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CONTROLLABILITY OF SECOND ORDER DISCRETE-TIME DESCRIPTOR SYSTEMS

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Abstract. This paper is mainly devoted to the controllability of second order discrete-time descriptor systems. Characterizations for different controllability concepts are derived and feedback designs are investigated by transforming the system into an appropriate form and then making use of novel methods. It shows how classical rank conditions for first order systems can be generalized to second order systems. This work extends and complements the researches about controllability of high-order descriptor systems.

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1. Introduction

In this paper we study the second order descriptor system in discrete-time

$$\begin{aligned} &Mx(n+2)+Dx(n+1)+Kx(n)=Bu(n) \;\text{ for all } n \geqslant n_0,\\ &x(n_0)=x_0,\; x(n_0+1)=x_1, \end{aligned} \tag{1.1}$$

where the system coefficients are M, D, $K \in \mathbb{R}^{d,d}$, $B \in \mathbb{R}^{d,m}$, $C \in \mathbb{R}^{p,d}$, and d, m, $p \in \mathbb{N}$. Notice that the matrices M, D, K can be either singular or nonsingular. Here x_0, x_1 are real-valued vectors in $\mathbb{R}^d$, $u(n)$ is a real-valued vector in $\mathbb{R}^m$, and $x = \{x(n)\}_{n\geqslant n_0}$, $u = \{u(n)\}_{n\geqslant n_0}$ are vector sequences. As said before, the system can be regular or singular depending on the invertibility of M. While regular system is well-understood, in this paper we focus on the remaining case. Singular systems of the form (1.1) arise as mathematical models in many fields such as population dynamics, economics, [1–3]. They are also obtained as the discretization of some differential-algebraic equations (DAEs) or partial differential equations (PDEs), or from sampling in dynamical systems, see *e.g.* [4, 5]. Recently, solvability and stability of second order singular difference equations (SiDEs) have been investigated in [6–8]. However, controllability for these systems has not been fully characterized even though it has been well-studied for both continuous-time and discrete-time singular systems of first order [9–11].

There are a few references for the controllability of second order descriptor systems in continuous-time [12–14]. However, comparable results for discrete-time systems are still missing. More importantly, understanding

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of discrete-time systems is necessary to study the invariance of certain structural properties, in particular controllability, under discretization or sampling. It is the ultimate goal of this work, that we want to set up a comparable framework for the controllability of discrete-time systems. We emphasize, that we only keep the spirit of the research [12–14], which is an *algebraic approach*, but not merely translate the results there from the continuous-time to the discrete-time case. Compared to [12], substantial modifications have been made and the advantages of the proposed technique are discussed in Remarks 3.7 and 4.11.

It should be noted, that some results in this paper (in particular the ones in Sect. 3) also carry over to systems with complex-valued coefficients or systems of order higher than two. However, for notational convenience, and because that this is the most important case in practice, we restrict to real-valued systems of second order.

This paper is organized as follows. In Section 2, we review some basic concepts and lemmas, and we explore the connection between causality, strict causality and strangeness-free form for second order systems. In Section 3, we introduce two useful condensed forms for system (1.1), and we discuss how to achieve causal controllability by applying different kinds of feedbacks. We also show the benefit of using acceleration feedback for enhancing the causal controllability of the system. Section 4 utilizes the condensed forms to derive characterizations for additional notions of controllability for system (1.1). We discuss key differences between continuous-time and discrete-time systems, particularly how C-controllability does not necessarily imply Y-controllability for second-order systems. Additionally, we address the minimal time required to achieve a desired state, as detailed in Corollaries 4.3 and 4.13. We conclude with some remarks.

