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OPTIMAL CONTROL OF THIRD GRADE FLUIDS WITH MULTIPLICATIVE NOISE

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Abstract. This work aims to control the dynamics of certain non-Newtonian fluids in a bounded domain of $\mathbb{R}^d$, $d = 2, 3$ perturbed by a multiplicative Wiener noise, the control acts as a predictable distributed random force, and the goal is to achieve a predefined velocity profile under a minimal cost. Due to the strong nonlinearity of the stochastic state equations, strong solutions are available just locally in time, and the cost functional includes an appropriate stopping time. First, we show the existence of an optimal pair. Then, we show that the solution of the stochastic forward linearized equation coincides with the Gâteaux derivative of the control-to-state mapping, after establishing some stability results. Next, we analyse the backward stochastic adjoint equation; where the uniqueness of solution holds only when $d = 2$. Finally, we establish a duality relation and deduce the necessary optimality conditions.

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1. Introduction

The purpose of this article is to control the random velocity field y of a non-Newtonian fluid, which fills a bounded domain $D \subset \mathbb{R}^d, d = 2, 3$ with smooth boundary, and moves under the action of random forces. More precisely, we consider a tracking problem and the goal is to minimize the cost functionals of the following form $\mathscr{J}_M(U, y) = f_M(y, \tau_M^U) + \dfrac{\lambda}{p}\mathbb{E}\int_0^T \|U\|^p_{(H^1(D))^d} dt$, where f_M is a given functional of the physical state y, which lives up to a certain stopping time τ_M^U, $\mathbb{E}$ denotes the average over all possible outcomes of the physical system, the constant $\lambda > 0$ sets the intensity of the cost and $p > 2(d + 1)$. The control acts through the external force U, and the random velocity field y is constrained to satisfy the incompressible third grade fluid equations driven by a multiplicative Wiener-noise, see (2.1) and Section 2 for the precise mathematical framework.

Most studies on fluid dynamics have been devoted to Newtonian fluids, which are characterized by the classical Newton's law of viscosity. However, there exist many real and industrial fluids with nonlinear viscoelastic behavior that does not obey Newton's law of viscosity, and consequently cannot be described by the classical viscous Newtonian fluids model. Among these fluids, we can find natural biological fluids, geological flows, industrial oils, fluids arising in food processing and cosmetic production, and many others, see *e.g* [1, 2].

Keywords and phrases: Third grade fluid, Navier-slip boundary conditions, stochastic PDE, optimal control, necessary optimality condition, multiplicative noise.

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Therefore, it is necessary to consider more general fluid models. Recently, the class of non-Newtonian fluids of differential type has received a special attention since it includes second grade fluids, which can be related to Camassa and Holm equations, possessing interesting geometric properties. The third grade fluids correspond to the next step of modeling fluids of differential type, which includes larger class of fluids, see *e.g.* [3, 4]. Consequently, the mathematical analysis of third grade fluid equations should be relevant to predict and control the behavior of these fluids, in order to design optimal flows that can be successfully used and applied in the industry. From mathematical point of view, third grade fluids constitute an hierarchy of fluids with increasing complexity and more nonlinear terms, which is more complex and require more involved analysis.

Without exhaustiveness, the third grade fluid equations with Dirichlet boundary condition were studied in [5, 6]. Later on [7, 8], supplementing the equation with a Navier-slip boundary condition, the authors established the existence of a global solution for initial conditions in H^2 and proved that uniqueness holds in 2D. In [9], the authors extended the later deterministic results to stochastic models in 2D. Recently, we proved in [10] the existence and uniqueness of local adapted solution to the stochastic third grade fluids equations with Navier-slip boundary conditions. The sample paths take values in the Sobolev space H^3 and are defined up to a certain positive stopping time, in 2D and 3D. See also [11] for ergodicity properties.

The control problems for Newtonian fluids (first grade fluids), where the flows are described by Navier–Stokes equations, have been extensively studied in the literature, let us refer for instance to [12–14]. We emphasize that directing the velocity field to a desired velocity profile over time, through a tracking-type cost functional, has a wide range of applications. To the best of our knowledge, the control problem for the second grade fluids (differential type) has been addressed for the first time in [15], where the authors proved the existence of a 2D-deterministic optimal control and deduced the first order optimality conditions. Later on, the control problem for 2D-stochastic second grade fluid models have been studied in [16, 17]. Recently in [18], the authors tackled the control problem for 2D-deterministic third grade fluids. Namely, they showed the existence of an optimal solution, they deduced the first order optimality conditions and proved the uniqueness of the coupled system.

It is worth mentioning that the majority of the works in the literature considered the optimal control problem in 2D. The 3D-problem is much more difficult due to the fact that the existence and uniqueness of the solution of Newtonian or Non-Newtonian fluid equations holds just locally in time (defined only up to a certain stopping time in the stochastic case). Taking into account the existence and uniqueness result of the solution, we define the cost functional including the stopping time and obtain a nonconvex optimization problem, see [19]. In order to establish the existence of the optimal control and to derive the optimality system, it is necessary to analyze the dependence of the cost functional on the stopping time, which is not an easy issue. In [19], the authors proved the well-posedeness of local mild solution to the stochastic Navier–Stokes equations with multiplicative Levy noise. Then, they proved the existence and uniqueness of optimal control, by analyzing some cost functional depending on stopping time. Recently, they studied in [20], the optimality system and established the necessary and sufficient optimality conditions with a multiplicative noise driven by a Q-Wiener process. It is worth mentioning that the approach in [19, 20] was based on the theory of semigroups.

Here, we aim to control stochastic non-Newtonian fluids is 2D and 3D. Namely, we will solve a stochastic optimal control problem constrained by the stochastic third grade fluid equations. We will show the existence of the optimal control, and establish the optimality system *via* variational approach. The stochastic third grade fluid equations is highly nonlinear which requires a subtil analysis, as well as the corresponding stochastic linearized and backward stochastic adjoint equations. Moreover, we should say that our work improves substantially the 2D result obtained in [16] in the case $\beta = 0$, since one can perform the analysis of the optimality system without the special weights used in [16]. Finally, we show that a similar analysis applies to derive the optimality system for cost functional including velocity field derivatives, which can be relevant to control the turbulence.

The article is organized as follows: in Section 2, we precise the mathematical framework and the assumptions on the data. Then, we recall the defintion of stochastic local solution and a result about the existence and uniqueness. Section 3 is devoted to establish some stability results. In Section 4, we formulate the optimal control problem and we state the main results of this work. Then, we show the existence and uniqueness of optimal solution. Section 5 is devoted to show the existence and the uniqueness of the solution to the stochastic linearized state equation. In Section 6, we show that the Gâteaux derivative of the control-to-state mapping concides with the solution of the linearized equations. Then, we write the variation of the cost functional. In Section 7, we study the backward stochastic adjoint equations in 2D and 3D, where the uniqueness holds only in 2D. Section 8 concerns the proof of a duality relation between the solution of the linearized equation and the adjoint state. Then, we deduce the first order optimality condition. We propose in Section 9 the extension of our analysis to the case of cost functional including derivatives. Finally, we gather in Section 10 some technical lemmas, that we used repeatedly in our analysis.

2. Content of the study

Let $\mathscr{W}$ be a cylindrical Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, P)$, endowed with a right-continuous filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$[1]. We assume that $\mathscr{F}_0$ contains all the P-null subset of Ω (see Sect. 2.2 for the assumptions on the noise). The goal is to study the optimal control of a third grade fluid where the control is introduced *via* the external forces. The fluid fills a bounded and simply connected domain $D \subset \mathbb{R}^d$, $d = 2, 3$, with regular boundary ∂D, and its dynamics is governed by the following equations

$$\begin{cases} d(v(y)) = \big(-\nabla \mathbf{P} + \nu\Delta y - (y\cdot\nabla)v(y) - \sum_j v^j(y)\nabla y^j + (\alpha_1+\alpha_2)\operatorname{div}(A^2) \\ \qquad\qquad +\beta\operatorname{div}(|A|^2A) + U\big)dt + G(\cdot, y)d\mathscr{W} & \text{in } D\times(0,T)\times\Omega, \\ \operatorname{div}(y) = 0 & \text{in } D\times(0,T)\times\Omega, \\ y\cdot\eta = 0, \quad (\eta\cdot\mathbb{D}(y))\big|_{\tan} = 0 & \text{on } \partial D\times(0,T)\times\Omega, \\ y(x,0) = y_0(x) & \text{in } D\times\Omega, \end{cases} \tag{2.1}$$

where $y := (y^1, \ldots, y^d)$ is the velocity of the fluid, $\mathbf{P}$ is the pressure and U corresponds to the external force. The operators v, A, $\mathbb{D}$ are defined by $v(y) = y - \alpha_1\Delta y := (y^1 - \alpha_1\Delta y^1, \ldots, y^d - \alpha_1\Delta y^d)$ and $A := A(y) = \nabla y + \nabla y^T = 2\mathbb{D}(y)$. The vector η denotes the outward normal to the boundary ∂D and $u|_{\tan}$ represents the tangent component of a vector u defined on ∂D. In addition, ν denotes the viscosity of the fluid and α_1, α_2, β are material moduli satisfying

$$\nu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1+\alpha_2| \leq \sqrt{24\nu\beta}, \quad \beta \geq 0. \tag{2.2}$$

It is worth noting that (2.2) allows the motion of the fluid to be compatible with thermodynamic laws, namely it ensures that the Helmholtz free energy density be at a minimum value when the fluid is locally at rest, we refer to [2] for more details. The diffusion coefficient G will be specified in Subsection 2.2.

2.1. Function spaces and notations

For a Banach space E, we define

$$(E)^k := \{(f_1, \cdots, f_k) : f_l \in E, \quad l = 1, \cdots, k\} \text{ for positive integer } k.$$

Let us introduce the following spaces:

$$\begin{aligned} H &= \{y \in (L^2(D))^d \mid \operatorname{div}(y) = 0 \text{ in } D \text{ and } y\cdot\eta = 0 \text{ on } \partial D\}, \\ V &= \{y \in (H^1(D))^d \mid \operatorname{div}(y) = 0 \text{ in } D \text{ and } y\cdot\eta = 0 \text{ on } \partial D\}, \\ W &= \{y \in V\cap(H^2(D))^d \mid (\eta\cdot\mathbb{D}(y))\big|_{\tan} = 0 \text{ on } \partial D\}, \quad \widetilde{W} = (H^3(D))^d \cap W. \end{aligned}$$

[1] *e.g.* the one generated by $\{\mathscr{W}(t)\}_{t\in[0,T]}$.

We recall that for $\mathbf{X} = \{y \in (L^2(D))^d : \operatorname{div}(y) \in L^2(D)\}$, the trace mapping $y \in \mathbf{X} \mapsto y.\eta \in H^{-1/2}(\partial D)$ is a bounded linear mapping.

First, we recall the Leray-Helmholtz projector $\mathbb{P} : (L^2(D))^d \to H$, which is a linear bounded operator characterized by the following L^2-orthogonal decomposition $v = \mathbb{P}v + \nabla\varphi,\ \varphi \in H^1(D)$.

Now, let us introduce the scalar product between two matrices $A : B = tr(AB^T)$ and denote $|A|^2 := A : A$. The divergence of a matrix $A \in \mathscr{M}_{d\times d}(E)$ is given by $(\operatorname{div}(A)_i)_{i=1}^{i=d} = (\sum_{j=1}^{d} \partial_j a_{ij})_{i=1}^{i=d}$. The space H is endowed with the L^2-inner product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|_2$. We recall that

$$(u,v) = \sum_{i=1}^{d}\int_D u_i v_i dx, \quad \forall u,v \in (L^2(D))^d, \quad (A,B) = \int_D A : B dx; \quad \forall A,B \in \mathscr{M}_{d\times d}(L^2(D)).$$

On the function spaces V, W and $\widetilde{W}$, we will consider the following inner products

$$\begin{aligned} (u,z)_V &:= (v(u),z) = (u,z) + 2\alpha_1(\mathbb{D}u,\mathbb{D}z),\\ (u,z)_W &:= (u,z)_V + (\mathbb{P}v(u),\mathbb{P}v(z)),\\ (u,z)_{\widetilde{W}} &:= (u,z)_V + (\operatorname{curl} v(u),\operatorname{curl} v(z)), \end{aligned} \tag{2.3}$$

and denote by $\|\cdot\|_V, \|\cdot\|_W$ and $\|\cdot\|_{\widetilde{W}}$ the corresponding norms. The usual norms on the classical Lebesgue and Sobolev spaces $L^p(D)$ and $W^{m,p}(D)$ will be denoted by $\|\cdot\|_p$ and $\|\cdot\|_{W^{m,p}}$, respectively. In addition, given a Banach space X, we will denote by X' its dual.

For the sake of simplicity, we do not distinguish between scalar, vector or matrix-valued notations when it is clear from the context. In particular, $\|\cdot\|_E$ should be understood as follows

- $\|f\|_E^2 = \|f_1\|_E^2 + \cdots + \|f_d\|_E^2$ for any $f = (f_1,\cdots,f_d) \in (E)^d$.
- $\|f\|_E^2 = \sum_{i,j=1}^{d} \|f_{ij}\|_E^2$ for any $f \in \mathscr{M}_{d\times d}(E)$.

Throughout the article, we consider $Q = D\times[0,T]$, $\Omega_T = \Omega\times[0,T]$, and denote by $C, C_i, i \in \mathbb{N}$, generic constants, which may vary from line to line.
The results on the following modified Stokes problem will be very usefull to our analysis

$$\begin{cases} h - \alpha_1\Delta h + \nabla\mathbf{P} = f, \quad \operatorname{div}(h) = 0 & \text{in } D,\\ h\cdot\eta = 0, \quad (\eta\cdot\mathbb{D}(h))\big|_{\tan} = 0 & \text{on } \partial D. \end{cases} \tag{2.4}$$

The solution h will be denoted by $h = (I - \alpha_1\mathbb{P}\Delta)^{-1}f$. We recall the existence and the uniqueness results, as well as the regularity of the solution $(h,\mathbf{P})$.

Theorem 2.1. *([3], Thm. 3) Suppose that $f \in (H^m(D))^d$, $m \in \mathbb{N}$. Then there exists a unique (up to a constant for $\boldsymbol{P}$) solution $(h,\boldsymbol{P}) \in (H^{m+2}(D))^d \times H^{m+1}(D)$ of the Stokes problem (2.4) such that*

$$\|h\|_{H^{m+2}} + \|\boldsymbol{P}\|_{H^{m+1}} \le C(m)\|f\|_{H^m}, \quad \textit{where } C(m) \textit{ is a positive constant.}$$

We also consider the trilinear form

$$b(\phi,z,y)=(\phi\cdot\nabla z,y)=\int_D(\phi\cdot\nabla z)\cdot y dx,\quad \forall\phi,z,y\in(H^1(D))^d,$$

which verifies $b(y,z,\phi)=-b(y,\phi,z),\quad \forall y\in V;\forall z,\phi\in(H^1(D))^d$.

2.2. The stochastic setting

Let us consider a cylindrical Wiener process $\mathscr{W}$ defined on $(\Omega,\mathscr{F},P)$, which can be written as $\mathscr{W}(t)=\sum_{\Bbbk\geq1}e_{\Bbbk}\beta_{\Bbbk}(t)$, where $(\beta_{\Bbbk})_{\Bbbk\geq1}$ is a sequence of mutually independent real valued standard Wiener processes and $(e_{\Bbbk})_{\Bbbk\geq1}$ is a complete orthonormal system in a separable Hilbert space $\mathbb{H}$. Recall that the sample paths of $\mathscr{W}$ take values in a larger Hilbert space H_0 such that $\mathbb{H}\hookrightarrow H_0$ defines a Hilbert–Schmidt embedding. For example, the space H_0 can be defined as follows

$$H_0=\Big\{u=\sum_{\Bbbk\geq1}\gamma_{\Bbbk}e_{\Bbbk}\mid\sum_{\Bbbk\geq1}\frac{\gamma_{\Bbbk}^2}{\Bbbk^2}<\infty\Big\}\text{ endowed with the norm }\|u\|_{H_0}^2=\sum_{\Bbbk\geq1}\frac{\gamma_{\Bbbk}^2}{\Bbbk^2},\ u=\sum_{\Bbbk\geq1}\gamma_k e_{\Bbbk}.$$

Hence, P-a.s. the trajectories of $\mathscr{W}$ belong to the space $C([0,T],H_0)$ (*cf.* [21], Chap. 4).

In order to define the stochastic integral in the infinite dimensional framework, let us consider another Hilbert space E and denote by $L_2(\mathbb{H},E)$ the space of Hilbert-Schmidt operators from $\mathbb{H}$ to E, which is the subspace of the linear operators defined as follows

$$L_2(\mathbb{H},E):=\Big\{G:\mathbb{H}\to E\mid\quad\|G\|_{L_2(\mathbb{H},E)}^2:=\sum_{\Bbbk\geq1}\|Ge_{\Bbbk}\|_E^2<\infty\Big\}.$$

Given a $L_2(\mathbb{H},E)-$valued predictable[2] process $G\in L^2(\Omega;L^2(0,T;L_2(\mathbb{H},E)))$, and taking $\sigma_{\Bbbk}=Ge_{\Bbbk}$, we may define the Itô stochastic integral by $\displaystyle\int_0^t Gd\mathscr{W}=\sum_{\Bbbk\geq1}\int_0^t\sigma_{\Bbbk}d\beta_k,\ \forall t\in[0,T]$.

Next, we will precise the assumptions on the data.

2.3. Definition of the diffusion coefficient and assumptions

Let us consider a family of Carathéodory functions $\sigma_{\Bbbk}:[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}^d,\ \Bbbk\in\mathbb{N}$, which satisfies $\sigma_{\Bbbk}(t,0)=0$, and there exists $L>0$ such that for a.e. $t\in(0,T)$,

$$\sum_{\Bbbk\geq1}\left|\sigma_{\Bbbk}(t,\lambda)-\sigma_{\Bbbk}(t,\mu)\right|^2\leq L|\lambda-\mu|^2;\quad\forall\lambda,\mu\in\mathbb{R}^d. \tag{2.5}$$

In addition, we assume that there exists a sequence $(a_k)\subset\mathbb{R}_0^+$ such that

$$|\nabla\sigma_{\Bbbk}(\cdot,\cdot)|\leq a_k,\quad\sum_{\Bbbk\geq1}a_k^2<\infty, \tag{2.6}$$

[2] $\mathscr{P}_T:=\sigma(\{]s,t]\times F_s|0\leq s<t\leq T,F_s\in\mathscr{F}_s\}\cup\{\{0\}\times F_0|F_0\in\mathscr{F}_0\})$ (see [22], p. 33). Then, a process defined on Ω_T with values in a given space E is predictable if it is $\mathscr{P}_T$-measurable.

and for any $\mu \in \mathbb{R}^d$, the mapping $(t,\lambda) \to \nabla\sigma_{\Bbbk}(t,\lambda)\mu$ is also a Carathéodory function.

We notice that, in particular, (2.5) gives $\mathbb{G}^2(t,\lambda) := \sum_{\Bbbk\geq 1} |\sigma_{\Bbbk}(t,\lambda)|^2 \leq L\,|\lambda|^2$.

For each $t \in [0,T]$ and $y \in V$, we introduce the Hilbert-Schmidt operator

$$G(t,y) : \mathbb{H} \to (H^1(D))^d, \qquad G(t,y)e_{\Bbbk} = \{x \mapsto \sigma_{\Bbbk}\big(t,y(x)\big)\}, \quad \Bbbk \geq 1.$$

Since G satisfies $\sigma_{\Bbbk} = Ge_{\Bbbk}$, the stochastic integral $\displaystyle\int_0^t G(\cdot,y)d\mathscr{W} = \sum_{\Bbbk\geq 1}\int_0^t \sigma_{\Bbbk}(\cdot,y)d\beta_{\Bbbk}$ is a well-defined $(\mathscr{F}_t)_{t\in[0,T]}$-martingale with values in $(H^1(D))^d$ (resp. $(L^2(D))^d$).

$\mathscr{H}_0$: In addition, for any $t \in [0,T]$, we assume that $\sigma_{\Bbbk}(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is Gâteaux differentiable, satisfying for any $v \in \mathbb{R}^d$

$$\frac{1}{\delta}[\sigma_{\Bbbk}(t,y+\delta v) - \sigma_{\Bbbk}(t,y)] = \nabla_y\sigma_{\Bbbk}(t,y)v + R^{\delta}_{\Bbbk}(t,v), \quad \forall\,\Bbbk \geq 1, \tag{2.7}$$

with $|R^{\delta}_{\Bbbk}(t,v)| \leq b_{\Bbbk}|v|\delta^{\gamma}, \gamma > 0$, where $(b_k) \subset \mathbb{R}^+_0$ and $\sum_{\Bbbk\geq 1} b^2_{\Bbbk} < \infty$.

Remark 2.2. The additional assumption $\mathscr{H}_0$ on $(\sigma_{\Bbbk})_{\Bbbk}$ is assumed to obtain the necessary optimality condition (see Sect. 6 and Sect. 8).

2.4. Assumptions on the data

$\mathscr{H}_1$: we consider $y_0 : \Omega \to \widetilde{W}$ and $U : \Omega \times [0,T] \to (H^1(D))^d$ such that

- y_0 is $\mathscr{F}_0$-measurable and U is predictable.
- $U \in L^p(\Omega\times(0,T);(H^1(D))^d), \quad y_0 \in L^p(\Omega,\widetilde{W}), \quad \text{for fixed} \quad p > 2(d+1).$ (2.8)

Remark 2.3. The assumption $p > 2(d+1)$ ensures uniform estimates for the solution of finite dimensional linearized and adjoint equations without the special weights used in [16].

Unless otherwise stated, p always satisfies $p > 2(d+1)$ in the following.

2.5. Existence of local strong solution

Let us recall the notion of the local solution and the pathwise uniqueness. This subsection is based on the results from [10].

Definition 2.4. Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},P)$ be a stochastic basis and $\mathscr{W}(t)$ be a $(\mathscr{F}_t)$-cylindrical Wiener process. We say that a pair (y,τ) is a local strong (pathwise) solution to (2.1) iff:

- τ is an a.s. strictly positive $(\mathscr{F}_t)$-stopping time.
- The velocity y is a W-valued predictable process satisfying

$$y(\cdot\wedge\tau) \in L^p(\Omega;\mathscr{C}([0,T];W^{2,4}(D))) \cap L^p_w(\Omega;L^\infty(0,T;\widetilde{W}))^3, \quad \text{for } p > 4.$$

[3] $L^p_w(\Omega;L^\infty(0,T;\widetilde{W})) = \{u : \Omega \to L^\infty(0,T;\widetilde{W})$ is weakly-* measurable and $E\|u\|^p_{L^\infty(0,T;\widetilde{W})} < \infty\}$.

- P-a.s. for all $t \in [0,T]$

$$(y(t\wedge\tau),\phi)_V = (y_0,\phi)_V + \int_0^{t\wedge\tau} \big(\nu\Delta y - (y\cdot\nabla)v(y) - \sum_j v(y)^j\nabla(y)^j + (\alpha_1+\alpha_2)\text{div}[A(y)^2]$$
$$+ \beta\text{div}[|A(y)|^2A(y)] + U,\phi\big)ds + \int_0^{t\wedge\tau}(G(\cdot,y),\phi)d\mathscr{W} \text{ for all } \phi \in V.$$

Definition 2.5. i) We say that local pathwise uniqueness holds if for any given pair $(y^1,\tau^1),(y^2,\tau^2)$ of local strong solutions of (2.1), we have

$$P\big(y^1(t) = y^2(t);\ \forall t \in [0,\tau^1\wedge\tau^2]\big) = 1.$$

ii) We say that $((y_M)_{M\in\mathbb{N}},(\tau_M)_{M\in\mathbb{N}},\mathbf{t})$ is a maximal strong local solution to (2.1) if for each $M \in \mathbb{N}$, the pair (y_M,τ_M) is a local strong solution, $(\tau_M)_M$ is an increasing sequence of stopping times such that $\mathbf{t} := \lim_{M\to\infty}\tau_M > 0$, P-a.s. and

$$\sup_{t\in[0,\tau_M]}\|y(t)\|_{W^{2,4}} \geq M \text{ on } \quad \{\mathbf{t} < T\}, \quad \forall M \in \mathbb{N}, \ \text{ P-a.s.} \tag{2.9}$$

Theorem 2.6. *There exists a unique maximal strong (pathwise) local solution to* (2.1) *with*

$$\tau_M = \inf\{t \geq 0 : \|y(t)\|_{W^{2,4}} \geq M\}\wedge T; \quad M \in \mathbb{N}. \tag{2.10}$$

Remark 2.7. We recall that the local strong solution of (2.1) have been constructed by using y_M (see [10], Lem. 19). In other words, we have: $\forall M \in \mathbb{N} : y(t) := y_M(t),\ \forall t \in [0,\tau_M]$, P-a.s.

