$ and $\tilde{V} := V \cap \mathbb{B}(\bar{y}, \nu)$ with $\nu := \frac{\gamma\tau}{\tau+1}$. Then $\tilde{U}, \tilde{V}$ are neighborhoods of $\bar{x}, \bar{y}$, respectively. Moreover, we have $x \in U \cap \mathscr{C}$ and $y \in V \cap \mathscr{D}$ whenever $x \in \tilde{U} \cap \mathscr{C}$ and $y \in \tilde{V} \cap \mathscr{D}$. Since $\|x - \bar{x}\| \leq \nu < \gamma$ for any $x \in \tilde{U}$, we achieve $d(x, F^{-1}(\bar{y}) \cap \mathscr{C}) < \gamma$. From (iii), we have $d(\bar{y}, F(x) \cap \mathscr{D}) \leq \tau d(x, F^{-1}(\bar{y}) \cap \mathscr{C})$ for any $x \in \tilde{U} \cap \mathscr{C}$. To prove (i), it is sufficient to show that

$$d(y, F(x) \cap \mathscr{D}) \leq \tau d(x, F^{-1}(y) \cap \mathscr{C}) \ \forall y \in \tilde{V} \cap \mathscr{D}, x \in \tilde{U} \cap \mathscr{C} \text{ with } d(x, F^{-1}(y) \cap \mathscr{C}) > \gamma.$$

Indeed, taking arbitrarily $y \in \tilde{V} \cap \mathscr{D}$ and $x \in \tilde{U} \cap \mathscr{C}$ with $d(x, F^{-1}(y) \cap \mathscr{C}) > \gamma$, we have

$$\begin{aligned} d(y, F(x) \cap \mathscr{D}) &\leq d(\bar{y}, F(x) \cap \mathscr{D}) + \|y - \bar{y}\| \\ &\leq \tau d(x, F^{-1}(\bar{y}) \cap \mathscr{C}) + \|y - \bar{y}\| \\ &\leq \tau \|x - \bar{x}\| + \|y - \bar{y}\| \\ &\leq \nu(\tau + 1) \\ &\leq \tau\gamma < \tau d(x, F^{-1}(y) \cap \mathscr{C}). \end{aligned}$$

Thus (i) holds.

"(i) $\Rightarrow$ (ii)". Let F^{-1} be metrically regular relative to $\mathscr{D} \times \mathscr{C} \subset \mathbb{R}^n \times \mathbb{R}^m$ around $(\bar{y}, \bar{x})$ with modulus τ. Then there exist neighborhoods V of $\bar{y}$ and U of $\bar{x}$ such that

$$\tau d(y, F(x) \cap \mathscr{D}) \leq d(x, F^{-1}(y) \cap \mathscr{C}) \ \forall y \in V \cap \mathscr{D}, x \in U \cap \mathscr{C}. \tag{3.2}$$

Pick $\epsilon > 0$ arbitrarily. For any $x, u \in U \cap \mathscr{C}$, we consider the following two cases.

Case 1. $F^{\mathscr{D}}(u) \cap V = \emptyset$. In this case, it is clearly that $F^{\mathscr{D}}(u) \cap V \subset F^{\mathscr{D}}(x) + \tau\|x - u\|\mathbb{B}$.

Case 2. $F^{\mathscr{D}}(u) \cap V \neq \emptyset$. For any $y \in F^{\mathscr{D}}(u) \cap V$, we have $u \in F^{-1}(y) \cap \mathscr{C}$. According to (3.2), we derive that

$$d(y, F^{\mathscr{D}}(x)) \leq \tau d(x, F^{-1}(y) \cap \mathscr{C}) \leq \tau\|x - u\|,$$

which implies that $F^{\mathscr{D}}(u) \cap V \subset F^{\mathscr{D}}(x) + (\tau + \epsilon)\|x - u\|\mathbb{B}$. Passing to the limit as $\epsilon \to 0^+$, we have

$$F^{\mathscr{D}}(u) \cap V \subset F^{\mathscr{D}}(x) + \tau\|x - u\|\mathbb{B},$$

F satisfies the Aubin property relative to $\mathscr{C} \times \mathscr{D}$ around $(\bar{x}, \bar{y})$ with modulus τ.

Let $\mathscr{C} \subset F^{-1}(\mathscr{D})$. We next prove "(ii) $\Rightarrow$ (iii)". Assume that F satisfies the Aubin property relative to $\mathscr{C} \times \mathscr{D}$ around $(\bar{x}, \bar{y})$ with a modulus $\tau > 0$. We find neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F^{\mathscr{D}}(u) \cap V \subset F^{\mathscr{D}}(x) + \tau\|x - u\|\mathbb{B} \ \text{ for any } x, u \in U \cap \mathscr{C}. \tag{3.3}$$

Since $\mathscr{C} \subset F^{-1}(\mathscr{D})$, we have $F^{\mathscr{D}}(x) \neq \emptyset$ for all $x \in \mathscr{C}$. Taking (3.3) into account, we obtain

$$d(y, F^{\mathscr{D}}(x)) \leq \tau\|u - x\| \ \forall x, u \in U \cap \mathscr{C} \text{ and } y \in F^{\mathscr{D}}(u) \cap V.$$

Let $\theta > 0$ satisfy $\bar{x} + \theta\mathbb{B} \subset U$, and put $\tilde{U} := \bar{x} + \frac{\theta}{2}\mathbb{B}$. Let $\frac{\theta}{6} > \epsilon > 0$. For any $x \in \tilde{U} \cap \mathscr{C}, y \in V \cap \mathscr{D}$ satisfying $d(x, F^{-1}(y) \cap \mathscr{C}) \leq \frac{\theta}{3}$, we find $u \in F^{-1}(y) \cap \mathscr{C}$ such that

$$\|x - u\| \leq d(x, F^{-1}(y) \cap \mathscr{C}) + \epsilon < \frac{\theta}{3} + \frac{\theta}{6} = \frac{\theta}{2},$$

which implies that $u \in U \cap \mathscr{C}$. Hence,

$$d(y, F^{\mathscr{D}}(x)) \leq \tau\|x - u\| \leq \tau(d(x, F^{-1}(y) \cap \mathscr{C}) + \epsilon) = \tau d(x, F^{-1}(y) \cap \mathscr{C}) + \tau\epsilon.$$

Passing to the limit as $\epsilon \to 0^+$ yields $d(y, F(x) \cap \mathscr{D}) \leq \tau d(x, F^{-1}(y) \cap \mathscr{C})$ for all $x \in \tilde{U} \cap \mathscr{C}$ and $y \in V \cap \mathscr{D}$. □

We now consider the stability of single-valued mappings. Note that the *local Lipschitz continuity relative to a set* of single-valued mappings was presented in [[20], Definition 9.1] (as *strict continuity relative to a set*) as follows.

Definition 3.4 ([20], Definition 9.1 (b))**.** Consider the mapping $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in \operatorname{dom} f$. Let $\mathscr{C} \subset \mathbb{R}^n$ be locally closed around $\bar{x}$ and satisfy $f \in \mathscr{F}_{\mathscr{C}}(\bar{x})$. Then f is said to be *locally Lipschitz continuous relative to* $\mathscr{C}$ around $\bar{x}$ if

$$\operatorname{lip}_{\mathscr{C}} f(\bar{x}) := \limsup_{\substack{x,u \xrightarrow{\mathscr{C}} \bar{x} \\ x \neq u}} \frac{|f(x) - f(u)|}{\|x - u\|} < \infty. \tag{3.4}$$

Here, $\operatorname{lip}_{\mathscr{C}} f(\bar{x})$ is called the *exact Lipschitzian constant relative to* $\mathscr{C}$ of f around $\bar{x}$. If (3.4) holds with $\mathscr{C} = \operatorname{dom} f$, then f is said to be *relatively locally Lipschitz continuous* around $\bar{x}$.

Remark 3.5. (i) It is clearly that the indicator function δ_Ω with a nonempty closed set $\Omega \in \mathbb{R}^n$ is a locally Lipschitz continuous relative to Ω at any bounded point of Ω, but it is not continuous at these points.

(ii) It is known (*cf.* [[18], p. 457]) that f is locally Lipschitz continuous relative to $\mathscr{C}$ around $\bar{x}$ if and only if $\mathscr{E}f$ satisfies the Aubin property relative to $\mathscr{C} \times \mathbb{R}$ around $(\bar{x}, f(\bar{x}))$. A differently equivalent characteristic of the locally Lipschitz continuity relative to a set of single-valued mappings was given in Proposition 3.6.

Proposition 3.6. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in \operatorname{dom} f$. Let $\mathscr{C} \subset \mathbb{R}^n$ be locally closed around $\bar{x}$ and satisfy $f \in \mathscr{F}_{\mathscr{C}}(\bar{x})$. Then f is locally Lipschitz continuous relative to $\mathscr{C}$ around $\bar{x}$ if and only if the set-valued mapping $F_f : \mathbb{R}^n \rightrightarrows \mathbb{R}$ defined by*

$$F_f(x) = \begin{cases} \{f(x)\} & \text{if } x \in \operatorname{dom} f, \\ \emptyset & \text{otherwise} \end{cases} \tag{3.5}$$

satisfies the Aubin property relative to $\mathscr{C} \times \mathbb{R}$ around $(\bar{x}, f(\bar{x}))$.