2. Preliminaries

First let us briefly recall some important concepts for a first order descriptor system

$$E\xi(n+1) - A\xi(n) = B_1 u(n) \quad \text{for all } n \geqslant n_0, \tag{2.1}$$

where $E,\ A \in \mathbb{R}^{\tilde{d},\tilde{d}}$, $B_1 \in \mathbb{R}^{\tilde{d},\tilde{m}}$ for some $\tilde{d}, \tilde{m} \in \mathbb{N}$. Within this section we use $\tilde{d}$ instead of d to denote the dimension of system (2.1), in order to avoid confusion later, where we want to reformulate a second order system as a first order one. Here, we notice that the matrix E may be rank deficient, and the matrix pair (E, A) is *regular*, *i.e.*, $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. It is well-known, that the regularity of the pair (E, A) is the necessary and sufficient condition for the existence and uniqueness of a solution to (2.1), see, *e.g.* [10]. Moreover, the regular pair (E, A) can be transformed to Kronecker–Weierstraß canonical form (see, *e.g.* [8]), *i.e.*, there exist nonsingular matrices U, V such that

$$UEV = \begin{bmatrix} I_{\tilde{d}_1} & 0 \\ 0 & N \end{bmatrix},\ UAV = \begin{bmatrix} J & 0 \\ 0 & I_{\tilde{d}_2} \end{bmatrix},\ UB_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \tag{2.2}$$

where N is a nilpotent matrix of nilpotency index ν, *i.e.*, $N^\nu = 0$ and $N^i \neq 0$ for $i = 1, \ldots, \nu - 1$. We call ν the index of the pair (E, A) (and also of system (2.1)), and write $\operatorname{ind}(E, A) = \nu$. In particular, when $N = 0$ we have $\operatorname{ind}(E, A) = 1$. Due to [3], $\tilde{d}_1$ (resp. $\tilde{d}_2$) is called the dimension of the dynamic part (resp. algebraic part), and $\tilde{d}_1 = \deg\det(\lambda E - A)$. We recall one basic result related to the row partition of system (2.1), see *e.g.* [3].

Lemma 2.1. *A regular system (2.1) is of index 1 if and only if there exists a nonsingular matrix $W \in \mathbb{R}^{\tilde{d},\tilde{d}}$ such that by multiplying this system with W (from the left) we obtain the new system of the form*

$$\begin{bmatrix} \mathbf{E}_1 \\ 0 \end{bmatrix} \xi(n+1) - \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \xi(n) = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} u(n) \quad \text{for all } n \geqslant 0, \tag{2.3}$$

where $\left[\begin{smallmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{smallmatrix}\right]$ is nonsingular.

System (2.3) is called the *strangeness-free formulation*, which can be achieved for an arbitrarily high index system (2.1) by applying an index reduction process. For more details, see [3], Chapter 3.

The explicit solution of (2.1) is of the form $\xi(n) = V \left[\begin{smallmatrix} \xi_1(n) \\ \xi_2(n) \end{smallmatrix}\right]$ with

$$\begin{aligned} \xi_1(n+1) &= J^{n-n_0+1}\, x(n_0) + \sum_{i=0}^{n-n_0} J^i\, B_{11}\, u(n-i) \\ \xi_2(n) &= -\sum_{i=0}^{\nu-1} N^i\, B_{12}\, u(n+i), \quad \text{for all } n \geqslant n_0. \end{aligned} \tag{2.4}$$

Clearly, an initial condition $\xi(n_0)$ could not be arbitrarily taken. In fact, for a given input $u = \{u(n)\}_{n \geqslant n_0}$, the set of all *consistent initial conditions (with respect to u)* is

$$\mathcal{S}_0 = \left\{ V \begin{bmatrix} \xi_1(n_0) \\ \xi_2(n_0) \end{bmatrix} \;\middle|\; \xi_1(n_0) \in \mathbb{R}^{\tilde{d}_1},\ \xi_2(n_0) = -\sum_{i=0}^{\nu-1} N^i\, B_{12}\, u(n_0+i) \right\}.$$

The *reachable set* $\mathcal{R}$ of (2.1) is the set of all vector $\xi(n)$ that can be reached from some input sequence $u = \{u(n)\}_{n \geqslant n_0}$ and some initial vector $\xi(n_0)$ consistent to u. Due to the variable transformation $\xi(n) = V \left[\begin{smallmatrix} \xi_1(n) \\ \xi_2(n) \end{smallmatrix}\right]$, we see that $\mathcal{R} = V\tilde{\mathcal{R}}$, where $\tilde{\mathcal{R}}$ is the reachable set of system (2.4). In fact, it is well-known (*e.g.* [15]) that

$$\tilde{\mathcal{R}} = \mathbb{R}^{\tilde{d}_1} \oplus \operatorname{Im}\mathcal{K}(N, B_{12}),$$

where $\mathcal{K}(N, B_{12}) := \left[B_{12},\ NB_{12},\ \ldots,\ N^{\nu-1}B_{12}\right]$. Here $\oplus$ denotes the direct sum of two vector spaces. In particular, the following corollary can be formulated.