Following [10], we have the following estimates for the solution of (2.1).

Proposition 2.8. *There exists* $K := K(L,M,T,\|y_0\|_{L^p(\Omega;\widetilde{W})},\|U\|_{L^p(\Omega\times[0,T];(H^1(D))^d)}) > 0$ *such that*

$$\mathbb{E}\sup_{s\in[0,\tau_M]}\|y\|_V^2 + 4\nu\mathbb{E}\int_0^{\tau_M}\|Dy\|_2^2dt + \frac{\beta}{2}\mathbb{E}\int_0^{\tau_M}\int_D|A(y)|^4dxdt \leq e^{cT}(\mathbb{E}\|y_0\|_V^2 + \mathbb{E}\int_0^T\|U\|_2^2dt),$$
$$\mathbb{E}\sup_{s\in[0,\tau_M]}\|y\|_{\widetilde{W}}^2 := \mathbb{E}\sup_{s\in[0,\tau_M]}[\|\mathit{curl}\ v(y)\|_2^2 + \|y\|_V^2] \leq K,$$
$$\mathbb{E}\sup_{[0,\tau_M]}\|y\|_{\widetilde{W}}^p \leq K(M,T,p)(1+\mathbb{E}\|y_0\|_{\widetilde{W}}^p + \mathbb{E}\int_0^T\|U\|_2^pds + \mathbb{E}\int_0^T\|\ \mathit{curl}\ U\|_2^pds), \quad \forall p > 2.$$

3. Stability results

In this section, we assume that the initial data y_0 and the force U satisfy $\mathscr{H}_1$, and consider two strong local solutions (y_1,τ_M^1) and (y_2,τ_M^2) to (2.1) in the sense of Definition 2.4 with the initial conditions y_0^1, y_0^2 and the forces U_1, U_2, respectively. In addition, we denote $y = y_1 - y_2, y_0 = y_0^1 - y_0^2$ and $U = U_1 - U_2$.

First, we state a result, which follows from [10], Lemma 18.

Lemma 3.1. *For all* $p \geq 2$, *there exists* $C(M,L,T,p) > 0$ *such that*

$$\mathbb{E}\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_1(s)-y_2(s)\|_V^p \leq C(M,L,T,p)\big[\mathbb{E}\|y_0^1-y_0^2\|_V^p + \mathbb{E}\int_0^{\tau_M^1\wedge\tau_M^2}\|U_1(s)-U_2(s)\|_2^pds\big].$$

Proof. Let us consider $1 \leq p < \infty$ and $t \in [0,T]$. For any $s \in [0, t\wedge \tau_M^1 \wedge \tau_M^2]$, from the proof of [10], Lemma 18, there exists $M_0 > 0$ such that

$$\|y(s)\|_V^2 + 4\nu \int_0^s \|\mathbb{D}y\|_2^2 dr \leq \|y_0\|_V^2 + M_0 \int_0^s (\|y_1\|_{W^{2,4}} + \|y_2\|_{W^{2,4}} + 1)\|y\|_V^2 dr + \int_0^s \|U_1 - U_2\|_2^2 dr + 2\int_0^s (G(\cdot, y_1) - G(\cdot, y_2), y) d\mathscr{W}. \tag{3.1}$$

Therefore, we have

$$\|y(s)\|_V^{2p} \leq C(p, \tau_M^1 \wedge \tau_M^2)\big\{\|y_0\|_V^{2p} + M_0 \int_0^s (\|y_1\|_{W^{2,4}} + \|y_2\|_{W^{2,4}} + 1)^p \|y\|_V^{2p} dr + \int_0^s \|U\|_2^{2p} dr + |\int_0^s (G(\cdot, y_1) - G(\cdot, y_2), y) d\mathscr{W}|^p\big\}, \quad \forall s \in [0, t\wedge \tau_M^1 \wedge \tau_M^2]. \tag{3.2}$$

Using Burkholder-Davis-Gundy inequality, we infer that

$$\mathbb{E} \sup_{s\in[0,t\wedge\tau_M^1\wedge\tau_M^2]} \|y(s)\|_V^{2p} \leq C(p, L, T)\{\mathbb{E}\|y_0\|_V^{2p} + \mathbb{E}\int_0^{t\wedge\tau_M^1\wedge\tau_M^2} \|U\|_2^{2p} ds + M_0(2M+1)^p \int_0^t \mathbb{E} \sup_{s\in[0,r\wedge\tau_M^1\wedge\tau_M^2]} \|y(s)\|_V^{2p} dr\}.$$

Finally, Grönwall's inequality ensures the result of Lemma 3.1. □

In order to derive the necessary optimality condition, we need a stability result with respect to W-norm. This is the aim of the next proposition.

Proposition 3.2. *There exists $K(M, \epsilon, p) > 0$, which depends only on the data such that*

$$\mathbb{E} \sup_{s\in[0,\tau_M^1\wedge\tau_M^2]} \|(y_1 - y_2)(s)\|_W^{2p} \leq K(M, \epsilon)\big\{\mathbb{E}\|y_0^1 - y_0^2\|_W^{2p} + \mathbb{E}\int_0^{\tau_M^1\wedge\tau_M^2} \|U_1 - U_2\|_2^{2p} ds + \|y_1\|^{2p(d+\epsilon)}_{L^{2p(d+1+\epsilon)}(\Omega\times(0,\tau_M^1\wedge\tau_M^2);(H^3)^d)} \|y\|^{2p}_{L^{2p(d+1+\epsilon)}(\Omega\times(0,\tau_M^1\wedge\tau_M^2);V)}\big\}, \tag{3.3}$$

where $p \in [1, \infty[$ and $\epsilon \in]0, 1]$.

Proof. For any $t \in [0, \tau_M^1 \wedge \tau_M^2]$, we have

$$\begin{aligned} v(y(t)) - v(y_0) = &- \int_0^t \nabla(p_1 - p_2) ds + \nu \int_0^t \Delta y - [(y_1\cdot\nabla)v(y_1) - (y_2\cdot\nabla)v(y_2)] ds \\ &- \sum_{j=1}^d \int_0^t [v(y_1)^j \nabla y_1^j - v(y_2)^j \nabla y_2^j] ds + (\alpha_1 + \alpha_2) \int_0^t [\mathrm{div}(A(y_1)^2) - \mathrm{div}(A(y_2)^2)] ds \\ &+ \beta \int_0^t [\mathrm{div}(|A(y_1)|^2 A(y_1)) - \mathrm{div}(|A(y_2)|^2 A(y_2))] ds + \int_0^t U ds + \int_0^t [G(\cdot, y_1) - G(\cdot, y_2)] d\mathscr{W}(s). \end{aligned} \tag{3.4}$$

From (3.1) we have

$$d\|y\|_V^2 + 4\nu\|D(y)\|_2^2 dt \leq K(1 + \|y_1\|_{W^{2,4}}^2 + \|y_2\|_{W^{2,4}}^2)\|y\|_W^2 dt + 2(G(\cdot, y_1) - G(\cdot, y_2), y) d\mathscr{W} + \|U\|_2^2 dt. \tag{3.5}$$

On the other hand, applying the operator $\mathbb{P}$ to (3.4) and using Itô formula, we derive

$$\begin{aligned}
d\|\mathbb{P}v(y)\|_2^2 &= 2\nu\big(\Delta(y),\mathbb{P}v(y)\big)dt - 2\big([(y_1\cdot\nabla)v(y_1)-(y_2\cdot\nabla)v(y_2)],\mathbb{P}v(y)\big)dt \\
&- 2\big(\sum_j[v(y_1)^j\nabla y_1^j-\theta_v(y_2)^j\nabla y_2^j],\mathbb{P}v(y)\big)dt + \big(\alpha_1+\alpha_2)([\mathrm{div}(A(y_1)^2)-\mathrm{div}(A(y_2)^2)],\mathbb{P}v(y)\big)dt \\
&+ 2\beta\big([\mathrm{div}(|A(y_1)|^2A(y_1))-\mathrm{div}(|A(y_2)|^2A(y_2))],\mathbb{P}v(y)\big)dt + 2\big(U,\mathbb{P}v(y)\big)dt \\
&+ \sum_{\mathbb{k}\geq 1}\|\mathbb{P}[\sigma_{\mathbb{k}}(\cdot,y_1)-\sigma_{\mathbb{k}}(\cdot,y_1)]\|_2^2dt + 2\big(G(\cdot,y_1)-G(\cdot,y_2),\mathbb{P}v(y)\big)d\mathscr{W}.
\end{aligned}$$

Let us estimate the terms in the right hand side of this equation. First, note that

$$2\nu\big(\mathbb{P}\Delta(y),\mathbb{P}v(y)\big)\leq \frac{-2\nu}{\alpha_1}\|\mathbb{P}v(y)\|_2^2+\frac{2\nu}{\alpha_1}\|y\|_W^2 \quad \text{and} \quad 2\big(U,\mathbb{P}v(y)\big)\leq \|U\|_2^2+\|y\|_{H^2}^2.$$

The properties of the projection $\mathbb{P}$ and (2.5) give $\sum_{\mathbb{k}\geq 1}\|\mathbb{P}[\sigma_{\mathbb{k}}(\cdot,y_1)-\sigma_{\mathbb{k}}(\cdot,y_1)]\|_2^2\leq L\|y_1-y_2\|_2^2$. Next, Sobolev embedding theorem (see *e.g.* [23], Ch. 1) and [8], Lemma 5 ensure that

$$\begin{aligned}
&\big(\{(y_1\cdot\nabla)v(y_1)-(y_2\cdot\nabla)v(y_2)\},\mathbb{P}v(y)\big) = [b(y,v(y_1),\mathbb{P}v(y))+b(y_2,v(y)-\mathbb{P}v(y),\mathbb{P}v(y))] \\
&\leq \|y\|_\infty\|y_1\|_{H^3}\|y\|_{H^2}+C\|y_2\|_\infty\|y\|_{H^2}\|y\|_{H^2}\leq \|y\|_\infty\|y_1\|_{H^3}\|y\|_{H^2}+C\|y_2\|_{W^{2,4}}\|y\|_W^2,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^d\big([v(y_1)^j\nabla y_1^j-v(y_2)^j\nabla y_2^j],\mathbb{P}v(y)\big) = b(\mathbb{P}v(y),y_2,v(y))+b(\mathbb{P}v(y),y,v(y_1)) \\
&\quad\leq \|y\|_{H^2}\|y_2\|_{W^{1,\infty}}\|y\|_{H^2}+\|y\|_{H^2}\|y\|_{W^{1,4}}\|y_1\|_{W^{2,4}}\leq K(\|y_1\|_{W^{2,4}}+\|y_2\|_{W^{2,4}})\|y_1-y_2\|_W^2.
\end{aligned}$$

On the other hand, using the embeddings $H^2(D)\hookrightarrow W^{1,4}(D)$, $W^{2,4}(D)\hookrightarrow W^{1,\infty}(D)$, we show the existence of $K>0$ such that

$$\begin{aligned}
&\big(\mathrm{div}(A(y_1)^2)-\big(\mathrm{div}(A(y_2)^2),\mathbb{P}v(y)\big) = \big(\ \mathrm{div}(A(y)A(y_1))+\ \mathrm{div}(A(y_2)A(y)),\mathbb{P}v(y))\big) \\
&\leq K(\|y_1\|_{W^{2,4}}+\|y_2\|_{W^{2,4}})\|y\|_W^2.
\end{aligned}$$

The Hölder inequality and the embeeding $W^{2,4}(D)\hookrightarrow W^{1,\infty}(D)$ yield

$$\begin{aligned}
&\big(\mathrm{div}(|A(y_1)|^2A(y_1))-\mathrm{div}(|A(y_2)|^2A(y_2)),\mathbb{P}v(y)\big) \\
&=\big(\mathrm{div}(|A(y_1)|^2A(y)),\mathbb{P}v(y)\big)+\big(\mathrm{div}([A(y_1):A(y)+A(y):A(y_2)]A(y_2)),\mathbb{P}v(y)\big) \\
&\leq K(\|y_1\|_{W^{2,4}}^2+\|y_2\|_{W^{2,4}}^2)\|y\|_W^2.
\end{aligned}$$

Hence

$$\begin{aligned}
d\|\mathbb{P}v(y)\|_2^2+\frac{2\nu}{\alpha_1}\|\mathbb{P}v(y)\|_2^2dt &\leq K(1+\|y_1\|_{W^{2,4}}^2+\|y_2\|_{W^{2,4}}^2)\|y\|_W^2 \\
&+2(G(\cdot,y_1)-G(\cdot,y_2),\mathbb{P}v(y))d\mathscr{W}+\|U\|_2^2dt+K\|y\|_\infty\|y_1\|_{H^3}\|y\|_{H^2}dt. \qquad (3.6)
\end{aligned}$$

Gathering (3.5) and (3.6), for any $t \in [0, \tau_M^1 \wedge \tau_M^2]$, we deduce the following relation

$$\|y(t)\|_W^2 + \min(1, \frac{1}{2\alpha_1})4\nu \int_0^t \|y\|_W^2 ds \leq \|y_0\|_W^2 + K\int_0^t (1 + \|y_1\|_{W^{2,4}}^2 + \|y_2\|_{W^{2,4}}^2)\|y\|_W^2 ds$$
$$+ 2|\int_0^t (G(\cdot, y_1) - G(\cdot, y_2), \mathbb{P}v(y) + y)d\mathscr{W}| + 2\int_0^t \|U\|_2^2 ds + K\int_0^t \|y\|_\infty \|y_1\|_{H^3}\|y\|_{H^2} ds. \tag{3.7}$$

Let $p \geq 1$, thanks to the Burkholder-Davis-Gundy inequality, for any $\delta > 0$, we have

$$2\mathbb{E} \sup_{s\in[0,t\wedge\tau_M^1\wedge\tau_M^2]} \Big| \int_0^s (G(\cdot, y_1) - G(\cdot, y_1), y + \mathbb{P}v(y))d\mathscr{W}\Big|^p$$
$$\leq \delta\mathbb{E} \sup_{s\in[0,t\wedge\tau_M^1\wedge\tau_M^2]} \|y\|_2^{2p} + \delta\mathbb{E} \sup_{s\in[0,t\wedge\tau_M^1\wedge\tau_M^2]} \|\mathbb{P}v(y)\|_2^{2p} + C_\delta \mathbb{E}\int_0^{t\wedge\tau_M^1\wedge\tau_M^2} \|y\|_2^{2p} dr,$$

Taking the p^{th} power of (3.7) and the expectation, we write

$$\mathbb{E} \sup_{s\in[0,t\wedge\tau_M^1\wedge\tau_M^2]} \|y(s)\|_W^{2p} \leq \mathbb{E}\|y_0\|_W^{2p} + K(\delta)\mathbb{E}\int_0^{t\wedge\tau_M^1\wedge\tau_M^2} (1 + \|y_1\|_{W^{2,4}}^2 + \|y_2\|_{W^{2,4}}^2)^p \|y\|_W^{2p} ds$$
$$+ \delta\mathbb{E} \sup_{s\in[0,t\wedge\tau_M^1\wedge\tau_M^2]} \|y\|_W^{2p} + K(\delta)\mathbb{E}\int_0^{\tau_M^1\wedge\tau_M^2} \|U\|_2^{2p} ds + K\mathbb{E}\int_0^{t\wedge\tau_M^1\wedge\tau_M^2} (\|y\|_\infty \|y_1\|_{H^3}\|y\|_{H^2})^p ds.$$

Since $\|y_i(t)\|_{W^{2,4}}^{i=1,2} \leq M, \forall t \in [0, \tau_M^1 \wedge \tau_M^2]$. An appropriate choice of δ and Lemma 10.2 ensure

$$\mathbb{E} \sup_{s\in[0,t\wedge\tau_M^1\wedge\tau_M^2]} \|y(s)\|_W^{2p} \leq \mathbb{E}\|y_0\|_W^{2p} + K(\delta, M, \epsilon)\mathbb{E}\int_0^{t\wedge\tau_M^1\wedge\tau_M^2} \|y\|_W^{2p} ds$$
$$+ K(\delta)\mathbb{E}\int_0^{\tau_M^1\wedge\tau_M^2} \|U\|_2^{2p} ds + K(\epsilon)\|y_1\|_{L^{2p(d+1+\epsilon)}(\Omega\times(0,\tau_M^1\wedge\tau_M^2);(H^3)^d)}^{2p(d+\epsilon)} \|y\|_{L^{2p(d+1+\epsilon)}(\Omega\times(0,\tau_M^1\wedge\tau_M^2);V)}^{2p}.$$

Therefore, Grönwall's lemma yields (3.3). □

As a consequence of Proposition 3.2 and Lemma 3.1, we establish the following corollary:

Corollary 3.3. *Consider U and y_0 satisfying (2.8) and $\psi \in L^p((\Omega_T, \mathscr{P}_T); (H^1(D))^d)$. Defining $U_\rho = U + \rho\psi$, $\rho \in (0, 1)$, let (y, τ_M) and (y_ρ, τ_M^ρ) be the solutions of (2.1) associated with (U, y_0) and (U_ρ, y_0), respectively. Then, there exist $C, K > 0$ such that*

$$\mathbb{E} \sup_{s\in[0,\tau_M\wedge\tau_M^\rho]} \|y_\rho(s) - y(s)\|_V^p \leq C(M, L, T, p)\rho^p \mathbb{E}\int_0^{\tau_M\wedge\tau_M^\rho} \|\psi(s)\|_2^p ds \leq C_\psi(M)\rho^p, \quad \textit{for } p \geq 2,$$
$$\mathbb{E} \sup_{s\in[0,\tau_M\wedge\tau_M^\rho]} \|(y_\rho - y)(s)\|_W^q \leq K(M, \epsilon)\{\rho^q \mathbb{E}\int_0^{\tau_M\wedge\tau_M^\rho} \|\psi\|_2^q ds + \|y\|_{L^{qd}(\Omega_\tau;(H^3)^d)}^{q(d+\epsilon)} \|y_\rho - y\|_{L^{qd}(\Omega_\tau;V)}^q\},$$

for any $q \in [2,\infty[$ *and* $\epsilon \in]0,1]$. *In particular, for* $q = 2+\epsilon$, *we have*

$$\mathbb{E}\sup_{s\in[0,\tau_M\wedge\tau_M^\rho]}\|(y_\rho - y)(s)\|_W^{2+\epsilon} \leq K(M,\epsilon)\{\rho^{2+\epsilon}\mathbb{E}\int_0^{\tau_M\wedge\tau_M^\rho}\|\psi\|_2^{2+\epsilon}ds + C\|y_\rho - y\|^{2+\epsilon}_{L^{(2+\epsilon)(d+1+\epsilon)}(\Omega_\tau;V)}\}$$
$$\leq K(M,\epsilon)\rho^{2+\epsilon} \text{ where } \Omega_\tau = \Omega\times(0,\tau_M\wedge\tau_M^\rho),\ q_d = q(d+1+\epsilon).$$

4. The optimal control problem and main results

4.1. Cost functional

Our aim is to control the solution of the system (2.1) through a stochastic distributed force U, which belongs to the admissible set $\mathscr{U}_{ad}^p$ defined as a nonempty, bounded, closed and convex subset of $L^p((\Omega_T,\mathscr{P}_T);(H^1(D))^d)$, such that $0\in\mathscr{U}_{ad}^p$[4].

Let $(y(t,U))_{t\in[0,\tau_M^U)}$ be the local pathwise solution of (2.1). We recall that $y := y(t,U) = y_M(t,U)$, $t\in[0,\tau_M^U]$, and the stopping times $(\tau_M^U)_{M\in\mathbb{N}}$ depending on the control $U\in\mathscr{U}_{ad}^p$. Therefore, the solution $y(t,U)$ is defined up to a certain stopping time. Thus, we introduce the cost functional $J_M : \mathscr{U}_{ad}^p \to \mathbb{R}$ defined by

$$J_M(U,y) = \frac{1}{2}\mathbb{E}\int_0^{\tau_M^U}\|y-y_d\|_2^2dt + \frac{\lambda}{p}\mathbb{E}\int_0^T\|U\|^p_{(H^1(D))^d}dt, \quad \lambda>0. \tag{4.1}$$

With a desired target field $y_d\in L^2(0,T;H)$[5]. Our goal is to solve the following problem

$$\mathscr{P}\left\{\min_U\{J_M(U,y) :\ U\in\mathscr{U}_{ad}^p \text{ and } y \text{ is the solution of (2.1) for the minimizing } U\}.\right. \tag{4.2}$$

In other words, we intend to find $U_M^*\in\mathscr{U}_{ad}^p$ such that $J_M(U_M^*,y(U_M^*)) = \min_{U\in\mathscr{U}_{ad}^p} J_M(U,y)$.

Remark 4.1. We wish to draw the reader's attention to the fact that with the inclusion of τ_M^U in the definition of the cost functional, we are able to consider 2D and 3D domains. On the other hand, even if $d=2$ and $\beta\neq 0$, we do not know if $\tau_M^U = T$. In the case $\beta=0$ and $d=2$, we get $\tau_M^U=T$ and we can recover the stochastic optimal control problem studied in [16]. Moreover, in the two-dimensional deterministic case *i.e.* $G\equiv 0$, one recovers the deterministic optimal control problem studied in [18] with $\tau_M^U = T$.

4.2. Main results

Here, we present the main results of our article, which show the existence of a solution to the control problem and establish the first order optimality conditions.

Theorem 4.2. *Assume* $\mathscr{H}_1$. *Then the control problem* (4.2) *admits a unique optimal solution* $(\widetilde{U_M},\tilde{y})\in\mathscr{U}_{ad}^p\times L^p(\Omega;L^p(0,T;\widetilde{W}))$, *where* $\tilde{y} := y(\widetilde{U_M})$ *is the unique solution of* (2.1) *with* $U=\widetilde{U_M}$. *Moreover, under the assumption* $\mathscr{H}_0$*:*

- *there exists a unique solution* $\tilde{z}$ *of the linearized equations* (5.3), *in 2D and 3D with* $y=\tilde{y}$ *and* $\psi = \psi - \widetilde{U_M}$*;*
- *if* $d=2$, *there exists a unique solution* $(\tilde{\mathfrak{p}},\tilde{\mathfrak{q}})$ *of the stochastic backward adjoint equation* (7.2), *with forces* $g=\tilde{y}-y_d$*;*
- *if* $d=3$, *there exists, at least, one solution* $(\tilde{\mathbf{p}},\tilde{\mathbf{q}})$ *of the stochastic backward adjoint equation* (7.23), *with forces* $g=\tilde{y}-y_d$.

[4] For example, $\mathscr{U}_{ad}^p = \{u\in L^p((\Omega_T,\mathscr{P}_T);(H^1(D))^d) := Y;\quad \|u\|_Y\leq K\},\quad 0<K<\infty$.

[5] In general, y_d is smoother with respect to the space variables and satisfies the boundary conditions. Here, we assume what is necessary to perform the analysis.

