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NEW MINIMAX THEOREMS FOR LOWER SEMICONTINUOUS FUNCTIONS AND APPLICATIONS

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Abstract. The classical Fountain Theorem is extended to the case of functionals I which are the sum of a C^1 term and of a convex lower semicontinuous functional. In this setting a suitable nonsmooth version of a Heinz's result has been proved in order to obtain a dual version of the main Fountain's type result. Moreover, as a byproduct of the theoretical arguments presented here, some applications concerning the existence of infinitely many solutions for a wide class of differential problems are also presented. More precisely, elliptic problems involving either a logarithmic nonlinearity or driven by the 1-Laplacian operator have been studied.

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1. Introduction

Several studies in critical point theory concern the existence and multiplicity of critical points for different classes of functionals that are differentiable in some sense. For instance, the classical Bartsch's Fountain Theorem ensures the existence and multiplicity of critical points for $\mathbb{Z}_2$ symmetric C^1 functionals; see [1], Theorem 2.5 and [2], Theorem 3.6.

Starting from the Bartsch's theorem, many authors were interested in finding critical points of real-valued functional Φ defined on an infinite dimensional Banach space X, obtaining several generalizations of this result, which allow to solve wide classes of ordinary or partial differential equations. For instance, in 1995, again Bartsch and Willem studied the existence of multiple solutions for a Dirichlet problem involving concave and convex nonlinearities by proving a dual and more general version of the Fountain Theorem (see [3]).

Some years later, in 2013, [4], Theorem 12 established a Fountain-type result for indefinite functionals $\Phi \in C^1(X,\mathbb{R})$ making use of the τ topology introduced by Kryszewski and Szulkin in [5] and by proving a suitable extension of the classical Borsuk-Ulam Theorem (see [4], Thm. 3) to a suitable class of admissible functions through degree theory methods (see [2], Chap. 6).

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We emphasize that, by arguing as in [5], a key restrictive assumption in [4], Theorem 12 is the τ-upper semi-continuity of the functional Φ. However, by using some ideas developed in [6], in 2016, Gu and Zhou [7], Theorem 1.2, are able to prove a more general version of [4], Theorem 12 without any request on the τ-upper semi-continuity of Φ.

Finally, some Fountain-type results was established in [8], Theorems 2.1 and 2.2 without the compactness Palais–-Smale's condition. As a meaningful consequence of their abstract results, in [8], Theorems 3.1, 3.2 and 3.3 the authors studied the existence of infinitely many solutions for some nonlinear Schrödinger equations and Dirichlet boundary value problems without the Ambrosetti–Rabinowitz's superquadraticity condition on the nonlinear datum.

It should be noted that the regularity assumption on the functional Φ plays an important role for the validity of the Fountain-type results cited above as well as their applications; see, among others, the papers [3, 4, 7–10]. However, the above results fail treating several mechanical and engineering questions, where the relevant energy functionals are neither convex nor smooth (the so-called super-potentials).

In 1981, through techniques of nonsmooth analysis previously introduced by Clarke (see [11]), Chang (in [12]) treated the case of functionals that are only locally Lipschitz continuous, generalizing both the famous Mountain Pass Theorem (briefly, MPT) of Ambrosetti–Rabinowitz [13], Theorem 2.2 and the Saddle Point Theorem [13], Theorem 4.6 to this more general framework; see Theorems 3.4 and 3.3 of [12], respectively (see also [14, 15] for related topics).

Later on, Szulkin in [16] extend the aforementioned results to real functionals $I : X \to (-\infty, +\infty]$ having the following structure

(H_0) $I := \Phi + \Psi$, *with* $\Phi \in C^1(X, \mathbb{R})$ *and* $\Psi : X \to (-\infty, +\infty]$ *is a convex lower semicontinuous functional and proper, i.e.* $\Psi \not\equiv \infty$.

From now on, a functional $I : X \to (-\infty, +\infty]$ is said to be of Szulkin-type if its structure is given as in (H_0).

In Section 2, we briefly recall some notions of nonsmooth critical point theory (in the Szulkin sense) needed in the sequel and then illustrate the functional framework we will move in. We also refer the interested reader to the following sources [17–22] for some theoretical aspects and direct applications.

Motivated by this wide interest in the current literature, the main purpose of this paper is to solve the following question

(Q_1) *Is it possible to prove a Fountain-type Theorem for Szulkin-type functionals?*

A complete and positive answer for (Q_1) in the pure locally Lipschitz continuous setting (*i.e.* $\Psi \equiv 0$) is given by Dai in [23], Theorem 3.1. Concerning the general case, to the best of our knowledge, only a partial and again affirmative answer can be found in [19]. More precisely, in [19], Theorem 1.1, by adapting the arguments of [1], Theorem 2.5, studied the existence of fountain solutions for the following logarithmic Schrödinger equation

$$-\Delta u + V(x)u = u \log u^2, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where V is a continuous function and $\lim_{|x|\to+\infty} V(x) = +\infty$.

Indeed, the energy functional associated to problem (1.1) is given by

$$J_0(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x) + 1)u^2)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x, \quad u \in E, \tag{1.2}$$

where E is the subspace of $H^1(\mathbb{R}^N)$ defined by

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x) + 1)^+ u^2)\mathrm{d}x < +\infty \right\},$$

see [19], Section 1 for details. We emphasize that depending on the choice of V the functional J_0 may or may not take the value $+\infty$ in E, in particular $J_0 \notin C^1(E,\mathbb{R})$. Moreover, as made in [17, 18, 22], the functional J_0 can be decomposed as a sum of a functional $J_1 \in C^1(E,\mathbb{R})$ and of a convex lower semi-continuous functional $J_2 : E \to (-\infty, +\infty]$. Consequently, J_0 is a Szulkin-type functional.

In order to prove [19], Theorem 1.1 the authors firstly show that J_0 satisfies the fountain geometry. Successively, a suitable deformation lemma given in [19], Proposition 3.4 has been proved by adapting the ideas previously developed by Squassina and Szulkin in [22]. Finally, a careful analysis of the aforementioned deformation lemma ensures that the fountain minimax levels are also critical values for the functional J_0; see [19], Lemma 3.6 and Proposition 3.7. For the sake of completeness we point out that a key point along the proof of [19], Theorem 1.1 is given by the smoothness of the energy functional J_0 when the integrals in (1.2) are taken on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, that is $J_0 \in C^1(H^1(\Omega),\mathbb{R})$; see Lemma 4.2 below, as well as [19], Lemma 2.2 and [22], Lemma 2.2.

In this paper, a complete and positive answer to (Q_1) is given by proving a nonsmooth version of Theorem 2.5 in [1] for Szulkin-type functionals; see Theorem 3.6 below.

To this aim, in Lemma 3.4 we prove a suitable equivariant version of the deformation lemma established by Szulkin in [16], Proposition 2.3. Moreover, since Lemma 3.4 deals with a symmetry assumption that verifies a specific condition of admissibility, this notion has been introduced in (G_0) and briefly discussed in Section 2. The special case due to the antipodal action of $\mathbb{Z}_2$ on X was considered in [16], Corollary 2.4.

A second question that naturally arises in this nonsmooth setting is the following

(Q_2) *Is it possible to prove a dual Fountain-type Theorem for Szulkin-type functionals*?

A careful analysis of the proof of the classical dual Fountain Theorem can be found in [2], Theorem 3.18. The main basic idea due to Bartsch and Willem consists in applying Theorem 2.5 of [1] to the functional $-I$ obtaining a real sequence (c_j) of negative critical values of I such that $c_j \to 0$ as $j \to \infty$. However, when I is a Szulkin-type functional it is easily seen that this procedure cannot be used in general as in the smooth case, because when I is a Szulkin-type functional we do not know, in general, if the functional $-I$ is also a Szulkin-type functional. Due to this main difficulty, in Theorem 3.8 a nonsmooth version of a Heinz's Theorem (see [24], Prop. 2.2) has been proved for Szulkin-type functionals by using the notions and definitions given and discussed in [16], Section 4.

We notice that Proposition 2.2 in [24] has been proved through a suitable version of the Ljusternik-Schnirelman theory developed by Clark in [25]. As in the classical dual Fountain Theorem (see [2], Thm. 3.18), under some technical assumptions involving the topological notion of genus, the original Heinz's result ensures the existence of a sequence (c_j) of negative critical values for a functional $J \in C^1(X,\mathbb{R})$ such that $c_j \to 0$ as $j \to \infty$.

Finally, by adapting the arguments used along the proof of the main Theorem 3.8, we are able to show a more precise version of [16], Corollary 4.8 obtaining a sequence of critical levels (c_j) for a Szulkin-type functional such that $c_j \to \infty$ as $j \to \infty$; see in Theorem 3.9 below.

The new minimax theorems mentioned above are used to establish the existence of infinitely many solutions for some classes of elliptic problems involving a logarithmic nonlinearity. As a first application we study the following inclusion problem

$$\begin{cases} -\Delta u + u + \partial F(x,u) \ni u \log u^2, & \text{a.e. in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_1}$$

where $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a N-measurable function satisfying some natural growth conditions, $F(x,t) := \int_0^t f(x,t)\, ds$ is its locally Lipschitz continuous potential and, as usual, $\partial F(x,t)$ denotes the generalized gradient of F with respect to the variable $t \in \mathbb{R}$ at the point $x \in \mathbb{R}^N$; see Section 2 for details.

Now, the natural candidate for the energy functional associated to problem (P_1) is given by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)\mathrm{d}x + \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - -\int_{\mathbb{R}^N} L(u)\mathrm{d}x, \;\; u \in H^1(\mathbb{R}^N), \tag{1.3}$$

where

$$L(t) := \int_0^t s\log s^2\, ds = -\frac{t^2}{2} + \frac{t^2\log t^2}{2}, \quad t \in \mathbb{R}.$$

Standard computations ensure that, in general, the functional I fails to be finite and C^1 on $H^1(\mathbb{R}^N)$. Due to this loss of smoothness, in order to study the existence of infinitely many solutions of problem (P_1), we exploit the fact that the associated energy I is a Szulkin-type functional. We would like to stress that the approach adopted in [19] does not work well studying problem (P_1). Indeed, the functional

$$\Psi_2(u) := \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x$$

cannot be of class C^1 even in $H^1(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Hence, also the technical trick in [19], Lemma 2.2 cannot be used in our setting recovering locally smoothness. Hence, the nonsmooth Fountain Theorem proved in Theorem 3.6 seems to be a crucial tool to prove that I has a sequence of critical points (u_k) such that $I(u_k) \to \infty$ as $k \to \infty$; see Theorem 4.9.

A further application of the main results is given in Theorem 4.19. In this theorem the existence of infinitely many solutions for a concave perturbation of a logarithmic Schrödinger-type equation is established. More precisely, by using the nonsmooth version of the Heinz's result proved in Theorem 3.8, we study the following problem

$$\begin{cases} -\Delta u + u = u\log u^2 + \lambda h(x)|u|^{q-2}u \;\text{ in }\; \mathbb{R}^N \\ \;u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_2}$$

where λ is a positive parameter, $q \in (1,2)$ and $h : \mathbb{R}^N \to \mathbb{R}$ is a real function that verifies a suitable technical condition. Clearly, with the above notations, the energy functional associated to (P_2) is given by

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)\mathrm{d}x - \int_{\mathbb{R}^N} L(u)\mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)|u|^q, \;\; u \in H^1(\mathbb{R}^N).$$

Moreover, also in this case, I_λ is a Szulkin-type functional and Theorem 3.8 yields the existence of a sequence of critical points (u_k) with $I_\lambda(u_k) \to 0$ as $k \to \infty$ provided that the parameter λ is sufficiently small.

Finally, in Theorem 4.26, we employ Theorem 3.9 to establish the existence of infinitely many solutions for the following problem

$$\begin{cases} -\Delta_1 u = |u|^{p-2}u, \;\text{ in }\; \Omega, \\ \;u|_\Omega = 0, \;\text{ on }\; \partial\Omega \end{cases} \tag{P_3}$$

where $\Omega \subset \mathbb{R}^N$ (with $N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$, $p \in (1, 1^*)$ and Δ_1 denotes the so-called *1-Laplacian operator* that, in a formal sense, can be defined by $\Delta_1 u := \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right)$.

We notice that the notion of solution for problem (P_3) is quite delicate and will be discussed in Section 4.3 according to Kawohl and Schuricht [26]. Moreover, elliptic problems driven by the 1-Laplacian operator have been intensively studied in the current literature; see, for instance, [27–31] and [26, 32–35], as well as the references therein.

The paper is organized as follows. In Section 2, some basic notions of nonsmooth analysis are briefly recalled. Furthermore, theoretical arguments on topological group actions on Banach spaces are discussed. Taking the advantage of the above notions, in Section 3 the Fountain-type results given in Theorems 3.6, 3.8 and 3.9 have been proved. Finally, in Section 4 we provide some meaningful applications of the main results; see Theorem 4.9, 4.19 and 4.26.

Notations

- $H^1_{\rm rad}(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is radial}\}$.
- $C^\infty_{0,\,\rm rad}(\mathbb{R}^N) := \{u \in C^\infty_0(\mathbb{R}^N) : u \text{ is radial}\}$.
- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space, with norm $\|u\|_p := \left(\int_{\mathbb{R}^N} |u|^p \mathrm{d}x\right)^{1/p}$, $1 \le p < 1$, and $\|u\|_\infty := \operatorname{esssup}_{x\in\mathbb{R}^N} |u(x)|$.
- If $\Omega \subset \mathbb{R}^N$ is a measurable set, we simply write $\int_\Omega f$ instead of $\int_\Omega f(x)\,\mathrm{d}x$ for any measurable real-values function f defined on Ω.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$
- $C(x_1, ..., x_n)$ denotes a positive constant that depends on $x_1, ..., x_n$.
- $1^* := \dfrac{N}{N-1}$, if $N \ge 2$.
- $2^* := \dfrac{2N}{N-2}$, if $N \ge 3$ and $2^* := \infty$ if either $N = 1$ or $N = 2$.

2. Nonsmooth analysis, group actions and preliminary results

In this section we briefly recall some basic notions from nonsmooth analysis and group representation theory needed in the sequel and then illustrate the functional framework we will move in. We refer the interested reader to the sources [2, 14, 15, 36, 37] for detailed derivations of the geometric aspects, their motivation and further applications. Moreover, here and in the sequel, $(X, \|\cdot\|)$ denotes a real Banach space and $(X^*, \|\cdot\|_*)$ its topological dual, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X and X^*.

2.1. A primer on nonsmooth analysis

Let us first consider some notions related to locally Lipschitz continuous functionals. Detailed proofs and remarks can be found in Chang [12] and Clarke [11, 38], as well as Carl, Le and Motreanu [14] and Motreanu and Panagiotopoulos [15], Chaps. 1–2. A real-valued functional $\varphi : X \to \mathbb{R}$ is called *locally Lipschitz continuous* (briefly $\varphi \in \mathrm{Lip}_{\rm loc}(X, \mathbb{R})$) when to every $u \in X$ there correspond a neighbourhood $V := V_u$ of u and a constant $K := K_u > 0$ such that

$$|\varphi(v) - \varphi(w)| \le K\|v - w\|, \quad \forall v, w \in V.$$

The *generalized directional derivative* of $\varphi \in \mathrm{Lip}_{\rm loc}(X, \mathbb{R})$ at u along the direction $v \in X$ is defined by

$$\varphi^\circ(u; v) := \limsup_{w\to u,\, t\to 0^+} \frac{\varphi(w + tv) - \varphi(w)}{t}.$$

The *generalized gradient* of the function $\varphi \in \mathrm{Lip}_{\rm loc}(X, \mathbb{R})$ in u is the set

$$\partial\varphi(u) = \{\phi \in X^* : \varphi^\circ(u; v) \ge \langle \phi, v \rangle\,, \forall\, v \in X\}.$$

Proposition 2.1.2 of [11] ensures that $\partial\varphi(u)$ turns out nonempty, convex, in addition to weak* compact, and that

$$\varphi^\circ(u;v) := \max\{\langle\eta, v\rangle : \eta \in \partial\varphi(u)\}.$$

In the sequel we say that a point $u \in X$ is a *critical point* of $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$ if $0 \in \partial\varphi(u)$. We also recall that, when a functional $\eta : X \to \mathbb{R}$ is convex, the *subdifferential* of η at u is the set

$$\partial_s\eta(u) := \{\phi \in X^* : \eta(v) - \eta(u) \geq \langle \phi, v-u\rangle, \forall\, v \in X\}. \tag{2.1}$$

If $\eta \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$ then $\partial_s\eta(u) = \partial\eta(u)$.

