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THE RELATIONSHIP BETWEEN MAXIMUM PRINCIPLE AND DYNAMIC PROGRAMMING PRINCIPLE FOR STOCHASTIC RECURSIVE CONTROL PROBLEM WITH RANDOM COEFFICIENTS

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Abstract. This paper aims to explore the relationship between maximum principle and dynamic programming principle for stochastic recursive control problem with random coefficients. Under certain regular conditions for the coefficients, the relationship between the Hamiltonian system with random coefficients and stochastic Hamilton–Jacobi–Bellman equation is obtained. It is very different from the deterministic coefficients case since stochastic Hamilton–Jacobi–Bellman equation is a backward stochastic partial differential equation with solution being a pair of random fields rather than a deterministic function. A linear quadratic recursive optimization problem is given as an explicitly illustrated example based on this kind of relationship.

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1. Introduction

As we all know, Pontryagin maximum principle (MP) and Bellman dynamic programming principle (DPP) serve as the most two important methods in solving optimal control problems. Both of them aim to obtain some necessary conditions of optimal controls. Hence it is natural to think that they have some kind of relationship, although they have been developed separately and independently in literature to a great extent. In general, the MP gives a necessity condition of the optimal control by the Hamiltonian system which is a forward–backward equation consisting of the optimal state equation, the adjoint equation and optimality condition. On the other hand, the DPP characterizes the optimal control by the Hamilton–Jacobi–Bellman (HJB) equation, to which the value function is a solution. Therefore, the relationship between Hamiltonian system and HJB equation can be thought as a relationship between MP and DPP.

Keywords and phrases: Maximum principle, dynamic programming principle, random coefficient, stochastic recursive control, backward stochastic partial differential equation.

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For the deterministic control system, the Hamiltonian system is an ordinary differential equation and the HJB equation is a first-order partial differential equation (PDE), whose connection was first given by Pontryagin, Boltyanski, Gamkrelidze and Mischenko [1] in 1962. Since the value function V is not always smooth, some nonsmooth versions of the relationship were studied by using nonsmooth analysis and generalized derivatives. For example, an attempt to relate these two methods without assuming the smoothness of the value function was done by Barron and Jensen [2], where the viscosity solution was used to derive the MP from the DPP. The relationship in deterministic case is known as

$$\Psi_t = -V_x(t,\bar{X}_t) \quad \text{and} \quad V_t(t,\bar{X}_t) = H(t,\bar{X}_t,\bar{u}_t,\Psi_t),$$

where $\bar{u}$ is the optimal control, $\bar{X}$ is the optimal state, Ψ is the adjoint variable, H is the Hamiltonian function, and V is the value function, respectively. For the stochastic control system whose state equation is a stochastic differential equation (SDE) with deterministic coefficients, the Hamiltonian system is a forward–backward stochastic differential equation (FBSDE) with deterministic coefficients, the HJB equation is a second-order fully nonlinear PDE, and their connection was given by Bismut [3] and Bensoussan [4]. As for nonsmooth value function, Zhou [5, 6] obtained the relationship between them in the viscosity sense of HJB equation. Also a recent progress for such a relationship in an infinite-dimensional space is made by Chen and Lu [7]. The relationship in this case can be summarized as

$$p_t = -V_x(t,\bar{X}_t), \quad q_t = -V_{xx}(t,\bar{X}_t)\sigma(t,\bar{X}_t,\bar{u}_t),$$

and

$$V_t(t,\bar{x}_t) = G\left(t,\bar{X}_t,\bar{u}_t,-V_x(t,\bar{X}_t),-V_{xx}(t,\bar{X}_t)\right),$$

where σ is the diffusion coefficient, (p,q) is the adjoint pair and G is the generalized Hamiltonian function.

However, when the state equation is a SDE with random coefficients, things are much different. Bear in mind that HJB equation in this case is a backward stochastic partial differential equation (BSPDE) with a pair of adapted solution, rather than a deterministic PDE with a deterministic solution. There should be also a relationship between MP and DPP, as well as between FBSDE with random coefficients and stochastic HJB equation, but no existing literature is concerned with this issue as far as we know.

The relationship between MP and DPP not only demonstrates the connection between two main methods of control theory, but also plays a very important role in economic theory as pointed out in Yong and Zhou [8]. Moreover, the relationship can be regarded as an extension of Feynman–Kac formula to fully nonlinear PDE, if one notices that the Hamiltonian system is a stochastic forward–backward ordinary differential equation and HJB equation is a fully nonlinear PDE in a stochastic control system with deterministic coefficients. For the random coefficients settings, Feynman–Kac formula is further extended to non-Markovian framework and fully nonlinear BSPDE. The reader can refer to [9–12] for related studies.

The control system we study for the relationship between MP and DPP is a stochastic recursive one with a general cost functional, which is governed by the following controlled FBSDE:

$$\begin{cases} \mathrm{d}X_s &= b(s,X_s,u_s)\mathrm{d}s + \sigma\big(s,X_s,u_s\big)\mathrm{d}W_s \\ X_0 &= x, \\ \mathrm{d}Y_s &= -f(s,X_s,Y_s,Z_s,u_s)\mathrm{d}s + Z_s\mathrm{d}W_s \\ Y_T &= h(X_T), \end{cases}$$

and the following cost functional:

$$J(0,x;u) \triangleq Y_0^{0,x;u}.$$

The above stochastic recursive control system was given by Peng [13] to establish DPP in the Lipschitz setting of the generator and explore the connection between its value function and HJB equation. In the meantime, Duffie and Epstein [14] studied such a control system from mathematical finance point of view, and they put forward the concept of stochastic (recursive) differential utility which is actually a solution of FBSDE.

From MP point of view, Peng [15] studied the above recursive control system and derived a local MP by representing the adjoint equation as a FBSDE, in which the control domain is convex. For the general settings that the control domain is nonconvex and the diffusion depends on control, the Ekeland variational principle was applied to obtain the MP in Wu [16] and Yong [17] by treating the second solution and the terminal condition in backward stochastic differential equation (BSDE) as a control and a constraint, respectively. By introducing new and general first-order and second-order adjoint equations, Hu [18] obtained the MP for the recursive stochastic optimal control problem without unknown parameters. These results, especially Hu [18], eventually solved the long-standing open problem put forward in Peng [19].

There has been results on the relationship between MP and DPP for stochastic recursive optimal control system with deterministic coefficients. With sufficiently regular assumptions on the coefficients, Shi [20] and Shi and Yu [21] first demonstrated this relationship. Nie, Shi and Wu [22, 23] studied the relationship between MP and DPP in the sense of viscosity solution of HJB equation. The relationship is summarized as follows:

$$\begin{cases} p_t^* = V_x(t,\bar{X}_t)^\top k_t^*, \\ q_t^* = \big[V_{xx}(t,\bar{X}_t)\sigma(t,\bar{X}_t,\bar{u}_t) + V_x(t,\bar{X}_t) \\ \qquad \times f_z\left(t,\bar{X}_t,-V(t,\bar{X}_t),-V_x(t,\bar{X}_t)\sigma(t,\bar{X}_t,\bar{u}_t),\bar{u}_t\right)\big]\, k_t^*, \end{cases}$$

and

$$V_t(t,\bar{X}_t) = G\left(t,\bar{X}_t,-V\left(t,\bar{X}_t\right),-V_x(t,\bar{X}_t),-V_{xx}(t,\bar{X}_t),\bar{u}_t\right),$$

where (p^*,q^*) is the adjoint pair of the forward part, k^* is the adjoint process of the backward part in stochastic recursive control system and G is the corresponding generalized Hamiltonian function.

In our paper, the most important feature is that the coefficients of the system we consider are random. We emphasize that this is an essential difference from existing literature. In 1992, Peng [13] studied the optimal control problem of non-Markovian stochastic systems using dynamic programming. Compared with the optimal control problem of Markovian stochastic systems, the value function is no longer a deterministic function, but a random field. In other words, it is a family of semi-martingales. Furthermore, the HJB equation derived from Bellman's principle of optimality is no longer a second-order fully nonlinear PDE, but a second-order fully nonlinear BSPDE, whose solution is a pair of random fields as BSDE's. To distinguish it from the classical HJB equation, we call it the stochastic HJB equation. As in the deterministic case, the existence of the solution for stochastic HJB equation is a very hard problem. The solvability has only been proved for a few cases, see [24–28] for instance. One contribution of our paper is to show that the value function of the recursive optimal control problem will be the classical solution of stochastic HJB equation, if the needed regularity is satisfied. It can be seen as a general form of Feynman–Kac representation. In this sense, our work extends the result of Tang [12], in which the author used a forward–backward system to represent semilinear BSPDE. In fact, our proof is partly inspired from that work, *i.e.* we also use the random field generated by the controlled SDE. Furthermore, we proved a verification theorem to show that the solution of stochastic HJB equation gives the optimal control. Another contribution of our paper is to show the relationship between the MP and the DPP. The relationship is very different from that for stochastic recursive optimal control system with deterministic coefficients due to the appearance of stochastic HJB equation. Note that we also assume that the value function is smooth to obtain the desired result, but how to deal with nonsmooth case is still unsolved. Actually, the solvability for the stochastic HJB equation in a general form is a long-existing open problem.

