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LINEAR-CONVEX PARTIALLY OBSERVED OPTIMAL CONTROL PROBLEM WITH MARKOV CHAIN AND INPUT CONSTRAINT

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Abstract. In this paper, we study a linear–convex problem for partially observed forward–backward stochastic control system with Markov chain and input constraints. The observation is assumed to be controlled and follows a regime-switching stochastic differential equation, whose drift term is linear with respect to the state process x and control strategy u. Firstly, for the general case, by using the backward separate approach to decompose the state and observation, we obtain the optimal control strategy by virtue of stochastic maximum principle. We then prove the well-posedness of the stochastic Hamiltonian system using the method of continuity. Secondly, for the linear-quadratic case under linear subspace constraints, we present the feedback representation of the optimal control strategy. Finally, we apply our theoretical results to an asset-liability problem to demonstrate their practical significance.

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1. Introduction

In recent decades, regime-switching models have gained increasing attentions from researchers due to their extensive applications and important theoretical significance in various fields, such as economics, information technology, engineering and system control, see [1–3]. Markovian regime-switching models are particularly useful in describing changes in the model state, such as the transition between bull market and bear market, see [4, 5].

Stochastic control problem is a fundamental part of control theory, and the stochastic maximum principle (SMP) is a powerful tool for solving these problems by providing the necessary condition for optimal control. It was first formulated by Pontryagin in 1950s and it converted the optimization problems into maximization of corresponding Hamiltonian functions. Many Researchers have made significant contributions to the SMP, including Bismut [6] who used linear backward stochastic differential equation (BSDE) as adjoint equation, and Peng [7] who extended the SMP to non-convex domain with general BSDE as second-order adjoint equation. Tang and Li [8] studied the optimal control problem with Poisson random measure, while Xu [9] studied the optimal control problem of forward–backward stochastic control system. With the development of Markovian regime-switching model, significant progress has been made in the corresponding SMP and optimal control problems. Donnelly [10] established a sufficient SMP for the optimal control of a regime-switching model, she also gave the connection to dynamic programming. Tao and Wu [11] derived the necessary and sufficient SMP for

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forward–backward Markovian regime-switching system. Zhang *et al.* [12] presented a global SMP for a Markovian regime-switching mean-field model, and Nguyen *et al.* [13] derived a general SMP for a regime-switching diffusion model with mean-field interactions.

The linear-quadratic (LQ) optimal control problem is a classical problem that plays a vital role in the field of control. In recent decades, both deterministic and stochastic LQ problems have been greatly developed. The stochastic linear-quadratic (SLQ) problem was first addressed by Kushner [14] using the dynamic programming method. Bismut [15] then studied the SLQ optimal control problem with random coefficients and obtained the random feedback representation of optimal control. The SLQ problem with indefinite control weight was first proposed by Chen *et al.* [16], and since then, numerous SLQ problems for complex models have been widely investigated. The SLQ problem involving Poisson random measure or regime-switching jumps have widely application in various fields, the reader who is interested may refer to [1, 3, 17].

We note that all the literatures mentioned above, the decision-makers could only make their decisions with full information. However, in practical problems, we often lack complete information and must make decisions based on only partial information. As a result, there are many literature on stochastic control problem with partial information, see [18–20]. Karatzas and Ocone [20] investigated the partially observed stochastic control problem on an infinite horizon by the dynamic programming principle. Li and Tang [21] obtained the general SMP by a pure probabilistic approach for partially observed systems, whose diffusion term contains control, and the observation also depends on control. In their model, the observation process is typically assumed to be an uncontrolled Brownian motion, and Girsanov transformation plays a crucial role. More details can be seen [22–25]. But in many practical problems, it is more reasonable to assume that the observation is a general stochastic process, resulting in a circular dependency between the observation and control strategy. The Wonham separation principle, discussed in [26], plays a crucial role in handing this issue. This theorem allows to first compute the filtering of state, and then to solve fully observed optimal control problems driven by the filtering states. But in many cases, the mean square error of state estimate still depends on the control variable, making the Wonham separation principle invalid. To overcome this difficulty, Wang and Wu [27] proposed a backward separation technique to solve some partially observed stochastic control problems by first decomposing the state and observation, and then computing the filtering. Wang *et al.* [28] studied an LQ optimal control problem of forward–backward stochastic control system with partial observation using the backward separation approach.

In this paper, we investigate a linear–convex (LC) optimal control problem for a partially observed forward–backward stochastic control system with Markov chain and input constraints. Our study is mainly motivated by an asset-liability management problem. Specifically, let $l_\cdot$ denote the company's liability process, which is governed by the following regime-switching stochastic differential equation (SDE),

$$-\mathrm{d}l_t = \big[B_{(t,\alpha_t)}u_t - b_{(t,\alpha_t)}\big]\mathrm{d}t + \sigma_{(t,\alpha_t)}\mathrm{d}W_t + \bar{\sigma}_{(t,\alpha_t)}\mathrm{d}\overline{W}_t.$$

Here $u_\cdot$ is the decision-maker's strategy, which means that the decision-maker can achieve a specific goal by injecting or withdrawing funds. The Markov chain $\alpha_\cdot$ represent the changes in market trends, thus Markov chain $\alpha_\cdot$ can be directly observed by some way. Let $b_{(\cdot,j)} > 0$, $j \in \mathcal{M}$ denote the expected liability rate. $\sigma_{(\cdot,j)} > 0$ and $\bar{\sigma}_{(\cdot,j)} > 0$ are the volatility of liability. Suppose the initial endowment of this company is x_0 and it can only invest its assets in a riskless bond whose interest rate is $r_{(\cdot,j)} > 0$. Thus the cash balance process $x_\cdot$ of the company satisfies the following form,

$$x_t = \mathrm{e}^{\int_0^t r_{(s,\alpha_s)}\mathrm{d}s}\left(x_0 - \int_0^t \mathrm{e}^{-\int_0^s r_{(v,\alpha_v)}\mathrm{d}v}\mathrm{d}l_s\right).$$

Applying Itô's formula to above equation, we obtain

$$\begin{cases}\mathrm{d}x_t = \big\{r_{(t,\alpha_t)}x_t + B_{(t,\alpha_t)}u_t - b_{(t,\alpha_t)}\big\}\mathrm{d}t + \sigma_{(t,\alpha_t)}\mathrm{d}W_t + \bar{\sigma}_{(t,\alpha_t)}\mathrm{d}\overline{W}_t,\\ x_0 = x_0, \qquad \alpha_0 = i.\end{cases} \tag{1.1}$$

Because of information asymmetry and other reasons, the decision-maker can only observe the cash balance by the following stock price,

$$\begin{cases} \mathrm{d}S_t = \big\{K_{(t,\alpha_t)}x_t + g_{(t,\alpha_t)} + \dfrac{1}{2}\tilde{\sigma}^2_{(t,\alpha_t)}\big\}S_t\mathrm{d}t + \tilde{\sigma}_{(t,\alpha_t)}S_t\mathrm{d}W_t, \\ S_0 = 1, \qquad \alpha_0 = i. \end{cases}$$

Then all information can be obtained by decision-maker is $\sigma\{S_s; 0 \le s \le t\}$ rather than $\mathcal{F}_t$ at time t. Let $Y_t = \ln S_t$, we get

$$\begin{cases} \mathrm{d}Y_t = \big\{K_{(t,\alpha_t)}x_t + g_{(t,\alpha_t)}\big\}\mathrm{d}t + \tilde{\sigma}_{(t,\alpha_t)}\mathrm{d}W(t), \\ Y_0 = 0, \qquad \alpha_0 = i. \end{cases} \tag{1.2}$$

It is obvious that $\mathcal{F}^Y_t = \sigma\{Y_s, \alpha_s; 0 \le s \le t\} = \sigma\{S_s, \alpha_s; 0 \le s \le t\}$ and then we definite $\mathbb{F}^Y := \{\mathcal{F}^Y_t\}_{0\le t\le T}$.

As we all know, the general nonlinear BSDE was originally introduced by Pardoux and Peng [29], while Duffie and Epstein [30] independently introduced the BSDE in the economic context. In their work, they present a stochastic differential representation of recursive utility, which is the generalization of the standard additive utility. The recursive utility depends not only on current consumption but also on future utility. EI Karoui *et al.* [31] later demonstrate that the recursive utility process can be represented by the solution to a BSDE, more details can be seen [32–34]. Next we consider the generalized stochastic recursive utility problem as follows.

Problem (RU) Find an $\mathbb{F}^Y$ adapted control process $\bar{u}.$ such that

$$\bar{J}(\bar{u}.) = \inf_u \bar{J}(u.) = \frac{1}{2}\mathbb{E}\Big[\int_0^T R_{(t,\alpha_t)}\big[u_t - a_{(t,\alpha_t)}\big]^2\mathrm{d}t + L_{(T,\alpha_T)}\big[x_T - \bar{l}_{(T,\alpha_T)}\big]^2 + 2ny_0\Big], \tag{1.3}$$

subject to (1.1), (1.2) and

$$\begin{cases} \mathrm{d}y_t = -\tilde{f}\big(t, x_t, y_t, z_t, k_t, \zeta_t, u_t, \alpha_t\big)\mathrm{d}t + z_t\mathrm{d}W_t + k_t\mathrm{d}\overline{W}_t + \displaystyle\sum_{i,j=1}^m \zeta^{ij}_t\mathrm{d}\widetilde{V}^{ij}_t, \\ y_T = x_T, \end{cases} \tag{1.4}$$

where $\zeta. = (\zeta^{ij}_{\cdot})_{m\times m}$. The first term in the cost functional (1.3) calculates the difference between the control strategy $u.$ and a given benchmark $a_{(\cdot,\alpha.)}$. The risk of terminal wealth is measured by the second term. And the last term denotes the stochastic recursive utility.

Motivated by the considerations mentioned above, we investigate a class of LC problem of a partially observed forward–backward stochastic control system with Markov chain and input constraints in this paper. Unlike traditional settings, the observation process in our model is a controlled stochastic process that is linear with respect to the state and control strategy, rather than a Brownian motion. This results in a circular dependency between the observation and control strategy, which we address by employing a backward separation approach to decompose the state and observation. For the LC case, we obtain the optimal control strategy by virtue of SMP. A non-linear fully coupled regime-switching FBSDE has been obtained and its well-posedness has also been proved. For the LQ case, we obtain the feedback representation of the optimal control strategy through the Riccati equation. Finally, we apply our results to solve the asset-liability management problem.

The main contributions of this paper comparing with existing literature can be summarized as follows:

(1) A class of new partially observed LC model with Markov chain and input constraints is introduced. The dynamic of the state and observation are governed by a regime-switching FBSDE and a regime-switching SDE, respectively. Moreover, the cost functional is a convex function, which is distinct from most existing literatures on LQ models. Our LC model is a general model that can be degenerated into an LQ model. As we all know,

there is almost no literature that investigates LC optimal problems in partially observed stochastic control models, we need use a range of adjusted assumptions and techniques to address this issue.

(2) The dynamic of the state cannot be directly observed, decision-makers can observe a noise process and Markov chain α which reflects the state changes of the model, (*e.g.*, trends in a financial market). In this context, the observation is a controlled stochastic process whose dynamic depends on both the state x and control strategy u, resulting in a circular dependency between the control strategy and observation. We note that all coefficients depend on Markov chain α, that is, our partially observed model is a LC problem with random coefficients, which is essentially different from the existing literature (see [28]). Additionally, the input constraints are also considered in our model.

(3) We present a novel approach to obtain the optimal control strategy for the general LC case. By utilizing the SMP and convex analysis, we derive the optimal control strategy through a stochastic Hamiltonian system (3.6), which is a fully coupled nonlinear regime-switching FBSDE in double dimensions with conditional expectation. Although there are extensive literature about fully coupled FBSDEs (see [35–37]), the regime-switching FBSDE (3.6) is a new type of FBSDE due to the introduction of Markov chain and filtering. Furthermore, we establish the well-posedness of the regime-switching FBSDE with filtering by the method of continuation with randomized domination-monotonicity conditions in Appendix A. Compared with the classical case without filtering (see [38]), we proposed a new form of randomized domination-monotonicity conditions (see Assumption (S)) to study the regime-switching FBSDE with filtering.

(4) In the LQ case, we obtain the feedback representation of the optimal control strategy when the input belongs to some closed linear space by using decoupling method and Riccati equation. To demonstrate the practical significance of our theoretical results, we solve an asset-liability management problem.

The rest part is structured as follows. Section 2 introduces the partially observed LC problem and provides several useful lemmas. The optimal control strategy of Problem (LC) is derived in Section 3. For the LQ case, the feedback representation of the optimal control is obtained in Section 4. In Section 5, we present an application to the financial problem discussed at the beginning of this paper, demonstrating the practical significance of our research.

2. Problem formulation and preliminary

2.1. Notations and terminology

Given $T > 0$ as a fixed time horizon. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be complete filtered probability space. On this probability space, there is a $2d$-dimensional $\mathcal{F}_t$-adapted standard Brownian motion $(W, \overline{W})$ with $W_0 = \overline{W}_0 = 0$ and a continuous-time stationary Markov chain α taking value in finite state space $\mathcal{M} = \{1, 2, \ldots, m\}$ adapted to $\mathcal{F}_t$. We assume that Brownian motion $(W, \overline{W})$ and Markov chain α are independent. Moreover, the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ is generated by W, $\overline{W}$ and α, that is

$$\mathcal{F}_t := \sigma\{W_s, \overline{W}_s; 0 \le s \le t\} \vee \sigma\{\alpha_s; 0 \le s \le t\} \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets. Then $\mathbb{F}$ satisfies the usual condition. Throughout the paper, we denote the k-dimensional Euclidean space by $\mathbb{R}^k$ with standard Euclidean norm $|\cdot|$ and standard Euclidean inner product $\langle \cdot, \cdot \rangle$. Moreover, $\mathbb{S}^n$ denotes the set of symmetric $n \times n$ matrices with real elements, and $\|\cdot\|_Q^2 := \langle Q\cdot, \cdot \rangle$ for any $Q \in \mathbb{S}^n$. If $M \in \mathbb{S}^n$ is positive definite (positive semi-definite, uniformly positive definite), we write $M > (\ge, \gg)\ 0$.

The generator of α is $Q = (q_{ij})_{m \times m}$. Define function $f^i : \mathcal{M} \to \mathbb{R}, f^i(x) = I_{\{i\}}(x)$, for $i \in \mathcal{M}$. Then $f^i(\alpha_t)$ has the following semimartingale decomposition,

$$f^i(\alpha_t) = f^i(\alpha_0) + \int_0^t \sum_{j=1}^m q_{\alpha_s j} f^i(j) \mathrm{d}s + M_t^i = f^i(\alpha_0) + \int_0^t q_{\alpha_s i} \mathrm{d}s + M_t^i, \tag{2.1}$$

where $M^i_\cdot$ is a martingale satisfying $\mathbb{E}[|M^i_t|^2]<\infty$. We define $V^{ij}_t=\sum_{0<s\le t} f^i(\alpha_{s-})\,f^j(\alpha_s)$, $i\neq j$, which counts the number of times that α jumps from i to j up to time t. Since $i\neq j$, we can obtain

$$V^{ij}_t=\sum_{0<s\le t} f^i(\alpha_{s-})f^j(\alpha_s)=\sum_{0<s\le t} f^i(\alpha_{s-})\left(\Delta f^j(\alpha)\right)_s=\int_0^t f^i(\alpha_{s-})\mathrm{d}f^j(\alpha_s).$$

According to (2.1), we obtain

$$V^{ij}_t=\int_0^t f^i(\alpha_{s-})\mathrm{d}f^j(\alpha_s)=\int_0^t f^i(\alpha_s)q_{\alpha_s j}\mathrm{d}s+\int_0^t f^i(\alpha_{s-})\mathrm{d}M^j_s.$$

Then we define $\widetilde{V}^{ij}_t:=V^{ij}_t-\int_0^t\lambda^{ij}_s\mathrm{d}s$ with $\lambda^{ij}_t=q_{\alpha_t j}f^i(\alpha_t)$. Now we introduce some spaces of stochastic processes and random variables, let $L^2_{\mathcal{F}_T}(\mathbb{H})$ denote the set of $\mathbb{H}$-valued, $\mathcal{F}_T$-measurable, square-integrable random variable; $L^\infty(0,T;\mathbb{H})$ denote the set of $\mathbb{H}$-valued, deterministic, uniformly bounded function; $S^2_{\mathbb{F}}(0,T;\mathbb{H})$ denote the set of $\mathbb{H}$-valued, $\mathbb{F}$-adapted, càdlàg process such that $\mathbb{E}[\sup_{0\le t\le T}|\phi_t|^2]<\infty$; $H^2_{\mathbb{F}}(0,T;\mathbb{H})$ denote the set of $\mathbb{H}$-valued, $\mathbb{F}$-predictable process such that $\mathbb{E}[\int_0^T|\phi_t|^2\mathrm{d}t]<\infty$; $M^2_{\mathbb{F}}(0,T;\mathbb{H})$ denote the set of $\mathbb{H}$-valued, $\mathbb{F}$-predictable process such that $\mathbb{E}[\sum_{i,j=1}^m\int_0^T|\phi^{ij}_t|^2\lambda^{ij}_t\mathrm{d}t]<\infty$.