In addition, the duality property

$$\mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(\psi-\widetilde{U_M},\tilde{p})ds=\mathbb{E}\int_0^T\mathbb{I}_{[0,\tau_M[}(s)(y(\widetilde{U_M})-y_d,z)ds=\mathbb{E}\int_0^T\mathbb{I}_{[0,\tau_M[}(s)(\psi-\widetilde{U_M},\tilde{\mathbf{p}})dt \tag{4.3}$$

is valid for any $\psi\in\mathscr{U}_{ad}^p$, and the following optimality condition holds, for any $\psi\in\mathscr{U}_{ad}^p$

$$\begin{aligned}&\mathbb{E}\int_0^T\big(\lambda\|\widetilde{U_M}\|_{(H^1(D))^2}^{p-2}(\widetilde{U_M},\psi-\widetilde{U_M})_{(H^1(D))^2}+\mathbb{I}_{[0,\tau_M[}(\psi-\widetilde{U_M},\tilde{p})\big)ds\geq 0;\\&\mathbb{E}\int_0^T\big(\lambda\|\widetilde{U_M}\|_{(H^1(D))^3}^{p-2}(\widetilde{U_M},\psi-\widetilde{U_M})_{(H^1(D))^3}+\mathbb{I}_{[0,\tau_M[}(\psi-\widetilde{U_M},\tilde{\mathbf{p}})\big)ds\geq 0.\end{aligned} \tag{4.4}$$

Proof. The proof of Theorem 4.2 results from the combination of Theorem 4.7, Theorem 5.3, Theorem 7.3, Theorem 7.7 and Section 8. □

Remark 4.3. In Theorem 4.2, we did not distinguish between the notations in 2D and 3D for z, $\tilde{y}-y_d$, the control U and ψ, since it is clear from the context.

4.3. Existence and uniqueness of optimal control

Let us notice that the velocity field and the stopping times are non-convex with respect to the control U, then (4.2) is a non-convex optimization problem. Our strategy to show the existence and uniqueness of the optimal solution relies on the lower semi-continuity of the cost functional and the application of a result from [24] (see Thm. 4.7). First, let us prove the following results, which will play an important role in the proof of existence of optimal control associated with the cost funtional (4.1).

Lemma 4.4. *For fixed $M\in\mathbb{N}$, let $(y_M(t;U))_{t\in[0,T]}$ be the solution to (2.1) in the sense of Definition 2.4 corresponding to the control $U\in L^p(\Omega_T,(H^1(D))^d)$ and the stopping time τ_M^U. Then we have*

$$\lim_{U_1\to U_2}P(\tau_M^{U_1}\neq\tau_M^{U_2})=0,\ \textit{where } U_1\to U_2 \textit{ means } \|U_1-U_2\|_{L^p(\Omega_T,(H^1(D))^d)}\to 0$$

$$\textit{and}\qquad \tau_M^{U_i}=\tau_M^i=\inf\{t\geq 0:\|y_i(t)\|_{W^{2,4}}\geq M\}\wedge T. \tag{4.5}$$

Proof. Let us consider $\epsilon,\delta,\gamma>0$ and two solutions $(y_M(t,U_i))_{t\in[0,T]}^{i=1,2}$ of equation (2.1), corresponding to $(U_i)^{i=1,2}$, respectively, with the same initial data y_0. Standard computations give

$$P\big(\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_M(s,U_1)-y_M(s,U_2)\|_{W^{2,4}}\geq\delta\big)\leq\frac{1}{\delta^{4(1+\gamma)}}\mathbb{E}\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_M(s,U_1)-y_M(s,U_2)\|_{W^{2,4}}^{4(1+\gamma)}.$$

On the one hand, due to the Hölder inequality there exists $C>0$ such that

$$\begin{aligned}&\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_M(s,U_1)-y_M(s,U_2)\|_{W^{2,4}}^{4(1+\gamma)}\\&\quad\leq C\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_M(s,U_1)-y_M(s,U_2)\|_{W}^{1+\gamma}\cdot\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_M(s,U_1)-y_M(s,U_2)\|_{W^{2,6}}^{3(1+\gamma)}.\end{aligned}$$

We recall that $p > 2(d+1)$. Therefore the Cauchy Schwarz inequality yields

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_M(s,U_1)-y_M(s,U_2)\|_{W^{2,4}}^{4(1+\gamma)}\\
&\leq \big(\mathbb{E}\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_M(s,U_1)-y_M(s,U_2)\|_{W}^{2(1+\gamma)}\big)^{1/2}\cdot\big(\mathbb{E}\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_M(s,U_1)-y_M(s,U_2)\|_{W^{2,6}}^{6(1+\gamma)}\big)^{1/2}\\
&\leq C(M,\epsilon)\sqrt{\mathbb{E}\int_0^{\tau_M^1\wedge\tau_M^2}\|U_1-U_2\|_2^{2(1+\gamma)}ds + C\|U_1-U_2\|^{2(1+\gamma)}_{L^{2(1+\gamma)(d+1+\epsilon)}(\Omega\times(0,\tau_M^1\wedge\tau_M^2);L^2)}} := A_M(\epsilon,\gamma),
\end{aligned}$$

where we used Proposition 3.2 and Proposition 2.8 to deduce the last inequality. Therefore

$$P\big(\sup_{s\in[0,\tau_M^1\wedge\tau_M^2]}\|y_M(s,U_1)-y_M(s,U_2)\|_{W^{2,4}}\geq\delta\big)\leq\frac{A_M(\epsilon,\gamma)}{\delta^{4(1+\gamma)}}\overset{U_1\to U_2}{\longrightarrow}0. \tag{4.6}$$

On the other hand, assume that $\lim\limits_{U_1\to U_2} P(\tau_M^{U_1} > \tau_M^{U_2}) > 0$. From the definition of the stopping times, we get $\lim\limits_{U_1\to U_2} P(\{\|y_M(\tau_M^{U_2},U_2)\|_{W^{2,4}} \geq M\}\cap\{\|y_M(\tau_M^{U_2},U_1)\|_{W^{2,4}} < M\}) > 0$. Therefore, there exists $\delta_0 > 0$ such that $\lim\limits_{U_1\to U_2} P(\|y_M(\tau_M^{U_2},U_2)\|_{W^{2,4}} - \|y_M(\tau_M^{U_2},U_1)\|_{W^{2,4}} \geq \delta_0) > 0$, which gives $\lim\limits_{U_1\to U_2} P\big(\|y_M(\tau_M^{U_2},U_2) - y_M(\tau_M^{U_2},U_1)\|_{W^{2,4}} \geq \delta_0\big) > 0$, which contradicts (4.6). Hence $\lim\limits_{U_1\to U_2} P(\tau_M^{U_1} > \tau_M^{U_2}) = 0$. A similar reasoning yields $\lim\limits_{U_1\to U_2} P(\tau_M^{U_2} > \tau_M^{U_1}) = 0$. □

Next, we establish a result which will be useful in the analysis of the variation of the cost functional (4.1).

Corollary 4.5. *Let us consider U and y_0 satisfying (2.8) and $\psi \in L^p((\Omega_T,\mathscr{P}_T);(H^1(D))^d)$. Defining $U_\rho = U + \rho\psi$, $\rho\in(0,1)$, let (y,τ_M) and (y_ρ,τ_M^ρ) be the solutions of (2.1) associated with (U,y_0) and (U_ρ,y_0), respectively. Then we have $\lim_{\rho\to 0}\dfrac{P(\tau_M^U\neq\tau_M^{U_\rho})}{\rho}=0$.*

Proof. Let $\gamma > 0$. Using similar arguments as in the proof of Lemma 4.4, we can show that $P\big(\sup_{s\in[0,\tau_M\wedge\tau_M^\rho]}\|y_\rho-y\|_{W^{2,4}}\geq\delta\big)\leq\dfrac{C(M,\epsilon)}{\delta^{4(1+\gamma)}}\rho^{1+\gamma}$, which implies the desired result. □

In order to simplify the notation, we set $X := L^p((\Omega_T,\mathscr{P}_T);(H^1(D))^d))$ and denote by $\|\cdot\|_X$ the natural norm on X. The subset $\mathscr{U}_{ad}^p\subset X$ is endowed with the induced topology.

Lemma 4.6. *For fixed $M\in\mathbb{N}$. Let $y_d\in L^2(0,T;H)$ fixed, and $y(U)=(y_M(t;U),\tau_M^U)$, $t\in[0,T]$, be the local pathwise solution of (2.1) associated with the control $U\in\mathscr{U}_{ad}^p$ and stopping times $(\tau_M^U)_{M\in\mathbb{N}}$. Then the mapping $f_M:\mathscr{U}_{ad}^p\to\mathbb{R}$ given by $f_M(U,y(U))=\mathbb{E}\displaystyle\int_0^{\tau_M^U}\|y(U)-y_d\|_2^2dt$ is continuous.*

Proof. Let $y_M(t;U_i))$, $t\in[0,T]$, be the strong solution of (2.1), in the sense of Definition 2.4, corresponding to the control $U_i\in\mathscr{U}_{ad}^p$, $i=1,2$, respectively. We introduce the stopping times $\underline{\tau_M}=\tau_M^{U_1}\wedge\tau_M^{U_2}$, $\overline{\tau_M}=\tau_M^{U_1}\vee\tau_M^{U_2}$[6] and the control $\bar{u}\in\mathscr{U}_{ad}^p$ defined by $\bar{u}=\begin{cases}U_1 & \text{if } \overline{\tau_M}=\tau_M^{U_1},\\ U_2 & \text{if } \overline{\tau_M}=\tau_M^{U_2}.\end{cases}$

[6]We recall that $a\wedge b=\min(a,b)$ and $a\vee b=\max(a,b)$ for $a,b\in\mathbb{R}$.

The triangular inequality and Lemma 3.1 yield

$$|f_M(U_1, y(U_1)) - f_M(U_2, y(U_2))| \leq C\mathbb{E}\int_0^{\tau_M} \|y(U_1) - y(U_2)\|_2^2 dt + \mathbb{E}\int_{\underline{\tau_M}}^{\overline{\tau_M}} \|y(\bar{u}) - y_d\|_2^2 dt$$
$$\leq C(M, L)\mathbb{E}\int_0^{\tau_M} \|U_1(s) - U_2(s)\|_2^2 ds + \int_0^T P(\underline{\tau_M} < t \leq \overline{\tau_M})(C^2M^2 + \|y_d\|_2^2)dt.$$

Taking into account Lemma 4.4, we deduce $\lim\limits_{\|U_1-U_2\|_X \to 0} |f_M(U_1, y(U_1)) - f_M(U_2, y(U_2))| = 0$, which gives the continuity of f_M. □

Next, we establish a main result on the existence and uniqueness of optimal control for (4.1).

Theorem 4.7. *Let us consider the cost functional $J_M : \mathscr{U}_{ad}^p \to \mathbb{R}$ given by (4.1) for fixed $M \in \mathbb{N}$. Then there exists a unique optimal control $\widetilde{U_M} \in \mathscr{U}_{ad}^p$.*

Proof. Notice that X is a uniformly convex Banach space. The set $\mathscr{U}_{ad}^p$ is bounded and closed convex subset of X. Thanks to Lemma 4.6, for $\lambda > 0$ we know that $\frac{p}{2\lambda} f_M : \mathscr{U}_{ad}^p \to \mathbb{R}$ is continuous and $f_M(U) \geq 0$ for every $U \in \mathscr{U}_{ad}^p$. By applying [24], Theorem 4, we infer the existence of a dense subset $M_0 \subset \mathscr{U}_{ad}^p$ such that for any $v \in M_0$, the functional $\frac{p}{2\lambda} f_M(U) + \|U - v\|_X^p$ attains its minimum over $\mathscr{U}_{ad}^p$. In other words, there exists a unique $U_M(v) \in \mathscr{U}_{ad}^p$ (by taking $\alpha = p$ in [24], Thm. 4) such that $(\frac{p}{2\lambda} f_M(U_M(v)) + \|U_M(v) - v\|_X^p) = \inf\limits_{U \in \mathscr{U}_{ad}^p} (\frac{p}{2\lambda} f_M(U) + \|U - v\|_X^p)$. Additionally, $v \mapsto U_M(v)$ is continuous, see [19], Proposition 5.6. Since $0 \in \mathscr{U}_{ad}^p$, there exists a sequence $(v_k)_{k\in\mathbb{N}} \subset M_0$ such that $\lim\limits_{k\to\infty} \|v_k\|_X = 0$. Now, set $\widetilde{U_M} = \lim\limits_{k\to\infty} U_M(v_k)$ and notice that

$$\frac{p}{\lambda} J_M(\widetilde{U_M}) = \lim_{k\to\infty} (\frac{p}{2\lambda} f_M(U_M(v_k)) + \|U_M(v_k) - v_k\|_X^p) = \lim_{k\to\infty} \inf_{U\in\mathscr{U}_{ad}^p} (\frac{p}{2\lambda} f_M(U) + \|U - v_k\|_X^p)$$
$$= \inf_{U\in\mathscr{U}_{ad}^p} (\frac{p}{2\lambda} f_M(U) + \|U\|_X^p) = \inf_{U\in\mathscr{U}_{ad}^p} \frac{p}{\lambda} J_M(U),$$

where the last inequalities hold thanks to the continuity properties of $v \mapsto \inf\limits_{U\in\mathscr{U}_{ad}} \frac{p}{2\lambda} f_M(U) + \|U - v\|_X^p$. The uniqueness of $\widetilde{U_M}$ follows from the uniqueness of the limit. □

5. Linearized state equation

This section is devoted to the study of the linearized state equation. The existence of the solution is based on the Faedo-Galerkin's approximation, which relies on a special basis, in order to derive the uniform estimates in V and W.
Let us consider $\psi : D\times]0, T[\times\Omega \to \mathbb{R}^d, d = 2, 3$ such that

$$\psi \in L^p((\Omega_T, \mathscr{P}_T), (H^1(D))^d). \tag{5.1}$$

Taking into account the assumptions (2.6) and $\mathscr{H}_0$, let us set

$$\begin{aligned}\nabla_y G(t, y) : (L^2(D))^d &\to L_2(\mathbb{H}, (L^2(D))^d)\\ v \mapsto \nabla_y G(t, y)v \;&=\; \{e_{\Bbbk} \to \nabla_y G(t, y)ve_{\Bbbk} := \nabla_y \sigma_{\Bbbk}(t, y)v\}.\end{aligned} \tag{5.2}$$

Our goal is to show the existence of the solution z to the following problem:

$$\begin{cases} dv(z)+\mathbb{I}_{[0,\tau_M[}\{-\nu\Delta z+(y\cdot\nabla)v(z)+(z\cdot\nabla)v(y)+\sum_j v(z)^j\nabla y^j+\sum_j v(y)^j\nabla z^j \\ \quad -(\alpha_1+\alpha_2)\text{div}[A(y)A(z)+A(z)A(y)]-\beta\ \text{div}\big[|A(y)|^2A(z)\big]-2\beta\ \text{div}\left[(A(z):A(y))\,A(y)\right]\}dt \\ \quad =\{\psi-\nabla\pi\}\mathbb{I}_{[0,\tau_M[}dt+\mathbb{I}_{[0,\tau_M[}\nabla_y G(\cdot,y)zd\mathscr{W} & \text{in } D\times\Omega\times(0,T), \\ \text{div}(z)=0 & \text{in } D\times\Omega\times(0,T), \\ z\cdot\eta=0 \quad [\eta\cdot\mathbb{D}(z)]\cdot\tau=0 & \text{on } \partial D\times\Omega\times(0,T), \\ z(0)=0 & \text{in } D\times\Omega, \end{cases} \tag{5.3}$$

where $(\tau_M)_M$ is the sequence of stopping times introduced in (2.10). Since the solution of (2.1) is defined up to the stopping times $(\tau_M)_{M\in\mathbb{N}}$.Then we define the solution of (5.3) accordingly.

Definition 5.1. A stochastic process z is a solution of (5.3) if and only if:

- z is predictable with values in W and $v(z)\in\mathscr{C}_w([0,T];(L^2(D))^d)$[7] P-a.s.
- $z\in L^p((\Omega_T,\mathscr{P}_T),V)\cap L^2_w(\Omega;L^\infty(0,T;W))$.
- For any $t\in[0,T]$ and P-a.s. $\omega\in\Omega$, the following equality holds

$$\begin{aligned} &(v(z(t)),\phi)+\int_0^t\mathbb{I}_{[0,\tau_M[}\{2\nu(\mathbb{D}z,\mathbb{D}\phi)+b(y,v(z),\phi)+b(z,v(y),\phi)+b(\phi,y,v(z))+b(\phi,z,v(y))\}ds \\ &\quad+\int_0^t\mathbb{I}_{[0,\tau_M[}\{(\alpha_1+\alpha_2)\big(A(y)A(z)+A(z)A(y),\nabla\phi\big)+\beta\big(|A(y)|^2A(z),\nabla\phi\big)\}ds \\ &\quad+\int_0^t 2\beta\mathbb{I}_{[0,\tau_M[}\big((A(z):A(y))A(y),\nabla\phi\big)ds \\ &\quad=\int_0^t\mathbb{I}_{[0,\tau_M[}(\psi,\phi)ds+\int_0^t\mathbb{I}_{[0,\tau_M[}(\nabla_yG(\cdot,y)z,\phi)d\mathscr{W}(s),\quad\forall\phi\in V. \end{aligned} \tag{5.4}$$

Remark 5.2. Due to the presence of τ_M in the system (5.3), the solution z depends on M.

Theorem 5.3. *Assume that ψ satisfy (5.1). Then there exists a unique solution z to (5.3), in the sense of Definition 5.1, satisfying the following estimates*

$$\mathbb{E}\sup_{s\in[0,T]}\|z(s)\|_V^p\le C(M)\int_0^{\tau_M}\|\psi\|_2^p ds \text{ and } \mathbb{E}\sup_{s\in[0,T]}\|z(s)\|_W^2\le C(\alpha_1,\alpha_2,\beta,\nu,T,M), \tag{5.5}$$

where $C(M)>0$ and $C(\alpha_1,\alpha_2,\beta,\nu,T,M)>0$.

5.1. Approximation

Following the same strategy as in [16], let us consider an orthonormal basis $\{h_i\}_{i\in\mathbb{N}}\subset(H^4(D)^d)\cap W$ in V, which satisfies

$$(v,h_i)_W=\mu_i(v,h_i)_V,\ \forall v\in W,\ \mu_i>0,\forall i\in\mathbb{N},\ \text{and}\ \ \mu_i\to\infty,\ \text{as } i\to\infty. \tag{5.6}$$

As a consequence of (5.6), the sequence $\{\tilde{h}_i=\frac{1}{\sqrt{\mu_i}}h_i\}$ is an orthonormal basis in W. Let us introduce the Galerkin approximations of (5.3). Consider $W_n=\text{span}\{h_1,\cdots,h_n\}$ and define $z_n(t)=\sum_{i=1}^n c_i(t)h_i$ for each

[7] For a Banach space X, $\mathscr{C}_w([0,T];X)$ denotes the space of weakly continuous functions on $[0,T]$ with values in X.

$t \in [0,T]$. The approximated problem for (5.3) reads, for $\phi \in W_n$,

$$\begin{aligned}
&(v(z_n(t)),\phi) + \int_0^t \mathbb{I}_{[0,\tau_M[}\{2\nu(\mathbb{D}z_n,\mathbb{D}\phi) + b(y,v(z_n),\phi) + b(z_n,v(y),\phi) + b(\phi,y,v(z_n)) + b(\phi,z_n,v(y))\}ds \\
&\quad + \int_0^t \mathbb{I}_{[0,\tau_M[}\{(\alpha_1+\alpha_2)\big(A(y)A(z_n)+A(z_n)A(y),\nabla\phi\big) + \beta\big(|A(y)|^2A(z_n),\nabla\phi\big)\}ds \qquad (5.7)\\
&\quad + \int_0^t 2\beta\mathbb{I}_{[0,\tau_M[}\big((A(z_n):A(y))A(y),\nabla\phi\big)ds = \int_0^t \mathbb{I}_{[0,\tau_M[}(\psi,\phi)ds + \int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y G(\cdot,y)z_n,\phi)d\mathscr{W}.
\end{aligned}$$

(5.7) defines a system of n-stochastic linear ODE, by using *e.g.* a *"Banach fixed point theorem"*, it follows that (5.7) has a unique solution z_n such that z_n is a predictable process and satisfies

$$z_n \in L^2(\Omega;\mathscr{C}([0,T],W_n)). \qquad (5.8)$$

5.2. Uniform estimates

5.2.1. Estimate in the space V for z_n

For any $t\in[0,T]$, setting $\phi = h_i$ in (5.7), and applying Itô's formula for the function $x\mapsto x^2$, we infer

$$\begin{aligned}
&(z_n(t),h_i)_V^2 + \int_0^t 2\mathbb{I}_{[0,\tau_M[}(z_n,h_i)_V\{(\alpha_1+\alpha_2)\big(A(y)A(z_n)+A(z_n)A(y),\nabla h_i\big) + \beta\big(|A(y)|^2A(z_n),\nabla h_i\big)\}ds \\
&+ \int_0^t 2\mathbb{I}_{[0,\tau_M[}(z_n,h_i)_V\{2\nu(\mathbb{D}z_n,\mathbb{D}h_i) + b(y,v(z_n),h_i) + b(z_n,v(y),h_i) + b(h_i,y,v(z_n)) + b(h_i,z_n,v(y))\}ds \\
&\quad + \int_0^t 4\beta\mathbb{I}_{[0,\tau_M[}(z_n,h_i)_V\big((A(z_n):A(y))A(y),\nabla h_i\big)ds = \int_0^t 2\mathbb{I}_{[0,\tau_M[}(z_n,h_i)_V(\psi,h_i)ds \\
&\qquad + \int_0^t 2\mathbb{I}_{[0,\tau_M[}(z_n,h_i)_V(\nabla_y G(\cdot,y)z_n,h_i)d\mathscr{W} + \sum_{\mathbb{k}\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y\sigma_{\mathbb{k}}(\cdot,y)z_n,h_i)^2ds.
\end{aligned}$$

By summing the last equalities from $i=1$ to n, we obtain

$$\begin{aligned}
&\|z_n(t)\|_V^2 + \int_0^t 2\mathbb{I}_{[0,\tau_M[}\{(\alpha_1+\alpha_2)\big(A(y)A(z_n)+A(z_n)A(y),\nabla z_n\big) + \beta\big(|A(y)|^2A(z_n),\nabla z_n\big)\}ds \\
&+ \int_0^t 2\mathbb{I}_{[0,\tau_M[}\{2\nu\|\mathbb{D}z_n\|_2^2 + b(y,v(z_n),z_n) + b(z_n,v(y),z_n) + b(z_n,y,v(z_n)) + b(z_n,z_n,v(y))\}ds \\
&\quad + \int_0^t 4\beta\mathbb{I}_{[0,\tau_M[}\big((A(z_n):A(y))A(y),\nabla z_n\big)ds = \int_0^t 2\mathbb{I}_{[0,\tau_M[}(\psi,z_n)ds \\
&\qquad + 2\int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y G(\cdot,y)z_n,z_n)d\mathscr{W} + \sum_{i=1}^n\sum_{\mathbb{k}\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y\sigma_{\mathbb{k}}(\cdot,y)z_n,h_i)^2ds.
\end{aligned}$$

Thanks to Lemma 10.1 (see Sect. 10), we have

$$\begin{aligned}
&\int_0^t \mathbb{I}_{[0,\tau_M[}\{b(y,v(z_n),z_n) + b(z_n,v(y),z_n) + b(z_n,y,v(z_n)) + b(z_n,z_n,v(y))\}ds \\
&= \int_0^t \mathbb{I}_{[0,\tau_M[}(-b(y,z_n,v(z_n)) + b(z_n,y,v(z_n)))ds \leq C\int_0^t \mathbb{I}_{[0,\tau_M[}\|y\|_{W^{2,4}}\|z_n\|_V^2ds \\
&\leq CM\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_V^2ds. \qquad (5.9)
\end{aligned}$$