The rest of this article is organized as follows. In Section 2, we introduce some notations and the basic setup of our problem. We characterize the optimal control by DPP, *i.e.* the connection between the value function and stochastic HJB equation in Section 3. In Section 4, the optimal control is characterized by MP, *i.e.* the stochastic Hamiltonian system. In Section 5, we show the connection between the MP and the DPP. As an application we discuss a linear quadratic (LQ) recursive utility portfolio optimization problem with the random coefficients in Section 6, in which the state feedback optimal control is obtained by both MP and DPP methods, respectively and the relationship is demonstrated explicitly.

2. Notations and statement of the problem

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, and $\{W_t, 0 \le t \le T\}$ is a one-dimensional standard Brownian motion on it generating a right-continuous filtration $\mathbb{F} = \{\mathscr{F}_t\}_{0 \le t \le T}$. Let E be an Euclidean space, and its inner product and norm are denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively. For a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$, we denote by ϕ_x its gradient and by ϕ_{xx} its Hessian (a symmetric matrix). If $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ $(k \ge 2)$, $\phi_x = (\frac{\partial \phi_i}{\partial x_j})$ is the corresponding $k \times n$ Jacobian matrix.

Next we introduce some useful spaces of random variables and stochastic processes. For any $\beta \in (0, \infty)$ and $t \in [0, T]$, define

- $M^\beta_{\mathscr{F}}(t, T; E)$: space of $\mathbb{F}$-adapted processes $f : \Omega \times [t, T] \to E$ with $\|f\|_{M^\beta_{\mathscr{F}}(t,T;E)} \triangleq \left(\mathbb{E} \int_t^T |f_s|^\beta \mathrm{d}s \right)^{1 \wedge \frac{1}{\beta}} < \infty$;
- $S^\beta_{\mathscr{F}}(t, T; E)$: space of $\mathbb{F}$-adapted continuous processes $f : \Omega \times [t, T] \to E$ with $\|f\|_{S^\beta_{\mathscr{F}}(t,T;E)} \triangleq \left(\mathbb{E} \sup_{s \in [t,T]} |f_s|^\beta \right)^{1 \wedge \frac{1}{\beta}} < \infty$;
- $L^\beta(\Omega, \mathscr{F}_t; E)$: space of $\mathscr{F}_t$-measurable random variables $\xi : \Omega \to E$ with $\|\xi\|_{L^\beta(\Omega, \mathscr{F}_t; E)} \triangleq \left(\mathbb{E}|\xi|^\beta \right)^{1 \wedge \frac{1}{\beta}} < \infty$.

For any $t, s \in [0, T]$ with $t \le s$, we define the admissible control set $\mathcal{U}^2[t, s] = M^2_{\mathscr{F}}(t, s; U)$ with U being a closed convex subset of $\mathbb{R}^k$. Given $x \in \mathbb{R}^n$ and $u \in \mathcal{U}^2[0, T]$, we consider the following FBSDE

$$\left\{ \begin{array}{lcl} \mathrm{d}X^{0,x;u}_s &=& b(s, X^{0,x;u}_s, u_s)\mathrm{d}s + \sigma\big(s, X^{0,x;u}_s, u_s)\mathrm{d}W_s \\ X^{0,x;u}_0 &=& x, \\ \mathrm{d}Y^{0,x;u}_s &=& -f(s, X^{0,x;u}_s, Y^{0,x;u}_s, Z^{0,x;u}_s, u_s)\mathrm{d}s + Z^{0,x;u}_s \mathrm{d}W_s \\ Y^{0,x;u}_T &=& h(X^{0,x;u}_T), \end{array} \right. \tag{2.1}$$

with the cost functional

$$J(0, x; u) \triangleq Y^{0,x;u}_0,$$

where $b : \Omega \times [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : \Omega \times [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}$, $h : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

We need the following assumptions on coefficients (b, σ, f, h).

Assumption 2.1. For any $(\omega, t, x, u) \in \Omega \times [0, T] \times \mathbb{R}^n \times U$, $b(\cdot, x, u)$ and $\sigma(\cdot, x, u)$ are $\mathbb{F}$-adapted processes; b, σ are differentiable with respect to u and twice differentiable with respect to x; $b_u(t, x, u)$, $\sigma_u(t, x, u)$, $b_{xx}(t, x, u)$ and $\sigma_{xx}(t, x, u)$ are continuous in (x, u); there exists a constant K such that

$$|b(t, x, u)|, |\sigma(t, x, u)| \le K(1 + |x| + |u|) \quad \textit{and} \quad |b_x|, |b_u|, |b_{xx}|, |\sigma_x|, |\sigma_u|, |\sigma_{xx}| \le K.$$

Assumption 2.2. For any $(\omega, t, x, x_1, x_2, y, z, u, u_1, u_2) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times U \times U \times U$, $f(\cdot, x, y, z, u)$ is an $\mathbb{F}$-adapted process and $h(x)$ an $\mathscr{F}_T$-measurable random variable; f is differentiable with respect to (x, y, z, u) and h is differentiable with respect to x; $f_x(t, x, y, z, u)$, $f_y(t, x, y, z, u)$, $f_z(t, x, y, z, u)$,

$f_u(t,x,y,z,u)$ are continuous in (x,y,z,u), $h_x(x)$ is continuous in x; there exists a constant $K \geq 0$ such that for $\gamma \in [0,1)$

$$|f(t,x,y,z,u) - f(t,x,y,0,u)| \leq K|z|^\gamma, \quad |h(x)| \leq K(1+|x|^2),$$

$$|f(t,0,0,0,0)|, |f_y|, |f_z| \leq K, |f_x(t,x,y,z,u)| + |f_u(t,x,y,z,u)| \leq K(1+|x|+|u|),$$

$$|h_x(x)| \leq K(1+|x|),$$

and

$$\begin{aligned} &|f(t,x_1,y,z,u_1) - f(t,x_2,y,z,u_2)| + |h(x_1) - h(x_2)| \\ &\leq K(1+|x_1|+|x_2|)(|x_1-x_2|) + K(1+|u_1|+|u_2|)|u_1-u_2|. \end{aligned}$$

Under Assumption 2.1, we can see that, for any given admissible control u, the forward part of FBSDE (2.1) admits a unique strong solution $X^u \in S^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$. Thus, the terminal $h(X^u_T)$ is only L^1-integrable due to the quadratic growth of h with respect to x, so the classic result for the solvability of BSDE is no longer applicable. The assumptions on f in Assumption 2.2 guarantee the conditions of Theorem 6.3 in [29] are satisfied, so we can apply this theorem to the backward part of FBSDE (2.1), which is a BSDE with L^1-integrable terminal value, to get the unique solution $(Y^u, Z^u) \in S^\beta_{\mathscr{F}}(0,T;\mathbb{R}) \times M^\beta_{\mathscr{F}}(0,T;\mathbb{R})$ for any $\beta \in (0,1)$. It is easy to check that $|J(0,x;u)| < \infty$. Then, we put forward the optimal control problem.

Problem 2.3. Find an admissible control $\bar{u}$ such that

$$J(0,x;\bar{u}) = \inf_{u \in \mathcal{U}^2[0,T]} J(0,x;u). \tag{2.2}$$

Any $\bar{u} \in \mathcal{U}^2[0,T]$ satisfying (2.2) is called an optimal control process of Problem 2.3. With $\bar{u}$, the solution $(\bar{X}, \bar{Y}, \bar{Z})$ of the state equation (2.1) is called the optimal state process, and consequently $(\bar{u}; \bar{X}, \bar{Y}, \bar{Z})$ is called an optimal pair of Problem 2.3.

3. The dynamic programming principle for stochastic recursive control problem

In this section, we are concerned with the DPP and the corresponding stochastic HJB equation for stochastic recursive control Problem 2.3. We shall show that, if the value function is a random field with some regularities, it will be the solution for the stochastic HJB equation. To this end, for $t \in [0,T]$, $\zeta \in L^2(\Omega, \mathscr{F}_t; \mathbb{R}^n)$ and $u \in \mathcal{U}^2[t,T]$, we consider the following parameterized FBSDE:

$$\left\{ \begin{array}{lcl} \mathrm{d}X^{t,\zeta;u}_s &=& b(s, X^{t,\zeta;u}_s, u_s)\mathrm{d}s + \sigma(s, X^{t,\zeta;u}_s, u_s)\mathrm{d}W_s \\ \mathrm{d}X^{t,\zeta;u}_t &=& \zeta, \\ Y^{t,\zeta;u}_s &=& -f(s, X^{t,\zeta;u}_s, Y^{t,\zeta;u}_s, Z^{t,\zeta;u}_s, u_s)\mathrm{d}s + Z^{t,\zeta;u}_s \mathrm{d}W_s \\ Y^{t,\zeta;u}_T &=& h(X^{t,\zeta;u}_T). \end{array} \right. \tag{3.1}$$

Similar to the solvability of FBSDE (2.1), under Assumptions 2.1 and 2.2, by Theorem 6.3 in [29] again, FBSDE (3.1) admits a unique strong solution $\Theta^{t,\zeta;u} = (X^{t,\zeta;u}, Y^{t,\zeta;u}, Z^{t,\zeta;u}) \in S^2_{\mathscr{F}}(t,T;\mathbb{R}^n) \times S^\beta_{\mathscr{F}}(t,T;\mathbb{R}) \times M^\beta_{\mathscr{F}}(t,T;\mathbb{R})$ for any $\beta \in (0,1)$. We call $\Theta^{t,\zeta;u}$, or $\Theta = (X,Y,Z)$ whenever its dependence on u and (t,ζ) is clear from context, the state process and $(u;\Theta)$ the admissible pair.

