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MINIMAL TIME SAMPLED-DATA CONTROLS FOR THE ORDINARY DIFFERENTIAL SYSTEM

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Abstract. In this paper, we design a minimal time optimal control problem for an ordinary differential system with sampled-data controls, and use it to approximate a time optimal control problem for the ordinary differential system with distributed controls. We find connections among this problem, a minimal norm sampled-data control problem and a minimization problem, and obtain some properties on these problems. Based on these, we not only build up error estimates for optimal time and optimal controls between the minimal time sampled-data control problem and the minimal time distributed control problem, in terms of the sampling period, but we also prove that such estimates are optimal in some sense.

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1. Introduction

1.1. Motivation

If you are driving a car and want to know your speed, you do not have to watch the speedometer continuously. It is sufficient that you monitor the continuously indicated velocity by an occasional glance, and that is "Sampling". Likewise the continuously changing voter opinion of political parties is sampled every few years by an election and the composition of the congress is then fixed for the following term of office. Stock market quotations are set every working day, and the temperature of a sick person is checked several times per day (see [1]).

Sampled-data controllers can also be applied to systems with continuous measurement and actuation. Probably the oldest sampled-data controller is the Gouy controller for the temperature control of an oven. A metal bar dips periodically into the mercury of a thermometer. This contact drives a relay which opens and closes the circuit for the heating. If the temperature of the oven produces a higher mercury level, then the heating is turned off for a longer time interval. Sampling is also necessary whenever one wants to use an expensive device which must perform different tasks in sequence. The main reason for the recent wide spread applications of sampled-data control is the availability of cheap microprocessors (see [1]). Hence, in practical application, it is more convenient to use controls that vary only finite times. (see [1–5]).

Keywords and phrases: Minimal time control, sampled-data control, ordinary differential system.

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Minimal time sampled-data control problem is an interesting topic. It is to ask for a sampled-data control from a constraint set so that the corresponding solution to the system (with an initial state) reaches the target in the shortest time. To the best of our knowledge, this kind of problem is seldom studied. In [6], the authors design a time optimal control problem for the heat equation with sampled-data controls, and then use it to approximate a time optimal control problem for the heat equation with distributed controls. The study of such a time optimal sampled-data control problem is not easy, because it may have infinitely many optimal controls. They find connections among this problem, a minimal norm sampled-data control problem and a minimization problem, and obtain some properties on these problems. Based on these, they not only build up error estimates for optimal time and optimal controls between the time optimal sampled-data control problem and the time optimal distributed control problem, in terms of the sampling period, but also they smartly prove that such estimates are optimal in some sense. Motivated by [6], we discuss a minimal time optimal sampled-data control problem for a kind of ordinary differential system in this paper. By using the strategy in [6], we establish error estimates for optimal time and optimal controls between the time optimal sampled-data control problem and the time optimal distributed control problem, in terms of the sampling period. From the viewpoint of numerical experiment, this work is meaningful. Indeed, for the numerical calculation of optimal control problem governed by partial differential equation, an essential step is to discretize the original problem to an optimal control problem governed by ordinary differential equation. It should be mentioned that in [7], the authors establish error estimate between the optimal distributed control and the optimal sampled-data control for an optimal control problem governed by periodic heat equation.

1.2. Formulation of problems

This subsection formulates problems studied in this paper. We begin with introducing some notations and the controlled system. $\mathbb{N}^+ \triangleq \{1, 2, 3, ...\}$; For each measurable set E in $\mathbb{R}$, $|E|$ denotes its Lebesgue measure; $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the usual inner product and Euclidean norm of $\mathbb{R}^\ell$ ($\ell \in \mathbb{N}^+$), respectively; For each $D \in \mathbb{R}^{s\times\ell}$ ($s, \ell \in \mathbb{N}^+$), $D^\top$ and $\|D\|$ denote the transpose and the spectral norm of D, respectively, more precisely, $\|D\| \triangleq \sup\{\|Dx\| : x \in \mathbb{R}^\ell, \|x\| = 1\}$; $C(\dots)$ denotes a positive constant dependent on what are enclosed in the bracket. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be two given constant matrices ($n, m \in \mathbb{N}^+$). Let $B_r(0)$ be the closed ball in $\mathbb{R}^n$, centered at 0 and of radius $r > 0$. In this paper, we study the following controlled linear ordinary differential system:

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \quad t > 0, \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $y_0 \in \mathbb{R}^n$ and $u \in L^2(0, +\infty; \mathbb{R}^m)$ is a control. Write $y(\cdot; y_0, u)$ for the solution of the system (1.1). It is well known that for each $T > 0$, $y(\cdot; y_0, u) \in C([0, T]; \mathbb{R}^n)$.

Given $M > 0$ and $y_0 \in \mathbb{R}^n \setminus B_r(0)$, we define control constraint set $\mathscr{U}^M$ and the minimal time optimal control problem $(\mathcal{TP})^M$ as follows:

$$\begin{aligned} &\mathscr{U}^M \triangleq \left\{u \in L^2(0, +\infty; \mathbb{R}^m) : \|u\|_{L^2(0,+\infty;\mathbb{R}^m)} \le M\right\}; \\ (\mathcal{TP})^M : \quad &\mathcal{T}(M) \triangleq \inf\left\{t > 0 : \exists u \in \mathscr{U}^M \text{ s.t. } y(t; y_0, u) \in B_r(0)\right\}. \end{aligned}$$

About Problem $(\mathcal{TP})^M$, some concepts are given in order: we call $\mathcal{T}(M)$ the optimal time; we call $\widehat{u} \in \mathscr{U}^M$ an admissible control if $y(\widehat{t}; y_0, \widehat{u}) \in B_r(0)$ for some $\widehat{t} > 0$; we call $u^* \in \mathscr{U}^M$ an optimal control if $y(\mathcal{T}(M); y_0, u^*) \in B_r(0)$.

Next, we are going to design a minimal time optimal sampled-data control problem for (1.1). To this end, for each $\delta > 0$, we define the space of sampled-data controls:

$$L^2_\delta(0,+\infty;\mathbb{R}^m) \triangleq \left\{ u_\delta \in L^2(0,+\infty;\mathbb{R}^m) : u_\delta(t) \triangleq \sum_{i=1}^{\infty} \chi_{((i-1)\delta,i\delta]}(t)u^i \text{ and } u^i \in \mathbb{R}^m \text{ for each } i \in \mathbb{N}^+ \right\},$$

where $\chi_{((i-1)\delta,i\delta]}(\cdot)$ $(i \in \mathbb{N}^+)$ denotes the characteristic function of the interval $((i-1)\delta, i\delta]$. The numbers $\delta, 2\delta, ..., i\delta, ...$ are called the sampling instants, while δ is called the sampling period. Each u_δ in space $L^2_\delta(0,+\infty;\mathbb{R}^m)$ is called a sampled-data control. Given $M > 0, \delta > 0$ and $y_0 \in \mathbb{R}^n \setminus B_r(0)$, we define control constraint set $\mathscr{U}^M_\delta$ and the minimal time optimal sampled-data control problem $(\mathcal{TP})^M_\delta$ as follows:

$$\mathscr{U}^M_\delta \triangleq \left\{ u_\delta \in L^2_\delta(0,+\infty;\mathbb{R}^m) : \|u_\delta\|_{L^2(0,+\infty;\mathbb{R}^m)} \leq M \right\};$$

$$(\mathcal{TP})^M_\delta : \quad \mathcal{T}_\delta(M) \triangleq \inf \left\{ k\delta : \exists k \in \mathbb{N}^+ \text{ and } u_\delta \in \mathscr{U}^M_\delta \text{ s.t. } y(k\delta; y_0, u_\delta) \in B_r(0) \right\}. \tag{1.2}$$

It is clear that $\mathscr{U}^M_\delta \subset \mathscr{U}^M$ and $\mathcal{T}_\delta(M) \geq \mathcal{T}(M)$ for each $M, \delta > 0$.

About $(\mathcal{TP})^M_\delta$, some concepts are given in order: we call $\mathcal{T}_\delta(M)$ the optimal time; we call $\widehat{u}_\delta \in \mathscr{U}^M_\delta$ an admissible control if there exists $\widehat{k} \in \mathbb{N}^+$ so that $y(\widehat{k}\delta; y_0, \widehat{u}_\delta) \in B_r(0)$; we call $u_\delta \in \mathscr{U}^M_\delta$ an optimal control if $y(\mathcal{T}_\delta(M); y_0, u_\delta) \in B_r(0)$; we call u^*_δ the optimal control with the minimal norm if u^*_δ is an optimal control and $\|u^*_\delta\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} \leq \|v_\delta\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}$ for any optimal control v_δ.

1.3. Minimal norm control problems and minimization problems

Our studies on time optimal control problems $(\mathcal{TP})^M$ and $(\mathcal{TP})^M_\delta$ are related to the following two norm optimal control problems. The first one corresponding to $(\mathcal{TP})^M$ reads:

$$(\mathcal{NP})^T : \mathcal{N}(T) \triangleq \inf\{\|v\|_{L^2(0,T;\mathbb{R}^m)} : v \in L^2(0,T;\mathbb{R}^m), y(T; y_0, v) \in B_r(0)\}, \tag{1.3}$$

where $T > 0$. The second one corresponding to $(\mathcal{TP})^M_\delta$ reads

$$(\mathcal{NP})^{k\delta}_\delta : \mathcal{N}_\delta(k\delta) \triangleq \inf\{\|v_\delta\|_{L^2(0,k\delta;\mathbb{R}^m)} : v_\delta \in L^2_\delta(0,k\delta;\mathbb{R}^m), y(k\delta; y_0, v_\delta) \in B_r(0)\}, \tag{1.4}$$

where $(\delta, k) \in (0,+\infty) \times \mathbb{N}^+$. It is clear that $\mathcal{N}_\delta(k\delta) \geq \mathcal{N}(k\delta)$ for each $(\delta, k) \in (0,+\infty) \times \mathbb{N}^+$.

About the above two norm optimal control problems, two notes are given in order.

- (a_1) In the problem $(\mathcal{NP})^T$, we call $\mathcal{N}(T)$ the optimal norm; we call $v \in L^2(0,T;\mathbb{R}^m)$ an admissible control if $y(T; y_0, v) \in B_r(0)$; we call v^* an optimal control if v^* is an admissible control and $\|v^*\|_{L^2(0,T;\mathbb{R}^m)} = \mathcal{N}(T)$.
- (a_2) In the problem $(\mathcal{NP})^{k\delta}_\delta$, we call $\mathcal{N}_\delta(k\delta)$ the optimal norm; we call $v_\delta \in L^2_\delta(0,k\delta;\mathbb{R}^m)$ an admissible control if $y(k\delta; y_0, v_\delta) \in B_r(0)$; we call v^*_δ an optimal control if v^*_δ is an admissible control and $\|v^*_\delta\|_{L^2(0,k\delta;\mathbb{R}^m)} = \mathcal{N}_\delta(k\delta)$.

We mention that both $(\mathcal{NP})^T$ and $(\mathcal{NP})^{k\delta}_\delta$ have unique nonzero solutions (see Thms. 4.1 and 4.3). Inspired by [8], we study the above two minimal norm control problems by two minimization problems. The first one corresponding to $(\mathcal{NP})^T$ is:

$$(\mathcal{JP})^T : V(T) \triangleq \inf_{z \in \mathbb{R}^n} \mathcal{J}^T(z) \triangleq \inf_{z \in \mathbb{R}^n} \left[\frac{1}{2}\|B^\top \varphi(\cdot; T, z)\|^2_{L^2(0,T;\mathbb{R}^m)} + \langle y_0, \varphi(0; T, z)\rangle + r\|z\| \right], \tag{1.5}$$

where $\varphi(\cdot;T,z)\in C([0,T];\mathbb{R}^n)$ is the unique solution to the adjoint system:

$$\begin{cases}\varphi'(t)=-A^\top\varphi(t), \quad t\in[0,T],\\ \varphi(T)=z.\end{cases} \tag{1.6}$$

The second minimization problem corresponding to $(\mathcal{NP})_\delta^{k\delta}$ is:

$$(\mathcal{JP})_\delta^{k\delta}: V_\delta(k\delta)\triangleq \inf_{z\in\mathbb{R}^n}\mathcal{J}_\delta^{k\delta}(z)\triangleq\inf_{z\in\mathbb{R}^n}\left[\frac{1}{2}\|B^\top\overline{\varphi}_\delta(\cdot;k\delta,z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)}+\langle y_0,\varphi(0;k\delta,z)\rangle+r\|z\|\right], \tag{1.7}$$

where $\overline{\varphi}_\delta(\cdot;k\delta,z)$ is defined by:

$$\overline{\varphi}_\delta(t;k\delta,z)\triangleq\sum_{i=1}^{k}\chi_{((i-1)\delta,i\delta]}(t)\frac{1}{\delta}\int_{(i-1)\delta}^{i\delta}\varphi(s;k\delta,z)\mathrm{d}s \ \text{ for each } t\in(0,k\delta]. \tag{1.8}$$

We mention that both $(\mathcal{JP})^T$ and $(\mathcal{JP})_\delta^{k\delta}$ have unique nonzero minimizers (see Thms. 4.1 and 4.3).

1.4. Hypotheses and Main results

Hypotheses. Our main theorems are based on the following two hypotheses.

(H_1) $rank(B,AB,...,A^{n-1}B)=n$;
(H_2) There is a positive constant σ_0 so that

$$\|\mathrm{e}^{tA}\|\le\mathrm{e}^{-\sigma_0 t} \ \text{ for each } t\in[0,+\infty).$$

Remark 1.1. If $A\in\mathbb{R}^{n\times n}$ is a negative definite matrix, we can easily check that A satisfies (H_2) (where $\sigma_0=|\lambda_{\max}(A)|$ and $\lambda_{\max}(A)$ is the maximal eigenvalue of A).

The main results are stated as follows.

Theorem 1.2. *Suppose that (H_1) and (H_2) hold. Then*
(i) there is a constant $\delta_0\triangleq\delta_0(M,y_0,r,A,B)>0$ so that

$$0\le\mathcal{T}_\delta(M)-\mathcal{T}(M)<2\delta \ \textit{ for each } \delta\in(0,\delta_0); \tag{1.9}$$

(ii) for each $\eta\in(0,1)$, there exists a measurable set $\mathcal{A}_{M,\eta}$ (depending also on y_0, r, A and B) with $\lim_{h\to0^+}\frac{1}{h}|\mathcal{A}_{M,\eta}\cap(0,h)|=\eta$ so that

$$\delta>\mathcal{T}_\delta(M)-\mathcal{T}(M)>(1-\eta)\delta \ \textit{ for each } \delta\in\mathcal{A}_{M,\eta}. \tag{1.10}$$

Theorem 1.3. *Suppose that (H_1) and (H_2) hold. Let u^* be the optimal control to $(\mathcal{TP})^M$. For each $\delta>0$, let u_δ^* be the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$. Then*
(i) there is a constant $C\triangleq C(M,y_0,r,A,B)>0$ so that

$$\|u_\delta^*-u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}\le C\delta \ \textit{ for each } \delta>0; \tag{1.11}$$

(ii) for each $\eta \in (0,1)$, there is a measurable set $\mathcal{A}_{M,\eta}$ (depending also on y_0, r, A and B) with $\lim_{h\to 0^+}\frac{1}{h}|\mathcal{A}_{M,\eta}\cap(0,h)| = \eta$ so that

$$\|u_\delta^* - u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} > \frac{\sigma_0^{3/2} r}{2\|B\|}(1-\eta)\delta \quad \text{for each } \delta \in \mathcal{A}_{M,\eta}. \tag{1.12}$$

Theorem 1.4. *Suppose that (H_1) and (H_2) hold. Let u^* be the optimal control to $(\mathcal{TP})^M$. For each $\delta > 0$, let u_δ be any optimal control to $(\mathcal{TP})_\delta^M$. Then*

(i) there is a constant $C \triangleq C(M, y_0, r, A, B) > 0$ so that

$$\|u_\delta - u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} \le C\sqrt{\delta} \quad \text{for each } \delta > 0; \tag{1.13}$$

(ii) for each $\eta \in (0,1)$, there exist a constant $C \triangleq C(M, y_0, r, A, B) > 0$ and a measurable set $\mathcal{A}_{M,\eta}$ (depending also on y_0, r, A and B) with $\lim_{h\to 0^+}\frac{1}{h}|\mathcal{A}_{M,\eta}\cap(0,h)| = \eta$ so that

$$\|\hat{u}_\delta - u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} > C\sqrt{(1-\eta)\delta} \tag{1.14}$$

for each $\delta \in \mathcal{A}_{M,\eta}$ and for some optimal control $\hat{u}_\delta$ to $(\mathcal{TP})_\delta^M$.

The rest of this paper is organized as follows: Section 2 presents two observability estimates. Section 3 gives the existence and uniqueness of optimal control to $(\mathcal{TP})^M$, and optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$. Section 4 shows some connections among time optimal control problems, norm optimal control problems and minimization problems. Section 5 presents several auxiliary estimates. Section 6 proves the main results.

2. Observability estimates

In this section, we prove two observability estimates, which will play an important role in getting the main results.

For each $\ell \in \mathbb{N}^+$ and $(\delta, k) \in (0, +\infty) \times \mathbb{N}^+$, we define

$$\bar{f}_\delta(t) \triangleq \sum_{i=1}^{k} \chi_{((i-1)\delta, i\delta]}(t)\frac{1}{\delta}\int_{(i-1)\delta}^{i\delta} f(s)\mathrm{d}s \quad \text{for each } t \in (0, k\delta], \tag{2.1}$$

where $f \in L^2(0, k\delta; \mathbb{R}^\ell)$.

Lemma 2.1. *For each $\ell \in \mathbb{N}^+$ and $(\delta, k) \in (0, +\infty) \times \mathbb{N}^+$, the following equalities hold:*

$$\langle \bar{f}_\delta, g\rangle_{L^2(0,k\delta;\mathbb{R}^\ell)} = \langle f, \bar{g}_\delta\rangle_{L^2(0,k\delta;\mathbb{R}^\ell)} = \langle \bar{f}_\delta, \bar{g}_\delta\rangle_{L^2(0,k\delta;\mathbb{R}^\ell)}, \tag{2.2}$$

where $f, g \in L^2(0, k\delta; \mathbb{R}^\ell)$.

Proof. For each $f, g \in L^2(0, k\delta; \mathbb{R}^\ell)$, it follows from (2.1) that

$$\begin{aligned}
\langle \bar{f}_\delta, g\rangle_{L^2(0,k\delta;\mathbb{R}^\ell)} &= \sum_{i=1}^{k}\left\langle \bar{f}_\delta(i\delta), \int_{(i-1)\delta}^{i\delta} g(t)\mathrm{d}t\right\rangle \\
&= \sum_{i=1}^{k}\delta\langle \bar{f}_\delta(i\delta), \bar{g}_\delta(i\delta)\rangle = \sum_{i=1}^{k}\langle \bar{f}_\delta, \bar{g}_\delta\rangle_{L^2((i-1)\delta, i\delta;\mathbb{R}^\ell)},
\end{aligned}$$

which indicates (2.2). □

The first observability estimate is as follows.