2.2. Problem formulation

In this subsection, motivated by the financial example, we consider the general SLQ problem of partially observed forward–backward stochastic control system with Markov chain as follows,

$$\begin{cases}\mathrm{d}x_t=\left\{A_{(t,\alpha_t)}x_t+B_{(t,\alpha_t)}u_t+b_{(t,\alpha_t)}\right\}\mathrm{d}t+\sigma_{(t,\alpha_t)}\mathrm{d}W_t+\bar{\sigma}_{(t,\alpha_t)}\mathrm{d}\overline{W}_t,\\ \mathrm{d}y_t=\left\{C_{(t,\alpha_t)}x_t+D_{(t,\alpha_t)}y_t+F_{(t,\alpha_t)}z_t+G_{(t,\alpha_t)}k_t+H_{(t,\alpha_t)}u_t+f_{(t,\alpha_t)}\right\}\mathrm{d}t\\ \qquad\qquad+z_t\mathrm{d}W_t+k_t\mathrm{d}\overline{W}_t+\sum_{i,j=1}^m\zeta^{ij}_t\mathrm{d}\widetilde{V}^{ij}_t,\\ x_0=x,\qquad \alpha_0=i,\qquad y_T=M_{(T,\alpha_T)}x_T+m_{(T,\alpha_T)},\end{cases}\tag{2.2}$$

where $x\in\mathbb{R}^n$ and the coefficients $A_{(\cdot,j)}$, $B_{(\cdot,j)}$, $C_{(\cdot,j)}$, $D_{(\cdot,j)}$, $F_{(\cdot,j)}$, $G_{(\cdot,j)}$, $H_{(\cdot,j)}$, $b_{(\cdot,j)}$, $f_{(\cdot,j)}$, $\sigma_{(\cdot,j)}$, $\bar{\sigma}_{(\cdot,j)}$, $M_{(T,j)}$, $m_{(T,j)}$, $j\in\mathcal{M}$ are given deterministic matrix-value functions of proper dimensions. In fact, the state process cannot be directly observed, decision-makers can observe a related process which governed by following regime-switching SDE,

$$\begin{cases}\mathrm{d}Y_t=\left\{K_{(t,\alpha_t)}x_t+I_{(t,\alpha_t)}u_t+g_{(t,\alpha_t)}\right\}\mathrm{d}t+\tilde{\sigma}_{(t,\alpha_t)}\mathrm{d}W_t,\\ Y_0=0,\qquad \alpha_0=i,\end{cases}\tag{2.3}$$

where $K_{(\cdot,j)}$, $I_{(\cdot,j)}$, $g_{(\cdot,j)}$, $\tilde{\sigma}_{(\cdot,j)}$, $j\in\mathcal{M}$ are given deterministic matrix-value functions of proper dimensions. Markov chain α is often used to describe the changes in market trends, such as bear market or bull market. Thus we also assume that Markov chain α can be directly observed by some way. Then we introduce the following assumptions.

Assumption (H1) For each $j\in\mathcal{M}$, the coefficients $A_{(\cdot,j)}$, $C_{(\cdot,j)}$, $D_{(\cdot,j)}$, $F_{(\cdot,j)}$, $G_{(\cdot,j)}$, $K_{(\cdot,j)}\in L^\infty(0,T;\mathbb{R}^{n\times n})$, $B_{(\cdot,j)}$, $H_{(\cdot,j)}$, $I_{(\cdot,j)}\in L^\infty(0,T;\mathbb{R}^{n\times m})$, $b_{(\cdot,j)}$, $f_{(\cdot,j)}$, $g_{(\cdot,j)}\in L^\infty(0,T;\mathbb{R}^n)$, $\sigma_{(\cdot,j)}$, $\bar{\sigma}_{(\cdot,j)}$, $\tilde{\sigma}_{(\cdot,j)}\in L^\infty(0,T;\mathbb{R}^{n\times d})$, $M_{(T,j)}\in\mathbb{S}^n$, $m_{(T,j)}\in\mathbb{R}^n$. $\tilde{\sigma}_{(\cdot,j)}$ satisfies the non-degenerate condition and $x\in\mathbb{R}^n$ is a given constant.

It is worth mentioning that the observation $Y_\cdot$ is a controlled stochastic process, whose drift term is linear with respect to the state $x_\cdot$ and control strategy $u_\cdot$. Naturally, decision-makers should formulate corresponding strategies based on the observed information. Then the circular dependency between the observation $Y_\cdot$ and control $u_\cdot$ arises. Thus we cannot directly define the admissible control set according to the filtration generated by observation $Y_\cdot$ and Markov chain α. Then we would like to use the so-called backward separation approach to overcome this difficulty. Specifically, we consider to separate the state $(x_\cdot, y_\cdot, z_\cdot, k_\cdot, \zeta_\cdot)$ and observation $Y_\cdot$ into two parts,

$$(x_\cdot, y_\cdot, z_\cdot, k_\cdot, \zeta_\cdot) = (x^0_\cdot, y^0_\cdot, z^0_\cdot, k^0_\cdot, \zeta^0_\cdot) + (x^1_\cdot, y^1_\cdot, z^1_\cdot, k^1_\cdot, \zeta^1_\cdot) \quad \text{and} \quad Y_\cdot = Y^0_\cdot + Y^1_\cdot. \tag{2.4}$$

Here, the two processes $(x^0_\cdot, y^0_\cdot, z^0_\cdot, k^0_\cdot, \zeta^0_\cdot)$ and $(x^1_\cdot, y^1_\cdot, z^1_\cdot, k^1_\cdot, \zeta^1_\cdot)$ of the state have the following form,

$$\begin{cases} \mathrm{d}x^0_t = A_{(t,\alpha_t)}x^0_t\mathrm{d}t + \sigma_{(t,\alpha_t)}\mathrm{d}W_t + \bar{\sigma}_{(t,\alpha_t)}\mathrm{d}\overline{W}_t, \\ \mathrm{d}y^0_t = \big\{C_{(t,\alpha_t)}x^0_t + D_{(t,\alpha_t)}y^0_t + F_{(t,\alpha_t)}z^0_t + G_{(t,\alpha_t)}k^0_t\big\}\mathrm{d}t + z^0_t\mathrm{d}W_t + k^0_t\mathrm{d}\overline{W}_t + \sum_{i,j=1}^m \zeta^{0,ij}_t\mathrm{d}\widetilde{V}^{ij}_t, \\ x^0_0 = x, \qquad \alpha_0 = i, \qquad y^0_T = M_{(T,\alpha_T)}x^0_T, \end{cases} \tag{2.5}$$

and

$$\begin{cases} \mathrm{d}x^1_t = \big\{A_{(t,\alpha_t)}x^1_t + B_{(t,\alpha_t)}u_t + b_{(t,\alpha_t)}\big\}\mathrm{d}t, \\ \mathrm{d}y^1_t = \big\{C_{(t,\alpha_t)}x^1_t + D_{(t,\alpha_t)}y^1_t + F_{(t,\alpha_t)}z^1_t + G_{(t,\alpha_t)}k^1_t + H_{(t,\alpha_t)}u_t + f_{(t,\alpha_t)}\big\}\mathrm{d}t \\ \qquad\qquad + z^1_t\mathrm{d}W_t + k^1_t\mathrm{d}\overline{W}_t + \sum_{i,j=1}^m \zeta^{1,ij}_t\mathrm{d}\widetilde{V}^{ij}_t, \\ x^1_0 = 0, \qquad \alpha_0 = i, \qquad y^1_T = M_{(T,\alpha_T)}x^1_T + m_{(T,\alpha_T)}, \end{cases} \tag{2.6}$$

where $u_\cdot \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$. In addition, the two separate observations $Y^0_\cdot$ and $Y^1_\cdot$ are given by the following regime-switching SDEs,

$$\begin{cases} \mathrm{d}Y^0_t = K_{(t,\alpha_t)}x^0_t\mathrm{d}t + \tilde{\sigma}_{(t,\alpha_t)}\mathrm{d}W_t, \\ Y^0_0 = 0, \qquad \alpha_0 = i, \end{cases} \tag{2.7}$$

and

$$\begin{cases} \mathrm{d}Y^1_t = \big\{K_{(t,\alpha_t)}x^1_t + I_{(t,\alpha_t)}u_t + g_{(t,\alpha_t)}\big\}\mathrm{d}t, \\ Y^1_0 = 0, \qquad \alpha_0 = i, \end{cases} \tag{2.8}$$

respectively. It should be point out that the separate dynamic $(x^0_\cdot, y^0_\cdot, z^0_\cdot, \bar{z}^0_\cdot, \zeta^0_\cdot)$ and observation $Y^0_\cdot$ are independent of the control strategy $u_\cdot$. Moreover, the separate dynamic $(x^1_\cdot, y^1_\cdot, z^1_\cdot, \bar{z}^1_\cdot, \zeta^1_\cdot)$ and observation $Y^1_\cdot$ still depend on the control strategy $u_\cdot$. This decomposed process is the key to overcoming the difficulty of circular dependency.

Let Assumption (H1) hold, for any $u_\cdot \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, it is easy to know that the regime-switching FBSDEs (2.5)–(2.6) and the regime-switching SDEs (2.7)–(2.8) admit unique solutions $(x^i_\cdot, y^i_\cdot, z^i_\cdot, k^i_\cdot, \zeta^i_\cdot) \in \mathcal{M}_{\mathbb{F}}(0,T) := S^2_{\mathbb{F}}(0,T;\mathbb{R}^n) \times S^2_{\mathbb{F}}(0,T;\mathbb{R}^n) \times H^2_{\mathbb{F}}(0,T;\mathbb{R}^{n\times d}) \times H^2_{\mathbb{F}}(0,T;\mathbb{R}^{n\times d}) \times M^2_{\mathbb{F}}(0,T;\mathbb{R}^{m\times m})$ and $Y^i_\cdot \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $i = 0,1$, respectively. Furthermore, by the decomposition (2.4), we can also know that the state equation (2.2) and the observation (2.3) admit unique solutions $(x_\cdot, y_\cdot, z_\cdot, k_\cdot, \zeta_\cdot) \in \mathcal{M}_{\mathbb{F}}(0,T)$ and $Y_\cdot \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, respectively.

Based on the above argument, we would like to introduce the admissible control set. Set

$$\mathcal{F}^Y_t = \sigma\{Y_s, \alpha_s; 0 \le s \le t\} \qquad \text{and} \qquad \mathcal{F}^{Y^0}_t = \sigma\{Y^0_s, \alpha_s; 0 \le s \le t\}.$$

Let Γ be a nonempty closed convex set of $\mathbb{R}^m$, and it means that the input control should satisfy some constraints, such as no-short and proportional investment. Then we define the following control set:

$$\mathcal{U}_{ad}^0 = \left\{ u.|u_t \text{ is } \mathbb{F}_t^{Y^0} \text{ progressive, with values in } \Gamma, \text{ such that } \mathbb{E}\left[\sup_{0\le t\le T} |u_t|^2 \right] < \infty \right\}.$$

Remark 2.1. A natural definition of admissible control is $u. \in L^2_{\mathbb{F}^Y}(0,T;\mathbb{R}^m)$. It implies that we may determine the control $u.$ by observation $Y.$. However, the circular dependency between the control and observation leads to an immediate difficulty in determining an admissible control. This is the main reason that (2.2) and (2.3) are split in two parts.

Remark 2.2. The study of optimal control problems with input control constraints has attracted consistent and extensive research attention because of the broad practical applications in various fields, likes finance and economics. For example, the no-shorting constraint in Markovitz optimal portfolio (*i.e.* $\Gamma = \mathbb{R}^m_+$, see, *e.g.*, [39]), the optimal investment problems with linear portfolio constraints (see, *e.g.*, [40]), and mean-field game with input control constraints (see, *e.g.*, [41, 42]).

Then we give the definition of the admissible control set:

Definition 2.3. A control strategy $u.$ is called admissible, if $u. \in \mathcal{U}_{ad}^0$ is $\mathbb{F}^Y$ progressive. The set of all admissible controls is denoted by $\mathcal{U}_{ad}$.

Moreover, inspired by [38], we consider the following cost functional,

$$J(u.) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left\{ q(t,x_t,\alpha_t) + p(t,y_t,\alpha_t) + \langle R_{(t,\alpha_t)}u_t + 2r_{(t,\alpha_t)}, u_t\rangle \right\}\mathrm{d}t + l(x_T,\alpha_T) + n(y_0) \right], \tag{2.9}$$

where $q : [0,T]\times\mathbb{R}^n\times\mathcal{M} \to \mathbb{R}$, $p : [0,T]\times\mathbb{R}^n\times\mathcal{M} \to \mathbb{R}$, $l : \mathbb{R}^n\times\mathcal{M} \to \mathbb{R}$ and $n : \mathbb{R}^n \to \mathbb{R}$ satisfying the following assumption.

Assumption (H2) The functions q, p, r, l, n are all twice continuously differentiable with respect to their elements. For any (t,x,y), $x \to q(t,x,\alpha)$, $y \to p(t,y,\alpha)$, $x \to l(x,\alpha)$ and $y \to n(y)$ are all convex. q, p, l and n are quadratic growth on (x,y) and lower bounded by K. Their first order partial derivatives have linear growth and are Lipschitz continuous on (x,y) with Lipschitz constant C_0. Furthermore, we assume that the control weight matrices $R_{(\cdot,j)} \in L^\infty(0,T;\mathbb{S}^m)$ and $r_{(\cdot,j)} \in L^\infty(0,T;\mathbb{R}^m)$ satisfy $R_{(\cdot,j)} \gg 0$, $j\in\mathcal{M}$.

Now, the LC optimal control problem for partially observed Markovian regime-switching stochastic control system is stated as follows.

Problem (LC) Find an admissible control $\bar{u}. \in \mathcal{U}_{ad}$ such that

$$J(\bar{u}.) = \inf_{u\in\mathcal{U}_{ad}} J(u.).$$

Remark 2.4. It is worth pointing out that we consider the LC problem, a more general model, in this paper, and it can easily degenerate into the LQ model in form (4.1), which will be studied in Section 4.

Problem (LC) is called well-posed if the infimum of $J(u.)$ over the admissible control set is finite. If Problem (LC) is well-posed and the infimum of the cost functional can be achieved, then Problem (LC) is said to be solvable. Any $\bar{u}.$ satisfying the equality is called an optimal control of Problem (LC), and $(\bar{x}., \bar{y}., \bar{z}., \bar{k}., \bar{\zeta}.)$ and $J(\bar{u}.)$ are called the optimal state and the optimal cost functional, respectively. Obviously, under Assumptions (H1)-(H2), we can know that the cost functional (2.9) is well-defined for any admissible control $u. \in \mathcal{U}_{ad}$. Furthermore, Problem (LC) is well-posed, see [38].

2.3. Preliminary result

In this subsection, in order to find the optimal control strategy $\bar{u}_\cdot$ of Problem (LC), we give some useful lemmas, which are the key to the subsequent proof. We first present the relationship between the filtration generated by the observation $Y_\cdot$ and the filtration generated by the decomposed process $Y_\cdot^0$ for any given control process $u_\cdot \in \mathcal{U}_{ad}$.

Lemma 2.5. *For any $u_\cdot \in \mathcal{U}_{ad}$, $\mathcal{F}_t^Y = \mathcal{F}_t^{Y^0}$.*

Proof. For any $u_\cdot \in \mathcal{U}_{ad}$, since u_t is $\mathcal{F}_t^{Y^0}$ adapted, then it follows from (2.6) that x_t^1 is $\mathcal{F}_t^{Y^0}$ adapted, so does Y_t^1. Because $Y_t = Y_t^0 + Y_t^1$, we know that Y_t is $\mathcal{F}_t^{Y^0}$ adapted. It implies $\mathcal{F}_t^Y \subseteq \mathcal{F}_t^{Y^0}$. In a similar way, we get $\mathcal{F}_t^{Y^0} \subseteq \mathcal{F}_t^Y$ *via* $Y_t^0 = Y_t - Y_t^1$. Then the proof is completed. □

Then we give the L^2 estimates of (2.2), which are derived by Itô's formula and Gronwall inequality, [43].