For a given control process $u \in \mathcal{U}^2[t,T]$, we define the associated cost functional as follows:

$$J(t,x;u) \triangleq Y_t^{t,x;u}, \quad (t,x) \in [0,T] \times \mathbb{R}^n. \tag{3.2}$$

From Theorem A.2 in [30], we get the following relation

$$J(t,\zeta;u) = Y_t^{t,\zeta;u}.$$

For $\zeta = x \in \mathbb{R}^n$, we define the value function

$$V(t,x) \triangleq \operatorname*{essinf}_{u \in \mathcal{U}^2[t,T]} J(t,x;u), \quad (t,x) \in [0,T] \times \mathbb{R}^n,$$

where the notation essinf stands for the essential infimum.

Now we discuss a generalized DPP for our stochastic optimal control problem. For this purpose, we define the family of (backward) semigroups associated with FBSDE (3.1), which was first introduced by Peng [30]. Given the initial data (t,x), a positive number $\delta \leq T - t$, an admissible control process $u \in \mathcal{U}^2[t, t+\delta]$ and a real-valued random variable $\eta \in L^2(\Omega, \mathscr{F}_{t+\delta}; \mathbb{R})$, we put

$$G_{s,t+\delta}^{t,x;u}(\eta) := \tilde{Y}_s^{t,x;u}, \quad s \in [t, t+\delta],$$

where $(X_\cdot^{t,x,u}, \tilde{Y}_\cdot^{t,x;u}, \tilde{Z}_\cdot^{t,x;u})$ is the solution of the following FBSDE with the time horizon $t+\delta$,

$$\begin{cases} \mathrm{d}X_s^{t,x;u} &= b(s, X_s^{t,x;u}, u_s)\mathrm{d}s + \sigma\big(s, X_s^{t,x;u}, u_s)\mathrm{d}W_s \\ X_t^{t,x;u} &= x, \\ \mathrm{d}\tilde{Y}_s^{t,x;u} &= -f(s, X_s^{t,x;u}, \tilde{Y}_s^{t,x;u}, \tilde{Z}_s^{t,x;u}, u_s)\mathrm{d}s + \tilde{Z}_s^{t,x;u}\mathrm{d}W_s \\ Y_{t+\delta}^{t,x;u} &= \eta. \end{cases}$$

Obviously, for any admissible control pair $(u; X^{t,x,u}, Y^{t,x;u}, Z^{t,x;u})$, we have

$$\begin{aligned} G_{t,T}^{t,x;u}\big(h(X_T^{t,x;u})\big) &= G_{t,t+\delta}^{t,x;u}(Y_{t+\delta}^{t,x;u}) \\ &= G_{t,t+\delta}^{t,x;u}(Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;u};u}) = G_{t,t+\delta}^{t,x;u}(J(t+\delta, X_{t+\delta}^{t,x;u}; u)). \end{aligned}$$

Moreover, the following DPP holds by a similar proof as in [30].

Theorem 3.1. *Under Assumptions 2.1 and 2.2, the value function $V(t,x)$ obeys the following DPP: for any $0 \leq t < t+\delta \leq T, x \in \mathbb{R}^n$,*

$$V(t,x) = \inf_{u \in \mathcal{U}^2[t,t+\delta]} G_{t,t+\delta}^{t,x;u}\big(V(t+\delta, X_{t+\delta}^{t,x;u})\big).$$

Next we shall show the relationship between the value function and stochastic HJB equation. For this purpose, the following lemma is needed.

Lemma 3.2. *([12]) For any fixed admissible control u, set $\mathbb{X}_s^x$ to be the solution of the following SDE:*

$$\begin{cases} \mathrm{d}X_s = b(s, X_s, u_s)\mathrm{d}s + \sigma(s, X_s, u_s)\mathrm{d}W_s \\ X_0 = x. \end{cases} \tag{3.3}$$

Then, almost surely, for each $s \in [0,T]$, $\mathbb{X}_s^{\cdot}$ *is a diffeomorphism of* C^1. *The gradient* $\partial \mathbb{X}_s^x$ *satisfies the following SDE:*

$$\begin{cases} \mathrm{d}\partial \mathbb{X}_s^x = b_x(s,\mathbb{X}_s^x,u_s)\partial \mathbb{X}_s^x \mathrm{d}s + \sigma_x(s,\mathbb{X}_s^x,u_s)\partial \mathbb{X}_s^x \mathrm{d}W_s \\ \partial \mathbb{X}_0^x = I. \end{cases}$$

Moreover, from the boundedness of the derivatives, a classical estimation for SDE yields that

$$\mathbb{E}\left[\sup_{s\in[0,T]} |\partial \mathbb{X}_s^x|^4\right] \leq M,$$

where M *is a constant independent of* x.

Then main result of this section is presented below.

Proposition 3.3. *In additional to Assumptions 2.1 and 2.2, we also assume that the control region* $U \subset \mathbb{R}^k$ *is bounded and for each* $t \in [0,T]$, $x \in \mathbb{R}^n$, *the infimum of the cost functional* $J(t,x;\cdot)$ *is attained by an optimal control* $u^{*,t,x}$. *Moreover, assume that the value function* V *admits the following semimartingale decomposition:*

$$V(t,x) = h(x) + \int_t^T \Gamma(s,x)\mathrm{d}s - \int_t^T \Psi(s,x)\mathrm{d}W_s, \quad t \in [0,T], \tag{3.4}$$

where the $\mathbb{R}$*-valued function* $\Gamma(t,\cdot)$ *and* $\Psi(t,\cdot)$ *are* $\mathscr{F}_t \times \mathcal{B}(\mathbb{R}^n)$ *measurable for each* $t \in [0,T]$ *and* V, Γ, Ψ *satisfy the following assumptions:*

(i) $(t,x) \longmapsto V(t,x)$ *is continuous a.s.,*
(ii) $x \longmapsto V(t,x)$ *is* C^2 *for each* $t \in [0,T]$ *a.s.,*
(iii) $x \longmapsto \Gamma(t,x)$ *is continuous for each* $t \in [0,T]$ *a.s.,*
(iv) $x \longmapsto \Psi(t,x)$ *is is* C^1 *for each* $t \in [0,T]$ *a.s.,*
(v) There exists $K \in M^2_{\mathscr{F}}(0,T;\mathbb{R}^+)$ *such that*

$$|V(t,x)|,\ |h(x)|,\ |\Gamma(t,x)|,\ |\Psi(t,x)| \leq K_t(1+|x|^2),$$
$$|\partial_x V(t,x)|,\ |\partial_x \Psi(t,x)| \leq K_t(1+|x|),$$
$$|\partial_{xx} V(t,x)| \leq K_t,$$
$$|\Gamma(t,x) - \Gamma(t,y)| \leq K_t(1+|x|+|y|)|x-y|.$$

Then, the value function V, *together with* Ψ, *constitutes a pair of solution of the stochastic HJB equation*

$$\begin{cases} dV(t,x) = -\inf_u G\big(t,x,V(t,x),\Psi(t,x),V_x(t,x),\Psi_x(t,x),V_{xx}(t,x),u\big)\mathrm{d}t + \Psi(t,x)\mathrm{d}W_t \\ V(T,x) = h(x), \end{cases} \tag{3.5}$$

where

$$G(t,x,y,z,p,q,A,u) = \langle p, b(t,x,u)\rangle + \langle q, \sigma(t,x,u)\rangle + \frac{1}{2}tr\big((\sigma\sigma^*)(t,x,u)A\big) + f(t,x,y,\sigma^* p + z, u).$$

Proof. The proof of this proposition is similar to Proposition 4.1 in Meng, Dong, Shen and Tang [31]. Here we only give a brief proof for the convenience of the readers. For a detailed proof, please refer to [31].