Theorem 2.2. *Suppose that (H_1) hold. Then there is a constant $C \triangleq C(A,B)>0$ so that*

$$\|z\| \le \mathrm{e}^{C(T+\frac{1}{T})}\|B^\top\varphi(\cdot;T,z)\|_{L^2(0,T;\mathbb{R}^m)} \;\; \textit{for each } T>0 \;\; \textit{and} \;\; z\in\mathbb{R}^n. \tag{2.3}$$

Proof. Note that

$$rank(B, AB, ..., A^{n-1}B) = n \Leftrightarrow rank(B, -AB, ..., (-A)^{n-1}B) = n.$$

This, along with (H_1), implies that

$$rank(B, -AB, ..., (-A)^{n-1}B) = n.$$

It follows from the latter and Proposition 2.4 of [9] that there is a constant $\widehat{C} \triangleq \widehat{C}(A,B)>0$ so that

$$\|\mathrm{e}^{-TA^\top}z\| \le \widehat{C}\mathrm{e}^{\frac{\widehat{C}}{T}}\int_0^T \|B^\top\mathrm{e}^{-tA^\top}z\|\mathrm{d}t \;\; \text{for each } T>0 \;\; \text{and} \;\; z\in\mathbb{R}^n,$$

which indicates that

$$\|z\| \le \widehat{C}\mathrm{e}^{\frac{\widehat{C}}{T}}\int_0^T \|B^\top\mathrm{e}^{(T-t)A^\top}z\|\mathrm{d}t \le \widehat{C}\mathrm{e}^{\frac{\widehat{C}}{T}}\sqrt{T}\|B^\top\varphi(\cdot;T,z)\|_{L^2(0,T;\mathbb{R}^m)}.$$

Hence, (2.3) holds. □

Before presenting the other observability estimate, we need some preparations. Let

$$T^* \triangleq \min\{t>0 \;:\; \|\mathrm{e}^{tA}y_0\| \le r\}, \tag{2.4}$$

$$\mathcal{P} \triangleq \left\{(\delta,k)\in(0,+\infty)\times\mathbb{N}^+ : \mathrm{e}^{2C(k\delta+\frac{1}{k\delta})}k\delta^3\|AB\|^2 \le \frac{3}{4}\right\} \tag{2.5}$$

and

$$\mathcal{P}_{T^*} \triangleq \left\{(\delta,k)\in(0,+\infty)\times\mathbb{N}^+ : \mathrm{e}^{2C(k\delta+\frac{1}{k\delta})}k\delta^3\|AB\|^2 \le \frac{3}{4},\; k\delta<T^*\right\}, \tag{2.6}$$

where $C=C(A,B)>0$ is given by Theorem 2.2. Three observations are listed in order.

(b_1) $0<T^*<\infty$. Indeed, this follows from (H_2) and the fact that $y_0\in\mathbb{R}^n\setminus B_r(0)$.
(b_2) $\mathcal{P}_{T^*}\subset\mathcal{P}$.
(b_3) $\mathcal{P}_{T^*}$ is not an empty set. For example, let

$$0<\delta\le\min\left(T^*/2, \sqrt{3/\left(4\mathrm{e}^{2C(T^*+\frac{2}{T^*})}T^*\|AB\|^2\right)}\right).$$

It is clear that there exists $k_\delta \in \mathbb{N}^+$ satisfying $k_\delta - 1 < T^*/(2\delta) \leq k_\delta$, which indicates $T^*/2 \leq k_\delta\delta < T^*/2 + \delta \leq T^*$. This implies that

$$\mathrm{e}^{2C(k_\delta\delta + \frac{1}{k_\delta\delta})} k_\delta \delta^3 \|AB\|^2 \leq \mathrm{e}^{2C(T^* + \frac{2}{T^*})} T^* \delta^2 \|AB\|^2 \leq \frac{3}{4}.$$

Hence, $(\delta, k_\delta) \in \mathcal{P}_{T^*}$.

The second observability estimate reads as follows.

Theorem 2.3. *Suppose that* (H_1) *and* (H_2) *hold. For each* $(\delta, k) \in \mathcal{P}$, *we have that*

$$\|z\| \leq 2\mathrm{e}^{C(k\delta + \frac{1}{k\delta})} \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|_{L^2(0,k\delta;\mathbb{R}^m)} \quad \text{for each } z \in \mathbb{R}^n, \tag{2.7}$$

where $C = C(A, B) > 0$ *is given by Theorem 2.2.*

Proof. Arbitrarily fix $z \in \mathbb{R}^n$. We first claim that for each $(\delta, k) \in (0, +\infty) \times \mathbb{N}^+$,

$$\|B^\top \varphi(\cdot; k\delta, z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} - \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} \leq k\delta^3 \|AB\|^2 \|z\|^2. \tag{2.8}$$

Throughout the proof of this Theorem, we simply write $\varphi(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$ for $\varphi(\cdot; k\delta, z)$ (see (1.6)) and $\overline{\varphi}_\delta(\cdot; k\delta, z)$ (see (1.8)), respectively. Indeed, by (2.1), (1.8) and Lemma 2.1 (where $\bar{f}_\delta \triangleq B^\top \overline{\varphi}_\delta$ and $g \triangleq B^\top \varphi - B^\top \overline{\varphi}_\delta$), we get that

$$\langle B^\top \overline{\varphi}_\delta, B^\top \varphi - B^\top \overline{\varphi}_\delta \rangle_{L^2(0,k\delta;\mathbb{R}^m)} = 0,$$

which indicates that

$$\begin{aligned}
&\|B^\top \varphi\|^2_{L^2(0,k\delta;\mathbb{R}^m)} - \|B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)} \\
=&\|B^\top \varphi - B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)} + 2\langle B^\top \overline{\varphi}_\delta, B^\top \varphi - B^\top \overline{\varphi}_\delta \rangle_{L^2(0,k\delta;\mathbb{R}^m)} \\
=&\int_0^{k\delta} \|B^\top \varphi(t) - B^\top \overline{\varphi}_\delta(t)\|^2 \mathrm{d}t.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|B^\top \varphi\|^2_{L^2(0,k\delta;\mathbb{R}^m)} - \|B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)} =& \sum_{j=1}^{k} \int_{(j-1)\delta}^{j\delta} \left\| B^\top \varphi(t) - \frac{1}{\delta} \int_{(j-1)\delta}^{j\delta} B^\top \varphi(s) \mathrm{d}s \right\|^2 \mathrm{d}t \\
=& \sum_{j=1}^{k} \int_{(j-1)\delta}^{j\delta} \left\| \frac{1}{\delta} \int_{(j-1)\delta}^{j\delta} \int_s^t B^\top \varphi'(\tau) \mathrm{d}\tau \mathrm{d}s \right\|^2 \mathrm{d}t \\
\leq& \sum_{j=1}^{k} \int_{(j-1)\delta}^{j\delta} \left(\int_{(j-1)\delta}^{j\delta} \|B^\top \varphi'(\tau)\| \mathrm{d}\tau \right)^2 \mathrm{d}t.
\end{aligned}$$

This, along with (H_2), implies that

$$\|B^\top \varphi\|^2_{L^2(0,k\delta;\mathbb{R}^m)} - \|B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)} \leq \sum_{j=1}^{k} \delta \left(\int_{(j-1)\delta}^{j\delta} \|B^\top A^\top \mathrm{e}^{(k\delta-\tau)A^\top} z\| \mathrm{d}\tau \right)^2$$
$$\leq \sum_{j=1}^{k} \delta \left(\delta \|B^\top A^\top\| \|z\| \right)^2,$$

which leads to (2.8).

Next, for each $(\delta, k) \in \mathcal{P}$, it follows from Theorem 2.2 (where $T = k\delta$) and (2.8) that

$$\|z\|^2 \leq \mathrm{e}^{2C(k\delta + \frac{1}{k\delta})} \|B^\top \varphi(\cdot; k\delta, z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)}$$
$$\leq \mathrm{e}^{2C(k\delta + \frac{1}{k\delta})} \left(\|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} + k\delta^3 \|AB\|^2 \|z\|^2 \right),$$

where $C = C(A, B) > 0$ is given by Theorem 2.2. The latter inequality, together with (2.5), yields that

$$\frac{1}{4}\|z\|^2 \leq \mathrm{e}^{2C(k\delta + \frac{1}{k\delta})} \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)},$$

which indicates (2.7).

This completes the proof. □

Remark 2.4. In the poof of Theorem 2.3, (H_2) can be weakened as follows:

$(H_2)'$ For each eigenvalue λ of the matrix A, $Re(\lambda) < 0$.

3. Existence and uniqueness of optimal controls

In this section, we assume that conditions (H_1) and $(H_2)'$ always hold. We will prove that for each $M > 0$, $(\mathcal{TP})^M$ has a unique optimal control; for each $M, \delta > 0$, $(\mathcal{TP})^M_\delta$ has a unique optimal control with the minimal norm; for some $M, \delta > 0$, $(\mathcal{TP})^M_\delta$ has infinitely many optimal controls.

Theorem 3.1. *Let $M > 0$. The following conclusions are true:*

(i) The problem $(\mathcal{TP})^M$ has a unique optimal control;

(ii) For each $\delta > 0$, $(\mathcal{TP})^M_\delta$ has a unique optimal control with the minimal norm;

(iii) Let u^*_δ (with $\delta > 0$) be the optimal control with the minimal norm to $(\mathcal{TP})^M_\delta$. Then $u^*_\delta|_{(0,\mathcal{T}_\delta(M))}$ is an optimal control to $(\mathcal{NP})^{\mathcal{T}_\delta(M)}_\delta$ and the $L^2(0, \mathcal{T}_\delta(M); \mathbb{R}^m)$-norm of u^*_δ is $\mathcal{N}_\delta(\mathcal{T}_\delta(M))$.

Proof. Arbitrarily fix $M > 0$. We will prove Theorem 3.1 one by one.

(i) Since $(H_2)'$ holds, $y(t; y_0, 0) \to 0$ as $t \to +\infty$. This implies that 0 is an admissible control to $(\mathcal{TP})^M$. By a standard argument (*i.e.*, taking a minimization sequence), we can easily check that $(\mathcal{TP})^M$ has at least one optimal control. To show the uniqueness of the optimal control to $(\mathcal{TP})^M$, we first note that each optimal control u^* to $(\mathcal{TP})^M$ has the following property:

$$\|u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} = M. \tag{3.1}$$

((3.1) can be proved in the same way as that used to show Lemma 4.3 of [10].) Furthermore, let u^*_1 and u^*_2 be two optimal controls to $(\mathcal{TP})^M$. It is clear that $(u^*_1 + u^*_2)/2$ is also an optimal control to $(\mathcal{TP})^M$. By (3.1), we have that

$$\|u_1^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} = \|u_2^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} = \|(u_1^*+u_2^*)/2\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)},$$

which, combined with the parallelogram law, indicates that

$$\|u_1^* - u_2^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} = 0.$$

Hence, $u_1^*(t) = u_2^*(t)$ a.e. $t \in (0, T(M))$.

(ii) Arbitrarily fix $\delta > 0$. Since $(H_2)'$ holds, $y(t; y_0, 0) \to 0$ as $t \to +\infty$. This implies that 0 is an admissible control to $(\mathcal{TP})_\delta^M$. Then there exists a minimization sequence $\{k_i\}_{i\geq 1} \subset \mathbb{N}^+$ and $\{u_\delta^i\}_{i\geq 1} \subset \mathscr{U}_\delta^M$ so that

$$\mathcal{T}_\delta(M) = \lim_{i\to+\infty} k_i\delta \text{ and } y(k_i\delta; y_0, u_\delta^i) \in B_r(0). \tag{3.2}$$

It follows from (3.2) that

$$\mathcal{T}_\delta(M) = k_{i_0}\delta \text{ and } y(\mathcal{T}_\delta(M); y_0, u_\delta^{i_0}) \in B_r(0) \text{ for some } i_0 \in \mathbb{N}^+,$$

which indicates that $u_\delta^{i_0}$ is an optimal control to $(\mathcal{TP})_\delta^M$. Thus, by a standard argument (*i.e.*, taking a minimization sequence), we can show that $(\mathcal{TP})_\delta^M$ has at least one optimal control with the minimal norm. Furthermore, the uniqueness of optimal control with the minimal norm follows from similar arguments as those in (i).

(iii) Let u_δ^* be the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$. Then

$$u_\delta^* \in \mathscr{U}_\delta^M,\ y(\mathcal{T}_\delta(M); y_0, u_\delta^*) \in B_r(0) \text{ and } \mathcal{T}_\delta(M) = \hat{k}\delta \text{ for some } \hat{k} \in \mathbb{N}^+. \tag{3.3}$$

These yield that

$$u_\delta^*|_{(0,\mathcal{T}_\delta(M))} \text{ is an admissible control to } (\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)} \tag{3.4}$$

and

$$\mathcal{N}_\delta(\mathcal{T}_\delta(M)) \leq \|u_\delta^*|_{(0,\mathcal{T}_\delta(M))}\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} \leq M. \tag{3.5}$$

By (3.4) and a standard argument (*i.e.*, taking a minimization sequence), we have that $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$ has at least one optimal control, denoted by $v_\delta^* \in L_\delta^2(0, \mathcal{T}_\delta(M); \mathbb{R}^m)$. Define the zero extension of v_δ^* over $(0, +\infty)$ as $\widetilde{v}_\delta^*$. It is clear that

$$y(\mathcal{T}_\delta(M); y_0, \widetilde{v}_\delta^*) \in B_r(0) \text{ and } \|\widetilde{v}_\delta^*\|_{L^2(0,+\infty;\mathbb{R}^m)} = \mathcal{N}_\delta(\mathcal{T}_\delta(M)). \tag{3.6}$$

These, along with (3.5), imply that $\widetilde{v}_\delta^*$ is an optimal control to $(\mathcal{TP})_\delta^M$. Since u_δ^* is the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$, it follows from (3.5) and the second equality in (3.6) that

$$\mathcal{N}_\delta(\mathcal{T}_\delta(M)) \leq \|u_\delta^*\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} \leq \|\widetilde{v}_\delta^*\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} = \mathcal{N}_\delta(\mathcal{T}_\delta(M)). \tag{3.7}$$

Hence, by (3.7) and (3.4), we have that

$$\|u_\delta^*\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} = \mathcal{N}_\delta(\mathcal{T}_\delta(M)),$$

and $u_\delta^*|_{(0,\mathcal{T}_\delta(M))}$ is an optimal control to $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$.

In summary, we finish the proof of Theorem 3.1. □

Lemma 3.2. *For each* $(M,\delta)\in(0,+\infty)\times(0,+\infty)$*, it stands that*

$$\mathcal{N}_\delta(\mathcal{T}_\delta(M))\leq M<\mathcal{N}_\delta(\mathcal{T}_\delta(M)-\delta). \tag{3.8}$$

(Here, we agree that $\mathcal{N}_\delta(0)\triangleq+\infty$*).*

Proof. Let $(M,\delta)\in(0,+\infty)\times(0,+\infty)$. According to (ii) and (iii) of Theorem 3.1, there is $\hat{k}\in\mathbb{N}^+$ so that

$$\mathcal{T}_\delta(M)=\hat{k}\delta \text{ and } \mathcal{N}_\delta(\mathcal{T}_\delta(M))=\|u_\delta^*|_{(0,\hat{k}\delta)}\|_{L^2(0,\hat{k}\delta;\mathbb{R}^m)}\leq M. \tag{3.9}$$

Here, u_δ^* is the unique optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$.

Next we show that

$$M<\mathcal{N}_\delta(\mathcal{T}_\delta(M)-\delta). \tag{3.10}$$

By contradiction, we suppose that

$$\mathcal{N}_\delta(\mathcal{T}_\delta(M)-\delta)=\mathcal{N}_\delta((\hat{k}-1)\delta)\leq M, \tag{3.11}$$

which shows that $(\mathcal{NP})_\delta^{(\hat{k}-1)\delta}$ has an admissible control. By a standard argument, we have that $(\mathcal{NP})_\delta^{(\hat{k}-1)\delta}$ has an optimal control, denoted by v_δ^*. Then

$$\|v_\delta^*\|_{L^2_\delta(0,(\hat{k}-1)\delta;\mathbb{R}^m)}=\mathcal{N}_\delta((\hat{k}-1)\delta) \text{ and } y((\hat{k}-1)\delta;y_0,v_\delta^*)\in B_r(0). \tag{3.12}$$

Define the zero extension of v_δ^* over $(0,+\infty)$ as $\widetilde{v}_\delta^*$. Then we would get from (3.11) and (3.12) that $\widetilde{v}_\delta^*$ is an admissible control to $(\mathcal{TP})_\delta^M$ and $\mathcal{T}_\delta(M)\leq(\hat{k}-1)\delta$, which contradicts the equality in (3.9). Hence, (3.10) holds.