Corollary 2.2. *Consider a first order descriptor system of the form (2.3), where $\left[\begin{smallmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{smallmatrix}\right]$ is nonsingular. Furthermore, assume that the matrix $\mathbf{B}_2$ has full row rank. Then the reachable subspace $\mathcal{R}$ is the whole space $\mathbb{R}^{\tilde{d}}$.*

Moreover, if $\operatorname{ind}(E, A) = 1$, by the Laurent series expansion about infinity of the resolvent matrix (see [16, 17]), we have

$$(\lambda E - A)^{-1} = z^{-1} \sum_{k=-1}^{\infty} \Phi_k \lambda^{-k},$$

where the sequence $\Phi = \{\Phi_k\}_{k \geqslant -1}$ is the fundamental matrix sequence. Using the formula of the fundamental solution matrix (see, *e.g.* [18]), we can compute

$$\Phi_{-1} = QG^{-1}, \Phi_0 = (I + QG^{-1}A)G^{-1}, \Phi_i = (\Phi_0 A)^i, \text{ for all } i \geqslant 1,$$

where Q is a projector onto kernel(E) with $\dim \operatorname{kernel}(E) = \tilde{d}_2$, $P = I_{\tilde{d}} - Q$ and $G = E - AQ$. Then (see [11]) we obtain that $\tilde{\mathcal{R}} = \mathbb{R}^{\tilde{d}_1} \oplus \operatorname{Im}(\Phi_{-1}B_1)$, and the reachable set from $\xi(0) = 0$ of system (2.4) is

$$\tilde{\mathcal{R}}(0) = \left(\sum_{i=0}^{\tilde{d}_1 - 1} \operatorname{Im}(\Phi_i B_1) \right) \oplus \operatorname{Im}(\Phi_{-1} B_1). \tag{2.5}$$

To follow [10], system (2.1) is called *causal* if the state $\xi(n)$ is uniquely determined by the initial condition $\xi(n_0)$ and inputs $u(i)$ with $i = n_0, n_0 + 1 \ldots, n$. It is easy to see that if $\operatorname{ind}(E, A) = 1$ then system (2.1) is causal.

Furthermore, system (2.1) is called *strictly causal* if for each $n \geqslant n_0 + 1$, $x(n)$ depends only on an input at past time, *i.e.*, $u(n_0)$,...,$u(n-1)$.

Definition 2.3. System (2.1) is called

(i) *completely controllable or C-controllable* if for any consistent initial condition $\xi_0 \in \mathbb{R}^{\tilde{d}}$ and any $\xi_f \in \mathbb{R}^{\tilde{d}}$ there exist a finite time n_f and an input u such that $\xi(n_f) = \xi_f$.
(ii) *controllable on a reachable set or R-controllable* if for any consistent initial condition $\xi_0 \in \mathbb{R}^{\tilde{d}}$ and any $\xi_f \in \mathcal{R}$ there exist a finite time n_f and an input u such that $\xi(n_f) = \xi_f$.
(iii) *Y-controllable* if there exists a feedback $u(k) = F\xi(k)$ such that the closed-loop system $E\xi(k+1) = (A + B_1F)\xi(k)$ is regular and of index 1.
(iv) *normalizable* if there exists a feedback $u(k) = F\xi(k+1)$ such that the closed-loop system $(E - B_1F)\xi(k+1) = A\xi(k)$ is an explicit difference equation, *i.e.*, $E - B_1F$ is nonsingular.