Proof. Let F_f satisfy the Aubin property relative to $\mathscr{C} \times \mathbb{R}$ around $(\bar{x}, f(\bar{x}))$, which means that there exist $\tau > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F_f(x) \cap V \subset F_f(u) + \tau\|u - x\|\mathbb{B}, \ \forall x, u \in \mathscr{C} \cap U. \tag{3.6}$$

Taking $x = \bar{x}$ in (3.6), we have $\{f(\bar{x})\} \subset \{f(u)\} + \tau\|u - \bar{x}\|\mathbb{B}$, which implies that $\|f(u) - f(\bar{x})\| \le \tau\|u - \bar{x}\|$, $\forall u \in \mathscr{C} \cap U$. Thus we can find a neighborhood $\tilde{U}$ of $\bar{x}$ such that $f(u) \in V$ for all $u \in \mathscr{C} \cap U \cap \tilde{U}$. Set $\bar{U} := U \cap \tilde{U}$. Then $\bar{U}$ is a neighborhood of $\bar{x}$ and $f(u) \in V$ for all $u \in \mathscr{C} \times \bar{U}$. In view of (3.6), one has $F_f(x) \subset F_f(u) + \tau\|x - u\|\mathbb{B}$, $\forall x, u \in \mathscr{C} \cap \bar{U}$, which is equivalent to $|f(x) - f(u)| \le \tau\|u - x\|$ for all $x, u \in \mathscr{C} \cap \bar{U}$. It follows that

$$\limsup_{\substack{x,u \xrightarrow{\mathscr{C}} \bar{x} \\ x \neq u}} \frac{|f(x) - f(u)|}{\|x - u\|} < \infty,$$

so f is locally Lipschitz continuous relative to $\mathscr{C}$ around $\bar{x}$. Let f be locally Lipschitz continuous relative to $\mathscr{C}$ around $\bar{x}$, which means that

$$0 \le \tau := \limsup_{\substack{x,u \xrightarrow{\mathscr{C}} \bar{x} \\ x \neq u}} \frac{|f(x) - f(u)|}{\|x - u\|} < \infty.$$

For any $\epsilon > 0$, one sees that there exists $\delta > 0$ such that

$$\frac{|f(x) - f(u)|}{\|x - u\|} \le \tau + \epsilon, \ \forall x \neq u \in \mathscr{C} \cap \mathbb{B}(\bar{x}, \delta). \tag{3.7}$$

Taking $x = \bar{x}$ in (3.7), we have $|f(\bar{x}) - f(u)| \le (\tau + \epsilon)\|\bar{x} - u\|$ for all $u \in \mathscr{C} \cap \mathbb{B}(\bar{x}, \delta)$. Setting $V := \mathbb{B}(f(\bar{x}), r)$ with $r := (\tau + \epsilon)\delta > 0$, we derive that $f(u) \in V$ for all $u \in \mathscr{C} \cap \mathbb{B}(\bar{x}, \delta)$. Taking (3.7) into account, we obtain

$$\{f(x)\} \cap V \subset \{f(u)\} + (\tau + \epsilon)\|u - x\|\mathbb{B}, \ \forall x, u \in \mathscr{C} \cap \mathbb{B}(\bar{x}, \delta),$$

which yields $F_f(x) \cap V \subset F_f(u) + (\tau + \epsilon)\|u - x\|\mathbb{B}$, $\forall x, u \in \mathscr{C} \cap \mathbb{B}(\bar{x}, \delta)$. Hence, F_f satisfies the Aubin property relative to $\mathscr{C} \times \mathbb{R}$ around $\bar{x}$. □

Definition 3.7. Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in \operatorname{dom} f$. Let $\mathscr{C} \subset \mathbb{R}^n, \mathscr{D} \subset \mathbb{R}$ be locally closed around $\bar{x}$ and $f(\bar{x})$, respectively. Assume that $f \in C_{\mathscr{C}}(\bar{x})$. Then f is said to be *metrically regular relative to* $\mathscr{C} \times \mathscr{D}$ around $\bar{x}$ if there exists $\tau > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y} := f(\bar{x})$ such that $d(x, f^{-1}(y) \cap \mathscr{C}) \le \tau\|y - f(x)\|$ for all $x \in U \cap \mathscr{C}$ and $y \in V \cap \mathscr{D}$.

Remark 3.8. (i) In Definition 3.7, if $\mathscr{C}$ and $\mathscr{D}$ are neighborhoods of $\bar{x}$ and $f(\bar{x})$, respectively, then the metric regularity relative to $\mathscr{C} \times \mathscr{D}$ around $\bar{x}$ reduces to the metric regularity around $\bar{x}$, *i.e.*, (see [[10], p. 20]) there exist $\tau > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y} := f(\bar{x})$ such that $d(x, f^{-1}(y)) \le \tau\|y - f(x)\|$ for all $x \in U$, $y \in V$.

(ii) f is metrically regular relative to $\mathscr{C} \times \mathscr{D}$ around $\bar{x}$ if and only if the set-valued mapping F_f as in (3.5) is metrically regular relative to $\mathscr{C} \times \mathscr{D}$ around $(\bar{x}, f(\bar{x}))$.

4. Limiting coderivative relative to a set and the stability of set-valued mappings

As it was mentioned, the limiting coderivative relative to a set, introduced in [19], provides an equivalent characteristic for the Aubin property relative to a set for set-valued mappings. However, a similar result for the metric regularity of set-valued mappings has not been established yet. We now introduce a new limiting coderivative relative to a set, which is a slight improvement of that in [19]. With this new limiting coderivative, we provide equivalent characteristics not only for the Aubin property relative to a set but also for the metrically regular property relative to a set of the set-valued mappings.

Definition 4.1. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $\emptyset \neq \mathscr{X} \subset \mathbb{R}^n \times \mathbb{R}^m$ be locally closed around $(\bar{x}, \bar{y}) \in \operatorname{gph} F \cap \mathscr{X}$. The $\mathscr{X}$-*coderivative* of F at $(\bar{x}, \bar{y})$ is a multifunction $D^*_{\mathscr{X}} F(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$D^*_{\mathscr{X}} F(\bar{x}, \bar{y})(y^*) = \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in N_{\mathscr{X}}((\bar{x}, \bar{y}), \operatorname{gph} F)\}, \ \forall y^* \in \mathbb{R}^m.$$

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$. We write $D^* f(\bar{x})(y^*)$ instead of $D^* F_f(\bar{x}, f(\bar{x}))(y^*)$ for all $y^* \in \mathbb{R}$, where F_f is given as (3.5).

Remark 4.2. (i) If $\mathscr{X} := \mathscr{C} \times \mathbb{R}^m$ in Definition 4.1, then $D^*_{\mathscr{X}} F(\bar{x}, \bar{y})$ is reduced to $D^*_{\mathscr{C}} F(\bar{x}, \bar{y})$ in [19].

(ii) In general, the $\mathscr{C} \times \mathbb{R}^m$-coderivative is not equal the normal coderivative relative to $\mathscr{C}$ in [22]. This is demonstrated in the following example.

Example 4.3. Consider $\mathscr{C} := \{(x_1, x_2) \mid (x_1, x_2) = (0, 0) \text{ or } x_1 \neq 0, x_2 = x_1 \sin(\log |x_1|)\}$ and the multifunction $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ defined by

$$F(x_1, x_2) := \begin{cases} 0 & \text{if } x_1 = x_2 = 0; \\ \emptyset & \text{otherwise.} \end{cases}$$

In this case, we have $\text{gph} F \cap \mathscr{C} \times \mathbb{R} = \{(0, 0, 0)\}$. Moreover, we can see

$$\mathscr{R}((0, 0, 0), \mathscr{C} \times \mathbb{R}) = \{(0, 0)\} \times \mathbb{R} \text{ and } T((0, 0, 0), \mathscr{C} \times \mathbb{R}) = \{(u_1, u_2) \mid |u_2| \leq |u_1|\}.$$

Thus we obtain

$$N_{\mathscr{C} \times \mathbb{R}}((0, 0, 0), \text{gph} F) = \{(0, 0)\} \times \mathbb{R}, \ \tilde{N}_{\mathscr{C} \times \mathbb{R}}((0, 0, 0), \text{gph} F) = \{(u_1, u_2) \mid |u_2| \leq |u_1|\} \times \mathbb{R},$$

where $\tilde{N}_{\mathscr{C} \times \mathbb{R}}((0, 0, 0), \text{gph} F)$ is the limiting normal cone with respect to $\mathscr{C} \times \mathbb{R}$ to $\text{gph} F$ at $(0, 0, 0)$ in sense of [22]. It follows that

$$D^*_{\mathscr{C} \times \mathbb{R}} F(0, 0, 0)(z^*) = \{(0, 0)\} \subsetneq \tilde{D}^*_{\mathscr{C}} F(0, 0, 0)(z^*) = \{(u_1, u_2) \mid |u_2| \leq |u_1|\}$$

for all $z^* \in \mathbb{R}$, where $\tilde{D}^*_{\mathscr{C}} F(0, 0, 0)$ is the normal corderivative relative to $\mathscr{C}$ of F at $(0, 0, 0)$ in seems of [22].

We now provide the necessary and sufficient conditions for the Aubin property relative to a set of multifunctions due to their limiting coderivative relative to that set as follows.

Theorem 4.4. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, and let $\mathscr{C} \subset \mathbb{R}^n$ and $\mathscr{D} \subset \mathbb{R}^m$ be convex and locally closed around $\bar{x}$ and $\bar{y}$, respectively with $(\bar{x}, \bar{y}) \in \text{gph} F \cap (\mathscr{C} \times \mathscr{D})$. Assume that $\operatorname{gph} F^{\mathscr{D}}_{\mathscr{C}}$ is locally closed around $(\bar{x}, \bar{y})$. Then the following statements are equivalent:*

(i) *F satisfies the Aubin property with respect to $\mathscr{C} \times \mathscr{D}$ around $(\bar{x}, \bar{y})$.*

(ii) *For any $\kappa \geq \text{lip}^{\mathscr{D}}_{\mathscr{C}} F(\bar{x}, \bar{y})$,*

$$\|x^*\| \leq \kappa \|y^*\| \ \text{ for all } (y^*, x^*) \in \operatorname{gph} D^*_{\mathscr{C} \times \mathbb{R}^m} F^{\mathscr{D}}(\bar{x}, \bar{y}). \tag{4.1}$$

(iii) $D^*_{\mathscr{C} \times \mathbb{R}^m} F^{\mathscr{D}}(\bar{x}, \bar{y})(0) = \{0\}$.

(iv) $D^*_{\mathscr{C} \times \mathscr{D}} F(\bar{x}, \bar{y})(0) = \{0\}$.