Finally, (3.8) follows from the inequality in (3.9) and (3.10). □

Theorem 3.3. *There are sequences* $\{M_\ell\}_{\ell\geq1}$ *and* $\{\delta_\ell\}_{\ell\geq1}\subset(0,+\infty)$*, with* $\lim_{\ell\to\infty}\delta_\ell=0$*, so that for each* $\ell\in\mathbb{N}^+$*, the problem* $(\mathcal{TP})_{\delta_\ell}^{M_\ell}$ *has infinitely many different optimal controls.*

Proof. Arbitrarily fix $M_0>0$. Let T^* be given by (2.4). Choose a sequence $\{M_\ell\}_{\ell\geq1}$ so that

$$\{M_\ell\}_{\ell\geq1}\subset(0,M_0]\setminus\{\mathcal{N}_{T^*/k}(jT^*/k):k,j\in\mathbb{N}^+\}. \tag{3.13}$$

According to (i) and (ii) of Theorem 3.1, it follows from (2.4) that

$$0<\mathcal{T}(M_0)\leq\mathcal{T}(M_\ell)\leq\mathcal{T}_{T^*/k}(M_\ell)\leq T^* \text{ for each } k,\ell\in\mathbb{N}^+. \tag{3.14}$$

Let $k_0\in\mathbb{N}^+$ satisfy

$$T^*/k_0\leq\sqrt{3/\left(4e^{2C(T^*+\frac{1}{\mathcal{T}(M_0)})}T^*\|AB\|^2\right)}, \tag{3.15}$$

where $C=C(A,B)$ is given by Theorem 2.2. Write $\delta_\ell\triangleq T^*/(k_0-1+\ell)$ $(\ell\geq1)$. This, along with (3.15), (3.14) and (2.5), yields that

$$(\delta_\ell,\mathcal{T}_{\delta_\ell}(M_\ell)/\delta_\ell)\in\mathcal{P} \text{ for each } \ell\in\mathbb{N}^+. \tag{3.16}$$

Moreover, by (3.8), we have that

$$\mathcal{N}_{\delta_\ell}(\mathcal{T}_{\delta_\ell}(M_\ell)) \leq M_\ell < \mathcal{N}_{\delta_\ell}(\mathcal{T}_{\delta_\ell}(M_\ell) - \delta_\ell) \text{ for each } \ell \in \mathbb{N}^+,$$

which, combined with (3.13), indicates that

$$\mathcal{N}_{\delta_\ell}(\mathcal{T}_{\delta_\ell}(M_\ell)) < M_\ell < \mathcal{N}_{\delta_\ell}(\mathcal{T}_{\delta_\ell}(M_\ell) - \delta_\ell) \text{ for each } \ell \in \mathbb{N}^+. \tag{3.17}$$

The key to show Theorem 3.3 is to claim that for each $\ell \in \mathbb{N}^+$,

$$(\mathcal{TP})_{\delta_\ell}^{M_\ell} \text{ has at least two different optimal controls.} \tag{3.18}$$

When (3.18) is proved, Theorem 3.3 holds immediately (since any convex combination of optimal controls to $(\mathcal{TP})_{\delta_\ell}^{M_\ell}$ is still an optimal control to $(\mathcal{TP})_{\delta_\ell}^{M_\ell}$). We use a contradiction argument to show (3.18). Suppose that for some $\ell_0 \in \mathbb{N}^+$, $(\mathcal{TP})_{\delta_{\ell_0}}^{M_{\ell_0}}$ has a unique optimal control, denoted by $u^*_{\delta_{\ell_0}}$. Define two convex subsets in $\mathbb{R}^n$ as follows:

$$A_{\ell_0} \triangleq \left\{ y(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0}); y_0, u_{\delta_{\ell_0}}) : \|u_{\delta_{\ell_0}}\|_{L^2_{\delta_{\ell_0}}(0,+\infty;\mathbb{R}^m)} \leq \mathcal{N}_{\delta_{\ell_0}}(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0})) \right\};$$
$$B_{\ell_0} \triangleq \left\{ y(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0}); 0, v_{\delta_{\ell_0}}) : \|v_{\delta_{\ell_0}}\|_{L^2_{\delta_{\ell_0}}(0,+\infty;\mathbb{R}^m)} \leq M_{\ell_0} - \mathcal{N}_{\delta_{\ell_0}}(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0})) \right\}.$$

On one hand, by (ii) and (iii) of Theorem 3.1, we have that

$$y(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0}); y_0, u^*_{\delta_{\ell_0}}) \in B_r(0) \text{ and } \|u^*_{\delta_{\ell_0}}\|_{L^2_{\delta_{\ell_0}}(0,\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0});\mathbb{R}^m)} = \mathcal{N}_{\delta_{\ell_0}}(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0})),$$

which implies that

$$A_{\ell_0} \cap B_r(0) \neq \emptyset. \tag{3.19}$$

On the other hand, according to definitions of A_{ℓ_0} and B_{ℓ_0}, it is clear that

$$A_{\ell_0} \subset (A_{\ell_0} + B_{\ell_0})$$

and

$$(A_{\ell_0} + B_{\ell_0}) \cap B_r(0) = \left\{ y(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0}); y_0, u_{\delta_{\ell_0}}) : u_{\delta_{\ell_0}} \text{ is an optimal control to } (\mathcal{TP})_{\delta_{\ell_0}}^{M_{\ell_0}} \right\}.$$

These, along with (3.19) and the assumption that $(\mathcal{TP})_{\delta_{\ell_0}}^{M_{\ell_0}}$ had a unique optimal control, yield that

$$A_{\ell_0} \cap B_r(0) = (A_{\ell_0} + B_{\ell_0}) \cap B_r(0) = \{\hat{\eta}\} \text{ for some } \hat{\eta} \in \mathbb{R}^n. \tag{3.20}$$

By (3.20), the convexity of $A_{\ell_0} + B_{\ell_0}$ and Hahn-–Banach Separation Theorem, there is a vector $\eta^* \in \mathbb{R}^n$, with $\|\eta^*\| = r > 0$, so that

$$\sup_{w \in A_{\ell_0} + B_{\ell_0}} \langle w, \eta^* \rangle \leq \inf_{z \in B_r(0)} \langle z, \eta^* \rangle.$$

This, together with (3.20), implies that $\langle w+\hat{\eta},\eta^*\rangle \le \langle \hat{\eta},\eta^*\rangle$ for each $w\in B_{\ell_0}$, *i.e.*,

$$\langle w,\eta^*\rangle \le 0 \text{ for each } w\in B_{\ell_0}. \tag{3.21}$$

In the rest proof of this theorem, we simply write $\varphi(\cdot)$ and $\overline{\varphi}_{\delta_{\ell_0}}(\cdot)$ for $\varphi(\cdot;\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0}),\eta^*)$ (see (1.6)) and $\overline{\varphi}_{\delta_{\ell_0}}(\cdot;\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0}),\eta^*)$ (see (1.8)), respectively. Arbitrarily fix $u_{\delta_{\ell_0}}\in L^2_{\delta_{\ell_0}}(0,+\infty;\mathbb{R}^m)$. By the definition of B_{ℓ_0} and (3.17), we observe that

$$y(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0});0,u_{\delta_{\ell_0}})\in \lambda B_{\ell_0} \ \text{ with } \ \lambda=\frac{\|u_{\delta_{\ell_0}}\|_{L^2_{\delta_{\ell_0}}(0,+\infty;\mathbb{R}^m)}}{M_{\ell_0}-\mathcal{N}_{\delta_{\ell_0}}(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0}))}.$$

It follows from the latter and (3.21) that

$$\langle y(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0});0,u_{\delta_{\ell_0}}),\eta^*\rangle \le 0. \tag{3.22}$$

Since $\langle u_{\delta_{\ell_0}},B^\top\varphi\rangle_{L^2(0,\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0});\mathbb{R}^m)}=\langle y(\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0});0,u_{\delta_{\ell_0}}),\eta^*\rangle$, (3.22) shows

$$\langle u_{\delta_{\ell_0}},B^\top\varphi\rangle_{L^2(0,\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0});\mathbb{R}^m)}\le 0,$$

which indicates that

$$B^\top\overline{\varphi}_{\delta_{\ell_0}}(t)=0 \ \text{ for a.e. } \ t\in(0,\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0})). \tag{3.23}$$

By (3.16) and (3.23), we can apply Theorem 2.3 (where $(\delta,k)=(\delta_{\ell_0},\mathcal{T}_{\delta_{\ell_0}}(M_{\ell_0})/\delta_{\ell_0})$) to get that $\eta^*=0$. This leads to a contradiction. Hence, (3.18) holds.

In summary, we finish the proof of Theorem 3.3. □

4. Connections among different problems

This section presents connections among $(\mathcal{TP})^M_\delta,(\mathcal{NP})^{k\delta}_\delta$ and $(\mathcal{JP})^{k\delta}_\delta$ (and among $(\mathcal{TP})^M,(\mathcal{NP})^T$ and $(\mathcal{JP})^T$). We start with giving connections between $(\mathcal{NP})^T$ and $(\mathcal{JP})^T$. They can be proved by same methods used in Lemma 3.6 and Proposition 3.7 in [11]. We omit their proofs here.

Theorem 4.1. *Suppose that (H_1) and (H_2) hold. Let $T\in(0,T^*)$ with T^* given by (2.4). Then*
(i) the problem $(\mathcal{JP})^T$ has a unique nonzero minimizer z^*;
(ii) the problem $(\mathcal{NP})^T$ has a unique optimal control $v^*\in L^2(0,T;\mathbb{R}^m)$, which satisfies that

$$v^*(t)=B^\top\varphi(t;T,z^*), \ \text{a.e. } t\in(0,T) \tag{4.1}$$

and that

$$y(T;y_0,v^*)=-rz^*/\|z^*\|; \tag{4.2}$$

(iii) it holds that $V(T)=-\frac{1}{2}\mathcal{N}(T)^2=-\frac{1}{2}\|B^\top\varphi(\cdot;T,z^*)\|^2_{L^2(0,T;\mathbb{R}^m)}$.

Connections between $(\mathcal{TP})^M$ and $(\mathcal{NP})^T$ can be shown by same arguments used in Theorem 1.1 and Theorem 2.1 of [12], Theorem 5.2 in [13] and Lemma 1 in [14]. We still omit their proofs here.

Theorem 4.2. *Suppose that* (H_1) *and* (H_2) *hold. Then*

(i) the function $T \to \mathcal{N}(T)$ *is strictly decreasing and continuous from* $(0, T^*)$ *onto* $(0, +\infty)$*;*

(ii) when $M > 0$ *and* $T \in (0, T^*)$, $\mathcal{N}(\mathcal{T}(M)) = M$ *and* $\mathcal{T}(\mathcal{N}(T)) = T$*;*

(iii) the function $M \to \mathcal{T}(M)$ *is strictly decreasing and continuous from* $(0, +\infty)$ *onto* $(0, T^*)$*;*

(iv) for each $M > 0$*, the optimal control to* $(\mathcal{TP})^M$*, when restricted on* $(0, \mathcal{T}(M))$*, is the optimal control to* $(\mathcal{NP})^{\mathcal{T}(M)}$*; for each* $T \in (0, T^*)$*, the zero extension of the optimal control to* $(\mathcal{NP})^{\mathcal{T}(M)}$ *is the optimal control to* $(\mathcal{TP})^M$*.*

The next theorem deals with connections between $(\mathcal{NP})_\delta^{k\delta}$ and $(\mathcal{JP})_\delta^{k\delta}$.

Theorem 4.3. *Suppose that* (H_1) *and* (H_2) *hold. Let* $(\delta, k) \in \mathcal{P}_{T^*}$ *(given by (2.6)). Then*

(i) the problem $(\mathcal{JP})_\delta^{k\delta}$ *has a unique nonzero minimizer* z_δ^**;*

(ii) the problem $(\mathcal{NP})_\delta^{k\delta}$ *has a unique optimal control* $v_\delta^* \in L_\delta^2(0, k\delta; \mathbb{R}^m)$*, which satisfies*

$$v_\delta^*(t) = B^\top \overline{\varphi}_\delta(t; k\delta, z_\delta^*), \text{ a.e. } t \in (0, k\delta) \tag{4.3}$$

and

$$y(k\delta; y_0, v_\delta^*) = -r z_\delta^* / \|z_\delta^*\|; \tag{4.4}$$

(iii) it holds that $V_\delta(k\delta) = -\frac{1}{2}\mathcal{N}_\delta(k\delta)^2 = -\frac{1}{2}\|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_\delta^*)\|^2_{L^2(0,k\delta;\mathbb{R}^m)}$*.*

Proof. Arbitrarily fix $(\delta, k) \in \mathcal{P}_{T^*}$. We will prove conclusions (i)–(iii) in Theorem 4.3 one by one.

(i) The continuity of $\mathcal{J}_\delta^{k\delta}(\cdot)$ is obvious. Firstly, we show the existence of minimizers of $(\mathcal{JP})_\delta^{k\delta}$. Since $\mathcal{P}_{T^*} \subset \mathcal{P}$ (given by (2.5)), it follows from (H_2) and (2.7) that for each $z \in \mathbb{R}^n$,

$$\|\varphi(0; k\delta, z)\| \le \|e^{k\delta A^\top}\| \|z\| \le 2e^{C(k\delta + \frac{1}{k\delta})} \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|_{L^2(0,k\delta;\mathbb{R}^m)},$$

where $C = C(A, B) > 0$ is given by Theorem 2.2. This, together with (1.7), implies that for each $z \in \mathbb{R}^n$,

$$\begin{aligned} \mathcal{J}_\delta^{k\delta}(z) &\ge \frac{1}{2}\|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} - \|y_0\| \|\varphi(0; k\delta, z)\| + r\|z\| \\ &\ge \frac{1}{2}\|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} - 2e^{C(k\delta + \frac{1}{k\delta})}\|y_0\| \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|_{L^2(0,k\delta;\mathbb{R}^m)} + r\|z\| \\ &\ge -2e^{2C(k\delta + \frac{1}{k\delta})}\|y_0\|^2 + r\|z\|, \end{aligned} \tag{4.5}$$

which leads to the coercivity of $\mathcal{J}_\delta^{k\delta}(\cdot)$ over $\mathbb{R}^n$. Hence, $\mathcal{J}_\delta^{k\delta}(\cdot)$ has at least one minimizer in $\mathbb{R}^n$.

Secondly, we claim that 0 is not a minimizer of $\mathcal{J}_\delta^{k\delta}(\cdot)$. By contradiction, we suppose that 0 is a minimizer of $\mathcal{J}_\delta^{k\delta}(\cdot)$. Then we would find from (1.7) that for all $z \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$0 \le \frac{\mathcal{J}_\delta^{k\delta}(\varepsilon z) - \mathcal{J}_\delta^{k\delta}(0)}{\varepsilon} = \frac{\varepsilon}{2}\|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} + \langle y_0, \varphi(0; k\delta, z)\rangle + r\|z\|.$$

Passing $\varepsilon \to 0^+$ in the above inequality, we would have that

$$\langle e^{k\delta A} y_0, z\rangle + r\|z\| = \langle y_0, \varphi(0; k\delta, z)\rangle + r\|z\| \ge 0 \text{ for each } z \in \mathbb{R}^n.$$

This yields that

$$r\|z\| \ge |\langle e^{k\delta A} y_0, z\rangle| \text{ for each } z \in \mathbb{R}^n. \tag{4.6}$$

Taking $z = \mathrm{e}^{k\delta A}y_0$ in (4.6), we obtain that $\|\mathrm{e}^{k\delta A}y_0\| \le r$, which contradicts the assumption that $(\delta, k) \in \mathcal{P}_{T^*}$ (see (2.4) and (2.6)). Thus, 0 is not a minimizer of $\mathcal{J}_\delta^{k\delta}(\cdot)$.

Finally, we prove the uniqueness of the minimizer of $\mathcal{J}_\delta^{k\delta}(\cdot)$. To this end, it suffices to show that the function $z \mapsto \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2(0,k\delta;\mathbb{R}^m)}$ is strictly convex. It is clear that for each $z_1, z_2 \in \mathbb{R}^n$ and $\lambda \in (0,1)$,

$$\begin{aligned}
&\|B^\top \overline{\varphi}_\delta(\cdot; k\delta, \lambda z_1 + (1-\lambda) z_2)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} \\
=&\lambda^2 \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_1)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} + (1-\lambda)^2 \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_2)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} \\
&+ 2\lambda(1-\lambda)\langle B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_1), B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_2)\rangle_{L^2(0,k\delta;\mathbb{R}^m)} \\
\le&\lambda \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_1)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} + (1-\lambda) \|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_2)\|^2_{L^2(0,k\delta;\mathbb{R}^m)}.
\end{aligned}$$

We observe that the above inequality becomes equality if and only if

$$\|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_1) - B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_2)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} = 0,$$

i.e.,

$$\|B^\top \overline{\varphi}_\delta(\cdot; k\delta, z_1 - z_2)\|^2_{L^2(0,k\delta;\mathbb{R}^m)} = 0. \tag{4.7}$$

Since $\mathcal{P}_{T^*} \subset \mathcal{P}$, it follows from (2.7) and (4.7) that $z_1 = z_2$.

Hence, $\mathcal{J}_\delta^{k\delta}(\cdot)$ has a unique nonzero minimizer.

(ii) Let z_δ^* be the minimizer of $\mathcal{J}_\delta^{k\delta}(\cdot)$. Let v_δ^* be given by (4.3). From now on and throughout the proof of Theorem 4.3, we simply write $\varphi(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$ for $\varphi(\cdot; k\delta, z_\delta^*)$ and $\overline{\varphi}_\delta(\cdot; k\delta, z_\delta^*)$, respectively.

Firstly, we show that v_δ^* is an admissible control to $(\mathcal{NP})_\delta^{k\delta}$ and satisfies (4.4). On one hand, since z_δ^* is the minimizer of $\mathcal{J}_\delta^{k\delta}(\cdot)$, we have that

$$\frac{\mathcal{J}_\delta^{k\delta}(z_\delta^* + \lambda z) - \mathcal{J}_\delta^{k\delta}(z_\delta^*)}{\lambda} \ge 0 \ \text{ for each } \ \lambda > 0 \ \text{ and } \ z \in \mathbb{R}^n.$$

Passing to the limit for $\lambda \to 0^+$ in the above inequality, we have that

$$\langle B^\top \overline{\varphi}_\delta, B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\rangle_{L^2(0,T;\mathbb{R}^m)} + \langle y_0, \varphi(0; k\delta, z)\rangle + r\frac{\langle z_\delta^*, z\rangle}{\|z_\delta^*\|} = 0 \ \text{ for each } z \in \mathbb{R}^n. \tag{4.8}$$

On the other hand, it follows from (2.1), (1.8) and Lemma 2.1 (where $\bar{f}_\delta \triangleq B^\top \overline{\varphi}_\delta$ and $g \triangleq B^\top \varphi(\cdot; k\delta, z)$) that

$$\langle B^\top \overline{\varphi}_\delta, B^\top \varphi(\cdot; k\delta, z)\rangle_{L^2(0,k\delta;\mathbb{R}^m)} = \langle B^\top \overline{\varphi}_\delta, B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\rangle_{L^2(0,k\delta;\mathbb{R}^m)} \ \text{ for each } z \in \mathbb{R}^n.$$

This, along with (4.3), implies that for each $z \in \mathbb{R}^n$,

$$\langle B^\top \overline{\varphi}_\delta, B^\top \overline{\varphi}_\delta(\cdot; k\delta, z)\rangle_{L^2(0,k\delta;\mathbb{R}^m)} = \langle v_\delta^*, B^\top \varphi(\cdot; k\delta, z)\rangle_{L^2(0,k\delta;\mathbb{R}^m)} = \langle y(k\delta; 0, v_\delta^*), z\rangle,$$

which, combined with (4.8), indicates that

$$y(k\delta; y_0, v_\delta^*) + r z_\delta^*/\|z_\delta^*\| = 0.$$

Hence, (4.4) holds and v_δ^* is an admissible control to $(\mathcal{NP})_\delta^{k\delta}$.

Secondly, we claim that

$$\|B^\top \overline{\varphi}_\delta\|_{L^2(0,k\delta;\mathbb{R}^m)} \neq 0. \tag{4.9}$$

By contradiction, we suppose that $\|B^\top \overline{\varphi}_\delta\|_{L^2(0,k\delta;\mathbb{R}^m)} = 0$. By the fact that $\mathcal{P}_{T^*} \subset \mathcal{P}$ and Theorem 2.3, we would have that

$$\|z_\delta^*\| \leq 2\mathrm{e}^{C(k\delta + \frac{1}{k\delta})}\|B^\top \overline{\varphi}_\delta\|_{L^2(0,k\delta;\mathbb{R}^m)} = 0,$$

which shows that $z_\delta^* = 0$ and leads to a contradiction.