Some usual properties of the generalized directional derivative as well of the generalized gradient are listed below.

Lemma 2.1. *Let $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$, then*

i) the map $(u,v) \mapsto \varphi^\circ(u,v)$ is an upper semicontinuous functional, i.e. if $(u_j, v_j) \to (u,v)$ then

$$\limsup \varphi^\circ(u_j, v_j) \leq \varphi^\circ(u,v);$$

ii) $\varphi^\circ(u,-v) = (-\varphi)^\circ(u,v)$.

Lemma 2.2. *If ψ is continuously Fréchet differentiable in an open neighborhood of $u \in X$, then $\partial\psi(u) = \{\psi'(u)\}$.*

Lemma 2.3. *If φ, $\psi \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$, then for each $u \in X$ one has*

i) $\partial(\varphi+\psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u)$;
ii) $\partial(\varphi+\psi)(u) = \{\varphi'(u)\} + \partial\psi(u)$, provided that $\varphi \in C^1(X,\mathbb{R})$.

In the next lemma we report an important property between $\varphi^\circ(u,v)$ and the Gâteaux derivatives of φ at $u \in X$ along $v \in X$, *i.e.*

$$\frac{\partial\varphi}{\partial v}(u) := \lim_{t\to 0^+} \frac{\varphi(u+tv) - \varphi(u)}{t}. \tag{2.2}$$

Lemma 2.4. *If $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$ is convex, then $\dfrac{\partial\varphi}{\partial v}(u)$ exists for any $u, v \in X$ and*

$$\frac{\partial\varphi}{\partial v}(u) = \varphi^\circ(u,v).$$

From now on, we say that a functional $I : X \to (-\infty, +\infty]$ is a *Szulkin-type functional* if

(H_0) *$I := \Phi + \Psi$, with $\Phi \in C^1(X,\mathbb{R})$ and $\Psi : X \to (-\infty,+\infty]$ is a convex lower semicontinuous functional and proper, i.e. $\Psi \not\equiv \infty$.*

The *effective domain* of I is defined by

$$D(I) := \{u \in X : I(u) < +\infty\},$$

and so, for a Szulkin-type functional I one has that $D(I) = D(\Psi)$. For each $u \in D(I)$, we say that the *subdifferential* of I at u is the set

$$\partial I(u) := \{\varphi \in X^* : \langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \geq \langle\varphi, v-u\rangle\,, \forall v \in X\}. \tag{2.3}$$

Definition 2.5. Suppose that I is a Szulkin-type functional Then

i) a point $u \in X$ is called a critical point of I if $0 \in \partial I(u)$, or more precisely, $u \in D(I)$ and

$$\langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in X,$$

ii) a sequence (u_n) is called a Palais–Smale sequence (briefly (PS) sequence) for I at level $c \in \mathbb{R}$ if $I(u_n) \to c$ and

$$\langle \Phi'(u_n), v-u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v-u_n\|, \quad \forall v \in X,$$

with $\varepsilon_n \to 0^+$, or equivalently (see [16], Prop. 1.2)

$$\langle \Phi'(u_n), v-u_n\rangle + \Psi(v) - \Psi(u_n) \geq \langle w_n, v-u_n\rangle, \quad \forall v \in X,$$

where $w_n \in X^*$ with $w_n \to 0$ in X^*;

iii) I satisfies the Palais-Smale condition (briefly (PS) condition) at level $c \in \mathbb{R}$ when each (PS) sequence (u_n) at level c has a convergent subsequence. If I verifies the (PS) condition for all level c, we say simply that I satisfies the (PS) condition.

For further details about the critical point theory described above see Szulkin [16], Carl, Le and Motreanu [14], as well as Motreanu and Panagiotopoulos [15].

Now, let us recall the classical Ekeland's Variational Principle [36], Theorem 1 that will be useful in the sequel.

Theorem 2.6. *Let (Y, d) be a complete metric space. Suppose that $\varphi : Y \to (-\infty, \infty]$ is a proper lower semicontinuous functional bounded from below. Given δ, $\tau > 0$ and $u_0 \in Y$ such that*

$$\inf_{u \in Y} \varphi(u) \leq \varphi(u_0) \leq \inf_{u \in Y} \varphi(u) + \delta, \tag{2.4}$$

then, there exists $v_0 \in Y$ verifying

i) $\varphi(v_0) \leq \varphi(u_0)$, $d(v_0, u_0) \leq 1/\tau$;

ii) $\varphi(v) - \varphi(v_0) \geq -\delta\tau d(v, v_0)$, $\forall v \in Y$.

2.2. Group actions

The notions described in this subsection follow closely the presentation in [2], Sections 1.6 and 3.2; see also Bartsch [39] for additional comments and remarks. Let G be a topological group with neutral element e. An action of G on X is a continuous function

$$\begin{aligned} \phi : G \times X &\to X \\ (g, v) &\mapsto \phi(g, v) = gv \end{aligned}$$

such that

(G_1) $ev = v$, $\forall x \in X$;

(G_2) $(gh)v = g(hv)$, $\forall v \in X$, $\forall g, h \in G$;

(G_3) *For each $g \in G$ the map*

$$\begin{aligned} \phi_g : X &\to X \\ v &\mapsto \phi_g(v) = gv \end{aligned}$$

is linear.

If in addition to the above condition, the following relation holds

(G_4) $\|gv\| = \|v\|, \ \forall v \in X, \ \forall g \in G$

the map ϕ is said to be an isometric action. According to the above definitions, we say that G acts isometrically on X when $(G_1) - (G_4)$ hold.

The subspace of *invariant elements* of X is defined by

$$Fix(G) := \{u \in X : \ gu = u \ \ \forall g \in G\}.$$

Example 2.7. Let $Id : X \to X$ be the identity map on X and consider the usual representation $\mathbb{Z}_2 = \{Id, -Id\}$. Standard computations ensure that the group $\mathbb{Z}_2$ acts isometrically on X.

A subset A of X is said to be G-*invariant* if $gA = A$ for every $g \in G$, where $gA := \{gx : \ x \in A\}$. Also, when $A \subset X$ is a G-invariant set, a map $\gamma : A \to X$ is called *equivariant map* if

$$\gamma(gx) = g\gamma(x) \ \ \forall x \in A, \forall g \in G.$$

If a functional (not necessarily linear) φ defined on X satisfies $\varphi(gx) = \varphi(x)$ for any $x \in X$ and $g \in G$, we say that φ is a G-invariant functional.

Notation: $\Gamma_G(A) := \{\gamma \in C(A, X) : \ \gamma \text{ is equivariant}\}$.

By following [2], Section 3.2 and [3], the notion of admissible action is given below.

Definition 2.8. Let Y be a finite dimensional vector space. Moreover, let us assume that G is a compact topological group that acts diagonally on Y^k, that is

$$gv = g(v_1, ..., v_k) = (gv_1, ..., gv_k),$$

for every $v = (v_1, ..., v_k) \in Y^k$ and each $g \in G$. The action of G on Y is said to be admissible if, for each equivariant map $\gamma : \partial U \to Y^{k-1}$, where $k \geq 2$ and U is a bounded G-invariant open set of Y^k with $0 \in U$, there is $u \in \partial U$ such that $\gamma(u) = 0$.

For our goals we will consider a special condition on a decomposition of space X with respect to action of G on X as follows:

(G_0) *G is a compact group that acts isometrically on*

$$X = \overline{\bigoplus_{j \in \mathbb{N}} X_j},$$

where every X_j is a G-invariant subspace of X such that $X_j \cong Y$, being Y a finite dimensional vector space for which the action of G is admissible.

We also emphasize that a key point in our approach is given by the *Haar integral* on a topological group G. For the sake of completeness a brief description of this abstract concept will be given below; see Nachbin [40] for additional comments and details. Suppose that G is a locally compact group and μ a positive measure on G. According to the classical literature on the subject, $\mathcal{L}(G, \mu)$ denotes here the space of the integrable functions $f : G \longrightarrow \mathbb{R}$ with respect to the measure μ, and μ is a *left invariant* measure when

$$\int_G f(g^{-1}y) d\mu = \int_G f(y) d\mu, \ \forall g \in G, \tag{2.5}$$

for every $f \in \mathcal{L}(G,\mu)$.

The next result assures the existence of a left invariant measure on a locally compact topological group G.

Theorem 2.9 (Haar)**.** *Let G be a locally compact group. Then, there exists at least one left invariant positive measure $\mu_0 \neq 0$. Moreover, the measure $\mu_0(G)$ is unique except for a strictly positive factor of proportionality, i.e. if μ_1 is a left invariant positive measure on G, there exists $c > 0$ such that $\mu_1 = c\mu_0(G)$. Finally*

$$\mu_0(G) < \infty \Leftrightarrow \ G \text{ is compact.}$$

See [40], Chapter II, Sections 4 and 5 for a detailed proof.

Corollary 2.10 (Normalized Haar measure)**.** *Let G be a compact group. Then, there exists a left invariant positive measure μ on G such that $\mu(G) = 1$.*

Proof. Take $\mu := \dfrac{1}{\mu_0(G)}\mu_0$, with μ_0 given in the Theorem 2.9. □

Remark 2.11. The integral associated to μ_0 in the Theorem 2.9 is the so called Haar's integral. From now on, we will denote by μ the measure with the property given in Corollary 2.10. We would like to point out that the Haar's integral as defined above can be extended for X-valued measurable functions, that is, for functions $f : G \longrightarrow X$. To see why, it is enough to apply the procedure in [41], Appendix and references therein to the measure μ, which permits to get a X-valued version of the integral associated with μ. Notice that the property in (2.5) still holds in the case of the X-valued version of the Haar's integral.

Let $\beta : X \to X$ be a continuous map and let G be a compact topological group. By the left invariance property of μ, if $\eta : X \to X$ is the map given by

$$\eta(u) := \int_G g\beta(g^{-1}u)d\mu, \ u \in X, \tag{2.6}$$

then $\eta \in \Gamma_G(X)$. This fact will be useful in the sequel.

We finish this subsection by recalling an important result due to Kobayashi-Ôtani which generalizes the Principle of Symmetric Criticality due to Palais; see [2], Theorem 1.28.

Theorem 2.12. *Let X be a reflexive Banach space and let G be a compact topological group that acts isometrically on X. If $I = \Phi + \Psi$ be a Szulkin-type functional with Φ and Ψ being G-invariant, then*

$$0 \in \partial(I|_Z)(u) \implies 0 \in \partial I(u), \tag{2.7}$$

for any $u \in Z := Fix(G)$.

An exhaustive proof of Theorem 2.12 is given in [20], Theorem 3.16. We emphasize here that (2.7) ensures that every critical point of $I|_Z$ is a critical point of I in the sense given in Definition 2.5.

2.3. *G*-index theory

Now, we introduce the notion of the G-index that will be required in our abstract results. The reader can consult [1] for a discussion in a more general situation. Let Σ be the class of subsets of $(X - \{0\})$ that are G-invariant and closed in X. Let us assume that the condition (G_0) holds and let Y be the vector space fixed in that condition.

Definition 2.13. The G-index of $A \in \Sigma \setminus \{\emptyset\}$ is defined as

$$\gamma_G(A) := \min\{k \in \mathbb{N} \setminus \{0\} : \exists\phi : A \to Y^k \setminus \{0\}, \, \phi \in \Gamma_G(A)\}$$

if such integer exists and $\gamma_G(A) := +\infty$ otherwise. Finally, we also set $\gamma_G(\emptyset) := 0$.

Remark 2.14. Note that when $G = \mathbb{Z}_2$ the G-index introduced above coincides with the genus of symmetric subset of $(X - \{0\})$; details and useful remarks on genus theory can be found in [13].

Denote by $\mathcal{C}$ the collection of all nonempty closed and bounded subsets of X. In $\mathcal{C}$ we put the Hausdorff metric d_H given by

$$d_H(A, B) := \max\left\{\sup_{x\in A} d(x, B),\, \sup_{y\in B} d(y, A)\right\}, \quad A, B \in \mathcal{C},$$

where d denotes the usual distance on X. It is well known that $(\mathcal{C}, d_H)$ is a complete metric space. Denote by $\mathcal{D}_G$ the subcollection of $\mathcal{C}$ of all nonempty compact G-invariant subset of X. By following the ideas in [16], Section 4 the reader is invited to note that $(\mathcal{D}_G, d_H)$ is a complete metric space. By a similar way, we notice that, setting

$$\Gamma_j := \overline{\{A \in \mathcal{D}_G;\, 0 \notin A,\, \gamma_G(A) \geq j\}}^{d_H},$$

the reasoning made in [16] can be adapted to show that the space (Γ_j, d_H) is a complete metric space. The next properties can be proved by using an analogous reasoning as made in [13].

Proposition 2.15. *For every $A, B \in \Sigma$ the following facts hold:*

i) *If there exists $\phi : A \to B$, $\phi \in \Gamma_G(A)$, then $\gamma_G(A) \leq \gamma_G(B)$;*
ii) *$A \subset B$ implies that $\gamma_G(A) \leq \gamma_G(B)$;*
iii) *$\gamma_G(\overline{A \cup B}) \leq \gamma_G(A) + \gamma_G(B)$;*
iv) *$\gamma_G(\overline{A \setminus B}) \geq \gamma_G(A) - \gamma_G(B)$, provided $\gamma_G(B) < \infty$;*
v) *If A is a compact set, then $\gamma_G(A) < \infty$.*
vi) *If A is a compact set, then we have*

$$\gamma_G(N_\delta(A)) = \gamma_G(A),$$

$\delta \approx 0^+$, *where*

$$N_\delta(A) := \{x \in X :\, d(x, A) \leq \delta\}.$$

Proof. The proof of $i) - iv)$ and $vi)$ follows using the same type of argument as made in [13]. To see that $v)$ holds, for each $x \in A$ consider the G-*orbit* $Gx := \{gx;\, g \in G\}$. We may fix $\phi = \phi_x : Gx \longrightarrow Y \setminus \{0\}$ an equivariant continuous map. Since Gx is a closed subset of A, we can extend ϕ to $\tilde{\phi} : U \longrightarrow Y \setminus \{0\}$, with $U = U_x$ an equivariant neighborhood of Gx, and $\tilde{\phi} \in \Gamma_G(U)$. By repeating this procedure for each $x \in A$, by the compactness of A it is possible to find $U_1, ..., U_k$ a finite list of equivariant closed sets and equivariant maps $\tilde{\phi}_j : U_j \longrightarrow Y \setminus \{0\}$, $j \in \{1, ..., k\}$, $A \subset \bigcup_j U_j$. Arguing as in [39], Sections 2.3 and 2.4, by considering an G-invariant partition of unity subordinate to $\{U_j\}_{1\leq j\leq k}$, one can obtain $\gamma : A \longrightarrow Y^k \setminus \{0\}$, $\gamma \in \Gamma_G(A)$. So, the item $v)$ holds and the proof is now complete. □

Finally, by following the idea in [16], Proposition 4.2, we can prove the property below.