For a fixed admissible control u and x, we abbreviate X for $X^{0,x;u}$ for simplicity. Applying Itô-Ventzell formula to $V(t,X_t)$, we have

$$\begin{aligned}
& V(t,X_t)\\
=\ & V(t+\delta,X_{t+\delta})\\
& + \int_t^{t+\delta}\Big(\Gamma(s,X_s)-G(s,X_s,V(s,X_s),\Psi(s,X_s),V_x(s,X_s),\Psi_x(s,X_s),V_{xx}(s,X_s),u_s)\\
& + f(s,X_s,V(s,X_s),Z'_s,u_s)\Big)\mathrm{d}s - \int_t^{t+\delta} Z'_s\mathrm{d}W_s,
\end{aligned}$$

where $Z'_s=\sigma^*V_x(s,X_s)+\Psi(s,X_s)$. From condition (v), it can be verified that $Z'\in M^2_{\mathscr{F}}(0,T;\mathbb{R})$. Consider the following BSDE

$$Y_r = V(t+\delta,X_{t+\delta}) + \int_r^{t+\delta} f(s,X_s,Y_s,Z_s,u_s)\mathrm{d}s - \int_r^{t+\delta} Z_s\mathrm{d}W_s.$$

From the DPP, we shall have $Y_t\geq V(t,X_t)$. From this, one can get that

$$\mathbb{E}\left[\int_t^{t+\delta}\Delta(s,X_s,u_s)\mathrm{d}s\middle|\mathcal{F}_t\right]\leq 0, \tag{3.6}$$

where

$$\Delta(s,x,u)\triangleq\Gamma(s,x)-G(s,x,V(s,x),\Psi(s,x),V_x(s,x),\Psi_x(s,x),V_{xx}(s,x),u).$$

Due to the arbitrariness of δ and t, it implies that

$$\Delta(s,X_s,u_s)\leq 0 \quad \text{for a.a. } s\in[0,T],\ \text{a.s.}$$

Note that $\mathbb{X}^x_s$ is the stochastic flow generated by the SDE (3.3). Previous argument shows that

$$\Delta(s,\mathbb{X}^x_s,u_s)\leq 0 \quad \text{for a.a. } s\in[0,T],\ \text{a.s.}$$

From Lemma 3.2, with probability 1, for each s, $\mathbb{X}^\cdot_s$ is a diffeomorphism of class C^1. Hence, we also have that

$$\Delta(s,x,u_s)\leq 0 \quad \text{for a.a. } s\in[0,T],\ \text{a.s.}$$

From the arbitrariness of u, we get that

$$\sup_u \Delta(s,x,u)\leq 0 \quad \text{for all } x\in\mathbb{R}^n,\ \text{a.a. } s\in[0,T],\ \text{a.s.}$$

On the other hand, note that the optimal control $u^{*,t,x}_\cdot$ and its corresponding state denoted by $X^{u^{*,t,x}}_\cdot$ exist. For simplicity, we abbreviate the pair as $(u^*_\cdot,X^*_\cdot)$. Following the same argument as previous, it holds that

$$\Delta(s,X^*_s;u^*_s)=0,\ \text{for a.a. } s\in[0,T],\ \text{a.s.}$$

This will imply that

$$\sup_u \Delta(s,x,u) = 0 \quad \text{for all } x \in \mathbb{R}^n, \text{ a.a. } s \in [0,T], \text{ a.s.}$$

which gives us the desired result. □

In above, we have proved that the value function is the solution of the stochastic HJB equation under suitable conditions. We will then prove the converse result.

Proposition 3.4. *[Stochastic Verification Theorem] Let (Φ, Ψ) be the solution of stochastic HJB equation (3.5) and assume that they satisfy the regularity assumptions in Proposition 3.3. Then, for any (t,x) and $u \in \mathcal{U}^2[t,T]$, we have*

$$\Phi(t,x) \le J(t,x;u).$$

Moreover, if there exists an admissible control $u \in \mathcal{U}^2[t,T]$ such that, for a.a. $s \in [t,T]$, a.s. $\omega \in \Omega$,

$$\begin{aligned} &G\big(s, X_s^{t,x;u}, \Phi(t,X^{t,x;u}), \Psi(t,X^{t,x;u}), \Phi_x(t,X^{t,x;u}), \Psi_x(t,X^{t,x;u}), \Phi_{xx}(t,X^{t,x;u}), u_s\big) \\ =& \inf_v G\big(s, X_s^{t,x;u}, \Phi(t,X^{t,x;u}), \Psi(t,X^{t,x;u}), \Phi_x(t,X^{t,x;u}), \Psi_x(t,X^{t,x;u}), \Phi_{xx}(t,X^{t,x;u}), v\big), \end{aligned}$$

u is the optimal control.

Proof. The result is obtained by applying Itô formula to $\Phi(s, X_s^{t,x;u})$ and comparing it with $Y_s^{t,x;u}$. Since the calculation is almost the same to previous proposition, we omit the proof here. □

4. The Maximum Principle for stochastic recursive control problem

In this section, we derive the stochastic maximum principle of Problem 2.3. We first define the Hamiltonian function $H : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$ by

$$H(t,x,y,z,p,q,k,u) = \langle p, b(t,x,u)\rangle + \langle q, \sigma(t,x,u)\rangle - kf(t,x,y,z,u).$$

To simplify our argument, we introduce some abbreviated notations. For this, let $(\bar{u}; \bar{X}, \bar{Y}, \bar{Z})$ be an optimal 4-tuple of Problem 2.3. For $\varphi = b, \sigma, b_x, b_u, \sigma_x, \sigma_u$, define

$$\bar{\varphi}(t) \triangleq \varphi(t, \bar{X}_t, \bar{u}_t),$$

for $\varphi = f, f_x, f_y, f_z, f_u$, define

$$\bar{\varphi}(t) \triangleq \varphi(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t),$$

and for h, define

$$\bar{h}(T) \triangleq h(\bar{X}_T), \quad \bar{h}_x(T) \triangleq h_x(\bar{X}_T).$$

Now we are ready to give the necessary conditions of optimality for the optimal control of Problem 2.3. Let $(\bar{u}; \bar{\Theta}) = (\bar{u}; \bar{X}, \bar{Y}, \bar{Z})$ be an optimal 4-tuple. Fix any admissible control $u \in \mathcal{U}^2[0,T]$. Consider $u^1 \in M_{\mathscr{F}}^\infty(0,T;\mathbb{R}^k)$ as $u_t^1 = \frac{u_t - \bar{u}_t}{|u_t - \bar{u}_t| \vee 1}$. For any $\varepsilon \in [0,1]$, we construct a perturbed admissible control as below

$$u^\varepsilon = \bar{u} + \varepsilon u^1.$$

Since u^ε is still a convex combination of u and $\bar{u}$, u^ε is an admissible control, and moreover, the boundedness of u^1 is used in the deductions in the rest of this section. Denote by $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ the corresponding state equation and consider the following variational equations:

$$\begin{cases} \mathrm{d}X_t^1 = \Big[\bar{b}_x(t)X_t^1 + \bar{b}_u(t)u_t^1\Big]\mathrm{d}t + \Big[\bar{\sigma}_x(t)X_t^1 + \bar{\sigma}_u(t)u_t^1\Big]\mathrm{d}W_t \\ X_0^1 = 0, \\ \mathrm{d}Y^1(t) = -\Big[\bar{f}_x(t)X_t^1 + \bar{f}_y(t)Y_t^1 + \bar{f}_z(t)Z_t^1 + \bar{f}_u(t)u_t^1\Big]\mathrm{d}t + Z_t^1\mathrm{d}W_t \\ Y_T = \bar{h}_x(T)X_T^1. \end{cases} \tag{4.1}$$

Since h_x is of linear growth with respect to x, the terminal $\bar{h}_x(T)X_T^1$ is not L^2-integrable in general. Thus, the solvability of (4.1) is not obvious. To solve it, we shall introduce the following result for BSDE with L^p terminal value proved in [29].

Lemma 4.1. *Consider the following BSDE*

$$\begin{cases} \mathrm{d}Y_t = -f(t, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t \\ Y_T = \xi, \end{cases}$$

with f being uniformly Lipschitz continuous with respect to (y,z) and ξ being L^p-integrable for some $p > 1$. There exists a unique solution (Y,Z), and for some constant $\tilde{C}$,

$$\|Y\|_{\mathcal{S}^p}^p + \|Z\|_{M^p}^p \le \tilde{C}\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f(t,0,0)|\mathrm{d}t\right)^p\right].$$

Moreover, we have the following lemmas.

Lemma 4.2. *Under Assumption 2.1, it holds that*

$$\mathbb{E}\sup_{0\le t\le T}|X_t^\varepsilon - \bar{X}_t|^p = O(\varepsilon^p) \tag{4.2}$$

and

$$\mathbb{E}\sup_{0\le t\le T}|X_t^\varepsilon - \bar{X}_t - \varepsilon X_t^1|^p = o(\varepsilon^p), \tag{4.3}$$

for any $p > 1$.