Finally, we prove that v_δ^* is the unique optimal control to $(\mathcal{NP})_\delta^{k\delta}$. To this end, we arbitrarily fix an admissible control v_δ to $(\mathcal{NP})_\delta^{k\delta}$. We note that

$$\left\langle v_\delta, B^\top \varphi \right\rangle_{L^2(0,k\delta;\mathbb{R}^m)} = \langle y(k\delta; y_0, v_\delta), z_\delta^* \rangle - \langle y_0, \varphi(0) \rangle.$$

This, together with (4.8), implies that

$$\left\langle v_\delta, B^\top \varphi \right\rangle_{L^2(0,k\delta;\mathbb{R}^m)} - \langle y(k\delta; y_0, v_\delta), z_\delta^* \rangle = \|B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)} + r\|z_\delta^*\|.$$

Since $\|y(k\delta; y_0, v_\delta)\| \leq r$, it follows from the latter equality that

$$\left\langle v_\delta, B^\top \varphi \right\rangle_{L^2(0,k\delta;\mathbb{R}^m)} - \|B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)} = \langle y(k\delta; y_0, v_\delta), z_\delta^* \rangle + r\|z_\delta^*\| \geq 0,$$

which, combined with (4.3), indicates that

$$\|v_\delta^*\|_{L^2(0,k\delta;\mathbb{R}^m)}\|B^\top \overline{\varphi}_\delta\|_{L^2(0,k\delta;\mathbb{R}^m)} \leq \langle v_\delta, B^\top \varphi \rangle_{L^2(0,k\delta;\mathbb{R}^m)}. \tag{4.10}$$

Since v_δ is the piecewise constant function, we can apply Lemma 2.1 (where $\bar{f}_\delta \triangleq v_\delta$ and $g \triangleq B^\top \varphi$) to get that

$$\langle v_\delta, B^\top \varphi \rangle_{L^2(0,k\delta;\mathbb{R}^m)} = \langle v_\delta, B^\top \overline{\varphi}_\delta \rangle_{L^2(0,k\delta;\mathbb{R}^m)}.$$

This, along with (4.10) and (4.9), yields that $\|v_\delta^*\|_{L^2(0,k\delta;\mathbb{R}^m)} \leq \|v_\delta\|_{L^2(0,k\delta;\mathbb{R}^m)}$. Thus, v_δ^* is an optimal control to $(\mathcal{NP})_\delta^{k\delta}$. The uniqueness of the optimal control to $(\mathcal{NP})_\delta^{k\delta}$ follows from the same arguments as those in (i) of Theorem 3.1.

(iii) Taking $z = z_\delta^*$ in (4.8) leads to

$$\langle y_0, \varphi(0) \rangle + r\|z_\delta^*\| = -\|B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)}.$$

Since z_δ^* is the minimizer of $\mathcal{J}_\delta^{k\delta}(\cdot)$, it follows from the above equality that

$$V_\delta(k\delta) = \mathcal{J}_\delta^{k\delta}(z_\delta^*) = -\frac{1}{2}\|B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)}.$$

This, together with the conclusion (ii), implies that

$$V_\delta(k\delta) = -\frac{1}{2}\|B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)} = -\frac{1}{2}\mathcal{N}_\delta(k\delta)^2.$$

In summary, we end the proof of Theorem 4.3. $\square$

5. Several auxiliary estimates

In this section, we will present several estimates of minimizers of $\mathcal{J}^T(\cdot)$ and $\mathcal{J}_\delta^{k\delta}(\cdot)$, the minimal norm functions and the minimal time functions one by one. These estimates will be useful for proofs of the main results in this paper.

The first one is concerned with estimates of minimizers of $\mathcal{J}^T(\cdot)$ and $\mathcal{J}_\delta^{k\delta}(\cdot)$.

Theorem 5.1. *Suppose that* (H_1) *and* (H_2) *hold. Let* $T \in (0, T^*)$ *(given by (2.4)) and* $(\delta, k) \in \mathcal{P}_{T^*}$ *(given by (2.6)). Then there is a constant* $C_1 \triangleq C_1(A, B) > 0$ *so that*

$$\|z^*\| \le \mathrm{e}^{C_1(T+\frac{1}{T})}\|y_0\|^2 r^{-1} \tag{5.1}$$

and

$$\|z_\delta^*\| \le \mathrm{e}^{C_1(k\delta+\frac{1}{k\delta})}\|y_0\|^2 r^{-1}, \tag{5.2}$$

where z^* *is the minimizer of* $\mathcal{J}^T(\cdot)$ *(see (i) of Theorem 4.1) and* z_δ^* *is the minimizer of* $\mathcal{J}_\delta^{k\delta}(\cdot)$ *(see (i) of Theorem 4.3).*

Proof. Arbitrarily fix $T \in (0, T^*)$ and $(\delta, k) \in \mathcal{P}_{T^*}$. Taking $z = z_\delta^*$ in (4.5), we find that

$$\mathcal{J}_\delta^{k\delta}(z_\delta^*) \ge -2\mathrm{e}^{2C(k\delta+\frac{1}{k\delta})}\|y_0\|^2 + r\|z_\delta^*\|,$$

where $C = C(A, B) > 0$ is given by Theorem 2.2. Similarly, we have that

$$\mathcal{J}^T(z^*) \ge -\frac{\mathrm{e}^{2C(T+\frac{1}{T})}}{2}\|y_0\|^2 + r\|z^*\|.$$

Since z^* is the minimizer of $\mathcal{J}^T(\cdot)$ and z_δ^* is the minimizer of $\mathcal{J}_\delta^{k\delta}(\cdot)$, the above two inequalities imply that

$$-\frac{\mathrm{e}^{2C(T+\frac{1}{T})}}{2}\|y_0\|^2 + r\|z^*\| \le \mathcal{J}^T(0) = 0$$

and

$$-2\mathrm{e}^{2C(k\delta+\frac{1}{k\delta})}\|y_0\|^2 + r\|z_\delta^*\| \le \mathcal{J}_\delta^{k\delta}(0) = 0,$$

which indicate (5.1) and (5.2). □

The second one is related to an estimate about the minimal norm function $\mathcal{N}(\cdot)$.

Theorem 5.2. *Suppose that* (H_1) *and* (H_2) *hold. Let* (T_1, T_2) *satisfy* $0 < T_1 \le T_2 < T^*$ *(given by (2.4)). Then there is a constant* $C_2 \triangleq C_2(A, B) > 0$ *so that*

$$\frac{\sigma_0^{3/2} r}{\|B\|}(T_2 - T_1) \le \mathcal{N}(T_1) - \mathcal{N}(T_2) \le \frac{\mathrm{e}^{C_2(T_1+\frac{1}{T_1})}\|y_0\|^4 r^{-2}}{\mathcal{N}(T_1)}(T_2 - T_1). \tag{5.3}$$

Proof. Arbitrarily fix a pair (T_1, T_2) with $0 < T_1 \le T_2 < T^*$. The proof will be carried out by the following two steps.

Step 1. We show that

$$\mathcal{N}(T_1) - \mathcal{N}(T_2) \geq \frac{\sigma_0^{3/2} r}{\|B\|}(T_2 - T_1). \tag{5.4}$$

Indeed, it follows from (i) and (ii) of Theorem 4.2 that

$$M_1 \triangleq \mathcal{N}(T_1) \geq \mathcal{N}(T_2) \triangleq M_2 \tag{5.5}$$

and

$$0 < \mathcal{T}(M_1) = T_1 \leq T_2 = \mathcal{T}(M_2) < T^*. \tag{5.6}$$

Let u_1^* be the unique optimal control to $(\mathcal{TP})^{M_1}$ (see (i) of Thm. 3.1). It is clear that

$$\|y(\mathcal{T}(M_1); y_0, u_1^*)\| \leq r \text{ and } \|u_1^*\|_{L^2(0.+\infty;\mathbb{R}^m)} \leq M_1. \tag{5.7}$$

By the first inequality in (5.7) and (5.5), we get that

$$\begin{aligned}\left\|y\Big(T(M_1); y_0, \frac{M_2}{M_1}u_1^*\Big)\right\| &\leq \left\|y\Big(\mathcal{T}(M_1); y_0, \frac{M_2}{M_1}u_1^*\Big) - y(\mathcal{T}(M_1); y_0, u_1^*)\right\| + \|y(\mathcal{T}(M_1); y_0, u_1^*)\| \\ &\leq \left\|\frac{M_2 - M_1}{M_1}\int_0^{\mathcal{T}(M_1)} \mathrm{e}^{(\mathcal{T}(M_1)-t)A}Bu_1^*(t)\mathrm{d}t\right\| + r \\ &\leq \frac{M_1 - M_2}{M_1}\|B\|\int_0^{\mathcal{T}(M_1)} \|\mathrm{e}^{(\mathcal{T}(M_1)-t)A}\|\|u_1^*(t)\|\mathrm{d}t + r.\end{aligned}$$

The above inequality, together with (H_2), Hölder's inequality and the second inequality in (5.7), yields that

$$\left\|y\Big(\mathcal{T}(M_1); y_0, \frac{M_2}{M_1}u_1^*\Big)\right\| \leq r + \frac{M_1 - M_2}{M_1}\frac{\|B\|}{\sqrt{2\sigma_0}}M_1 \leq r + (M_1 - M_2)\frac{\|B\|}{\sqrt{\sigma_0}}. \tag{5.8}$$

Define a control u_2 as follows:

$$u_2(t) \triangleq \begin{cases} \frac{M_2}{M_1}u_1^*(t), & t \in (0, \mathcal{T}(M_1)], \\ 0, & t \in (\mathcal{T}(M_1), \infty). \end{cases} \tag{5.9}$$

This, along with second inequality in (5.7), implies that

$$\|u_2\|_{L^2(0.+\infty;\mathbb{R}^m)} \leq M_2. \tag{5.10}$$

Meanwhile, we denote

$$\widehat{T} \triangleq \frac{1}{\sigma_0}\ln\left(1 + \frac{\|B\|}{r\sqrt{\sigma_0}}(M_1 - M_2)\right). \tag{5.11}$$

From (5.9), (H_2), (5.8) and (5.11) it follows that

$$\|y(\mathcal{T}(M_1) + \widehat{T}; y_0, u_2)\| = \|\mathrm{e}^{A\widehat{T}}y(\mathcal{T}(M_1); y_0, u_2)\| \leq \mathrm{e}^{-\sigma_0\widehat{T}}\left(r + (M_1 - M_2)\frac{\|B\|}{\sqrt{\sigma_0}}\right) = r,$$

which, combined with (5.10), indicates that u_2 is an admissible control to $(\mathcal{TP})^{M_2}$ and

$$\mathcal{T}(M_2) \le \mathcal{T}(M_1) + \widehat{T}.$$

This, together with (5.6) and (5.11), yields that

$$T_2 - T_1 = \mathcal{T}(M_2) - \mathcal{T}(M_1) \le \widehat{T} \le \frac{\|B\|}{\sigma_0^{3/2} r}(M_1 - M_2).$$

Hence, (5.4) follows from the latter and (5.5) immediately.

Step 2. We claim that

$$\mathcal{N}(T_1) - \mathcal{N}(T_2) \le \frac{\mathrm{e}^{C_2(T_1 + \frac{1}{T_1})}\|y_0\|^4 r^{-2}}{\mathcal{N}(T_1)}(T_2 - T_1). \tag{5.12}$$

Let z_1^* be the unique minimizer of $\mathcal{J}^{T_1}(\cdot)$ (see (i) of Thm. 4.1). On one hand, since

$$\begin{aligned}
&\|B^\top \varphi(t; T_2, z_1^*)\|^2_{L^2(0,T_2;\mathbb{R}^m)} - \|B^\top \varphi(t; T_1, z_1^*)\|^2_{L^2(0,T_1;\mathbb{R}^m)} \\
=& \int_0^{T_2} \|B^\top \mathrm{e}^{(T_2 - t)A^\top} z_1^*\|^2 \mathrm{d}t - \int_{T_2 - T_1}^{T_2} \|B^\top \mathrm{e}^{(T_2 - t)A^\top} z_1^*\|^2 \mathrm{d}t \\
=& \int_0^{T_2 - T_1} \|B^\top \mathrm{e}^{(T_2 - t)A^\top} z_1^*\|^2 \mathrm{d}t,
\end{aligned}$$

it follows from (H_2) that

$$\|B^\top \varphi(t; T_2, z_1^*)\|^2_{L^2(0,T_2;\mathbb{R}^m)} - \|B^\top \varphi(t; T_1, z_1^*)\|^2_{L^2(0,T_1;\mathbb{R}^m)} \le (T_2 - T_1)\|B\|^2 \|z_1^*\|^2. \tag{5.13}$$

Meanwhile, we observe that

$$\begin{aligned}
&\langle y_0, \varphi(0; T_2, z_1^*)\rangle - \langle y_0, \varphi(0; T_1, z_1^*)\rangle \\
&= \langle y_0, \varphi(0; T_2, z_1^*) - \varphi(T_2 - T_1; T_2, z_1^*)\rangle \\
&\le \|y_0\| \left\| \int_0^{T_2 - T_1} \varphi'(t; T_2, z_1^*) \mathrm{d}t \right\| \\
&\le \|y_0\| \int_0^{T_2 - T_1} \|A^\top \mathrm{e}^{(T_2 - t)A^\top} z_1^*\| \mathrm{d}t,
\end{aligned}$$

which, combined with (H_2), indicates that

$$\langle y_0, \varphi(0; T_2, z_1^*)\rangle - \langle y_0, \varphi(0; T_2, z_1^*)\rangle \le (T_2 - T_1)\|y_0\|\|A\|\|z_1^*\|.$$

This, along with (5.13) and (5.1), yields that

$$\begin{aligned}\mathcal{J}^{T_2}(z_1^*) - \mathcal{J}^{T_1}(z_1^*) =& \frac{1}{2}\|B^\top \varphi(t;T_2,z_1^*)\|^2_{L^2(0,T_2;\mathbb{R}^m)} + \langle y_0, \varphi(0;T_2,z_1^*)\rangle + r\|z_1^*\| \\ &- \frac{1}{2}\|B^\top \varphi(t;T_1,z_1^*)\|^2_{L^2(0,T_1;\mathbb{R}^m)} - \langle y_0, \varphi(0;T_1,z_1^*)\rangle - r\|z_1^*\| \\ \leq& \frac{1}{2}(T_2-T_1)\|B\|^2\|z_1^*\|^2 + (T_2-T_1)\|y_0\|\|A\|\|z_1^*\| \\ \leq& (T_2-T_1)\left(\frac{1}{2}\|B\|^2 + \|A\|\right)(\mathrm{e}^{2C_1(T_1+\frac{1}{T_1})}\|y_0\|^4 r^{-2} + \mathrm{e}^{C_1(T_1+\frac{1}{T_1})}\|y_0\|^3 r^{-1}),\end{aligned}$$

where $C_1 = C_1(A,B) > 0$ is given by Theorem 5.1. Since $\|y_0\| > r$, we obtain from the above inequality that

$$\mathcal{J}^{T_2}(z_1^*) - \mathcal{J}^{T_1}(z_1^*) \leq (T_2-T_1)\left(\|B\|^2 + 2\|A\|\right)\mathrm{e}^{2C_1(T_1+\frac{1}{T_1})}\|y_0\|^4 r^{-2}. \tag{5.14}$$

One the other hand, by (iii) of Theorem 4.1 and (5.5), we have that

$$V(T_1) = -\frac{1}{2}\mathcal{N}(T_1)^2 \leq -\frac{1}{2}\mathcal{N}(T_2)^2 = V(T_2). \tag{5.15}$$

Since z_1^* is the minimizer of $\mathcal{J}^{T_1}(\cdot)$, it follows from (5.15) and (1.5) that

$$0 \leq V(T_2) - V(T_1) \leq \mathcal{J}^{T_2}(z_1^*) - \mathcal{J}^{T_1}(z_1^*).$$

This, together with (5.15) and (5.14), implies that

$$\begin{aligned}\mathcal{N}(T_1) - \mathcal{N}(T_2) =& \frac{2}{\mathcal{N}(T_2)+\mathcal{N}(T_1)}(V(T_2)-V(T_1)) \\ \leq& \frac{2}{\mathcal{N}(T_1)}(T_2-T_1)\left(\|B\|^2+2\|A\|\right)\mathrm{e}^{2C_1(T_1+\frac{1}{T_1})}\|y_0\|^4 r^{-2},\end{aligned}$$

which leads to (5.12) for some constant $C_2 \triangleq C_2(A,B) > 0$.

In summary, we finish the proof of Theorem 5.2. □

The third one is about estimates between minimal norm functions $\mathcal{N}_\delta(\cdot)$ and $\mathcal{N}(\cdot)$.

Theorem 5.3. *Suppose that* (H_1) *and* (H_2) *hold. Let* T^* *be given by (2.4) and* $\mathcal{P}_{T^*}$ *be given by (2.6). Then there is a constant* $C_3 \triangleq C_3(A,B) > 0$ *so that, for each* $(\delta,k) \in \mathcal{P}_{T^*}$*,*

$$0 \leq \mathcal{N}_\delta(k\delta) - \mathcal{N}(k\delta) \leq \mathrm{e}^{C_3(k\delta+\frac{1}{k\delta}+\frac{1}{T^*-k\delta})}\|y_0\|^4 r^{-3}\delta^2. \tag{5.16}$$

Proof. Arbitrarily fix $(\delta,k) \in \mathcal{P}_{T^*}$. Let z_δ^* be the minimizer of $\mathcal{J}_\delta^{k\delta}(\cdot)$ (see (i) of Thm. 4.3). Throughout the proof of Theorem 5.3, we simply write $\varphi(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$ for $\varphi(\cdot;k\delta,z_\delta^*)$ and $\overline{\varphi}_\delta(\cdot;k\delta,z_\delta^*)$, respectively. Since $\mathcal{N}(k\delta) \leq \mathcal{N}_\delta(k\delta)$, it follows from (iii) of Theorem 4.1 and (iii) of Theorem 4.3 that

$$V_\delta(k\delta) = -\frac{1}{2}\mathcal{N}_\delta(k\delta)^2 \leq -\frac{1}{2}\mathcal{N}(k\delta)^2 = V(k\delta). \tag{5.17}$$

By (5.17), (1.5) and (1.7), we get that

$$0 \leq V(k\delta) - V_\delta(k\delta) \leq J^{k\delta}(z_\delta^*) - J_\delta^{k\delta}(z_\delta^*) = \frac{1}{2}(\|B^\top \varphi\|^2_{L^2(0,k\delta;\mathbb{R}^m)} - \|B^\top \overline{\varphi}_\delta\|^2_{L^2(0,k\delta;\mathbb{R}^m)}).$$

This, together with (2.8), yields that

$$0 \leq V(k\delta) - V_\delta(k\delta) \leq \frac{1}{2} k\delta^3 \|AB\|^2 \|z_\delta^*\|^2. \tag{5.18}$$

Meanwhile, by (i) of Theorem 4.2, we have that

$$\lim_{T_2 \to T^{*-}} \mathcal{N}(T_2) = 0,$$

which, along with the first inequality in (5.3) (where $T_1 = k\delta$), indicates

$$\mathcal{N}(k\delta) = \lim_{T_2 \to T^{*-}} (\mathcal{N}(k\delta) - \mathcal{N}(T_2)) \geq \sigma_0^{3/2} r(T^* - k\delta)/\|B\|. \tag{5.19}$$

Since

$$\sigma_0 \geq \mathrm{e}^{-\frac{1}{\sigma_0}} \text{ and } T^* - k\delta \geq \mathrm{e}^{-\frac{1}{T^*-k\delta}},$$

we obtain from (5.19) that

$$\mathcal{N}(k\delta) \geq \mathrm{e}^{-\frac{3}{2\sigma_0} - \frac{1}{T^*-k\delta}} r/\|B\|. \tag{5.20}$$

Then it follows from (5.17), (5.18), (5.20) and (5.2) that

$$\begin{aligned} 0 \leq \mathcal{N}_\delta(k\delta) - \mathcal{N}(k\delta) &= \frac{2V(k\delta) - 2V_\delta(k\delta)}{\mathcal{N}(k\delta) + \mathcal{N}_\delta(k\delta)} \\ &\leq k\delta^3 \|AB\|^2 \|z_\delta^*\|^2 \|B\| \mathrm{e}^{\frac{3}{2\sigma_0} + \frac{1}{T^*-k\delta}} r^{-1} \\ &\leq \delta^2 \mathrm{e}^{k\delta} \|AB\|^2 \mathrm{e}^{2C_1(k\delta + \frac{1}{k\delta})} \|y_0\|^4 r^{-3} \|B\| \mathrm{e}^{\frac{3}{2\sigma_0} + \frac{1}{T^*-k\delta}}, \end{aligned}$$

where $C_1 = C_1(A, B) > 0$ is given by Theorem 5.1. The latter inequality implies (5.16) for some $C_3 \triangleq C_3(A, B) > 0$. □

The fourth one deals with some estimates about minimal time functions $\mathcal{T}_\delta(\cdot)$ and $\mathcal{T}(\cdot)$.