Lemma 2.6. *Let Assumption (H1) hold. For any $u_\cdot^i \in \mathcal{U}_{ad}, i = 1, 2$, let $(x_\cdot^i, y_\cdot^i, z_\cdot^i, k_\cdot^i, \zeta_\cdot^i)$ be the solution to regime-switching FBSDE (2.2) corresponding to $u_\cdot^i$. Then there exists a constant $\widetilde{C} > 0$ such that*

$$\sup_{0\le t\le T} \mathbb{E}\big[|x_t^1 - x_t^2|^2\big] \le \widetilde{C}\mathbb{E}\bigg[\int_0^T |u_t^1 - u_t^2|^2 \mathrm{d}t\bigg],$$

$$\sup_{0\le t\le T} \mathbb{E}\big[|y_t^1 - y_t^2|^2\big] \le \widetilde{C}\bigg\{\sup_{0\le t\le T} \mathbb{E}\big[|x_t^1 - x_t^2|^2\big] + \mathbb{E}\bigg[\int_0^T |u_t^1 - u_t^2|^2 \mathrm{d}t\bigg]\bigg\}.$$

Inspired by [44] and [28], we have the following lemma, which is the key of this article.

Lemma 2.7. *Under Assumptions (H1)–(H2), we obtain*

$$\inf_{\tilde{u}\in\mathcal{U}_{ad}} J(\tilde{u}_\cdot) = \inf_{u\in\mathcal{U}_{ad}^0} J(u_\cdot).$$

Proof. We have $\mathcal{U}_{ad} \subseteq \mathcal{U}_{ad}^0$ by Definition 2.3, thus

$$\inf_{u\in\mathcal{U}_{ad}^0} J(u_\cdot) \le \inf_{\tilde{u}\in\mathcal{U}_{ad}} J(\tilde{u}_\cdot)$$

Next, let's prove the reverse inequality by the following three steps.

Step 1: $\mathcal{U}_{ad}$ is dense in $\mathcal{U}_{ad}^0$ under the metric of $L^2_{\mathbb{F}^{Y^0}}(0, T; \mathbb{R}^m)$.

For any $u_\cdot \in \mathcal{U}_{ad}^0$, define

$$u_{n,s} = \begin{cases} u_0, & \text{for } 0 \le s \le \delta_n, \\ \dfrac{1}{\delta_n}\displaystyle\int_{(j-1)\delta_n}^{j\delta_n} u_t \mathrm{d}t, & \text{for } j\delta_n < s \le (j+1)\delta_n, \end{cases}$$

where $u_0 \in \mathbb{R}^m$, j, n are natural numbers, $1 \le j \le n-1$, and $\delta_n = \frac{T}{n}$. Then $u_{n,s}$ is $\mathcal{F}_{j\delta_n}^{Y^0}$ measurable for any $j\delta_n < s \le (j+1)\delta_n$, and for any n

$$\sup_{0\le s\le T} |u_{n,s}| \le |u_0| + \sup_{0\le s\le T} |u_s|.$$

Thus, $u_{n,\cdot} \in \mathcal{U}_{ad}^0$. Let $(x_\cdot^n, y_\cdot^n, z_\cdot^n, k_\cdot^n, \zeta_\cdot^n)$ and $Y_\cdot^n$ be the trajectories of (2.2) and (2.3).

For any $0 \le s \le \delta_n$, we can know that $u_{n,s}$ is $\mathcal{F}_0^{Y^0} = \mathcal{F}_0^{Y^n}$ measurable. By virtue of Lemma 2.5, we have

$$\mathcal{F}_s^{Y^0} = \mathcal{F}_s^{Y^n}, \qquad 0 \le s \le \delta_n.$$

Next, for any $\delta_n \le s \le 2\delta_n$, we know that $u_{n,s}$ is $\mathcal{F}_{\delta_n}^{Y^0}$ measurable. Obviously, $u_{n,s}$ is also $\mathcal{F}_{\delta_n}^{Y^n}$ measurable, then we also have

$$\mathcal{F}_s^{Y^0} = \mathcal{F}_s^{Y^n}, \qquad \delta_n \le s \le 2\delta_n.$$

If for any $j\delta_n \le s \le (j+1)\delta_n$, such that $u_{n,s}$ is $\mathcal{F}_{j\delta_n}^{Y^0} = \mathcal{F}_{j\delta_n}^{Y^n}$ measurable. By the similar argument, we have

$$\mathcal{F}_s^{Y^0} = \mathcal{F}_s^{Y^n}, \qquad j\delta_n \le s \le (j+1)\delta_n.$$

We can know that $u_{n,\cdot}$ is adapted to $\mathcal{F}_\cdot^{Y^0}$ and $\mathcal{F}_\cdot^{Y^n}$, and $\mathcal{F}_s^{Y^0} = \mathcal{F}_s^{Y^n}$, $s \in [0,T]$. Then $u_n \in \mathcal{U}_{ad}$, thus (2.2) admits a unique solution $(x_\cdot^n, y_\cdot^n, z_\cdot^n, k_\cdot^n, \zeta_\cdot^n) \in \mathcal{M}_{\mathbb{F}}(0,T)$. In addition, $u_{n,\cdot}$ converges to $u_\cdot$ in probability when n tends to infinity. Using the integrability condition of $u_\cdot$ in $\mathcal{U}_{ad}^0$, we get $u_{n,\cdot} \to u_\cdot$ as $n \to +\infty$ in $L^2_{\mathbb{F}^{Y^0}}(0,T;\mathbb{R}^m)$, that is, $\mathcal{U}_{ad}$ is dense in $\mathcal{U}_{ad}^0$.

Step 2: $\lim_{n\to+\infty} J(u_{n,\cdot}) = J(u_\cdot)$, where $u_\cdot, u_{n,\cdot}$ and $(x_\cdot^n, y_\cdot^n, z_\cdot^n, k_\cdot^n, \zeta_\cdot^n)$ are defined as in Step 1. Noticing the Lipschitz condition in Assumption (H2) and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
2|J(u_{n,\cdot}) - J(u_\cdot)| \le{}& C_0\sqrt{\mathbb{E}\Big[\int_0^T |x_s^n + x_s + 1|^2 \mathrm{d}s\Big]}\sqrt{\mathbb{E}\Big[\int_0^T |x_s^n - x_s|^2 \mathrm{d}s\Big]} \\
&+ C_0\sqrt{\mathbb{E}\Big[\int_0^T |y_s^n + y_s + 1|^2 \mathrm{d}s\Big]}\sqrt{\mathbb{E}\Big[\int_0^T |y_s^n - y_s|^2 \mathrm{d}s\Big]} \\
&+ \sqrt{\mathbb{E}\Big[\int_0^T |R_{(s,\alpha_s)}(u_{n,s} + u_s) + 2r_{(s,\alpha_s)}|^2 \mathrm{d}s\Big]}\sqrt{\mathbb{E}\Big[\int_0^T |u_{n,s} - u_s|^2 \mathrm{d}s\Big]} \\
&+ C_0\sqrt{\mathbb{E}\big[|x_T^n + x_T + 1|^2\big]}\sqrt{\mathbb{E}\big[|x_T^n - x_T|^2\big]} \\
&+ C_0\sqrt{\mathbb{E}\big[|y_0^n + y_0 + 1|^2\big]}\sqrt{\mathbb{E}\big[|y_0^n - y_0|^2\big]}.
\end{aligned}$$

According to Lemma 2.6, we have $J(u_{n,\cdot}) \to J(u_\cdot)$ as $n \to +\infty$.

Step 3: $\inf_{u\in\mathcal{U}_{ad}^0} J(u_\cdot) \ge \inf_{\tilde{u}\in\mathcal{U}_{ad}} J(\tilde{u}_\cdot)$.

Since $u_{n,\cdot} \in \mathcal{U}_{ad}$, then

$$J(u_{n,\cdot}) \ge \inf_{\tilde{u}\in\mathcal{U}_{ad}} J(\tilde{u}_\cdot).$$

We have $J(u_\cdot) \ge \inf_{\tilde{u}\in\mathcal{U}_{ad}} J(\tilde{u}_\cdot)$ as $n \to +\infty$. Since $u_\cdot$ is arbitrary, we can get the result directly. □

Remark 2.8. Lemma 2.7 plays a key role in solving Problem (LC), which implies that we can find an optimal control $u_\cdot \in \mathcal{U}_{ad}^0$ instead of the optimal control $u_\cdot \in \mathcal{U}_{ad}$ to minimize $J(u_\cdot)$.

3. Optimal control of the LC case

In this section, we first establish a necessary condition for optimality of Problem (LC).

Theorem 3.1. *Let Assumptions (H1)–(H2) hold. Suppose that* $\bar{u}.$ *is the optimal control of Problem (LC) and* $(\bar{x}., \bar{y}., \bar{z}., \bar{k}., \bar{\zeta}.)$ *is the corresponding optimal trajectories. Then the following regime-switching FBSDE*

$$\begin{cases} \mathrm{d}\varphi_t = \big\{ -D^{\top}_{(t,\alpha_t)}\varphi_t + p_y(t,\bar{y}_t,\alpha_t)\big\}\mathrm{d}t - F^{\top}_{(t,\alpha_t)}\varphi_t \mathrm{d}W_t - G^{\top}_{(t,\alpha_t)}\varphi_t \mathrm{d}\overline{W}_t, \\ \mathrm{d}\xi_t = -\big\{A^{\top}_{(t,\alpha_t)}\xi_t + q_x(t,\bar{x}_t,\alpha_t) - C^{\top}_{(t,\alpha_t)}\varphi_t\big\}\mathrm{d}t + \theta_t \mathrm{d}W_t + \vartheta_t \mathrm{d}\overline{W}_t + \sum_{i,j=1}^{m} \eta_t^{ij}\mathrm{d}\widetilde{V}_t^{ij}, \\ \varphi_0 = n_y(\bar{y}_0), \qquad \alpha_0 = i, \qquad \xi_T = l_x(\bar{x}_T,\alpha_T) + M^{\top}_{(T,\alpha_T)}\varphi_T, \end{cases} \tag{3.1}$$

admits a unique solution $(\varphi., \xi., \theta., \vartheta., \eta.) \in \mathcal{M}_{\mathbb{F}}(0,T)$ *such that*

$$\big\langle R^{\frac{1}{2}}_{(t,\alpha_t)}\big\{R^{-1}_{(t,\alpha_t)}\big[H^{\top}_{(t,\alpha_t)}\mathbb{E}[\varphi_t|\mathcal{F}_t^{\bar{Y}}] - B^{\top}_{(t,\alpha_t)}\mathbb{E}[\xi_t|\mathcal{F}_t^{\bar{Y}}] - r_{(t,\alpha_t)}\big] - \bar{u}_t\big\}, R^{\frac{1}{2}}_{(t,\alpha_t)}(v_t - \bar{u}_t)\big\rangle \leq 0, \tag{3.2}$$

for any $v. \in \mathcal{U}^0_{ad}$*, where* $\mathcal{F}_t^{\bar{Y}} = \sigma\{\bar{Y}_s, \alpha_s; 0 \leq s \leq t\}$ *with the observation* $\bar{Y}.$ *related to the optimal control* $\bar{u}..$

Proof. According to Lemma 2.7, if $\bar{u}.$ is an optimal control of Problem (LC), then

$$J(\bar{u}.) = \inf_{u \in \mathcal{U}^0_{ad}} J(u.).$$

For any $v. \in \mathcal{U}^0_{ad}$, let $(x^{\epsilon}_., y^{\epsilon}_., z^{\epsilon}_., k^{\epsilon}_., \zeta^{\epsilon}_.) \in \mathcal{M}_{\mathbb{F}}(0,T)$ be the solution to regime-switching FBSDE (2.2) corresponding to $u^{\epsilon}_. = \bar{u}. + \epsilon(v. - \bar{u}.), 0 < \epsilon < 1$. Then, we introduce the following variation equation

$$\begin{cases} \mathrm{d}\breve{x}_t = \big\{A_{(t,\alpha_t)}\breve{x}_t + B_{(t,\alpha_t)}(v_t - \bar{u}_t)\big\}\mathrm{d}t, \\ \mathrm{d}\breve{y}_t = \big\{C_{(t,\alpha_t)}\breve{x}_t + D_{(t,\alpha_t)}\breve{y}_t + F_{(t,\alpha_t)}\breve{z}_t + G_{(t,\alpha_t)}\breve{k}_t + H_{(t,\alpha_t)}(v_t - \bar{u}_t)\big\}\mathrm{d}t \\ \qquad\quad + \breve{z}_t \mathrm{d}W_t + \breve{k}_t \mathrm{d}\overline{W}_t + \sum_{i,j=1}^{m} \breve{\zeta}_t^{ij}\mathrm{d}\widetilde{V}_t^{ij}, \\ \breve{x}_0 = 0, \qquad \alpha_0 = i, \qquad \breve{y}_T = M_{(T,\alpha_T)}\breve{x}_T, \end{cases} \tag{3.3}$$

which admits a unique solution $(\breve{x}., \breve{y}., \breve{z}., \breve{k}., \breve{\zeta}.) \in \mathcal{M}_{\mathbb{F}}(0,T)$. By the usual method and Gronwall's inequalities (see [11], Lem. 3.1), we obtain

$$\lim_{\epsilon \to 0} \mathbb{E}\bigg[\sup_{0 \leq t \leq T}\bigg|\frac{x^{\epsilon}_t - \bar{x}_t}{\epsilon} - \breve{x}_t\bigg|^2\bigg] = 0, \qquad \text{and} \qquad \lim_{\epsilon \to 0} \mathbb{E}\bigg[\sup_{0 \leq t \leq T}\bigg|\frac{y^{\epsilon}_t - \bar{y}_t}{\epsilon} - \breve{y}_t\bigg|^2\bigg] = 0.$$

Moreover, we also have

$$\begin{aligned} 0 \leq \lim_{\epsilon \to 0} \frac{J(u^{\epsilon}_.) - J(\bar{u}.)}{\epsilon} = \mathbb{E}\bigg[\int_0^T \Big\{&\langle q_x(t,\bar{x}_t,\alpha_t), \breve{x}_t\rangle + \langle p_y(t,\bar{y}_t,\alpha_t), \breve{y}_t\rangle + \langle R_{(t,\alpha_t)}\bar{u}_t \\ &+ r_{(t,\alpha_t)}, v_t - \bar{u}_t\rangle\Big\}\mathrm{d}t + \langle l_x(\bar{x}_T,\alpha_T), \breve{x}_T\rangle + \langle n_y(\bar{y}_0), \breve{y}_0\rangle\bigg]. \end{aligned} \tag{3.4}$$

Applying Itô's formula to $\langle \xi_t, \breve{x}_t\rangle$ and $\langle \varphi_t, \breve{y}_t\rangle$, we have

$$\begin{aligned}&\mathbb{E}\bigg[\langle l_x(\bar{x}_T,\alpha_T),\breve{x}_T\rangle+\langle n_y(\bar{y}_0),\breve{y}_0\rangle+\int_0^T\big\{\langle q_x(t,\bar{x}_t,\alpha_t),\breve{x}_t\rangle+\langle p_y(t,\bar{y}_t,\alpha_t),\breve{y}_t\rangle\big\}\mathrm{d}t\bigg]\\&=\mathbb{E}\bigg[\int_0^T\langle B^\top_{(t,\alpha_t)}\xi_t-H^\top_{(t,\alpha_t)}\varphi_t,v_t-\bar{u}_t\rangle\mathrm{d}t\bigg].\end{aligned}$$

Then the inequality (3.4) can be rewritten as

$$\mathbb{E}\bigg[\int_0^T\langle R_{(t,\alpha_t)}\bar{u}_t+B^\top_{(t,\alpha_t)}\xi_t-H^\top_{(t,\alpha_t)}\varphi_t+r_{(t,\alpha_t)},v_t-\bar{u}_t\rangle\mathrm{d}t\bigg]\geq 0.$$

Noticing $\bar{u}_\cdot$, $v_\cdot\in\mathcal{U}^0_{ad}$ (*i.e.*, $\bar{u}_t$ and v_t are $\mathcal{F}^{Y^0}_t$ adapted) and $R_{(\cdot,j)}\gg 0$, $j\in\mathcal{M}$, it implies

$$\big\langle R^{\frac12}_{(t,\alpha_t)}\big[R^{-1}_{(t,\alpha_t)}\big[H^\top_{(t,\alpha_t)}\mathbb{E}[\varphi_t|\mathcal{F}^{Y^0}_t]-B^\top_{(t,\alpha_t)}\mathbb{E}[\xi_t|\mathcal{F}^{Y^0}_t]-r_{(t,\alpha_t)}\big]-\bar{u}_t\big],R^{\frac12}_{(t,\alpha_t)}(v_t-\bar{u}_t)\big\rangle\leq 0.$$