Now, similarly to [18], Section 4.2, we derive the following estimates:

$$\begin{aligned}
&\left|\int_0^t 2(\alpha_1+\alpha_2)\mathbb{I}_{[0,\tau_M[}\int_D [A(y)A(z_n)+A(z_n)A(y)] : \nabla z_n dxds\right| \leq C_2\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_{H^1}^2\|y\|_{W^{1,\infty}}ds\\
&\quad \leq C_2\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_{V}^2\|y\|_{W^{2,4}}ds \leq C_2M\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_{V}^2 ds;\\
&\left|\int_0^t 4\beta\mathbb{I}_{[0,\tau_M[}\int_D \big[(A(z_n)\cdot A(y))\,A(y)\big] : \nabla z_n dxds\right| \leq C_3\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_{H^1}^2\|y\|_{W^{1,\infty}}^2ds\\
&\quad \leq C_3\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_{V}^2\|y\|_{W^{2,4}}^2ds \leq C_3M^2\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_{V}^2 ds. \qquad (5.10)
\end{aligned}$$

Since $A(z_n)$ is symmetric, we have the relation

$$\int_0^t 2\beta\mathbb{I}_{[0,\tau_M[}\big(|A(y)|^2A(z_n),\nabla z_n\big)ds = \beta\int_0^t \mathbb{I}_{[0,\tau_M[}\int_D |A(y)|^2|A(z_n)|^2dxds \geq 0. \qquad (5.11)$$

Let $\Bbbk\in\mathbb{N}^*$, denote by $\bar{\sigma_\Bbbk}$ the solution of (2.4) with $f=\nabla_y\sigma_\Bbbk(\cdot,y)z_n\in(L^2(D))^d$. Therefore

$$\begin{aligned}
&\sum_{i=1}^n\sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y\sigma_\Bbbk(\cdot,y)z_n,h_i)^2ds = \sum_{i=1}^n\sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}(\bar{\sigma_\Bbbk},h_i)_V^2ds = \sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}\|\bar{\sigma_\Bbbk}\|_V^2ds\\
&\leq C\sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}\|\nabla_y\sigma_\Bbbk(\cdot,y)z_n\|_2^2ds \leq C\sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}a_k^2\|z_n\|_2^2ds \leq C_*\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_2^2ds.
\end{aligned}$$

Gathering the previous estimates, there exists $C>0$ independent of n such that the following inequality holds

$$\begin{aligned}
\|z_n(t)\|_V^2+4\nu\int_0^t \mathbb{I}_{[0,\tau_M[}\|\mathbb{D}z_n\|_2^2ds &\leq C(1+M^2)\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_V^2ds+\int_0^{\tau_M}\|\psi\|_2^2ds\\
&\quad +2\int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_yG(\cdot,y)z_n,z_n)d\mathscr{W}.
\end{aligned}$$

Letting $p\geq 1$ and taking the p^{th} power of each term of this inequality, we obtain

$$\begin{aligned}
\|z_n(t)\|_V^{2p} &\leq C(p)T^{p-1}[(1+M^2)^p\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_V^{2p}ds+\int_0^t \mathbb{I}_{[0,\tau_M[}\|\psi\|_2^{2p}ds]\\
&\quad +C(p)|\int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_yG(\cdot,y)z_n,z_n)d\mathscr{W}|^p. \qquad (5.12)
\end{aligned}$$

For fixed $N\in\mathbb{N}$, let us introduce the stopping time $\mathbf{t}_N^n=\inf\{t\in[0,T]:\|z_n(t)\|_V\geq N\}$. For $t\in[0,T]$, and any $\delta>0$, the Burkholder-Davis-Gundy inquality gives

$$\mathbb{E}\sup_{s\in[0,t\wedge\mathbf{t}_N^n]}|\int_0^s \mathbb{I}_{[0,\tau_M[}(\nabla_yG(\cdot,y)z_n,z_n)d\mathscr{W}|^p \leq \delta\mathbb{E}\sup_{s\in[0,t\wedge\mathbf{t}_N^n]}\|z_n(s)\|_V^{2p}+C_\delta\mathbb{E}\int_0^{t\wedge\mathbf{t}_N^n}\mathbb{I}_{[0,\tau_M[}\|z_n\|_2^{2p}ds.$$

Considering the supremum in the inequality (5.12), for $s\in[0,t\wedge \mathbf{t}_N^n]$, and inserting the estimate of the stochastic term, after an appropriate choice of δ, we deduce

$$\mathbb{E}\sup_{s\in[0,t\wedge \mathbf{t}_N^n]}\|z_n(s)\|_V^{2p}\leq C(p)T^{p-1}[(1+M^2)^p\mathbb{E}\int_0^t \mathbb{I}_{[0,\tau_M[}\sup_{r\in[0,s\wedge \mathbf{t}_N^n]}\|z_n(r)\|_V^{2p}ds+\mathbb{E}\int_0^{\tau_M}\|\psi\|_2^{2p}ds].$$

Using the Grönwall's inequality, we obtain $\mathbb{E}\sup_{s\in[0,\mathbf{t}_N^n]}\|z_n(s)\|_V^{2p}\leq C(M)\mathbb{E}\int_0^{\tau_M}\|\psi\|_2^{2p}ds$, where $C(M)=C(p)T^{p-1}[(1+M^2)^pe^{C(p)T^p[(1+M^2)^p}$. Note that $(\mathbf{t}_N^n)_N$ is an increasing positive sequence of stopping times and $\mathbf{t}_N^n\to T$ in probability. Thus, the monotone convergence theorem ensures

$$\mathbb{E}\sup_{s\in[0,T]}\|z_n(s)\|_V^{2p}\leq C(M)\mathbb{E}\int_0^{\tau_M}\|\psi\|_2^{2p}ds,\quad \forall t\in[0,T],\quad \forall p\geq 1.$$

Next, we will establish a W-uniform estimate for z_n. Before entering in details, we recall a useful equality for divergence free vector fields tangent to the boundary.

Lemma 5.4. *We have:* $(curl v(y)\times u,\phi)=b(\phi,u,v(y))-b(u,\phi,v(y))$, $\forall u,y\in\widetilde{W}$ *and* $\phi\in V$.

5.2.2. Estimate in the space W for z_n

As a consequence of Lemma 5.4, we get

$$((y\cdot\nabla)v(z)+(z\cdot\nabla)v(y)+\sum_j v(z)^j\nabla y^j+\sum_j v(y)^j\nabla z^j,\phi)=(curl v(y)\times z,\phi)+(curl v(z)\times y,\phi),$$

for any $y\in\widetilde{W}$ and $z,\phi\in W$. Setting $\phi=h_i$ in (5.7) we write

$$(z_n(t),h_i)_V+\int_0^t\mathbb{I}_{[0,\tau_M[}(f_n,h_i)ds=\sum_{\Bbbk\geq 1}\int_0^t\mathbb{I}_{[0,\tau_M[}(\nabla_y\sigma_{\Bbbk}(\cdot,y)z_n,h_i)d\beta_{\Bbbk},\quad\forall t\in[0,T],\tag{5.13}$$

where $f_n=-\nu\Delta z_n+curl v(y)\times z_n+curl v(z_n)\times y-(\alpha_1+\alpha_2)\text{div}[A(y)A(z_n)+A(z_n)A(y)]$
$-\beta\ \text{div}\big[|A(y)|^2A(z_n)\big]-2\beta\ \text{div}\left[(A(z_n):A(y))\,A(y)\right]-\psi.$

Let $\widetilde{f}_n$ and $\tilde{\sigma}_{\Bbbk}^n$ be the solutions of (2.4) for $f=f_n$ and $f=\nabla_y\sigma_{\Bbbk}(\cdot,y)z_n,\forall\,\Bbbk\in\mathbb{N}$. We have

$$(f_n,h_i)=(\widetilde{f}_n,h_i)_V,\ (\nabla_y\sigma_{\Bbbk}(\cdot,y)z_n,h_i)=(\tilde{\sigma}_k^n,h_i)_V.\tag{5.14}$$

By multiplying (5.13) by μ_i and using (5.6), we derive

$$(z_n(t),h_i)_W+\int_0^t\mathbb{I}_{[0,\tau_M[}(\widetilde{f}_n,h_i)_Wds=\sum_{\Bbbk\geq 1}\int_0^t\mathbb{I}_{[0,\tau_M[}(\tilde{\sigma}_k^n,h_i)_Wd\beta_{\Bbbk}.$$

Using Ito's formula, next multiplying by $\frac{1}{\mu_i}$ and summing over $i=1,\cdots,n$ we are able to infer

$$\|z_n(t)\|_W^2+2\int_0^t\mathbb{I}_{[0,\tau_M[}(\widetilde{f}_n,z_n)_Wds=2\sum_{\Bbbk\geq 1}\int_0^t\mathbb{I}_{[0,\tau_M[}(\tilde{\sigma}_k^n,z_n)_Wd\beta_{\Bbbk}+\sum_{\Bbbk\geq 1}\int_0^t\mathbb{I}_{[0,\tau_M[}\|\tilde{\sigma}_k^n\|_W^2ds.$$

By using the definition of inner product in the space W and using the properties of $\mathbb{P}$, we deduce

$$\|z_n(t)\|_W^2 + 2\int_0^t \mathbb{I}_{[0,\tau_M[}(f_n, z_n)ds + 2\int_0^t \mathbb{I}_{[0,\tau_M[}(f_n, \mathbb{P}v(z_n))ds - \sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}\|\tilde{\sigma}_k^n\|_W^2 ds$$
$$= 2\sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y \sigma_\Bbbk(\cdot,y)z_n, z_n)d\beta_\Bbbk + 2\sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y \sigma_\Bbbk(\cdot,y)z_n, \mathbb{P}v(z_n))d\beta_\Bbbk. \tag{5.15}$$

Arguments already detailed in (5.9)–(5.11) yield

$$2\Big|\int_0^t \mathbb{I}_{[0,\tau_M[}(f_n, z_n)ds\Big| \leq 4\nu\int_0^t \mathbb{I}_{[0,\tau_M[}\|\mathbb{D}z_n\|_2^2 ds + C(1+M^2)\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_V^2 ds + \int_0^{\tau_M}\|\psi\|_2^2 ds.$$

Concerning the third term of left hand side of (5.15), we have

$$\begin{aligned} 2\int_0^t \mathbb{I}_{[0,\tau_M[}(f_n, \mathbb{P}v(z_n))ds &= 2\nu\int_0^t \mathbb{I}_{[0,\tau_M[}(-\Delta z_n, \mathbb{P}v(z_n))ds \\ &+ 2\int_0^t \mathbb{I}_{[0,\tau_M[}((y\cdot\nabla)v(z_n) + (z_n\cdot\nabla)v(y) + \sum_j v(z_n)^j\nabla y^j + \sum_j v(y)^j\nabla z_n^j, \mathbb{P}v(z_n))ds \\ &- 2\int_0^t \mathbb{I}_{[0,\tau_M[}(\psi, \mathbb{P}v(z_n))ds - 4\beta\int_0^t \mathbb{I}_{[0,\tau_M[}\operatorname{div}\left[(A(z_n):A(y))\,A(y)\right], \mathbb{P}v(z_n))ds \\ &+ 2\int_0^t \mathbb{I}_{[0,\tau_M[}(-(\alpha_1+\alpha_2)\operatorname{div}[A(y)A(z_n)+A(z_n)A(y)] - \beta\operatorname{div}\big[|A(y)|^2A(z_n)\big], \mathbb{P}v(z_n))ds. \end{aligned}$$

Using the properties of the trilinear form b, [8], Lemma 5 and $W^{2,4}(D)\hookrightarrow W^{1,\infty}(D)$, we deduce

$$\begin{aligned} &2((y\cdot\nabla)v(z_n) + (z_n\cdot\nabla)v(y) + \sum_j v(z_n)^j\nabla y^j + \sum_j v(y)^j\nabla z_n^j, \mathbb{P}v(z_n))ds \\ &\leq 2\big(\|y\|_\infty\|v(z_n)-\mathbb{P}v(z_n)\|_{H^1}\|z_n\|_W + \|z_n\|_\infty\|y\|_{H^3}\|z_n\|_W + \|y\|_{W^{1,\infty}}\|z_n\|_W^2 + \|y\|_{W^{2,4}}\|z_n\|_{W^{1,4}}\|z_n\|_W\big) \\ &\leq C\|y\|_{W^{2,4}}\|z_n\|_W^2 + 2\|z_n\|_\infty\|y\|_{H^3}\|z_n\|_W. \end{aligned}$$

In addition, we notice that $2\nu(-\Delta z_n, \mathbb{P}v(z_n)) - 2(\psi, \mathbb{P}v(z_n)) \leq (2\nu+1)\|z_n\|_W^2 + \|\psi\|_2^2$. On the other hand, standard computations ensure

$$\begin{aligned} &2(-(\alpha_1+\alpha_2)\operatorname{div}[A(y)A(z_n)+A(z_n)A(y)] - \beta\operatorname{div}\big[|A(y)|^2A(z_n)\big] - 2\beta\operatorname{div}\left[(A(z_n):A(y))\,A(y)\right], \mathbb{P}v(z_n)) \\ &\leq C\|y\|_{W^{2,4}}\|z_n\|_W^2 + C\|y\|_{W^{2,4}}^2\|z_n\|_W^2 \leq C(\alpha_1,\alpha_2,\beta)(\|y\|_{W^{2,4}}^2+1)\|z_n\|_W^2. \end{aligned}$$

Therefore, there exists $\mathbf{C}>0$ such that

$$|2(f_n, \mathbb{P}v(z_n))| \leq C(\alpha_1,\alpha_2,\beta,\nu)(\|y\|_{W^{2,4}}^2+1)\|z_n\|_W^2 + \|\psi\|_2^2 + 2\|z_n\|_\infty\|y\|_{H^3}\|z_n\|_W. \tag{5.16}$$

Taking into account the properties of the solution of (2.4), we infer that

$$\sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}\|\tilde{\sigma}_k^n\|_W^2 ds \leq C\sum_{\Bbbk\geq 1}\int_0^t \mathbb{I}_{[0,\tau_M[}\|\nabla_y\sigma_\Bbbk(\cdot,y)z_n\|_2^2 ds \leq C_M\int_0^t \mathbb{I}_{[0,\tau_M[}\|z_n\|_2^2 ds.$$

Concerning the stochastic integrals, let us consider $q \geq 1$ and define the sequence of stopping times $\mathbf{t}_N^n = \inf\{t \in [0,T] : \|z_n(t)\|_W \geq N\}$, $N \in \mathbb{N}$. For any $t \in [0,T]$ and $\delta > 0$, the Burkholder-Davis-Gundy inquality gives

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t\wedge\mathbf{t}_N^n]} |\int_0^s \mathbb{I}_{[0,\tau_M[}(\nabla_y G(\cdot,y)z_n, z_n) d\mathscr{W}|^q + \mathbb{E} \sup_{s\in[0,t\wedge\mathbf{t}_N^n]} |\int_0^s \mathbb{I}_{[0,\tau_M[} \sum_{\mathbb{k}\geq 1} (\nabla_y \sigma_{\mathbb{k}}(\cdot,y)z_n, \mathbb{P}v(z_n)) d\beta_{\mathbb{k}}|^q \\
&\quad \leq \delta \mathbb{E} \sup_{s\in[0,t\wedge\mathbf{t}_N^n]} \|\mathbb{P}v(z_n)(s)\|_2^{2q} + \delta \mathbb{E} \sup_{s\in[0,t\wedge\mathbf{t}_N^n]} \|z_n(s)\|_V^{2q} + C_\delta \mathbb{E} \int_0^{t\wedge\mathbf{t}_N^n} \mathbb{I}_{[0,\tau_M[} \|z_n\|_2^{2q} ds.
\end{aligned}$$

Now, taking the q^{th}power of relation (5.15), computing the supremum over the interval $[0, t \wedge \mathbf{t}_N^n]$, taking the expectation and gathering the previous estimates, we infer that

$$\begin{aligned}
(1-2\delta)\mathbb{E} \sup_{s\in[0,t\wedge\mathbf{t}_N^n]} \|z_n(s)\|_W^{2q} &\leq C(\alpha_1,\alpha_2,\beta,\nu,T,M,q,\delta) \int_0^t \mathbb{E} \sup_{r\in[0,s\wedge\mathbf{t}_N^n]} \|z_n(r)\|_W^{2q} ds \\
&\quad + C(T)\mathbb{E} \int_0^{\tau_M} \|\psi\|_2^{2q} ds + C(q,T)\mathbb{E} \int_0^{t\wedge\mathbf{t}_N^n} (\|z_n\|_\infty \|y\|_{H^3} \|z_n\|_W)^q ds.
\end{aligned} \tag{5.17}$$

Thanks to Lemma 10.2 (see Sect. 10), for any $q \geq 1$ and $\epsilon \in]0,1]$ we have

$$\begin{aligned}
&\mathbb{E} \int_0^{t\wedge\mathbf{t}_N^n} (\|z_n\|_\infty \|y\|_{H^3} \|z_n\|_W)^q ds \\
&\quad \leq C_\epsilon(D) [\int_0^t \mathbb{E} \sup_{r\in[0,s\wedge\mathbf{t}_N^n]} \|z_n(r)\|_W^{2q} ds + \|y\|_{L^{2q(d+1+\epsilon)}(\Omega\times(0,\tau_M);H^3)}^{2q(d+\epsilon)} \|z_n\|_{L^{2q(d+1+\epsilon)}(\Omega\times(0,T);V)}^{2q}],
\end{aligned}$$

Using this inequality and choosing δ small enough in (5.17), we are able to apply Grönwall's inequality to show the existence of a constant $C := C(\alpha_1,\alpha_2,\beta,\nu,T,M,q,\delta,\epsilon) > 0$ such that

$$\mathbb{E} \sup_{s\in[0,t\wedge\mathbf{t}_N^n]} \|z_n(s)\|_W^{2q} \leq C\mathbb{E} \int_0^{\tau_M} \|\psi\|_2^{2q} ds + C_\epsilon(D) \|y\|_{L^{2q(d+1+\epsilon)}(\Omega\times(0,\tau_M);H^3)}^{2q(d+\epsilon)} \|z_n\|_{L^{2q(d+1+\epsilon)}(\Omega\times(0,T);V)}^{2q}.$$

In particular, for $q = 1$ and $t = T$ we obtain

$$\mathbb{E} \sup_{s\in[0,\mathbf{t}_N^n]} \|z_n(s)\|_W^2 \leq C\mathbb{E} \int_0^{\tau_M} \|\psi\|_2^2 ds + C_\epsilon(D) \|y\|_{L^{2(d+1+\epsilon)}(\Omega\times(0,\tau_M);H^3)}^{2(d+\epsilon)} \|z_n\|_{L^{2(d+1+\epsilon)}(\Omega\times(0,T);V)}^2.$$

From Subsection 5.2.1, we know that $\mathbb{E} \sup_{s\in[0,T]} \|z_n(s)\|_V^p \leq C(M)\mathbb{E} \int_0^{\tau_M} \|\psi\|_2^p ds, \quad \forall p \geq 2.$

Since $p > 2(d+1)$, for $\epsilon = \dfrac{p-2(d+1)}{2}$, there exists $C(M) > 0$ such that

$$C_\epsilon(D) \|y\|_{L^{2(d+1+\epsilon)}(\Omega\times(0,\tau_M);H^3)}^{2(d+\epsilon)} \|z_n\|_{L^{2(d+1+\epsilon)}(\Omega\times(0,T);V)}^2 \leq C(M).$$

Since $(\mathbf{t}_N^n)_N$ is non-decreasing positive sequence of stopping time and $\mathbf{t}_N^n \to T$ in probability, the monotone convergence theorem ensures $\mathbb{E} \sup_{s\in[0,T]} \|z_n(s)\|_W^2 \leq C(\alpha_1,\alpha_2,\beta,\nu,T,M)$. Thus

Proposition 5.5. *Let z_n be the solution to (5.7). Then, $z_n \in \mathscr{C}([0,T], W_n)$ and there exist $C(M) > 0$ and $C(\alpha_1, \alpha_2, \beta, \nu, T, M) > 0$, independents of n such that*

$$\mathbb{E} \sup_{s\in[0,T]} \|z_n(s)\|_V^p \leq C(M)\mathbb{E}\int_0^{\tau_M} \|\psi\|_2^p ds \text{ and } \mathbb{E} \sup_{s\in[0,T]} \|z_n(s)\|_W^2 \leq C(\alpha_1,\alpha_2,\beta,\nu,T,M).$$

5.3. Proof of Theorem 5.3

5.3.1. 1^{st}*step*

Using Proposition 5.5, we deduce the existence of a subsequence of $(z_n)_n$, still denoted by $(z_n)_n$, and a predictable stochastic process $z \in L^2(\Omega, L^q(0,T;W))$ such that

$$z_n \rightharpoonup z \text{ weakly in } L^2(\Omega, L^q(0,T;W)), \quad \text{for any } 2 \leq q < \infty. \tag{5.18}$$

On the other hand, the sequence (z_n) is bounded in $L^2(\Omega, L^\infty(0,T;W))$, thus in $L^2_w(\Omega, L^\infty(0,T;W)) \simeq (L^2(\Omega, L^1(0,T;W')))'$, where w stands for the weak-* measurability (see *e.g.* [25], Thm. 8.20.3, [26], Rem. 2.1). Hence, Banach-–Alaoglu theorem's ensures $z \in L^2_w(\Omega, L^\infty(0,T;W)$.

Recall that the stochastic Itô integral is linear and continuous. Thanks to [27], Proposition 21.27, p. 261 and (5.18), we infer that

$$\int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y G(\cdot,y)z_n,\phi)d\mathscr{W} \rightharpoonup \int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y G(\cdot,y)z,\phi)d\mathscr{W} \text{ weakly in } L^2(\Omega\times[0,T]). \tag{5.19}$$

5.3.2. 2^{nd}*step*

For $t \in [0,T]$, let us set

$$B_n(t) := v(z_n(t)) - \int_0^t \mathbb{I}_{[0,\tau_M[}\nabla_y G(\cdot,y)z_n d\mathscr{W}, \quad B(t) := v(z(t)) - \int_0^t \mathbb{I}_{[0,\tau_M[}\nabla_y G(\cdot,y)z d\mathscr{W}.$$

Due to (5.7) we have

$$\begin{aligned}\frac{d}{dt}(B_n,\phi) = &-\mathbb{I}_{[0,\tau_M[}\{2\nu(\mathbb{D}z_n,\mathbb{D}\phi) + b(y,v(z_n),\phi) + b(z_n,v(y),\phi) + b(\phi,y,v(z_n)) + b(\phi,z_n,v(y))\}\\ &- \mathbb{I}_{[0,\tau_M[}\{(\alpha_1+\alpha_2)\big(A(y)A(z_n)+A(z_n)A(y),\nabla\phi\big) + \beta\big(|A(y)|^2A(z_n),\nabla\phi\big)\}\\ &- 2\beta\mathbb{I}_{[0,\tau_M[}\big((A(z_n):A(y))A(y),\nabla\phi\big) + \mathbb{I}_{[0,\tau_M[}(\psi,\phi), \quad \forall\phi\in W_n.\end{aligned} \tag{5.20}$$

Let $A \in \mathscr{F}$ and $\xi \in \mathscr{D}(0,T)$[8], by multiplying (5.20) by $\mathbb{I}_A\xi$ and integrating over Ω_T we derive

$$\begin{aligned}&-\int_A\int_0^T \big[(B_n,\phi)\frac{d\xi}{ds}\big]dsdP \\ &= -\int_A\int_0^T \mathbb{I}_{[0,\tau_M[}\{2\nu(\mathbb{D}z_n,\mathbb{D}\phi) + b(y,v(z_n),\phi) + b(z_n,v(y),\phi) + b(\phi,y,v(z_n)) + b(\phi,z_n,v(y))\}\xi dsdP\\ &\quad -\int_A\int_0^T \mathbb{I}_{[0,\tau_M[}\{(\alpha_1+\alpha_2)\big(A(y)A(z_n)+A(z_n)A(y),\nabla\phi\big) + \beta\big(|A(y)|^2A(z_n),\nabla\phi\big)\}\xi dsdP\\ &\quad - 2\beta\int_A\int_0^T \mathbb{I}_{[0,\tau_M[}\big((A(z_n):A(y))A(y),\nabla\phi\big)\xi dsdP + \int_A\int_0^T \mathbb{I}_{[0,\tau_M[}(\psi,\phi)\xi dsdP, \quad \forall\phi\in W_n.\end{aligned} \tag{5.21}$$

[8] $\mathscr{D}(0,T)$ denotes the space of $\mathscr{C}^\infty$-functions with compact suport in $]0,T[$.