Proposition 2.16. *If $A \in \Gamma_j$ is such that $0 \notin A$, then $\gamma_G(A) \geq j$.*

3. Deformation Lemmas and minimax theorems for lower semicontinuous functionals

In order to prove the main variant of the classical Fountain Theorem given in Theorem 3.6 below, at the beginning of this section, we recall a suitable version of the standard deformation lemma valid for Szulkin-type functionals; see [16], Proposition 2.3. In addition, in Lemma 3.4 an equivariant version of the aforementioned result has been established. Finally, in the last subsection two abstract results have been proved. More precisely, [24], Proposition 2.2 due to Heinz has been extended to Szulkin-type functionals as well as a new version of [16], Corollary 4.8 is given in Theorem 3.9. For the sake of completeness, we emphasize that, on the contrary of [16], Corollary 4.8, the conclusion of Theorem 3.9 ensures that the obtained critical levels c_k satisfy $c_k \to \infty$ as $k \to \infty$. From a theoretical point of view the results obtained here complete the picture made by Szulkin in the seminal paper [16].

Let $I := \Phi + \Psi$ be a Szulkin-type functional and let us denote by I^c, K and K_c respectively, the following sets

$$I^c := I^{-1}((-\infty, c]) \text{ for every } c \in \mathbb{R},$$

$$K := \{u \in X : u \text{ is a critical point of } I\},$$

and

$$K_c := \{u \in K : I(u) = c\}.$$

3.1. Deformation lemmas and Fountain Theorem

As usual, a *deformation* is a family of maps of the form

$$\alpha_s := \alpha(s, \cdot) : W \subset X \to X, \; s \in [0, s_0]$$

such that $\alpha_0 \equiv Id|_W$, with $\alpha \in C([0, s_0] \times W, X)$ and $\mathcal{I}d|_W$ denotes the restriction of the identity map $\mathcal{I}d$ on X to W.

The next result has been proved by Szulkin in [16], Proposition 2.3.

Lemma 3.1. *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds and let N be a neighbourhood of K_c. Then, fixed $\varepsilon_0 > 0$, there is $\varepsilon \in (0, \varepsilon_0)$ such that, for each compact set $A \subset X \setminus N$ with*

$$c \leq \sup_{u \in A} I(u) \leq c + \varepsilon,$$

there exist a closed set W, with $A \subset int(W)$, and a deformation $\alpha_s : W \to X$, with $0 \leq s \leq s_0 \approx 0^+$, such that

i) $\|\alpha_s(u) - u\| \leq s, \;\; \forall u \in W$;
ii) *There is a number $\delta = \delta_\varepsilon \approx 0^+$ such that*

$$I(\alpha_s(u)) - I(u) \leq s + \delta s \quad \forall u \in W,$$

and

$$I(\alpha_s(u)) - I(u) \leq -3\varepsilon s + \delta s \quad \forall u \in W, \; I(u) \geq c - \varepsilon.$$

Moreover, by ii) it follows that

iii) $I(\alpha_s(u)) - I(u) \leq 2s, \ \ \forall u \in W;$
iv) $I(\alpha_s(u)) - I(u) \leq -2\varepsilon s, \ \ \forall u \in W, \ I(u) \geq c - \varepsilon;$
v) $\sup_{u\in A} I(\alpha_s(u)) - \sup_{u\in A} I(u) \leq -2\varepsilon s.$
vi) $I(\alpha_s(u)) - I(u) \leq 0, \ \ \forall u \in W \cap C,$ *for each closed set verifying* $C \cap K = \emptyset$.

We would like to point out that *ii)* is not contained in the statement of [16], Proposition 2.3. However, the sufficiently small constant $\delta > 0$ in *ii)* explicitly appears along the proof of the cited proposition.

Now, we are able to prove an equivariant version of Lemma 3.1 making use of the next notion that involves a functional $\Psi : X \to (-\infty, +\infty]$ as well as the action of a compact topological group G on X.

Definition 3.2. Let $\Psi : X \to (-\infty, +\infty]$ be a functional and let G be a compact topological group that acts on X. We say that Ψ is compatible with the action of G on X (briefly G-compatible) if the following inequality holds

$$\Psi\left(\int_G g^{-1}\beta(gu)d\mu\right) \leq \int_G \Psi(g^{-1}\beta(gu))d\mu, \tag{3.1}$$

for every fixed $u \in X$, $\beta \in C(Gu, X)$, where $Gu := \{gu;\, g \in G\}$ and μ denotes the normalized Haar measure on G.

The inequality in (3.1) is verified in some meaningful cases and some of them are briefly discussed in the next example.

Example 3.3. By using the usual notations, let us restrict our attention to the following cases:

1) Let $\Psi \equiv \|\cdot\| : X \to \mathbb{R}$ be the norm defined on X. Fixed $u \in X$ and a map $\beta \in C(Gu, X)$, let $\eta \in C(G, X)$ be given by $\eta(g) := g^{-1}\beta(gu)$. Next, let (β_n) be a sequence of simple functions with

$$\int_G \beta_n(g)d\mu \to \int_G \eta(g)d\mu \quad \text{and} \quad \int_G \|\beta_n(g)\|d\mu \to \int_G \|\eta(g)\|d\mu. \tag{3.2}$$

Each function β_n can be written as a finite sum:

$$\beta_n = \sum_i \chi_{A_i} v_i \quad \text{where} \quad A_i := \beta_n^{-1}(\{v_i\}) \quad \text{and} \quad v_i \in X.$$

Since μ is the normalized Haar measure on G ($\mu(G) = 1$), we have $\sum_i \mu(A_i) = 1$ and

$$\left\|\int_G \beta_n(g)d\mu\right\| = \left\|\sum_i \mu(A_i)v_i\right\| \leq \sum_i \mu(A_i)\|v_i\| = \int_G \|\beta_n(g)\|d\mu,$$

for every $n \in \mathbb{N}$. Consequently, by using (3.2) it follows that

$$\left\|\int_G \eta(g)d\mu\right\| \leq \int_G \|\eta(g)\|d\mu,$$

that is,

$$\left\|\int_G g^{-1}\beta(gu)d\mu\right\| \leq \int_G \left\|g^{-1}\beta(gu)\right\| d\mu$$

So $\|\cdot\|$ is compatible with the action of G on X. In general, the result is still true for an arbitrary convex continuous function $\Psi : X \to \mathbb{R}$.

2) Let us assume that $G := \{g_1, ..., g_k\}$ is a finite group and let $\Psi : X \to (-\infty, +\infty]$ be a convex functional. Since

$$\sum_{i=1}^{k} \mu(\{g_i\}) = 1,$$

for each $u \in X$ and $\beta \in C(Gu, X)$ the integral $\int_G g^{-1}\beta(gu)d\mu$ can be written as a finite convex combination of vectors of X. More precisely, one has

$$\int_G \beta(g)d\mu = \sum_{i=1}^{k} \mu(\{g_i\})v_i,$$

where $v_i := g_i^{-1}\beta(g_i u)$.
Then, since Ψ is convex,

$$\Psi\left(\int_G g^{-1}\beta(gu)d\mu\right) = \Psi\left(\sum_{i=1}^{k} \mu(\{g_i\})v_i\right) \leq \sum_{i=1}^{k} \mu(\{g_i\})\Psi(v_i) = \int_G \Psi(g^{-1}\beta(gu))d\mu,$$

i.e. Ψ is compatible with the action of G on X.

The next result (Equivariant Deformation Lemma) is a more general form of Corollary 2.4 in [16]. This preparatory property can be also viewed as a complement of Lemma 5.1 proved by Bereanu and Jebelean in [42].

Lemma 3.4. *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds. Assume that Φ and Ψ are G-invariant functionals and Ψ is compatible with the action of the compact topological group G on X. Moreover, suppose that G acts isometrically on X. Under the hypothesis of Lemma 3.1, the same conclusions hold with $\alpha_s : W \to X$ equivariant in A, whenever A is a G-invariant set.*

Proof. Denote by β_s the deformation of Lemma 3.1 and set

$$\alpha_s(u) := \int_G g^{-1}\beta_s(gu)d\mu. \tag{3.3}$$

Thanks to (2.6), we observe that $\alpha_s \in \Gamma_G(A)$. Now, let us prove that the function α_s verifies all the assumptions of Lemma 3.1. More precisely, since $iii)$, $iv)$ and $v)$ are a direct consequence of $ii)$, it is enough to show $i)$ and $ii)$. By Lemma 3.1, Part - $i)$, it follows that

$$\begin{aligned} \|\alpha_s(u) - u\| &= \left\| \int_G g^{-1}\beta_s(gu)d\mu - \int_G (g^{-1}g)ud\mu \right\| \\ &\leq \int_G \left\| g^{-1}(\beta_s(gu) - gu) \right\| d\mu \\ &\leq \int_G s d\mu = s \quad \text{for every } u \in W, \end{aligned} \tag{3.4}$$

i.e. α_s verifies $i)$ as claimed.

In order to prove *ii)* let us write $\beta_s(u) = u + h_s(u)$, so that $\alpha_s(u) = u + w_s(u)$, where $w_s(u) = \int_G g^{-1}h_s(gu)d\mu$. Consequently, the Taylor's formula immediately yields

$$I(\alpha_s(u)) = \{\Phi(u) + \langle\Phi'(u), w_s(u)\rangle + r(s)\} + \Psi(\alpha_s(u)), \quad \frac{r(s)}{s} = o_s(1). \tag{3.5}$$

Now, the compatibility condition of Ψ gives

$$I(\alpha_s(u)) \leq \int_G (\Phi(u) + \langle\Phi'(u), g^{-1}h_s(gu)\rangle)d\mu + \int_G \Psi(g^{-1}\beta_s(gu))d\mu + \frac{\delta}{2}s, \tag{3.6}$$

for $s \approx 0^+$. Moreover, since

$$\langle\Phi'(u), g^{-1}h_s(gu)\rangle = \langle\Phi'(gu), h_s(gu)\rangle,$$

the G-invariance of Φ and the Taylor's expansion applied to $I(\beta_s(gu))$ give

$$\begin{aligned} I(\alpha_s(u)) &\leq \int_G (\Phi(gu) + \langle\Phi'(gu), h_s(gu)\rangle)d\mu + \int_G \Psi(\beta_s(gu))d\mu + \frac{\delta}{2}s \\ &= \int_G (I(\beta_s(gu)) - \rho(s))d\mu + \frac{\delta}{2}s \leq \int_G I(\beta_s(gu))d\mu + \delta s. \end{aligned} \tag{3.7}$$

Here, we have used ρ as being the rest in the Taylor's expansion. Finally, by Lemma 3.1, Part - *ii)* and (3.7), it follows that

$$I(\alpha_s(u)) \leq \int_G I(gu)d\mu + s + 2\delta s \leq I(u) + s + 2\delta s, \tag{3.8}$$

for every $u \in W$. Similarly

$$I(\alpha_s(u)) \leq I(u) - 3\varepsilon s + 2\delta s, \quad \text{for every} \quad u \in W \quad \text{and} \quad I(u) \geq c - \varepsilon. \tag{3.9}$$

Inequalities (3.8) and (3.9) ensure that α_s satisfies *ii)* provided that δ is sufficiently small. □

For the sake of completeness, let us recall now the notion of *homotopy*. Let B be a subset of X and $f, g \in C(B, X)$. As usual, we say that f *is homotopic to* g if there is $h \in C([0,1] \times B, X)$ satisfying

$$h(0, \cdot) \equiv f \quad \text{and} \quad h(1, \cdot) \equiv g. \tag{3.10}$$

The map h is called a *homotopy* between f and g. We denote $f \approx g$ to designate that f is homotopic to g by an equivariant homotopy, *i.e.*, there exists $h \in C([0,1] \times B, X)$ satisfying (3.10) with $h(t, \cdot) \in \Gamma_G(B)$ for any $t \in [0,1]$. It easily seen that $\approx$ is an equivalence relation in $C(B, X)$.

In what follows, for each $k \in \mathbb{N}$, we set

i) $Y_k := \bigoplus_{j=1}^{k} X_j$ and $Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j}$;
ii) $B_k := \{u \in Y_k;\ \|u\| \leq \rho_k\}$ and $N_k := \{u \in Z_k;\ \|u\| = r_k\}$, with $\rho_k > r_k > 0$.

Finally, let us recall the Intersection Lemma proved in [2], Lemma 3.4; see also [3], Theorem 2 for additional comments and remarks.

Lemma 3.5. *Assume that* (G_0) *holds. If* $\gamma \in C(B_k, X) \cap \Gamma_G(B_k)$ *and* $\gamma|\partial B_k \equiv \mathcal{I}d|_{\partial B_k}$, *then* $\gamma(B_k) \cap N_k \neq \emptyset$.

Now, we are ready to show a version of the classical Fountain Theorem due to Bartsch [1] that is valid for Szulkin-type functionals.

Theorem 3.6. *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds with $I(0) = 0$. Assume that Φ and Ψ are G-invariant functionals with Ψ compatible with respect to the action of a compact topological group G on X. Moreover, assume that (G_0) holds as well as*

i) $a_k := \sup_{u\in Y_k, \|u\|=\rho_k} I(u) \leq 0$;
ii) $b_k := \inf_{u\in Z_k, \|u\|=r_k} I(u) \to \infty$,

for every $k \geq 2$. Finally, set $c_k := \inf_{\gamma\in\Theta_k} \sup_{u\in B_k} I(\gamma(u)) < \infty$, where

$$\Theta_k := \{\gamma \in \Gamma_G(B_k);\ \ \gamma|_{\partial B_k} \equiv Id|_{\partial B_k}\}. \tag{3.11}$$

Then, the functional I has infinitely many critical points (u_k) such that $I(u_k) = c_k \to \infty$.