Theorem 5.4. *Suppose that (H_1) and (H_2) hold. For each $M > 0$ and $\eta \in (0, 1)$, there is a measurable subset $\mathcal{A}_{M,\eta}$ (depending also on y_0, r, A and B) with $\lim_{h \to 0^+} \frac{1}{h} |\mathcal{A}_{M,\eta} \cap (0, h)| = \eta$ so that, for each $\delta \in \mathcal{A}_{M,\eta}$, there is $a_\delta \in (0, \eta)$ so that*

$$\mathcal{T}_\delta(M) - \mathcal{T}(M) = (1 - a_\delta)\delta \text{ and } M > \mathcal{N}_\delta(\mathcal{T}_\delta(M)) + \frac{\sigma_0^{3/2} r}{2\|B\|}(1 - \eta)\delta. \tag{5.21}$$

Proof. Arbitrarily fix $M > 0$ and $\eta \in (0, 1)$. For each $k \in \mathbb{N}^+$ and $a \in (0, \eta)$, we define a subset of $\mathbb{R}$ in the following manner:

$$\mathcal{B}_{M,\eta}^{k,a} \triangleq \{\delta : (k + a)\delta = \mathcal{T}(M)\}. \tag{5.22}$$

Then we define another subset of $\mathbb{R}$ as follows:

$$\mathcal{B}_{M,\eta} \triangleq \bigcup_{k\in\mathbb{N}^+} \bigcup_{a\in(0,\eta)} \mathcal{B}^{k,a}_{M,\eta}. \tag{5.23}$$

The proof is split into the follows three steps.

Step 1. We show that

$$\lim_{h\to 0^+} \frac{1}{h} |\mathcal{B}_{M,\eta} \cap (0,h)| = \eta. \tag{5.24}$$

Indeed, it follows from (5.22) that

$$\bigcup_{a\in(0,\eta)} \mathcal{B}^{k,a}_{M,\eta} = (\mathcal{T}(M)/(k+\eta), \mathcal{T}(M)/k) \quad \text{for each } k\in\mathbb{N}^+.$$

This, along with (5.23), yields that

$$\mathcal{B}_{M,\eta} = \bigcup_{k\in\mathbb{N}^+} \left(\mathcal{T}(M)/(k+\eta), \mathcal{T}(M)/k\right). \tag{5.25}$$

For each $j\in\mathbb{N}^+$, we let $h_j \triangleq \mathcal{T}(M)/j$. For each $h\in(0,\mathcal{T}(M))$, we let $j(h)$ be the integer that $h_{j(h)+1} \leq h < h_{j(h)}$. Then by (5.25), we have that

$$\begin{aligned}
\frac{|\mathcal{B}_{M,\eta}\bigcap(0,h_{j(h)})|}{h_{j(h)}} &= \frac{1}{h_{j(h)}} \left| \bigcup_{k\in\mathbb{N}^+} \left(\frac{\mathcal{T}(M)}{k+\eta}, \frac{\mathcal{T}(M)}{k}\right) \bigcap \left(0, \frac{\mathcal{T}(M)}{j(h)}\right)\right| \\
&= \frac{1}{h_{j(h)}} \sum_{k\in\mathbb{N}^+} \left| \left(\frac{\mathcal{T}(M)}{k+\eta}, \frac{\mathcal{T}(M)}{k}\right) \bigcap \left(0, \frac{\mathcal{T}(M)}{j(h)}\right)\right| \\
&= \frac{1}{h_{j(h)}} \sum_{k=j(h)}^{\infty} \left| \left(\frac{\mathcal{T}(M)}{k+\eta}, \frac{\mathcal{T}(M)}{k}\right) \bigcap \left(0, \frac{\mathcal{T}(M)}{j(h)}\right)\right|,
\end{aligned}$$

which indicates

$$\frac{|\mathcal{B}_{M,\eta}\bigcap(0,h_{j(h)})|}{h_{j(h)}} = \frac{1}{h_{j(h)}} \sum_{k=j(h)}^{\infty} \left| \left(\frac{\mathcal{T}(M)}{k+\eta}, \frac{\mathcal{T}(M)}{k}\right)\right| = \sum_{k=j(h)}^{\infty} \frac{j(h)\eta}{k(k+\eta)}.$$

Hence, we get that

$$\lim_{h\to 0^+} \frac{|\mathcal{B}_{M,\eta}\bigcap(0,h_{j(h)})|}{h_{j(h)}} = \lim_{j(h)\to+\infty} \sum_{k=j(h)}^{\infty} \frac{j(h)\eta}{k(k+\eta)} = \eta. \tag{5.26}$$

Since

$$\lim_{h\to 0^+} \frac{h_{j(h)+1}}{h} = \lim_{h\to 0^+} \frac{h_{j(h)}}{h} = 1$$

and

$$\frac{h_{j(h)+1}}{h}\frac{|\mathcal{B}_{M,\eta}\bigcap(0,h_{j(h)+1})|}{h_{j(h)+1}} \leq \frac{|\mathcal{B}_{M,\eta}\bigcap(0,h)|}{h} \leq \frac{|\mathcal{B}_{M,\eta}\bigcap(0,h_{j(h)})|}{h_{j(h)}}\frac{h_{j(h)}}{h},$$

we can obtain (5.24) from (5.26) immediately.

Step 2. By (5.22) and (5.23), we can easily find that for each $\delta\in\mathcal{B}_{M,\eta}$, there is a unique pair (k_δ, a_δ) so that

$$(k_\delta + a_\delta)\delta = \mathcal{T}(M) \text{ with } k_\delta \in \mathbb{N}^+ \text{ and } a_\delta \in (0,\eta). \tag{5.27}$$

We claim that there exists $\delta_{M,\eta}>0$ (which will be precised later) so that

$$M > \mathcal{N}_\delta((k_\delta+1)\delta) + \frac{\sigma_0^{3/2} r}{2\|B\|}(1-\eta)\delta \;\text{ for each } \delta\in\mathcal{B}_{M,\eta}\cap(0,\delta_{M,\eta}). \tag{5.28}$$

To this end, we note that $0<\mathcal{T}(M)<T^*$ (see (iii) of Thm. 4.2). Denote

$$\delta^0_{M,\eta} \triangleq \min\left\{\left(T^*-\mathcal{T}(M)\right)/2, \sqrt{3/\left(4\mathrm{e}^{2C(T^*+\frac{1}{\mathcal{T}(M)})}T^*\|AB\|^2\right)}\right\}, \tag{5.29}$$

where $C=C(A,B)>0$ is given by Theorem 2.2. It follows from (5.27) and (5.29) that for each $\delta\in\mathcal{B}_{M,\eta}\cap(0,\delta^0_{M,\eta})$,

$$\mathcal{T}(M) < (k_\delta+1)\delta < \mathcal{T}(M)+\delta < \left(T^*+\mathcal{T}(M)\right)/2 < T^*, \tag{5.30}$$

and

$$\mathrm{e}^{2C\left[(k_\delta+1)\delta+\frac{1}{(k_\delta+1)\delta}\right]}(k_\delta+1)\delta^3\|AB\|^2 \leq \frac{3}{4}.$$

These, along with (2.6), yield that

$$(\delta, k_\delta+1)\in\mathcal{P}_{T^*}. \tag{5.31}$$

By (5.31), (5.30), Theorem 5.3 (where $(\delta,k)=(\delta,k_\delta+1)$) and Theorem 5.2 (where $T_1=\mathcal{T}(M)$ and $T_2=(k_\delta+1)\delta$), we get that

$$\begin{aligned}\mathcal{N}_\delta((k_\delta+1)\delta) &\leq \mathcal{N}((k_\delta+1)\delta) + \mathrm{e}^{C_3\left[(k_\delta+1)\delta+\frac{1}{(k_\delta+1)\delta}+\frac{1}{T^*-(k_\delta+1)\delta}\right]}\|y_0\|^4 r^{-3}\delta^2\\ &\leq \mathcal{N}(\mathcal{T}(M)) - \frac{\sigma_0^{3/2}r}{\|B\|}\big((k_\delta+1)\delta-\mathcal{T}(M)\big)\\ &\quad + \mathrm{e}^{C_3\left[(k_\delta+1)\delta+\frac{1}{(k_\delta+1)\delta}+\frac{1}{T^*-(k_\delta+1)\delta}\right]}\|y_0\|^4 r^{-3}\delta^2,\end{aligned} \tag{5.32}$$

where $C_3\triangleq C_3(A,B)>0$ is given by Theorem 5.3. Meanwhile, by (5.27), we find that

$$(k_\delta+1)\delta-\mathcal{T}(M) > (1-\eta)\delta.$$

This, together with (5.32), (ii) of Thm. 4.2 and (5.30), implies that for each $\delta \in \mathcal{B}_{M,\eta} \cap (0, \delta^0_{M,\eta})$,

$$\mathcal{N}_\delta((k_\delta+1)\delta) < M - \frac{\sigma_0^{3/2} r}{\|B\|}(1-\eta)\delta + \mathrm{e}^{C_3\left[T^* + \frac{1}{\mathcal{T}(M)} + \frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^4 r^{-3}\delta^2. \tag{5.33}$$

Let

$$\delta_{M,\eta} \triangleq \min\left\{\delta^0_{M,\eta}, \frac{\sigma_0^{3/2} r^4}{2\|B\|}(1-\eta)\mathrm{e}^{-C_3\left[T^* + \frac{1}{\mathcal{T}(M)} + \frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^{-4}\right\}.$$

Then (5.28) follows from (5.33) at once.

Step 3. End of the proof.

Define the set $\mathcal{A}_{M,\eta}$ in the following manner:

$$\mathcal{A}_{M,\eta} \triangleq \mathcal{B}_{M,\eta} \cap (0, \delta_{M,\eta}). \tag{5.34}$$

By (5.24), one can directly check that $\lim_{h\to 0^+} \frac{1}{h}|\mathcal{A}_{M,\eta} \cap (0,h)| = \eta$. For each $\delta \in \mathcal{A}_{M,\eta}$, let u_δ be the unique optimal control to $(\mathcal{NP})^{(k_\delta+1)\delta}_\delta$ (see (5.31) and (ii) of Thm. 4.3) and $\tilde{u}_\delta$ be the zero extension of u_δ over $(0,+\infty)$. Since $\mathcal{N}_\delta((k_\delta+1)\delta) \le M$ (see (5.28)), one can easily check that $\tilde{u}_\delta$ is an admissible control to $(\mathcal{TP})^M_\delta$, which drives the solution to $B_r(0)$ at time $(k_\delta+1)\delta$. This, along with the optimality of $\mathcal{T}_\delta(M)$, yields that

$$\mathcal{T}_\delta(M) \le (k_\delta+1)\delta. \tag{5.35}$$

Since $\mathcal{T}(M) \le \mathcal{T}_\delta(M)$, we obtain from (5.27) that $\mathcal{T}_\delta(M) > k_\delta\delta$. This, along with (5.35), implies that

$$\mathcal{T}_\delta(M) = (k_\delta+1)\delta \ \text{ for each } \ \delta \in \mathcal{A}_{M,\eta}. \tag{5.36}$$

Thus, (5.21) follows from (5.36), (5.27) and (5.28) immediately.

In summary, we finish the proof of Theorem 5.4. □

The last one gives a lower bound for the diameter of the subset $\mathcal{O}_\delta$ (in $L^2(0,\mathcal{T}(M);\mathbb{R}^m)$), which is defined by (5.38) below.

Theorem 5.5. *For each $M > 0$ and $\eta \in (0,1)$, there are two constants $C_4 \triangleq C_4(M, y_0, r, A, B) > 0$ and $\widehat{\delta}_0 \triangleq \widehat{\delta}_0(M, y_0, r, A, B) > 0$ so that for each $\delta \in \mathcal{A}_{M,\eta} \cap (0, \widehat{\delta}_0)$ (where $\mathcal{A}_{M,\eta}$ is given by Thm. 5.4),*

$$\operatorname{diam} \mathcal{O}_\delta \triangleq \sup\{\|u_\delta - v_\delta\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} : u_\delta, v_\delta \in \mathcal{O}_\delta\} > C_4\sqrt{(1-\eta)\delta}, \tag{5.37}$$

where $\mathcal{O}_\delta$ is given by

$$\mathcal{O}_\delta \triangleq \{u_\delta|_{(0,\mathcal{T}(M))} : u_\delta \ \text{ is an optimal control to } \ (\mathcal{TP})^M_\delta\}. \tag{5.38}$$

Proof. Arbitrarily $M > 0$, $\eta \in (0,1)$ and $\delta \in \mathcal{A}_{M,\eta}$. Let T^* be given by (2.4) and $\mathcal{P}_{T^*}$ be given by (2.6). Recall the definition of $\mathcal{A}_{M,\eta}$ (see (5.34)). By (5.36), (5.30) and (5.31), we have that

$$\mathcal{T}(M) \le \mathcal{T}_\delta(M) < T^*, \tag{5.39}$$

and

$$(\delta, \mathcal{T}_\delta(M)/\delta) \in \mathcal{P}_{T^*}. \tag{5.40}$$

Since $\delta \in \mathcal{A}_{M,\eta}$, it follows from Theorem 5.4 that

$$M_\delta \triangleq \mathcal{N}_\delta(\mathcal{T}_\delta(M)) < M - \frac{\sigma_0^{3/2} r}{2\|B\|}(1-\eta)\delta. \tag{5.41}$$

Let u_δ^* be the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$ (see (ii) of Thm. 3.1) and z_δ^* be the minimizer of $(\mathcal{JP})_\delta^{\mathcal{T}_\delta(M)}$ (see (i) of Thm. 4.3). Arbitrarily fix $\hat{v}_\delta \in L^2_\delta(0, \mathcal{T}_\delta(M); \mathbb{R}^m)$ so that

$$\begin{cases} \text{supp } \hat{v}_\delta \subset (0, \mathcal{T}(M)), \\ \langle \hat{v}_\delta, u_\delta^* \rangle_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} = 0, \\ \|\hat{v}_\delta\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} = 1, \\ \langle y(\mathcal{T}_\delta(M); 0, u_\delta^*), y(\mathcal{T}_\delta(M); 0, \hat{v}_\delta) \rangle \le 0, \end{cases} \tag{5.42}$$

where the existence of such $\hat{v}_\delta$ can be easily verified. Let $\mathcal{O}'_\delta$ be the set of solutions u_δ to the following problem:

$$(\mathcal{O}'_\delta) \qquad \begin{cases} u_\delta = \alpha u_\delta^* + \beta \hat{v}_\delta, \alpha, \beta \in \mathbb{R}, \\ \alpha^2 M_\delta^2 + \beta^2 \le M^2, \\ a_\delta^2 \beta^2 + 2(\alpha-1) b_\delta \beta + (\alpha-1)^2 c_\delta^2 - 2(\alpha-1)\dfrac{r M_\delta^2}{\|z_\delta^*\|} \le 0. \end{cases} \tag{5.43}$$

Here, the pair $(a_\delta, b_\delta, c_\delta)$ is defined by

$$\begin{cases} a_\delta \triangleq \|y(\mathcal{T}_\delta(M); 0, \hat{v}_\delta)\|, \\ b_\delta \triangleq \langle y(\mathcal{T}_\delta(M); 0, u_\delta^*), y(\mathcal{T}_\delta(M); 0, \hat{v}_\delta) \rangle, \\ c_\delta \triangleq \|y(\mathcal{T}_\delta(M); 0, u_\delta^*)\|. \end{cases} \tag{5.44}$$

From now on and throughout the proof of Theorem 5.5, we simply write $\|\cdot\|_{L^2(0,\mathcal{T})}$ and $\|\cdot\|_{L^2(0,\mathcal{T}_\delta)}$ for $\|\cdot\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}$ and $\|\cdot\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}$, respectively; simply write $\langle\cdot,\cdot\rangle_{L^2(0,\mathcal{T})}$ and $\langle\cdot,\cdot\rangle_{L^2(0,\mathcal{T}_\delta)}$ for $\langle\cdot,\cdot\rangle_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}$ and $\langle\cdot,\cdot\rangle_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}$, respectively. To show (5.37), we organize the proof by three steps as follows.

Step 1. We prove that

$$\mathcal{O}''_\delta \subset \mathcal{O}'_\delta, \tag{5.45}$$

where $\mathcal{O}''_\delta$ is defined as the set of solutions u_δ to the following problem:

$$(\mathcal{O}''_\delta) \qquad \begin{cases} u_\delta = \alpha u_\delta^* + \beta \hat{v}_\delta, \alpha = 1 + \hat{\lambda}\dfrac{M - M_\delta}{M_\delta}, \beta > 0, \\ \beta^2 \le (1-\hat{\lambda}) M (M - M_\delta), \\ a_\delta^2 \beta^2 \le \dfrac{\hat{\lambda} r M_\delta (M - M_\delta)}{\|z_\delta^*\|}, \end{cases} \tag{5.46}$$

where $M_\delta(< M)$ is given by (5.41) and the number $\hat{\lambda} \in (0,1)$ is given by

$$\hat{\lambda} \triangleq \min\left\{\frac{rM_\delta^3}{\|z_\delta^*\| c_\delta^2 (M - M_\delta)}, \frac{1}{2}\right\}. \tag{5.47}$$

(Here, we agree that $\frac{1}{0} = \infty$.)

Arbitrarily fix $u_\delta = \alpha u_\delta^* + \beta \hat{v}_\delta \in \mathcal{O}_\delta''$. Since $M > M_\delta$ and $\hat{\lambda} \in (0,1)$, it follows from (5.46) that

$$\begin{aligned} \alpha^2 M_\delta^2 + \beta^2 \le\ & \Big(M_\delta + \hat{\lambda}(M - M_\delta)\Big)^2 + (1-\hat{\lambda})M(M - M_\delta) \\ =\ & \Big(M - (1-\hat{\lambda})\,(M - M_\delta)\Big)^2 + (1-\hat{\lambda})M(M - M_\delta) \\ =\ & M^2 - (1-\hat{\lambda})(M - M_\delta)\Big(M - (1-\hat{\lambda})(M - M_\delta)\Big) \le M^2. \end{aligned} \tag{5.48}$$

Meanwhile, since $b_\delta \le 0$ (see (5.44) and (5.42)), it follows from (5.46) and (5.47) that

$$a_\delta^2\beta^2 + 2(\alpha - 1)b_\delta\beta \le a_\delta^2\beta^2 \le \frac{\hat{\lambda} r M_\delta (M - M_\delta)}{\|z_\delta^*\|} = (\alpha - 1)\frac{rM_\delta^2}{\|z_\delta^*\|} \le 2(\alpha - 1)\frac{rM_\delta^2}{\|z_\delta^*\|} - (\alpha - 1)^2 c_\delta^2.$$

This, along with (5.48) and (5.43), leads to (5.45).