Moreover,

$$\text{lip}_{\mathscr{C} \times \mathbb{R}} F(\bar{x}, \bar{y}) = \|D^*_{\mathscr{C} \times \mathbb{R}^m} F^{\mathscr{D}}(\bar{x}, \bar{y})\| := \sup \left\{ \|x^*\| \mid y^* \in \mathbb{B}, x^* \in D^*_{\mathscr{C} \times \mathbb{R}^m} F^{\mathscr{D}}(\bar{x}, \bar{y})(y^*) \right\} \tag{4.2}$$

$$= \|D^*_{\mathscr{C} \times \mathscr{D}} F(\bar{x}, \bar{y})\| := \sup \left\{ \|x^*\| \mid y^* \in \mathbb{B}, x^* \in D^*_{\mathscr{C} \times \mathscr{D}} F(\bar{x}, \bar{y})(y^*) \right\} \tag{4.3}$$

Proof. The process of the proof is as follows: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). We first prove (i) $\Rightarrow$ (ii). Let F satisfy the Aubin property with respect to $\mathscr{C}\times\mathscr{D}$ around $(\bar{x},\bar{y})$. Then $F^{\mathscr{D}}$ satisfies the Aubin property with respect to $\mathscr{C}\times\mathbb{R}^m$ around $(\bar{x},\bar{y})$. Pick $x^*\in D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})(y^*)$ with some $y^*\in\mathbb{R}^m$. Using the definition, one sees that there exist sequences $(x_k,y_k)\xrightarrow{\operatorname{gph}F^{\mathscr{D}}\cap\mathscr{C}\times\mathbb{R}^m}(\bar{x},\bar{y})$ and $(x_k^*,y_k^*)\to(x^*,y^*)$ such that

$$(x_k^*,-y_k^*)\in N^p_{\mathscr{C}\times\mathbb{R}^m}((x_k,y_k),\operatorname{gph}F^{\mathscr{D}}).$$

It follows from (2.4) in Proposition 2.5 that there exists $\alpha_k>0$ for every $k\in\mathbb{N}$ satisfying

$$(x_k,y_k)+\alpha_k(x_k^*,-y_k^*)\in\mathscr{C}\times\mathbb{R}^m$$

and the following inequality holds for all $(x,y)\in\operatorname{gph}F^{\mathscr{D}}\cap(\mathscr{C}\times\mathbb{R}^m)$

$$\langle x^*,x-x_k\rangle-\langle y_k^*,y-y_k\rangle\le\frac{1}{2\alpha_k}\|(x,y)-(x_k,y_k)\|^2. \tag{4.4}$$

Besides, from the Aubin property with respect to $\mathscr{C}\times\mathscr{D}$ around $(\bar{x},\bar{y})$ with the constant κ of F, we find neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F^{\mathscr{D}}(x)\cap V\subset F^{\mathscr{D}}(z)+\kappa\|z-x\|\mathbb{B}\ \text{ for all } x,z\in U\cap\mathscr{C}. \tag{4.5}$$

Take $\eta\in(0,\frac{1}{2})$ satisfying $\bar{x}+3\eta\mathbb{B}\subset U$ and $\bar{y}+2\eta\mathbb{B}\subset V$, and pick $k_0\in\mathbb{N}$ such that $z_k:=x_k+\alpha_kx_k^*, x_k\in\bar{x}+\eta\mathbb{B}$ and $y_k\in\bar{y}+\eta\mathbb{B}$ for all $k\ge k_0$. For any $\epsilon>0$, we further take

$$\lambda_k:=\min\{\eta,\frac{2\epsilon\alpha_k(1+\kappa)}{1+\kappa^2}\}\in(0,1).$$

Then, for any $x\in\mathbb{B}(x_k,\lambda_k)$, one has $x\in x_k+\eta\mathbb{B}\subset\bar{x}+2\eta\mathbb{B}\subset U$. Pick $\tilde{x}_k:=\lambda_kz_k+(1-\lambda_k)x_k$. It follows from $\eta<\dfrac{1}{2}$ that

$$\tilde{x}_k\in\lambda_k(x_k+2\eta\mathbb{B})+(1-\lambda_k)x_k\subset x_k+\lambda_k\mathbb{B}.$$

Moreover, by the convexity of $\mathscr{C}$, we obtain $\tilde{x}_k\in\mathscr{C}$, so $\tilde{x}_k\in(x_k+\lambda_k\mathbb{B})\cap\mathscr{C}$. Taking $x=x_k$ and $z=\tilde{x}_k$ in (4.5), we obtain

$$y_k\in F^{\mathscr{D}}(\tilde{x}_k)+\kappa\|x_k-\tilde{x}_k\|\mathbb{B},$$

which implies the existence of $\tilde{y}_k\in F^{\mathscr{D}}(\tilde{x}_k)$ and $b_k\in\mathbb{B}$ satisfies

$$y_k=\tilde{y}_k+\kappa\|x_k-\tilde{x}_k\|b_k.$$

Taking (4.4) into account with $x=\tilde{x}_k, y=\tilde{y}_k$, we obtain

$$\begin{aligned}\langle x_k^*,\tilde{x}_k-x_k\rangle+\langle y_k^*,\kappa\|\tilde{x}_k-x_k\|b_k\rangle&\le\frac{1}{2\alpha_k}\left(\|\tilde{x}_k-x_k\|^2+\kappa^2\|\tilde{x}_k-x_k\|^2\|b_k\|^2\right)\\&\le\frac{1}{2\alpha_k}(1+\kappa^2)\|\tilde{x}_k-x_k\|^2\\&\le\epsilon(1+\kappa)\|\tilde{x}_k-x_k\|.\end{aligned}$$

Note that $\tilde{x}_k - x_k = \lambda_k(z_k - x_k) = \lambda_k \alpha_k x_k^*$. It follows that

$$\lambda_k \langle x_k^*, \alpha_k x_k^* \rangle + \lambda_k \langle y_k^*, \kappa \alpha_k \|x_k^*\| b_k \rangle \leq \lambda_k \epsilon (1+\kappa) \alpha_k \|x_k^*\|,$$

which gives us that

$$\|x_k^*\| \leq \kappa \|y_k^*\| + \epsilon(1+\kappa).$$

Passing to the limit as $\epsilon \to 0$ and $k \to \infty$, we further conclude $\|x^*\| \leq \kappa \|y^*\|$.

(ii) $\Rightarrow$ (i) Let relation (4.1) hold for some $\kappa \geq 0$. We prove that F satisfies the Aubin property with respect to $\mathscr{C} \times \mathscr{D}$ around $(\bar{x}, \bar{y})$ with $0 \leq \operatorname{lip}_{\mathscr{C}\times\mathbb{R}} F(\bar{x}, \bar{y}) \leq \kappa$. Indeed, if this assertion is not true, then (3.1) is not satisfied for some $\tilde{\kappa} > \kappa$. Thus there exist $x_k', x_k'' \xrightarrow{\operatorname{dom} F_{\mathscr{C}}^{\mathscr{D}}} \bar{x}$ with $x_k' \neq x_k''$ and $y_k'' \xrightarrow{F^{\mathscr{D}}(x_k'')} \bar{y}$ such that

$$d\left(y_k'', F^{\mathscr{D}}(x_k')\right) > \tilde{\kappa} \|x_k' - x_k''\|,$$

which implies that $y_k'' \notin F^{\mathscr{D}}(x_k')$, so $y_k'' \notin F(x_k')$, which is due to the fact that $y_k'' \in \mathscr{D}$. Since $\operatorname{gph} F_{\mathscr{C}}^{\mathscr{D}}$ is locally closed around $(\bar{x}, \bar{y})$, one sees that there exists a $r > 0$ such that $\Omega := \operatorname{gph} F_{\mathscr{C}}^{\mathscr{D}} \cap \big(\mathbb{B}(\bar{x}, 4r) \times \mathbb{B}(\bar{y}, 2r(\tilde{\kappa}+1)\big)$ is a closed set, which implies that Ω is compact. Setting $r_k := \max\{\|x_k' - \bar{x}\|, \|x_k'' - \bar{x}\|, \|y_k'' - \bar{y}\|\}$, we have $r_k \to 0^+$ and $x_k', x_k'' \in \mathbb{B}(\bar{x}, r_k)$ and $y_k'' \in \mathbb{B}(\bar{y}, r_k)$. Without loss of generality, we may assume that $r_k < r$ for all k. Let $y_k' \in \Pi(y_k''; F^{\mathscr{D}}(x_k'))$. Then $y_k' \neq y_k''$ and

$$\|y - y_k''\| \geq \|y_k' - y_k''\| > \tilde{\kappa} \|x_k'' - x_k'\| \ \forall y \in F^{\mathscr{D}}(x_k'). \tag{4.6}$$

For each $k \in \mathbb{N}$, we define the extended real-valued function as follows:

$$\varphi_k(x, y) := \tilde{\kappa} \|x - x_k'\| + \|y - y_k''\| + \delta_\Omega(x, y) \ \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Then φ_k is lower semi-continuous, level-bounded, and proper. By using [[20], Theorem 1.9], we have that there exists $(\hat{x}_k, \hat{y}_k) \in \Omega$, which is a global minimizer to φ_k. We next prove that $(\hat{x}_k, \hat{y}_k) \xrightarrow{\mathscr{C}\times\mathscr{D}} (\bar{x}, \bar{y})$. Since $(\hat{x}_k, \hat{y}_k) \in \Omega \subset \mathscr{C} \times \mathscr{D}$, it is enough to prove that $\hat{x}_k \to \bar{x}$ and $\hat{y}_k \to \bar{y}$. Indeed, we first see that

$$\tilde{\kappa} \|x_k'' - x_k'\| = \varphi_k(x_k'', y_k'') \geq \varphi_k(\hat{x}_k, \hat{y}_k) \geq \tilde{\kappa} \|\hat{x}_k - x_k'\|.$$

By the facts that $x_k', x_k'' \to \bar{x}$ and

$$\begin{aligned} \|\hat{x}_k - \bar{x}\| &\leq \|\hat{x}_k - x_k'\| + \|x_k' - \bar{x}\| \\ &\leq \|x_k'' - x_k'\| + \|x_k' - \bar{x}\| \\ &\leq 2\|x_k' - \bar{x}\| + \|x_k'' - \bar{x}\| \leq 3r_k, \end{aligned} \tag{4.7}$$

we obtain $\hat{x}_k \to \bar{x}$.

On the other hand, since $(\hat{x}_k, \hat{y}_k)$ is the global minimizer to φ_k, one has

$$\varphi_k(\hat{x}_k, \hat{y}_k) = \tilde{\kappa} \|\hat{x}_k - x_k'\| + \|y_k'' - \hat{y}_k\| \leq \varphi_k(x_k'', y_k'') = \tilde{\kappa} \|x_k' - x_k''\|,$$

which follows from $\|\hat{x}_k - x_k'\| \to 0$ and $\|x_k' - x_k''\| \to 0$ that $\|y_k'' - \hat{y}_k\| \to 0$. According to $y_k'' \to \bar{y}$, we derive that $\hat{y}_k \to \bar{y}$.