Since $\bar{u}_\cdot\in\mathcal{U}_{ad}$, it follows from Lemma 2.5 that $\mathcal{F}^{Y^0}_t=\mathcal{F}^{\bar{Y}}_t$, then $\bar{u}_\cdot$ is an optimal control of Problem (LC) if and only if (3.2) hold. □

In virtue of SMP (Thm. 3.1) and by the well-known results of convex analysis [45], Theorem 5.2, we can know that the optimal control $\bar{u}_\cdot$ satisfies the following equation,

$$\bar{u}_t=\mathbf{P}_\Gamma\big[R^{-1}_{(t,\alpha_t)}\big\{H^\top_{(t,\alpha_t)}\mathbb{E}[\varphi_t|\mathcal{F}^{\bar{Y}}_t]-B^\top_{(t,\alpha_t)}\mathbb{E}[\xi_t|\mathcal{F}^{\bar{Y}}_t]-r_{(t,\alpha_t)}\big\}\big],\quad t\in[0,T],\tag{3.5}$$

where $\mathbf{P}_\Gamma[\cdot]$ is the projection mapping from $\mathbb{R}^m$ to its closed convex subset Γ under the norm $|\cdot|^2_R:=\langle R^{\frac12}\cdot,R^{\frac12}\cdot\rangle$, [41, 42]. Thus, combining (2.2), (3.1) with (3.2), we obtain the following stochastic Hamiltonian system:

$$\begin{cases}\mathrm{d}\bar{x}_t=\big\{A_{(t,\alpha_t)}\bar{x}_t+B_{(t,\alpha_t)}\bar{u}_t+b_{(t,\alpha_t)}\big\}\mathrm{d}t+\sigma_{(t,\alpha_t)}\mathrm{d}W_t+\bar{\sigma}_{(t,\alpha_t)}\mathrm{d}\overline{W}_t,\\ \mathrm{d}\bar{y}_t=\big\{C_{(t,\alpha_t)}\bar{x}_t+D_{(t,\alpha_t)}\bar{y}_t+F_{(t,\alpha_t)}\bar{z}_t+G_{(t,\alpha_t)}\bar{k}_t+H_{(t,\alpha_t)}\bar{u}_t+f_{(t,\alpha_t)}\big\}\mathrm{d}t\\ \qquad\qquad+\bar{z}_t\mathrm{d}W_t+\bar{k}_t\mathrm{d}\overline{W}_t+\displaystyle\sum_{i,j=1}^m\bar{\zeta}^{ij}_t\mathrm{d}\widetilde{V}^{ij}_t,\\ \mathrm{d}\varphi_t=\big\{p_y(t,\bar{y}_t,\alpha_t)-D^\top_{(t,\alpha_t)}\varphi_t\big\}\mathrm{d}t-F^\top_{(t,\alpha_t)}\varphi_t\mathrm{d}W_t-G^\top_{(t,\alpha_t)}\varphi_t\mathrm{d}\overline{W}_t,\\ \mathrm{d}\xi_t=-\big\{A^\top_{(t,\alpha_t)}\xi_t+q_x(t,\bar{x}_t,\alpha_t)-C^\top_{(t,\alpha_t)}\varphi_t\big\}\mathrm{d}t+\theta_t\mathrm{d}W_t+\vartheta_t\mathrm{d}\overline{W}_t+\displaystyle\sum_{i,j=1}^m\eta^{ij}_t\mathrm{d}\widetilde{V}^{ij}_t,\\ \bar{u}_t=\mathbf{P}_\Gamma\big[R^{-1}_{(t,\alpha_t)}\big\{H^\top_{(t,\alpha_t)}\mathbb{E}[\varphi_t|\mathcal{F}^{Y^0}_t]-B^\top_{(t,\alpha_t)}\mathbb{E}[\xi_t|\mathcal{F}^{Y^0}_t]-r_{(t,\alpha_t)}\big\}\big],\\ \bar{x}_0=x,\qquad\alpha_0=i,\qquad\bar{y}_T=M_{(T,\alpha_T)}\bar{x}_T+m_{(T,\alpha_T)},\\ \varphi_0=n_y(\bar{y}_0),\qquad\xi_T=l_x(\bar{x}_T,\alpha_T)+M^\top_{(T,\alpha_T)}\varphi_T,\end{cases}\tag{3.6}$$

which is a fully coupled nonlinear regime-switching FBSDE in double dimensions. Since the stochastic Hamiltonian system (3.6) includes both the nonlinear terms (projection mapping and convex functions), conditional expectations and Markov chain, it is hard to obtain the existence and uniqueness of this system. The existence and uniqueness of solutions to fully coupled FBSDE have been studied in extensive literature, such as [35–37]. However, their results cannot be directly used to study the well-posedness of regime-switching FBSDE (3.6). Next we would like to illustrate this point. In [36], the authors studied a class of fully coupled FBSDE with an arbitrarily large time duration and the dimensions of SDE and BSDE can be different. In their paper, the monotonicity condition holds and the continuation method plays an important role. In [35], a class of fully coupled FBSDE with small time horizon and weak coupling condition has been studied. The discounting method was applied and the well-posedness of this type of FBSDE was obtained. In [37], the existence and uniqueness of a kind of FBSDE with double dimensions had been obtained under some monotonicity conditions. As a contrast, due to the introduction of Markov chain and conditional expectation term, the above results cannot be applied to regime-switching FBSDE (3.6). In virtue of method of continuation under domination-monotonicity conditions, we obtain the well-posedness of the regime-switching FBSDE with filtering by Theorem A.3.

Then, based on the stochastic Hamiltonian system (3.6), we achieve the following results about Problem (LC).

Theorem 3.2. *Under Assumptions (A1)–(A2), Problem (LC) admits a unique optimal control*

$$\bar{u}_t = \mathbf{P}_\Gamma\left[R^{-1}_{(t,\alpha_t)}\left\{H^\top_{(t,\alpha_t)}\mathbb{E}[\varphi_t|\mathcal{F}^{\bar{Y}}_t] - B^\top_{(t,\alpha_t)}\mathbb{E}[\xi_t|\mathcal{F}^{\bar{Y}}_t] - r_{(t,\alpha_t)}\right\}\right], \qquad t\in[0,T], \tag{3.7}$$

where $(\varphi_\cdot, \xi_\cdot) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ *is given by the stochastic Hamiltonian system* (3.6).

Proof. On one hand, we show that $\bar{u}_\cdot$ is an optimal control of Problem (LC). For any given $v_\cdot \in \mathcal{U}^0_{ad}$, noting the definition of $\mathbf{P}_\Gamma$, we have

$$\mathbb{E}\left[\int_0^T \langle R_{(t,\alpha_t)}\bar{u}_t + B^\top_{(t,\alpha_t)}\xi_t - H^\top_{(t,\alpha_t)}\varphi_t + r_{(t,\alpha_t)}, v_t - \bar{u}_t\rangle \mathrm{d}t\right] \geq 0.$$

Based on it, we could calculate the difference between $J(v_\cdot)$ and $J(\bar{u}_\cdot)$ by the similar argument in Theorem 3.1, we derive $J(v_\cdot) - J(\bar{u}_\cdot) \geq 0$, which implies $\bar{u}_\cdot$ given by (3.7) is an optimal control.

On the other hand, let $\bar{v}_\cdot \in \mathcal{U}_{ad}$ be another optimal control and the corresponding state be $(\bar{x}^v_\cdot, \bar{y}^v_\cdot, \bar{z}^v_\cdot, \bar{k}^v_\cdot, \bar{\zeta}^v_\cdot)$. Similar to Theorem 3.1, we introduce the corresponding adjoint process $(\varphi^v_\cdot, \xi^v_\cdot, \theta^v_\cdot, \vartheta^v_\cdot, \eta^v_\cdot)$ in the form of regime-switching FBSDE (3.1). Then, combining the state process and adjoint process, they should satisfy the stochastic Hamiltonian system (3.6). Moreover, according to the existence and uniqueness, we can obtain that the stochastic Hamiltonian system (3.6) has a unique solution, *i.e.*, two related Hamiltonian systems of two control variables $\bar{v}_\cdot$ and $\bar{u}_\cdot$ are equal. Thus, by noting the unique property of the projection mapping [45], Theorem 5.2, we obtain $\bar{v}_\cdot = \bar{u}_\cdot$. Then the proof is completed. □

4. Optimal control for the LQ case

In this section, in order to obtain the feedback representation of optimal control strategy $\bar{u}_\cdot$, we consider the LQ case, *i.e.*, the convex cost functional (2.9) becomes quadratic form as follows,

$$\begin{aligned} J(u_\cdot) = \frac{1}{2}\mathbb{E}\bigg[\int_0^T &\Big\{\langle Q_{(t,\alpha_t)}x_t + 2q_{(t,\alpha_t)}, x_t\rangle + \langle P_{(t,\alpha_t)}y_t + p_{(t,\alpha_t)}, y_t\rangle + 2\langle S_{(t,\alpha_t)}u_t, x_t\rangle \\ &+ \langle R_{(t,\alpha_t)}u_t + 2r_{(t,\alpha_t)}, u_t\rangle\Big\}\mathrm{d}t + \langle L_{(T,\alpha_T)}x_T + 2l_{(T,\alpha_T)}, x_T\rangle + \langle Ny_0 + 2n, y_0\rangle\bigg], \end{aligned} \tag{4.1}$$

where $Q_{(\cdot,j)}$, $P_{(\cdot,j)}$, $R_{(\cdot,j)}$, $S_{(\cdot,j)}$, $q_{(\cdot,j)}$, $p_{(\cdot,j)}$, $r_{(\cdot,j)}$, $L_{(T,j)}$, $l_{(T,j)}$, $j\in\mathcal{M}$, N and n are given deterministic matrix-value functions of proper dimensions. Then for the cost functional (4.1), we give the following hypothesis.

Assumption (H3) For $j\in\mathcal{M}$, the coefficients $Q_{(\cdot,j)}$, $P_{(\cdot,j)} \in L^\infty(0,T;\mathbb{S}^n)$, $R_{(\cdot,j)} \in L^\infty(0,T;\mathbb{S}^m)$, $S_{(\cdot,j)} \in L^\infty(0,T;\mathbb{R}^{n\times m})$, $q_{(\cdot,j)}$, $p_{(\cdot,j)} \in L^\infty(0,T;\mathbb{R}^n)$, $r_{(\cdot,j)} \in L^\infty(0,T;\mathbb{R}^m)$, $L_{(T,j)}$, $N \in \mathbb{S}^n$, $l_{(T,j)}$, $n\in\mathbb{R}^n$ such that $Q_{(\cdot,j)} \geq 0$, $P_{(\cdot,j)} \geq 0$, $L_{(T,j)} \geq 0$ and $R_{(\cdot,j)} \gg 0$.

Now, the LQ optimal control problem for partially observed Markovian regime-switching stochastic control system is stated as follows.
Problem (LQ) Find an admissible control $\bar{u}. \in \mathcal{U}_{ad}$ such that

$$J(\bar{u}.) = \inf_{u \in \mathcal{U}_{ad}} J(u.).$$

Obviously, under Assumptions (H1) and (H3), the Problem (LQ) is well-defined and has the following corollary (noticing Thm. 3.1).

Corollary 4.1. *Let Assumptions (H1) and (H3) hold. Suppose that $\bar{u}.$ is the optimal control of Problem (LQ), then it satisfies the following stationary condition,*

$$\bar{u}_t = \mathbf{P}_\Gamma\big[R^{-1}_{(t,\alpha_t)}\big\{H^\top_{(t,\alpha_t)}\mathbb{E}[\varphi_t|\mathcal{F}^{\bar{Y}}_t] - B^\top_{(t,\alpha_t)}\mathbb{E}[\xi_t|\mathcal{F}^{\bar{Y}}_t] - S^\top_{(t,\alpha_t)}\mathbb{E}[\bar{x}_t|\mathcal{F}^{\bar{Y}}_t] - r_{(t,\alpha_t)}\big\}\big], \tag{4.2}$$

where $(\bar{x}., \varphi., \xi.)$ satisfies the following stochastic Hamiltonian system

$$\begin{cases} \mathrm{d}\bar{x}_t = \big\{A_{(t,\alpha_t)}\bar{x}_t + B_{(t,\alpha_t)}\bar{u}_t + b_{(t,\alpha_t)}\big\}\mathrm{d}t + \sigma_{(t,\alpha_t)}\mathrm{d}W_t + \bar{\sigma}_{(t,\alpha_t)}\mathrm{d}\overline{W}_t, \\ \mathrm{d}\bar{y}_t = \big\{C_{(t,\alpha_t)}\bar{x}_t + D_{(t,\alpha_t)}\bar{y}_t + F_{(t,\alpha_t)}\bar{z}_t + G_{(t,\alpha_t)}\bar{k}_t + H_{(t,\alpha_t)}\bar{u}_t + f_{(t,\alpha_t)}\big\}\mathrm{d}t \\ \qquad\quad + \bar{z}_t\mathrm{d}W_t + \bar{k}_t\mathrm{d}\overline{W}_t + \displaystyle\sum_{i,j=1}^m \bar{\zeta}^{ij}_t\mathrm{d}\widetilde{V}^{ij}_t, \\ \mathrm{d}\varphi_t = \big\{-D^\top_{(t,\alpha_t)}\varphi_t + P_{(t,\alpha_t)}\bar{y}_t + p_{(t,\alpha_t)}\big\}\mathrm{d}t - F^\top_{(t,\alpha_t)}\varphi_t\mathrm{d}W_t - G^\top_{(t,\alpha_t)}\varphi_t\mathrm{d}\overline{W}_t, \\ \mathrm{d}\xi_t = -\big\{A^\top_{(t,\alpha_t)}\xi_t - C^\top_{(t,\alpha_t)}\varphi_t + Q_{(t,\alpha_t)}\bar{x}_t + S_{(t,\alpha_t)}\bar{u}_t + q_{(t,\alpha_t)}\big\}\mathrm{d}t \\ \qquad\quad + \theta_t\mathrm{d}W_t + \vartheta_t\mathrm{d}\overline{W}_t + \displaystyle\sum_{i,j=1}^m \eta^{ij}_t\mathrm{d}\widetilde{V}^{ij}_t, \\ \bar{x}_0 = x, \qquad \alpha_0 = i, \qquad \bar{y}_T = M_{(T,\alpha_T)}\bar{x}_T + m_{(T,\alpha_T)}, \\ \varphi_0 = N\bar{y}_0 + n, \qquad \xi_T = L_{(T,\alpha_T)}\bar{x}_T + M^\top_{(T,\alpha_T)}\varphi_T + l_{(T,\alpha_T)}. \end{cases} \tag{4.3}$$

Now, in order to obtain the feedback representation of the optimal control strategy $\bar{u}.$, we consider a special case, closed linear subspace, of primal problem, and our main objective is to exploit the linearity of the projection operators $\mathbf{P}_\Gamma$ to derive more explicit results. More precisely, inspired by [40, 45], the control constraint Γ is a linear subspace of $\mathbb{R}^m$. Let the columns of O be bases for Γ, and then the orthogonal projectors onto Γ is $\mathbf{P}_\Gamma := P_\Gamma = O(O^\top O)^{-1}O^\top$, where P_Γ is the related projector matrix and it easy to check that it satisfies $P_\Gamma^2 = P_\Gamma = P_\Gamma^\top$. Next we here assume P_Γ and R are commutative, which will be in force throughout this section. Then we have the following main conclusion in this section about the feedback representation optimal control strategy.