Step 2. We claim that

$$\mathcal{O}_\delta' \subset \mathcal{O}_\delta. \tag{5.49}$$

Indeed, since u_δ^* is the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$, it follows from (iii) of Theorem 3.1 that the restriction of u_δ^* on $(0, \mathcal{T}_\delta(M))$ is an optimal control to $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$. Then

$$\|u_\delta^*\|_{L^2(0,\mathcal{T}_\delta)} = \mathcal{N}_\delta(\mathcal{T}_\delta(M)) \triangleq M_\delta. \tag{5.50}$$

Arbitrarily fix $u_\delta = \alpha u_\delta^* + \beta \hat{v}_\delta \in \mathcal{O}_\delta'$. We have that

$$\|u_\delta\|^2_{L^2(0,\mathcal{T}_\delta)} = \alpha^2 ||u_\delta^*||^2_{L^2(0,\mathcal{T}_\delta)} + \beta^2 ||\hat{v}_\delta||^2_{L^2(0,\mathcal{T}_\delta)} + 2\alpha\beta\langle u_\delta^*, \hat{v}_\delta\rangle_{L^2(0,\mathcal{T}_\delta)}.$$

This, along with (5.42), (5.50) and the second inequality of (5.43), yields that

$$\|u_\delta\|^2_{L^2(0,\mathcal{T}_\delta)} = \alpha^2 M_\delta^2 + \beta^2 \le M^2. \tag{5.51}$$

By (5.40) and (4.4) (where $(\delta, k) = (\delta, \mathcal{T}_\delta(M)/\delta)$), we obtain that

$$\|y(\mathcal{T}_\delta(M); y_0, u_\delta^*)\| = r \tag{5.52}$$

and

$$\langle y(\mathcal{T}_\delta(M); y_0, u_\delta^*), y(\mathcal{T}_\delta(M); 0, u_\delta^*)\rangle = -\frac{r}{\|z_\delta^*\|}\langle B^\top \varphi(\cdot; \mathcal{T}_\delta(M), z_\delta^*), u_\delta^*\rangle_{L^2(0,\mathcal{T}_\delta)}. \tag{5.53}$$

Meanwhile, by Lemma 2.1 (where $(f,\bar{g}_\delta)\triangleq(B^\top\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*),u_\delta^*)$), we get that

$$\langle B^\top\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*),u_\delta^*\rangle_{L^2(0,\mathcal{T}_\delta)}=\langle B^\top\overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*),u_\delta^*\rangle_{L^2(0,\mathcal{T}_\delta)},$$

which, combined with (5.53), (4.3) and (5.50), indicates that

$$\langle y(\mathcal{T}_\delta(M);y_0,u_\delta^*),y(\mathcal{T}_\delta(M);0,u_\delta^*)\rangle=-\frac{rM_\delta^2}{\|z_\delta^*\|}. \tag{5.54}$$

Similarly, we have that

$$\begin{aligned}\langle y(\mathcal{T}_\delta(M);y_0,u_\delta^*),y(\mathcal{T}_\delta(M);0,\hat{v}_\delta)\rangle&=\left\langle -r\frac{z_\delta^*}{\|z_\delta^*\|},y(\mathcal{T}_\delta(M);0,\hat{v}_\delta)\right\rangle\\&=-\frac{r}{\|z_\delta^*\|}\langle B^\top\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*),\hat{v}_\delta\rangle_{L^2(0,\mathcal{T}_\delta)}\\&=-\frac{r}{\|z_\delta^*\|}\langle u_\delta^*,\hat{v}_\delta\rangle_{L^2(0,\mathcal{T}_\delta)}.\end{aligned}$$

This, together with the second equality in (5.42), yields that

$$\langle y(\mathcal{T}_\delta(M);y_0,u_\delta^*),y(\mathcal{T}_\delta(M);0,\hat{v}_\delta)\rangle=0. \tag{5.55}$$

Note that $u_\delta=\alpha u_\delta^*+\beta\hat{v}_\delta$. It follows from (5.52), (5.44), (5.54), (5.55) and the third inequality in (5.43) that

$$\begin{aligned}\|y(\mathcal{T}_\delta(M);y_0,u_\delta)\|^2&=\|y(\mathcal{T}_\delta(M);y_0,u_\delta^*)+(\alpha-1)y(\mathcal{T}_\delta(M);0,u_\delta^*)+\beta y(\mathcal{T}_\delta(M);0,\hat{v}_\delta)\|^2\\&=r^2+(\alpha-1)^2c_\delta^2+\beta^2a_\delta^2+2(\alpha-1)\left(-\frac{rM_\delta^2}{\|z_\delta^*\|}\right)+2(\alpha-1)\beta b_\delta\\&\le r^2.\end{aligned}$$

Hence, (5.49) follows from the above inequality and (5.51) immediately.

Step 3. End of the proof.

To this end, let

$$\widehat{\delta}_0\triangleq\min\left\{\frac{1}{2}M^2\mathrm{e}^{-C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}\|y_0\|^{-4}r^2,1\right\}, \tag{5.56}$$

where $C_2=C_2(A,B)>0$ is given by Theorem 5.2.

We first prove the following inequalities one by one:

$$a_\delta\le\frac{\|B\|}{\sqrt{\sigma_0}}\quad\text{and}\quad c_\delta\le\frac{\|B\|}{\sqrt{\sigma_0}}M; \tag{5.57}$$

$$M_\delta>M/2\quad\text{for each }\delta\in\mathcal{A}_{M,\eta}\cap(0,\widehat{\delta}_0); \tag{5.58}$$

$$\|z_\delta^*\|\le\mathrm{e}^{C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2r^{-1}, \tag{5.59}$$

where $C_1 = C_1(A,B) > 0$ is given by Theorem 5.1. Indeed, by (5.44), (H_2), Hölder inequality and (5.42), we have that

$$a_\delta \leq \int_0^{\mathcal{T}_\delta(M)} \left\| e^{(\mathcal{T}_\delta(M)-t)A} B\hat{v}_\delta(t) \right\| dt \leq \|B\| \int_0^{\mathcal{T}_\delta(M)} e^{-(\mathcal{T}_\delta(M)-t)\sigma_0} \|\hat{v}_\delta(t)\| dt \leq \frac{\|B\|}{\sqrt{\sigma_0}},$$

and

$$c_\delta \leq \int_0^{\mathcal{T}_\delta(M)} \left\| e^{(\mathcal{T}_\delta(M)-t)A} Bu_\delta^*(t) \right\| dt \leq \|B\| \int_0^{\mathcal{T}_\delta(M)} e^{-(\mathcal{T}_\delta(M)-t)\sigma_0} \|u_\delta^*(t)\| dt \leq \frac{\|B\|}{\sqrt{\sigma_0}} M.$$

The above two inequalities imply (5.57). By (5.40) and Theorem 5.3 (where $(\delta, k) = (\delta, \mathcal{T}_\delta(M)/\delta)$), we observe that $\mathcal{N}_\delta(\mathcal{T}_\delta(M)) \geq \mathcal{N}(\mathcal{T}_\delta(M))$. This, along with (ii) of Theorem 4.2 and (5.41), implies that

$$0 < M - M_\delta = \mathcal{N}(\mathcal{T}(M)) - \mathcal{N}_\delta(\mathcal{T}_\delta(M)) \leq \mathcal{N}(\mathcal{T}(M)) - \mathcal{N}(\mathcal{T}_\delta(M)). \tag{5.60}$$

Meanwhile, by (5.39), Theorem 5.2 (where $T_1 = \mathcal{T}(M)$ and $T_2 = \mathcal{T}_\delta(M)$) and (ii) of Theorem 4.2, we get that

$$\mathcal{N}(\mathcal{T}(M)) - \mathcal{N}(\mathcal{T}_\delta(M)) \leq \frac{e^{C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}\|y_0\|^4 r^{-2}}{M} (\mathcal{T}_\delta(M) - \mathcal{T}(M)). \tag{5.61}$$

Since $\delta \in \mathcal{A}_{M,\eta}$, we can apply Theorem 5.4 to get that $\mathcal{T}_\delta(M) - \mathcal{T}(M) < \delta$. This, together with (5.60) and (5.61), yields that

$$M - M_\delta < e^{C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}\|y_0\|^4 r^{-2} M^{-1}\delta,$$

which, combined with (5.56), leads to (5.58). Since z_δ^* is the minimizer of $(\mathcal{JP})_\delta^{\mathcal{T}_\delta(M)}$, we obtain (5.59) from (5.40), (5.2) and (5.39) directly.

Next, we turn to the proof of (5.37). Since $\mathcal{O}_\delta'' \subset \mathcal{O}_\delta$ (see (5.45) and (5.49)) and

$$\operatorname{diam} \mathcal{O}_\delta \geq \sup\{\|u_\delta - u_\delta^*\|_{L^2(0,\mathcal{T})} : u_\delta \in \mathcal{O}_\delta\},$$

it follows from (5.46) and (5.42) that

$$\operatorname{diam} \mathcal{O}_\delta \geq \sup\{\|u_\delta - u_\delta^*\|_{L^2(0,\mathcal{T})} : u_\delta \in \mathcal{O}_\delta''\} \geq \beta\|\hat{v}_\delta\|_{L^2(0,\mathcal{T})} = \beta,$$

for each β satisfying

$$0 < \beta \leq \min\left\{ \sqrt{M(1-\hat{\lambda})(M-M_\delta)}, \sqrt{\frac{\hat{\lambda} r M_\delta (M-M_\delta)}{a_\delta^2 \|z_\delta^*\|}} \right\}.$$

Then we have that

$$\operatorname{diam} \mathcal{O}_\delta \geq \min\left\{ \sqrt{M(1-\hat{\lambda})(M-M_\delta)}, \sqrt{\frac{\hat{\lambda} r M_\delta (M-M_\delta)}{a_\delta^2 \|z_\delta^*\|}} \right\}.$$

This, together with (5.47), yields that

$$\operatorname{diam}\mathcal{O}_\delta \geq \min\left\{\sqrt{\frac{M(M-M_\delta)}{2}}, \sqrt{\frac{rM_\delta(M-M_\delta)}{2a_\delta^2\|z_\delta^*\|}}, \frac{rM_\delta^2}{a_\delta c_\delta\|z_\delta^*\|}\right\}. \tag{5.62}$$

For each $\delta \in \mathcal{A}_{M,\eta} \cap (0, \widehat{\delta}_0)$, it follows from (5.41), (5.58), (5.57), (5.59) and (5.56) that

$$\sqrt{\frac{M(M-M_\delta)}{2}} > \sqrt{\frac{M\sigma_0^{3/2} r(1-\eta)\delta}{4\|B\|}}\ ,$$

$$\sqrt{\frac{rM_\delta(M-M_\delta)}{2a_\delta^2\|z_\delta^*\|}} > \sqrt{\frac{rM(M-M_\delta)}{4a_\delta^2\|z_\delta^*\|}} > \sqrt{\frac{M\sigma_0^{5/2}(1-\eta)\delta}{8\|B\|^3 \mathrm{e}^{C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2 r^{-3}}}\ ,$$

and

$$\frac{rM_\delta^2}{a_\delta c_\delta\|z_\delta^*\|} > \frac{M\sigma_0}{4\|B\|^2 \mathrm{e}^{C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2 r^{-2}} > \frac{M\sigma_0\sqrt{(1-\eta)\delta}}{4\|B\|^2 \mathrm{e}^{C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2 r^{-2}}\ .$$

These, along with (5.62), imply that

$$\operatorname{diam}\mathcal{O}_\delta > C_4\sqrt{(1-\eta)\delta},$$

where $C_4 \triangleq C_4(M, y_0, r, A, B) > 0$ is given by

$$C_4 \triangleq \min\left\{\sqrt{\frac{M\sigma_0^{3/2} r}{4\|B\|}}, \sqrt{\frac{M\sigma_0^{5/2}}{8\|B\|^3 \mathrm{e}^{C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2 r^{-3}}}, \frac{M\sigma_0}{4\|B\|^2 \mathrm{e}^{C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2 r^{-2}}\right\}.$$

Hence, (5.37) holds.

In summary, we finish the proof of Theorem 5.5. □

6. Proofs of main Theorems

In this section, we will give proofs of Theorems 1.2–1.4 one by one.

6.1. Proof of Theorem 1.2

Note that (ii) follows from the equality in (5.21) directly. It suffices to show (i).

Arbitrarily fix $M > 0$. Let T^* be given by (2.4). It follows from (iii) of Theorem 4.2 that $\mathcal{T}(M) < T^*$. Denote

$$\delta_0 \triangleq \min\left\{\frac{T^*-\mathcal{T}(M)}{4}, \sqrt{\frac{3}{4\mathrm{e}^{2C(T^*+\frac{1}{\mathcal{T}(M)})}T^*\|AB\|^2}}, \frac{\sigma_0^{3/2}}{2\|B\|}\mathrm{e}^{-C_3\left[T^*+\frac{1}{\mathcal{T}(M)}+\frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^{-4} r^4\right\}, \tag{6.1}$$

where $C = C(A, B) > 0$ and $C_3 = C_3(A, B) > 0$ are given by Theorem 2.2 and Theorem 5.3, respectively. For each $\delta \in (0, \delta_0)$, let $\hat{k}_\delta \in \mathbb{N}^+$ satisfy that

$$(\hat{k}_\delta - 1)\delta < \mathcal{T}(M) \leq \hat{k}_\delta \delta. \tag{6.2}$$

It is clear that

$$0 < \mathcal{T}(M) < (\hat{k}_\delta + 1)\delta < \mathcal{T}(M) + 2\delta < (T^* + \mathcal{T}(M))/2 < T^*, \tag{6.3}$$

and

$$e^{2C\left[(\hat{k}_\delta+1)\delta + \frac{1}{(\hat{k}_\delta+1)\delta}\right]}(\hat{k}_\delta + 1)\delta^3\|AB\|^2 \leq \frac{3}{4}.$$

These, along with (2.6), yield that

$$(\delta, \hat{k}_\delta + 1) \in \mathcal{P}_{T^*}. \tag{6.4}$$

On one hand, by (6.4) and (6.3), we can apply Theorem 5.3 (where $(\delta, k) = (\delta, \hat{k}_\delta + 1)$) and Theorem 5.2 (where $T_1 = \mathcal{T}(M)$ and $T_2 = (\hat{k}_\delta + 1)\delta$) to get that

$$\begin{aligned}\mathcal{N}_\delta((\hat{k}_\delta + 1)\delta) \leq & \mathcal{N}((\hat{k}_\delta + 1)\delta) + e^{C_3\left[(\hat{k}_\delta+1)\delta + \frac{1}{(\hat{k}_\delta+1)\delta} + \frac{1}{T^*-(\hat{k}_\delta+1)\delta}\right]}\|y_0\|^4 r^{-3}\delta^2 \\ \leq & \mathcal{N}(\mathcal{T}(M)) - \frac{\sigma_0^{3/2} r}{\|B\|}\big((\hat{k}_\delta + 1)\delta - \mathcal{T}(M)\big) + e^{C_3\left[(\hat{k}_\delta+1)\delta + \frac{1}{(\hat{k}_\delta+1)\delta} + \frac{1}{T^*-(\hat{k}_\delta+1)\delta}\right]}\|y_0\|^4 r^{-3}\delta^2.\end{aligned}$$

This, together with (ii) of Theorem 4.2, (6.2) and (6.3), implies that

$$\mathcal{N}_\delta((\hat{k}_\delta + 1)\delta) \leq M - \frac{\sigma_0^{3/2} r}{\|B\|}\delta + e^{C_3\left[T^* + \frac{1}{\mathcal{T}(M)} + \frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^4 r^{-3}\delta^2.$$

It follows from the above inequality and (6.1) that

$$M \geq \mathcal{N}_\delta((\hat{k}_\delta + 1)\delta) + \frac{\sigma_0^{3/2} r}{2\|B\|}\delta \quad \text{for each } \delta \in (0, \delta_0). \tag{6.5}$$

On the other hand, let u_δ be the zero extension of the optimal control to $(\mathcal{NP})_\delta^{(\hat{k}_\delta+1)\delta}$ over $(0, +\infty)$. By (6.5), one can easily check that u_δ is an admissible control to $(\mathcal{TP})_\delta^M$, which drives the solution to $B_r(0)$ at time $(\hat{k}_\delta + 1)\delta$. Then we have that

$$\mathcal{T}(M) \leq \mathcal{T}_\delta(M) \leq (\hat{k}_\delta + 1)\delta \quad \text{for each } \delta \in (0, \delta_0). \tag{6.6}$$

Hence, (1.9) follows from (6.2) and (6.6) immediately.

In summary, we finish the proof of Theorem 1.2. □

6.2. Proof of Theorem 1.3

For each $M > 0$ and $\delta > 0$, let u^* and u^*_δ be the optimal control to $(\mathcal{TP})^M$ and the optimal control with the minimal norm to $(\mathcal{TP})^M_\delta$ (see (i) and (ii) of Thm. 3.1), respectively.

(i) Let $\delta_0 = \delta_0(M, y_0, r, A, B) > 0$ be given by Theorem 1.2 (see (6.1)) and $C_2 = C_2(A, B) > 0$ be given by Theorem 5.2. Denote

$$\delta_1 \triangleq \min\left\{\delta_0, \frac{1}{4}M^2 \mathrm{e}^{-C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}\|y_0\|^{-4}r^2, 1\right\}. \tag{6.7}$$

For each $\delta > 0$, two cases may occur: Case 1. $\delta \geq \delta_1$; Case 2. $0 < \delta < \delta_1$.

Case 1. In this case, we have that

$$\|u^* - u^*_\delta\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} \leq \|u^*\|_{L^2(0,+\infty;\mathbb{R}^m)} + \|u^*_\delta\|_{L^2(0,+\infty;\mathbb{R}^m)} \leq 2M \leq 2M\delta_1^{-1}\delta,$$

which indicates (1.11) with $C \triangleq 2M\delta_1^{-1}$.

Case 2. In this case, the proof will be split into eight steps as follows:

Step 1. We present some preparations.