Proof. The proof is rather standard. For (4.2), by the L^p estimate for SDE (see Prop. 2.1 in [32]) and Assumptions 2.1, we have

$$\begin{aligned} \mathbb{E}\Big(\sup_{0\le t\le T}|X_t^\varepsilon - \bar{X}_t|^p\Big) \le C\Big[&\mathbb{E}\Big(\int_0^T |b(t,\bar{X}_t,u_t^\varepsilon) - b(t,\bar{X}_t,\bar{u}_t)|\mathrm{d}t\Big)^p \\ &+\mathbb{E}\Big(\int_0^T |\sigma(t,\bar{X}_t,u_t^\varepsilon) - \sigma(t,\bar{X}_t,\bar{u}_t)|^2\mathrm{d}t\Big)^{p/2}\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq C\mathbb{E}\Big(\int_0^T |u_t^\varepsilon - \bar{u}_t|^2 \mathrm{d}t\Big)^{p/2}\\
&= C\mathbb{E}\Big(\int_0^T |\varepsilon u_t^1|^2 \mathrm{d}t\Big)^{p/2}\\
&= C\varepsilon^p\mathbb{E}\Big(\int_0^T |u_t^1|^2 \mathrm{d}t\Big)^{p/2}\\
&= O(\varepsilon^p).
\end{aligned}
$$

For (4.3), denote $\delta X \triangleq X^\varepsilon - \bar{X} - \varepsilon X^1$. Then we have

$$
\begin{cases}
\mathrm{d}\delta X_t = & \bar{b}_x(t)\delta X_t + (\tilde{b}_x(t) - \bar{b}_x(t))(X_t^\varepsilon - \bar{X}_t)\mathrm{d}t + \bar{\sigma}_x(t)\delta X_t \\
& +(\tilde{\sigma}_x(t) - \bar{\sigma}_x(t))(X_t^\varepsilon - \bar{X}_t)\mathrm{d}W_t \\
\delta X_0 = & 0,
\end{cases}
$$

with

$$
\tilde{b}_x(t) \triangleq \int_0^1 b_x(t, \bar{X}_t + \lambda(X_t^\varepsilon - \bar{X}_t), \bar{u}_t + \lambda\varepsilon u_t^1)\mathrm{d}\lambda
$$

and

$$
\tilde{\sigma}_x(t) \triangleq \int_0^1 \sigma_x(t, \bar{X}_t + \lambda(X_t^\varepsilon - \bar{X}_t), \bar{u}_t + \lambda\varepsilon u_t^1)\mathrm{d}\lambda.
$$

Hence (4.3) follows from the previous estimation for $X^\varepsilon - \bar{X}$ and a standard estimation for SDE. The proof is completed. □

It is also easy to show that X_T^1 is L^p-integrable for any $p > 1$, and then the terminal value $\bar{h}_x(T)X_T^1$ is L^p-integrable for any $p \in (1,2)$, which together with Lemma 4.1 leads to

Lemma 4.3. *Under Assumptions 2.1 and 2.2, FBSDE (4.1) admits a unique solution (X^1, Y^1, Z^1). Moreover, $X^1 \in S^{p_1}$ and $(Y^1, Z^1) \in S^{p_2} \times M^{p_2}$ for any $p_1 > 1$ and $p_2 \in (1,2)$.*

Next we prove the expansion for $\bar{Y}$.

Lemma 4.4. *Under Assumptions 2.1 and 2.2, we have for any $p \in (1,2)$*

$$
\lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\leq t\leq T}|\frac{Y_t^\varepsilon - \bar{Y}_t}{\varepsilon} - Y_t^1|^p = 0. \tag{4.4}
$$

Proof. A direct calculation gives

$$\begin{aligned}
&Y_t^\varepsilon - \bar{Y}_t - \varepsilon Y_t^1 \\
=& \tilde{h}_x(T)(X_T^\varepsilon - \bar{X}_T) + \bar{h}_x(T)\delta X(T) \\
&+ \int_t^T \tilde{f}_x(s)(X_s^\varepsilon - \bar{X}_s - \varepsilon X_s^1)\mathrm{d}s + \int_t^T \tilde{f}_y(s)(Y_s^\varepsilon - \bar{Y}_s - \varepsilon Y_s^1)\mathrm{d}s \\
&+ \int_t^T \tilde{f}_z(s)(Z_s^\varepsilon - \bar{Z}_s - \varepsilon Z_s^1)\mathrm{d}s + \int_t^T (\tilde{f}_x(s) - \bar{f}_x(s))\varepsilon X_s^1 \mathrm{d}s \\
&+ \int_t^T (\tilde{f}_y(s) - \bar{f}_y(s))\varepsilon Y_s^1 \mathrm{d}s + \int_t^T (\tilde{f}_z(s) - \bar{f}_z(s))\varepsilon Z_s^1 \mathrm{d}s \\
&+ \varepsilon \int_0^t (\tilde{f}_u(s) - \bar{f}_u(s))u_s^1 \mathrm{d}s + \int_t^T (Z^\varepsilon(s) - \bar{Z}(s) - \varepsilon Z_s^1)\mathrm{d}W_s,
\end{aligned}$$

with

$$\tilde{f}_x(t) \triangleq \int_0^1 f_x(t, \bar{X}_t + \lambda(X_t^\varepsilon - \bar{X}_t), \bar{Y}(t) + \lambda(Y_t^\varepsilon - \bar{Y}_t), \bar{Z}_t + \lambda(Z_t^\varepsilon - \bar{Z}_t), \bar{u}_t + \lambda\varepsilon u_t^1)\mathrm{d}\lambda,$$

and $\tilde{f}_y$, $\tilde{f}_z$, $\tilde{h}_x$ similarly defined. Combining Lemma 4.1 and Lemma 4.3, we have

$$\mathbb{E} \sup_{0\le t\le T} |Y_t^\varepsilon - \bar{Y}_t - \varepsilon Y_t^1|^p = o(\varepsilon^p),$$

which is equivalent to (4.4). □

Finally, we have the following MP based on a stochastic Hamiltonian system.

Theorem 4.5. *Under Assumptions 2.1 and 2.2, set* $(\bar{u};\bar{\Theta}) = (\bar{u};\bar{X},\bar{Y},\bar{Z})$ *to be an optimal 4-tuple of Problem 2.3. Then, we have for a.a.* $t \in [0,T]$*, a.s.* $\omega \in \Omega$*,*

$$H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, p_t, q_t, k_t, \bar{u}_t)(u - \bar{u}_t) \ge 0, \quad \text{for any } u \in U, \tag{4.5}$$

where $\Lambda = (p,q,k)$ *is the solution to the following FBSDE:*

$$\begin{cases} dp_t = -\bar{H}_x(t)\mathrm{d}t + q_t \mathrm{d}W_t \\ p_T = -\bar{h}_x^*(T)k_T, \\ dk_t = -\bar{H}_y(t)\mathrm{d}t - \bar{H}_z(t)\mathrm{d}W_t \\ k_0 = -1, \quad 0 \le t \le T, \end{cases} \tag{4.6}$$

with $\eta(t) = H, H_x, H_y, H_z, H_u$ *defined as*

$$\bar{\eta}(t) \triangleq \eta(t, \bar{\Theta}_t, \Lambda_t, \bar{u}_t).$$

Proof. Fix any admissible control $u \in \mathcal{U}^2[0,T]$. For any $\varepsilon \in [0,1]$, we construct a perturbed admissible control

$$u^\varepsilon = \bar{u} + \varepsilon u^1,$$

with $u_t^1 = \frac{u_t - \bar{u}_t}{|u_t - \bar{u}_t| \vee 1}$, and the corresponding state equation is denoted by $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$. Let (X^1, Y^1, Z^1) be the solution of FBSDE (4.1). From Lemma 4.4, we have for any $p \in (1, 2)$,

$$\lim_{\varepsilon \longrightarrow 0} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \frac{Y_t^\varepsilon - \bar{Y}_t}{\varepsilon} - Y_t^1 \right|^p \right] = 0.$$

Then, it holds that

$$Y_0^1 = \lim_{\varepsilon \to 0^+} \frac{Y_0^\varepsilon - \bar{Y}_0}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{J(0, x, u^\varepsilon) - J(0, x, \bar{u})}{\varepsilon} \geq 0. \tag{4.7}$$

Applying Itô formula to $\langle Y_t^1, k_t \rangle + \langle X_t^1, p_t \rangle$, we have

$$Y_0^1 = \quad E \int_0^T H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t, p_t, q_t, k_t) u_t^1 \mathrm{d}t.$$

Thus, by the variational inequality (4.7), we have

$$E \int_0^T H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t, p_t, q_t, k_t) u_t^1 \mathrm{d}t \geq 0,$$

which is equivalent to

$$E \int_0^T H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t, p_t, q_t, k_t) \frac{u_t - \bar{u}_t}{|u_t - \bar{u}_t| \vee 1} \mathrm{d}t \geq 0,$$

for any $u \in \mathcal{U}^2[0, T]$. Due to the arbitrariness of u^1, we shall get that

$$H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t, p_t, q_t, k_t) \frac{u - \bar{u}_t}{|u - \bar{u}_t| \vee 1} \geq 0,$$

for any $u \in U$. This will implies (4.5). □

5. The relationship between MP and DPP

In this section, we will state the relationship between MP and DPP for the recursive optimization problem.