We next demonstrate that $\hat{x}_k \neq x_k'$. Indeed, in the opposite that $\hat{x}_k = x_k'$, one has $\hat{y}_k \in F^{\mathscr{D}}(x_k')$. Thus

$$\varphi_k(\hat{x}_k, \hat{y}_k) = \|\hat{y}_k - y_k''\| \geq \|y_k' - y_k''\| > \tilde{\kappa} \|x_k' - x_k''\| = \varphi_k(x_k'', y_k''),$$

which contradicts the fact that $(\hat{x}_k, \hat{y}_k)$ is a global minimizer to φ_k. Define, for each $k \in \mathbb{N}$,

$$\phi_k(x,y) := \tilde{\kappa}\|x - x'_k\| + \|y - y''_k\| \text{ for all } (x,y) \in \mathbb{R}^n \times \mathbb{R}^m$$

and $\nu_k := \frac{r_k}{4}\|x'_k - \hat{x}_k\| > 0$. For any $(x,y) \in \mathbb{B}(\hat{x}_k, \nu_k) \times \mathbb{B}(\hat{y}_k, \nu_k)$, we have

$$\|x - x'_k\| \geq \|x'_k - \hat{x}_k\| - \|x - \hat{x}_k\| \geq (1 - \frac{r_k}{4})\|x'_k - \hat{x}_k\| > 0. \tag{4.8}$$

It implies that $x \neq x'_k$, so $\|x - x'_k\|$ is twice continuously differentiable at x. This combines with the convexity of $\|y - y''_k\|$ indicates

$$\partial^p \phi_k(x,y) = \left(\tilde{\kappa}\frac{x - x'_k}{\|x - x'_k\|}, 0\right) + \begin{cases} \left(0, \frac{y - y''_k}{\|y - y''_k\|}\right) & \text{if } y \neq y''_k; \\ \{0\} \times \mathbb{B}^m & \text{if } y = y''_k \end{cases}$$

due to [[25], Corollary 2.6] and Proposition 2.11. Since $(\hat{x}_k, \hat{y}_k)$ is a global minimizer of φ_k, it is also a global minimizer of ϕ_k over Ω. Thus there exist, by [[25], Exercise 8.4 (b)], $(x^i_k, y^i_k) \in \Omega \cap [\mathbb{B}(\hat{x}_k, \nu_k) \times \mathbb{B}(\hat{y}_k, \nu_k)]$ for $i = 1, 2$ such that

$$\begin{aligned}(0,0) &\in \partial^p \phi_k(x^1_k, y^1_k) + N^p((x^2_k, y^2_k), \Omega) + \nu_k(\mathbb{B}^n \times \mathbb{B}^m) \\ &\subset \left\{\tilde{\kappa}\frac{x^1_k - x'_k}{\|x^1_k - x'_k\|}\right\} \times \mathbb{B}^m + N^p((x^2_k, y^2_k), \Omega) + \nu_k(\mathbb{B}^n \times \mathbb{B}^m).\end{aligned}$$

Thus there exists $(b_1, b_2) \in \mathbb{B}^n \times \mathbb{B}^m$ such that

$$(u,v) := \left(\tilde{\kappa}\frac{x'_k - x^1_k}{\|x^1_k - x'_k\|} - \nu_k b_1, (1 + \nu_k) b_2\right) \in N^p((x^2_k, y^2_k), \Omega) \subset \hat{N}((x^2_k, y^2_k), \Omega). \tag{4.9}$$

Putting, for each $k \in \mathbb{N}$, $x^*_k := \tilde{\kappa}\frac{x'_k - x^2_k}{\|x'_k - x^2_k\|}$ and $y^*_k := (1 - \nu_k) b_2$, we prove that

$$(x^*_k, y^*_k) \in \widehat{N}^{r_k(1+\tilde{\kappa})}_{\mathscr{C} \times \mathbb{R}^m}((x^2_k, y^2_k), \operatorname{gph} F^{\mathscr{D}}).$$

Indeed, since $x^i_k \in \mathbb{B}(\hat{x}_k, \nu_k)$ for $i = 1, 2$, we have $\|x^i_k - \hat{x}_k\| \leq \nu_k = \frac{r_k}{4}\|x'_k - \hat{x}_k\|$, which together with (4.8) indicates that

$$\left\|\frac{x^i_k - \hat{x}_k}{\|x'_k - \hat{x}_k\|}\right\| \leq \frac{r_k}{4} \text{ and } \left\|\frac{x^i_k - x'_k}{\|x^i_k - x'_k\|} - \frac{x^i_k - x'_k}{\|\hat{x}_k - x'_k\|}\right\| \leq 1 - (1 - \frac{r_k}{4}) = \frac{r_k}{4} \;\forall i = 1, 2.$$

By the triangular inequality, we have

$$\begin{aligned}\left\|\frac{x^2_k - x'_k}{\|x^2_k - x'_k\|} - \frac{x^1_k - x'_k}{\|x^1_k - x'_k\|}\right\| &\leq \left\|\frac{x^2_k - x'_k}{\|x^2_k - x'_k\|} - \frac{\hat{x}_k - x'_k}{\|\hat{x}_k - x'_k\|}\right\| + \left\|\frac{x^1_k - x'_k}{\|x^1_k - x'_k\|} - \frac{\hat{x}_k - x'_k}{\|\hat{x}_k - x'_k\|}\right\| \\ &\leq \left\|\frac{x^2_k - x'_k}{\|x^2_k - x'_k\|} - \frac{x^2_k - x'_k}{\|\hat{x}_k - x'_k\|}\right\| + \left\|\frac{x^2_k - \hat{x}_k}{\|\hat{x}_k - x'_k\|}\right\| \\ &\quad + \left\|\frac{x^1_k - x'_k}{\|x^1_k - x'_k\|} - \frac{x^1_k - x'_k}{\|\hat{x}_k - x'_k\|}\right\| + \left\|\frac{x^1_k - \hat{x}_k}{\|\hat{x}_k - x'_k\|}\right\| \\ &\leq \frac{r_k}{4} + \frac{r_k}{4} + \frac{r_k}{4} + \frac{r_k}{4} = r_k.\end{aligned}$$

So we can find $b_3 \in \mathbb{B}^n$ such that

$$x_k^* = \tilde{\kappa}\frac{x_k^1 - x_k'}{\|x_k^1 - x_k'\|} + \tilde{\kappa} r_k b_3 = u + \nu_k b_1 + \tilde{\kappa} r_k b_3.$$

Otherwise, we also have $y_k^* = v - 2\nu_k b_2$, so the following relations

$$\begin{aligned}\frac{\langle (x_k^*, y_k^*), (x,y) - (x_k^2, y_k^2)\rangle}{\|(x,y) - (x_k^2, y_k^2)\|} &= \frac{\langle (u + \nu_k b_1 + \tilde{\kappa} r_k b_3, v - 2\nu_k b_2), (x,y) - (x_k^2, y_k^2)\rangle}{\|(x,y) - (x_k^2, y_k^2)\|} \\ &\leq \frac{\langle (u,v), (x,y) - (x_k^2, y_k^2)\rangle}{\|(x,y) - (x_k^2, y_k^2)\|} + \|(\nu_k b_1 + \tilde{\kappa} r_k b_3, -2\nu_k b_2)\| \\ &< \frac{\langle (u,v), (x,y) - (x_k^2, y_k^2)\rangle}{\|(x,y) - (x_k^2, y_k^2)\|} + r_k(1+\tilde{\kappa})\end{aligned}$$

hold for all $(x,y) \in \Omega$. Passing to the superior limit and taking (4.9) into account, we obtain

$$\begin{aligned}&\limsup_{(x,y)\xrightarrow{\Omega}(x_k^2, y_k^2)} \frac{\langle (x_k^*, y_k^*), (x,y) - (x_k^2, y_k^2)\rangle}{\|(x,y) - (x_k^2, y_k^2)\|} \\ &\qquad \leq \limsup_{(x,y)\xrightarrow{\Omega}(x_k^2, y_k^2)} \frac{\langle (u,v), (x,y) - (x_k^2, y_k^2)\rangle}{\|(x,y) - (x_k^2, y_k^2)\|} + r_k(1+\tilde{\kappa}) \leq \epsilon_k := r_k(1+\tilde{\kappa})\end{aligned}$$

which implies that

$$(x_k^*, y_k^*) \in \hat{N}_{\epsilon_k}((x_k^2, y_k^2), \Omega) = \hat{N}_{\epsilon_k}((x_k^2, y_k^2), \operatorname{gph} F^{\mathscr{D}} \cap (\mathscr{C} \times \mathbb{R}^m)). \tag{4.10}$$

Moreover, it is easy to see from the notation of (x_k^*, y_k^*) that

$$(x_k^2, y_k^2) + \frac{\|x_k^2 - x_k'\|}{\tilde{\kappa}}(x_k^*, y_k^*) = \left(x_k', y_k^2 + \frac{\|x_k^2 - x_k'\|}{\tilde{\kappa}}(1-\nu_k) b_2\right) \in \mathscr{C} \times \mathbb{R}^m,$$

which means that $(x_k^*, y_k^*) \in \mathscr{R}((x_k^2, y_k^2), \mathscr{C} \times \mathbb{R}^m)$. From (4.10), we further obtain

$$(x_k^*, y_k^*) \in \widehat{N}^{\epsilon_k}_{\mathscr{C}\times\mathbb{R}^m}((x_k^2, y_k^2), \operatorname{gph} F^{\mathscr{D}}). \tag{4.11}$$

We have from (4.7) and the fact $\hat{y}_k \to \bar{y}$ that

$$\|x_k^2 - \bar{x}\| \leq \|x_k^2 - \hat{x}_k\| + \|\hat{x}_k - \bar{x}\| \leq r_k + 3r_k = 4r_k \to 0^+, \tag{4.12}$$

$$\|y_k^2 - \bar{y}\| \leq \|y_k^2 - \hat{y}_k\| + \|\hat{y}_k - \bar{y}\| \leq r_k + \|\hat{y}_k - \bar{y}\| \to 0^+. \tag{4.13}$$

Moreover, without loss of generality, we can assume that $x_k^* \to x^*$ for some $x^* \in \mathbb{R}^n$ with $\|x^*\| = \tilde{\kappa}$ and $y_k^* \to y^*$ for some $y^* \in \mathbb{B}^m$. Since $\epsilon_k \to 0^+$, we have by Proposition 2.5 (v) that

$$(x^*, y^*) \in \mathop{\text{Lim sup}}_{\substack{\epsilon \to 0^+ \\ (x,y)\xrightarrow{\operatorname{gph} F_{\mathscr{C}}}(\bar{x},\bar{y})}} \hat{N}^{\epsilon}_{\mathscr{C}\times\mathbb{R}^m}((x,y), \operatorname{gph} F) = N_{\mathscr{C}\times\mathbb{R}^m}((\bar{x},\bar{y}), \operatorname{gph} F^{\mathscr{D}}),$$

so $x^* \in D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})(-y^*)$. However, it is easy to see that

$$\|x^*\| = \tilde{\kappa} > \frac{\kappa+\tilde{\kappa}}{2} \geq \frac{\kappa+\tilde{\kappa}}{2}\|-y^*\|,$$

which contradicts (4.1). Thus F has to satisfy the Aubin property with respect to $\mathscr{C}\times\mathscr{D}$ around $(\bar{x},\bar{y})$ for any constant $\tilde{\kappa} > \kappa$. Moreover, we also have $\text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y}) \leq \kappa$. So (i) $\Leftrightarrow$ (ii).