Let T^* be given by (2.4) and $\mathcal{P}_{T^*}$ be given by (2.6). By (6.6), (6.3) and (6.1), we have that

$$0 < \mathcal{T}(M) \leq \mathcal{T}_\delta(M) < T^* \tag{6.8}$$

and

$$\mathrm{e}^{2C(\mathcal{T}_\delta(M)+\frac{1}{\mathcal{T}_\delta(M)})}\mathcal{T}_\delta(M)\delta^2\|AB\|^2 \leq \mathrm{e}^{2C(T^*+\frac{1}{\mathcal{T}(M)})}T^*\|AB\|^2\delta^2 \leq 3/4, \tag{6.9}$$

where $C = C(A, B) > 0$ is given by Theorem 2.2. Note that $\mathcal{T}_\delta(M)$ is a multiple of δ (see (1.2)). It follows from (6.8), (6.9) and (2.6) that

$$(\delta, \mathcal{T}_\delta(M)/\delta) \triangleq (\delta, k_\delta) \in \mathcal{P}_{T^*} \text{ with } k_\delta \in \mathbb{N}^+. \tag{6.10}$$

Let $z^* \neq 0$ and $z^*_\delta \neq 0$ be the unique minimizer of $\mathcal{J}^{\mathcal{T}(M)}(\cdot)$ and $\mathcal{J}^{\mathcal{T}_\delta(M)}_\delta(\cdot)$, respectively (see (i) of Thm. 4.1 and (i) of Thm. 4.3). Define

$$\hat{z}^* \triangleq z^*/\|z^*\| \quad \text{and} \quad \hat{z}^*_\delta \triangleq z^*_\delta/\|z^*_\delta\|. \tag{6.11}$$

Since u^* is the optimal control to $(\mathcal{TP})^M$, it follows from (iv) of Theorem 4.2 that the restriction of u^* on $(0, \mathcal{T}(M))$ is the optimal control to $(\mathcal{NP})^{\mathcal{T}(M)}$. Then by (6.8), (ii) of Theorem 4.1, (ii) of Theorem 4.2 and (6.11), we have that

$$u^*(t) = B^\top\varphi(t; \mathcal{T}(M), z^*) = M\frac{B^\top\varphi(t; \mathcal{T}(M), \hat{z}^*)}{\|B^\top\varphi(\cdot; \mathcal{T}(M), \hat{z}^*)\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}} \text{ for a.e. } t \in (0, \mathcal{T}(M)). \tag{6.12}$$

Similarly, since u_δ^* is the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$, it follows from (iii) of Theorem 3.1 that the restriction of u_δ^* on $(0,\mathcal{T}_\delta(M))$ is an optimal control to $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$. Then

$$M_\delta \triangleq \mathcal{N}_\delta(\mathcal{T}_\delta(M)) = \|u_\delta^*\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} \leq M. \tag{6.13}$$

Moreover, by (6.10), (ii) of Theorem 4.3 (where $(\delta,k)=(\delta,k_\delta)$) and (6.11), we get that

$$u_\delta^*(t) = B^\top \overline{\varphi}_\delta(t;\mathcal{T}_\delta(M),z_\delta^*) = M_\delta \frac{B^\top \overline{\varphi}_\delta(t;\mathcal{T}_\delta(M),\hat{z}_\delta^*)}{\|B^\top \overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),\hat{z}_\delta^*)\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}} \text{ for a.e. } t\in(0,\mathcal{T}_\delta(M)). \tag{6.14}$$

Define an affiliated control $\hat{u}_\delta$ from $(0,+\infty)$ to $\mathbb{R}^m$ by

$$\hat{u}_\delta(t) \triangleq \begin{cases} M\dfrac{B^\top\varphi(t;\mathcal{T}(M),\hat{z}_\delta^*)}{\|B^\top\varphi(\cdot;\mathcal{T}(M),\hat{z}_\delta^*)\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}}, & t\in(0,\mathcal{T}(M)),\\ 0, & t\in[\mathcal{T}(M),+\infty). \end{cases} \tag{6.15}$$

Step 2. We show that

$$\|y(\mathcal{T}(M);y_0,u_\delta^*)\| \leq \|y_0\| + \sigma_0^{-1}\|B\|^2\|z_\delta^*\|. \tag{6.16}$$

Indeed, by (H_2), we have that, for each $i\in\{1,2,...,k_\delta\}$,

$$\left\|\frac{1}{\delta}\int_{(i-1)\delta}^{i\delta}\varphi(s;k_\delta\delta,z_\delta^*)\mathrm{d}s\right\| \leq \frac{1}{\delta}\int_{(i-1)\delta}^{i\delta}\|\mathrm{e}^{(k_\delta\delta-s)A^\top}z_\delta^*\|\mathrm{d}s \leq \|z_\delta^*\|. \tag{6.17}$$

This, along with (6.14), (6.10) and (1.8), yields that

$$\|Bu_\delta^*(t)\| = \|BB^\top\overline{\varphi}_\delta(t;k_\delta\delta,z_\delta^*)\| \leq \|B\|^2\|z_\delta^*\| \text{ for a.e. } t\in(0,\mathcal{T}_\delta(M)),$$

which indicates

$$\|Bu_\delta^*\|_{L^\infty(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} \leq \|B\|^2\|z_\delta^*\|. \tag{6.18}$$

Since $\mathcal{T}(M)\leq\mathcal{T}_\delta(M)$, it follows from (H_2) that

$$\begin{aligned}\|y(\mathcal{T}(M);y_0,u_\delta^*)\| \leq& \|\mathrm{e}^{\mathcal{T}(M)A}y_0\| + \left\|\int_0^{\mathcal{T}(M)}\mathrm{e}^{(\mathcal{T}(M)-s)A}Bu_\delta^*(s)\mathrm{d}s\right\|\\ \leq& \|y_0\| + \|Bu_\delta^*\|_{L^\infty(0,\mathcal{T}(M);\mathbb{R}^m)}\int_0^{\mathcal{T}(M)}\mathrm{e}^{-(\mathcal{T}(M)-s)\sigma_0}\mathrm{d}s\\ \leq& \|y_0\| + \sigma_0^{-1}\|Bu_\delta^*\|_{L^\infty(0,\mathcal{T}(M);\mathbb{R}^m)}.\end{aligned}$$

Hence, we obtain (6.16) from the latter inequality and (6.18).

Step 3. We prove that

$$\|\varphi(\cdot;\mathcal{T}(M),z_\delta^*) - \overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} \leq \big(\mathcal{T}_\delta(M)-\mathcal{T}(M)+\delta\big)\|A\|\|z_\delta^*\|\sigma_0^{-1/2}. \tag{6.19}$$

On one hand, for each $t \in (0, \mathcal{T}(M))$, we have that

$$\begin{aligned}&\varphi(t;\mathcal{T}(M),z_\delta^*)-\varphi(t;\mathcal{T}_\delta(M),z_\delta^*)\\=&\ \varphi(\mathcal{T}_\delta(M);\mathcal{T}_\delta(M),\mathrm{e}^{(\mathcal{T}(M)-t)A^\top}z_\delta^*)-\varphi(\mathcal{T}(M);\mathcal{T}_\delta(M),\mathrm{e}^{(\mathcal{T}(M)-t)A^\top}z_\delta^*)\\=&\ \int_{\mathcal{T}(M)}^{\mathcal{T}_\delta(M)}\varphi'(s;\mathcal{T}_\delta(M),\mathrm{e}^{(\mathcal{T}(M)-t)A^\top}z_\delta^*)\mathrm{d}s,\end{aligned}$$

which, combined with (H_2), indicates that

$$\begin{aligned}&\|\varphi(t;\mathcal{T}(M),z_\delta^*)-\varphi(t;\mathcal{T}_\delta(M),z_\delta^*)\|\\=&\left\|\int_{\mathcal{T}(M)}^{\mathcal{T}_\delta(M)}A^\top\mathrm{e}^{(\mathcal{T}_\delta(M)-s)A^\top}\mathrm{e}^{(\mathcal{T}(M)-t)A^\top}z_\delta^*\mathrm{d}s\right\|\\\le&\big(\mathcal{T}_\delta(M)-\mathcal{T}(M)\big)\|A\|\mathrm{e}^{-\sigma_0(\mathcal{T}(M)-t)}\|z_\delta^*\|.\end{aligned}$$

This implies that

$$\begin{aligned}&\|\varphi(\cdot;\mathcal{T}(M),z_\delta^*)-\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}\\\le&\big(\mathcal{T}_\delta(M)-\mathcal{T}(M)\big)\|A\|\|z_\delta^*\|\left(\int_0^{\mathcal{T}(M)}\mathrm{e}^{-2(\mathcal{T}(M)-t)\sigma_0}\mathrm{d}t\right)^{1/2}\\\le&\big(\mathcal{T}_\delta(M)-\mathcal{T}(M)\big)\|A\|\|z_\delta^*\|\sigma_0^{-1/2}.\end{aligned}\tag{6.20}$$

On the other hand, since $\mathcal{T}(M) \le \mathcal{T}_\delta(M)$ and $\mathcal{T}_\delta(M) = k_\delta\delta$ (see (6.10)), it follows from (1.8) that

$$\begin{aligned}&\|\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*)-\overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|^2_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}\\\le&\int_0^{\mathcal{T}_\delta(M)}\|\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*)-\overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|^2\mathrm{d}t\\=&\sum_{j=1}^{k_\delta}\int_{(j-1)\delta}^{j\delta}\left\|\varphi(t;\mathcal{T}_\delta(M),z_\delta^*)-\frac{1}{\delta}\int_{(j-1)\delta}^{j\delta}\varphi(s;\mathcal{T}_\delta(M),z_\delta^*)\mathrm{d}s\right\|^2\mathrm{d}t,\end{aligned}$$

which indicates that

$$\begin{aligned}&\|\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*)-\overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|^2_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}\\\le&\sum_{j=1}^{k_\delta}\delta\left(\int_{(j-1)\delta}^{j\delta}\|\varphi'(\tau;\mathcal{T}_\delta(M),z_\delta^*)\|\mathrm{d}\tau\right)^2\\\le&\delta^2\|\varphi'(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|^2_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}.\end{aligned}$$

This, along with (H_2), yields that

$$\begin{aligned}&\|\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*)-\overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}\\\le&\delta\left(\int_0^{\mathcal{T}_\delta(M)}\|A^\top\mathrm{e}^{(\mathcal{T}_\delta(M)-t)A^\top}z_\delta^*\|^2\mathrm{d}t\right)^{1/2}\\\le&\delta\|A\|\|z_\delta^*\|\sigma_0^{-1/2}.\end{aligned}$$

Hence, (6.19) follows from the latter inequality and (6.20) immediately.

Step 4. We show that

$$|\mathcal{T}_\delta(M)-\mathcal{T}(M)|+|M-M_\delta|+\delta\le \widetilde{C}_1\delta, \tag{6.21}$$

where $\widetilde{C}_1\triangleq 3\mathrm{e}^{C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}(M^{-1}\|y_0\|^4r^{-2}+1)$.

Since $\mathcal{T}_\delta(M)=k_\delta\delta$ (see (6.10)), we have that $\mathcal{N}_\delta(\mathcal{T}_\delta(M))\ge\mathcal{N}(\mathcal{T}_\delta(M))$. Then by (6.13) and (ii) of Theorem 4.2, we get that

$$0\le M-M_\delta=\mathcal{N}(\mathcal{T}(M))-\mathcal{N}_\delta(\mathcal{T}_\delta(M))\le\mathcal{N}(\mathcal{T}(M))-\mathcal{N}(\mathcal{T}_\delta(M)). \tag{6.22}$$

Meanwhile, it follows from (6.8) and Theorem 5.2 (where $T_1=\mathcal{T}(M)$ and $T_2=\mathcal{T}_\delta(M)$) that

$$\mathcal{N}(\mathcal{T}(M))-\mathcal{N}(\mathcal{T}_\delta(M))\le\frac{\mathrm{e}^{C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}\|y_0\|^4r^{-2}}{\mathcal{N}(\mathcal{T}(M))}(\mathcal{T}_\delta(M)-\mathcal{T}(M)).$$

The above inequality, along with (6.22) and (ii) of Theorem 4.2, yields that

$$0\le M-M_\delta\le M^{-1}\mathrm{e}^{C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}\|y_0\|^4r^{-2}(\mathcal{T}_\delta(M)-\mathcal{T}(M)).$$

Since $0<\delta<\delta_1\le\delta_0$ (see (6.7)), we obtain from the latter inequality and (i) of Theorem 1.2 that

$$0\le M-M_\delta\le 2M^{-1}\mathrm{e}^{C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}\|y_0\|^4r^{-2}\delta, \tag{6.23}$$

and

$$M_\delta\ge M-2M^{-1}\mathrm{e}^{C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}\|y_0\|^4r^{-2}\delta\ge M/2. \tag{6.24}$$

Hence, (6.21) follows from (6.23) and (i) of Theorem 1.2 immediately.

Step 5. We show that

$$\|\hat{u}_\delta-u^*_\delta\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}\le\left(2\|B\|\|A\|\|z^*_\delta\|\sigma_0^{-1/2}+\frac{2\|z^*_\delta\|^2\|B\|^2}{M}+1\right)\widetilde{C}_1\delta. \tag{6.25}$$

Recall (6.11) for the definition of $\hat{z}^*_\delta$. In this step, we simply write $\varphi(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$ for $\varphi(\cdot;\mathcal{T}(M),\hat{z}^*_\delta)$ and $\overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),\hat{z}^*_\delta)$, respectively. Meanwhile, we simply write $\|\cdot\|_{L^2(0,\mathcal{T})}$, $\|\cdot\|_{L^2(0,\mathcal{T}_\delta)}$ and $\|\cdot\|_{L^2(\mathcal{T},\mathcal{T}_\delta)}$ for $\|\cdot\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}$, $\|\cdot\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}$ and $\|\cdot\|_{L^2(\mathcal{T}(M),\mathcal{T}_\delta(M);\mathbb{R}^m)}$, respectively. By (6.15) and (6.14), we obtain that

$$\begin{aligned}&\|\hat{u}_\delta-u^*_\delta\|_{L^2(0,\mathcal{T})}\\ =&\left\|M\frac{B^\top\varphi}{\|B^\top\varphi\|_{L^2(0,\mathcal{T})}}-M_\delta\frac{B^\top\overline{\varphi}_\delta}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}}\right\|_{L^2(0,\mathcal{T})}\\ \le& M_\delta\left\|\frac{B^\top\varphi}{\|B^\top\varphi\|_{L^2(0,\mathcal{T})}}-\frac{B^\top\overline{\varphi}_\delta}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}}\right\|_{L^2(0,\mathcal{T})}+|M-M_\delta|.\end{aligned}$$

Then

$$
\begin{aligned}
&\|\hat{u}_\delta - u_\delta^*\|_{L^2(0,\mathcal{T})} \\
\leq & M_\delta \left\| \frac{B^\top\varphi(\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)} - \|B^\top\varphi\|_{L^2(0,\mathcal{T})})}{\|B^\top\varphi\|_{L^2(0,\mathcal{T})}\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}} + \frac{B^\top\varphi - B^\top\overline{\varphi}_\delta}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}} \right\|_{L^2(0,\mathcal{T})} + |M - M_\delta| \\
\leq & M_\delta \frac{\left|\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)} - \|B^\top\varphi\|_{L^2(0,\mathcal{T})}\right| + \|B^\top\varphi - B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T})}}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}} + |M - M_\delta|.
\end{aligned}
\tag{6.26}
$$

Since $\mathcal{T}(M) \leq \mathcal{T}_\delta(M)$, one can directly check that

$$
\begin{aligned}
\left|\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)} - \|B^\top\varphi\|_{L^2(0,\mathcal{T})}\right| &\leq \frac{\left|\|B^\top\overline{\varphi}_\delta\|^2_{L^2(0,\mathcal{T})} - \|B^\top\varphi\|^2_{L^2(0,\mathcal{T})}\right| + \|B^\top\overline{\varphi}_\delta\|^2_{L^2(\mathcal{T},\mathcal{T}_\delta)}}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)} + \|B^\top\varphi\|_{L^2(0,\mathcal{T})}} \\
&\leq \frac{\left|\|B^\top\overline{\varphi}_\delta\|^2_{L^2(0,\mathcal{T})} - \|B^\top\varphi\|^2_{L^2(0,\mathcal{T})}\right|}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T})} + \|B^\top\varphi\|_{L^2(0,\mathcal{T})}} + \frac{\|B^\top\overline{\varphi}_\delta\|^2_{L^2(\mathcal{T},\mathcal{T}_\delta)}}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}} \\
&\leq \|B^\top\overline{\varphi}_\delta - B^\top\varphi\|_{L^2(0,\mathcal{T})} + \frac{\|B^\top\overline{\varphi}_\delta\|^2_{L^2(\mathcal{T},\mathcal{T}_\delta)}}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}}.
\end{aligned}
$$

This, along with (6.26), yields that

$$
\|\hat{u}_\delta - u_\delta^*\|_{L^2(0,\mathcal{T})} \leq \frac{M_\delta}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}} \left(2\|B^\top\varphi - B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T})} + \frac{\|B^\top\overline{\varphi}_\delta\|^2_{L^2(\mathcal{T},\mathcal{T}_\delta)}}{\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}} \right) + |M - M_\delta|.
$$

From (6.14), (6.11) and the latter inequality, it follows that $M_\delta = \|z_\delta^*\|\|B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T}_\delta)}$ and

$$
\begin{aligned}
\|\hat{u}_\delta - u_\delta^*\|_{L^2(0,\mathcal{T})} \leq & \|z_\delta^*\| \left(2\|B^\top\varphi - B^\top\overline{\varphi}_\delta\|_{L^2(0,\mathcal{T})} + \frac{\|z_\delta^*\|}{M_\delta}\|B^\top\overline{\varphi}_\delta\|^2_{L^2(\mathcal{T},\mathcal{T}_\delta)} \right) + |M - M_\delta| \\
\leq & 2\|B^\top\varphi(\cdot;\mathcal{T}(M), z_\delta^*) - B^\top\overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M), z_\delta^*)\|_{L^2(0,\mathcal{T})} \\
& + \frac{1}{M_\delta}\|B^\top\overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M), z_\delta^*)\|^2_{L^2(\mathcal{T},\mathcal{T}_\delta)} + |M - M_\delta|.
\end{aligned}
\tag{6.27}
$$

Note that $\|\overline{\varphi}_\delta(t;\mathcal{T}_\delta(M), z_\delta^*)\| \leq \|z_\delta^*\|$ for each $t \in (0, \mathcal{T}_\delta(M))$ (see (1.8) and (6.17)). Then by (6.27) and (6.19), we have that

$$
\begin{aligned}
\|\hat{u}_\delta - u_\delta^*\|_{L^2(0,\mathcal{T})} &\leq 2\big(\mathcal{T}_\delta(M) - \mathcal{T}(M) + \delta\big)\|B\|\|A\|\|z_\delta^*\|\sigma_0^{-1/2} + \frac{\|z_\delta^*\|^2\|B\|^2}{M_\delta}(\mathcal{T}_\delta(M) - \mathcal{T}(M)) + |M - M_\delta| \\
&\leq \left(2\|B\|\|A\|\|z_\delta^*\|\sigma_0^{-1/2} + \frac{\|z_\delta^*\|^2\|B\|^2}{M_\delta} + 1 \right) (|\mathcal{T}_\delta(M) - \mathcal{T}(M)| + |M - M_\delta| + \delta),
\end{aligned}
$$

which, combined with (6.24) and (6.21), indicates (6.25).