Proof. Let us argue by contradiction. In such a case, we may assume that $K_{c_k} = \emptyset$ for some $k \geq 2$. Now, if k is large enough, by Lemma 3.5, one has $c_k \geq b_k > 0$. Thus, we are in position to apply Lemma 3.1 with $N = \emptyset$ and $\varepsilon_0 = c_k$. By fixing $\varepsilon \in (0, c_k)$ given in Lemma 3.1, we will get a contradiction. Indeed, let us define

$$\tilde{\Theta}_k := \left\{\gamma \in \Gamma_G(B_k);\ \gamma|_{\partial B_k} \approx \mathcal{I}d|_{\partial B_k} \text{ in } I^{c_k-\frac{\varepsilon}{4}} \text{ and } (I\circ\gamma)|_{\partial B_k} \leq \left(c_k - \frac{\varepsilon}{2}\right)\right\}. \tag{3.12}$$

Thanks to conditions $i)$ and $ii)$, if $\gamma \in \Theta_k$ and $u \in \partial B_k$, we derive

$$I(\gamma(u)) = I(u) \leq 0 < c_k - \frac{\varepsilon}{2} < c_k - \frac{\varepsilon}{4}.$$

Hence $\Theta_k \subset \tilde{\Theta}_k$ and

$$\tilde{c}_k := \inf_{\gamma\in\tilde{\Theta}_k} \sup_{u\in B_k} I(\gamma(u)) \leq c_k. \tag{3.13}$$

If $\tilde{c}_k < c_k$, it easily seen that there exists $\gamma_0 \in \tilde{\Theta}_k$ such that

$$m_0 := \sup_{u\in B_k} I(\gamma_0(u)) < c_k.$$

Moreover, by (3.12), there exists a homotopy $H \in C\left([0,1] \times \partial B_k, I^{c_k-\frac{\varepsilon}{4}}\right)$ such that

$$H(0,\cdot) \equiv \gamma_0|_{\partial B_k} \text{ and } H(1,\cdot) \equiv \mathcal{I}d|_{\partial B_k}, \tag{3.14}$$

with $H(t,\cdot)$ equivariant for every $t \in [0,1]$. Since B_k is a ball of radius ρ_k each point $u \in B_k$ can be represented as $u \equiv (s, \tilde{u})$, $s \in [0, \rho_k]$, $\tilde{u} \in \partial B_k$; polar coordinates of u. Hence, if $u \in \partial B_k$ then $u \equiv (\rho_k, u)$. Now, define $\gamma_1 : B_K \to X$ by

$$\gamma_1(s,v) := \begin{cases} \gamma_0(s,v) & s \in \left[0,\ \frac{\rho_k}{2}\right] \\ H\left(\frac{2}{\rho_k}s - 1, v\right) & s \in \left[\frac{\rho_k}{2},\ \rho_k\right]. \end{cases} \tag{3.15}$$

According to (3.14), when $s = \rho_k/2$ it holds $H(2s/\rho_k - 1, \cdot) = H(0, \cdot) \equiv \gamma_0$, which assures that γ_1 is well defined and $\gamma_1 \in \Gamma_G(B_k)$, since γ_0 and $H(t, \cdot)$ are equivariants. By using again (3.14), if $u \in \partial B_k$ one has

$$\gamma_1(u) = H(1, u) = \mathcal{I}d|_{\partial B_k}(u),$$

so that $\gamma_1 \in \Theta_k$, and

$$\sup_{u \in B_k} I(\gamma_1(u)) \leq \max\left\{m_0, c_k - \frac{\varepsilon}{4}\right\} < c_k,$$

against the definition of c_k. This contradiction assures that $\tilde{c}_k = c_k$ in (3.13). Consequently, we can work with $\tilde{\Theta}_k$ instead Θ_k.

Now, let us observe that the collection $\tilde{\Theta}_k$ is a (complete) metric subspace of the complete metric space $C(B_k, X)$ endowed by $d(f, g) := \sup_{u \in B_k} \|f(u) - g(u)\|$. Indeed, suppose that $\gamma_n \to \gamma$ in $C(B_k, X)$ with $\gamma_n \in \tilde{\Theta}_k$. The semicontinuity of I yields

$$I(\gamma(u)) \leq \liminf I(\gamma_n(u)) \leq c_k - \frac{\varepsilon}{2}, \ \ u \in \partial B_k.$$

Moreover, the action properties give

$$\gamma(gu) = \lim \gamma_n(gu) = g \lim \gamma_n(u) \ \ \forall u \in B_k, \ \forall g \in G,$$

so that $\gamma \in \Gamma_G(B_k)$. On the other hand, thanks to the continuity of Φ, it is possible to find a sequence of positive numbers $\tau_n = o_n(1)$ such that

$$\Phi(t\gamma_n(u) + (1-t)\gamma(u)) \leq t\Phi(\gamma_n(u)) + (1-t)\Phi(\gamma(u)) + \tau_n \ \ \forall u \in \partial B_k, \ \forall t \in [0,1]. \tag{3.16}$$

More precisely $\tau_n := 2\max\{\tau_n^1, \tau_n^2\}$ with

$$\tau_n^1 := \sup_{u \in B_k,\, t \in [0,1]} |\Phi(t\gamma_n(u) + (1-t)\gamma(u)) - \Phi(\gamma(u))|$$

and

$$\tau_n^2 := \sup_{u \in B_k} |\Phi(\gamma_n(u)) - \Phi(\gamma(u))|.$$

Inequality (3.16) associated to the convexity of Ψ implies

$$\begin{aligned} I(t\gamma_n(u) + (1-t)\gamma(u)) &\leq tI(\gamma_n(u)) + (1-t)I(\gamma(u)) + \tau_n \\ &\leq c_k - \frac{\varepsilon}{2} + \tau_n \leq c_k - \frac{\varepsilon}{4} \quad \forall u \in \partial B_k, \ \forall t \in [0,1], \end{aligned} \tag{3.17}$$

for n sufficiently large.

Thus $\gamma_n|_{\partial B_k} \approx \gamma|_{\partial B_k}$ *via* the equivariant homotopy $F(t, \cdot) := t\gamma_n(\cdot) + (1-t)\gamma(\cdot)$. Consequently $\gamma|_{\partial B_k} \approx \mathcal{I}d|_{\partial B_k}$, so that $\tilde{\Theta}_k$ is a complete metric subspace of $C(B_k, X)$ as claimed. Hence, the conclusion follows arguing as in [16], Theorem 3.2.

Now, since I is a lower semicontinuous functional, by using [16], Lemma 3.1 and the definition of c_k, we have that the functional $\varphi : \tilde{\Theta}_k \to (-\infty, +\infty]$ defined by

$$\varphi(\gamma) := \sup_{u \in B_k} I(\gamma(u))$$

is lower semicontinuous and bounded from below. Since $\tilde{\Theta}_k$ is a complete metric space, we can apply the classical Ekeland's Variational Principle recalled in Theorem 2.6, to the functional φ with $\delta = \varepsilon$ and $\tau = 1$. Then, we may take $\gamma \in \tilde{\Theta}_k$ such that $\varphi(\gamma) \leq c_k + \varepsilon$, and

$$\varphi(\eta) - \varphi(\gamma) \geq -\varepsilon d(\eta, \gamma) \quad \forall \eta \in \tilde{\Theta}_k. \tag{3.18}$$

It follows that $A := \gamma(B_k)$ is a compact equivariant set with

$$\sup_{v \in A} I(v) = \sup_{u \in B_k} I(\gamma(u)) \leq c_k + \varepsilon,$$

so that A verifies all the assumptions of the equivariant deformation lemma given in Lemma 3.4. Hence, let $\eta := \alpha_s \circ \gamma$, where α_s is the equivariant deformation given in Lemma 3.4 and let us prove that $\eta \in \tilde{\Theta}_k$ for $s \approx 0^+$. Indeed $\eta \in \Gamma_G(B_k)$ and if $u \in \partial B_k$, by $iii)$ and $iv)$ in Lemma 3.1, it follows that

$$\begin{cases} I(\eta(u)) = I(\alpha_s(\gamma(u))) \leq I(\gamma(u)) \leq c_k - \dfrac{\varepsilon}{2}, & I(\gamma(u)) \in \left(c_k - \varepsilon, c_k - \dfrac{\varepsilon}{2}\right] \\ I(\eta(u)) \leq I(\gamma(u)) + 2s \leq c_k - \dfrac{\varepsilon}{2}, & I(u) \leq c_k - \varepsilon, \end{cases} \tag{3.19}$$

so that

$$(I \circ \eta)|_{\partial B_k} \leq c_k - \frac{\varepsilon}{2}.$$

Now, since $\alpha_s \circ \gamma$ can be viewed as an equivariant homotopy such that $(\alpha_s \circ \gamma)|_{\partial B_k} \approx \gamma|_{\partial B_k}$ in $I^{c_k - \frac{\varepsilon}{2}}$, it follows that

$$\eta|_{\partial B_k} \approx (\alpha_s \circ \gamma)|_{\partial B_k} \approx \mathcal{I}d|_{\partial B_k} \quad \text{in} \quad I^{c_k - \frac{\varepsilon}{4}},$$

taking into account that $\gamma|_{\partial B_k} \approx \mathcal{I}d|_{\partial B_k}$.

Finally, since $\eta \in \tilde{\Theta}_k$, by using $i)$ and $v)$ of Lemma 3.1 and (3.18), one has

$$\begin{aligned} -\varepsilon s &\leq \varphi(\eta) - \varphi(\gamma) \\ &= \sup_{u \in B_k} I(\alpha_s(\gamma(u))) - \sup_{u \in B_k} I(\gamma(u)) \leq -2\varepsilon s, \end{aligned} \tag{3.20}$$

which is an absurd. Hence, there exists a positive integer k_0 such that $K_{c_k} \neq \emptyset$ for $k \geq k_0$. The proof is complete since, by construction, one clearly has $c_k \geq b_k$. □

3.2. Minimax results involving the G-index theory

Let A be a compact set of a real Banach space X and $\delta > 0$. Let us recall the notation

$$N_\delta(A) := \{x \in X : d(x, A) \leq \delta\}.$$

The next technical result will be useful in the sequel.

Lemma 3.7. *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds. Moreover, let (c_j) be a real sequence such that $c_j \to c \in \mathbb{R}$. Then, given $\delta > 0$, there exists $j_0 \in \mathbb{N}$ such that*

$$K_{c_j} \subset N_\delta(K_c),$$

for every $j \geq j_0$.

Proof. Arguing by contradiction, assume that there exist a subsequence (c_{j_k}) of (c_j), a number $\delta_0 > 0$, and a sequence (u_k) with $u_k \in K_{c_{j_k}}$ such that

$$d(u_k, K_c) > \delta_0, \ \ \forall k \in \mathbb{N}. \tag{3.21}$$

The definition of $K_{c_{j_k}}$ immediately yields

$$\langle \Phi'(u_k), v - u_k \rangle + \Psi(v) - \Psi(u_k) \geq 0, \quad \forall v \in X, \tag{3.22}$$

as well as

$$I(u_k) = c_{j_k} \to c,$$

so that (u_k) is a $(\mathrm{PS})_\mathrm{c}$ sequence for the functional I. Now, the (PS) condition ensures the existence of $u_0 \in X$ and a subsequence of (u_k), still denoted again by (u_k), such that

$$u_k \to u_0 \quad \text{in} \quad X.$$

Now, taking $v = u_0$ in (3.22), we get $\limsup \Psi(u_k) \leq \Psi(u_0)$. The last inequality in addition to the semicontinuity property of Ψ gives $\lim \Psi(u_k) = \Psi(u_0)$, so that $u_0 \in K_c$. Hence $d(u_k, K_c) \to 0$ as $k \to \infty$, against (3.21). □

The next result extends [24], Proposition 2.2 to Szulkin-type functionals.

Theorem 3.8. *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds and such that $I(0) = 0$. Assume that Φ and Ψ are G-invariant functionals with Ψ compatible with respect to the action of a compact topological group G on X. Finally, for every $j \in \mathbb{N}$, set*

$$c_j := \inf_{A \in \Gamma_j} \sup_{u \in A} I(u),$$

and assume that the following conditions are verified:

- *i)* $-\infty < c_j$ *for every* $j \in \mathbb{N}$;
- *ii)* *Given* $j \in \mathbb{N}$, *there exists* $A \in \Sigma$ *such that*

$$\gamma_G(A) \geq j \quad \text{and} \quad \sup_{u \in A} I(u) < 0,$$

where $A \neq \emptyset$ is a compact set.

Then, the numbers c_j are negative critical values of I and $c_j \to 0$ as $j \to \infty$.

Proof. We first notice that conditions *i)* and *ii)* imply that $-\infty < c_j < 0$. Now, a careful analysis of the arguments in [16], Theorem 4.3 ensures that the sequence (c_j) consists of critical values of I. In fact, the proof of [16], Theorem 4.3 only depends on the properties *i)* − *vi)* in Proposition 2.15 with $G = \mathbb{Z}_2$ and where γ_G coincides with the genus of a symmetric set as in Remark 2.14. In view of Proposition 2.15, the argument used in [16], Theorem 4.3 can be adopted in our case. It remains to show that $c_j \to 0$ as $j \to \infty$. To this aim let us observe that, the definition of c_j yields

$$c_j \leq c_{j+1}, \;\; \forall j \in \mathbb{N}.$$

Arguing by contradiction, if $c_j \nrightarrow 0$ for $j \to \infty$, there exists $c < 0$ such that $c_j \to c$. The (PS) condition ensures that K_c is compact. Moreover, the assumptions on I yields that K_c is G-invariant and $0 \notin K_c$. Thereby, $K_c \in \Sigma$ and, by following the idea of Lemma 3.7, as $c_j \to c$ and $K_{c_j} \neq \emptyset$, one has that $K_c \neq \emptyset$. By $vi)$ of Proposition 2.15 there is $\delta > 0$ such that $\gamma_G(N_{2\delta}(K_c)) = \gamma_G(K_c)$; note that $N_\delta(K_c) \neq \emptyset$. Since K_c is a compact set, by Proposition 2.15-item $v)$, we may fix $p \in \mathbb{N}$ with $\gamma_G(K_c) = p$.

Define, for each $j \in \mathbb{N}$,

$$\begin{aligned} \varphi_j : \Gamma_j &\to \;\; (-\infty, +\infty] \\ A &\longmapsto \varphi_j(A) := \sup_{u \in A} I(u). \end{aligned}$$

Clearly φ_j is lower semicontinuous functional since I is too. Set

$$\varepsilon_0 := \min\{1, \delta, -c\}$$

and take $\varepsilon \in (0, \varepsilon_0)$ as in Lemma 3.1. Now, let $A_1 \in \Gamma_{j+p}$ be such that

$$c_{j+p} \leq \varphi_{j+p}(A_1) < c_{j+p} + \frac{\varepsilon^2}{2}.$$

Since $c_j \to c$, it follows that, for a convenient $j_0 \in \mathbb{N}$,

$$\varphi_{j+p}(A_1) < c_{j+p} + \frac{\varepsilon^2}{2} \leq c + \frac{\varepsilon^2}{2} \leq c_j + \varepsilon^2 < c_j + \varepsilon < 0,$$

for $j \geq j_0$. Hence, by fixing $j = j_0$, we get $0 \notin A_1$ and $\gamma_G(A_1) \geq j_0 + p$ by Proposition 2.16. If we set $A_2 := A_1 \setminus N_{2\delta}(K_c)$ we also have

$$\sup_{u \in A_2} I(u) \leq \sup_{u \in A_1} I(u) < c_{j_0} + \varepsilon^2 < 0,$$

so that $0 \notin A_2$ and $\gamma_G(A_2) \geq (j_0 + p) - p = j_0$ by Proposition 2.15, Part - $iv)$. Consequently $A_2 \in \Gamma_{j_0}$. Now, Theorem 2.6 applied to the function $\varphi_{j_0} : \Gamma_{j_0} \to (-\infty, +\infty]$ (note that Γ_{j_0} is complete) yields the existence of $A \in \Gamma_{j_0}$ such that

$$c_{j_0} \leq \sup_{u \in A} I(u) = \varphi_{j_0}(A) \leq \varphi_{j_0}(A_2) < c_{j_0} + \varepsilon, \;\; d_H(A, A_2) \leq \varepsilon$$

as well as

$$\varphi_{j_0}(B) - \varphi_{j_0}(A) \geq -\varepsilon d_H(A, B) \;\; \forall B \in \Gamma_{j_0}. \tag{3.23}$$

Since Lemma 3.7 gives $K_{c_{j_0}} \subset N_\delta(K_c)$ for $j_0 \approx \infty$, by setting $N = N_\delta(K_c)$ we derive $A \cap N = \emptyset$, taking into account that $\varepsilon < \delta$. These informations ensure that A, N and $K_{c_{j_0}}$ verify the hypothesis of the deformation result given in Lemma 3.1.

Thus by Lemma 3.4 the existence of an equivariant deformation α_s is obtained. In this way, if we set $B := \alpha_s(A)$, on account of Proposition 2.15, Part - i), one has $B \in \Gamma_{j_0}$. Now, combining the properties of α_s with (3.23) we derive the contradiction

$$-2\varepsilon s \geq \varphi(B) - \varphi(A) \geq -\varepsilon s.$$

This completes the proof. □

The last result can be viewed as a complement of Corollary 4.8 proved by Szulkin in [16].