(ii) $\Leftrightarrow$ (iii) is obvious. We continue to prove (iii) $\Leftrightarrow$ (iv). To do this, we need to show that

$$\{0\} \subset D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})(0) \subset D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})(0) = \{0\}. \tag{4.14}$$

Since $\{0\} \subset D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})(0)$ and $D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})(0) = \{0\}$, it is sufficient to show that

$$D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})(y^*) \subset D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})(y^*) \text{ for all } y^* \in \mathbb{R}^m. \tag{4.15}$$

Let $x^* \in D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})(y^*)$. Then there exist sequences $(x_k,y_k) \xrightarrow{\text{gph}F^{\mathscr{D}}_{\mathscr{C}}} (\bar{x},\bar{y})$ and $(x^*_k,y^*_k) \to (x^*,y^*)$ such that the following relations

$$\begin{aligned}(x^*_k,-y^*_k) \in N^p_{\mathscr{C}\times\mathscr{D}}((x_k,y_k),\text{gph}\,F) &= N^p((x_k,y_k),\text{gph}\,F\cap(\mathscr{C}\times\mathscr{D}))\cap\mathscr{R}((x_k,y_k),\mathscr{C}\times\mathscr{D})\\ &\subset N^p((x_k,y_k),\text{gph}\,F^{\mathscr{D}}\cap(\mathscr{C}\times\mathbb{R}^m))\cap\mathscr{R}((x_k,y_k),\mathscr{C}\times\mathbb{R}^m)\\ &= N^p_{\mathscr{C}\times\mathbb{R}^m}\Big((x_k,y_k),\text{gph}\,F^{\mathscr{D}}\Big)\end{aligned}$$

hold for all $k \in \mathbb{N}$. Thus $(x^*,-y^*) \in N_{\mathscr{C}\times\mathbb{R}^m}((\bar{x},\bar{y}),\text{gph}\,F^{\mathscr{D}})$, which is due to $(x_k,y_k) \xrightarrow{\text{gph}F^{\mathscr{D}}_{\mathscr{C}}} (\bar{x},\bar{y})$ and $(x^*_k,y^*_k) \to (x^*,y^*)$. This implies that $x^* \in D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})(y^*)$ and then (4.15) holds. Thus, we obtain (4.14).

Let us now prove formulas (4.2) and (4.3). To do this, we prove that

$$\text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y}) \geq \|D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})\| \geq \|D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})\| \geq \text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y}).$$

First, let us prove $\text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y}) \geq \|D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})\|$. Taking an arbitrarily $\kappa \geq \text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y})$, one sees that F has the Aubin property around $(\bar{x},\bar{y})$ with respect to $\mathscr{C}\times\mathscr{D}$, so inequality (4.1) holds with κ, which means that $\|x^*\| \leq \kappa\|y^*\|$ for all $(y^*,x^*) \in \text{gph}\,D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})$. Thus, for any $y^* \in \mathbb{B}$ and $x^* \in D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})(y^*)$, we have $\|x^*\| \leq \kappa$, which follows that

$$\|D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})\| := \sup\left\{\|x^*\| \mid y^* \in \mathbb{B}, x^* \in D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})(y^*)\right\} \leq \kappa.$$

Since κ is arbitrarily taken in $\big[\text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y}),\infty\big)$, we obtain $\text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y}) \geq \|D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})\|$. From (4.15), we have

$$\|D^*_{\mathscr{D}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})\| \geq \|D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})\|.$$

It remains to prove $\|D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})\| \geq \text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y})$. Suppose by the contradiction that $\|D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})\| < \text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y})$. Taking $\kappa := \frac{\|D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})\|+\text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y})}{2}$, we obtain $\|D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})\| < \kappa < \text{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y})$, so F does not satisfy the Aubin property with respect to $\mathscr{C}\times\mathscr{D}$ around $(\bar{x},\bar{y})$ with constant κ. This implies that there exists $x^* \neq 0$ such that $x^* \in D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})(0)$. We obtain by setting $\tilde{x}^* = (\kappa+1)\frac{x^*}{\|x^*\|}$ that

$$(0,\tilde{x}^*) \in \text{gph}\,D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y}) \text{ and } \|\tilde{x}^*\| = \kappa+1 > \kappa.$$

Hence,

$$\|D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})\| := \sup\{\|x^*\| \mid y^*\in\mathbb{B}, (y^*,x^*)\in \operatorname{gph} D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})\} \geq \|\tilde{x}^*\| > \kappa,$$

which reaches a contradiction. Thus $\|D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})\| \geq \operatorname{lip}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y})$. The proof of the theorem is completed. □

Remark 4.5. (i) Although we have an equivalent characteristic between the Aubin property relative to $\mathscr{C}\times\mathscr{D}$ around $(\bar{x},\bar{y})$ of F and the equality $D^*_{\mathscr{C}\times\mathbb{R}^m}F^{\mathscr{D}}(\bar{x},\bar{y})(0)=\{0\}$, this approach does not help in establishing an equivalent characteristic for metric regularity relative to $\mathscr{C}\times\mathscr{D}$ of set-valued mappings. Meanwhile, characterizing the Aubin property relative to $\mathscr{C}\times\mathscr{D}$ through the $\mathscr{C}\times\mathscr{D}$-coderivative, $D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})(0)=\{0\}$ is an important basis for achieving the characteristic of metric regularity relative to $\mathscr{C}\times\mathscr{D}$ of set-valued mappings.

(ii) Taking $\mathscr{D}=\mathbb{R}^m$, we see that Theorem 4.4 reduces to [[19], Theorem 3]. In addition, by taking $\mathscr{C}:=\operatorname{dom}F$ and $\mathscr{D}=\mathbb{R}^m$ in Theorem 4.4, we obtain a necessary and sufficient for a multifunction with the relative Aubin property.

We next provide coderivative characteristics for the metric regularity relative to a set of set-valued mappings.

Theorem 4.6 (Coderivative criterion for metric regularity relative to a set)**.** *Let $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, and let $(\bar{x},\bar{y})\in$ gph$F$. Assume that $\operatorname{gph}F^{\mathscr{D}}_{\mathscr{C}}$ and $\mathscr{C}\times\mathscr{D}$ are locally closed around $(\bar{x},\bar{y})$. Suppose, in addition, $\mathscr{C}$ and $\mathscr{D}$ are convex. For the following statements:*

(i) *F is metrically regular relative to $\mathscr{C}\times\mathscr{D}$ around $(\bar{x},\bar{y})$;*

(ii) *F^{-1} satisfies the Aubin property relative to $\mathscr{D}\times\mathscr{C}$ around $(\bar{y},\bar{x})$;*

(iii) $D^*_{\mathscr{D}\times\mathscr{C}}F^{-1}(\bar{y},\bar{x})(0)=\{0\}$;

(iv) $\ker D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})=\{0\}$,

it holds that (i) ⇒ (ii) ⇔ (iii) ⇔ (iv)*. In addition, if $\mathscr{D}\subset F(\mathscr{C})$, then the above implications are equivalent to each other. In this case, the regular constant can be computed by*

$$\operatorname{reg}^{\mathscr{D}}_{\mathscr{C}}F(\bar{x},\bar{y}) = \|D^*_{\mathscr{D}\times\mathscr{C}}F^{-1}(\bar{y},\bar{x})\|. \tag{4.16}$$

Proof. (i) ⇒ (ii) ⇔ (iii) directly implies from Theorems 3.3 and 4.4. We now justify (iii) ⇔ (iv). It is sufficient to demonstrate that

$$(y^*,x^*)\in\operatorname{gph} D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y}) \text{ if and only if } (-x^*,-y^*)\in\operatorname{gph} D^*_{\mathscr{D}\times\mathscr{C}}F^{-1}(\bar{y},\bar{x}).$$

Indeed, let $(y^*,x^*)\in\operatorname{gph} D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})$. By the definition, one sees that there exist sequences $(x_k,y_k)\xrightarrow{\operatorname{gph}F\cap\mathscr{C}\times\mathscr{D}}(\bar{x},\bar{y})$ and $(x_k^*,y_k^*)\to(x^*,y^*)$ such that $(x_k^*,-y_k^*)\in N^p_{\mathscr{C}\times\mathscr{D}}((x_k,y_k),\operatorname{gph}F)$, which is equivalent to the fact that there exists $t>0$ such that $(x_k,y_k)+t(x_k^*,-y_k^*)\in\Pi^{-1}((x_k,y_k),\operatorname{gph}F\cap\mathscr{C}\times\mathscr{D})\cap\mathscr{C}\times\mathscr{D}$. Otherwise, since $(x,y)\in\operatorname{gph}F$ if and only if $(y,x)\in\operatorname{gph}F^{-1}$, one has that $(x_k,y_k)\xrightarrow{\operatorname{gph}F\cap\mathscr{C}\times\mathscr{D}}(\bar{x},\bar{y})$ is equivalent to $(y_k,x_k)\xrightarrow{\operatorname{gph}F^{-1}\cap\mathscr{D}\times\mathscr{C}}(\bar{y},\bar{x})$ and

$$(x_k,y_k)+t(x_k^*,-y_k^*)\in\Pi^{-1}((x_k,y_k),\operatorname{gph}F\cap\mathscr{C}\times\mathscr{D})\cap\mathscr{C}\times\mathscr{D}$$

if and only if

$$(y_k,x_k)+t(-y_k^*,x_k^*)\in\Pi^{-1}((y_k,x_k),\operatorname{gph}F^{-1}\cap\mathscr{D}\times\mathscr{C})\cap\mathscr{D}\times\mathscr{C}.$$

This means that $(x_k^*,-y_k^*)\in N^p_{\mathscr{C}\times\mathscr{D}}((x_k,y_k),\operatorname{gph}F)$ if and only if $(-y_k^*,x_k^*)\in N^p_{\mathscr{D}\times\mathscr{C}}((y_k,x_k),\operatorname{gph}F^{-1})$, which is equivalent to $(-y^*,x^*)\in N_{\mathscr{D}\times\mathscr{C}}((\bar{y},\bar{x}),\operatorname{gph}F^{-1})$. Thus

$$(-x^*,-y^*)\in\operatorname{gph} D^*_{\mathscr{D}\times\mathscr{C}}F^{-1}(\bar{y},\bar{x}),$$

which indicates that $0\in D^*_{\mathscr{C}\times\mathscr{D}}F(\bar{x},\bar{y})(y^*)$ if and only if $-y^*\in D^*_{\mathscr{D}\times\mathscr{C}}F^{-1}(\bar{y},\bar{x})(0)$. Hence, one has (iii) ⇔ (iv).