Theorem 5.1. *Under Assumptions 2.1, 2.2 and the assumption that the value function admits the following form*

$$V(t, x) = h(x) + \int_t^T \Gamma(s, x) \mathrm{d}s - \int_t^T \Psi(s, x) \mathrm{d}W_s, \quad t \in [0, T], \tag{5.1}$$

we have for a.a. $s \in [t, T]$, a.s. $\omega \in \Omega$,

$$\begin{aligned} \Gamma(s, \bar{X}_s^{t,x}) =& G\big(s, \bar{X}_s^{t,x}, \bar{u}_s, V(s, \bar{X}_s^{t,x}), \Psi(s, \bar{X}_s^{t,x}), V_x(s, \bar{X}_s^{t,x}), \Psi_x(s, \bar{X}_s^{t,x}), V_{xx}(s, \bar{X}_s^{t,x})\big) \\ =& \inf_{u \in U} G\big(s, \bar{X}_s^{t,x}, u, V(s, \bar{X}_s^{t,x}), \Psi(s, \bar{X}_s^{t,x}), V_x(s, \bar{X}_s^{t,x}), \Psi_x(s, \bar{X}_s^{t,x}), V_{xx}(s, \bar{X}_s^{t,x})\big). \end{aligned}$$

Moreover, if $V \in C^{1,3}([0,T]\times\mathbb{R}^n)$ *and* $\Gamma_x, \Psi_x \in C^{0,0}([0,T]\times\mathbb{R}^n)$*, we have*

$$
\begin{aligned}
p_s =& - V_x(s,\bar{X}_s^{t,x})k_s,\\
q_s =& - \Big[V_x(s,\bar{X}_s^{t,x})f_z\big(s,\bar{X}_s^{t,x},V(s,\bar{X}_s^{t,x}),\sigma^*(s,\bar{X}_s^{t,x},\bar{u}_s)V_x(s,\bar{X}_s^{t,x})\\
&+V_{xx}(s,\bar{X}_s^{t,x})\sigma(s,\bar{X}_s^{t,x},\bar{u}_s)+\Psi(s,\bar{X}_s^{t,x}),\bar{u}_s\big)+\Psi_x(s,\bar{X}_s^{t,x})\Big]k_s,
\end{aligned}
\tag{5.2}
$$

for a.a. $s\in[0,T]$*, a.s.* $\omega\in\Omega$*, where* k_s *satisfies* $k_0=-1$ *and*

$$
\begin{aligned}
dk_s =& f_y(s,\bar{X}_s^{t,x},\bar{u}_s,V(s,\bar{X}_s^{t,x}),\sigma(s,\bar{X}_s^{t,x},\bar{u}_s)V_x(s,\bar{X}_s^{t,x})+\Psi(s,\bar{X}_s^{t,x}))k_s\mathrm{d}s\\
&+f_z(s,\bar{X}_s^{t,x},\bar{u}_s,V(s,\bar{X}_s^{t,x}),\sigma(s,\bar{X}_s^{t,x},\bar{u}_s)V_x(s,\bar{X}_s^{t,x})+\Psi(s,\bar{X}_s^{t,x}))k_s\mathrm{d}W_s,\ \textit{for } s\in[t,T].
\end{aligned}
\tag{5.3}
$$

Proof. First note that there exists a unique solution of (5.3), since f is Lipschitz continuous with respect to y and z. Noticing that V satisfies HJB equation (3.5) and has a form as (3.4), we conclude

$$
\begin{aligned}
\Gamma(s,x) =& \inf_{u\in U} G\big(s,x,u,V(s,x),\Psi(s,x),V_x(s,x),\Psi_x(s,x),V_{xx}(s,x)\big)\\
\le& G\big(s,x,\bar{u}_s,V(s,x),\Psi(s,x),V_x(s,x),\Psi_x(s,x),V_{xx}(s,x)\big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
0 =& G\big(s,\bar{X}_s^{t,x},\bar{u}_s,V(s,\bar{X}_s^{t,x}),\Psi(s,\bar{X}_s^{t,x}),V_x(s,\bar{X}_s^{t,x}),\Psi_x(s,\bar{X}_s^{t,x}),V_{xx}(s,\bar{X}_s^{t,x})\big)-\Gamma(s,\bar{X}_s^{t,x})\\
\le& G\big(s,x,\bar{u}_s,V(s,x),\Psi(s,x),V_x(s,x),\Psi_x(s,x),V_{xx}(s,x)\big)-\Gamma(s,x).
\end{aligned}
$$

Bearing in mind that $V\in C^{1,3}([0,T]\times\mathbb{R}^n)$ and $\Gamma_x\in C^{0,0}([0,T]\times\mathbb{R}^n)$, we have

$$
\frac{\partial}{\partial x}\Big\{G\big(s,x,\bar{u}_s,V(s,x),\Psi(s,x),V_x(s,x),\Psi_x(s,x),V_{xx}(s,x)\big)-\Gamma(s,x)\Big\}_{x=\bar{X}_s^{t,x}}=0.
$$

This implies

$$
\begin{aligned}
&(\sigma_x)^*(s,\bar{X}_s^{t,x},\bar{u}_s)\big(V_{xx}(s,\bar{X}_s^{t,x})\sigma(s,\bar{X}_s^{t,x},\bar{u}_s)\big)+\frac{1}{2}tr\big((\sigma\sigma^*)(s,\bar{X}_s^{t,x},\bar{u}_s)V_{xxx}(s,\bar{X}_s^{t,x})\big)\\
&+b_x^*(s,\bar{X}_s^{t,x},\bar{u}_s)V_x(s,\bar{X}_s^{t,x})+V_{xx}(s,\bar{X}_s^{t,x})b(s,\bar{X}_s^{t,x},\bar{u}_s)+\sigma_x^*(s,\bar{X}_s^{t,x},\bar{u}_s)\Psi_x(s,\bar{X}_s^{t,x})\\
&+\Psi_{xx}(s,\bar{X}_s^{t,x})\sigma(s,\bar{X}_s^{t,x},\bar{u}_s)+f_x\big(s,\bar{X}_s^{t,x},V(s,\bar{X}_s^{t,x}),\Psi(s,\bar{X}_s^{t,x})+\sigma^*V_x(s,\bar{X}_s^{t,x}),\bar{u}_s\big)\\
&+f_y\big(s,\bar{X}_s^{t,x},V(s,\bar{X}_s^{t,x}),\Psi(s,\bar{X}_s^{t,x})+\sigma^*V_x(s,\bar{X}_s^{t,x}),\bar{u}_s\big)V_x(s,\bar{X}_s^{t,x})\\
&+f_z\big(s,\bar{X}_s^{t,x},V(s,\bar{X}_s^{t,x}),\Psi(s,\bar{X}_s^{t,x})+\sigma^*V_x(s,\bar{X}_s^{t,x}),\bar{u}_s\big)\Psi_x(s,\bar{X}_s^{t,x})\\
&+f_z\big(s,\bar{X}_s^{t,x},V(s,\bar{X}_s^{t,x}),\Psi(s,\bar{X}_s^{t,x})+\sigma^*V_x(s,\bar{X}_s^{t,x}),\bar{u}_s\big)\sigma_x^*(s,\bar{X}_s^{t,x},\bar{u}_s)V_x(s,\bar{X}_s^{t,x})\\
&+f_z\big(s,\bar{X}_s^{t,x},V(s,\bar{X}_s^{t,x}),\Psi(s,\bar{X}_s^{t,x})+\sigma^*V_x(s,\bar{X}_s^{t,x}),\bar{u}_s\big)\sigma^*(s,\bar{X}_s^{t,x},\bar{u}_s)V_{xx}(s,\bar{X}_s^{t,x})-\Gamma_x(s,\bar{X}_s^{t,x})\\
&=0.
\end{aligned}
\tag{5.4}
$$

Here and in the rest of this paper,

$$
\frac{1}{2}tr\big((\sigma\sigma^*)V_{xxx}\big)\triangleq\Big(tr\big(\sigma\sigma^*(V_x)_{xx}^1\big),tr\big(\sigma\sigma^*(V_x)_{xx}^2\big),\cdots,tr\big(\sigma\sigma^*(V_x)_{xx}^n\big)\Big)^*.
$$

On the other hand, from (5.1), we have

$$V_x(t,x)=h_x(x)+\int_t^T\Gamma_x(s,x)\mathrm{d}s-\int_t^T\Psi_x(s,x)\mathrm{d}W_s,\quad t\in[0,T].$$