The convergences (5.18)–(5.19) allow to pass to the limit, as $n \to \infty$, in (5.21) and deduce

$$
\begin{aligned}
&-\int_A \int_0^T \big[(B,\phi)\frac{d\xi}{ds}\big] ds dP \qquad (5.22)\\
&= -\int_A \int_0^T \mathbb{I}_{[0,\tau_M[}\{2\nu(\mathbb{D}z,\mathbb{D}\phi) + b(y,v(z),\phi) + b(z,v(y),\phi) + b(\phi,y,v(z)) + b(\phi,z,v(y))\}\xi ds dP\\
&\quad -\int_A \int_0^T \mathbb{I}_{[0,\tau_M[}\{(\alpha_1+\alpha_2)\big(A(y)A(z)+A(z)A(y),\nabla\phi\big) + \beta\big(|A(y)|^2A(z),\nabla\phi\big)\}\xi ds dP \qquad (5.23)\\
&\quad -2\beta\int_A \int_0^T \mathbb{I}_{[0,\tau_M[}\big((A(z):A(y))A(y),\nabla\phi\big)\xi ds dP + \int_A \int_0^T \mathbb{I}_{[0,\tau_M[}(\psi,\phi)\xi ds dP, \quad \forall \phi \in V.
\end{aligned}
$$

Then, taking into account the result of Proposition 5.5, we infer that the distributional derivative $\dfrac{dB}{dt}$ belongs to the space $L^2(\Omega; L^2(0,T;V'))$. Recalling that $B \in L^2(\Omega; L^2(0,T;(L^2(D))^d)$, we conclude that (see [28]) $B(\cdot) = v(z(\cdot)) - \int_0^{\cdot} \mathbb{I}_{[0,\tau_M[}\nabla_y G(s,y) z d\mathscr{W}(s) \in L^2(\Omega;\mathscr{C}([0,T];V')$.

Considering the properties of the Itô's integral, we conclude that $v(z) \in L^2(\Omega;\mathscr{C}([0,T];V')$. Thus $v(z) \in L^2(\Omega;\mathscr{C}_w([0,T];(L^2(D))^d)$, thanks to [29], Lemma 1.4, p. 263. In order to identify the initial datum, let $\phi \in V$ and $\xi \in \mathscr{C}^\infty([0,t])$ for $t \in]0,T]$ and note that the following integration by parts formula holds

$$
\int_0^t \langle \frac{dB(s)}{ds}, \phi\xi\rangle_{V^*,V} ds = -\int_0^t [(B(s),\phi)\frac{d\xi}{ds}]ds + (B(t),\phi)\xi(t) - (z(0),\phi)_V \xi(0) \qquad (5.24)
$$

First, we multiply (5.20) by $\mathbb{I}_A\xi$ and integrate over Ω_T. Then, we pass to the limit as $n \to \infty$ and we use (5.24) to obtain for all $t \in [0,T]$, $\quad v(z_n(t)) \rightharpoonup v(z(t))$ in $L^2(\Omega,(L^2(D))^d)$, as $n \to \infty$, (see *e.g.* [30], Prop. 3 for similar arguments). Hence the proof of Theorem 5.3 is completed.

6. Gâteaux differentiability of the control-to-state mapping

In this section, we will prove that the Gâteaux derivative of the control-to-state mapping is provided by the solution of the linearized equations, given by (5.3).

Proposition 6.1. *Let us consider U and y_0 satisfying (2.8) and $\psi \in L^p((\Omega_T,\mathscr{P}_T);(H^1(D))^d)$. Defining $U_\rho = U + \rho\psi$, $\quad \rho \in (0,1)$, let (y,τ_M) and (y_ρ,τ_M^ρ) be the solutions of (2.1) associated with (U,y_0) and (U_ρ,y_0), respectively, then the following representaion holds*

$$
y_\rho = y + \rho z + \rho\delta_\rho \quad \textit{with} \quad \lim_{\rho\to 0}\mathbb{E}\sup_{s\in[0,\tau_M\wedge\tau_M^\rho]}\|\delta_\rho(s)\|_V^2 = 0, \qquad (6.1)
$$

where z is the solution of (5.3), in the sense of Definition 5.1 and satisfying the estimate (5.5).

Proof. Let $t \in [0,\tau_M\wedge\tau_M^\rho]$, by using (2.1), we derive

$$
\begin{aligned}
d(v(y_\rho - y)) &= \{-\nabla(\mathbf{P}_\rho - \mathbf{P}) + \nu\Delta(y_\rho - y) - ((y_\rho\cdot\nabla)v_\rho - (y\cdot\nabla)v) - \sum_j (v_\rho^j\nabla y_\rho^j - v^j\nabla y^j)\\
&+ (\alpha_1+\alpha_2)\mathrm{div}(A_\rho^2 - A^2) + \beta\ \mathrm{div}\big(|A(y_\rho)|^2A(y_\rho) - |A(y)|^2A(y)\big) + \rho\psi\}dt + (G(\cdot,y_\rho) - G(\cdot,y))d\mathscr{W},
\end{aligned}
$$

where $v_\rho = v(y_\rho), \quad v = v(y), \quad A_\rho = A(y_\rho), \quad A = A(y).$
Setting $z_\rho = \dfrac{y_\rho - y}{\rho}$, $\pi_\rho = \dfrac{\mathbf{P}_\rho - \mathbf{P}}{\rho}$, we notice that z_ρ is the unique solution to

$$\begin{aligned}
&d(v(z_\rho)) = \big\{\psi - \nabla\pi_\rho + \nu\Delta z_\rho - [(z_\rho\cdot\nabla)v(y_\rho) + (y\cdot\nabla)v(z_\rho)] \\
&-\textstyle\sum_j [v^j(z_\rho)\nabla y_\rho^j + v^j(y)\nabla z_\rho^j] + \beta\mathrm{div}\big(|A(y_\rho)|^2 A(z_\rho) + [A(z_\rho) : A(y_\rho) \\
&+A(y) : A(z_\rho)]A(y)\big) + (\alpha_1+\alpha_2)\mathrm{div}\,[A(z_\rho)A(y_\rho) + A(y)A(z_\rho)]\,\big\}dt + \frac{1}{\rho}(G(\cdot,y_\rho) - G(\cdot,y))d\mathscr{W}.
\end{aligned}$$

Defining $\delta_\rho = z_\rho - z$, the following equation hold

$$\begin{aligned}
&d(v(\delta_\rho)) = \big\{-\nabla(\pi_\rho - \pi) + \nu\Delta\delta_\rho - [(y\cdot\nabla)v(\delta_\rho) + (\delta_\rho\cdot\nabla)v(y_\rho)] - (z\cdot\nabla)v(y_\rho - y) \\
&-\textstyle\sum_j [v^j(y)\nabla\delta_\rho^j + v^j(\delta_\rho)\nabla y_\rho^j + v^j(z)\nabla(y_\rho - y)^j] + (\alpha_1+\alpha_2)\mathrm{div}\big[A(y)A(\delta_\rho) + A(\delta_\rho)A(y_\rho) \\
&+A(z)A(y_\rho - y)\big] + \beta\ \mathrm{div}\,\big\{(A(y) : A(\delta_\rho))\,A(y) + |A(y_\rho)|^2 A(\delta_\rho)\big\} \\
&+\beta\ \mathrm{div}([A(y_\rho - y) : A(y_\rho) + A(y) : A(y_\rho - y)]\,A(z)]) \\
&+\beta\ \mathrm{div}\,[(A(\delta_\rho) : A(y_\rho) + A(z) : A(y_\rho - y))\,A(y)]\,\big\}dt \\
&+\{\frac{1}{\rho}(G(\cdot,y_\rho) - G(\cdot,y)) - \nabla_y G(\cdot,y)z\}d\mathscr{W} =: g(\delta_\rho)dt + Rd\mathscr{W} = g(\delta_\rho)dt + \textstyle\sum_{k\geq 1} R_{k} d\beta_{k}.
\end{aligned}$$

By applying $(I - \alpha_1\mathbb{P}\Delta)^{-1}$ to the last equations and using Itô formula for $\|\delta_\rho\|_V^2$, we deduce

$$d\|\delta_\rho\|_V^2 = 2(g(\delta_\rho),\delta_\rho)dt + 2(R,\delta_\rho)d\mathscr{W} + \sum_{k\geq 1}\|\tilde{\sigma}_k^\rho - \tilde{\sigma}_k^z\|_V^2 dt, \tag{6.2}$$

where $\tilde{\sigma}_k^\rho$ and $\tilde{\sigma}_k^z$ are the solutions of (2.4) with f replaced by $\{\frac{1}{\rho}(\sigma_{k}(\cdot,y_\rho) - \sigma_{k}(\cdot,y))$ and $\nabla_y\sigma_{k}(\cdot,y)z$, respectively. Thanks to ([3], Thm. 3), there exists $C >$ such that

$$\sum_{k\geq 1}\|\tilde{\sigma}_k^\rho - \tilde{\sigma}_k^z\|_V^2 \leq C\sum_{k\geq 1}\|\frac{1}{\rho}(\sigma_{k}(\cdot,y_\rho) - \sigma_{k}(\cdot,y)) - \nabla_y\sigma_{k}(\cdot,y)z\|_2^2$$

Using the assumption $\mathscr{H}_0$ (see (2.7)), we have

$$\frac{1}{\rho}(\sigma_{k}(\cdot,y_\rho) - \sigma_{k}(\cdot,y)) = \frac{1}{\rho}[\sigma_{k}(\cdot,y+\rho z_\rho) - \sigma_{k}(\cdot,y)] = \nabla_y\sigma_{k}(\cdot,y)z_\rho + R_{k}^\rho(\cdot,z_\rho), \quad \forall k \geq 1.$$

Therefore

$$\begin{aligned}
\sum_{k\geq 1}\|\tilde{\sigma}_k^\rho - \tilde{\sigma}_k^z\|_V^2 &\leq C\sum_{k\geq 1}\|\nabla_y\sigma_{k}(\cdot,y)\delta_\rho + R_{k}^\rho(\cdot,z_\rho)\|_2^2 \leq C\|\delta_\rho\|_2^2 + \sum_{k\geq 1} b_{k}^2\|z_\rho\|_2^2\rho^{2\gamma} \\
&\leq C\|\delta_\rho\|_2^2 + C\|z+\delta_\rho\|_2^2\rho^{2\gamma} \leq C\|\delta_\rho\|_2^2 + C\|z\|_2^2\rho^{2\gamma},
\end{aligned}$$

where we used (2.6) and (2.7) to get the last estimate. In order to estimate the first term in the right hand side of (6.2), let us write $2(g(\delta_\rho),\delta_\rho)=-4\nu\|\mathbb{D}\delta_\rho\|_2^2+R_1+R_2+R_3$, where

$$\begin{aligned} R_1 &= 2b(y,\delta_\rho,v(\delta_\rho))-2b(\delta_\rho,y_\rho,v(\delta_\rho))-[2b(\delta_\rho,v(y_\rho),\delta_\rho)+2b(z,v(y_\rho-y),\delta_\rho)]\\ &\quad -2b(\delta_\rho,\delta_\rho,v(y))-2b(\delta_\rho,(y_\rho-y),v(z)). \end{aligned}$$

Lemma 10.1 ensures $b(\delta_\rho,y_\rho,v(\delta_\rho))\leq C\|y_\rho\|_{W^{2,4}}\|\delta_\rho\|_V^2$ and $|b(y,\delta_\rho,v(\delta_\rho))|\leq C\|y\|_{W^{2,4}}\|\delta_\rho\|_V^2$. Consequently, with the help of Hölder inequality we obtain

$$R_1\leq C(\|y\|_{W^{2,4}}+\|y_\rho\|_{W^{2,4}})\|\delta_\rho\|_V^2+C\|\delta_\rho\|_V^2+C\|z\|_{H^2}^2\|y_\rho-y\|_{H^2}^2.$$

Using the Stokes theorem and the boundary conditions for δ_ρ, we deduce

$$\begin{aligned} R_2 &= -2(\alpha_1+\alpha_2)\int_D[A(y)A(\delta_\rho)+A(\delta_\rho)A(y_\rho)+A(z)A(y_\rho-y)]:\nabla\delta_\rho dx\\ &\leq C\|y\|_{W^{1,\infty}}\|\delta_\rho\|_V^2+C\|y_\rho\|_{W^{1,\infty}}\|\delta_\rho\|_V^2+C\|z\|_{W^{1,4}}\|y_\rho-y\|_{W^{1,4}}\|\delta_\rho\|_V\\ &\leq C(\|y\|_{W^{2,4}}+\|y_\rho\|_{W^{2,4}})\|\delta_\rho\|_V^2+C\|\delta_\rho\|_V^2+C\|z\|_{H^2}^2\|y_\rho-y\|_{H^2}^2. \end{aligned}$$

Analogous arguments give $R_3=-2\beta\int_D\left\{A(y):A(\delta_\rho)A(y)+|A(y_\rho)|^2A(\delta_\rho)\right\}:\nabla\delta_\rho dx$

$$\begin{aligned} &-2\beta\int_D\{[A(y_\rho-y):A(y_\rho)+A(y):A(y_\rho-y)]\,A(z)\}:\nabla\delta_\rho dx\\ &-2\beta\int_D[(A(\delta_\rho):A(y_\rho)+A(z):A(y_\rho-y))\,A(y)]:\nabla\delta_\rho dx\\ &\leq C\left(\|y\|_{W^{2,4}}^2+\|y_\rho\|_{W^{2,4}}^2\right)\|\delta_\rho\|_V^2+C\|z\|_{H^2}^2\|y_\rho-y\|_{H^2}^2, \end{aligned}$$

where we used Hölder and Young inequalities to deduce the last estimate. Summing up, we obtain the existence of constant $C_*>0$ such that

$$\begin{aligned} d\|\delta_\rho\|_V^2+4\nu\|\mathbb{D}\delta_\rho\|_2^2dt &\leq C_*\left(1+\|y\|_{W^{2,4}}^2+\|y_\rho\|_{W^{2,4}}^2\right)\|\delta_\rho\|_V^2dt\\ &\quad +C_*\|z\|_{H^2}^2\|y_\rho-y\|_{H^2}^2dt+C_*\|z\|_2^2\rho^{2\gamma}dt+2(R,\delta_\rho)d\mathscr{W}. \end{aligned} \tag{6.3}$$

Regarding the last term in the right hand side of (6.2), we write

$$\mathbb{E}\sup_{t\in[0,\tau_M\wedge\tau_M^\rho]}|\int_0^t(R,\delta_\rho)d\mathscr{W}(s)|=\mathbb{E}\sup_{t\in[0,\tau_M\wedge\tau_M^\rho]}|\int_0^t\sum_{\Bbbk\geq1}(\frac{1}{\rho}(\sigma_\Bbbk(\cdot,y_\rho)-\sigma_\Bbbk(\cdot,y))-\nabla_y\sigma_\Bbbk(\cdot,y)z,\delta_\rho)d\beta_\Bbbk|.$$

By using $\mathscr{H}_0$ (see (2.7)), we have $\frac{1}{\rho}(\sigma_\Bbbk(\cdot,y_\rho)-\sigma_\Bbbk(\cdot,y))=\nabla_y\sigma_\Bbbk(\cdot,y)z_\rho+R_\Bbbk^\rho(\cdot,z_\rho),\forall\,\Bbbk\geq1$.

Hence, for any $t\in[0,T]$ and $\epsilon>0$, the Burkholder–Davis–Gundy and Young inequalities ensure

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,t\wedge\tau_M\wedge\tau_M^\rho]}|\int_0^s(R,\delta_\rho)d\mathscr{W}|=\mathbb{E}\sup_{s\in[0,t\wedge\tau_M\wedge\tau_M^\rho]}|\int_0^s\sum_{\Bbbk\geq1}(\nabla_y\sigma_\Bbbk(\cdot,y)\delta_\rho+R_\Bbbk^\rho(\cdot,z_\rho),\delta_\rho)d\beta_\Bbbk|\\ &\leq C\mathbb{E}\big[\sum_{\Bbbk\geq1}\int_0^{t\wedge\tau_M\wedge\tau_M^\rho}\|\nabla_y\sigma_\Bbbk(\cdot,y)\delta_\rho+R_\Bbbk^\rho(\cdot,z_\rho)\|_2^2\|\delta_\rho\|_2^2ds\big]^{1/2} \end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon \mathbb{E} \sup_{s\in[0,t\wedge\tau_M\wedge\tau_M^\rho]} \|\delta_\rho(s)\|_V^2 + C_\epsilon \mathbb{E}\int_0^{t\wedge\tau_M\wedge\tau_M^\rho} \|z+\delta_\rho\|_2^2\rho^{2\gamma}ds + C_\epsilon\mathbb{E}\int_0^{t\wedge\tau_M\wedge\tau_M^\rho}\|\delta_\rho\|_2^2 ds\\
&\leq \epsilon \mathbb{E} \sup_{s\in[0,t\wedge\tau_M\wedge\tau_M^\rho]} \|\delta_\rho(s)\|_V^2 + C_\epsilon \mathbb{E}\int_0^{t\wedge\tau_M\wedge\tau_M^\rho} \|z\|_2^2\rho^{2\gamma}ds + C_\epsilon\mathbb{E}\int_0^{t\wedge\tau_M\wedge\tau_M^\rho}\|\delta_\rho\|_2^2 ds,
\end{aligned}
$$

where (2.7) and ([3], Thm. 3) ensure the last inequality. Taking the supremum of (6.3) over the time interval $[0, t\wedge\tau_M\wedge\tau_M^\rho]$, next applying the expectation, we obtain

$$
\begin{aligned}
(1-\epsilon)\mathbb{E} \sup_{s\in[0,t\wedge\tau_M\wedge\tau_M^\rho]} \|\delta_\rho(s)\|_V^2 &\leq C_*\mathbb{E}\int_0^{t\wedge\tau_M\wedge\tau_M^\rho}\|z\|_{H^2}^2\|y_\rho-y\|_{H^2}^2ds + C_*\mathbb{E}\int_0^{t\wedge\tau_M\wedge\tau_M^\rho}\|z\|_2^2\rho^{2\gamma}ds\\
&\quad + C_\epsilon(M)\mathbb{E}\int_0^t \sup_{r\in[0,s\wedge\tau_M\wedge\tau_M^\rho]}\|\delta_\rho(r)\|_V^2 ds, \quad \forall t\in[0,T].
\end{aligned}
$$

Now, we choose $\epsilon = \dfrac{1}{2}$ and apply Grönwall's inequality to obtain

$$
\begin{aligned}
\mathbb{E} \sup_{s\in[0,\tau_M\wedge\tau_M^\rho]} \|\delta_\rho(s)\|_V^2 &\leq C_* e^{C(M)\tau_M}\{\mathbb{E}\int_0^{\tau_M\wedge\tau_M^\rho}\|z\|_{H^2}^2\|y_\rho-y\|_{H^2}^2 ds + \rho^{2\gamma}\mathbb{E}\int_0^{\tau_M\wedge\tau_M^\rho}\|z\|_2^2 ds\}\\
&:= A_1 + A_2.
\end{aligned}
$$

The right hand side of the last inequality converges to 0 as $\rho\to 0$. Indeed, it is clear that $\lim\limits_{\rho\to 0} A_2 = 0$. On the other hand, let us introduce the following stopping time

$$
\tau_N^z = \inf\{t\in[0,T] : \int_0^t \|z\|_{H^2}^2 ds \geq N\}\wedge \tau_M.
$$

By using Corollary 3.3, we get $\lim\limits_{N\to\infty}\lim\limits_{\rho\to 0}\mathbb{E}\int_0^{\tau_M\wedge\tau_M^\rho\wedge\tau_N^z}\|z\|_{H^2}^2\|y_\rho-y\|_{H^2}^2 ds = 0$. Note that $(\tau_N^z)_N$ is non-decreasiong positive sequence of stopping time and converges to τ_M in probability, thanks to Theorem 5.3. Hence, monotone convergence theorem ensures

$$
\lim_{N\to\infty}\mathbb{E}\int_0^{\tau_M\wedge\tau_M^\rho\wedge\tau_N^z}\|z\|_{H^2}^2\|y_\rho-y\|_{H^2}^2 ds = \mathbb{E}\int_0^{\tau_M\wedge\tau_M^\rho}\|z\|_{H^2}^2\|y_\rho-y\|_{H^2}^2 ds.
$$

Now, a standard argument allows to conclude that $\lim\limits_{\rho\to 0} A_1 = 0$. □

Remark 6.2. We would like to mention that when dealing with certain control problems, the implicit function theorem can be applied to study the differentiability of the control-to-state mapping. However, in our framework the nonlinearity in state equation (2.1) depends on the gradient and second order derivatives of the solution of the state equation, and the stopping time (the local time existence) depends on the control, then it is not possible to use the implicit function theorem to get the differentiability of the control-to-state mapping. Furthermore, it is worth to recall that even in simpler system where the principal part depends on the gradient of the solution, the implicit function theorem is not applicable because of norm gap issues, we refer *e.g.* to [31] for an example and more details.

6.1. Variation of the cost functional

Proposition 6.3. *Let J_M be given by (4.1) and consider U, y_0, ψ, $U_\rho = U + \rho\psi$ verifying the hypothesis of Proposition 6.1. Then*

$$J_M(U_\rho, y_\rho) = J_M(U, y) + \rho\lambda\mathbb{E}\int_0^T \|U\|^{p-2}_{(H^1(D))^d}(U,\psi)_{(H^1(D))^d}dt + \rho\mathbb{E}\int_0^{\tau_M^U}(y-y_d, z)dt + o(\rho),$$

where y_ρ, y are the solutions of (2.1), corresponding to (U_ρ, y_0) and (U, y_0), respectively, τ_M^U is the stopping time correponding to the solution y and z is the solution of (5.3) associated to ψ.