If, in addition, $\mathscr{D} \subset F(\mathscr{C})$, then we have by using Theorem 3.3 again that (i) $\Leftrightarrow$ (ii), so (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Note that formula (4.16) is from Theorems 3.3 and 4.4. Hence, the proof of the theorem is completed. ☐

Remark 4.7. In Theorem 4.6, we does not need the assumption that $\mathscr{C} = F^{-1}(\mathscr{D})$, which is essential in [[22], Theorem 3.9]. Moreover, it is different to [[22], Theorem 3.9], which used the relative mirror contingent coderivative to state the metrically regular property a set. In Theorem 4.6, we use the coderivative in Definition 4.1 to characterize for the metric regularity relative to a set.

Now, it is the time to consider examples to illustrate how Theorems 4.4 and 4.6 are used to check the Aubin property and metric regularity relative to a set of set-valued mappings.

Example 4.8 (Relative Aubin property but no metric regularity relative to image)**.** Consider the multifunction $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ given by

$$F(x,y) = \left\{(u,v)^\top \in \mathbb{R}^2_+ \mid M(u,v)^\top + (x,y)^\top \in \mathbb{R}^2_+, \langle (u,v)^\top, M(u,v)^\top + (x,y)^\top \rangle = 0\right\},$$

where

$$M := \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } (x,y)^\top := \begin{pmatrix} x \\ y \end{pmatrix}.$$

It follows that

$$\begin{aligned} \operatorname{dom} F = \mathbb{R}_+ \times \mathbb{R}, \quad \operatorname{rge} F = \mathbb{R}_+ \times \mathbb{R}_+ \ \text{ and} \\ \operatorname{gph} F = \{(x,y,u,v)^\top \in \mathbb{R}^4 \mid x,u,v \geq 0, x \geq u, y+u+v \geq 0 \\ u(x-u) = 0, v(u+v+y) = 0\}. \end{aligned}$$

Picking $\bar{z} := (\bar{x},\bar{y}) = (0,0), \bar{w} := (\bar{u},\bar{v}) = (0,0)$, and $\mathscr{C} := \operatorname{dom} F$ and $\mathscr{D} := \operatorname{rge} F$, we have $\operatorname{gph} F^{\mathscr{D}}_{\mathscr{C}} = \operatorname{gph} F$.

- It is known from [[19], Ex. 3] that $D_{\mathscr{C}\times\mathbb{R}^2} F(\bar{x},\bar{w})(0,0) = \{(0,0)\}$, which means that F satisfies the relative Aubin property around $(\bar{z},\bar{w})$.
- We now justify that F is not metrically regular relative to image around $(\bar{z},\bar{w})$ due to $(1,1) \in \ker D^*_{\mathbb{R}^2\times\mathscr{D}} F(\bar{z},\bar{w})$. To see this, we take $(\frac{1}{k},\frac{1}{k},0,0) \xrightarrow{\operatorname{gph} F \cap (\mathbb{R}^2\times\mathscr{D})} (\bar{z},\bar{w})$ and prove that

  $$(0,0,-1,-1) \in N^p_{\mathbb{R}^2\times\mathscr{D}}((\frac{1}{k},\frac{1}{k},0,0), \operatorname{gph} F).$$

  Indeed, for any $(x,y,u,v) \in \operatorname{gph} F^{\mathscr{D}} \cap \mathbb{B}((\frac{1}{k},\frac{1}{k},0,0),\frac{1}{2k})$, we have

  $$\langle (0,0,-1,-1), (x,y,u,v) - (\frac{1}{k},\frac{1}{k},0,0)\rangle = -u-v \leq 0 \leq \frac{1}{2}\|(x,y,u,v) - (\frac{1}{k},\frac{1}{k},0,0)\|^2,$$

  which implies that $(0,0,-1,-1) \in N_{\mathbb{R}^2\times\mathscr{D}}((\bar{z},\bar{w}), \operatorname{gph} F)$. Thus

  $$(1,1) \in \ker D^*_{\mathbb{R}^2\times\mathscr{D}} F(\bar{z},\bar{w}).$$

Using Theorem 4.6, we derive that F is not metrically regular relative to image around $(\bar{z},\bar{w})$.

Example 4.9. Let $\Omega = \mathbb{R}_+$. We consider the normal cone mapping to Ω, $\mathscr{N} : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$\mathscr{N}(x) = \begin{cases} \mathbb{R}_- & \text{if } x = 0, \\ \{0\} & \text{if } x > 0, \\ \emptyset & \text{if } x < 0. \end{cases}$$

We have $\operatorname{dom} \mathscr{N} = \mathbb{R}_+$, $\operatorname{rge} \mathscr{N} = \mathbb{R}_-$, and $\operatorname{gph} \mathscr{N} = (\{0\} \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \{0\})$. We find $D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{N}(0,0)$ with $\mathscr{C} := \operatorname{dom} \mathscr{N}$ and $\mathscr{D} := \operatorname{rge} \mathscr{N}$. By direct computation, we have

$$N^p_{\mathscr{C}\times\mathscr{D}}((x,y), \operatorname{gph} \mathscr{N}) = \begin{cases} \{(0,0)\} & \text{if } x = 0, y = 0; \\ \mathbb{R}_+ \times \{0\} & \text{if } x = 0, y < 0; \\ \{0\} \times \mathbb{R}_- & \text{if } x > 0, y = 0; \\ \emptyset & \text{otherwise,} \end{cases}$$

which implies that $N_{\mathscr{C}\times\mathscr{D}}((0,0), \operatorname{gph} \mathscr{N}) = (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_-)$. Therefore,

- $D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{N}(0,0)(0) = \mathbb{R}_+$, which means that $\mathscr{N}$ does not satisfy the relative Aubin property at $(0,0)$.
- $\ker D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{N}(0,0) = \mathbb{R}_-$, which follows that $\mathscr{N}$ is not metrically regular relative to image around $(0,0)$.

Let us consider $\bar{\mathscr{D}} := \{0\}$. Then $\bar{\mathscr{D}} \subset \mathscr{N}(\mathbb{R}_+)$. By direct computation, we have

$$N^p_{\mathbb{R}_+\times\bar{\mathscr{D}}}((x,y), \operatorname{gph} \mathscr{N}) = \begin{cases} \{(0,0)\} & \text{if } x > 0, y = 0; \\ \emptyset & \text{otherwise.} \end{cases}$$

It follows that $N_{\mathbb{R}_+\times\bar{\mathscr{D}}}((0,0), \operatorname{gph} \mathscr{N}) = \{(0,0)\}$, and then

$$\ker D^*_{\mathbb{R}_+\times\bar{\mathscr{D}}}\mathscr{N}(0,0) = \{0\} \text{ and } D^*_{\mathbb{R}_+\times\bar{\mathscr{D}}}\mathscr{N}(0,0)(0) = \{0\}.$$

This is equivalent to the fact that F is not only metrically regular relative to $\mathbb{R}_+ \times \{0\}$ around $(0,0)$, but also satisfies the Aubin property relative to $\mathbb{R}_+ \times \{0\}$ around that point.

5. Limiting subdifferential relative to a set and the stability of single-valued mappings

In this section, we first present the concept of subdifferentials relative to a set which is small improvements of those introduced in [19]. Then we use these subdifferentials to provide characteristics for the metric regularity and locally Lipschitz continuous property relative to a set of single-valued mappings.

Definition 5.1. Consider the extended real-valued function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$. Let $\mathscr{X} \subset \mathbb{R}^{n+1}$ and $\operatorname{epi} f \cap \mathscr{X}$ be locally closed around $(\bar{x}, f(\bar{x})) \in \operatorname{epi} f \cap \mathscr{X}$.

(i) The set $\partial_{\mathscr{X}} f(\bar{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, -1) \in N_{\mathscr{X}}((\bar{x}, f(\bar{x})), \operatorname{epi} f)\}$ is called the *$\mathscr{X}$-subdifferential* of f at $\bar{x}$. We also call every vector $x^* \in \partial_{\mathscr{X}} f(\bar{x})$ to be a *$\mathscr{X}$-subgradient* of f at $\bar{x}$.

(ii) The *$\mathscr{X}$-singular subdifferential* of f at $\bar{x}$, denoted $\partial^\infty_{\mathscr{X}} f(\bar{x})$, is defined by

$$\partial^\infty_{\mathscr{X}} f(\bar{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, 0) \in N_{\mathscr{X}}((\bar{x}, f(\bar{x})), \operatorname{epi} f)\}.$$

If $\bar{x} \notin \operatorname{dom} f$ or $(\bar{x}, f(\bar{x})) \notin \mathscr{X}$, then we put $\partial_{\mathscr{X}} f(\bar{x}) := \partial^\infty_{\mathscr{X}} f(\bar{x}) := \emptyset$.