Theorem 4.2. *Let Assumptions (H1) and (H3) hold. Suppose $P_{(\cdot,j)} \equiv 0$, $j \in \mathcal{M}$ and the projection operator $\mathbf{P}_\Gamma$ is a linear operator, then the optimal control $\bar{u}.$ of Problem (LQ) has the following feedback representation*

$$\bar{u}_t = P^R_{(t,\alpha_t)}\big\{[H^\top_{(t,\alpha_t)} - B^\top_{(t,\alpha_t)}\Delta_{(t,\alpha_t)}]\hat{\varphi}_t - [B^\top_{(t,\alpha_t)}\Pi_{(t,\alpha_t)} + S^\top_{(t,\alpha_t)}]\hat{\bar{x}}_t - B^\top_{(t,\alpha_t)}\pi_t - r_{(t,\alpha_t)}\big\},$$

where $P^R_{(t,\alpha_t)} = P_\Gamma R^{-1}_{(t,\alpha_t)} P_\Gamma$ *and* $\hat{\varpi}_\cdot = \mathbb{E}[\varpi_\cdot|\mathcal{F}^{\bar{Y}}_\cdot]$ *with* $\varpi = \bar{x}, \varphi$*. Here* $\Pi_{(\cdot,j)}$*,* $\Delta_{(\cdot,j)}$ *and* $\pi_\cdot$ *are respectively given by*

$$\begin{cases} \dot{\Pi}_{(t,j)} + A^\top_{(t,j)}\Pi_{(t,j)} + \Pi_{(t,j)}A_{(t,j)} - \big[\Pi_{(t,j)}B_{(t,j)} + S_{(t,j)}\big] \\ \qquad \times P^R_{(t,\alpha_t)}\big[\Pi_{(t,j)}B_{(t,j)} + S_{(t,j)}\big]^\top + Q_{(t,j)} + \sum_{k=1}^m q_{jk}\Pi_{(t,k)} = 0, \\ \Pi_{(T,j)} = L_{(T,j)}, \quad j \in \mathcal{M}, \end{cases} \tag{4.4}$$

$$\begin{cases} \dot{\Delta}_{(t,j)} + A^\top_{(t,j)}\Delta_{(t,j)} + \Delta_{(t,j)}D_{(t,j)} - \big[\Pi_{(t,j)}B_{(t,j)} + S_{(t,j)}\big] \\ \qquad \times P^R_{(t,\alpha_t)}\big[B^\top_{(t,j)}\Delta_{(t,j)} - H^\top_{(t,j)}\big] - C^\top_{(t,j)} + \sum_{k=1}^m q_{jk}\Delta_{(t,k)} = 0, \\ \Delta_{(T,j)} = M_{(T,j)}, \quad j \in \mathcal{M}, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \mathrm{d}\pi_t = -\big\{A^\top_{(t,\alpha_t)}\pi_t + \Pi_{(t,\alpha_t)}b_{(t,\alpha_t)} + \Delta_{(t,\alpha_t)}p_{(t,\alpha_t)} - \big[\Pi_{(t,\alpha_t)}B_{(t,\alpha_t)} + S_{(t,\alpha_t)}\big] \\ \qquad \times P^R_{(t,\alpha_t)}\big[B^\top_{(t,\alpha_t)}\pi_t + r_{(t,\alpha_t)}\big] + q_{(t,\alpha_t)}\big\}\mathrm{d}t + \sum_{j,k=1}^m \delta^{jk}_t \mathrm{d}\widetilde{V}^{jk}_t, \\ \pi_T = l_{(T,\alpha_T)}. \end{cases} \tag{4.6}$$

Proof. Firstly, suppose we have the following relation

$$\xi_t = \Pi_{(t,\alpha_t)}\bar{x}_t + \Delta_{(t,\alpha_t)}\varphi_t + \pi_t, \tag{4.7}$$

where $\Pi_{(\cdot,\cdot)} : [0,T]\times\mathcal{M} \to \mathbb{S}^n$ and $\Delta_{(\cdot,\cdot)} : [0,T]\times\mathcal{M} \to \mathbb{R}^{n\times n}$ are absolutely continuous with terminal condition $\Pi_{(T,j)} = L_{(T,j)}$ and $\Delta_{(T,j)} = M_{(T,j)}$, $j \in \mathcal{M}$, respectively. $\pi_\cdot$ satisfies the following regime-switching BSDE

$$\begin{cases} \mathrm{d}\pi_t = \varrho_t \mathrm{d}t + \varsigma_t \mathrm{d}W_t + \bar{\varsigma}_t \mathrm{d}\overline{W}_t + \sum_{i,j=1}^m \delta^{ij}_t \mathrm{d}\widetilde{V}^{ij}_t, \\ \pi_T = l_{(T,\alpha_T)}, \end{cases}$$

where $\varrho_\cdot$ is some $\mathcal{F}_t$-progressively measurable process to be determined.

Secondly, applying Itô's formula to (4.7), we have

$$\begin{aligned} \mathrm{d}\xi_t = \{&\dot{\Pi}_{(t,\alpha_t)}\bar{x}_t + \Pi_{(t,\alpha_t)}[A_{(t,\alpha_t)}\bar{x}_t + B_{(t,\alpha_t)}\bar{u}_t + b_{(t,\alpha_t)}] + \dot{\Delta}_{(t,\alpha_t)}\varphi_t + \Delta_{(t,\alpha_t)}[-D^\top_{(t,\alpha_t)}\varphi_t + P_{(t,\alpha_t)}\bar{y}_t \\ &+ p_{(t,\alpha_t)}] + \sum_{j=1}^m q_{\alpha_{t-}j}\Pi_{(t,j)}\bar{x}_t + \sum_{j=1}^m q_{\alpha_{t-}j}\Delta_{(t,j)}\varphi_t + \varrho_t\}\mathrm{d}t + [\Pi_{(t,\alpha_t)}\sigma_{(t,\alpha_t)} - \Delta_{(t,\alpha_t)}F^\top_{(t,\alpha_t)}\varphi_t + \varsigma_t]\mathrm{d}W_t \\ &+ [\Pi_{(t,\alpha_t)}\tilde{\sigma}_{(t,\alpha_t)} - \Delta_{(t,\alpha_t)}G^\top_{(t,\alpha_t)}\varphi_t + \bar{\varsigma}_t]\mathrm{d}\overline{W}_t + \sum_{i,j=1}^m [\delta^{ij}_t + \Pi_{(i,j)}\bar{x}_t + \Delta_{(i,j)}\varphi_t]\mathrm{d}\widetilde{V}^{ij}_t. \end{aligned} \tag{4.8}$$

Comparing (4.8) with (3.1), it yields, $\theta_t = \Pi_{(t,\alpha_t)}\sigma_{(t,\alpha_t)} - \Delta_{(t,\alpha_t)}F^\top_{(t,\alpha_t)}\varphi_t + \varsigma_t$, $\vartheta_t = \Pi_{(t,\alpha_t)}\tilde{\sigma}_{(t,\alpha_t)} - \Delta_{(t,\alpha_t)}G^\top_{(t,\alpha_t)}\varphi_t + \bar{\varsigma}_t$ and $\eta^{ij}_t = \Pi_{(i,j)}\bar{x}_t + \Delta_{(i,j)}\varphi_t + \delta^{ij}_t$. Taking $\mathbb{E}[\cdot|\mathcal{F}^{\bar{Y}}_\cdot]$ both on the drift terms of (4.8) with

(3.1), we obtain

$$
\begin{aligned}
0 = &\Big\{\dot{\Pi}_{(t,\alpha_t)} + A^\top_{(t,\alpha_t)}\Pi_{(t,\alpha_t)} + \Pi_{(t,\alpha_t)}A_{(t,\alpha_t)} - \big[\Pi_{(t,\alpha_t)}B_{(t,\alpha_t)} + S_{(t,\alpha_t)}\big]P^R_{(t,\alpha_t)}\big[\Pi_{(t,\alpha_t)}B_{(t,\alpha_t)} + S_{(t,\alpha_t)}\big]^\top \\
&+ Q_{(t,\alpha_t)} + \sum_{j=1}^m q_{\alpha_t-j}\Pi_{(t,j)}\Big\}\hat{\hat{x}}_t + \Big\{\dot{\Delta}_{(t,\alpha_t)} + A^\top_{(t,\alpha_t)}\Delta_{(t,\alpha_t)} + \Delta_{(t,\alpha_t)}D_{(t,\alpha_t)} - C^\top_{(t,\alpha_t)} + \sum_{j=1}^m q_{\alpha_t-j}\Delta_{(t,j)} \\
&- \big[\Pi_{(t,\alpha_t)}B_{(t,\alpha_t)} + S_{(t,\alpha_t)}\big]P^R_{(t,\alpha_t)}\big[B^\top_{(t,\alpha_t)}\Delta_{(t,\alpha_t)} - H^\top_{(t,\alpha_t)}\big]\Big\}\hat{\varphi}_t + \varrho_t + \big\{A^\top_{(t,\alpha_t)}\pi_t + \Pi_{(t,\alpha_t)}b_{(t,\alpha_t)} \\
&+ \Delta_{(t,\alpha_t)}p_{(t,\alpha_t)} - \big[\Pi_{(t,\alpha_t)}B_{(t,\alpha_t)} + S_{(t,\alpha_t)}\big]P^R_{(t,\alpha_t)}\big[B^\top_{(t,\alpha_t)}\pi_t + r_{(t,\alpha_t)}\big] + q_{(t,\alpha_t)}\big\}.
\end{aligned}
$$

Thus we obtain the Riccati equation (4.4), ordinary differential equation (ODE) (4.5) and the following regime-switching BSDE,

$$
\begin{cases}
\mathrm{d}\pi_t = -\big\{A^\top_{(t,\alpha_t)}\pi_t + \Pi_{(t,\alpha_t)}b_{(t,\alpha_t)} + \Delta_{(t,\alpha_t)}p_{(t,\alpha_t)} - \big[\Pi_{(t,\alpha_t)}B_{(t,\alpha_t)} + S_{(t,\alpha_t)}\big] \\
\qquad\qquad \times P^R_{(t,\alpha_t)}\big[B^\top_{(t,\alpha_t)}\pi_t + r_{(t,\alpha_t)}\big] + q_{(t,\alpha_t)}\big\}\mathrm{d}t + \varsigma_t\mathrm{d}W_t + \bar{\varsigma}_t\mathrm{d}\overline{W}_t + \displaystyle\sum_{i,j=1}^m \delta^{ij}_t\mathrm{d}\widetilde{V}^{ij}_t, \\
\pi_T = l_{(T,\alpha_T)}.
\end{cases}
$$

Since the terminal condition $l_{(T,j)}$, $j \in \mathcal{M}$ and all coefficients are $\mathbb{F}^\alpha$ adapted, $(\pi_\cdot, 0, 0, \delta_\cdot)$ is the unique solution to above regime-switching BSDE, where $(\pi_\cdot, \delta_\cdot)$ is given by regime-switching BSDE (4.6). □

Remark 4.3. Let Assumptions (H1) and (H3) hold. According to the property of projection operator, by virtue of [17], Theorem 6.3, we can know that the Riccati equation (4.4) admits a unique solution $\Pi_{(\cdot,\cdot)} \in C([0,T]\times\mathcal{M};\mathbb{S}^n)$. Once $\Pi_{(\cdot,\cdot)}$ is given, then the well-posedness of ODE (4.5) and regime-switching BSDE (4.6) is obvious.

5. Financial application

In this section, we would like to solve an asset-liability problem, which put forward at the beginning of this paper. Firstly, we give the following assumption.

Assumption (H4) Let $\tilde{f}(t,x,y,z,k,\zeta,u,\alpha) = -\tilde{D}_{(t,\alpha)}y - \tilde{H}_{(t,\alpha)}u$. In addition, the coefficients $\tilde{D}_{(\cdot,j)} \in L^\infty(0,T;\mathbb{R}^n)$, $\tilde{H}_{(\cdot,j)} \in L^\infty(0,T;\mathbb{R}^m)$.

Then the regime-switching BSDE (1.4) can be rewritten as follows,

$$
\begin{cases}
\mathrm{d}y_t = \big\{\tilde{D}_{(t,\alpha)}y_t + \tilde{H}_{(t,\alpha)}u_t\big\}\mathrm{d}t + z_t\mathrm{d}W_t + k_t\mathrm{d}\overline{W}_t + \displaystyle\sum_{i,j=1}^m \zeta^{ij}_t\mathrm{d}\widetilde{V}^{ij}_t, \\
y_T = x_T.
\end{cases}
$$

By simple calculation, we obtain

$$
\mathbb{E}\big[y_t\big] = \mathbb{E}\bigg[x_T\mathrm{e}^{-\int_t^T \tilde{D}_{(s,\alpha_s)}\mathrm{d}s} + \int_t^T \tilde{H}_{(s,\alpha_s)}\mathrm{e}^{\int_t^s \tilde{B}_{(r,\alpha_r)}\mathrm{d}r}u_s\mathrm{d}s\bigg], \qquad t\in[0,T].
$$

Assume that the decision-makers can only inject funds, that is, the control strategy can only take values in $\Gamma = \mathbb{R}^m_+$. Then the Problem (RU) in Section 1 is equivalent to the following problem,

Problem (LQU) Find an admissible control strategy $\bar{u}. \in \mathcal{U}_{ad}$ such that

$$\bar{J}(\bar{u}.) = \inf_{u \in \mathcal{U}_{ad}} \bar{J}(u.) = \frac{1}{2}\mathbb{E}\left[\int_0^T \langle R_{(t,\alpha_t)} u_t + 2\tilde{r}_{(t,\alpha_t)}, u_t\rangle \mathrm{d}t + \langle L_{(T,\alpha_T)} x_T + 2\tilde{l}_{(T,\alpha_T)}, x_T\rangle\right], \tag{5.1}$$

subject to (1.1) and (1.2) with

$$\tilde{r}_{(t,\alpha_t)} = -R_{(t,\alpha_t)} a_{(t,\alpha_t)} + n\tilde{H}_{(t,\alpha_t)} \mathrm{e}^{\int_0^t \tilde{B}_{(s,\alpha_s)}\mathrm{d}s}, \qquad \text{and} \qquad \tilde{l}_{(T,\alpha_T)} = -L_{(T,\alpha_T)} \bar{l}_{(T,\alpha_T)} + n\mathrm{e}^{-\int_0^T \tilde{D}_{(t,\alpha_t)}\mathrm{d}t}.$$

By virtue of Corollary 4.1, we can know that if $\bar{u}.$ is an optimal control strategy, then the following regime-switching FBSDE,

$$\begin{cases} \mathrm{d}\bar{x}_t = \left\{r_{(t,\alpha_t)}\bar{x}_t + B_{(t,\alpha_t)}\bar{u}_t - b_{(t,\alpha_t)}\right\}\mathrm{d}t + \sigma_{(t,\alpha_t)}\mathrm{d}W_t + \bar{\sigma}_{(t,\alpha_t)}\mathrm{d}\overline{W}_t, \\ \mathrm{d}\xi_t = -r^\top_{(t,\alpha_t)}\xi_t\mathrm{d}t + \theta_t\mathrm{d}W_t + \vartheta_t\mathrm{d}\overline{W}_t + \sum_{i,j=1}^m \eta_t^{ij}\mathrm{d}\widetilde{V}_t^{ij}, \\ \bar{x}_0 = x_0, \qquad \alpha_0 = i, \qquad \xi_T = L_{(T,\alpha_T)}\bar{x}_T + \tilde{l}_{(T,\alpha_T)}. \end{cases} \tag{5.2}$$

admits a unique solution $(\bar{x}., \bar{\xi}., \theta., \vartheta., \eta.) \in \mathcal{M}_{\mathbb{F}}(0,T)$ such that

$$\bar{u}_t = \mathbf{P}_\Gamma\left[-R^{-1}_{(t,\alpha_t)}\left(B^\top_{(t,\alpha_t)}\hat{\xi}_t + \tilde{r}_{(t,\alpha_t)}\right)\right] = \max\left\{0, -R^{-1}_{(t,\alpha_t)}\left(B^\top_{(t,\alpha_t)}\hat{\xi}_t + \tilde{r}_{(t,\alpha_t)}\right)\right\}. \tag{5.3}$$

In order to more clearly show the impact of market state changes and interest rate fluctuation on the optimal control strategy $\bar{u}.$ with constraints, we give the numerical analysis. We use bull market and bear market to describe market quotation, and they correspond to two states of Markov chain $\alpha.$, respectively. Suppose that the Markov chain $\alpha.$ takes values in $\mathcal{M} = \{1,2\}$ with following generator

$$Q = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Here, we assume that the market is bear market when $\alpha. = 1$ and it is bull market when $\alpha. = 2$. Let $\sigma_{(t,1)} = 0.2$, $\sigma_{(t,2)} = 0.4$, $L_{(t,1)} = L_{(t,2)} = 0.6$, $B_{(t,1)} = B_{(t,2)} = 0.2$ $R_{(t,1)} = R_{(t,2)} = 0.4$, $b_{(t,1)} = b_{(t,2)} = 0.2$, $\tilde{r}_{(t,1)} = \tilde{r}_{(t,2)} = 0$ and $\tilde{l}_{(t,1)} = \tilde{l}_{(t,2)} = 0$. Then we show the simulation of the optimal strategy with constraints as the market situation and interest rate level change on Fig. 1.

In this figure, when $\alpha_t = 1$, $r_{(t,\alpha_t)} = 0.05$ and $\alpha_t = 2$, $r_{(t,\alpha_t)} = 0.15$, the optimal strategy is respectively given by blue dashed line and cyan dashed line; when $\alpha_t = 1$, $r_{(t,\alpha_t)} = 0.1$ and $\alpha_t = 2$, $r_{(t,\alpha_t)} = 0.25$, the optimal strategy is respectively given by black solid line and magenta solid line; when $\alpha_t = 1$, $r_{(t,\alpha_t)} = 0.2$ and $\alpha_t = 2$, $r_{(t,\alpha_t)} = 0.35$, the optimal strategy is respectively given by red dotted line and green dotted line. It is easy to know that with the increase of interest rate, decision-makers would like to inject more funds into the company to invest.