Then by the application of Itô's formula to $-V_x(s,\bar{X}_s^{t,x})k_s$, it turns out from (5.4) that

$$\begin{aligned}
&-V_x(s,\bar{X}_s^{t,x})k_s\\
=&-V_x(T,\bar{X}_T^{t,x})k_T+\int_s^T k_r dV_x(r,\bar{X}_r^{t,x})+\int_s^T V_x(r,\bar{X}_r^{t,x})dk_r+\int_s^T dV_x(r,\bar{X}_r^{t,x})\cdot dk_r\\
=&-V_x(T,\bar{X}_T^{t,x})k_T+\int_s^T\Big[-\Gamma_x(r,\bar{X}_r^{t,x})+\frac{1}{2}tr\big((\sigma\sigma^*)(r,\bar{X}_r^{t,x},\bar{u}_r)V_{xxx}(r,\bar{X}_r^{t,x})\big)\\
&\qquad+V_{xx}(r,\bar{X}_r^{t,x})b(r,\bar{X}_r^{t,x},\bar{u}_r)+\Psi_{xx}(r,\bar{X}_r^{t,x})\sigma(r,\bar{X}_r^{t,x},\bar{u}_r)\Big]k_r dr\\
&+\int_s^T\Big[\Psi_x(r,\bar{X}_r^{t,x})+V_{xx}(r,\bar{X}_r^{t,x})\sigma(r,\bar{X}_r^{t,x},\bar{u}_r)\Big]k_r\mathrm{d}W_r\\
&+\int_s^T V_x(r,\bar{X}_r^{t,x})f_y(r,\bar{X}_r^{t,x},\bar{Y}_r^{t,x},\bar{Z}_r^{t,x},\bar{u}_r)k_r dr+\int_s^T V_x(r,\bar{X}_r^{t,x})f_z(r,\bar{X}_r^{t,x},\bar{Y}_r^{t,x},\bar{Z}_r^{t,x},\bar{u}_r)k_r\mathrm{d}W_r\\
&+\int_s^T\Big[\Psi_x(r,\bar{X}_r^{t,x})+V_{xx}(r,\bar{X}_r^{t,x})\sigma(r,\bar{X}_r^{t,x},\bar{u}_r)\Big]f_z(r,\bar{X}_r^{t,x},\bar{Y}_r^{t,x},\bar{Z}_r^{t,x},\bar{u}_r)k_r dr\\
=&-V_x(T,\bar{X}_T^{t,x})k_T\\
&+\int_s^T\Big[-(\sigma_x)^*(r,\bar{X}_r^{t,x},\bar{u}_r)\big(V_{xx}(r,\bar{x}_r)\sigma(r,\bar{X}_r^{t,x},\bar{u}_r)\big)-b_x^*(r,\bar{X}_r^{t,x},\bar{u}_r)V_x(r,\bar{x}_r)\\
&\quad-\sigma_x^*(r,\bar{X}_r^{t,x},\bar{u}_r)\Psi_x(r,\bar{x}_r)-f_x\big(r,\bar{X}_r^{t,x},V(r,\bar{X}_r^{t,x}),\Psi(r,\bar{X}_r^{t,x})+\sigma^*V_x(r,\bar{X}_r^{t,x}),\bar{u}_r\big)\\
&\quad-f_y\big(r,\bar{X}_r^{t,x},V(r,\bar{X}_r^{t,x}),\Psi(r,\bar{X}_r^{t,x})+\sigma^*V_x(r,\bar{X}_r^{t,x}),\bar{u}_r\big)V_x(r,\bar{X}_r^{t,x})\\
&\quad-f_z\big(r,\bar{X}_r^{t,x},V(r,\bar{X}_r^{t,x}),\Psi(r,\bar{X}_r^{t,x})+\sigma^*V_x(r,\bar{X}_r^{t,x}),\bar{u}_r\big)\Psi_x(r,\bar{X}_r^{t,x})\\
&\quad-f_z\big(r,\bar{X}_r^{t,x},V(r,\bar{X}_r^{t,x}),\Psi(r,\bar{X}_r^{t,x})+\sigma^*V_x(r,\bar{X}_r^{t,x}),\bar{u}_r\big)\sigma_x^*(r,\bar{X}_r^{t,x},\bar{u}_r)V_x(r,\bar{X}_r^{t,x})\\
&\quad-f_z\big(r,\bar{X}_r^{t,x},V(r,\bar{X}_r^{t,x}),\Psi(s,\bar{X}_r^{t,x})+\sigma^*V_x(r,\bar{X}_r^{t,x}),\bar{u}_r\big)\sigma^*(r,\bar{X}_r^{t,x},\bar{u}_r)V_{xx}(r,\bar{X}_r^{t,x})\Big]k_r dr\\
&+\int_s^T\Big[\Psi_x(r,\bar{X}_r^{t,x})+V_{xx}(r,\bar{X}_r^{t,x})\sigma(r,\bar{X}_r^{t,x},\bar{u}_r)\Big]k_r\mathrm{d}W_r\\
&+\int_s^T V_x(r,\bar{X}_r^{t,x})f_y(r,\bar{X}_r^{t,x},\bar{Y}_r^{t,x},\bar{Z}_r^{t,x},\bar{u}_r)k_r dr+\int_s^T V_x(r,\bar{X}_r^{t,x})f_z(r,\bar{X}_r^{t,x},\bar{Y}_r^{t,x},\bar{Z}_r^{t,x},\bar{u}_r)k_r\mathrm{d}W_r\\
&+\int_s^T\Big[\Psi_x(r,\bar{X}_r^{t,x})+V_{xx}(r,\bar{X}_r^{t,x})\sigma(r,\bar{X}_r^{t,x},\bar{u}_r)\Big]f_z(r,\bar{X}_r^{t,x},\bar{Y}_r^{t,x},\bar{Z}_r^{t,x},\bar{u}_r)k_r dr\\
=&-V_x(T,\bar{x}_T)k_T\\
&+\int_s^T\Big[-(\sigma_x)^*(r,\bar{X}_r^{t,x},\bar{u}_r)\big(V_{xx}(r,\bar{X}_r^{t,x})\sigma(r,\bar{X}_r^{t,x},\bar{u}_r)\big)-b_x^*(r,\bar{X}_r^{t,x},\bar{u}_r)V_x(r,\bar{X}_r^{t,x})\\
&\quad-\sigma_x^*(r,\bar{X}_r^{t,x},\bar{u}_r)\Psi_x(r,\bar{X}_r^{t,x})-f_x\big(r,\bar{X}_r^{t,x},V(r,\bar{X}_r^{t,x}),\Psi(r,\bar{X}_r^{t,x})+\sigma^*V_x(r,\bar{X}_r^{t,x}),\bar{u}_r\big)\\
&\quad-f_z\big(r,\bar{X}_r^{t,x},V(r,\bar{X}_r^{t,x}),\Psi(r,\bar{X}_r^{t,x})+\sigma^*V_x(r,\bar{X}_r^{t,x}),\bar{u}_r\big)\sigma_x^*(r,\bar{X}_r^{t,x},\bar{u}_r)V_x(r,\bar{X}_r^{t,x})\Big]k_r dr\\
&-\int_s^T-\Big[\Psi_x(r,\bar{X}_r^{t,x})+V_{xx}(r,\bar{X}_r^{t,x})\sigma(r,\bar{X}_r^{t,x},\bar{u}_r)+V_x(r,\bar{X}_r^{t,x})f_z(r,\bar{X}_r^{t,x},\bar{Y}_r^{t,x},\bar{Z}_r^{t,x},\bar{u}_r)\Big]k_r\mathrm{d}W_r.
\end{aligned}\tag{5.5}$$

Due to $h_x(\bar{X}_T^{t,x}) = V_x(T, \bar{X}_T^{t,x})$, by the uniqueness of the solution to FBSDE (4.6), we obtain (5.2). $\square$

6. An example: LQ problem

In this section, we take the LQ problem for example to show the relationship between stochastic MP and stochastic DDP. Consider the following forward–backward stochastic system:

$$\begin{cases} \mathrm{d}X_s = \big[A_s X_s + B_s u_s\big]\mathrm{d}s + \big[C_s X_s + D_s u_s\big]\mathrm{d}W_s \\ X_t = x, \\ \mathrm{d}Y_s = -\big[\lambda_s Y_s + \langle Q_s X_s, X_s\rangle + \langle R_s u_s, u_s\rangle\big]\mathrm{d}s + Z_s \mathrm{d}W_s \\ Y_T = \langle G X_T, X_T\rangle. \end{cases}$$

The admissible control set is $\mathcal{U}^2[t,T]$ and the cost functional is defined as (3.2). Note that in this section we always take the control domain $U = \mathbb{R}^k$. Although U is not bounded, all the results in Section 3 still hold due to the LQ structure (refer to *e.g.* Tang [24] and [25]).

We have the following assumptions for the coefficients.

Assumption 6.1.

1. The coefficients A, B, C, D, λ, Q, and R are all bounded $\mathbb{F}$-adapted processes;
2. The coefficients Q and R are uniformly positive definitive, *i.e.*, there exists a constant C such that

$$Q_s, R_s \geq CI, \text{ for all } s \in [t,T], \text{ a.s.,}$$

   where I is the identity matrix.

For any $u \in \mathcal{U}^2[t,T]$ and initial state x, we introduce the corresponding adjoint equation

$$\begin{cases} dp_s = -\big[A_s^* p_s + C_s^* q_s - 2k_s Q_s X_s\big]\mathrm{d}s + q_s \mathrm{d}W_s \\ p_T = -2k_T G X_T, \\ dk_s = \lambda_s k_s \mathrm{d}s \\ k_t = -1. \end{cases}$$

From stochastic Hamiltonian system we proved in Section 4, we shall have the following theorem.