Proof. Let us split the cost functional into two parts. Namely,

$$\begin{aligned} S_1 : \mathscr{U}^p_{ad} &\to \mathbb{R}; & S_2 : \mathscr{U}^p_{ad} &\to \mathbb{R} \\ U &\to S_1(U) = \frac{1}{2}\mathbb{E}\int_0^{\tau_M^U}\|y(U)-y_d\|_2^2dt; & U &\to S_2(U) = \frac{\lambda}{p}\mathbb{E}\int_0^T\|U\|^p_{(H^1(D))^d}dt. \end{aligned}$$

First, let us derive the Gâteaux derivative of S_1 at $U \in L^p((\Omega_T, \mathscr{P}_T); (H^1(D))^d)$ in direction of $\psi \in L^p((\Omega_T, \mathscr{P}_T); (H^1(D))^d)$. Let $\rho > 0$ and z be the solution of (5.3), we write

$$\Big|\frac{1}{2\rho}[S_1(U+\rho\psi) - S_1(U)] - \mathbb{E}\int_0^{\tau_M^U}(y-y_d, z)dt\Big| \le I_1(\rho) + I_2(\rho) + I_3(\rho) + I_4(\rho) + I_5(\rho),$$

where
$$\begin{aligned} I_1(\rho) &= \frac{1}{2\rho}\mathbb{E}\int_0^{\tau_M^U\wedge\tau_M^{U+\rho\psi}}\|y(U+\rho\psi) - y(U)\|_2^2dt; \\ I_2(\rho) &= |\mathbb{E}\int_0^{\tau_M^U\wedge\tau_M^{U+\rho\psi}}(y(U)-y_d, \frac{1}{\rho}[y(U+\rho\psi)-y(U)] - z(\psi))dt|; \\ I_3(\rho) &= |\mathbb{E}\int_{\tau_M^U\wedge\tau_M^{U+\rho\psi}}^{\tau_M^U}(y(U)-y_d, z(\psi))dt|; \quad I_4(\rho) = |\frac{1}{2\rho}\mathbb{E}\int_{\tau_M^U\wedge\tau_M^{U+\rho\psi}}^{\tau_M^U}\|y(U)-y_d\|_2^2dt|; \\ I_5(\rho) &= |\frac{1}{2\rho}\mathbb{E}\int_{\tau_M^U\wedge\tau_M^{U+\rho\psi}}^{\tau_M^{U+\rho\psi}}\|y(U+\rho\psi)-y_d\|_2^2dt|. \end{aligned}$$

Thanks to Corollary 3.3, we get $\lim_{\rho\to 0} I_1(\rho) = 0$. More precisely we have

$$I_1(\rho) \le \frac{C}{2\rho}\mathbb{E}\sup_{s\in[0,\tau_M^U\wedge\tau_M^{U+\rho\psi}]}\|y_\rho(s)-y(s)\|_V^2 \le C(M,L,T)\rho\mathbb{E}\int_0^{\tau_M^U\wedge\tau_M^{U+\rho\psi}}\|\psi(s)\|_2^2ds \overset{\rho\to 0}{\to} 0.$$

According to Proposition 6.1, we have $\frac{1}{\rho}[y(U+\rho\psi) - y(U)] - z(\psi) = \delta_\rho$. Therefore, using Cauchy-Schwarz inequality and (6.1), we deduce

$$\begin{aligned} I_2(\rho) &= |\mathbb{E}\int_0^{\tau_M^U\wedge\tau_M^{U+\rho\psi}}(y(U)-y_d, \delta_\rho)dt| \\ &\le 2\sqrt{T}\Big(\mathbb{E}\int_0^{\tau_M^U}(\|y(U)\|_2^2 + \|y_d\|_2^2)ds\Big)^{1/2}\Big(\mathbb{E}\sup_{s\in[0,\tau_M^U\wedge\tau_M^{U+\rho\psi}]}\|\delta_\rho(s)\|_2^2\Big)^{1/2} \overset{\rho\to 0}{\to} 0. \end{aligned}$$

On the other hand, the Cauchy-Schwarz inequality, the Fubini's theorem and Lemma 4.4 yield

$$\begin{aligned} I_3(\rho) &= |\mathbb{E}\int_{\tau_M^U\wedge\tau_M^{U+\rho\psi}}^{\tau_M^U}(y(U)-y_d,z(\psi))dt| \\ &\leq \sqrt{T}\Big(\int_0^T P(\tau_M^U\wedge\tau_M^{U+\rho\psi}<t\leq\tau_M^U)[M^2+\|y_d\|_2^2]ds\Big)^{1/2}\Big(\mathbb{E}\sup_{s\in[0,\tau_M^U]}\|z(s)\|_2^2\Big)^{1/2}\overset{\rho\to 0}{\to} 0. \end{aligned}$$

Finally, I_4 and I_5 can be estimated as follows

$$\begin{aligned} I_4(\rho) &= |\frac{1}{2\rho}\mathbb{E}\int_{\tau_M^U\wedge\tau_M^{U+\rho\psi}}^{\tau_M^U}\|y(U)-y_d\|_2^2dt| \leq \int_0^T\frac{P(\tau_M^U\wedge\tau_M^{U+\rho\psi}<\tau_M^U)}{\rho}[M^2+\|y_d\|_2^2]ds; \\ I_5(\rho) &= |\frac{1}{2\rho}\mathbb{E}\int_{\tau_M^U\wedge\tau_M^{U+\rho\psi}}^{\tau_M^{U+\rho\psi}}\|y(U+\rho\psi)-y_d\|_2^2dt| \leq \int_0^T\frac{P(\tau_M^U\wedge\tau_M^{U+\rho\psi}<\tau_M^{U+\rho\psi})}{\rho}[M^2+\|y_d\|_2^2]ds. \end{aligned}$$

Since $\lim_{\rho\to 0}(U+\rho\psi)=U$ in X, Corollary 4.5 ensures that $\lim_{\rho\to 0} I_4(\rho)+I_5(\rho)=0$. Consequently, $\lim_{\rho\to 0}|\frac{1}{2\rho}[S_1(U+\rho\psi)-S_1(U)]-\mathbb{E}\int_0^{\tau_M^U}(y-y_d,z)dt|=0$, and the Gâteaux derivative of S_1 at $U\in L^p((\Omega_T,\mathscr{P}_T);(H^1(D))^d)$ in direction of $\psi\in L^p((\Omega_T,\mathscr{P}_T);(H^1(D))^d)$ is given by $d^GS_1(U)[\psi]=\mathbb{E}\int_0^{\tau_M^U}(y-y_d,z)dt$. Notice that S_2 is given by the p^{th}-power of a norm in $L^p(\Omega\times(0,T);(H^1(D))^d)$. Therefore, by using standard arguments, we can show that the Gâteaux derivative of S_2 at $U\in L^p((\Omega_T,\mathscr{P}_T);(H^1(D))^d)$ in direction of $\psi\in L^p((\Omega_T,\mathscr{P}_T);(H^1(D))^d)$ corresponds to $d^GS_2(U)[\psi]=\lambda\mathbb{E}\int_0^T\|U\|_{(H^1(D))^d}^{p-2}(U,\psi)_{(H^1(D))^d}dt$. It is clear that $d^GS_1(U)$ and $d^GS_2(U)$ are linear and continuous (bounded) functionals, which ends the proof. □

7. Stochastic backward adjoint equations

This section is devoted to the formulation and study of the so-called adjoint equation, which is crucial for writing the first order optimality conditions for the system (2.1). It is worth to mention that the adjoint equation, as well as the optimality conditions, can be deduced by formally applying the Lagrange multiplier method, as explained in [32], Section 2.10. The equations, once formally deduced, should be rigorously justified. For this task, a suitable integration by parts should be performed. Since the 3D and 2D analysis are different, we will distinguish the two cases, by studying each one separately.

Let g be a predictable function satisfying $g\in L^p(\Omega;L^2(0,T;H))$. In particular, we will set later $g=y-y_d$, where y is the solution of (2.1) and $y_d\in L^2(0,T;H)$.
First, let us make some observations about the stochastic part of the problem.

Remark 7.1. Let us consider the following mapping

$$\begin{aligned} &\nabla_y G:(L^2(D))^d\to L_2(\mathbb{H};(L^2(D))^d), \\ &\qquad u\mapsto\nabla_y G(\cdot,y)u:\mathbb{H}\to(L^2(D))^d,\quad \nabla_y G(\cdot,y)ue_{\Bbbk}=\nabla_y\sigma_{\Bbbk}(\cdot,y)u,\quad\forall\,\Bbbk\geq 1. \end{aligned}$$

- We recall that $(L^2(D))^d$ and $L_2(\mathbb{H};(L^2(D))^d)$ are Hilbert spaces. Since $\nabla_y G$ is linear and continuous (bouned) operator, there exists a linear and bounded operator $G^*: L_2(\mathbb{H};(L^2(D))^d)\to(L^2(D))^d$ satisfying

$$(\nabla_y G u, q)_{L_2(\mathbb{H};(L^2(D))^d)} = (u, G^* q); \quad \forall q \in L_2(\mathbb{H};(L^2(D))^d), \forall u\in (L^2(D))^d. \tag{7.1}$$

- It follows from (7.1) that G^*q is predictable too for every $L_2(\mathbb{H};(L^2(D))^d)$-valued predictable process q. Moreover, for any $q\in L_2(\mathbb{H};(L^2(D))^d)$, $u\in(L^2(D))^d$, we have

$$(\nabla_y G u, q)_{L_2(\mathbb{H};(L^2(D))^d)} = \sum_{k\geq 1}(\nabla_y\sigma_k(\cdot,y)u, qe_k) = \sum_{k\geq 1}(u,(\nabla_y\sigma_k(\cdot,y))^T qe_k) = (u, G^*q),$$

where B^T denotes the transpose of the matrix B.

7.1. Existence and uniqueness of solution to the backward adjoint equation in 2D

Here we aim to show the well-posedness of solution (p,q) to the following 2D adjoint system

$$\begin{cases} -d(v(p)) + \mathbb{I}_{[0,\tau_M[}\{-\nu\Delta p - \text{curl } v(y)\times p + \text{curl } v(y\times p) - (\alpha_1+\alpha_2)\text{div}\big[A(y)A(p)+A(p)A(y)\big] \\ \quad -\beta\text{div}\big[|A(y)|^2A(p)\big] - 2\beta\text{div}\big[(A(y):A(p))A(y)\big]\}dt \\ \qquad = \mathbb{I}_{[0,\tau_M[}\{g - \nabla\pi - \sum_{k\geq 1}(\nabla_y\sigma_k(\cdot,y))^T qe_k\}dt + \sum_{k\geq 1} v(qe_k)d\beta_k & \text{in } D\times(0,T)\times\Omega, \\ \text{div}(p) = 0 & \text{in } D\times(0,T)\times\Omega, \\ p\cdot\eta = 0, \quad [\eta\cdot\mathbb{D}(p)]\cdot\tau = 0 & \text{on } \partial D\times(0,T)\times\Omega, \\ p(T) = 0 & \text{in } D\times\Omega, \end{cases} \tag{7.2}$$

where y is the solution of (2.1), in the sense of Definition 2.4 and τ_M is given by (2.10).

Let us introduce the notion of solution to the system (7.2), the so-called adjoint state.

Definition 7.2. A pair (p,q) of stochastic processes is a solution to (7.2) if it satisfies the following properties

i) p and q are predictable processes with values in W and $L_2(\mathbb{H};W)$, respectively.
ii) $p\in L^\infty(0,T;L^2(\Omega;W))$, $q\in L^2(\Omega_T;L_2(\mathbb{H};W))$ and $v(p)\in\mathscr{C}_w([0,T];L^2(\Omega;(L^2(D))^2))$.
iii) For any $t\in[0,T]$, P-a.s. in Ω and for any $\phi\in W$, the following equation holds

$$\begin{aligned} &(v(p(t)),\phi) + 2\nu\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(\mathbb{D}p,\mathbb{D}\phi)ds + 2\beta\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\big((A(p):A(y))\,A(y),\nabla\phi\big)ds \\ &\quad + \int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{-b(\phi,p,v(y)) + b(p,\phi,v(y)) + b(p,y,v(\phi)) - b(y,p,v(\phi))\Big\}ds \\ &\quad + \int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{(\alpha_1+\alpha_2)\big(A(y)A(p)+A(p)A(y),\nabla\phi\big) + \beta\big(|A(y)|^2A(p),\nabla\phi\big)\Big\}ds \\ &\quad = \int_t^T\mathbb{I}_{[0,\tau_M[}(s)(g - G^*q,\phi)ds + \sum_{k\geq 1}\int_t^T (v(qe_k),\phi)d\beta_k(s). \end{aligned} \tag{7.3}$$

Theorem 7.3. *There exists a unique pair* $(\mathscr{p},\mathscr{q})$ *in the sense of Definition 7.2, satisfying*

$$\sup_{s\in[0,T]}\mathbb{E}\|\mathscr{p}(s)\|_W^2\leq C(\alpha_1,\alpha_2,\beta,\nu,T,M)\ \textit{and}\ \mathbb{E}\int_0^T\|\mathscr{q}\|^2_{L_2(\mathbb{H};W)}ds\leq C(M), \tag{7.4}$$

where $C(M)>0$ *and* $C(\alpha_1,\alpha_2,\beta,\nu,T,M)>0$*. Moreover, for each fixed* $M\in\mathbb{N}$ *we have*

$$\mathbb{E}\sup_{s\in[\tau_M,T]}\|\mathscr{p}(s)\|_W^2=0\ \textit{and}\ \mathbb{E}\int_{\tau_M}^T\|\mathscr{q}\|^2_{L_2(\mathbb{H};W)}ds=0.$$

7.1.1. Finite dimensional approximation

Let us consider the orthonormal basis $\{h_i\}$ in V, given by (5.6) and the finite dimensional spaces W_n, already introduced in Section 5.1. Setting $p_n(t)=\sum_{i=1}^n d_i(t)h_i,\quad n\in\mathbb{N},\ d_i(t)\in\mathbb{R}$, and $q_n(t)\in L_2(\mathbb{H},W_n)$, the approximation for (7.2) reads

$$\begin{aligned}
&(v(p_n(t)),\phi)+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{2\nu(\mathbb{D}p_n,\mathbb{D}\phi)ds-b(\phi,p_n,v(y))\\
&+b(p_n,\phi,v(y))+b(p_n,y,v(\phi))-b(y,p_n,v(\phi))\Big\}ds\\
&+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{(\alpha_1+\alpha_2)\big(A(y)A(p_n)+A(p_n)A(y),\nabla\phi\big)+\beta\big(|A(y)|^2A(p_n),\nabla\phi\big)\Big\}ds\\
&+2\beta\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\big(\,(A(p_n):A(y))\,A(y),\nabla\phi\big)ds=\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(g-G^*q_n,\phi)ds\\
&+\sum_{\mathbb{k}\geq1}\int_t^T(v(q_ne_{\mathbb{k}}),\phi)d\beta_{\mathbb{k}}(s),\qquad\text{for any }\phi\in W_n.
\end{aligned}\tag{7.5}$$

From [33], Proposition 6.20, there exists a unique pair of predictable processes (p_n,q_n) such that (7.5) holds for any $t\in[0,T]$ and

$$p_n\in L^r_{\mathscr{P}_T}(\Omega;\mathscr{C}([0,T],W_n)),\quad q_n\in L^r(\Omega;L^2(0,T;L_2(\mathbb{H},W_n)),\ \text{for all }1\leq r\leq2(d+1). \tag{7.6}$$

7.1.2. Uniform estimates

Step 1. Estimate in the space V

Let us consider $0\leq t\leq T$. Setting $\phi=h_i$ in (7.5), we obtain

$$\begin{aligned}
&(v(p_n(t)),h_i)+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{(\alpha_1+\alpha_2)\big(A(y)A(p_n)+A(p_n)A(y),\nabla h_i\big)+\beta\big(|A(y)|^2A(p_n),\nabla h_i\big)\Big\}ds\\
&+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{2\nu(\mathbb{D}p_n,\mathbb{D}h_i)-b(h_i,p_n,v(y))+b(p_n,h_i,v(y))+b(p_n,y,v(h_i))-b(y,p_n,v(h_i))\Big\}ds\\
&+2\beta\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\big(\,(A(p_n):A(y))\,A(y),\nabla h_i\big)ds\\
&=\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(g-G^*q_n,h_i)ds+\sum_{\mathbb{k}\geq1}\int_t^T(v(q_ne_{\mathbb{k}}),h_i)d\beta_{\mathbb{k}}(s).
\end{aligned}$$

Applying the Itô's formula, integrating over the time interval $[t,T]$, and summing from $i=1$ to n, we deduce

$$\begin{aligned}
&\|p_n(t)\|_V^2 + \int_t^T 2\mathbb{I}_{[0,\tau_M[}(s)\Big\{(\alpha_1+\alpha_2)\big(A(y)A(p_n)+A(p_n)A(y),\nabla p_n\big)+\beta\big(|A(y)|^2A(p_n),\nabla p_n\big)\Big\}ds\\
&+\int_t^T 2\mathbb{I}_{[0,\tau_M[}(s)\Big\{2\nu(\mathbb{D}p_n,\mathbb{D}p_n)+b(p_n,y,v(p_n))-b(y,p_n,v(p_n))\Big\}ds\\
&+4\beta\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\big(\,(A(p_n):A(y))\,A(y),\nabla p_n\big)ds = \int_t^T 2\mathbb{I}_{[0,\tau_M[}(s)(g-G^*q_n,p_n)ds \qquad (7.7)\\
&+\sum_{\Bbbk\geq 1}\int_t^T 2(q_ne_k,p_n)_V d\beta_{\Bbbk}(s) - \sum_{i=1}^n\sum_{\Bbbk\geq 1}\int_t^T (q_ne_k,h_i)_V^2ds.
\end{aligned}$$

We notice that $\sum_{i=1}^n\sum_{\Bbbk\geq 1}\int_t^T (q_ne_{\Bbbk},h_i)_V^2ds = \sum_{\Bbbk\geq 1}\int_t^T \|q_ne_{\Bbbk}\|_V^2ds = \int_t^T \|q_n\|^2_{L_2(\mathbb{H},V)}ds.$

On the other hand, taking into account (7.6), we have

$$\mathbb{E}\sum_{\Bbbk\geq 1}\int_0^T (q_ne_k,p_n)_V^2ds<\infty \text{ thus } \mathbb{E}\sum_{\Bbbk\geq 1}\int_t^T 2(q_ne_k,p_n)_Vd\beta_{\Bbbk}(s)=0. \qquad (7.8)$$

Thanks to Lemma 10.1 (see Sect. 10), we derive

$$\begin{aligned}
&\int_t^T 2\mathbb{I}_{[0,\tau_M[}(s)\Big\{2\nu(\mathbb{D}p_n,\mathbb{D}p_n)+b(p_n,y,v(p_n))-b(y,p_n,v(p_n))\Big\}ds\\
&\leq 4\nu\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|\mathbb{D}p_n\|_2^2ds + C\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|y\|_{W^{2,4}}\|p_n\|_V^2ds\\
&\leq 4\nu\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|\mathbb{D}p_n\|_2^2ds + CM\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_V^2ds.
\end{aligned}$$

Now, using similar arguments to those used in [18], Section 4.2, we deduce the following estimates

$$\begin{aligned}
&\left|\int_t^T 2\mathbb{I}_{[0,\tau_M[}(s)\Big\{(\alpha_1+\alpha_2)\big(A(y)A(p_n)+A(p_n)A(y),\nabla p_n\big)\Big\}ds\right| \leq C_2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|^2_{H^1}\|y\|_{W^{1,\infty}}ds\\
&\text{and } |4\beta\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\big(\,(A(p_n):A(y))\,A(y),\nabla p_n\big)ds+\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\beta\big(|A(y)|^2A(p_n),\nabla p_n\big)ds|\\
&\qquad \leq C_3\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|^2_{H^1}\|y\|^2_{W^{1,\infty}}ds \leq C_3\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|^2_{H^1}\|y\|^2_{W^{2,4}}ds
\end{aligned}$$

Gathering the previous estimates, we show that there exists $C>0$, independent of n, such that

$$\begin{aligned}
\|p_n(t)\|_V^2 + \int_t^T \|q_n\|^2_{L_2(\mathbb{H};V)}ds + 4\nu\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|\mathbb{D}p_n\|_2^2ds &\leq C(M^2+1)\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_V^2ds\\
+\int_t^T 2\mathbb{I}_{[0,\tau_M[}(s)(g-G^*q_n,p_n)ds &+ \sum_{\Bbbk\geq 1}\int_t^T 2(q_ne_{\Bbbk},p_n)_Vd\beta_{\Bbbk}(s). \qquad (7.9)
\end{aligned}$$

Let $\delta>0$. Thanks to Remark 7.1 and since $V\hookrightarrow (L^2(D))^2$, one has

$$\begin{aligned}&|\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(g-G^*q_n,p_n)ds|\le C\int_t^{\tau_M}\|g\|_2^2ds+C_p\int_t^{\tau_M}\|p_n\|_2^2ds+C\int_t^{\tau_M}|(G^*q_n,p_n)|ds\\&\le C\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2ds+\delta C(G^*)[\int_t^T\|q_n\|_{L_2(\mathbb{H};V)}^2ds+C_\delta(T)\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_V^2ds].\end{aligned}$$

By taking the expectation in (7.9), using (7.8) and an appropriate choice of δ, we are able to deduce for any $0\le t<\tau_M$ that

$$\begin{aligned}\mathbb{E}\|p_n(t)\|_V^2+\mathbb{E}\big[\int_t^T\|q_n\|_{L_2(\mathbb{H};V)}^2ds\big]&\le C(G^*)(M^2+1)\mathbb{E}\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_V^2ds\\&\quad+C\mathbb{E}\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2ds.\end{aligned}$$

Therefore, Grönwall's inequality ensures the existence of $C(M)>0$ such that

$$\sup_{r\in[t,T]}\mathbb{E}\|p_n(r)\|_V^2+\mathbb{E}\big[\int_t^T\|q_n\|_{L_2(\mathbb{H};V)}^2ds\big]\le Ce^{C(G^*)(M^2+1)T}\mathbb{E}\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2ds:=C(M). \tag{7.10}$$

Step 2. Estimate in the space W

For any $t\in[0,T]$, the equation (7.5) gives

$$\begin{aligned}&(p_n(t),h_i)_V+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(F(p_n),h_i)ds\\&\qquad=\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(B(q_n),h_i)ds+\sum_{\Bbbk\ge1}\int_t^T(q_ne_{\Bbbk},h_i)_Vd\beta_{\Bbbk},\quad\forall 1\le i\le n,\end{aligned}\tag{7.11}$$

where $F(u):=-\nu\Delta u-\text{curl }v(y)\times u+\text{curl }v(y\times u)-(\alpha_1+\alpha_2)\text{div}\big[A(y)A(u)+A(u)A(y)\big]$
$-\beta\text{div}\big[|A(y)|^2A(u)\big]-2\beta\text{div}\big[(A(y):A(u))A(y)\big];\ B(q_n):=g-G^*q_n.$

Let $\widetilde{F}_n$ and $\widetilde{B}_n$ be the solutions of (2.4) for $f=F(p_n)$ and $f=B(q_n)$. Therefore

$$(F(p_n),h_i)=(\widetilde{F}_n,h_i)_V,\quad (B(q_n),h_i)=(\widetilde{B}_n,h_i)_V;\quad \forall 1\le i\le n \tag{7.12}$$

Multiplying (7.11) by μ_i and using (5.6), we deduce for $1\le i\le n$

$$(p_n(t),h_i)_W+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(\widetilde{F}_n,h_i)_Wds=\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(\widetilde{B}_n,h_i)_Wds+\sum_{\Bbbk\ge1}\int_t^T(q_ne_{\Bbbk},h_i)_Wd\beta_{\Bbbk}.$$

Now, using Ito's formula for the function $u\mapsto u^2$ and the process $(p_n(t),h_i)_W$, next multiplying by $\frac{1}{\mu_i}$ and summing over $i=1,\cdots,n$, we are able to infer that

$$\begin{aligned}&\|p_n(t)\|_W^2+2\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(\widetilde{F}_n,p_n)_Wds\\&=2\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(\widetilde{B}_n,p_n)_Wds+2\sum_{\Bbbk\ge1}\int_t^T(q_ne_{\Bbbk},p_n)_Wd\beta_{\Bbbk}-\sum_{i=1}^n\sum_{\Bbbk\ge1}\int_t^T(q_ne_{\Bbbk},\tilde{h}_i)_W^2ds.\end{aligned}$$

In addition, we observe that $\sum_{i=1}^{n}\sum_{\Bbbk\geq 1}\int_t^T (q_n e_{\Bbbk}, \tilde{h}_i)_W^2 ds = \sum_{\Bbbk\geq 1}\int_t^T \|q_n e_{\Bbbk}\|_W^2 ds = \int_t^T \|q_n\|^2_{L_2(\mathbb{H};W)}ds.$
Taking into account the definition of inner product in the space W, we deduce

$$\begin{aligned}
&\|p_n(t)\|_W^2 + \int_t^T \|q_n\|^2_{L_2(\mathbb{H};W)}ds = -2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(F(p_n),p_n)ds - 2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(F(p_n),\mathbb{P}v(p_n))ds\\
&+2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(g-G^*q_n,p_n)ds + 2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(g-G^*q_n,\mathbb{P}v(p_n))ds + 2\sum_{\Bbbk\geq 1}\int_t^T (q_n e_{\Bbbk}, p_n)_W d\beta_{\Bbbk}\\
&= I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