Next, we consider the case of unconstraint. By virtue of Theorem 4.2, we can obtain the feedback representation of the optimal strategy (5.3) without constraint as follows,

$$\bar{u}_t = -R^{-1}_{(t,\alpha_t)}\left[B^\top_{(t,\alpha_t)}\Pi_{(t,\alpha_t)}\hat{\bar{x}}_t + B^\top_{(t,\alpha_t)}\pi_t + \tilde{r}_{(t,\alpha_t)}\right], \tag{5.4}$$

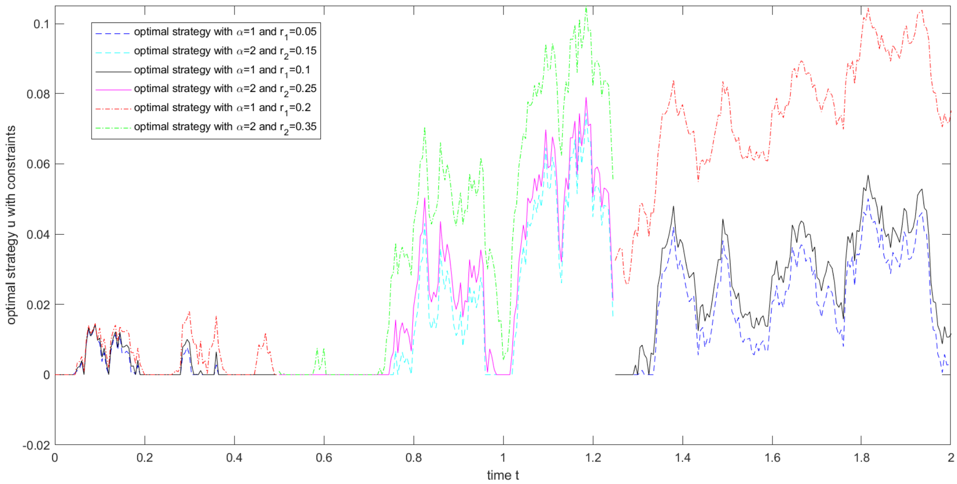

FIGURE 1. Constrained optimal strategy u with different interest rate.

where $\Pi_{(\cdot,j)}$, $j \in \mathcal{M}$ are the unique solutions to the following Riccati equations,

$$\begin{cases}\dot{\Pi}_{(t,j)}+r^\top_{(t,j)}\Pi_{(t,j)}+\Pi_{(t,j)}r_{(t,j)}-\Pi_{(t,j)}B_{(t,j)}R^{-1}_{(t,j)}B^\top_{(t,j)}\Pi^\top_{(t,j)}+\sum_{k=1}^m q_{jk}\Pi_{(t,k)}=0,\\ \Pi_{(T,j)}=L_{(T,j)},\end{cases}$$

The process $\pi.$ is the unique solution to the following regime-switching BSDE,

$$\begin{cases}\mathrm{d}\pi_t=-\big\{r^\top_{(t,\alpha_t)}\pi_t-\Pi_{(t,\alpha_t)}B_{(t,\alpha_t)}R^{-1}_{(t,\alpha_t)}\big(B^\top_{(t,\alpha_t)}\pi_t+\tilde{r}_{(t,\alpha_t)}\big)-\Pi_{(t,\alpha_t)}b_{(t,\alpha_t)}\big\}\mathrm{d}t+\sum_{j,k=1}^m\delta^{jk}_t\mathrm{d}\widetilde{V}^{jk}_t,\\ \pi_T=\tilde{l}_{(T,\alpha_T)},\end{cases}$$

and $\bar{x}.$ is the unique solution to the following regime-switching SDE,

$$\begin{cases}\mathrm{d}\bar{x}_t=\Big\{r_{(t,\alpha_t)}\bar{x}_t-B_{(t,\alpha_t)}R^{-1}_{(t,\alpha_t)}B^\top_{(t,\alpha_t)}\Pi_{(t,\alpha_t)}\hat{\bar{x}}_t-B_{(t,\alpha_t)}R^{-1}_{(t,\alpha_t)}B^\top_{(t,\alpha_t)}\pi_t\\ \qquad\quad -B_{(t,\alpha_t)}R^{-1}_{(t,\alpha_t)}\tilde{r}_{(t,\alpha_t)}-b_{(t,\alpha_t)}\Big\}\mathrm{d}t+\sigma_{(t,\alpha_t)}\mathrm{d}W_t+\bar{\sigma}_{(t,\alpha_t)}\mathrm{d}\overline{W}_t,,\\ \bar{x}_0=x_0,\qquad \alpha_0=i,\end{cases} \tag{5.5}$$

Applying Itô's formula to $(\bar{x}_t)^2$ with B-D-G inequality, we obtain $\mathbb{E}[\sup_{0\le t\le T}(\bar{x}_t)^2]<+\infty$. In addition, (5.4) is adapted to $\mathbb{F}^Y$ and to $\mathbb{F}^{Y^0}$. Obviously, the optimal control given by (5.4) is an admissible control, and thus regime-switching FBSDE (5.2) admits a unique solution $(\bar{x}.,\bar{\xi}.,\theta.,\vartheta.,\eta.)\in\mathcal{M}_{\mathbb{F}}(0,T)$. And Theorem 3.2 implies that the control strategy (5.4) is the unique optimal control strategy of Problem (LQU). Next we also show the simulation of the optimal strategy without constraints as the market situation and interest rate level change on Fig. 2.

From Fig. 2, we can see that the impact of interest rate change on the unconstrained optimal strategy $\bar{u}.$ is similar to the constrained case. And we can see more clearly that when the state of Markov chain $\alpha.$ changes from 1 to 2 or from 2 to 1, that is, when the market switches between bull market and bear market, the optimal

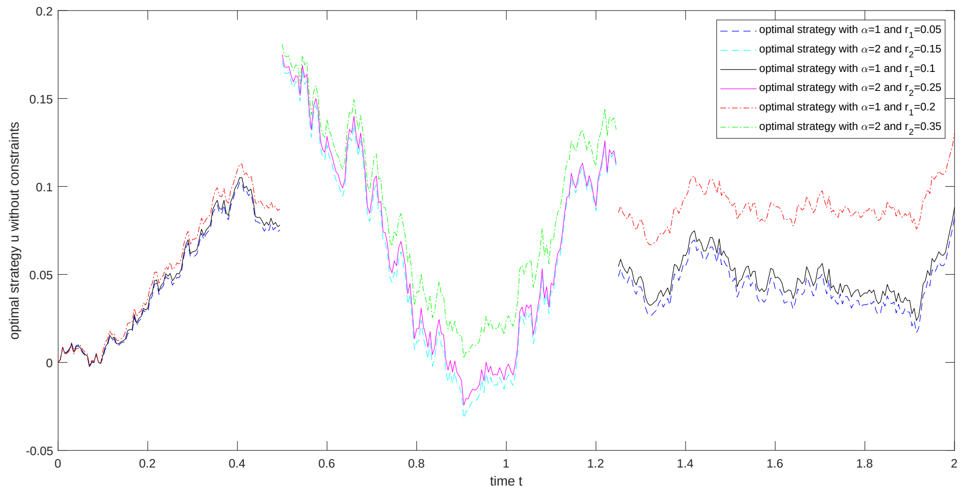

FIGURE 2. Unconstrained optimal strategy u with different interest rate.

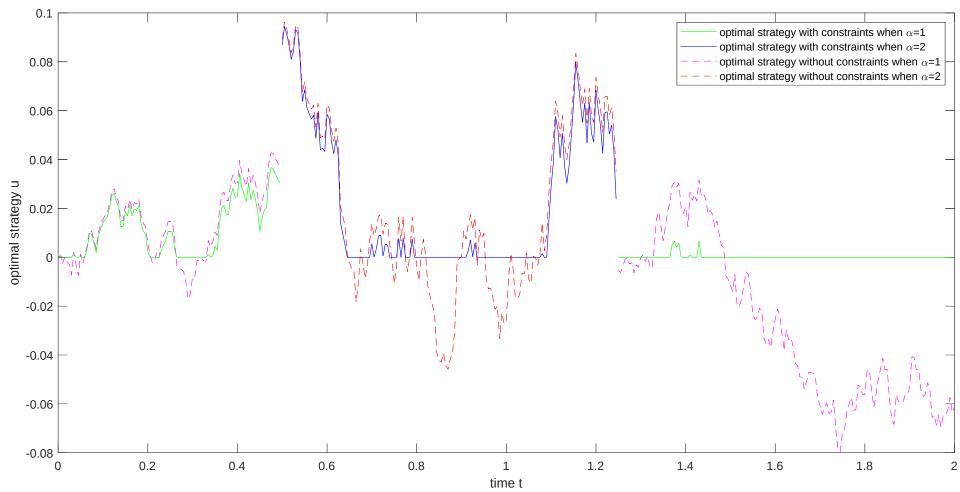

FIGURE 3. Optimal strategy with and without constraints.

strategy $\bar{u}$. will also change dramatically. Compared with the bear market, the expected return rate in the bull market is higher, thus the decision-makers are willing to inject more funds. But higher returns correspond to higher risks, thus the fluctuation of the optimal strategy in the bull market is more intense. In addition, it is worth pointing that the optimal strategy with constraints is not directly derived from the projection of the optimal strategy without constraints. In order to illustrate this point, we assume that $r_{(t,1)} = 0.1$, $r_{(t,2)} = 0.5$. Then we present the simulations of constrained and unconstrained optimal strategy on Fig. 3, respectively.

It can be seen from Fig. 3 that the non-negative constraints of control will affect the formulation of subsequent strategies. Thus, under the same conditions, the constrained optimal strategy cannot be directly obtained from the projection of unconstrained optimal strategy. Hence, studying the related problems with input constraints is awfully necessary and significant for us.

6. Conclusion

In this paper, a general LC optimal control problem for partially observed forward–backward stochastic control system with Markov chain and input constraints has been investigated. The observation is a controlled stochastic process, which is linear with respect to the state and control strategy. Applying backward separation approach to decompose the state and observation, then the circular dependency has been overcome. It is worth pointing out that the coefficients of this model are all dependent on Markov chain. For the case of LC model with input constraints, according to the SMP, we obtain the optimal control strategy by a stochastic Hamiltonian system and the property of projection operator. The well-posedness of the stochastic Hamiltonian system has been obtained by the method of continuation. For the LQ case, we obtain the feedback representation of optimal control strategy. As an application, an asset-liability management problem with input constraint has been studied. We can know that the optimal control strategy will change dramatically when the state of Markov chain changes. Moreover, three numerical examples has been presented, it can be seen from the figures that the research on optimal control problems with input constraint is very meaningful and necessary.

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Data availability statement

The research data associated with this article are included in the article.

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Appendix A. Unique solvability of regime-switching FBSDE with partial information

In this section, we would like to give the existence and uniqueness of a general FBSDE with regime-switching and filtering. For the brevity of the notation, we have a bit of abusive notation, and the symbols are unrelated to the previous sections. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be the filtered complete probability space. There are two independent d-dimensional Brownian motions W and $\overline{W}$ and a continuous-time stationary Markov chain α taking value in finite state space $\mathcal{M} = \{1, 2, \cdots, m\}$. We also assume that Brownian motion $(W, \overline{W})$ and Markov chain α are independent. Denote $\mathcal{F}$ (resp. $\mathcal{F}_t^{W,\overline{W}}$, $\mathcal{F}_t^{\alpha}$) as the natural filtration generated by $\{W_s, \overline{W}_s, \alpha_s; 0 \le s \le t\}$ (resp. $\{W_s, \overline{W}_s; 0 \le s \le t\}$, $\{\alpha_s; 0 \le s \le t\}$) satisfying usual condition. Let $\mathcal{G}_t := \mathcal{H}_t \vee \mathcal{F}_t^{\alpha}$, where $\mathcal{H}_t \subseteq \mathcal{F}_t^{W,\overline{W}}$ is the any sub-filtration. Consider the following fully coupled FBSDE with filtering and regime-switching as follows,

$$\begin{cases} \mathrm{d}X_t = b(t, \alpha_t, \Phi_t, \mathbb{E}[\Phi_t|\mathcal{G}_t])\mathrm{d}t + \sigma(t, \alpha_t, \Phi_t, \mathbb{E}[\Phi_t|\mathcal{G}_t])\mathrm{d}W_t + \bar{\sigma}(t, \alpha_t, \Phi_t, \mathbb{E}[\Phi_t|\mathcal{G}_t])\mathrm{d}\overline{W}_t, \\ \mathrm{d}Y_t = f(t, \alpha_t, \Phi_t, \mathbb{E}[\Phi_t|\mathcal{G}_t])\mathrm{d}t + Z_t\mathrm{d}W_t + K_t\mathrm{d}\overline{W}_t + \displaystyle\sum_{i,j=1}^{m} \zeta_t^{ij}\mathrm{d}\widetilde{V}_t^{ij}, \\ X_0 = g(Y_0), \qquad \alpha_0 = i, \qquad Y_T = h(\alpha_T, X_T, \mathbb{E}[X_T|\mathcal{G}_T]), \end{cases} \tag{A.1}$$

where $\Phi^\top = (X^\top, Y^\top, Z^\top, K^\top)$. Here $g(y)$ is deterministic for any $y \in \mathbb{R}^n$, $h(\alpha, x, \bar{x})$ is $\mathcal{F}_T$-measurable for any $x, \bar{x} \in \mathbb{R}^n$, $\Psi(\cdot, \alpha, \phi, \bar{\phi})$ is $\mathbb{F}$-progressive measurable for any $\phi \in \mathbb{R}^{n+n+n\times d+n\times d}$ with $\phi = (x, y, z, k)$ and $\Psi = b, \sigma, \bar{\sigma}, f$. g, h and Ψ are uniformly Lipschitz continuous with respect to y, $(x, \bar{x})$ and $(\phi, \bar{\phi})$, respectively. Moreover, $|g(0)|^2 < \infty$, $h(\alpha, 0, 0) \in L^2_{\mathcal{F}_T}(\mathbb{R}^n)$, $\psi_1(\cdot, \alpha, 0, 0) \in H^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$, $\psi_2(\cdot, \alpha, 0, 0) \in H^2_{\mathbb{F}}(0, T; \mathbb{R}^{n\times d})$ with $\psi_1 = b, f$ and $\psi_2 = \sigma, \tilde{\sigma}$. Now we introduce the following domination-monotonicity condition.

Assumption (S) There exist two constants $\mu \ge 0$, $\nu \ge 0$, some matrices $G \in \mathbb{R}^{m_1\times n}$, $H, \bar{H} \in \mathbb{R}^{m_2\times n}$, and some matrix-valued processes $A, \bar{A}, B, \bar{B}, C, \bar{C}, D, \bar{D} \in L^\infty(0, T; \mathbb{R}^{m_3\times n})$, (where $m_1, m_2, m_3 \in \mathbb{N}$ are given), such that the following conditions hold:

(i) One of the following cases holds. Case A: $\mu > 0$ and $\nu = 0$. Case B: $\mu = 0$ and $\nu > 0$.