Theorem 3.9. *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds and such that $I(0) = 0$. Assume that Φ and Ψ are G-invariant functionals with Ψ compatible with respect to the action of a compact topological group G on X and that the condition (G_0) holds.*

Finally, assume that there exist subspaces Y, Z of X such that $X = Y \oplus Z$, $\dim Y < \infty$, Z is closed and

- *i) There are numbers $r, \rho > 0$ such that $I|_{\partial B_r(0)\cap Z} \geq \rho$;*
- *ii) For each positive integer k there is a k-dimensional subspace X_k of X such that $I(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in X_k$.*

Then I has a sequence critical values (c_j) with $c_j \to \infty$ as $j \to \infty$.

In order to prove Theorem 3.9 some notations are introduced. To this aim, let us fix, given $k \in \mathbb{N}$, the set $M_k := \overline{B}_{R_k}(0) \cap X_k$ with $R_k > r$ and $I|_{\partial M_k} \leq -c_0$, for some fixed $c_0 > 0$ (this construction is possible in view of the assumption ii) in Thm. 3.9). Now, let us define the following sets

$$\mathcal{F} := \left\{ \eta \in \Gamma_G(M_k) : \begin{array}{c} \text{there exists}\, d = d(\eta) > 0,\ \ \eta|_{\partial M_k} \approx \mathcal{I}d|_{\partial M_k} \text{ in } I^{-d} \subset X \setminus \{0\} \\ \text{by an equivariant homotopy.} \end{array} \right\}.,$$

for each $j \in \mathbb{N}$ and $k \geq j$,

$$\tilde{\Lambda}_j^k := \left\{ \begin{array}{l} \eta(M_k \setminus U) : \eta \in \mathcal{F},\, U \text{ is } G\text{-invariant and open in } M_k,\, U \cap \partial M_k = \emptyset, \\ \qquad\qquad\qquad \text{with}\, \gamma_G(W) \leq k-j, \text{ for } W \in \Sigma,\ W \subset U. \end{array} \right\}$$

and

$$\tilde{\Lambda}_j := \bigcup_{k \geq j} \tilde{\Lambda}_j^k.$$

Finally, for each $j \in \mathbb{N}$, we fix

$$\Lambda_j := \{A \subset X : A \text{ is compact, } G\text{-invariant and for each open}\, U \supset A, \text{ there is}\, A_0 \in \tilde{\Lambda}_j,\ A_0 \subset U\}.$$

and

$$c_j := \inf_{A \in \Lambda_j} \sup_{u \in A} I(u).$$

By applying the same arguments used in [16], Theorem 4.4, Lemma 4.6 we can prove that Λ_j verifies the properties $i) - v)$ below (note that, in view of the Proposition 2.15, the arguments in [16], Theorem 4.4 can be applied to the G-index γ_G).

Lemma 3.10. *The sets Λ_j defined above satisfy the following claims:*

i) (Λ_j, d_H) *is a complete metric space;*
ii) $c_j \geq \rho$, *for all* $j > \text{dim}Y$;
iii) $\Lambda_{j+1} \subset \Lambda_j$;
iv) *Let* $A \in \Lambda_j$ *and* W *be a closed* G*-invariant set containing* A *in its interior. Moreover, if* $\alpha : W \to X$ *is an equivariant mapping such that*

$$\alpha|_{W\cap I^{-d}} \approx \mathcal{I}d|_{W\cap I^{-d}} \ \ in \ I^{-d},$$

for some $d > 0$, *by an equivariant homotopy, then* $\alpha(A) \in \Lambda_j$;
v) *For each compact* B *with* $B \in \Sigma$, $\gamma_G(B) \leq p$, $I|_B > 0$, *there exists a number* $\delta_0 > 0$ *such that* $A \setminus int(N_\delta(B)) \in \Lambda_j$, *for* $A \in \Lambda_{j+p}$, $\delta \in (0, \delta_0)$.

Part - *v)* in Lemma 3.10 is different with respect to the statement of [16], Lemma 4.6. However, the main assertion is a direct consequence of the arguments proved there.

Proof of Theorem 3.9. The first part of the proof can be derived by using similar arguments given in [16], Corollary 4.8. Hence, it remains to show that $c_j \to \infty$ as $j \to \infty$. Now, by Lemma 3.10, Part - *iii)*, it follows that

$$c_j \leq c_{j+1} \quad \forall j \in \mathbb{N}.$$

Thus, if $c_j \nrightarrow \infty$, by *ii)* of last lemma, there exists $c > 0$ such that $c_j \to c$. Arguing as in the proof of Theorem 3.8, we deduce that K_c is a compact G-invariant set with $0 \notin K_c$ and $K_c \neq \emptyset$. Hence, for a convenient $\delta > 0$, by Proposition 2.15-item *v)*, one has $\gamma_G(N_{2\delta}(K_c)) = \gamma_G(K_c) =: p \in \mathbb{N}$. Now, set $\varepsilon_0 := \min\{1, \delta\}$, take $\varepsilon \in (0, \varepsilon_0)$ as in Lemma 3.1 and define

$$\begin{aligned} \varphi_j : \Lambda_j &\to \quad (-\infty, +\infty] \\ A &\longmapsto \varphi(A) := \sup_{u\in A} I(u). \end{aligned}$$

Clearly φ_j is a lower semicontinuous functional that is bounded from below for every $j \in \mathbb{N}$. Hence, let $A_1 \in \Lambda_{j+p}$ be such that

$$\varphi_{j+p}(A_1) < c_{j+p} + \frac{\varepsilon^2}{2}.$$

Consequently, for some $j_0 \in \mathbb{N}$,

$$\varphi_{j+p}(A_1) < c_j + \varepsilon,$$

for $j \geq j_0$. Now, if $A_2 := \overline{A_1 \setminus \text{int}(N_{2\delta}(K_c))}$, by Part - *v)* of Lemma 3.10 we have $A_2 \in \Lambda_{j_0}$ and $\varphi_{j_0}(A_2) \leq \varphi_{j_0}(A_1)$. Moreover, by Theorem 2.6, there exists $A \in \Lambda_{j_0}$ such that

$$\varphi_{j_0}(A) \leq \varphi_{j_0}(A_2) < c_{j_0} + \varepsilon \quad d_H(A, A_2) \leq \varepsilon$$

as well as

$$\varphi_{j_0}(B) - \varphi_{j_0}(A) \geq -\varepsilon d_H(B, A) \quad \forall B \in \Lambda_{j_0}. \tag{3.24}$$

If we set $N := N_\delta(K_c)$, Lemma 3.7 implies that $K_{c_{j_0}} \subset N$ if $j_0 \approx \infty$. The definition of ε_0 yields $A \cap N = \emptyset$ and

$$c_{j_0} \leq \sup_{u \in A} I(u) < c_{j_0} + \varepsilon.$$

Then, we can apply Lemma 3.4 to obtain an equivariant deformation α_s, $s \in [0, s_0]$, for some $s_0 \approx 0^+$. If we set $B := \alpha_s(A)$, by Part - *iii*) of Lemma 3.1, it holds

$$I(\alpha_s(u)) \leq I(u) + 2s_0, \ \ \forall u \in A,$$

so $\alpha_s(u) \in I^{-d+2s_0}$, provided $u \in I^{-d}$. By setting $-\tilde{d} := -d + 2s_0$, we know that $-\tilde{d} < 0$ for a suitable s_0. Thus, the deformation α_s can be viewed as an equivariant homotopy in $I^{-\tilde{d}}$ verifying the Part - *iv*) of Lemma 3.10 and so one has $B \in \Lambda_{j_0}$. Finally, a contradiction is achieved by replacing B in (3.24) and arguing as in the proof of Theorem 3.8. □

4. Some applications to elliptic problems

In this section we illustrate how the abstract results of the previous section can be applied to establish the existence of infinitely many solutions for some classes of elliptic problems.

4.1. A logarithmic variational inclusion problem

In this subsection we study the existence of infinitely many solutions for the logarithmic inclusion problem

$$\begin{cases} -\Delta u + u + \partial F(x, u) \ni u \log u^2, & \text{in } \mathbb{R}^N \\ \ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_1}$$

where $F(x,t) := \displaystyle\int_0^t f(x,s)ds$ is a convex locally Lipschitz function with $F(x,\cdot) \geq 0$ for every $x \in \mathbb{R}^N$. We also require that the nonlinear term f is a N-measurable function that satisfies the following technical conditions:

(f_1) There is a nonnegative and radial function $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$|f(x,t)| \leq h(x)|t|, \ \ \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall t \in \mathbb{R}.$$

(f_2) $f(x,-t) = -f(x,t)$ and $f(|x|,t) = f(x,t)$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.
(f_3) There is $C > 0$ such that for any $\eta_t \in \partial F(x,t)$ it holds

$$F(x,u) - \frac{1}{2}\eta_t t \geq -Ch(x), \ \ \text{a.e } x \in \mathbb{R}^N, \forall\ t \in \mathbb{R}.$$

Example 4.1 (A function satisfying $(f_1) - (f_3)$)**.** Consider $F(x,t) := h(x)\displaystyle\int_0^t H(|s| - a)s\,ds$, where $a > 0$, $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is nonnegative and radial and H is the Heaviside function, *i.e.*,

$$H(t) := \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases}$$

In this case, we notice that

$$\partial F(x,s) = h(x)\begin{cases} \{s\} & |s| > a, \\ [-a,0] & s = -a, \\ [0,a] & s = a, \\ \{0\} & |s| < a. \end{cases}$$

Direct computations ensure that $(f_1) - (f_3)$ are verified.

Now, consider the energy functional associated to problem (P_1) given by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2) + \int_{\mathbb{R}^N} F(x,u) - \int_{\mathbb{R}^N} L(u), \;\; u \in H^1(\mathbb{R}^N),$$

where

$$L(t) := -\frac{t^2}{2} + \frac{t^2 \log t^2}{2} \quad \forall\, t \in \mathbb{R}.$$

Hereafter, we make use of the approach given in [17–19] to decompose I as a sum of a C^1 functional and a convex lower semicontinuous functional. To this aim, fixed $\delta > 0$ sufficiently small, we set

$$F_1(s) := \begin{cases} 0 & s = 0 \\ -\frac{1}{2}s^2 \log s^2 & 0 < |s| < \delta \\ -\frac{1}{2}s^2(\log \delta^2 + 3) + 2\delta|s| - \frac{\delta^2}{2} & |s| \geq \delta \end{cases}$$

and

$$F_2(s) := \begin{cases} 0 & s = 0 \\ -\frac{1}{2}s^2 \log\left(\frac{s^2}{\delta^2}\right) + 2\delta|s| - \frac{3}{2}s^2 - \frac{\delta^2}{2} & |s| \geq \delta \end{cases}$$

for every $s \in \mathbb{R}$. Therefore

$$F_2(s) - F_1(s) = \frac{1}{2}s^2 \log s^2 \;\; \forall s \in \mathbb{R},$$

and

$$I(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F(x,u) + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u) \;\; u \in H^1(\mathbb{R}^N), \tag{4.1}$$

where $\|\cdot\|$ denotes the norm in $H^1(\mathbb{R}^N)$ induced by the inner product given by

$$\langle u, v\rangle := \int_{\mathbb{R}^N}(\nabla u \cdot \nabla v + 2uv), \;\; \forall u, v \in H^1(\mathbb{R}^N).$$

According to [17], Sect. 2 and [19], Section 2 the functions F_1 and F_2 satisfy the following conditions:

(A_1) F_1 is an even function with $F_1'(s)s \geq 0$ and $F_1 \geq 0$. Moreover $F_1 \in C^1(\mathbb{R},\mathbb{R})$ and convex provided that $\delta \approx 0^+$;

(A_2) $F_2 \in C^1(\mathbb{R},\mathbb{R})$ and for each $p \in (2, 2^*)$, there exists $C = C_p > 0$ such that

$$|F_2'(s)| \leq C|s|^{p-1} \quad \forall s \in \mathbb{R}.$$

Now, by (A_1) and (A_2), it is easily seen that I is a Szulkin-type functional with

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F_2(u)$$

and

$$\Psi(u) := \int_{\mathbb{R}^N} F_1(u) + \int_{\mathbb{R}^N} F(x,u).$$

We notice that $\Psi = \Psi_1 + \Psi_2$, where

$$\Psi_1(u) := \int_{\mathbb{R}^N} F_1(u) \text{ and } \Psi_2(u) := \int_{\mathbb{R}^N} F(x,u).$$

Direct arguments and [17], Lemma 2.1 ensure the validity of the next result.

Lemma 4.2. *Let $\Psi_1 : H^1(\mathbb{R}^N) \to (-\infty, +\infty]$ be the functional defined above. Then*

i) $D(I) = D(\Psi_1)$, that is $I(u) < \infty$ if and only if $\Psi_1(u) < \infty$.

ii) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with regular boundary. Then the functional

$$\tilde{\Psi}_1(u) = \int_{\Omega} F_1(u) \tag{4.2}$$

belongs to $C^1(H^1(\Omega),\mathbb{R})$.

Moreover, according to [12], the structural conditions on the function F assure that the functional $\Psi_2 : H^1(\mathbb{R}^N) \to \mathbb{R}$ is convex and lower semicontinuous as well as $\Psi_2 \in \mathrm{Lip}_{\mathrm{loc}}(H^1(\mathbb{R}^N),\mathbb{R})$.

From now on, for each $u \in H^1(\mathbb{R}^N)$, let us consider the functional φ_1^u defined by

$$\langle \varphi_1^u, v\rangle := \int_{\mathbb{R}^N} F_1'(u)v, \quad \forall v \in C_0^\infty(\mathbb{R}^N). \tag{4.3}$$

If

$$\|\varphi_1^u\| := \sup_{v\in C_0^\infty(\mathbb{R}^N),\, \|v\|\leq 1} \langle \varphi_1^u, v\rangle < \infty,$$

then φ_1^u can be extended to a continuous linear functional on $H^1(\mathbb{R}^N)$.

Moreover, if $\tilde{I} : H^1(\mathbb{R}^N) \to (-\infty, +\infty]$ denotes the functional given by

$$\tilde{I}(u) := \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u),$$

then $\tilde{I}$ is a Szulkin-type functional and $I = \tilde{I} + \Psi_2$.

By [17] the following lemma holds.

Lemma 4.3. *If $u \in D(\tilde{I})$ and $\|\varphi_1^u\| < \infty$ then there is a unique functional in $\partial \tilde{I}(u)$, denoted by $\tilde{I}'(u)$, such that*

$$\tilde{I}'(u)(v) = \langle \Phi'(u), v\rangle + \int_{\mathbb{R}^N} F_1'(u)v \quad \forall v \in C_0^\infty(\mathbb{R}^N). \tag{4.4}$$

Furthermore, $F_1'(u)u \in L^1(\mathbb{R}^N)$, and

$$\tilde{I}'(u)(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \int_{\mathbb{R}^N} u^2 \log u^2, \tag{4.5}$$

as well as

$$\tilde{I}(u) - \frac{1}{2}\tilde{I}'(u)(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2. \tag{4.6}$$

Remark 4.4. Lemma 4.3 remains valid if we take $\tilde{J} := \tilde{I}|_{H^1_{\text{rad}}(\mathbb{R}^N)}$. Indeed, the arguments used in [17],] Lemma 2.2 and of Corollary 2.1 can be adapted to the radial space $H^1_{\text{rad}}(\mathbb{R}^N)$ by taking $\{\varphi_1^u\} \subset H^1_{\text{rad}}(\mathbb{R}^N)$ and

$$\langle \varphi_1^u, v\rangle = \int_{\mathbb{R}^N} F_1'(u)v \quad v \in C^\infty_{0,\,\text{rad}}(\mathbb{R}^N).$$

The notion of solution for problem (P_1) requires some comments. To this aim, let us define the functions

$$\underline{f}(x,t) := \lim_{r\downarrow 0} \operatorname{essinf}\{f(x,s) : |s-t| < r\} \tag{4.7}$$

and

$$\overline{f}(x,t) := \lim_{r\downarrow 0} \operatorname{esssup}\{f(x,s) : |s-t| < r\}. \tag{4.8}$$

According to [12], Section 2 if $F(x,t) = \displaystyle\int_0^t f(x,s)\,ds$, then

$$\partial F(x,t) = [\underline{f}(x,t), \overline{f}(x,t)].$$

The above remark makes sense to the following notion.