Let us focus on estimating I_i, $i=1,\dots,5$. By using the boundary conditions, we get

$$\begin{aligned}
&-I_1 = 2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(F(p_n),p_n)ds = \int_t^T 2\mathbb{I}_{[0,\tau_M[}(s)\Big\{2\nu(\mathbb{D}p_n,\mathbb{D}p_n) + b(p_n,y,v(p_n)) - b(y,p_n,v(p_n))\Big\}ds\\
&+\int_t^T 2\mathbb{I}_{[0,\tau_M[}(s)\Big\{(\alpha_1+\alpha_2)\big(A(y)A(p_n)+A(p_n)A(y),\nabla p_n\big) + \beta\big(|A(y)|^2A(p_n),\nabla p_n\big)\Big\}ds\\
&+4\beta\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\big(\,(A(p_n):A(y))\,A(y),\nabla p_n\big)ds.
\end{aligned}$$

Arguments already detailed in Subsubsection 7.1.2 yields the existence of $C, C_\delta>0$ such that

$$\begin{aligned}
&|I_1| \leq C(M^2+1)\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_V^2,\\
&|I_3| \leq C\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2 ds + \delta C[\int_t^T \|q_n\|^2_{L_2(\mathbb{H};V)}ds + C_\delta\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_V^2 ds], \quad \forall\delta>0.
\end{aligned}$$

By using (7.6), we get that $\mathbb{E}(I_5)=0$. Thanks to Remark 7.1, there exists $C>0$ such that

$$\begin{aligned}
|I_4| &= 2|\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(g-G^*q_n,\mathbb{P}v(p_n))ds|\\
&\leq C\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2ds + C\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|\mathbb{P}v(p_n)\|_2^2ds + C\int_t^T \mathbb{I}_{[0,\tau_M[}(s)|(G^*q_n,\mathbb{P}v(p_n))|ds\\
&\leq C\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2ds + \delta C[\int_t^T \|q_n\|^2_{L_2(\mathbb{H};W)}ds + \frac{C}{\delta}\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_W^2 ds], \quad \forall\delta>0,
\end{aligned}$$

where we used $W\hookrightarrow (L^2(D))^2$. Finally, we write

$$\begin{aligned}
I_2 = -2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(&-\nu\Delta p_n - \text{curl }v(y)\times p_n + \text{curl }v(y\times p_n) - (\alpha_1+\alpha_2)\text{div}\big[A(y)A(p_n)+A(p_n)A(y)\big]\\
&-\beta\text{div}\big[|A(y)|^2A(p_n)\big] - 2\beta\text{div}\big[(A(y):A(p_n))A(y)\big],\mathbb{P}v(p_n))ds = I_2^1+I_2^2+I_2^3+I_2^4.
\end{aligned}$$

We have $|I_2^1| = |2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(-\nu\Delta p_n, \mathbb{P}v(p_n))ds| \leq 2\nu\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_W^2 ds$. By performing standard calculations, we show that there exists $C>0$ such that

$$\begin{aligned}|I_2^4| = |2\int_t^T &\mathbb{I}_{[0,\tau_M[}(s)(-(\alpha_1+\alpha_2)\mathrm{div}\big[A(y)A(p_n)+A(p_n)A(y)\big] - \beta\mathrm{div}\big[|A(y)|^2A(p_n)\big] \\ &- 2\beta\mathrm{div}\big[(A(y):A(p_n))A(y)\big], \mathbb{P}v(p_n))ds| \\ \leq C\int_t^T &\mathbb{I}_{[0,\tau_M[}(s)\|y\|_{W^{2,4}}\|p_n\|_W^2 + C\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|y\|_{W^{2,4}}^2\|p_n\|_W^2 \leq C(1+M^2)\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_W^2.\end{aligned}$$

Concerning $I_2^2+I_2^3$, we invoke Lemma 10.3 (see Sect. 10) to get $C>0$ such that

$$\begin{aligned}|I_2^2+I_2^3| &= |2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)(\mathrm{curl}\ v(y\times p_n) - \mathrm{curl}\ v(y)\times p_n, \mathbb{P}v(p_n))ds| \\ &\leq C(1+\alpha_1)\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|y\|_{W^{2,4}}\|p_n\|_W\|\mathbb{P}v(p_n)\|_2 ds + 2\int_t^T \mathbb{I}_{[0,\tau_M[}(s)|b(y, v(p_n)-\mathbb{P}v(p_n), \mathbb{P}v(p_n))|ds.\end{aligned}$$

Thanks to [8], Lemma 5 and $W^{2,4}\hookrightarrow L^\infty$, there exists $C>0$ depending only on D such that

$$|b(y, v(p_n)-\mathbb{P}v(p_n), \mathbb{P}v(p_n))| \leq \|y\|_\infty\|v(p_n)-\mathbb{P}v(p_n)\|_{H^1}\|\mathbb{P}v(p_n)\|_2 \leq C\|y\|_{W^{2,4}}\|p_n\|_{H^2}\|\mathbb{P}v(p_n)\|_2.$$

Hence, there exists $C>0$ such that

$$|I_2^2+I_2^3| \leq C(1+\alpha_1)\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|y\|_{W^{2,4}}\|p_n\|_W\|\mathbb{P}v(p_n)\|_2 ds \leq CM(1+\alpha_1)\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_W^2 ds.$$

By taking the expectation and gathering the previous estimates, we obtain

$$\begin{aligned}&\mathbb{E}\|p_n(t)\|_W^2 + \mathbb{E}\int_t^T \|q_n\|_{L_2(\mathbb{H};W)}^2 ds \leq C(\alpha_1,\alpha_2,\beta)(1+M^2)\mathbb{E}\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_W^2 ds \\ &+ C\mathbb{E}\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2 ds + \delta C[\mathbb{E}\int_t^T \|q_n\|_{L_2(\mathbb{H};W)}^2 ds + \frac{C}{\delta}\mathbb{E}\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_W^2 ds], \quad \forall\delta>0.\end{aligned}$$

An appropriate choice of δ gives

$$\mathbb{E}\|p_n(t)\|_W^2 + \mathbb{E}\int_t^T \|q_n\|_{L_2(\mathbb{H};W)}^2 ds \leq C(1+M^2)\mathbb{E}\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|p_n\|_W^2 ds + C\mathbb{E}\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2 ds.$$

Now, Grönwall's inequality ensures

$$\mathbb{E}\|p_n(t)\|_W^2 + \mathbb{E}\int_t^T \|q_n\|_{L_2(\mathbb{H};W)}^2 ds \leq Ce^{C(\alpha_1,\alpha_2,\beta)(1+M^2)T}\mathbb{E}\int_t^T \mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2 ds \leq C(M), \tag{7.13}$$

$\forall t\in[0,T]$. Let $t\geq\tau_M$, it follows from (7.13) that $\mathbb{E}\|p_n(t)\|_W^2 + \mathbb{E}\int_t^T \|q_n\|_{L_2(\mathbb{H};W)}^2 ds = 0$, $\forall t\geq\tau_M$. As a conclusion, we state the next proposition.

Proposition 7.4. *There exist $C(M)>0$ and $C(\alpha_1,\alpha_2,\beta,\nu,T,M)>0$ such that*

$$\sup_{s\in[0,T]}\mathbb{E}\|p_n(s)\|_W^2\le C(\alpha_1,\alpha_2,\beta,\nu,T,M),\ \text{ and } \mathbb{E}\int_0^T\|q_n\|^2_{L_2(\mathbb{H};W)}ds\le C(M). \tag{7.14}$$

Moreover, for fixed $M\in\mathbb{N}$ we have $\mathbb{E}\sup_{s\in[\tau_M,T]}\|p_n(s)\|_W^2=0$ *and* $\mathbb{E}\int_{\tau_M}^T\|q_n\|^2_{L_2(\mathbb{H};W)}ds=0.$

7.1.3. Proof of Theorem 7.3

Here, we will adapt the reasoning previously used to prove the Theorem 5.3. Due to Proposition 7.4, we deduce the existence of subsequence (denoted by the same way) (p_n,q_n) and $(\mathscr{p},\mathscr{q})\in L^2(\Omega,L^2(0,T;W))\times L^2(\Omega,L^2(0,T;L_2(\mathbb{H};W)))$, where $\mathscr{p}$ and $\mathscr{q}$ are predictables on Ω_T such that

$$p_n\rightharpoonup \mathscr{p} \text{ weakly in } L^2(\Omega,L^2(0,T;W)), \tag{7.15}$$
$$q_n\rightharpoonup \mathscr{q} \text{ weakly in } L^2(\Omega,L^2(0,T;L_2(\mathbb{H};W))). \tag{7.16}$$

According to Remark 7.1 we also know that G^* defined by

$$G^*:L^2(\Omega\times[0,T];L_2(\mathbb{H};(L^2(D))^d))\to L^2(\Omega\times[0,T];\mathbb{R});\quad q\mapsto(G^*q,\phi)$$

is linear and bounded. Recall that the stochastic Itô integral is linear and continuous. Thus, using [27], Proposition 21.27, p. 261 and (7.16) we show that

$$\sum_{\mathscr{k}\ge1}\int_t^T(v(q_ne_{\mathscr{k}}),\phi)d\beta_{\mathscr{k}}(s)\rightharpoonup\sum_{\mathscr{k}\ge1}\int_t^T(v(\mathscr{q}e_{\mathscr{k}}),\phi)d\beta_{\mathscr{k}}(s) \text{ in } L^2(\Omega\times[0,T]),\quad\forall t\in[0,T], \tag{7.17}$$
$$(G^*q_n,\phi)\rightharpoonup(G^*\mathscr{q},\phi) \text{ in } L^2(\Omega\times[0,T]),\quad\forall t\in[0,T]. \tag{7.18}$$

Setting $M_n(t)=v(p_n(t))-\sum_{\mathscr{k}\ge1}\int_t^T v(q_ne_{\mathscr{k}})d\beta_{\mathscr{k}}(s)$, $t\in[0,T]$, the relation (7.5) gives

$$\begin{aligned}\frac{d}{dt}(M_n(t),\phi)&=\mathbb{I}_{[0,\tau_M[}\Big\{2\nu(\mathbb{D}p_n,\mathbb{D}\phi)ds-b(\phi,p_n,v(y))+b(p_n,\phi,v(y))+b(p_n,y,v(\phi))\\&-b(y,p_n,v(\phi))\Big\}+\mathbb{I}_{[0,\tau_M[}\Big\{(\alpha_1+\alpha_2)\big(A(y)A(p_n)+A(p_n)A(y),\nabla\phi\big)+\beta\big(|A(y)|^2A(p_n),\nabla\phi\big)\Big\}\\&+2\beta\mathbb{I}_{[0,\tau_M[}\big(\,(A(p_n):A(y))\,A(y),\nabla\phi\big)-\mathbb{I}_{[0,\tau_M[}(g-G^*q_n,\phi).\end{aligned} \tag{7.19}$$

With the help of (7.15), (7.17) and (7.18), we are able to pass to the limit in the weak sense in this equation, as $n\to\infty$, and deduce

$$\begin{aligned}\frac{d}{dt}(M(t),\phi)&=\mathbb{I}_{[0,\tau_M[}\Big\{2\nu(\mathbb{D}\mathscr{p},\mathbb{D}\phi)ds-b(\phi,\mathscr{p},v(y))+b(\mathscr{p},\phi,v(y))+b(\mathscr{p},y,v(\phi))\\&-b(y,\mathscr{p},v(\phi))\Big\}+\mathbb{I}_{[0,\tau_M[}\Big\{(\alpha_1+\alpha_2)\big(A(y)A(\mathscr{p})+A(\mathscr{p})A(y),\nabla\phi\big)+\beta\big(|A(y)|^2A(\mathscr{p}),\nabla\phi\big)\Big\}\\&+2\beta\mathbb{I}_{[0,\tau_M[}\big(\,(A(\mathscr{p}):A(y))\,A(y),\nabla\phi\big)-\mathbb{I}_{[0,\tau_M[}(g-G^*\mathscr{q},\phi),\end{aligned} \tag{7.20}$$

where $M(t)=v(\mathscr{p}(t))-\sum_{\mathscr{k}\ge1}\int_t^T v(\mathscr{q}e_{\mathscr{k}})d\beta_{\mathscr{k}}(s)$. Using Proposition 7.4, we can verify that the distributional derivative $\frac{dM}{dt}$ belongs to $L^2(0,T;L^2(\Omega;W'))$. On the other hand, Proposition 7.4 ensures that $M\in L^\infty(0,T;L^2(\Omega;(L^2(D))^d))$, then we infer that $M\in\mathscr{C}([0,T];L^2(\Omega;W'))$. Taking into account the properties of the stochastic integral, we conclude that $v(\mathscr{p})\in\mathscr{C}([0,T];L^2(\Omega;W'))$. Thus $v(\mathscr{p})\in\mathscr{C}_w([0,T];L^2(\Omega;(L^2(D))^d))$, thanks to [29], Lemma 1.4, p. 263. To verify that $p(T)=0$, let $\phi\in\overline{W_n}$ and $\xi\in\mathscr{C}^\infty([t,T])$ for $t\in[0,T]$ and note that

$$\int_t^T\langle\frac{dM(s)}{ds},\phi\xi\rangle_{W^*,W}ds=-\int_t^T[(M(s),\phi)\frac{d\xi}{ds}]ds+(M(T),\phi)_V\xi(T).\tag{7.21}$$

First, we multiply (7.19) by $\mathbb{I}_A\xi$, $\xi\in\mathscr{C}^\infty([0,t])$ for $t\in]0,T]$ and integrate over Ω_T. Then, we pass to the limit as $n\to\infty$ and we use (7.21). It follows by standard arguments (see *e.g.* [30], Prop. 3) that, for all $t\in[0,T]$, $v(p_n(t))\rightharpoonup v(p(t))$ in $L^2(\Omega,(L^2(D))^d)$. Hence the proof of Theorem 7.3 is completed.

Remark 7.5. *i)* Let us stress here that the main difference between the adjoint equation in 2D and 3D relies in the analysis of the term $b(p,y,v(\phi))-b(y,p,v(\phi))$. In the former case, we can take advantage of the equality $b(p,y,v(\phi))-b(y,p,v(\phi))=(\text{curl}v(y\times p),\phi)$, which plays a crucial role in the deduction of the W-estimate for the solution of (7.2) (see (7.11)–(7.12)). Instead, in 3D we have the more complicated relation

$$b(p,y,v(\phi))-b(y,p,v(\phi))=(\text{curl}v(y\times p),\phi)+I_{\partial D},$$

unfortunately, the boundary terms $I_{\partial D}$ (see (10.3)) resulting from integration by parts do not vanish, and are very difficult to handle in order to derive the W-estimate.
ii) We recall that the derivation of the first order optimality conditions can be performed by using the formal Lagrange method, where the *"formal Lagrangian function"* $L(y,U,p)$ becomes "meaningful" after only one integration by parts, and $D_yL(y,U,p)$[9] leads to the variational equation (7.3) and the "formal" optimality condition

$$D_UL(y(\widetilde{U_M}),\widetilde{U_M},p)(u-\widetilde{U_M})\geq 0,\quad\forall u\in\mathscr{U}^p_{ad}.\tag{7.22}$$

Therefore, in 2D it is the variational equation (7.3), which will play the most important role to rigorously prove the necessary first order optimality conditions (7.22) (see Sect. 8). This observation motivates the natural extension of the 2D adjoint equation to the 3D setting.

7.2. Existence of solution to the backward adjoint equation in 3D

Taking into account Remark 7.5, it is expected that the passage from the 2D to the 3D framework is accompanied by some loss of regularity for the solution of the adjoint system.

Definition 7.6. A stochastic processes $(\mathbf{p},\mathbf{q})$ is said to be an adjoint state for the control problem if the following properties hold

i) $\mathbf{p}$ and $\mathbf{q}$ are predictables processes with values in V and $L_2(\mathbb{H};V)$, respectively.
ii) $\mathbf{p}\in L^\infty(0,T;L^2(\Omega;V))$, $\mathbf{q}\in L^2(\Omega_T;L_2(\mathbb{H};V))$ and $\mathbf{p}\in\mathscr{C}_w([0,T];L^2(\Omega;V))$.
iii) For any $t\in[0,T]$, P-a.s. in Ω, the following adjoint equation is satisfied

$$\begin{aligned}&(\mathbf{p}(t),\phi)+2\alpha_1(\mathbb{D}\mathbf{p}(t),\mathbb{D}\phi)+2\beta\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\big((A(\mathbf{p}):A(y))\,A(y),\nabla\phi\big)ds\\&+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{2\nu(\mathbb{D}\mathbf{p},\mathbb{D}\phi)ds-b(\phi,\mathbf{p},v(y))+b(\mathbf{p},\phi,v(y))+b(\mathbf{p},y,v(\phi))-b(y,\mathbf{p},v(\phi))\Big\}ds\\&+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{(\alpha_1+\alpha_2)\big(A(y)A(\mathbf{p})+A(\mathbf{p})A(y),\nabla\phi\big)+\beta\big(|A(y)|^2A(\mathbf{p}),\nabla\phi\big)\Big\}ds\\&\quad=\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(g-G^*\mathbf{q},\phi)ds+\sum_{\Bbbk\geq1}\int_t^T(\mathbf{q}e_\Bbbk,\phi)_Vd\beta_\Bbbk(s),\quad\forall\phi\in W,\end{aligned}\tag{7.23}$$

where y is the solution of (2.1), in the sense of Definition 2.4 and τ_M is given by (2.10).

[9] D_vL denotes the derivative of L with respect to the variable v.

Theorem 7.7. *There exists, at least, a pair* $(\mathbf{q},\mathbf{q})$, *which is a solution of the adjoint equation* (7.23), *according to the Definition 7.6, verifying the properties*

$$\exists C(M)>0 \text{ such that } \sup_{s\in[0,T]}\mathbb{E}\|\mathbf{p}(s)\|_V^2\le C(M) \text{ and } \mathbb{E}\int_0^T\|\mathbf{q}\|^2_{L_2(\mathbb{H};V)}ds\le C(M). \tag{7.24}$$

Moreover, for fixed $M\in\mathbb{N}$ *we have* $\mathbb{E}\sup_{s\in[\tau_M,T]}\|\mathbf{p}(s)\|_V^2=0$ *and* $\mathbb{E}\int_{\tau_M}^T\|\mathbf{q}\|^2_{L_2(\mathbb{H};V)}ds=0.$

7.2.1. Proof of Theorem 7.7

As in the 2D case (see Sect. 7.1.1) we consider $p_n(t)=\sum_{i=1}^n d_i(t)h_i$ and $q_n(t)\in L_2(\mathbb{H},W_n)$, $t\in[0,T]$. The approximated problem for (7.23) reads

$$\begin{aligned}&(p_n(t),v(\phi))+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{2\nu(\mathbb{D}p_n,\mathbb{D}\phi)ds-b(\phi,p_n,v(y))+b(p_n,\phi,v(y))+b(p_n,y,v(\phi))-b(y,p_n,v(\phi))\Big\}ds\\&+\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\Big\{(\alpha_1+\alpha_2)\big(A(y)A(p_n)+A(p_n)A(y),\nabla\phi\big)+\beta\big(|A(y)|^2A(p_n),\nabla\phi\big)\Big\}ds\\&+2\beta\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\big(\left(A(p_n):A(y)\right)A(y),\nabla\phi\big)ds=\int_t^T\mathbb{I}_{[0,\tau_M[}(s)(g-G^*q_n,\phi)ds+\sum_{\Bbbk\ge1}\int_t^T(q_ne_{\Bbbk},v(\phi))d\beta_{\Bbbk}(s),\end{aligned}\tag{7.25}$$

for any $\phi\in W_n$. Arguments already detailed ensure the existence of a unique pair of predictable processes (p_n,q_n) such that (7.25) holds for any $t\in[0,T]$ and

$$p_n\in L^r(\Omega;\mathscr{C}([0,T],W_n)),\quad q_n\in L^r(\Omega;L^2(0,T;L_2(\mathbb{H},W_n)),\qquad\forall 1\le r\le 2(d+1). \tag{7.26}$$

Arguments already detailed (see Sect. 7.1.2) yield the existence of $C(M)>0$ such that

$$\sup_{r\in[t,T]}\mathbb{E}\|p_n(r)\|_V^2+\mathbb{E}\Big[\int_t^T\|q_n\|^2_{L_2(\mathbb{H};V)}ds\Big]\le Ce^{C(G^*)(M^2+1)T}\mathbb{E}\int_t^T\mathbb{I}_{[0,\tau_M[}(s)\|g\|_2^2ds,\forall t\in[0,T].$$

It follows that $\mathbb{E}\|p_n(t)\|_V^2+\mathbb{E}\int_t^T\|q_n\|^2_{L_2(\mathbb{H};V)}ds=0,\quad\forall t\ge\tau_M$. Finally, the proof follows from straightforward adaptation of the proof of Theorem 7.3. On the one hand, we notice that the *uniform estimates* allows to pass to the limit in (7.25). On the other hand, by standard arguments we get: $\forall t\in[0,T]$, $(p_n(t),\phi)_V\rightharpoonup(\mathbf{p}(t),\phi)_V$ in $L^2(\Omega)$, for $\phi\in V$, which allows to identify the terminal condition.