(ii) (Domination conditions). For any $\phi_j, \bar{\phi}_j$, $j = 1, 2$, and almost all $(t, \omega) \in [0, T] \times \Omega$, with $\delta\phi = \phi_1 - \phi_2$ and $\delta\bar{\phi} = \bar{\phi}_1 - \bar{\phi}_2$, we have

$$|g(y_1) - g(y_2)| \le \frac{1}{\mu}|G\delta y|, \qquad |h(\alpha, x_1, \bar{x}_1) - h(\alpha, x_2, \bar{x}_2)| \le \frac{1}{\nu}\left|\begin{pmatrix} H(\delta x - \delta\bar{x}) \\ \bar{H}\delta\bar{x} \end{pmatrix}\right|,$$

$$|f(t, \alpha, \phi_1, \bar{\phi}_1) - f(t, \alpha, \phi_2, \bar{\phi}_2)| \le \frac{1}{\nu}\left|\begin{pmatrix} A_t(\delta x - \delta\bar{x}) \\ \bar{A}_t\delta\bar{x} \end{pmatrix}\right|,$$

$$|\psi_3(t, \alpha, \phi_1, \bar{\phi}_1) - \psi_3(t, \alpha_t, \phi_2, \bar{\phi}_2)| \le \frac{1}{\mu}\left|\begin{pmatrix} B_t(\delta y - \delta\bar{y}) + C_t(\delta z - \delta\bar{z}) + D_t(\delta k - \delta\bar{k}) \\ \bar{B}_t\delta\bar{y} + \bar{C}_t\delta z + \bar{D}_t\delta k \end{pmatrix}\right|,$$

where $\psi_3 = b, \sigma, \bar{\sigma}$. Here, with a bit of abuse of notations, when $\mu = 0$ (resp. $\nu = 0$), $\frac{1}{\mu}$ (resp. $\frac{1}{\nu}$) means $+\infty$. That is, if $\mu = 0$ or $\nu = 0$, then the corresponding domination constraints will vanish.
(iii) (Monotonicity condition) For any $y_1,\ y_2 \in \mathbb{R}^n$,

$$\langle g(y_1) - g(y_2), \delta y\rangle \leq -\mu |G\delta y|^2.$$

For any $X_1,\ X_2 \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, define $\delta h(T) := h(\alpha_T, X_1, \mathbb{E}[X_1|\mathcal{G}_T]) - h(\alpha_T, X_2, \mathbb{E}[X_2|\mathcal{G}_T])$,

$$\mathbb{E}\left[\left\langle \begin{pmatrix} \delta h(T) - \mathbb{E}[\delta h(T)|\mathcal{G}_T] \\ \mathbb{E}[\delta h(T)|\mathcal{G}_T] \end{pmatrix}, \begin{pmatrix} \delta X - \mathbb{E}[\delta X|\mathcal{G}_T] \\ \mathbb{E}[\delta X|\mathcal{G}_T] \end{pmatrix} \right\rangle\right] \geq \nu \mathbb{E}\left[\left| \begin{pmatrix} H(\delta X - \mathbb{E}[\delta X|\mathcal{G}_T]) \\ \bar{H}\mathbb{E}[\delta X|\mathcal{G}_T] \end{pmatrix} \right|^2\right].$$

For almost all $t \in [0,T]$ and any $\Phi_j^\top = (X_j^\top, Y_j^\top, Z_j^\top, K_j^\top) \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^{n+n+n\times d+n\times d})$, $j = 1,2$, define $\delta\Psi(t) := \Psi(t,\alpha_t,\Phi_1,\mathbb{E}[\Phi_1|\mathcal{G}_t]) - \Psi(t,\alpha_t,\Phi_2,\mathbb{E}[\Phi_2|\mathcal{G}_t])$,

$$\begin{aligned}\mathbb{E}\left[\left\langle \begin{pmatrix} \delta\Psi(t) - \mathbb{E}[\delta\Psi(t)|\mathcal{G}_t] \\ \mathbb{E}[\delta\Psi(t)|\mathcal{G}_t] \end{pmatrix}, \begin{pmatrix} \delta\Phi - \mathbb{E}[\delta\Phi|\mathcal{G}_t] \\ \mathbb{E}[\delta\Phi|\mathcal{G}_t] \end{pmatrix} \right\rangle\right] &\leq -\nu \mathbb{E}\left[\left| \begin{pmatrix} A_t(\delta X - \mathbb{E}[\delta X|\mathcal{G}_t]) \\ \bar{A}_t\mathbb{E}[\delta X|\mathcal{G}_t] \end{pmatrix} \right|^2\right] \\ &- \mu \mathbb{E}\left[\left| \begin{pmatrix} B_t(\delta Y - \mathbb{E}[\delta Y|\mathcal{G}_t]) + C_t(\delta Z - \mathbb{E}[\delta Z|\mathcal{G}_t]) + D_t(\delta K - \mathbb{E}[\delta K|\mathcal{G}_t]) \\ \bar{B}_t\mathbb{E}[\delta Y|\mathcal{G}_t] + \bar{C}_t\mathbb{E}[\delta Z|\mathcal{G}_t] + \bar{D}_t\mathbb{E}[\delta K|\mathcal{G}_t] \end{pmatrix} \right|^2\right].\end{aligned}$$

For any $\xi \in \mathbb{R}^n$, $\eta \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ and $\varphi_{1,\cdot},\ \varphi_{4,\cdot} \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $\varphi_{2,\cdot},\ \varphi_{3,\cdot} \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n\times d})$, we introduce a family of regime-switching FBSDEs parameterized by $\gamma \in [0,1]$ as follows,

$$\begin{cases} \mathrm{d}X_t^\gamma = [b^\gamma(t,\alpha_t,\Phi_t^\gamma,\mathbb{E}[\Phi_t^\gamma|\mathcal{G}_t]) + \varphi_{1,t}]\mathrm{d}t + [\sigma^\gamma(t,\alpha_t,\Phi_t^\gamma,\mathbb{E}[\Phi_t^\gamma|\mathcal{G}_t]) + \varphi_{2,t}]\mathrm{d}W_t \\ \qquad\qquad + [\bar{\sigma}^\gamma(t,\alpha_t,\Phi_t^\gamma,\mathbb{E}[\Phi_t^\gamma|\mathcal{G}_t]) + \varphi_{3,t}]\mathrm{d}\overline{W}_t, \\ \mathrm{d}Y_t^\gamma = [f^\gamma(t,\alpha_t,\Phi_t^\gamma,\mathbb{E}[\Phi_t^\gamma|\mathcal{G}_t]) + \varphi_{4,t}]\mathrm{d}t + Z_t^\gamma \mathrm{d}W_t + K_t^\gamma \mathrm{d}\overline{W}_t + \sum_{i,j=1}^m \zeta_t^{\gamma,ij}\mathrm{d}\tilde{V}_t^{ij}, \\ X_0^\gamma = g^\gamma(Y_0^\gamma) + \xi, \qquad \alpha_0 = i, \qquad Y_T^\gamma = h_T^\gamma(X_T^\gamma,\mathbb{E}[X_T^\gamma|\mathcal{G}_T]) + \eta, \end{cases} \tag{A.2}$$

where $\Phi^\gamma = [(X^\gamma)^\top,(Y^\gamma)^\top,(Z^\gamma)^\top,(K^\gamma)^\top]^\top$ and

$$\begin{aligned} b^\gamma(t,\alpha,\phi,\bar{\phi}) &= \gamma b(t,\alpha,\phi,\bar{\phi}) - (1-\gamma)\mu\{B_t^\top[B_t(y-\bar{y}) + C_t(z-\bar{z}) + D_t(k-\bar{k})] + \bar{B}_t(\bar{B}_t\bar{y} + \bar{C}_t\bar{z} + \bar{D}_t\bar{k})\}, \\ \sigma^\gamma(t,\alpha,\phi,\bar{\phi}) &= \gamma\sigma(t,\alpha,\phi,\bar{\phi}) - (1-\gamma)\mu\{C_t^\top[B_t(y-\bar{y}) + C_t(z-\bar{z}) + D_t(k-\bar{k})] + \bar{C}_t^\top[\bar{B}_t\bar{y} + \bar{C}_t\bar{z} + \bar{D}_t\bar{k}]\}, \\ \bar{\sigma}^\gamma(t,\alpha,\phi,\bar{\phi}) &= \gamma\bar{\sigma}(t,\alpha,\phi,\bar{\phi}) - (1-\gamma)\mu\{D_t^\top[B_t(y-\bar{y}) + C_t(z-\bar{z}) + D_t(k-\bar{k})] + \bar{D}_t^\top[\bar{B}_t\bar{y} + \bar{C}_t\bar{z} + \bar{D}_t\bar{k}]\}, \\ f^\gamma(t,\alpha,\phi,\bar{\phi}) &= \gamma f(t,\alpha,\phi,\bar{\phi}) - (1-\gamma)\nu\{A_t^\top A_t(x-\bar{x}) + \bar{A}_t^\top\bar{A}_t\bar{x}\}, \\ h^\gamma(t,\alpha,\phi,\bar{\phi}) &= \gamma h(t,\alpha,x,\bar{x}) + (1-\gamma)\nu\{H^\top H(x-\bar{x}) + \bar{H}^\top H\bar{x}\}, \\ g^\gamma(y) &= \gamma g(y) - (1-\gamma)\mu G^\top G y. \end{aligned}$$

It is clear that, when $\gamma = 1$ and $(\xi,\eta,\varphi) = (0,0,0)$ with $\varphi_\cdot^\top = (\varphi_{1,\cdot}^\top,\varphi_{2,\cdot}^\top,\varphi_{3,\cdot}^\top,\varphi_{4,\cdot}^\top)$, regime-switching FBSDE (A.2) coincides with FBSDE (A.1). Moreover, when $\gamma = 0$, regime-switching FBSDE (A.2) becomes

$$\begin{cases} \mathrm{d}X_t^0 = [b^0(t,\alpha_t,\Phi_t^0,\mathbb{E}[\Phi_t^0|\mathcal{G}_t]) + \varphi_{1,t}]\mathrm{d}t + [\sigma^0(t,\alpha_t,\Phi_t^0,\mathbb{E}[\Phi_t^0|\mathcal{G}_t]) + \varphi_{2,t}]\mathrm{d}W_t \\ \qquad\qquad + [\bar{\sigma}^0(t,\alpha_t,\Phi_t^0,\mathbb{E}[\Phi_t^0|\mathcal{G}_t]) + \varphi_{3,t}]\mathrm{d}\overline{W}_t, \\ \mathrm{d}Y_t^0 = [f^0(t,\alpha_t,\Phi_t^0,\mathbb{E}[\Phi_t^0|\mathcal{G}_t]) + \varphi_{4,t}]\mathrm{d}t + Z_t^0 \mathrm{d}W_t + K_t^0 \mathrm{d}\overline{W}_t + \sum_{i,j=1}^m \zeta_t^{0,ij}\mathrm{d}\tilde{V}_t^{ij}, \\ X_0^0 = g^0(Y_0^0) + \xi, \qquad \alpha_0 = i, \qquad Y_T^0 = h_T^0(X_T^0,\mathbb{E}[X_T^0|\mathcal{G}_T]) + \eta, \end{cases} \tag{A.3}$$

Then we consider the well-posedness of regime-switching FBSDE (A.3) under two cases of Assumption (S) (i). Under Case A (*i.e.* $\mu > 0$ and $\nu = 0$), regime-switching FBSDE (A.3) is in a decoupled form. In fact, we can first solve the BSDE to obtain $(Y^0_\cdot, Z^0_\cdot, K^0_\cdot, \zeta^0_\cdot)$, which can be substituted into the SDE to solve $X^0_\cdot$. Under Case B (*i.e.* $\mu = 0$ and $\nu > 0$), regime-switching FBSDE (A.3) is also in a decoupled form. Then we can first solve the SDE to obtain $X^0_\cdot$, which can be substituted into the BSDE to obtain $(Y^0_\cdot, Z^0_\cdot, K^0_\cdot, \zeta^0_\cdot)$.

Next we will show that there exists a positive constant β_0, such that, if construct some $\gamma_0 \in [0,1)$, regime-switching (A.2) is unique solvable, then the same conclusion holds for γ_0 being replaced by $\gamma_0 + \beta \leq 1$ with $\beta \in [0, \beta_0]$. Then we can increase the parameter γ to $\gamma = 1$. In order to achieve this goal, we first establish a basic estimates of the solution to regime-switching FBSDE (A.2).

Lemma A.1. *Under Assumption (S), let $(X^j_\cdot, Y^j_\cdot, Z^j_\cdot, K^j_\cdot, \zeta^j_\cdot) \in \mathcal{M}_{\mathbb{F}}(0,T)$, $j = 1,2$ be the solution to regime-switching FBSDE (A.2) corresponding with $(\xi^j, \eta^j, \varphi^j_\cdot)$, respectively. Then the following estimation holds,*

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{0\leq t\leq T}\big|\delta X_t\big|^2 + \sup_{0\leq t\leq T}\big|\delta Y_t\big|^2 + \int_0^T \big|\delta Z_t\big|^2 \mathrm{d}t + \int_0^T \big|\delta K_t\big|^2 \mathrm{d}t + \sum_{i,j=1}^m \int_0^T |\zeta_t^{ij}|^2 \lambda_t^{ij} \mathrm{d}t\bigg] \\
&\leq \tilde{C}\mathbb{E}\bigg[|\delta\xi|^2 + |\delta\eta|^2 + \bigg(\int_0^T \big|\delta\varphi_{1,t}\big|\mathrm{d}t\bigg)^2 + \int_0^T \big|\delta\varphi_{2,t}\big|^2\mathrm{d}t + \int_0^T \big|\delta\varphi_{3,t}\big|^2\mathrm{d}t + \bigg(\int_0^T \big|\delta\varphi_{4,t}\big|\mathrm{d}t\bigg)^2\bigg],
\end{aligned}$$

where $\tilde{C} > 0$ is a constant depending on the Lipschitz constant, μ, ν and the bounds of all coefficients.

Proof. **Step 1**, we give some estimations under two cases, respectively.

Under Case A (*i.e.* $\mu > 0$ and $\nu = 0$), by the estimate of SDEs and the domination conditions of $b, \sigma, \bar{\sigma}$ and g, we get

$$\begin{aligned}
\mathbb{E}\bigg[\sup_{0\leq t\leq T}\big|\delta X_t\big|^2\bigg] \leq \bar{C}\mathbb{E}\bigg[&|\delta\xi|^2 + \bigg(\int_0^T \big|\delta\varphi_{1,t}\big|\mathrm{d}t\bigg)^2 + \int_0^T \big|\delta\varphi_{2,t}\big|^2\mathrm{d}t + \int_0^T \big|\delta\varphi_{3,t}\big|^2\mathrm{d}t + \big|G\delta Y_0\big|^2 \\
&+ \int_0^T \left|\begin{pmatrix} B_t(\delta Y_t - \mathbb{E}[\delta Y_t|\mathcal{G}_t]) + C_t(\delta Z_t - \mathbb{E}[\delta Z_t|\mathcal{G}_t]) + D_t(\delta K_t - \mathbb{E}[\delta K_t|\mathcal{G}_t]) \\ \bar{B}_t\mathbb{E}[\delta Y_t|\mathcal{G}_t] + \bar{C}_t\mathbb{E}[\delta Z_t|\mathcal{G}_t] + \bar{D}_t\mathbb{E}[\delta K_t|\mathcal{G}_t]\end{pmatrix}\right|^2 \mathrm{d}t\bigg],
\end{aligned}$$

where $\bar{C}$ is a generic constant which can be varied from line to line. By the estimate of BSDEs and the Lipschitz conditions of f and h, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{0\leq t\leq T}\big|\delta Y_t\big|^2 + \int_0^T \big|\delta Z_t\big|^2\mathrm{d}t + \int_0^T \big|\delta K_t\big|^2\mathrm{d}t + \sum_{i,j=1}^m \int_0^T |\zeta_t^{ij}|^2\lambda_t^{ij}\mathrm{d}t\bigg] \\
&\leq \bar{C}\mathbb{E}\bigg[\sum_{0\leq t\leq T}\big|\delta X_t\big|^2 + |\delta\eta|^2 + \bigg(\int_0^T \big|\delta\varphi_{4,t}\big|\mathrm{d}t\bigg)^2\bigg].
\end{aligned}$$

Under Case B (*i.e.* $\mu = 0$ and $\nu > 0$), by the estimate of SDEs and the Lipschitz conditions of $b, \sigma, \bar{\sigma}$ and g, we obtain

$$\begin{aligned}
\mathbb{E}\bigg[\sup_{0\leq t\leq T}\big|\delta X_t\big|^2\bigg] \leq \bar{C}\mathbb{E}\bigg[&|\delta\xi|^2 + \bigg(\int_0^T \big|\delta\varphi_{1,t}\big|\mathrm{d}t\bigg)^2 + \int_0^T \big|\delta\varphi_{2,t}\big|^2\mathrm{d}t + \int_0^T \big|\delta\varphi_{3,t}\big|^2\mathrm{d}t \\
&+ \sup_{0\leq t\leq T}\big|\delta Y_t\big|^2 + \int_0^T \big|\delta Z_t\big|^2\mathrm{d}t + \int_0^T \big|\delta K_t\big|^2\mathrm{d}t\bigg],
\end{aligned}$$

By the estimate of BSDEs and the domination condition of f and h, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{0\leq t\leq T}\big|\delta Y_t\big|^2 + \int_0^T \big|\delta Z_t\big|^2\mathrm{d}t + \int_0^T \big|\delta K_t\big|^2\mathrm{d}t + \sum_{i,j=1}^m \int_0^T |\zeta_t^{ij}|^2\lambda_t^{ij}\mathrm{d}t\bigg] \\
&\leq \bar{C}\mathbb{E}\bigg[|\delta\eta|^2 + \bigg(\int_0^T |\delta\varphi_{4,t}|\mathrm{d}t\bigg)^2 + \left|\begin{pmatrix} H(\delta X_T - \mathbb{E}[\delta X_T|\mathcal{G}_T]) \\ \bar{H}\mathbb{E}[\delta X_T|\mathcal{G}_T]\end{pmatrix}\right|^2 + \int_0^T \left|\begin{pmatrix} A_t(\delta X - \mathbb{E}[\delta X|\mathcal{G}_t]) \\ \bar{A}_t\mathbb{E}[\delta X|\mathcal{G}_t]\end{pmatrix}\right|^2 \mathrm{d}t\bigg].
\end{aligned}$$