Definition 4.5. A function $u \in H^1(\mathbb{R}^N)$ is said to be a solution of (P_1) if $u^2\log u^2 \in L^1(\mathbb{R}^N)$ and there exists $\rho \in L^2(\mathbb{R}^N)$ such that

$$\rho(x) \in [\underline{f}(x,u(x)), \overline{f}(x,u(x))] \quad \text{a.e in} \quad \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi + u\phi) + \int_{\mathbb{R}^N} \rho\phi = \int_{\mathbb{R}^N} u \log u^2 \phi, \;\; \forall \phi \in C_0^\infty(\mathbb{R}^N). \tag{4.9}$$

A proof of the next technical result can be found in [43], Lemma 4.1.

Lemma 4.6. *The functions $\underline{f}$ and $\overline{f}$ are N-measurable functions, $\Psi_2 \in \mathrm{Lip}_{\mathrm{loc}}(L^2(\mathbb{R}^N), \mathbb{R})$ and*

$$\partial \Psi_2(u) \subseteq \partial F(x,u) = [\underline{f}(x,u(x)), \overline{f}(x,u(x))], \tag{4.10}$$

for every $u \in L^2(\mathbb{R}^N)$.

The inclusion in (4.10) has the following meaning: for each $\eta \in \partial\Psi_2(u)$ there is a function $\tilde{\eta} \in L^2(\mathbb{R}^N)$ such that

i) $\eta(v) = \int_{\mathbb{R}^N} \tilde{\eta} v \quad \forall v \in L^2(\mathbb{R}^N)$;
ii) $\tilde{\eta}(x) \in [\underline{f}(x,u(x)), \overline{f}(x,u(x))]$ a.e. in $\mathbb{R}^N$.

Our next step is to prove that the critical points of I in the sense given in Definition 2.5 are solutions of (P_1).

Lemma 4.7. *Every critical point of the functional I is a solution of (P_1).*

Proof. Suppose that $u \in D(I)$ is a critical point of I, that is

$$\begin{aligned} \int_{\mathbb{R}^N} (\nabla u \cdot \nabla(v-u) + 2u(v-u)) + \int_{\mathbb{R}^N} (F(x,v) - F(x,u)) \\ \geq \int_{\mathbb{R}^N} F_2'(u)(v-u) - \int_{\mathbb{R}^N} (F_1(v) - F_1(u)), \end{aligned} \tag{4.11}$$

for every $v \in H^1(\mathbb{R}^N)$. The last sentence means that the functional $-\Phi'(u)$ belongs to $\partial\Psi(u)$. Hence, by choosing $v = u + t\phi$, $t > 0$, $\phi \in C_0^\infty(\mathbb{R}^N)$, we find

$$\int_{\mathbb{R}^N} \frac{1}{t}(F(x,u+t\phi) - F(x,u)) + \int_{\mathbb{R}^N} \frac{1}{t}(F_1(u+t\phi) - F_1(u)) \geq \langle -\Phi'(u), \phi \rangle, \tag{4.12}$$

which is equivalent to

$$\frac{1}{t}[\Psi_2(u+t\phi) - \Psi_2(u)] + \int_{\mathbb{R}^N} \frac{1}{t}(F_1(u+t\phi) - F_1(u)) \geq \langle -\Phi'(u), \phi \rangle. \tag{4.13}$$

As Ψ_2 is convex, when $t \to 0^+$, the Lemmas 2.4 and 4.2 imply that

$$\Psi_2^\circ(u,\phi) + \int_{\mathbb{R}^N} F_1'(u)\phi \geq \langle -\Phi'(u), \phi \rangle. \tag{4.14}$$

Replacing ϕ with $-\phi$ in (4.14) and by using Lemma 2.4 it follows that

$$\Psi_2^\circ(u,-\phi) - \langle \Phi'(u), \phi \rangle \geq \int_{\mathbb{R}^N} F_1'(u)\phi. \tag{4.15}$$

Then, according to the notation introduced in (4.3), one has

$$\Psi_2^\circ(u,-\phi) - \langle \Phi'(u), \phi \rangle \geq \langle \varphi_1^u, \phi \rangle. \tag{4.16}$$

The following claim will be crucial in the rest of the proof.

Claim 4.8. $\sup_{\phi \in C_0^\infty(\mathbb{R}^N),\, \|\phi\| \leq 1} \Psi_2^\circ(u,\phi) < \infty.$

Indeed, by Lemma 4.6, for each $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\|\phi\| \leq 1$, there is $\tilde{\eta}_\phi \in L^2(\mathbb{R}^N)$ such that $\tilde{\eta}_\phi(x) \in [\underline{f}(x,u(x)), \overline{f}(x,u(x))]$ and

$$\Psi_2^\circ(u,\phi) = \int_{\mathbb{R}^N} \tilde{\eta}_\phi \phi.$$

Now, by (f_1), there exists a constant $C := C(u,h) > 0$, independent of ϕ, such that

$$\left| \int_{\mathbb{R}^N} \tilde{\eta}_\phi \phi \right| \leq C\|\phi\|.$$

The above inequality ensures our assertion.

Now, Claim 4.8 in addition to inequality (4.16) ensures that

$$\sup_{\phi \in C_0^\infty(\mathbb{R}^N),\, \|\phi\| \leq 1} \langle \varphi_1^u, \phi \rangle < \infty.$$

Consequently, the classical Hahn-Banach's extension theorem ensures that the functional φ_1 admits an extension, still denoted by φ_1, to a continuous linear functional on $H^1(\mathbb{R}^N)$. Moreover, Lemma 2.1, inequality (4.14) and the density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$ yield

$$\langle -\Phi'(u) - \varphi_1^u, v \rangle \leq \Psi_2^\circ(u,v) \quad \forall v \in H^1(\mathbb{R}^N), \tag{4.17}$$

that is,

$$-\Phi'(u) - \varphi_1^u \in \partial \Psi_2(u). \tag{4.18}$$

Thus, there exists $\varphi_2 \in \partial\Psi_2(u)$ such that $-\Phi'(u) - \varphi_1^u = \varphi_2$. Now, by Lemma 4.6, there exists $\rho \in L^2(\mathbb{R}^N)$ such that $\rho(x) \in [\underline{f}(x,u(x)), \overline{f}(x,u(x))]$ a.e. in $\mathbb{R}^N$ and

$$\langle \varphi_2, v \rangle = \int_{\mathbb{R}^N} \rho v, \quad \forall v \in H^1(\mathbb{R}^N).$$

Hence

$$\langle -\Phi'(u), v \rangle = \langle \varphi_1^u, v \rangle + \int_{\mathbb{R}^N} \rho v \quad \forall v \in H^1(\mathbb{R}^N).$$

Taking $v = \phi \in C_0^\infty(\mathbb{R}^N)$ in the above equation, one has

$$\int_{\mathbb{R}^N} \rho\phi + \int_{\mathbb{R}^N} F_1'(u)\phi = \langle -\Phi'(u), \phi \rangle \quad \forall \phi \in C_0^\infty(\mathbb{R}^N), \tag{4.19}$$

which completes the proof. □

The above Lemma 4.7 is crucial in order to prove the next result.

Theorem 4.9. *The functional I has a sequence of critical points (u_n) such that $I(u_n) \to \infty$ as $n \to \infty$. Hence, problem (P_1) has infinitely many nontrivial solutions.*

The proof of Theorem 4.9 is divided into several preliminary results. To this goal, let $O(N)$ be the orthogonal group in $\mathbb{R}^N$. So, by using a standard change of variable, it is easy to check that the functional I is $O(N)$-invariant. Moreover, the space of invariant elements of $H^1(\mathbb{R}^N)$ under the natural action of $O(N)$ coincides with the subspace $H^1_{\rm rad}(\mathbb{R}^N)$ of radial functions of $H^1(\mathbb{R}^N)$. The classical Symmetric Criticality Principle recalled in Theorem 2.12 ensures that the critical points of $J := I|_{H^1_{\rm rad}(\mathbb{R}^N)}$ are also critical points of the functional I. We notice that Theorem 4.9 can be proved by using Theorem 3.6 due to the $\mathbb{Z}_2$-invariant of the even functional J; see Example 2.7 for related topics. A key ingredient along the proof of Theorem 4.9 is the Sobolev compact embedding

$$H^1_{\rm rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad \forall p \in (2, 2^*). \tag{4.20}$$

See [2], Corollary 1.26 for additional comments and remarks.

Let us prove the following technical result.

Lemma 4.10. *Let (u_n) be a (PS) sequence for the functional J at a level c and let $\varphi_1^{(n)} := \varphi_1^{u_n}$ as in (4.3). Then, $\|\varphi_1^{(n)}\| < \infty$ for any $n \in \mathbb{N}$ and there is a unique $w_n \in \partial J(u_n)$, which will be denoted by $J'(u_n)$, such that:*

i) For some $\varphi_2^{(n)} \in \partial\Psi_2(u_n)$ one has

$$J'(u_n)(v) = \langle \varphi_2^{(n)}, v\rangle + \langle \varphi_1^{(n)}, v\rangle + \langle \Phi'(u_n), v\rangle, \quad \forall v \in H^1_{rad}(\mathbb{R}^N).$$

ii) $J'(u_n)u_n = o_n(1)\|u_n\|$ with

$$J'(u_n)(u_n) \le \Psi_2^\circ(u_n, u_n) + \int_{\mathbb{R}^N} F_1'(u_n)u_n + \langle \Phi'(u_n), u_n\rangle, \quad \forall n \in \mathbb{N}.$$

Proof. Let (u_n) be a (PS) for the functional J. Then

$$\Psi_2(v) - \Psi_2(u_n) + \int_{\mathbb{R}^N} (F_1(v) - F_1(u_n)) \ge \langle -\Phi'(u_n), v - u_n\rangle + \langle w_n, v - u_n\rangle, \quad v \in H^1_{\rm rad}(\mathbb{R}^N), \tag{4.21}$$

with $w_n \in (H^1_{\rm rad}(\mathbb{R}^N))'$, and $w_n \to 0$. Set $\phi \in C^\infty_{0,\,\rm rad}(\mathbb{R}^N)$, and take $v := u_n + t\phi$, with $t > 0$. By Lemma 2.4 it follows that

$$\Psi_2^\circ(u_n, \phi) + \int_{\mathbb{R}^N} F_1'(u_n)\phi \ge \langle -\Phi'(u_n), \phi\rangle + \langle w_n, \phi\rangle \quad \forall \phi \in C^\infty_{0,\,\rm rad}(\mathbb{R}^N), \tag{4.22}$$

as $t \to 0^+$. Since

$$\langle \varphi_1^{(n)}, \phi\rangle = \int_{\mathbb{R}^N} F_1'(u_n)\phi \quad \phi \in C^\infty_{0,\,\rm rad}(\mathbb{R}^N),$$

arguing as in the proof of Lemma 4.7, one has

$$\sup_{\phi \in C^\infty_{0,\,\rm rad}(\mathbb{R}^N),\, \|\phi\| \le 1} \langle \varphi_1^{(n)}\, \phi\rangle < \infty. \tag{4.23}$$

Therefore, the functional φ_1^n can be extended to the whole $H^1_{\rm rad}(\mathbb{R}^N)$. By using (4.22), again as in Lemma 4.7, we get

$$-\Phi'(u_n) - \varphi_1^{(n)} + w_n \in \partial\Psi_2(u_n). \tag{4.24}$$

Consequently, by setting $J'(u_n) := w_n$, one has

$$J'(u_n) = \varphi_2^{(n)} + \varphi_1^{(n)} + \Phi'(u_n), \tag{4.25}$$

for some $\varphi_2^{(n)} \in \partial\Psi_2(u_n)$. Hence part *i)* has been proved. In order to show part *ii)*, let us observe that

$$J'(u_n)(u_n) = \langle w_n, u_n\rangle = o_n(1)\|u_n\|,$$

as $J'(u_n) \to 0$. Hence, by choosing $v := u_n + tu_n$ in (4.21), we have

$$J'(u_n)(u_n) \le \frac{1}{t}[\Psi_2(u_n+tu_n) - \Psi_2(u_n)] + \int_{\mathbb{R}^N} \frac{1}{t}[F_1(u_n+tu_n) - F_1(u_n)] + \langle \Phi'(u_n), u_n\rangle. \tag{4.26}$$

Since F_1 is convex, the map

$$t \longmapsto \frac{F_1(u_n+tu_n) - F_1(u_n)}{t}, \;\; t>0$$

is monotone and

$$\frac{F_1(u_n+tu_n) - F_1(u_n)}{t} \to F_1'(u_n)u_n,$$

as $t \to 0^+$. Now, Lemma 4.3 and (4.23) yields $F_1'(u_n)u_n \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \frac{F_1(u_n+tu_n) - F_1(u_n)}{t} \to \int_{\mathbb{R}^N} F_1'(u_n)u_n,$$

by using the classical Lebesgue's Dominated Convergence Theorem. In conclusion, as $t \to 0$ in (4.26), by Lemma 2.4, it follows that

$$J'(u_n)(u_n) \le \Psi_2^\circ(u_n, u_n) + \int_{\mathbb{R}^N} F_1'(u_n)u_n + \langle \Phi'(u_n), u_n\rangle.$$

This completes the proof. □

A consequence of Lemma 4.10 is the following result that will be useful in order to prove that any (PS) sequence for the functional J is bounded; see Lemma 4.13.

Lemma 4.11. *Let (u_n) be a (PS) sequence for the functional J at level c. Then*

$$\int_{\mathbb{R}^N} |u_n|^2 \le M + o_n(1)\|u_n\|, \;\; n \ge n_0 \tag{4.27}$$

for some $M > 0$ and $n_0 \in \mathbb{N}$.