8. Duality relation and optimality condition

Proposition 8.1. *Let* $\psi\in L^p((\Omega_T,\mathscr{P}_T),(H^1(D))^d)$ *and* $g=y-y_d$. *Let* z_n *be the solution of* (5.7), (p_n,q_n) *be the solution of* (7.5) *in 2D and* $(\mathbf{p}_n,\mathbf{q}_n)$ *be the solution of* (7.25) *in 3D. Then, we have*

$$\mathbb{E}\int_0^T\mathbb{I}_{[0,\tau_M[}(\psi,p_n)ds=\mathbb{E}\int_0^T\mathbb{I}_{[0,\tau_M[}(s)(g,z_n)ds=\mathbb{E}\int_0^T\mathbb{I}_{[0,\tau_M[}(s)(\psi,\mathbf{p}_n)dt. \tag{8.1}$$

Proof. Set $\phi = h_i$ in (5.7) and use that $(v(z_n(\cdot)), h_i) = (z_n(\cdot), h_i)_V$, we obtain

$$
\begin{aligned}
&(z_n(t), h_i)_V + \int_0^t \mathbb{I}_{[0,\tau_M[}\{2\nu(\mathbb{D}z_n, \mathbb{D}h_i) + b(y, v(z_n), h_i) + b(z_n, v(y), h_i) + b(h_i, y, v(z_n)) + b(h_i, z_n, v(y))\}ds \\
&+ \int_0^t \mathbb{I}_{[0,\tau_M[}\{(\alpha_1+\alpha_2)\big(A(y)A(z_n) + A(z_n)A(y), \nabla h_i\big) + \beta\big(|A(y)|^2 A(z_n), \nabla h_i\big)\}ds \\
&+ \int_0^t 2\beta \mathbb{I}_{[0,\tau_M[}\big((A(z_n):A(y))A(y), \nabla h_i\big)ds = \int_0^t \mathbb{I}_{[0,\tau_M[}(\psi, h_i)ds + \int_0^t \mathbb{I}_{[0,\tau_M[}(\nabla_y G(\cdot, y)z_n, h_i)d\mathscr{W}(s).
\end{aligned}
\tag{8.2}
$$

On the other hand, let $(\mathbf{p}_n, \mathbf{q}_n)$ be the solution of (7.5) (or (7.25)). By setting $\phi = h_i$ in (7.5) (or (7.25)) and using that $(\mathbf{p}_n(\cdot), v(h_i)) = (\mathbf{p}_n(\cdot), h_i)_V$, $(\mathbf{q}_n e_{\Bbbk}, v(h_i)) = (\mathbf{q}_n e_{\Bbbk}, h_i)_V$, we get

$$
\begin{aligned}
&(\mathbf{p}_n(t), h_i)_V \\
&+ \int_t^T \mathbb{I}_{[0,\tau_M[}\Big\{2\nu(\mathbb{D}\mathbf{p}_n, \mathbb{D}h_i)ds - b(h_i, \mathbf{p}_n, v(y)) + b(\mathbf{p}_n, h_i, v(y)) + b(\mathbf{p}_n, y, v(h_i)) - b(y, \mathbf{p}_n, v(h_i))\Big\}ds \\
&+ \int_t^T \mathbb{I}_{[0,\tau_M[}\Big\{(\alpha_1+\alpha_2)\big(A(y)A(\mathbf{p}_n) + A(\mathbf{p}_n)A(y), \nabla h_i\big) + \beta\big(|A(y)|^2 A(\mathbf{p}_n), \nabla h_i\big)\Big\}ds \\
&+ 2\beta \int_t^T \mathbb{I}_{[0,\tau_M[}\big(\,(A(\mathbf{p}_n):A(y))\,A(y), \nabla h_i\big)ds = \int_t^T \mathbb{I}_{[0,\tau_M[}(g - G^*\mathbf{q}_n, h_i)ds + \sum_{\Bbbk\geq 1}\int_t^T (\mathbf{q}_n e_{\Bbbk}, h_i)_V d\beta_{\Bbbk}(s).
\end{aligned}
\tag{8.3}
$$

The Itô formula and the symmetry of matrices $A(y)$, $A(\mathbf{p}_n)$, $A(z_n)$ yield

$$
\begin{aligned}
&(z_n(T), \mathbf{p}_n(T))_V - (z_n(0), \mathbf{p}_n(0))_V = \int_0^T \mathbb{I}_{[0,\tau_M[}(\psi, \mathbf{p}_n)ds + \int_0^T \mathbb{I}_{[0,\tau_M[}(\nabla_y G(\cdot, y)z_n, \mathbf{p}_n)d\mathscr{W}(s) \\
&- \int_0^T \mathbb{I}_{[0,\tau_M[}(s)(g, z_n)ds + \int_0^T \mathbb{I}_{[0,\tau_M[}(s)(G^*\mathbf{q}_n, z_n)ds - \sum_{\Bbbk\geq 1}\int_0^T (\mathbf{q}_n e_{\Bbbk}, z_n)_V d\beta_{\Bbbk}(s) \\
&- \sum_{\Bbbk\geq 1}\int_0^T \mathbb{I}_{[0,\tau_M[}(s)(\nabla_y \sigma_{\Bbbk}(\cdot, y)z_n, \mathbf{q}_n e_{\Bbbk})ds.
\end{aligned}
$$

By using Remark 7.1 and knowing that $p_n(T) = 0 = z_n(0)$, we obtain

$$
\begin{aligned}
&\int_0^T \mathbb{I}_{[0,\tau_M[}(\psi, \mathbf{p}_n)ds + \int_0^T \mathbb{I}_{[0,\tau_M[}(\nabla_y G(\cdot, y)z_n, \mathbf{p}_n)d\mathscr{W}(s) \\
&= \int_0^T \mathbb{I}_{[0,\tau_M[}(s)(g, z_n)ds + \sum_{\Bbbk\geq 1}\int_0^T (\mathbf{q}_n e_{\Bbbk}, z_n)_V d\beta_{\Bbbk}(s).
\end{aligned}
$$

Taking the expectation, we deduce $\mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(\psi, \mathbf{p}_n)ds = \mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(s)(g, z_n)ds.$ □

By passing to the limit, as $n \to \infty$, in (8.1), we establish the next result.

Corollary 8.2. *Consider $\psi \in L^p((\Omega_T, \mathscr{P}_T), (L^2(D))^d)$. Let z be the solution of (5.3), $(\mathcal{p}, \mathcal{q})$ be the solution of the adjoint equations in 2D (7.2)(or the solution of adjoint equations in 3D (7.23), which is denoted by $(\mathbf{p}, \mathbf{q})$).*

Then, we have

$$\mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(\psi, \boldsymbol{p})ds = \mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(s)(y-y_d, z)ds = \mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(\psi, \mathbf{p})ds.$$

8.1. A necessary optimality condition for (4.2)

Let $(\widetilde{U_M}, y(\widetilde{U_M}))$ be the optimal control pair. Consider $\psi \in \mathscr{U}_{ad}^p$ and define $U_\rho = \widetilde{U_M} + \rho(\psi - \widetilde{U_M})$. Thanks to Proposition 6.3, we get that the Gâteaux derivative of the cost functional J is given by

$$\lim_{\rho\to 0} \frac{J_M(U_\rho, y_\rho) - J_M(\widetilde{U_M}, y(\widetilde{U_M}))}{\rho}$$
$$= \lambda\mathbb{E}\int_0^T \|\widetilde{U_M}\|_{(H^1(D))^d}^{p-2}(\widetilde{U_M}, \psi - \widetilde{U_M})_{(H^1(D))^d}dt + \mathbb{E}\int_0^{\tau_M^{\widetilde{U_M}}} (y(\widetilde{U_M}) - y_d, z)dt \geq 0,$$

where z is the unique solution to the linearized problem (5.3) with ψ replaced by $\psi - \widetilde{U_M}$.

Let $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ be the unique solution of (7.2) (or (7.23) in 3D) with $g = y(\widetilde{U_M}) - y_d$. The application of Corollary 8.2 yields $\mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(s)(y(\widetilde{U_M}) - y_d, z)ds = \mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(s)(\psi - \widetilde{U_M}, \tilde{\mathbf{p}})dt$. Finally, we obtain the following optimality condition, for any $\psi \in \mathscr{U}_{ad}^p$:

$$\mathbb{E}\int_0^T \big(\lambda\|\widetilde{U_M}\|_{(H^1(D))^d}^{p-2}(\widetilde{U_M}, \psi - \widetilde{U_M})_{(H^1(D))^d} + \mathbb{I}_{[0,\tau_M[}(\psi - \widetilde{U_M}, \tilde{\mathbf{p}})\big)ds \geq 0.$$

9. Cost functional with derivatives

The control of the evolution of the velocity field derivatives is relevant in the study of turbulence, hydrodynamics and combustion theory. Therefore, it is important to consider cost functionals depending on the derivatives of the quantities of interest. For example, enstrophy $\mathscr{E}$ is one of the quantities that play a crucial role in the control of turbulent flows and is given by $\mathscr{E}(y) := \|\nabla y\|_2^2$. We refer to [34] for approaches that minimize the enstrophy. Our aim in this section is to propose an extension of our analysis to cost functionals including first-order derivatives of the velocity field. Let us consider the following problem

$$\widetilde{\mathscr{P}}\begin{cases} \min_U \{\frac{1}{2}\mathbb{E}\int_0^{\tau_M^U} \|y - y_d\|_V^2 dt + \frac{\lambda}{p}\mathbb{E}\int_0^T \|U\|_{(H^1(D))^d}^p dt, \quad \lambda > 0 : \\ \qquad U \in \mathscr{U}_{ad}^p \text{ and } y \text{ is the solution of (2.1) for the minimizing } U \in \mathscr{U}_{ad}^p, \end{cases} \tag{9.1}$$

with a desired target field $y_d \in L^2(0, T; W)$ and $(\tau_M^U)_{M\in\mathbb{N}}$ given by (2.10). A similar raisoning to that of Theorem 9.1 yields the following result.

Theorem 9.1. *Assume $\mathscr{H}_1$. Then the control problem (9.1) admits a unique optimal solution $(\widetilde{U_M}, \tilde{y}) \in \mathscr{U}_{ad}^p \times L^p(\Omega; L^p(0, T; \widetilde{W}))$, where $\tilde{y} := y(\widetilde{U_M})$ is the unique solution of (2.1) with $U = \widetilde{U_M}$. Moreover, under the assumption $\mathscr{H}_0$:*

- *there exists a unique solution $\tilde{z}$ of the linearized equations (5.3), in 2D and 3D with $y = \tilde{y}$ and $\psi = \psi - \widetilde{U_M}$;*
- *if $d = 2$, there exists a unique solution $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}})$ of the stochastic backward adjoint equation (7.2), with force $g = v(\tilde{y} - y_d)$;*
- *if $d = 3$, there exists, at least, one solution $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ of the stochastic backward adjoint equation (7.23), with force $g = v(\tilde{y} - y_d)$.*

In addition, the duality property

$$\mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(\psi-\widetilde{U_M},\tilde{\mathfrak{p}})ds=\mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(s)(y(\widetilde{U_M})-y_d,z)_V ds=\mathbb{E}\int_0^T \mathbb{I}_{[0,\tau_M[}(s)(\psi-\widetilde{U_M},\tilde{\mathbf{p}})dt$$

is valid for any $\psi\in\mathscr{U}_{ad}^p$, and the following optimality condition holds, for any $\psi\in\mathscr{U}_{ad}^p$

$$\mathbb{E}\int_0^T \big(\lambda\|\widetilde{U_M}\|^{p-2}_{(H^1(D))^d}(\widetilde{U_M},\psi-\widetilde{U_M})_{(H^1(D))^d}+\mathbb{I}_{[0,\tau_M[}(\psi-\widetilde{U_M},\tilde{\mathbf{p}})\big)ds\geq 0;$$
$$\mathbb{E}\int_0^T \big(\lambda\|\widetilde{U_M}\|^{p-2}_{(H^1(D))^d}(\widetilde{U_M},\psi-\widetilde{U_M})_{(H^1(D))^d}+\mathbb{I}_{[0,\tau_M[}(\psi-\widetilde{U_M},\tilde{\mathfrak{p}})\big)ds\geq 0.$$

10. Technical lemmas

In this section, we establish some lemmas which play an important role in the estimation of the nonlinear terms and the derivation of the optimality system. We recall that $D\subset\mathbb{R}^d, d=2,3$ is a bounded and simply connected domain with regular boundary ∂D and $\eta=(\eta_k)_{k=1}^d$ denote the outward normal to the boundary ∂D.

We start with a result, which can be deduced by closely following the analysis in [3, Appendix].

Lemma 10.1. *There exists $C:=C(D,\eta)>0$ such that the following inequalities hold*

$$|b(\delta,y,v(\delta))|\leq C\|y\|_{W^{2,4}}\|\delta\|_V^2 \text{ and } |b(y,\delta,v(\delta))|\leq C\|y\|_{W^{2,4}}\|\delta\|_V^2,\quad \forall y\in\widetilde{W},\forall\delta\in W.$$

Lemma 10.2. *Let $t>0$ and $0<\epsilon\leq 1$. There exists $C_\epsilon(D)>0$ such that*

$$\mathbb{E}\int_0^t |b(y,v(\phi),\mathbb{P}v(y))|^q\leq \mathbb{E}\int_0^t \big(\|y\|_\infty\|\phi\|_{H^3}\|y\|_W\big)^q ds$$
$$\leq C_\epsilon(D)\big(\mathbb{E}\int_0^t \|y\|_W^{2q}ds+\|\phi\|^{2q(d+\epsilon)}_{L^{2q(d+1+\epsilon)}(\Omega\times(0,t);H^3)}\|y\|^{2q}_{L^{2q(d+1+\epsilon)}(\Omega\times(0,t);V)}\big),$$

$\forall q\in[1,\infty[$, $y\in L^{2q}(\Omega\times(0,t);W)\cap L^{(d+1+\epsilon)2q}(\Omega\times(0,t);V)$ *and* $\phi\in L^{(d+1+\epsilon)2q}(\Omega\times(0,t);(H^3(D))^d)$.

Proof. Let us consider $(q,\epsilon)\in[1,\infty[\times]0,1]$, and recall that $b(y,v(\phi),\mathbb{P}v(y))=\displaystyle\sum_{i,j=1}^d\int_D y^i\frac{\partial v(\phi)^j}{\partial x_i}\mathbb{P}v(y)^j dx$. Applying the Hölder inequality, we derive

$$|b(y,v(\phi),\mathbb{P}v(y))|\leq C\|y\|_\infty\|\phi\|_{H^3}\|y\|_W\leq C_D\|y\|_{W^{1,d+\epsilon}}\|\phi\|_{H^3}\|y\|_W, \tag{10.1}$$

where $C_D>0$ is related to the Sobolev embedding $W^{1,d+\epsilon}(D)\hookrightarrow L^\infty(D)$ (see [23], Thm. 1.20). Applying again the Hölder inequality, and the Sobolev embedding $H^2(D)\hookrightarrow W^{1,2(d-1+\epsilon)}(D)$, we deduce

$$\|y\|_{W^{1,d+\epsilon}}\leq C\|y\|_{H^1}^{\frac{1}{d+\epsilon}}\|y\|^{\frac{d-1+\epsilon}{d+\epsilon}}_{W^{1,2(d-1+\epsilon)}}\leq C\|y\|_{H^1}^{\frac{1}{d+\epsilon}}\|y\|_{H^2}^{\frac{d-1+\epsilon}{d+\epsilon}}. \tag{10.2}$$

Inserting (10.2) in (10.1) and next taking the q^{th} power of the resulting inequality, we obtain

$$|b(y,v(\phi),\mathbb{P}v(y))|^q\leq C_D\|y\|_V^{\frac{q}{d+\epsilon}}\|\phi\|_{H^3}^q\|y\|_W^{q\frac{2d+2\epsilon-1}{d+\epsilon}}\leq C_\epsilon(D)\big(\|y\|_W^{2q}+\|y\|_V^{2q}\|\phi\|_{H^3}^{2(d+\epsilon)q}\big),$$

where we used Young inequality with $\gamma = \dfrac{2(d+\epsilon)}{2(d+\epsilon)-1} > 1$. Now, the Hölder inequality yields

$$\mathbb{E}\int_0^t \|y\|_V^{2q}\|\phi\|_{H^3}^{2(d+\epsilon)q}ds \leq \big(\mathbb{E}\int_0^t \|\phi\|_{H^3}^{2q(d+1+\epsilon)}ds\big)^{\frac{d+\epsilon}{d+1+\epsilon}}\big(\mathbb{E}\int_0^t \|y\|_V^{2q(d+1+\epsilon)}ds\big)^{\frac{1}{d+1+\epsilon}}$$
$$\leq \|\phi\|^{2q(d+\epsilon)}_{L^{2q(d+1+\epsilon)}(\Omega\times(0,t);H^3)}\|y\|^{2q}_{L^{2q(d+1+\epsilon)}(\Omega\times(0,t);V)}.$$

□

Convenient modifications in the proof of [15], Lemma 5.3 yield

Lemma 10.3. *Let $d=2$, $y,\psi\in\widetilde{W}$ and $\phi\in V$. Then there exists $C>0$ depending only on D such that* $|(curl(v(y\times\psi)) - curlv(y)\times\psi,\phi)| \leq C(1+\alpha_1)\|y\|_{W^{2,4}}\|\psi\|_W\|\phi\|_2 + \alpha_1|b(y,\Delta\psi,\phi)|$.

Lemma 10.4. *Let $y,p\in\widetilde{W}$ and $\phi\in W$. Then*

- *2D case: we have* $(curlv(y\times p),\phi) = b(p,y,v(\phi)) - b(y,p,v(\phi))$.
- *3D case: we have* $(curlv(y\times p),\phi) = b(p,y,v(\phi)) - b(y,p,v(\phi)) + \alpha_1 I_{\partial D}$,

$$where \qquad I_{\partial D} = \int_{\partial D}(curl(p\cdot\nabla y - y\cdot\nabla p)\cdot(\phi\times\eta)dS$$
$$- 2\int_{\partial D}(p\cdot\nabla y - y\cdot\nabla p)\cdot(\eta\times\sum_{k=1}^{3}\phi_k(\eta\times\nabla)\eta_k)dS. \tag{10.3}$$

Proof. Let $y,p\in\widetilde{W}$ and $\phi\in W$. Integrating twice by parts, we deduce

$$(curlv(y\times p),\phi) = b(p,y,v(\phi)) - b(y,p,v(\phi))$$
$$+\alpha_1[\int_{\partial D}(\text{curlcurl }(y\times p)\times\phi)\cdot\eta dS + \int_{\partial D}(\text{curl }(y\times p)\times\text{curl }\phi)\cdot\eta dS].$$

Since $div(y)=div(p)=0$, one has curl $(y\times p) = p\cdot\nabla y - y\cdot\nabla p$ and therefore

$$\text{curlcurl }(y\times p) = (d-1)[p\cdot\nabla curl(y) - y\cdot\nabla curl(p)]$$
$$-[curl(y)\cdot\nabla p - curl(p)\cdot\nabla y] + 2\sum_{k=1}^{d}\nabla p_k\times\nabla y_k.$$

We wish to draw the reader's attention to the well known explicit relation between the normal and tangent vectors to the boundary in 2D: $\eta=(\eta_1,\eta_2)$ and $\tau=(-\eta_2,\eta_1)$, which will play a crucial role to show that boundary terms vanish in 2D, we refer to [35], Lemma 3.6 for more details. Concerning the 3D case, the situation is more delicate. First, the term $curl(y)\cdot\nabla p - curl(p)\cdot\nabla y$ does not vanish as in 2D. Using [3], Proposition 2, we are able to infer

$$F := \int_{\partial D}(\text{curl }(y\times p)\times\text{curl }\phi)\cdot\eta dS = -2\int_{\partial D}(p\cdot\nabla y - y\cdot\nabla p)\cdot(\eta\times\sum_{k=1}^{3}\phi_k(\eta\times\nabla)\eta_k)dS.$$

On the other hand, we notice that $B := \int_{\partial D}(\text{curlcurl }(y \times p) \times \phi) \cdot \eta dS = \int_{\partial D} \text{curlcurl }(y \times p) \cdot (\phi \times \eta) dS$. Thus

$$F + B = \int_{\partial D} \text{curlcurl }(y \times p) \cdot (\phi \times \eta) dS - 2 \int_{\partial D} (p \cdot \nabla y - y \cdot \nabla p) \cdot (\eta \times \sum_{k=1}^{3} \phi_k (\eta \times \nabla) \eta_k) dS.$$

Finally, we deduce

$$F + B = \int_{\partial D} (\text{curl}(p \cdot \nabla y - y \cdot \nabla p) \cdot (\phi \times \eta) dS - 2 \int_{\partial D} (p \cdot \nabla y - y \cdot \nabla p) \cdot (\eta \times \sum_{k=1}^{3} \phi_k (\eta \times \nabla) \eta_k) dS.$$

□

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Data availability statement

The research data associated with this article are included in the article.

References

[1] J.E. Dunn and K.R. Rajagopal, Fluids of differential type: critical review and thermodynamical analysis. *Int. J. Eng. Sci.* **33** (1995) 689–729.
[2] R.L. Fosdick and K.R. Rajagopal, Thermodynamics and stability of fluids of third grade. *Proc. Roy. Soc. London Ser. A* **339** (1980) 351–377.
[3] A.V. Busuioc and T.S. Ratiu, The second grade fluid and averaged Euler equations with Navier-slip boundary conditions. *Nonlinearity* **16** (2003) 1119–1149.
[4] D.D. Holm, J.E. Marsden and T.S. Ratiu, Euler–Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett.* **80** (1998) 4173–4176.
[5] C. Amrouche and D. Cioranescu, On a class of fluids of grade 3. *Int. J. Non-Linear Mech.* **32** (1997) 73–88.
[6] A. Sequeira and J. Videman, Global existence of classical solutions for the equations of third grade fluids. *J. Math. Phys. Sci.* **29** (1995) 47–69.
[7] A.V. Busuioc and D. Iftimie, *Global existence and uniqueness of solutions for the equations of third grade fluids. Int. J. Non-Linear Mech.* **39** (2004) 1–12.
[8] A.V. Busuioc and D. Iftimie, A non-Newtonian fluid with Navier boundary conditions. *J. Dyn. Differ. Equ.* **18** (2007) 357–379.
[9] F. Cipriano, P. Didier and S. Guerra, Well-posedness of stochastic third grade fluid equation. *J. Differ. Equ.* **285** (2021) 496–535.
[10] Y. Tahraoui and F. Cipriano, Local strong solutions to the stochastic third grade fluid equations with Navier boundary conditions. *Stoch. PDE: Anal. Comp.* **12** (2024) 1699–1744.
[11] Y. Tahraoui and F. Cipriano, Invariant measures for a class of stochastic third-grade fluid equations in 2D and 3D bounded domains. *J. Nonlinear Sci.* **34** (2024) 107.
[12] F. Abergel and R. Temam, On some control problems in fluid mechanics. *Theor. Computat. Fluid Dyn.* **1** (1990) 303–325.
[13] J.C. De Los Reyes and R. Griesse, State-constrained optimal control of the three-dimensional stationary Navier–Stokes equations. *J. Math. Anal. Appl.* **343** (2008) 257–272.
[14] M. Hinze and K. Kunisch, Second order methods for optimal control of time-dependent fluid flow. *SIAM J. Control Optim.* **40** (2001) 925–946.
[15] N. Arada and F. Cipriano, Optimal control of non-stationary second grade fluids with Navier-slip boundary conditions. Preprint (2015). https://arxiv.org/abs/1511.01134

[16] N. Chemetov and F. Cipriano, Optimal control for two-dimensional stochastic second grade fluids. *Stoch. Processes Appl.* **128** (2018) 2710–2749.
[17] F. Cipriano and D. Pereira, On the existence of optimal and $\epsilon-$optimal feedback controls for stochastic second grade fluids. *J. Math. Anal. Appl.* **475** (2019) 1956–1977.
[18] Y. Tahraoui and F. Cipriano, Optimal control of two dimensional third grade fluids. *J. Math. Anal. Appl.* **523** (2023) 127032.
[19] P. Benner and C. Trautwein, Optimal control problems constrained by the stochastic Navier–-Stokes equations with multiplicative Lévy noise. *Math. Nachr.* **292** (2019) 1444–1461.
[20] P. Benner and C. Trautwein, A stochastic maximum principle for control problems constrained by the stochastic Navier–Stokes equations. *Appl. Math. Optim.* **84** (2021) 1001–1054.
[21] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions. Encyclopedia Math. Appl., Vol. 44. Cambridge University Press, Cambridge (1992).
[22] W. Liu and M. Rockner, Stochastic Partial Differential Equations: An Introduction. Springer International Publishing Switzerland (2015).
[23] T. Roubíček, Nonlinear partial differential equations with applications, Vol. 153 of *International Series of Numerical Mathematics*. Birkhäuser Verlag, Basel (2005).
[24] M. Goebel, On existence of optimal control. *Math. Nachr.* **93** (1979) 67–73.
[25] R.E. Edwards, Functional Analysis. Dover Publications Inc., New York (1995).
[26] G. Vallet and A. Zimmermann, Well-posedness for nonlinear SPDEs with strongly continuous perturbation. *Proc. Roy. Soc. Edinb. Sect. A: Math.* **151** (2021) 265–295.
[27] E. Zeidler, Nonlinear Functional Analysis and its Applications II/A: Linear Monotone Operators. Springer Verlag, New York–Berlin (1989).
[28] J. Simon, Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pur. Appl.* **146** (1987) 65–96.
[29] R. Temam, Navier–Stokes Equations. Theory and Numerical Analysis. North-Holland Publishing Co., Amsterdam/New York/Oxford (1977).
[30] G. Vallet and A. Zimmermann, Well-posedness for a pseudomonotone evolution problem with multiplicative noise. *J. Evol. Equ.* **19** (2019) 153–202.
[31] G. Wachsmuth, Differentiability of implicit functions: beyond the implicit function theorem. *J. Math. Anal. Appl.* **414** (2014) 259–272.
[32] F. Tröltzsch, Optimal Control of Partial Differential Equations: Theory, Methods, and Applications, (Vol. 112). American Mathematical Society (2010).
[33] G. Fabbri, F. Gozzi and A. Swiech, Stochastic Optimal Control in Infinite Dimension. Probability and Stochastic Modelling. Springer (2017).
[34] S.S. Sritharan, An Introduction to Deterministic and Stochastic Control of Viscous Flow. Optimal Control of Viscous Flow. SIAM, Philadelphia (1998) 1–42.
[35] N. Chemetov and F. Cipriano, Optimal control for two-dimensional stochastic second grade fluids. *Stoch. Processes Appl.* **128** (2018) 2710–2749.