Theorem 6.2. *If $(\bar{u}, \bar{X})$ is the optimal pair of LQ problem, $(\bar{u}, \bar{X})$ satisfies*

$$-2\bar{k}_s R_s \bar{u}_s + D_s^* \bar{q}_s + B_s^* \bar{p}_s = 0,$$

where $(\bar{p}, \bar{q}, \bar{k})$ is the solution to the corresponding adjoint equation with the optimal pair $(\bar{u}, \bar{X})$. Therefore, the optimal control has the dual presentation

$$\bar{u}_s = \frac{1}{2}\bar{k}_s^{-1} R_s^{-1}\big[D_s^* \bar{q}_s + B_s^* \bar{p}_s\big].$$

If we give an explicit presentation to (p, q, k), a further expression of optimal control can be demonstrated. For this, combining the adjoint system with the original controlled system, we have the following stochastic Hamiltonian system:

$$\begin{cases} \mathrm{d}X_s = \big[A_sX_s + B_su_s\big]\mathrm{d}s + \big[C_sX_s + D_su_s\big]\mathrm{d}W_s \\ X_0 = x, \\ \mathrm{d}Y_s = -\big[\lambda Y_s + \langle Q_sX_s, X_s\rangle + \langle R_su_s, u_s\rangle\big]\mathrm{d}s + Z_s\mathrm{d}W_s \\ Y_T = \langle GX_T, X_T\rangle, \\ dp_s = -\big[A_s^*p_s + C_s^*q_s - 2k_sQ_sX_s\big]\mathrm{d}s + q_s\mathrm{d}W_s \\ p_T = -2k_TGX_T, \\ dk_s = \lambda_sk_s\mathrm{d}s \\ k_0 = -1, \\ -2k_sR_su_s + D_s^*q_s + B_s^*p_s = 0. \end{cases}$$

In summary, the stochastic Hamiltonian system completely characterizes the optimal control in LQ problem. Therefore, solving LQ problem is equivalent to solving the stochastic Hamiltonian system. But this Hamiltonian system consists of coupled FBSDEs. Thus, this characterization is far from satisfactory. We then introduce the Riccati equation to give the state feedback representation of the optimal control and further discussion of stochastic Hamiltonian system.

Different from the Markovian case, the Riccati equation here is a BSDE due to the non-Markovian coefficients

$$\begin{cases} dP_s = -\{A_s^*P_s + P_sA_s + C_s^*P_sC_s + \lambda_sP_s + C_s^*L_s + L_sC_s + Q_s \\ \qquad\qquad -\big[P_sB_s + C_s^*P_sD_s + L_sD_s\big] \\ \qquad\qquad\quad \times\big[R_s + D_s^*P_sD_s\big]^{-1}\big[P_sB_s + C_s^*P_sD_s + L_sD_s\big]^*\}\mathrm{d}s + L_s\mathrm{d}W_s \\ P_T = G. \end{cases} \tag{6.1}$$

The solvability of (6.1) had been solved by Tang [25].

Theorem 6.3. *Under Assumption 6.1, the stochastic Riccati equation (6.1) has a unique solution (P, L), where P is a uniformly bounded and nonnegative matrix-valued process and L satisfies*

$$E\left(\int_0^T |L_s|^2\,\mathrm{d}s\right)^p < \infty,$$

for any $p > 1$.

For the concerned LQ problem, we still define its value function as

$$V(t,x) \triangleq \inf_{u\in\mathcal{A}} J(t,x;u_\cdot) = \inf_{u\in\mathcal{A}} Y_t^{t,x;u}.$$

Then, the corresponding stochastic HJB equation is

$$
\begin{aligned}
&V(t,x)\\
=&\ \langle Gx,x\rangle+\int_t^T \inf_u H\big(s,x,u,V(s,x),\Psi(s,x),V_x(s,x),\Psi_x(s,x),V_{xx}(s,x)\big)\mathrm{d}s\\
&-\int_t^T \Psi(s,x)\mathrm{d}W_s\\
=&\ \langle Gx,x\rangle+\int_t^T \inf_u\{\langle V_x(s,x),A_sx+B_su\rangle+\frac{1}{2}tr\big((C_sx+D_su)(C_sx+D_su)^*V_{xx}(s,x)\big)\\
&+\langle \Psi_x(s,x),C_sx+D_su\rangle+\lambda_sV(s,x)+\langle Q_sX_s,X_s\rangle+\langle R_su,u\rangle\}\mathrm{d}s\\
&-\int_t^T \Psi(s,x)\mathrm{d}W_s.
\end{aligned}
\tag{6.2}
$$

With the help of stochastic Riccati equation, we can obtain a solution of above stochastic HJB equation.

Proposition 6.4. *If (P,L) is the unique solution of the stochastic Riccati equation (6.1), $(\langle P_sx,x\rangle,\langle L_sx,x\rangle)$ is a classical solution of the stochastic HJB equation (6.2).*

Proof. Set

$$
v(s,x)=\langle P_sx,x\rangle,\qquad \psi(s,x)=\langle L_sx,x\rangle.
$$

First note that $v_x(s,x)=(P_s+P_s^*)x=2P_sx$, $\psi_x(s,x)=(L_s+L_s^*)x=2L_sx$, $v_{xx}(s,x)=P_s+P_s^*=2P_s$. Then we have

$$
\begin{aligned}
&\inf_u\{\langle v_x(s,x),A_sx+B_su\rangle+\frac{1}{2}tr\big((C_sx+D_su)(C_sx+D_su)^*v_{xx}(s,x)\big)+\langle \psi_x(s,x),C_sx+D_su\rangle\\
&\quad+\lambda_sv(s,x)+\langle Q_sx,x\rangle+\langle R_su,u\rangle\}\\
=&\ \inf_u\{\langle 2P_sx,A_sx+B_su\rangle+\frac{1}{2}tr\big((C_sx+D_su)(C_sx+D_su)^*2P_s\big)+\langle 2L_sx,C_sx+D_su\rangle\\
&\quad+\lambda_s\langle P_sx,x\rangle+\langle Q_sx,x\rangle+\langle R_su,u\rangle\}\\
=&\ \inf_u\{\langle x,P_sA_s+A_s^*P_s+Q_s+C_s^*P_sC_s+C_s^*L_s+L_sC_s+\lambda_sP_s)x\rangle\\
&\quad+2\langle u,\big[P_sB_s+C_s^*P_sD_s+L_sD_s\big]^*x\rangle+\langle u,(R_s+D_s^*P_sD_s)u\rangle\}\\
=&\ \langle\big[P_sA_s+A_s^*P_s+Q_s+C_s^*P_sC_s+C_s^*L_s+L_sC_s+\lambda_sP_s\big]x,x\rangle\\
&\ -\langle\big[P_sB_s+C_s^*P_sD_s+L_sD_s\big](R_s+D_s^*P_sD_s)^{-1}\big[P_sB_s+C_s^*P_sD_s+L_sD_s\big]^*x,x\rangle\}.
\end{aligned}
\tag{6.3}
$$

Thus, noticing (6.1), we have

$$
\begin{aligned}
\mathrm{d}\langle P_sx,x\rangle=&-\{\langle\big[P_sA_s+A_s^*P_s+Q_s+C_s^*P_sC_s+C_s^*L_s+L_sC_s+\lambda_sP_s\big]x,x\rangle\\
&-\langle\big[P_sB_s+C_s^*P_sD_s+L_sD_s\big](R_s+D_s^*P_sD_s)^{-1}\big[P_sB_s+C_s^*P_sD_s+L_sD_s\big]^*x,x\rangle\}\mathrm{d}s\\
&+\langle L_sx,x\rangle\mathrm{d}W_s.
\end{aligned}
$$

By the definition for (v,ψ), together with (6.3), it turns out that

$$
\begin{aligned}
dv(s,x)=&-\inf_u\{\langle v_x(s,x),A_sx+B_su\rangle+\frac{1}{2}tr\big((C_sx+D_su)(C_sx+D_su)^*v_{xx}(s,x)\big)\\
&+\langle\psi_x(s,x),C_sx+D_su\rangle+\lambda_sv(s,x)+\langle Q_sx,x\rangle+\langle R_su,u\rangle\}\mathrm{d}s+\psi(s,x)\mathrm{d}W_s,
\end{aligned}
$$

which demonstrates that (v,ψ) is the classical solution of the stochastic HJB equation. □

With a classical solution of stochastic HJB equation, we can further obtain the optimal control for the LQ problem.

Proposition 6.5. *The optimal control of LQ problem is given by*

$$\bar{u}_s = -(R_s + D_s^* P_s D_s)^{-1}\big[P_s B_s + C_s^* P_s D_s + L_s D_s\big]^* \bar{X}_s.$$

Proof. By Proposition 3.4, we see that the candidate control-state pair (u, X) for the optimal one is of the following feedback form:

$$u_s = -(R_s + D_s^* P_s D_s)^{-1}\big[P_s B_s + C_s^* P_s D_s + L_s D_s\big]^* X_s.$$

To show that it is indeed the optimal pair, one only need to prove that u is a admissible control, which is proved in Tang [25]. ☐

Finally, applying Itô formula to $dP_s X_s k_s$, we immediately have the desired relationship for LQ problem.

Theorem 6.6. *For LQ Problem, we have the relationship between stochastic Hamiltonian system and stochastic HJB equation below:*

$$\begin{aligned} \bar{p}_s &= -2P_s \bar{X}_s \bar{k}_s, \\ \bar{q}_s &= -2\big[P_s(C_s \bar{X}_s + D_s \bar{u}_s) + L_s \bar{X}_s\big]\bar{k}_s. \end{aligned}$$

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