Step 2, applying Itô's formula to $\langle \delta X_t, \delta Y_t\rangle$, we have

$$\begin{aligned}\mathbb{E}\big[\langle \delta h^\gamma(T), \delta X_T\rangle + \langle \delta\eta, \delta X_T\rangle\big] = \mathbb{E}\bigg[\int_0^T \Big\{\langle \delta\varphi_t, \delta\Phi_t\rangle + \langle \delta f^\gamma(t), \delta X_t\rangle + \langle \delta b^\gamma(t), \delta Y_t\rangle + \langle \delta\sigma^\gamma(t), \delta Z_t\rangle \\ + \langle \delta\bar\sigma^\gamma(t), \delta K_t\rangle\Big\}\mathrm{d}t + \langle \delta g^\gamma(0), \delta Y_0\rangle + \langle \delta\xi, \delta Y_0\rangle\bigg],\end{aligned} \tag{A.4}$$

where $\delta\Psi^\gamma(t) = \Psi^\gamma(t,\alpha,\phi^1,\bar\phi^1) - \Psi^\gamma(t,\alpha,\phi^2,\bar\phi^2)$ with $\Psi = b,\sigma,\bar\sigma, f$. According to the monotonicity condition in Assumption (S), we have $\langle \delta g^\gamma(0), \delta Y_0\rangle \le -\mu|G\delta Y_0|^2$,

$$\begin{aligned}\mathbb{E}[\langle \delta h^\gamma(T), \delta X_T\rangle] &= \gamma\mathbb{E}\left[\left\langle \begin{pmatrix}\delta h(T) - \mathbb{E}[\delta h(T)|\mathcal{G}_T] \\ \mathbb{E}[\delta h(T)|\mathcal{G}_T]\end{pmatrix}, \begin{pmatrix}\delta X_T - \mathbb{E}[\delta X_T|\mathcal{G}_T] \\ \mathbb{E}[\delta X_T|\mathcal{G}_T]\end{pmatrix}\right\rangle\right] \\ &\quad + (1-\gamma)\nu\mathbb{E}\left[\left|\begin{pmatrix}H(\delta X_T - \mathbb{E}[\delta X_T|\mathcal{G}_T]) \\ \bar H\mathbb{E}[\delta X_T|\mathcal{G}_T]\end{pmatrix}\right|^2\right] \ge \nu\mathbb{E}\left[\left|\begin{pmatrix}H(\delta X_T - \mathbb{E}[\delta X_T|\mathcal{G}_T]) \\ \bar H\mathbb{E}[\delta X_T|\mathcal{G}_T]\end{pmatrix}\right|^2\right],\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[\langle \delta\Psi^\gamma(t), \delta\Phi_t\rangle] &\le -\nu\mathbb{E}\left[\left|\begin{pmatrix}A_t(\delta X_t - \mathbb{E}[\delta X_t|\mathcal{G}_t]) \\ \bar A_t\mathbb{E}[\delta X_t|\mathcal{G}_t]\end{pmatrix}\right|^2\right] \\ &\quad - \mu\mathbb{E}\left[\left|\begin{pmatrix}B_t(\delta Y_t - \mathbb{E}[\delta Y_t|\mathcal{G}_t]) + C_t(\delta Z_t - \mathbb{E}[\delta Z_t|\mathcal{G}_t]) + D_t(\delta K_t - \mathbb{E}[\delta K_t|\mathcal{G}_t]) \\ \bar B_t\mathbb{E}[\delta Y_t|\mathcal{G}_t] + \bar C_t\mathbb{E}[\delta Z_t|\mathcal{G}_t] + \bar D_t\mathbb{E}[\delta K_t|\mathcal{G}_t]\end{pmatrix}\right|^2\right].\end{aligned}$$

Then equation (A.4) can be reduced to

$$\begin{aligned}&\mathbb{E}\Bigg[\mu|G\delta Y_0|^2 + \int_0^T \Bigg\{\mu\left|\begin{pmatrix}B_t(\delta Y_t - \mathbb{E}[\delta Y_t|\mathcal{G}_t]) + C_t(\delta Z_t - \mathbb{E}[\delta Z_t|\mathcal{G}_t]) + D_t(\delta K_t - \mathbb{E}[\delta K_t|\mathcal{G}_t]) \\ \bar B_t\mathbb{E}[\delta Y_t|\mathcal{G}_t] + \bar C_t\mathbb{E}[\delta Z_t|\mathcal{G}_t] + \bar D_t\mathbb{E}[\delta K_t|\mathcal{G}_t]\end{pmatrix}\right|^2 \\ &\qquad + \nu\left|\begin{pmatrix}A_t(\delta X_t - \mathbb{E}[\delta X_t|\mathcal{G}_t]) \\ \bar A_t\mathbb{E}[\delta X_t|\mathcal{G}_t]\end{pmatrix}\right|^2\Bigg\}\mathrm{d}t + \nu\left|\begin{pmatrix}H(\delta X_T - \mathbb{E}[\delta X_T|\mathcal{G}_T]) \\ \bar H\mathbb{E}[\delta X_T|\mathcal{G}_T]\end{pmatrix}\right|^2\Bigg] \\ &\le \mathbb{E}\bigg[\langle \delta\xi, \delta Y_0\rangle - \langle \delta\eta, \delta X_T\rangle + \int_0^T \langle \delta\varphi_t, \delta\Phi_t\rangle\mathrm{d}t\bigg].\end{aligned} \tag{A.5}$$

Step 3, under Case A or Case B, by combining Step 1 and inequality (A.5), we have

$$\begin{aligned}&\mathbb{E}\bigg[\sup_{0\le t\le T}\big|\delta X_t\big|^2 + \sup_{0\le t\le T}\big|\delta Y_t\big|^2 + \int_0^T \big|\delta Z_t\big|^2\mathrm{d}t + \int_0^T \big|\delta K_t\big|^2\mathrm{d}t + \sum_{i,j=1}^m \int_0^T |\zeta_t^{ij}|^2\lambda_t^{ij}\mathrm{d}t\bigg] \\ &\le \bar C\mathbb{E}\bigg[|\delta\xi|^2 + |\delta\eta|^2 + \langle \delta\xi, \delta Y_0\rangle - \langle \delta\eta, \delta X_T\rangle + \int_0^T \langle \delta\varphi_t, \delta\Phi_t\rangle\mathrm{d}t \\ &\quad + \bigg(\int_0^T |\delta\varphi_{1,t}|\mathrm{d}t\bigg)^2 + \int_0^T |\delta\varphi_{2,t}|^2\mathrm{d}t + \int_0^T |\delta\varphi_{3,t}|^2\mathrm{d}t + \bigg(\int_0^T |\delta\varphi_{4,t}|\mathrm{d}t\bigg)^2\bigg] \\ &\le \mathbb{E}\bigg[\bar C\bigg\{|\delta\xi|^2 + |\delta\eta|^2 + \bigg(\int_0^T |\delta\varphi_{1,t}|\mathrm{d}t\bigg)^2 + \int_0^T |\delta\varphi_{2,t}|^2\mathrm{d}t + \int_0^T |\delta\varphi_{3,t}|^2\mathrm{d}t + \bigg(\int_0^T |\delta\varphi_{4,t}|\mathrm{d}t\bigg)^2\bigg\} \\ &\quad + \bar C^2\bigg\{|\delta\xi|^2 + |\delta\eta|^2 + \bigg(\int_0^T |\delta\varphi_{1,t}|\mathrm{d}t\bigg)^2 + \bigg(\int_0^T |\delta\varphi_{4,t}|\mathrm{d}t\bigg)^2\bigg\} + \frac{\bar C^2}{2}\bigg\{\int_0^T |\delta\varphi_{2,t}|^2\mathrm{d}t + \int_0^T |\delta\varphi_{3,t}|^2\mathrm{d}t\bigg\} \\ &\quad \frac{1}{2}\bigg\{\sup_{0\le t\le T}\big|\delta X_t\big|^2 + \sup_{0\le t\le T}\big|\delta Y_t\big|^2 + \int_0^T \big|\delta Z_t\big|^2\mathrm{d}t + \int_0^T \big|\delta K_t\big|^2\mathrm{d}t\bigg\}\bigg],\end{aligned}$$

which implies the desired results. □

By virtue of Lemma A.1, we prove the following continuation lemma.

Lemma A.2. *Let Assumption (S) hold. There exists a positive constant β_0, such that, if for some $\gamma_0 \in [0,1)$, regime-switching FBSDE (A.2) is unique solvable, then the same conclusion holds for β_0 being replaced by $\gamma_0+\beta \leq 1$ with $\beta \in (0,\beta_0]$.*

Proof. Let $\beta_0 > 0$, which will be determined below, and $\beta \in (0,\beta_0]$. For any $\Theta^\top = (X_\cdot^\top, Y_\cdot^\top, Z_\cdot^\top, K_\cdot^\top, \zeta_\cdot^\top) \in \mathcal{M}_{\mathbb{F}}(0,T)$, we consider the following regime-switching FBSDE,

$$\begin{cases} \mathrm{d}\widetilde{X}_t = [b^{\gamma_0}(t,\alpha_t,\widetilde{\Phi}_t,\mathbb{E}[\widetilde{\Phi}_t|\mathcal{G}_t]) + \tilde{\varphi}_{1,t}]\mathrm{d}t + [\sigma^{\gamma_0}(t,\alpha_t,\widetilde{\Phi}_t,\mathbb{E}[\widetilde{\Phi}_t|\mathcal{G}_t]) + \tilde{\varphi}_{2,t}]\mathrm{d}W_t \\ \qquad\qquad + [\bar{\sigma}^{\gamma_0}(t,\alpha_t,\widetilde{\Phi}_t,\mathbb{E}[\widetilde{\Phi}_t|\mathcal{G}_t]) + \tilde{\varphi}_{3,t}]\mathrm{d}\overline{W}_t, \\ \mathrm{d}\widetilde{Y}_t = [f^{\gamma_0}(t,\alpha_t,\widetilde{\Phi}_t,\mathbb{E}[\widetilde{\Phi}_t|\mathcal{G}_t]) + \tilde{\varphi}_{4,t}]\mathrm{d}t + \widetilde{Z}_t\mathrm{d}W_t + \widetilde{K}_t\mathrm{d}\overline{W}_t + \sum_{i,j=1}^m \tilde{\zeta}_t^{ij}\mathrm{d}\tilde{V}_t^{ij}, \\ \widetilde{X}_0 = g^{\gamma_0}(\widetilde{Y}_0) + \tilde{\xi}, \qquad \alpha_0 = i, \qquad \widetilde{Y}_T = h^{\gamma_0}(\alpha_T,\widetilde{X}_T,\mathbb{E}[\widetilde{X}_T|\mathcal{G}_T]) + \tilde{\eta}, \end{cases} \tag{A.6}$$

where

$$\begin{aligned} \tilde{\varphi}_{1,t} &= \varphi_{1,t} + \beta b(t,\alpha_t,\Phi_t,\mathbb{E}[\Phi_t|\mathcal{G}_t]) + \beta\mu\{B_t^\top[B_t(Y_t - \mathbb{E}[Y_t|\mathcal{G}_t]) + C_t(Z_t - \mathbb{E}[Z_t|\mathcal{G}_t]) \\ &\qquad + D_t(K_t - \mathbb{E}[K_t|\mathcal{G}_t])] + \bar{B}_t^\top(\bar{B}_t\mathbb{E}[Y_t|\mathcal{G}_t] + \bar{C}_t\mathbb{E}[Z_t|\mathcal{G}_t] + \bar{D}_t\mathbb{E}[K_t|\mathcal{G}_t])\}, \\ \tilde{\varphi}_{2,t} &= \varphi_{2,t} + \beta\sigma(t,\alpha_t,\Phi_t,\mathbb{E}[\Phi_t|\mathcal{G}_t]) + \beta\mu\{C_t^\top[B_t(Y_t - \mathbb{E}[Y_t|\mathcal{G}_t]) + C_t(Z_t - \mathbb{E}[Z_t|\mathcal{G}_t]) \\ &\qquad + D_t(K_t - \mathbb{E}[K_t|\mathcal{G}_t])] + \bar{C}_t^\top(\bar{B}_t\mathbb{E}[Y_t|\mathcal{G}_t] + \bar{C}_t\mathbb{E}[Z_t|\mathcal{G}_t] + \bar{D}_t\mathbb{E}[K_t|\mathcal{G}_t])\}, \\ \tilde{\varphi}_{3,t} &= \varphi_{3,t} + \beta\bar{\sigma}(t,\alpha_t,\Phi_t,\mathbb{E}[\Phi_t|\mathcal{G}_t]) + \beta\mu\{D_t^\top[B_t(Y_t - \mathbb{E}[Y_t|\mathcal{G}_t]) + C_t(Z_t - \mathbb{E}[Z_t|\mathcal{G}_t]) \\ &\qquad + D_t(K_t - \mathbb{E}[K_t|\mathcal{G}_t])] + \bar{D}_t^\top(\bar{B}_t\mathbb{E}[Y_t|\mathcal{G}_t] + \bar{C}_t\mathbb{E}[Z_t|\mathcal{G}_t] + \bar{D}_t\mathbb{E}[K_t|\mathcal{G}_t])\}, \\ \tilde{\varphi}_{4,t} &= \varphi_{4,t} + \beta f(t,\alpha_t,\Phi_t,\mathbb{E}[\Phi_t|\mathcal{G}_t]) + \beta\nu\{A_t^\top A_t(X_t - \mathbb{E}[X_t|\mathcal{G}_t]) + \bar{A}_T^\top\bar{A}_t\mathbb{E}[X_t|\mathcal{G}_t]\}, \\ \tilde{\eta} &= \eta + \beta h(\alpha_T,X_T,\mathbb{E}[X_T|\mathcal{G}_T]) - \beta\nu\{H^\top H(X_T - \mathbb{E}[X_T|\mathcal{G}_T]) + \bar{H}^\top\bar{H}\mathbb{E}[X_T|\mathcal{G}_T]\}, \\ \tilde{\xi} &= \xi + \beta g(Y_0) + \beta\mu G^\top G Y_0. \end{aligned}$$

We can easily check that $\tilde{\xi} \in \mathbb{R}^n$, $\tilde{\eta} \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ and $\tilde{\varphi}_{1,\cdot}, \tilde{\varphi}_{4,\cdot} \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $\tilde{\varphi}_{2,\cdot}, \tilde{\varphi}_{3,\cdot} \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n\times d})$. Then we know that regime-switching FBSDE (A.6) admits a unique solution $\theta^\top = (\widetilde{X}_\cdot^\top, \widetilde{Y}_\cdot^\top, \widetilde{Z}_\cdot^\top, \widetilde{K}_\cdot^\top, \tilde{\zeta}_\cdot^\top) \in \mathcal{M}_{\mathbb{F}}(0,T)$. Then we would like to prove that, if β is sufficiently small, the mapping defined by

$$\theta_\cdot = I_{\gamma_0+\beta}(\Theta_\cdot) : \mathcal{M}_{\mathbb{F}}(0,T) \to \mathcal{M}_{\mathbb{F}}(0,T)$$

is a contraction. Similar to the proof of Lemma A.1, we get $\|\delta\theta_\cdot\|^2 \leq \bar{C}\beta^2\|\delta\Theta_\cdot\|^2$, where $\bar{C}$ is a constant independent of γ_0 and β. Thus we take $\gamma_0 > 0$ such that $\bar{C}\gamma_0 \leq \frac{1}{2}$. Then for any $\beta \in (0,\beta_0]$, we have $\|\delta\theta_\cdot\|^2 \leq \frac{1}{2}\|\delta\Theta_\cdot\|^2$, which implies that the mapping $I_{\gamma_0+\beta}$ is a contraction. Therefore, it has a unique fixed point, which is the unique solution to regime-switching FBSDE (A.6). □

Finally, we repeat this process n times with $1 < n\beta_0 < 1+\beta_0$. Then the regime-switching FBSDE (A.1) admits a unique solution with $\gamma = 1$ and $(\xi,\eta,\varphi) = (0,0,0)$. We get the main results of this Appendix.

Theorem A.3. *Under Assumption (S), regime-switching FBSDE (A.1) admits a unique solution $(X_\cdot, Y_\cdot, Z_\cdot, K_\cdot, \zeta_\cdot) \in \mathcal{M}_{\mathbb{F}}(0,T)$.*