The relation between $\partial_{\mathscr{C}\times\mathscr{D}} f(\bar{x})$ and $D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{E} f(\bar{x}, f(\bar{x}))$ is completely presented in the following proposition.

Proposition 5.2. *Let $f:\mathbb{R}^n\to\bar{\mathbb{R}}$, and let $\mathscr{C}\subset\mathbb{R}^n$ and $\mathscr{D}\subset\mathbb{R}$. Let $(\bar{x},f(\bar{x}))\in\mathscr{C}\times\mathscr{D}$. Suppose that $\operatorname{epi} f\cap\mathscr{C}\times\mathscr{D}$ is locally closed around $(\bar{x},f(\bar{x}))$. Assume, in addition, that $\mathscr{D}$ is convex. Then*

$$D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{E}f(\bar{x},f(\bar{x}))(\lambda)=\begin{cases}\lambda\partial_{\mathscr{C}\times\mathscr{D}}f(\bar{x}) & \text{if } \lambda>0;\\ \partial^\infty_{\mathscr{C}\times\mathscr{D}}f(\bar{x}) & \text{if } \lambda=0;\\ \emptyset & \text{otherwise.}\end{cases}$$

Proof. We consider the following three cases:

Case 1. $\lambda>0$. In this case,

$$D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{E}f(\bar{x},f(\bar{x}))(\lambda)=\left\{x^*\in\mathbb{R}^n\mid(x^*,-\lambda)\in N_{\mathscr{C}\times\mathscr{D}}\Big((\bar{x},f(\bar{x})),\operatorname{gph}\mathscr{E}f\Big)\right\}.$$

Notice that $\operatorname{gph}\mathscr{E}f^{\mathscr{D}}_{\mathscr{C}}=\{(x,r)\mid r\ge f(x),x\in\mathscr{C},r\in\mathscr{D}\}=\operatorname{epi} f\cap(\mathscr{C}\times\mathscr{D})$. Hence,

$$\begin{aligned}D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{E}f(\bar{x},f(\bar{x}))(\lambda)&=\left\{x^*\in\mathbb{R}^n\mid(\frac{x^*}{\lambda},-1)\in N_{\mathscr{C}\times\mathscr{D}}\Big((\bar{x},f(\bar{x})),\operatorname{epi} f\Big)\right\}\\&=\left\{x^*\in\mathbb{R}^n\mid\frac{x^*}{\lambda}\in\partial_{\mathscr{C}\times\mathscr{D}}f(\bar{x})\right\}.\end{aligned}$$

Therefore, $D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{E}f(\bar{x},f(\bar{x}))(\lambda)=\lambda\partial_{\mathscr{C}\times\mathscr{D}}f(\bar{x})$.

Case 2. $\lambda=0$. In this case,

$$\begin{aligned}D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{E}f(\bar{x},f(\bar{x}))(0)&=\left\{x^*\in\mathbb{R}^n\mid(x^*,0)\in N_{\mathscr{C}\times\mathscr{D}}\Big((\bar{x},f(\bar{x})),\operatorname{gph}\mathscr{E}f\Big)\right\}\\&=\left\{x^*\in\mathbb{R}^n\mid(x^*,0)\in N_{\mathscr{C}\times\mathscr{D}}\Big((\bar{x},f(\bar{x})),\operatorname{epi} f\Big)\right\}\\&=\left\{x^*\in\mathbb{R}^n\mid x^*\in\partial^\infty_{\mathscr{C}\times\mathscr{D}}f(\bar{x})\right\}.\end{aligned}$$

Hence, we obtain $D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{E}f(\bar{x},f(\bar{x}))(0)=\partial^\infty_{\mathscr{C}\times\mathscr{D}}f(\bar{x})$.

Case 3. $\lambda<0$. We assume the opposite that there exists $x^*\in D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{E}f(\bar{x},f(\bar{x}))(\lambda)$. We find, by the definition, sequences $(x_k,r_k)\xrightarrow{\operatorname{gph}\mathscr{E}f^{\mathscr{D}}_{\mathscr{C}}}(\bar{x},f(\bar{x}))$ and $(x_k^*,\lambda_k)\to(x^*,\lambda)$ such that $(x_k^*,\lambda_k)\in N^p_{\mathscr{C}\times\mathscr{D}}((x_k,r_k),\operatorname{gph}\mathscr{E}f)$. Thus, for each $k\in\mathbb{N}$, there exists, due to Proposition 2.5 (i), $p_k\to0^+$ such that $(x_k,r_k)+p_k(x_k^*,-\lambda_k)\in\mathscr{C}\times\mathscr{D}$ and

$$\langle x_k^*,x-x_k\rangle-\lambda_k(r-r_k)\le\frac{1}{2p_k}\left(\|x-x_k\|^2+(r-r_k)^2\right)\forall(x,r)\in\operatorname{gph}\mathscr{E}f\cap\mathscr{C}\times\mathscr{D}. \tag{5.1}$$

Moreover, by the fact that $\lambda_k\to\lambda<0$, without loss of generality, we can assume that $0<\frac{1}{k}<-\lambda_k$ for all $k\in\mathbb{N}$, which implies that

$$\left(1+\frac{1}{k\lambda_k}\right)r_k-\frac{1}{k\lambda_k}\left(r_k-\lambda_kp_k\right)=r_k+\frac{p_k}{k}\in\mathscr{D}$$

due to $r_k,r_k-\lambda_kp_k\in\mathscr{D}$ and the convexity of $\mathscr{D}$. Picking $x=x_k$ and $r=r_k+\dfrac{p_k}{k}$, one has $(x,r)\in\operatorname{gph}\mathscr{E}f^{\mathscr{D}}_{\mathscr{C}}$. Taking (5.1) into account, we conclude that $-\lambda_k\frac{p_k}{k}\le\frac{1}{2p_k}\frac{p_k^2}{k^2}$, which is equivalent to $\lambda_k\ge0$. This is a contradiction to $0<\frac{1}{k}<-\lambda_k$ for all $k\in\mathbb{N}$. Hence, $D^*_{\mathscr{C}\times\mathscr{D}}\mathscr{E}^f(\bar{x},f(\bar{x}))(\lambda)=\emptyset$. □

Let $f:\mathbb{R}^n\to\bar{\mathbb{R}}$ and $\mathscr{C}\subset\operatorname{dom} f$. Then $\mathscr{C}\subset F_f^{-1}(\mathbb{R})$, where F_f is defined as in (3.5). Moreover, it is known that f is the locally Lipschitz continuity relative to $\mathscr{C}$ around $\bar{x}$ if and only if $\mathscr{E}f$ satisfies the Aubin property relative to

$\mathscr{C} \times \mathbb{R}$ around $(\bar{x}, f(\bar{x}))$. Thus, by using Proposition 5.2 and Theorem 4.4, we obtain the necessary and sufficient condition for the local Lipschitz continuity relative to $\mathscr{C}$ of single-valued mappings *via* the $\mathscr{C} \times \mathbb{R}$-subdifferential and the $\mathscr{C} \times \mathbb{R}$-singular subdifferential as follows.

Theorem 5.3. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, and let $\mathscr{C}$ be convex and locally closed around $\bar{x} \in \mathscr{C} \cap \operatorname{dom} f$. Let $f \in \mathscr{F}_{\mathscr{C}}(\bar{x})$. For the following statements:*
(i) *f is locally Lipschitz continuous relative to $\mathscr{C}$ around $\bar{x}$;*
(ii) *there exists $\kappa > 0$ such that $\|x^*\| \le \kappa$ whenever $x^* \in \partial_{\mathscr{C}\times\mathbb{R}} f(\bar{x})$;*
(iii) *$\partial^{\infty}_{\mathscr{C}\times\mathbb{R}} f(\bar{x}) = \{0\}$;*
(iv) *$\partial_{\mathscr{C}\times\mathbb{R}} f(\bar{x})$ is nonempty compact set and $\operatorname{lip}_{\mathscr{C}} f(\bar{x}) = \max\{\|x^*\| \mid x^* \in \partial_{\mathscr{C}\times\mathbb{R}} f(\bar{x})\}$,*
it holds that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\implies$ (iv).

In what follow, we provide the subdifferential criterion for the metric regularity relative to a set of single-valued mappings.

Theorem 5.4. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\mathscr{C}, \mathscr{D}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}$, respectively with $\mathscr{D} \subset f(\mathscr{C})$. Let $f \in C_{\mathscr{C}}(\bar{x})$, and let $\mathscr{C} \times \mathscr{D}$ be convex and locally closed around $(\bar{x}, f(\bar{x}))$. For the following statements:*
(i) *f is metric regularity relative to $\mathscr{C} \times \mathscr{D}$ around $\bar{x}$;*
(ii) *$\ker D^*_{\mathscr{C}\times\mathscr{D}} f(\bar{x}) = \{0\}$;*
(iii) *$0 \notin \partial_{\mathscr{C}\times\mathscr{D}} f(\bar{x})$,*
it holds that (i) $\Leftrightarrow$ (ii). *In addition, if $\mathscr{D} = f(\mathscr{C})$, then* (i) *(or* (ii)*) implies* (iii).