Proof. Since $J(u_n) \to c$, there is $n_0 \in \mathbb{N}$ such that

$$J(u_n) \leq c + 1, \quad n \geq n_0. \tag{4.28}$$

By setting $\tilde{J} = \tilde{I}|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)}$, *i.e.*

$$\tilde{J}(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u) \quad u \in H^1_{\mathrm{rad}}(\mathbb{R}^N),$$

we can write $J = \tilde{J} + \Psi_2|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)}$. By Lemmas 4.3 and 4.10 Part - *ii*), one has

$$J'(u_n)(u_n) \leq \tilde{J}'(u_n)(u_n) + \Psi_2^\circ(u_n, u_n)$$

as well as

$$J(u_n) - \frac{1}{2}J'(u_n)(u_n) \geq \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2 + \left(\Psi_2(u_n) - \frac{1}{2}\Psi_2^\circ(u_n, u_n)\right). \tag{4.29}$$

Now, gathering $J'(u_n)u_n = o_n(1)\|u_n\|$ with (4.28) and (4.29), we get

$$c + 1 + o_n(1)\|u_n\| \geq \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2 + \left(\Psi_2(u_n) - \frac{1}{2}\Psi_2^\circ(u_n, u_n)\right), \quad \forall n \geq n_0.$$

In order to finish the proof, it is enough to show that there is $M > 0$ (independent of n) such that

$$\left(\Psi_2(u_n) - \frac{1}{2}\Psi_2^\circ(u_n, u_n)\right) \geq -M, \quad \forall n \in \mathbb{N}. \tag{4.30}$$

Bearing in mind the above computations, we employ Lemma 4.6 to obtain

$$\Psi_2(u_n) - \frac{1}{2}\Psi_2^\circ(u_n, u_n) = \int_{\mathbb{R}^N} F(x, u_n) - \frac{1}{2}\int_{\mathbb{R}^N} \eta^{(n)} u_n,$$

where $\eta^{(n)} \in L^2(\mathbb{R}^N)$ and $\eta^{(n)}(x) \in [\underline{f}(x, u_n(x)), \overline{f}(x, u_n(x))]$ a.e. in $\mathbb{R}^N$. Finally, the condition (f_3) yields

$$\int_{\mathbb{R}^N} F(x, u_n) - \frac{1}{2}\int_{\mathbb{R}^N} \eta^{(n)} u_n \geq -C\int_{\mathbb{R}^N} h(x) \geq -M,$$

for some $M = M_h > 0$. This completes the proof. □

Let us recall now the so-called logarithmic Sobolev inequality proved in [17], p. 144, as well as [19], Sentence (2.4) and the references therein. More precisely, for each $b > 0$, one has

$$\int_{\mathbb{R}^N} u^2 \log u^2 \leq \frac{b^2}{\pi}\|\nabla u\|_2^2 + (\log\|u\|_2^2 - N(1 + \log b))\|u\|_2^2 \tag{4.31}$$

for every $u \in H^1(\mathbb{R}^N)$.

An immediate consequence of (4.31) is given below.

Corollary 4.12. *There is $C > 0$ such that*

$$\int_{\mathbb{R}^N} u^2 \log u^2 \leq \frac{1}{2}\|\nabla u\|_2^2 + C(\log \|u\|_2^2) + 1)\|u\|_2^2,$$

for every $u \in H^1(\mathbb{R}^N)$.

The following results involve the notion of (PS) condition and will be proved as consequences of Corollary 4.12.

Lemma 4.13. *If (u_n) is a (PS) sequence for the functional J at level $c \in \mathbb{R}$, then (u_n) is bounded.*

Proof. By Lemma 4.11 and Corollary 4.12, for each $r \in (0,1)$ there is $C_1 > 0$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \log u_n^2 \leq \frac{1}{4}\|u\|^2 + C_1(1 + \|u_n\|^{1+r}).$$

Since $J(u_n) \to c$, there is $n_0 \in \mathbb{N}$ such that

$$c + 1 \geq J(u_n) \geq \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \log u_n^2, \quad n \geq n_0.$$

Then

$$c + 1 \geq \frac{1}{4}\|u_n\|^2 - C_1(1 + \|u_n\|^{1+r}),$$

for every $n \geq n_0$. The proof is complete. □

Lemma 4.14. *The functional J satisfies the* (PS) *condition.*

Proof. Let (u_n) be a (PS) sequence for J at level c. By Lemma 4.13, the sequence (u_n) is bounded. Consequently, the embedding (4.20) yields

i) $u_n \rightharpoonup u_0$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$;
ii) $u_n \to u_0 \in L^p(\mathbb{R}^N)$ with $p \in (2, 2^*)$;
iii) $\|u_n\| \to M$ and $u_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$.

As (u_n) is a (PS) sequence, we have that

$$\langle u_n, v - u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n\|v - u_n\| + \int_{\mathbb{R}^N} F_2'(u_n)(v - u_n), \quad \forall v \in H^1_{\mathrm{rad}}(\mathbb{R}^N), \tag{4.32}$$

with $\varepsilon_n \to 0^+$. If we take $v := u_0$ in (4.32), the boundedness of (u_n) and the subcritical growth of F_2 immediately give

$$\langle u_n, u_0 - u_n\rangle + \Psi(u_0) - \Psi(u_n) \geq o_n(1). \tag{4.33}$$

Hence, the lower semicontinuity property of Ψ combined with inequality (4.33) leads to

$$\|u_0\|^2 \geq \lim \|u_n\|^2 = M^2, \tag{4.34}$$

on account of $i), ii)$ and $iii)$. In conclusion $u_n \to u_0$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. □

In order to prove that J satisfies the hypotheses of the Fountain Theorem 3.6, a suitable splitting of the Sobolev space $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is necessary. To this aim, we first observe that by [44], Proposition 1.a.9 and Section 1.b, p. 8 and [19], Section 5 the next property holds.

Lemma 4.15. *Let A be a dense subset of $H^1(\mathbb{R}^N)$, then $H^1(\mathbb{R}^N)$ has an orthonormal hilbertian basis that is constituted by elements of A.*

Thanks to Lemma 4.15 the following result holds.

Corollary 4.16. *The space $H^1(\mathbb{R}^N)$ has an orthonormal hilbertian basis constituted by elements of $C_0^\infty(\mathbb{R}^N)$. Consequently, there exists a sequence $(v_j) \subset C_0^\infty(\mathbb{R}^N)$ such that*

$$H^1(\mathbb{R}^N) = \overline{\bigoplus_{j\in\mathbb{N}} X_j} \quad \textit{with} \quad X_j = \mathit{span}\{v_j\}, \tag{4.35}$$

and $\langle v_i, v_j\rangle = 0$, for every $i \neq j$.

Moreover, the same conclusion holds if we replace $H^1(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N)$ by $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and $C^\infty_{0,\,\mathrm{rad}}(\mathbb{R}^N)$ respectively.

From now on, let us consider

$$H^1_{\mathrm{rad}}(\mathbb{R}^N) = \overline{\bigoplus_{j\in\mathbb{N}} X_j} \tag{4.36}$$

and set

$$Y_k := \bigoplus_{j=1}^{k} X_j \quad \text{ as well as } \quad Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j}, \tag{4.37}$$

for every $k \in \mathbb{N}$.

Since the action of $\mathbb{Z}_2$ on $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ satisfies (G_0) with $X_j \cong \mathbb{R} =: V$ we only need to prove that the functional J satisfies the Parts - $i)$ and $ii)$ of Theorem 3.6.

To this aim, let us briefly recall the next fact.

Lemma 4.17. *Let β_k defined by*

$$\beta_k := \sup_{u\in Z_k,\|u\|=1} \|u\|_p. \tag{4.38}$$

Then $\beta_k \to 0$.

See [2], Lemma 3.8 as well as the proof of Proposition 3.7 in [19] for additional comments and remarks.

Taking into account Lemma 4.17, we are able to prove that the functional J satisfies the Fountain geometry.

Lemma 4.18. *The functional J verifies*

i) $\displaystyle\sup_{u\in Y_k,\|u\|=\rho_k} J(u) \leq 0;$

ii) $\displaystyle\inf_{u\in Z_k,\|u\|=r_k} J(u) \to \infty.$

Proof. We first recall that

$$J(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F(x,u) + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u), \quad \forall\, u \in H^1_{\mathrm{rad}}(\mathbb{R}^N).$$

Part - $i)$ By (f_1) one has

$$|F(x,s)| \le B|s|^2, \;\; \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall s \in \mathbb{R},$$

for some constant $B > 0$. Now, by definition, since $Y_k \subset C^\infty_{0,\,\mathrm{rad}}(\mathbb{R}^N)$ it follows that $Y_k \subset D(J)$ for each $k \in \mathbb{N}$. Hence

$$J(u) \le \frac{1}{2}\|u\|^2 + B\|u\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2, \tag{4.39}$$

for every $u \in Y_k$.

If we take $v := \dfrac{u}{\|u\|}$ for $u \neq 0$, it follows that

$$\begin{aligned} J(u) &\le \frac{1}{2}\|u\|^2 \left(1 + B - \int_{\mathbb{R}^N} v^2 \log(v^2\|u\|^2)\right) \\ &= \frac{1}{2}\|u\|^2 \left(1 + B - \int_{\mathbb{R}^N} v^2 \log v^2 - \log(\|u\|^2)\int_{\mathbb{R}^N} v^2\right), \end{aligned} \tag{4.40}$$

for every $u \in Y_k$. As $\dim Y_k < \infty$, all the norms on Y_k are equivalent. Hence, if $\|u\| = \rho_k \approx \infty$, one gets

$$1 + B - \int_{\mathbb{R}^N} v^2 \log v^2 - \log(\|u\|^2)\int_{\mathbb{R}^N} v^2 \le 0.$$

Then

$$\sup_{u \in Y_k, \|u\| = \rho_k} J(u) \le 0,$$

so that $i)$ is verified.

Part - $ii)$ By (A_2) for every $s \in \mathbb{R}$,

$$|F_2(s)| \le C|s|^p, \quad p \in (2, 2^*),$$

for some $C > 0$. Hence

$$J(u) \ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F_2(u) \ge \frac{1}{2}\|u\|^2 - \beta_k^p C\|u\|^p,$$

for every $u \in Z_k$. Moreover, by Lemma 4.17 one has $\beta_k \to 0$. Then, by choosing

$$r_k := (pC\beta_k^p)^{\frac{1}{2-p}},$$

it follows that $r_k \to \infty$ and

$$J(u) \ge \left(\frac{1}{2} - \frac{1}{p}\right) r_k^2.$$

In conclusion

$$\inf_{u\in Z_k,\, \|u\|=r_k} J(u) > 0,$$

for k sufficiently large. □

Conclusion of the proof of Theorem 4.9. First of all, we emphasize that, for every $k \in \mathbb{N}$, the minimax levels

$$c_k := \inf_{\gamma\in\Theta_k} \sup_{u\in B_k} J(\gamma(u))$$

are finite. Indeed, if we take $\tilde{\gamma} := \mathcal{I}d|_{B_k}$, by using the classical inequality

$$|t^2 \log t^2| \le C(|t| + |t|^p), \;\; p > 2 \quad \text{and} \quad \forall t \in \mathbb{R},$$

we infer that there exists $C_1 > 0$ such that

$$J(\tilde{\gamma}(u)) \le |J(u)| \le \frac{1}{2}\|u\|^2 + B\|u\|_2^2 + C_1(\|u\|_1 + \|u\|_p^p), \tag{4.41}$$

for every $u \in B_k \subset Y_k$. The equivalence of the norms in Y_k in addition to (4.41) guarantee that

$$c_k = \inf_{\gamma\in\Theta_k} \sup_{u\in B_k} J(\gamma(u)) \le \sup_{u\in B_k} J(\tilde{\gamma}(u)) < \infty.$$

Finally, we would like to point out that if $u \in H^1(\mathbb{R}^N)$ is a critical point of I, then there exists $\rho \in L^2(\mathbb{R}^N)$ with

$$\rho(x) \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \;\; \text{a.e. in} \;\; \mathbb{R}^N,$$

such that

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi + u\phi) + \int_{\mathbb{R}^N} \rho(x)\phi = \int_{\mathbb{R}^N} u^2 \log u\phi, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

Therefore, by elliptic regularity theory, there is $r \ge 1$ such that $u \in H^1(\mathbb{R}^N) \cap W^{2,r}_{\text{loc}}(\mathbb{R}^N)$ and

$$-\Delta u + u + \rho(x) = u \log u^2 \;\; \text{a.e. in} \;\; \mathbb{R}^N.$$

In conclusion

$$\Delta u - u + u \log u^2 \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \;\; \text{a.e. in} \;\; \mathbb{R}^N.$$

4.2. A concave perturbation of logarithmic equation

In this subsection we study the existence of solutions for the following class of problems

$$\begin{cases} -\Delta u + u = u \log u^2 + \lambda h(x)|u|^{q-2}u, \;\; \text{in} \;\; \mathbb{R}^N, \\ \;\; u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_2}$$

where λ is a positive parameter, $q \in (1,2)$ and $h : \mathbb{R}^N \to \mathbb{R}$ is chosen as in the condition (f_1) above. By using the same notations of the previous subsection, the energy functional associated to (P_2) is given by

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u) - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q h(x), \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{4.42}$$

Note that I_λ is a Szulkin-type functional, with $I_\lambda(u) = \Phi(u) + \Psi(u)$, where

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F_2(u) - \frac{\lambda}{q}\int_{\mathbb{R}^N} h|u|^q$$

and

$$\Psi(u) := \int_{\mathbb{R}^N} F_1(u).$$

In the sequel, we say that a function $u \in H^1(\mathbb{R}^N)$ is a *solution* of (P_2) if $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (\nabla u \nabla \phi + u\phi) = \int_{\mathbb{R}^N} (u \log u^2 \phi + \lambda h(x)|u|^{q-2}u\phi), \quad \forall \phi \in C_0^\infty(\mathbb{R}^N). \tag{4.43}$$

By Part - *ii)* of Lemma 4.2 it is possible to see that any critical point of the Szulkin-type functional I_λ is a solution of (P_2); see also [17], Lemma 2.1. Moreover, if $J_\lambda := I_\lambda|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)}$, again by Theorem 2.12, the critical points of J_λ are also critical points of the functional I_λ.

The main result this subsection reads as follows.

Theorem 4.19. *There exists $\lambda_0 > 0$ such that, for $\lambda \in (0, \lambda_0)$, the functional J_λ has infinitely many critical points (u_n) with $J_\lambda(u_n) \to 0$ as $n \to \infty$. Hence, for $\lambda \in (0, \lambda_0)$, the problem (P_2) has infinitely many nontrivial solutions.*

In order to prove Theorem 4.19, let us introduce a modified functional $\tilde{J}_\lambda$ which will be crucial in our approach. However, let us start by proving the following technical result.

Proposition 4.20. *If $\lambda \approx 0^+$, then there is a function*

$$g(t) := \frac{1}{2}t^2 - Bt^p - C\lambda t^q, \;\; t > 0,$$

with $p \in (2, 2^*)$ and $B, C > 0$, that attains a nonnegative maximum and

$$J_\lambda(u) \geq g(\|u\|), \quad \forall u \in H^1(\mathbb{R}^N).$$

Proof. Since $F_1 \geq 0$, we have that, for every $u \in H^1(\mathbb{R}^N)$

$$\begin{aligned} J_\lambda(u) \geq & \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F_2(u) - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q h(x) \\ & \geq \frac{1}{2}\|u\|^2 - C_1\|u\|^p - \lambda C_2 \|u\|^q \\ & =: g(\|u\|), \end{aligned}$$

for some $C_1 = C(p) > 0$ and $C_2 = C(h,q) > 0$. Moreover, if $\lambda \approx 0^+$ it is clearly seen that the function g attains a nonnegative maximum. □

Now, fix R_0, R_1 and R_2 positive constants satisfying:

(g_1) $g|_{[0,R_0]} \leq 0$ and $g(R_0) = 0$;
(g_2) $g|_{[R_0,R_2]} \geq 0$, $g|_{[R_2,\infty)} \leq 0$ and $g(R_2) = 0$, where $R_0 < R_1 < R_2$ and R_1 is the point in which g attains its maximum value; note that $g(t) \to -\infty$, as $t \to \infty$.

Moreover, take $\eta \in C^\infty([0,\infty))$ such that the following condition holds:

(η_1) η is a nonnegative and non-increasing function such that

$$\eta|_{[0,R_0]} \equiv 1 \quad \text{and} \quad \eta|_{[R_2,\infty)} \equiv 0.$$

Set $\varphi(u) := \eta(\|u\|)$. Arguing as in [45], let us consider the energy functional

$$\tilde{J}_\lambda(u) := \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_1(u) - \varphi(u)\int_{\mathbb{R}^N} F_2(u) - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q h(x), \tag{4.44}$$

for every $u \in H^1_{\text{rad}}(\mathbb{R}^N)$.