Step 6. We show that there exists $\widetilde{C}_2 \triangleq \widetilde{C}_2(M, A, B) > 0$ so that

$$
\|y(\mathcal{T}(M); y_0, \hat{u}_\delta) - y(\mathcal{T}_\delta(M); y_0, u_\delta^*)\| \leq \widetilde{C}_2\widetilde{C}_1(1 + \|y_0\| + \|z_\delta^*\|^2)\delta. \tag{6.28}
$$

Indeed, by (H_2), we have that

$$\begin{aligned}\|y(\mathcal{T}(M);y_0,\hat{u}_\delta)-y(\mathcal{T}(M);y_0,u_\delta^*)\| &= \left\|\int_0^{\mathcal{T}(M)} \mathrm{e}^{(\mathcal{T}(M)-t)A}B(\hat{u}_\delta(t)-u_\delta^*(t))\mathrm{d}t\right\| \\ &\leq \|B\|\int_0^{\mathcal{T}(M)} \mathrm{e}^{-(\mathcal{T}(M)-t)\sigma_0}\|\hat{u}_\delta(t)-u_\delta^*(t)\|\mathrm{d}t \\ &\leq \frac{\|B\|}{\sigma_0^{1/2}}\|\hat{u}_\delta-u_\delta^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}.\end{aligned} \tag{6.29}$$

Since $\mathcal{T}(M)\leq\mathcal{T}_\delta(M)$, it follows from ($H_2$) that for each $z\in\mathbb{R}^n$,

$$\|z-\mathrm{e}^{(\mathcal{T}_\delta(M)-\mathcal{T}(M))A^\top}z\| = \left\|\int_{\mathcal{T}(M)}^{\mathcal{T}_\delta(M)} \varphi'(t;\mathcal{T}_\delta(M),z)\mathrm{d}t\right\| \leq (\mathcal{T}_\delta(M)-\mathcal{T}(M))\|A\|\|z\|.$$

This, together with (6.18) and (H_2), yields that

$$\begin{aligned}&\|y(\mathcal{T}(M);y_0,u_\delta^*)-y(\mathcal{T}_\delta(M);y_0,u_\delta^*)\| \\ \leq\ &\|y(\mathcal{T}(M);y_0,u_\delta^*)-\mathrm{e}^{(\mathcal{T}_\delta(M)-\mathcal{T}(M))A}y(\mathcal{T}(M);y_0,u_\delta^*)\| + \left\|\int_{\mathcal{T}(M)}^{\mathcal{T}_\delta(M)} \mathrm{e}^{(\mathcal{T}_\delta(M)-t)A}Bu_\delta^*(t)\mathrm{d}t\right\| \\ \leq\ &(\mathcal{T}_\delta(M)-\mathcal{T}(M))\|A\|\|y(\mathcal{T}(M);y_0,u_\delta^*)\| + (\mathcal{T}_\delta(M)-\mathcal{T}(M))\|B\|^2\|z_\delta^*\|.\end{aligned}$$

Since $0<\delta<\delta_1\leq\delta_0$, it follows from the above inequality, (i) of Theorem 1.2 and (6.16) that

$$\|y(\mathcal{T}(M);y_0,u_\delta^*)-y(\mathcal{T}_\delta(M);y_0,u_\delta^*)\| \leq 2\delta\left(\|A\|\|y_0\|+\|A\|\sigma_0^{-1}\|B\|^2\|z_\delta^*\|+\|B\|^2\|z_\delta^*\|\right),$$

which, combined with (6.29) and (6.25), indicates that

$$\begin{aligned}&\|y(\mathcal{T}(M);y_0,\hat{u}_\delta)-y(\mathcal{T}_\delta(M);y_0,u_\delta^*)\| \\ \leq\ &\|y(\mathcal{T}(M);y_0,\hat{u}_\delta)-y(\mathcal{T}(M);y_0,u_\delta^*)\|+\|y(\mathcal{T}(M);y_0,u_\delta^*)-y(\mathcal{T}_\delta(M);y_0,u_\delta^*)\| \\ \leq\ &4\widetilde{C}_1\delta(1+\|y_0\|+\|z_\delta^*\|^2)\left(2\|B\|^2\|A\|\sigma_0^{-1}+\frac{\|B\|^3}{M\sigma_0^{1/2}}+\frac{\|B\|}{\sigma_0^{1/2}}+\|A\|+\|B\|^2\right).\end{aligned}$$

Hence, (6.28) holds at once, where

$$\widetilde{C}_2 \triangleq 8\|B\|^2\|A\|\sigma_0^{-1}+\frac{4\|B\|^3}{M\sigma_0^{1/2}}+\frac{4\|B\|}{\sigma_0^{1/2}}+4\|A\|+4\|B\|^2.$$

Step 7. We prove that

$$\|\hat{z}^*-\hat{z}_\delta^*\| \leq \frac{1}{r}\|y(\mathcal{T}(M);y_0,\hat{u}_\delta)-y(\mathcal{T}_\delta(M);y_0,u_\delta^*)\|. \tag{6.30}$$

To this end, by (6.12) and (6.15), one can directly check that

$$\begin{aligned}&\langle \hat{z}^* - \hat{z}^*_\delta, y(\mathcal{T}(M); y_0, u^*) - y(\mathcal{T}(M); y_0, \hat{u}_\delta)\rangle \\ =\ &\langle B^\top \varphi(\cdot; \mathcal{T}(M), \hat{z}^*) - B^\top \varphi(\cdot; \mathcal{T}(M), \hat{z}^*_\delta), u^* - \hat{u}_\delta\rangle_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} \geq 0.\end{aligned} \tag{6.31}$$

Recall that $u^*|_{(0,\mathcal{T}(M))}$ and $u^*_\delta|_{(0,\mathcal{T}_\delta(M))}$ are the unique optimal control to $(\mathcal{NP})^{\mathcal{T}(M)}$ and $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$, respectively. Since $\mathcal{T}(M) < T^*$ (see (6.8)) and $(\delta, \mathcal{T}_\delta(M)/\delta) \in \mathcal{P}_{T^*}$ (see (6.10)), it follows from (6.11), (4.2), (4.4) and (6.31) that

$$\begin{aligned}\|\hat{z}^* - \hat{z}^*_\delta\|^2 &= \left\langle \hat{z}^* - \hat{z}^*_\delta, -\frac{1}{r}\left(y(\mathcal{T}(M); y_0, u^*) - y(\mathcal{T}_\delta(M); y_0, u^*_\delta)\right)\right\rangle \\ &= -\frac{1}{r}\left\langle \hat{z}^* - \hat{z}^*_\delta, y(\mathcal{T}(M); y_0, u^*) - y(\mathcal{T}(M); y_0, \hat{u}_\delta) + y(\mathcal{T}(M); y_0, \hat{u}_\delta) - y(\mathcal{T}_\delta(M); y_0, u^*_\delta)\right\rangle \\ &\leq -\frac{1}{r}\langle \hat{z}^* - \hat{z}^*_\delta, y(\mathcal{T}(M); y_0, \hat{u}_\delta) - y(\mathcal{T}_\delta(M); y_0, u^*_\delta)\rangle,\end{aligned}$$

which indicates (6.30).

Step 8. End of the proof.

Recall (6.11) for the definition of $\hat{z}^*$ and $\hat{z}^*_\delta$. In this step, we simply write $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ for $\varphi(\cdot; \mathcal{T}(M), \hat{z}^*)$ and $\varphi(\cdot; \mathcal{T}(M), \hat{z}^*_\delta)$, respectively; simply write $\|\cdot\|_{L^2(0,\mathcal{T})}$ for $\|\cdot\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}$. By (6.12) and (6.15), we observe that

$$\begin{aligned}\|u^* - \hat{u}_\delta\|_{L^2(0,\mathcal{T})} &= M\left\|\frac{B^\top\varphi_1}{\|B^\top\varphi_1\|_{L^2(0,\mathcal{T})}} - \frac{B^\top\varphi_2}{\|B^\top\varphi_2\|_{L^2(0,\mathcal{T})}}\right\|_{L^2(0,\mathcal{T})} \\ &\leq M\frac{\left\|B^\top\varphi_1 - B^\top\varphi_2\right\|_{L^2(0,\mathcal{T})} + \left|\|B^\top\varphi_2\|_{L^2(0,\mathcal{T})} - \|B^\top\varphi_1\|_{L^2(0,\mathcal{T})}\right|}{\|B^\top\varphi_1\|_{L^2(0,\mathcal{T})}} \\ &\leq \frac{2M}{\|B^\top\varphi_1\|_{L^2(0,\mathcal{T})}}\left\|B^\top\varphi_1 - B^\top\varphi_2\right\|_{L^2(0,\mathcal{T})}.\end{aligned}$$

From (6.12), (6.11) and the latter inequality it follows that $M = \|z^*\|\|B^\top\varphi_1\|_{L^2(0,\mathcal{T})}$ and

$$\|u^* - \hat{u}_\delta\|_{L^2(0,\mathcal{T})} \leq 2\|z^*\|\|B\|\|\varphi(\cdot; \mathcal{T}(M), \hat{z}^* - \hat{z}^*_\delta)\|_{L^2(0,\mathcal{T})}. \tag{6.32}$$

Furthermore, from (H_2) we get that

$$\|\varphi(\cdot; \mathcal{T}(M), \hat{z}^* - \hat{z}^*_\delta)\|^2_{L^2(0,\mathcal{T})} = \int_0^{\mathcal{T}(M)} \|\mathrm{e}^{(\mathcal{T}(M)-t)A^\top}(\hat{z}^* - \hat{z}^*_\delta)\|^2 \mathrm{d}t \leq \|\hat{z}^* - \hat{z}^*_\delta\|^2 \sigma_0^{-1}.$$

This, together with (6.32), (6.30), (6.28) and (6.25), implies that

$$\begin{aligned}&\|u^* - u^*_\delta\|_{L^2(0,\mathcal{T})} \\ \leq\ &\|u^* - \hat{u}_\delta\|_{L^2(0,\mathcal{T})} + \|\hat{u}_\delta - u^*_\delta\|_{L^2(0,\mathcal{T})} \\ \leq\ &\frac{2\|z^*\|\|B\|}{r\sigma_0^{1/2}}\widetilde{C}_2\widetilde{C}_1(1 + \|y_0\| + \|z^*_\delta\|^2)\delta + \left(2\|B\|\|A\|\|z^*_\delta\|\sigma_0^{-1/2} + \frac{2\|z^*_\delta\|^2\|B\|^2}{M} + 1\right)\widetilde{C}_1\delta \\ \leq\ &2\widetilde{C}_1\left(\frac{\|z^*\|\|B\|}{r}\widetilde{C}_2 + \|B\|\|A\| + \frac{\|B\|^2}{M} + 1\right)(\sigma_0^{-1/2} + 1)(1 + \|y_0\| + \|z^*_\delta\|^2)\delta.\end{aligned} \tag{6.33}$$

Recall that z^* and z^*_δ are the minimizer of $\mathcal{J}^{\mathcal{T}(M)}(\cdot)$ and $\mathcal{J}_\delta^{\mathcal{T}_\delta(M)}(\cdot)$, respectively. It follows from (6.8), (6.10) and Theorem 5.1 that

$$\|z^*\| \leq \mathrm{e}^{C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2 r^{-1} \text{ and } \|z^*_\delta\| \leq \mathrm{e}^{C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2 r^{-1}. \tag{6.34}$$

Note that $\|y_0\| > r$. We obtain from (6.34) that

$$1 + \|y_0\| + \|z^*_\delta\|^2 \leq 1 + \|y_0\| + \mathrm{e}^{2C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^4 r^{-2} \leq 2\mathrm{e}^{2C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2 r^{-2}(1+\|y_0\|^2),$$

and

$$\frac{\|z^*\|\|B\|}{r}\widetilde{C}_2 + \|B\|\|A\| + \frac{\|B\|^2}{M} + 1 \leq \mathrm{e}^{C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2 r^{-2}\left(\|B\|\widetilde{C}_2 + \|B\|\|A\| + \frac{\|B\|^2}{M} + 1\right).$$

These, along with (6.33), yield that

$$\|u^* - u^*_\delta\|_{L^2(0,\mathcal{T})} \leq \widetilde{C}_3\delta,$$

with

$$\widetilde{C}_3 \triangleq 4\mathrm{e}^{3C_1(T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^4 r^{-4}\left(\|B\|\widetilde{C}_2 + \|B\|\|A\| + \frac{\|B\|^2}{M} + 1\right)(\sigma_0^{-1/2}+1)(1+\|y_0\|^2)\widetilde{C}_1,$$

where $C_1 = C_1(A,B) > 0$, $\widetilde{C}_1 = \widetilde{C}_1(M, y_0, r, A, B) > 0$ and $\widetilde{C}_2 = \widetilde{C}_2(M, A, B) > 0$ are given by Theorem 5.1, (6.21) and (6.28), respectively.

In summary, we finish the proof of the conclusion (i) in Theorem 1.3.

(ii) Arbitrarily fix $\eta \in (0,1)$ and $M > 0$. Let $\mathcal{A}_{M,\eta}$ be given by Theorem 5.4. According to Theorem 5.4, it is clear that

$$M - \mathcal{N}_\delta(\mathcal{T}_\delta(M)) > \frac{\sigma_0^{3/2} r}{2\|B\|}(1-\eta)\delta \text{ for each } \delta \in \mathcal{A}_{M,\eta}. \tag{6.35}$$

Note that u^* and u^*_δ are the optimal control to $(\mathcal{TP})^M$ and the optimal control with the minimal norm to $(\mathcal{TP})^M_\delta$, respectively. We obtain from (ii) and (iv) of Theorem 4.2, and (iii) of Theorem 3.1 that

$$\|u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} = \mathcal{N}(\mathcal{T}(M)) = M, \tag{6.36}$$

and

$$\|u^*_\delta\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)} = \mathcal{N}_\delta(\mathcal{T}_\delta(M)). \tag{6.37}$$

Since $\mathcal{T}(M) \leq \mathcal{T}_\delta(M)$, it follows from (6.36), (6.37) and (6.35) that

$$\|u^*_\delta - u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} \geq M - \mathcal{N}_\delta(\mathcal{T}_\delta(M)) > \frac{\sigma_0^{3/2} r}{2\|B\|}(1-\eta)\delta \;\text{ for each } \delta \in \mathcal{A}_{M,\eta},$$

which leads to (1.12). □

6.3. Proof of Theorem 1.4

Arbitrarily fix $M>0$. Let u^* and u^*_δ be the optimal control to $(\mathcal{TP})^M$ and the optimal control with the minimal norm to $(\mathcal{TP})^M_\delta$, respectively. Let u_δ be any optimal control to $(\mathcal{TP})^M_\delta$.

(i) Let $\delta_1=\delta_1(M,y_0,r,A,B)>0$ be given by (6.7). For each $\delta>0$, two cases may occur: Case 1. $\delta\geq\delta_1$; Case 2. $0<\delta<\delta_1$.

Case 1. In this case, we can obtain (1.13) by similar arguments as those used to prove (1.11).

Case 2. In this case, we arbitrarily fix $\delta\in(0,\delta_1)$. For each $\lambda\in(0,1)$, $\lambda u_\delta+(1-\lambda)u^*_\delta$ is also an optimal control to $(\mathcal{TP})^M_\delta$. Then

$$\begin{aligned}\|u^*_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}\leq&\|\lambda u_\delta+(1-\lambda)u^*_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}\\ \leq&\|u^*_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}+2\lambda\langle u_\delta-u^*_\delta,u^*_\delta\rangle_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}+\lambda^2\|u_\delta-u^*_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)},\end{aligned}$$

which indicates that

$$0\leq2\langle u_\delta-u^*_\delta,u^*_\delta\rangle_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}+\lambda\|u_\delta-u^*_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}.$$

Passing to the limit $\lambda\to0^+$ in the above inequality, we obtain that

$$\|u^*_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}\leq\langle u_\delta,u^*_\delta\rangle_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}.$$

This, along with (6.13), yields that

$$\begin{aligned}\|u_\delta-u^*_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}&=\|u_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}-2\langle u_\delta,u^*_\delta\rangle_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}+\|u^*_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}\\&\leq\|u_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}-\|u^*_\delta\|^2_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}\\&\leq2M(M-M_\delta).\end{aligned}\tag{6.38}$$

Since $\mathcal{T}(M)\leq\mathcal{T}_\delta(M)$, it follows from (i) of Theorem 1.3, (6.38), (6.7) and (6.23) that

$$\begin{aligned}\|u^*-u_\delta\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}&\leq\|u^*-u^*_\delta\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}+\|u^*_\delta-u_\delta\|_{L^2(0,\mathcal{T}_\delta(M);\mathbb{R}^m)}\\&\leq C\delta+\sqrt{2M(M-M_\delta)}\\&\leq\left(C+2e^{C_2(\mathcal{T}(M)+\frac{1}{\mathcal{T}(M)})}\|y_0\|^2r^{-1}\right)\sqrt{\delta},\end{aligned}$$

where $C=C(M,y_0,r,A,B)>0$ and $C_2=C_2(A,B)>0$ are given by Theorem 1.3 and Theorem 5.2, respectively. Hence, (1.13) holds.

(ii) Arbitrarily fix $\eta\in(0,1)$. Let $\mathcal{A}_{M,\eta}$ be given by Theorem 5.4. Let $C_4=C_4(M,y_0,r,A,B)>0$ and $\widehat{\delta}_0=\widehat{\delta}_0(M,y_0,r,A,B)>0$ be given by Theorem 5.5. Arbitrarily fix $\delta\in\mathcal{A}_{M,\eta}\bigcap(0,\widehat{\delta}_0)$. Now we claim that there is a control $\hat{u}_\delta\in\mathcal{O}_\delta$ (given by (5.38)) so that

$$\|\hat{u}_\delta-u^*_\delta\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}>C_4\sqrt{(1-\eta)\delta}/2.\tag{6.39}$$

By contradiction, we suppose that

$$\|u_\delta-u^*_\delta\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}\leq C_4\sqrt{(1-\eta)\delta}/2\ \text{ for each } u_\delta\in\mathcal{O}_\delta.$$

This implies that

$$\begin{aligned}\operatorname{diam}\mathcal{O}_\delta &= \sup\{\|u_\delta - v_\delta\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} : u_\delta, v_\delta \in \mathcal{O}_\delta\}\\ &\le 2\sup\{\|u_\delta - u_\delta^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} : u_\delta \in \mathcal{O}_\delta\}\\ &\le C_4\sqrt{(1-\eta)\delta},\end{aligned}$$

which contradicts (5.37). Thus, (6.39) is true.

Then by (6.39) and (i) of Theorem 1.3, we have that

$$\begin{aligned}\|\hat{u}_\delta - u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} &\ge \|\hat{u}_\delta - u_\delta^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} - \|u_\delta^* - u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)}\\ &> C_4\sqrt{(1-\eta)\delta}/2 - C\delta,\end{aligned} \tag{6.40}$$

where $C = C(M, y_0, r, A, B) > 0$ is given by (1.11). Define

$$\delta_\eta \triangleq \min\left\{\widehat{\delta}_0, \left(\frac{C_4}{4C}\right)^2(1-\eta)\right\} \text{ and } \widehat{\mathcal{A}}_{M,\eta} \triangleq \mathcal{A}_{M,\eta}\cap(0,\delta_\eta). \tag{6.41}$$

According to Theorem 5.4, it is clear that

$$\lim_{h\to 0^+}\frac{1}{h}|\widehat{\mathcal{A}}_{M,\eta}\cap(0,h)| = \lim_{h\to 0^+}\frac{1}{h}|\mathcal{A}_{M,\eta}\cap(0,h)| = \eta.$$

Moreover, it follows from (6.40) and (6.41) that

$$\|\hat{u}_\delta - u^*\|_{L^2(0,\mathcal{T}(M);\mathbb{R}^m)} > C_4\sqrt{(1-\eta)\delta}/4 \ \text{ for each } \delta \in \widehat{\mathcal{A}}_{M,\eta},$$

which indicates (1.14).

In summary, we finish the proof of Theorem 1.4. □

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