Proof. (i) $\Leftrightarrow$ (ii) is due to Remark 3.8 (ii) and Theorem 4.6. It remains to prove (ii) $\Rightarrow$ (iii) in the case of $\mathscr{D} = f(\mathscr{C})$. Indeed, $0 \in \partial_{\mathscr{C}\times\mathscr{D}} f(\bar{x})$, which means that $(0, -1) \in N_{\mathscr{C}\times\mathscr{D}}((\bar{x}, f(\bar{x})), \operatorname{gph} \mathscr{E} f)$. Thus there exist sequences $(x_k, y_k) \xrightarrow{\operatorname{gph} \mathscr{E} f^{\mathscr{D}}_{\mathscr{C}}} (\bar{x}, f(\bar{x}))$ and $(x_k^*, y_k^*) \to (0, -1)$ such that $(x_k^*, y_k^*) \in N^p_{\mathscr{C}\times\mathscr{D}}((\bar{x}, f(\bar{x})), \operatorname{gph} \mathscr{E} f)$. Thus we can find $\delta, p > 0$ such that

$$\langle (x_k^*, y_k^*), (x, y) - (x_k, y_k)\rangle \le p\|(x, y) - (x_k, y_k)\|^2 \ \forall (x, y) \in \operatorname{gph} \mathscr{E} f^{\mathscr{D}}_{\mathscr{C}} \cap \mathbb{B}((\bar{x}, f(\bar{x})), \delta). \tag{5.2}$$

Since $f \in C_{\mathscr{C}}(\bar{x})$ and $x_k \xrightarrow{\mathscr{C}} \bar{x}$, one has $f(x_k) \xrightarrow{\mathscr{D}} f(\bar{x})$ and then $(x_k, f(x_k)) \in \operatorname{gph} \mathscr{E} f^{\mathscr{D}}_{\mathscr{C}} \cap \mathbb{B}((\bar{x}, f(\bar{x})), \delta)$ for k large enough. If $y_k > f(x_k)$ for all sufficiently large k, then we have by taking $x = x_k, y = f(x_k)$ in (5.2) that

$$\langle y_k^*, f(x_k) - y_k\rangle \le p\|f(x_k) - y_k\|^2,$$

which implies that $y_k^* \ge p|f(x_k) - y_k| \to 0$ as $k \to \infty$. This contradicts $y_k^* \to -1$. Therefore, there exists a subsequence (x_{k_s}, y_{k_s}) of (x_k, y_k) satisfying $y_{k_s} = f(x_{k_s})$. It follows that

$$(x_{k_s}, y_{k_s}) \xrightarrow{\operatorname{gph} f \cap \mathscr{C}\times\mathscr{D}} (\bar{x}, f(\bar{x})).$$

Moreover, we also have

$$\begin{aligned}(x_{k_s}^*, y_{k_s}^*) &\in N^p((x_{k_s}, y_{k_s}), \operatorname{gph} \mathscr{E} f) \cap \mathscr{R}((x_{k_s}, y_{k_s}), \mathscr{C} \times \mathscr{D}) \\ &\subset N^p((x_{k_s}, y_{k_s}), \operatorname{gph} f) \cap \mathscr{R}((x_{k_s}, y_{k_s}), \mathscr{C} \times \mathbb{R}),\end{aligned}$$

which is due to the fact that $\operatorname{gph} f \subset \operatorname{gph} \mathscr{E} f$ and the property of proximal cones. This gives us that

$$(0, -1) \in N_{\mathscr{C}\times\mathscr{D}}((\bar{x}, f(\bar{x})), \operatorname{gph} f),$$

so $1 \in \ker D^*_{\mathscr{C}\times\mathscr{D}} f(\bar{x})$. This is a contradiction. Thus $0 \notin \partial_{\mathscr{C}\times\mathscr{D}} f(\bar{x})$ and hence (ii) holds. $\square$

We close this section *via* the following illustrated examples for Theorem 5.4.

Example 5.5. Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2-2x$ for all $x\in\mathbb{R}$.

- Let $\mathscr{C}=\mathbb{R}_+$. By the directly computation, we have

$$N_{\mathscr{C}\times\mathbb{R}}((\bar{x},f(\bar{x})),\operatorname{gph} f)=\left\{(u,v)\mid v\geq \frac{1}{2}u\geq 0\right\}\cup\mathbb{R}_+(-2,-1),$$

  which implies that $D^*_{\mathscr{C}\times\mathbb{R}}f(\bar{x})=\{0\}$. Thus f is locally Lipschitz continuous relative to $\mathscr{C}$ around $\bar{x}$.
- Let $\mathscr{C}=\mathbb{R}_+$ and $\mathscr{D}:=[-1,\infty)$. Then $\mathscr{D}=f(\mathscr{C})$. In view of

$$\begin{aligned}N_{\mathscr{C}\times\mathscr{D}}((\bar{x},f(\bar{x})),\operatorname{gph} f)&=N_{\mathscr{C}\times\mathbb{R}}((\bar{x},f(\bar{x})),\operatorname{gph} f)\\&=\left\{(u,v)\mid v\geq \frac{1}{2}u\geq 0\right\}\cup\mathbb{R}_+(-2,-1),\end{aligned}$$

  one has $\ker D^*_{\mathscr{C}\times\mathscr{D}}f(\bar{x})=\mathbb{R}_-$, so f is not metrically regular relative to $\mathscr{C}\times\mathscr{D}$ around $\bar{x}$.
  We next check this assertion by definition. Indeed, for any neighborhoods U and V of 0, we take $x=0$ and a sequence $y_k=\frac{1}{k}$. Then $y_k\in V$ for sufficiently large k and

$$f^{-1}(y_k)\cap\mathscr{C}=\left\{1+\sqrt{1+y_k}\right\}.$$

  Moreover, we also have $d(x,f^{-1}(y_k)\cap\mathscr{C})=|1+\sqrt{1+y_k}|>2$, while $|f(x)-f(y_k)|=\frac{1}{k}\to 0$. Thus there is no $\tau>0$ such that

$$d(x,f^{-1}(y_k)\cap\mathscr{C})\leq\tau|f(x)-f(y_k)|\ \forall k,$$

  so f is not metrically regular relative to $\mathscr{C}\times\mathbb{R}$ around $\bar{x}$.
- Let $\mathscr{C}:=\mathbb{R}$ and $\mathscr{D}:=[-1,\infty)$. Then $\mathscr{D}=f(\mathscr{C})$ and f is metrically regular relative to $\mathscr{C}\times\mathscr{D}$ around $\bar{x}$. Indeed, in this case, we have $N_{\mathscr{C}\times\mathscr{D}}((\bar{x},f(\bar{x})),\operatorname{gph} f)=\mathbb{R}(2,1)$, so $\ker D^*_{\mathscr{C}\times\mathscr{D}}f(\bar{x})(0)=\{0\}$.

6. Conclusion

New versions of the limiting coderivative and subdifferentials relative to a set were presented in this study. We characterized the Aubin property relative to a set and the metric regularity relative to a set of multifunctions based on the new limiting coderivative relative to a set. Additionally, the new subdifferentials relative to a set provide the characteristics for the metric regularity and the locally Lipschitz continuity relative to a set of single-valued mappings. The coderivative and subdifferential relative to a set characteristics in this work can be used to identify the Aubin property and the metric regularity of set-valued mappings as well as the locally Lipschitz continuity and the metric regularity relative to a set of single-valued mappings in non-Euclidean finite-dimensional spaces, which is because these stability properties do not depend on equivalent norms of finite-dimensional spaces.

Conflict of interests/competing interests

The authors declare that they have no conflict of interest.

References

[1] J.-P. Aubin and H. Frankowska, Set-valued Analysis. Birkhaüser, Boston (1991).
[2] M. Benko, H. Gfrerer and J.V. Outrata, Calculus for directional limiting normal cones and subdifferentials. *Set-Valued Variat. Anal.* **27** (2019) 713–745.
[3] J.F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems, 1st edn. Springer-Verlag, New York (2000).
[4] T.D. Chuong and D.S. Kim, Holder-like property and metric regularity of a positive-order for implicit multifunctions. *Math. Oper. Res.* **41** (2016) 596–611.

[5] A.L. Dontchev and R.T. Rockafellar, Implicit Functions and Solution Mappings, A View from Variational Analysis, 2nd edn. Springer, New York (2014).
[6] H. Gfrerer, On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs. *Set-Valued Variat. Anal.* **21** (2013) 151–176.
[7] H. Gfrerer and J.V. Outrata, On Lipschitzian properties of implicit multifunctions. *SIAM J. Optim.* **26** (2016) 2160–2189.
[8] A.D. Ioffe, On regularity concepts in variational analysis. *J. Fixed Point Theory Appl.* **8** (2010) 339–363.
[9] B.S. Mordukhovich, Complete characterization of openness, metric regularity, and Lipschitzian properties of multifunctions. *Trans. Amer. Math. Soc.* **340** (1993) 1–35.
[10] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation. I. Basic Theory. Springer-Verlag, Berlin (2006).
[11] R.A. Poliquin and R.T. Rockafellar, Tilt stability of a local minimum. *SIAM J. Optim.* **8** (1998) 287–299.
[12] R.T. Rockafellar, Convex Analysis. Princeton University Press, Princeton (1996).
[13] N.D. Yen, Lipschitz continuity of solutions of variational inequalities with a parametric polyhedral constraint. *Math. Oper. Res.* **20** (1995) 695–708.
[14] H.V. Ngai, N.H. Tron and M. Théra, Metric regularity relative to a cone. *Vietnam J. Math.* **47** (2019) 733–756.
[15] A.V. Arutyunov and A.F. Izmailov, Directional stability theorem and directional metric regularity. *Math. Oper. Res.* **31** (2006) 526–543.
[16] M. Durea, M. Panţiruc and R. Stugariu, A new type of directional regularity for mappings and applications to optimization. *SIAM J. Optim.* **27** (2017) 12004–1229.
[17] H.V. Ngai and M. Théra, Directional metric regularity of multifunctions. *Math. Oper. Res.* **40** (2015) 797–1088.
[18] K.W. Meng, M.H. Li, W.F. Yao and X.Q. Yang, Lipschitz-like property relative to a set and the generalized Mordukhovich criterion. *Math. Program.* **189** (2021) 45–489.
[19] X. Qin, V.D. Thinh and J.-C. Yao, On new generalized differentials relative to a set and their applications. *Set-Valued Variat. Anal.* **32** (2024) 27.
[20] R.T. Rockafellar and R.-B. Wet, Variational Analysis. Springer, Berlin (1998).
[21] W. Yao and X. Yang, Relative Lipschitz-like property of parametric systems *via* projectional coderivative. *SIAM J. Optim.* **3** (2023) 2021–2040.
[22] B.S. Mordukhovich, P. Wu and X. Yang, Relative well-posedness of constrained systems with applications to variational inequalities (2024), https://arxiv.org/abs/2212.02727
[23] V.D. Thinh, X. Qin and J.-C. Yao, Formulas for calculating of generalized differentials relative to a set and their applications (2023), https://arxiv.org/pdf/2311.08741.pdf
[24] V.D. Thinh, X. Qin and J.-C. Yao, Coderivatives relative to a set of the normal cone mappings and their applications (2023), http://arxiv.org/abs/2312.15834
[25] E.H. Clarke, Y.S. Ledyaev, R.J. Stem and R.R. Wolenski, Nonsmooth Analysis and Control Theory. Springer-Verlag, New York (1998).
[26] W. Yao, K. Meng, M. Li and X. Yang, Projectional coderivatives and calculus rules. *Set-Valued Variat. Anal.* **31** (2023) 36.