Lemma 4.21. *Let $\tilde{J}_\lambda$ be the functional given in (4.44). Then, the following facts hold:*

i) $\tilde{J}_\lambda \in (H_0)$ *with* $\tilde{J}_\lambda = \tilde{\Phi}_\lambda + \tilde{\Psi}$ *and* $\tilde{\Psi} = \Psi|_{H^1_{\text{rad}}(\mathbb{R}^N)}$;
ii) If $\tilde{J}_\lambda(u) < 0$ *then* $\|u\| < R_0$ *and* $\tilde{J}_\lambda(u) = J_\lambda(u)$;
iii) Let (u_n) *be a* $(\text{PS})_c$ *sequence for* $\tilde{J}_\lambda$ *with* $c < 0$ *then* (u_n) *is a* $(\text{PS})_c$ *sequence for* J_λ;
iv) If $u \in B_{R_0}(0)$ *is a critical point of* $\tilde{J}_\lambda$ *then* u *is a critical point of* J_λ.

Proof. Part - i) immediately follows by (η_1) and the definition of $\tilde{J}_\lambda$. Moreover, if $\lambda \approx 0^+$ then

$$\tilde{g}(t) := \frac{1}{2}t^2 - \lambda C_2 t^q \geq 0$$

for every $t \geq R_2$ and $\tilde{J}_\lambda(\|u\|) \geq \tilde{g}(\|u\|)$. Hence, Part - ii) holds. The rest of the proof is an easy consequence of i) and ii). ☐

By using the above notations and results we are able to prove Theorem 4.19.

Proof of Theorem 4.19. - By Lemma 4.21 it is sufficient to show that $\tilde{J}_\lambda$ has a sequence of critical points (u_n) with $u_n \in B_{R_0}(0)$ for every $n \in \mathbb{N}$. This will be done by showing that $\tilde{J}_\lambda$ satisfies the hypotheses of Theorem 3.8. To this aim, we first notice that $\tilde{J}_\lambda$ is even and $\tilde{J}_\lambda(0) = 0$. Therefore, we can apply Theorem 3.8 with $G = \mathbb{Z}_2$. In this way, $\gamma_G = \gamma$ is the genus of a symmetric closed set; see Remark 2.14. Moreover, $\tilde{J}_\lambda$ is a coercive functional and consequently any $(\text{PS})_c$ sequence for $\tilde{J}_\lambda$ is bounded. If (u_n) is a $(\text{PS})_c$ sequence for $\tilde{J}_\lambda$, with $c < 0$, then Lemma 4.21 ensures that (u_n) is also a $(\text{PS})_c$ sequence for J_λ. Finally, arguing as in Lemma 4.14, it easily seen that $\tilde{J}_\lambda$ satisfies the $(\text{PS})_c$ condition for $c < 0$. It remains to show that $\tilde{J}_\lambda$ satisfies i) and ii) of Theorem 3.8.

Part - i) Since $\tilde{J}_\lambda$ satisfies

$$\tilde{J}_\lambda(u) \geq g(\|u\|) \quad \forall u \in H^1(\mathbb{R}^N)$$

and $\tilde{J}_\lambda(u) \geq 0$ for every $\|u\| \geq R_2$, we conclude that $\tilde{J}_\lambda$ is bounded from below. Consequently

$$c_j := \inf_{A \in \Gamma_j} \sup_{u \in A} \tilde{J}_\lambda(u) > -\infty.$$

Part - *ii*) For each $k \in \mathbb{N}$, let us consider Y_k and Z_k as in (4.37). In this case $\dim Y_k < \infty$ and $Y_k \subset C_0^\infty(\mathbb{R}^N)$. Bearing in mind that

$$F_1(u) < \infty, \ \forall u \in Y_k,$$

we infer that $Y_k \subset D(\tilde{J}_\lambda)$ for any $k \in \mathbb{N}$. As $\tilde{J}_\lambda \equiv J_\lambda$ in B_{R_0}, one has

$$\tilde{J}_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q h(x).$$

Moreover, if $\delta \approx 0^+$

$$|t|^2 |\log t^2| \leq C_1(|t|^{2-\delta} + |t|^{2+\delta}), \quad \forall t \in \mathbb{R},$$

for some $C_1 = C_1(\delta) > 0$. Consequently

$$\tilde{J}_\lambda(u) \leq \frac{1}{2}\|u\|^2 + C\int_{\mathbb{R}^N} (|u|^{2-\delta} + |u|^{2+\delta}) - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q h(x),$$

for every $u \in B_{R_0}$. Now, if $u \in Y_k$ then $u \in L^r(\mathbb{R}^N)$ for every $r \in [1, 2)$. Since all the norms on Y_k are equivalent, one has

$$\tilde{J}_\lambda(u) \leq \frac{1}{2}\|u\|^2 + C_2(\|u\|^{2-\delta} + \|u\|^{2+\delta}) - C\|u\|^q, \tag{4.45}$$

for some constant $C_2 > 0$. Now, for each $k \in \mathbb{N}$, fix $A := S_\rho(0) \cap Y_k$ with $\rho \approx 0^+$. Then A is a closed and symmetric set with $\gamma(A) = k$. By choosing δ such that $2 - \delta > q$, on account of (4.45), it follows that

$$\sup_{u \in A} \tilde{J}_\lambda(u) < 0.$$

The proof is now complete. □

4.3. A problem involving the 1-Laplacian operator with subcritical growth

In this subsection we study the existence of infinitely many solutions for the following problem

$$\begin{cases} -\Delta_1 u = |u|^{p-2}u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_3}$$

where $\Omega \subset \mathbb{R}^N$ (with $N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$ and $p \in (1, 1^*)$. In order to simplify the notation, we set $q := p/(p-1)$.

From now on we denote by $\mathcal{M}(\Omega, \mathbb{R}^N)$ (briefly $\mathcal{M}(\Omega)$) the space of the vectorial Radon measures on Ω and by $BV(\Omega)$ the space of the functions $u : \Omega \to \mathbb{R}$ of bounded variation, that is

$$BV(\Omega) := \{u \in L^1(\Omega) : \ Du \in \mathcal{M}(\Omega)\},$$

where Du denotes the distributional derivative of $u \in L^1(\Omega)$. It is well known that $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and

$$\int_\Omega |Du| = \sup\left\{\int_\Omega u\mathrm{div}\phi : \phi \in C_0^1(\Omega,\mathbb{R}^N), \text{ and } \|\phi\|_\infty \le 1)\right\} < +\infty.$$

Moreover $BV(\Omega)$ is a Banach space endowed by the norm

$$\|u\|_{BV(\Omega)} := \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1},$$

where, as usual, $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. We also recall that the continuous embedding

$$BV(\Omega) \hookrightarrow L^r(\Omega),\ \ r \in [1, 1^*] \tag{4.46}$$

is compact provided that $r \in [1, 1^*)$; see [26, 46, 47] for advanced theoretical results on the subject.

According to Kawohl and Schuricht in [26], as well as Degiovanni in [29], the notion of solution for problem (P_3) can be formulated as follows.

Definition 4.22. We say that a function $u \in BV(\Omega)$ is a solution of (P_3) if there exists $z \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \le 1$, such that

$$\begin{cases} -\displaystyle\int_\Omega u\mathrm{div}z = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}, \ \ \mathrm{div}z \in L^q(\Omega), \\ -\mathrm{div}z = |u|^{p-2}u \ \text{ a.e. in } \ \Omega, \end{cases}$$

where $q := p/(p-1)$.

Now, let us consider the energy functional $I : L^p(\Omega) \to (-\infty, +\infty]$ given by

$$I(u) = \Phi(u) + \Psi(u), \tag{4.47}$$

where

$$\Phi(u) := -\frac{1}{p}\int_\Omega |u|^p$$

and

$$\Psi(u) := \begin{cases} \displaystyle\int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} & u \in BV(\Omega) \\ \infty & u \in L^p(\Omega) \setminus BV(\Omega) \end{cases},$$

for every $u \in L^p(\Omega)$.

It is easily seen that $\Phi \in C^1(L^p(\Omega), \mathbb{R})$ as well as Ψ is a convex and lower semicontinuous functional, so that I is a Szulkin-type functional. Consequently $D(I) = BV(\Omega)$ and, for each fixed $u \in BV(\Omega)$, the subdifferential $\partial\Psi(u)$ can be identified as a subset of $L^q(\Omega)$.

The next results will be crucial in the sequel.

Lemma 4.23. *If $u \in BV(\Omega)$ and $\partial\Psi(u) \neq \emptyset$ then $u \in L^\infty(\Omega)$.*

Proof. We first notice that $L^{1^*}(\Omega) \hookrightarrow L^p(\Omega)$, so that $L^q(\Omega) \hookrightarrow L^N(\Omega)$. Consequently, if $w \in \partial\Psi(u) \subset L^q(\Omega)$, one has that $w \in L^N(\Omega)$. The conclusion is achieved by arguing as in [29], Proposition 3.3. □

Lemma 4.24. *If $u \in BV(\Omega)$ then, for each $w \in \partial\Psi(u)$, there exists $z \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \leq 1$, such that*

$$\begin{cases} w = -\text{div}z \in L^q(\Omega) \\ -\displaystyle\int_\Omega u\text{div}z = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}. \end{cases}$$

Proof. Let us define

$$\tilde{\Psi}(u) := \begin{cases} \displaystyle\int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} & u \in BV(\Omega) \\ \infty & u \in L^{1^*}(\Omega) \setminus BV(\Omega) \end{cases},$$

and take $w \in \partial\Psi(u) \subset L^q(\Omega)$. Then $w \in L^N(\Omega)$ and

$$\tilde{\Psi}(v) - \tilde{\Psi}(u) = \Psi(v) - \Psi(u) \geq \int_\Omega w(v-u), \qquad \forall v \in BV(\Omega) = D(\tilde{\Psi}),$$

so that $w \in \partial\tilde{\Psi}(u)$. The conclusion follows by [26], Proposition 4.23. □

The next result connects critical points of the energy functional I with solutions of (P_3).

Lemma 4.25. *If $u \in BV(\Omega)$ is a critical point of the functional I then $u \in L^\infty(\Omega)$. Moreover, the function u is a solution of (P_3) in the sense of Definition 4.22.*

Proof. Let $u \in BV(\Omega)$ be a critical point of I. Then

$$-\Phi'(u) \in \partial\Psi(u) \subset L^q(\Omega).$$

Thereby, there exists $w \in \partial\Psi(u)$ such that

$$-\Phi'(u) = w \ \text{ in } \ L^q(\Omega).$$

Consequently, Lemma 4.24 and the definition of Φ yield the existence of $z \in L^\infty(\Omega, \mathbb{R}^N)$, with $\|z\|_\infty \leq 1$, such that $-\text{div}z = w$ in $L^q(\Omega)$ and

$$\begin{cases} -\displaystyle\int_\Omega u\text{div}z = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}, \ \text{div}z \in L^q(\Omega) \\ -\text{div}z = |u|^{p-2}u \ \text{ a.e. in } \ \Omega. \end{cases}$$

Moreover, Lemma 4.23 ensures that $u \in L^\infty(\Omega)$. The proof is now complete. □

By Lemmas 4.24 and 4.25 we are able to prove the main result of this subsection.

Theorem 4.26. *The functional I has infinitely many critical points (u_n) with $I(u_n) \to \infty$ as $n \to \infty$. Hence, problem (P_3) has infinitely many nontrivial solutions.*

Proof. Hereafter, we are going to prove that I verifies the assumptions of Theorem 3.9 with $Y = \{0\}$. We first prove that I satisfies the compactness (PS) condition. To this end, let (u_n) be a (PS) sequence for I. So, let $c \in \mathbb{R}$ such that

$$I(u_n) \to c,$$

and

$$\Psi(v) - \Psi(u_n) \geq \int_\Omega |u_n|^{p-2} u_n (v - u_n) + \int_\Omega w_n (v - u_n), \quad \forall v \in BV(\Omega),$$

where $w_n \in L^q(\Omega)$ and $w_n \to 0$ in $L^q(\Omega)$. The last inequality gives

$$|u_n|^{p-2} u_n + w_n \in \partial \Psi(u_n), \quad \forall n \in \mathbb{N}.$$

Hence, Lemma 4.24 yields

$$\Psi(u_n) = \int_\Omega |Du_n| + \int_{\partial\Omega} |u_n| d\mathcal{H}^{N-1} = \int_\Omega |u_n|^p + \int_\Omega w_n u_n, \quad \forall n \in \mathbb{N}.$$

If we set

$$A(u_n) := \Psi(u_n) - \int_\Omega |u_n|^p + \int_\Omega w_n u_n = 0,$$

the classical Hölder's inequality leads to

$$\begin{aligned} c + 1 &\geq I(u_n) - \frac{1}{r} A(u_n) \\ &\geq \left(1 - \frac{1}{r}\right) \Psi(u_n) + \left(\frac{1}{r} - \frac{1}{p}\right) \|u_n\|^p_{L^p(\Omega)} - \frac{1}{r} \|w_n\|_{L^q(\Omega)} \|u_n\|_{L^p(\Omega)} \\ &\geq C_1 \|u_n\|_{BV(\Omega)} + C_2 \left(\|u_n\|^p_{L^p(\Omega)} - \|u_n\|_{L^p(\Omega)} \right), \end{aligned}$$

for some $r < p$ and n large enough. Since the real function $h(t) := t^p - t$, for every $t \geq 0$, is bounded from below, the last inequality clearly implies that $\sup_{n \in \mathbb{N}} \|u_n\|_{BV(\Omega)} < \infty$.

Therefore the (PS) condition is verified, since the embedding $BV(\Omega) \hookrightarrow L^p(\Omega)$ is compact. Now, if $u \in BV(\Omega)$ is a critical point of I then

$$|u|^{p-2} u \in \partial \Psi(u).$$

Consequently, by Lemma 4.24, it follows that

$$\int_\Omega |u|^p = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}.$$

Thereby, by setting

$$B(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} - \int_\Omega |u|^p,$$

one has

$$I(u) = I(u) - \frac{1}{p}B(u) = \left(1 - \frac{1}{p}\right) \|u\|_{BV(\Omega)} \geq 0,$$

for every $u \in L^p(\Omega)$. Hence, the set I^{-c} has no critical points for any $c > 0$.

Finally, let us prove that the functional I satisfies conditions $i)$ and $ii)$ of Theorem 3.9.

Part - $i)$ Without loss of generality we can suppose $u \in BV(\Omega)$, otherwise $I(u) = \infty$. Now, if $u \in BV(\Omega)$, the embedding $BV(\Omega) \hookrightarrow L^p(\Omega)$ immediately yields

$$I(u) \geq C\|u\|_{L^p(\Omega)} - \frac{1}{p}\|u\|^p_{L^p(\Omega)},$$

for some constant $C > 0$. Since $p > 1$, if $\|u\|_{L^p(\Omega)} = r \approx 0^+$, we also have

$$I(u) \geq \rho,$$

for some $\rho > 0$. Thus, condition $i)$ of Theorem 3.9 is proved with $Z = L^p(\Omega)$.

Part - $ii)$ For each $k \in \mathbb{N}$, let us consider X_k be a k-dimensional subspace of $C_0^\infty(\Omega)$. Since all the norms are equivalent on X_k, it easily seen that

$$I(u) \leq C_k\|u\|_{L^p(\Omega)} - \frac{1}{p}\|u\|^p_{L^p(\Omega)} \quad \forall u \in X_k,$$

for a convenient $C_k > 0$. Thus

$$I(u) \to -\infty, \;\; \text{as} \;\; \|u\|_{L^p(\Omega)} \to \infty \, \text{and} \, u \in X_k.$$

The proof is now complete. □

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