Remark 2.4. One may argue that using a feedback $u(k) = Fx(k+1)$, which depends on future state, is conceptional problematic and not feasible. This argument is true in general. Nevertheless, for strictly-causal systems, this feedback still makes sense.

For most classical control design aim, typically, one or more of the following rank conditions are required

$$\begin{aligned}
&\mathbf{C0}: \quad \operatorname{rank}\left[\alpha E - \beta A,\ B_1\right] = \tilde{d} \ \text{ for all } (\alpha, \beta) \in \mathbb{C}^2 \backslash \{(0,0)\}, \\
&\mathbf{C1}: \quad \operatorname{rank}\left[\lambda E - A,\ B_1\right] = \tilde{d} \ \text{ for all } \lambda \in \mathbb{C}, \\
&\mathbf{C2}: \quad \operatorname{rank}\left[E,\ AS_\infty(E),\ B_1\right] = \tilde{d}, \\
&\mathbf{C3}: \quad \operatorname{rank}\left[E,\ B_1\right] = \tilde{d},
\end{aligned} \tag{2.6}$$

where $S_\infty(E)$ is a matrix whose columns span an orthogonal basis of kernel(E). Furthermore, it should be noted that **C0** = **C1** + **C3**. From characterizations of controllability in [9–11] and by the Kronecker–Weierstraßcanonical form (2.2) we can deduce

Proposition 2.5. *Consider the first order descriptor system* (2.1)*, whose the matrix pair* (E, A) *is regular. Then* (2.1) *is*

(i) C-controllable if and only if ***C0*** *holds.*
(ii) R-controllable if and only if ***C1*** *holds.*
(iii) Y-controllable if and only if ***C2*** *holds.*
(iv) normalizable if and only if ***C3*** *holds.*

Remark 2.6. In the case of continuous-time, first order descriptor systems, the comparable version for R- (resp. Y-controllability) is the *behavioral controllability* (resp. impulse controllability). The characterization for these controllability concepts are the same for both continuous and discrete time systems, see *e.g.* [10, 19].

For the physical meanings of these controllability concepts and their properties, we refer the interested readers to classical textbooks [20–23].

Furthermore, one important issue in the control theory of discrete-time system is about the minimal amount of time to reach a desired state. We recall the following result, see [11], Theorem 3.9, Remark 1.

Proposition 2.7. *The regular system* (2.1) *is C-controllable if and only if the following rank condition is satisfied.*

$$\operatorname{rank}\left[\Phi_{-1}B_1 \quad \Phi_0 B_1 \quad \dots \quad \Phi_{\tilde{d}_1-1}B_1\right] = \tilde{d}. \tag{2.7}$$

Furthermore, an arbitrary state $z \in \mathbb{R}^{\tilde{d}}$ *can be reached from any consistent initial condition* $x(n_0)$ *in* $\tilde{d}_1 = \deg\det(\lambda E - A)$ *steps.*

To study control properties of second order descriptor systems, the classical approach is to reformulate (1.1) in the first order form (2.1). Nevertheless, some critical difficulties may arise, as demonstrated in the following two examples.

Example 2.8. There are at least four ways to reformulate system (1.1) as follows

$$\begin{aligned}
\textbf{companion form}: & \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} x(n{+}1) \\ x(n{+}2) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}\begin{bmatrix} x(n) \\ x(n{+}1) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(n), \\
\textbf{2nd form}: & \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}\begin{bmatrix} x(n{+}1) \\ x(n{+}2) \end{bmatrix} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} x(n) \\ x(n{+}1) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(n), \\
\textbf{3rd form}: & \begin{bmatrix} D & M \\ -M & 0 \end{bmatrix}\begin{bmatrix} x(n{+}1) \\ x(n{+}2) \end{bmatrix} = \begin{bmatrix} -K & 0 \\ 0 & -M \end{bmatrix}\begin{bmatrix} x(n) \\ x(n{+}1) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(n), \\
\textbf{4th form}: & \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} x(n{+}1) \\ x(n{+}2) \end{bmatrix} = \begin{bmatrix} 0 & -K \\ -K & -D \end{bmatrix}\begin{bmatrix} x(n) \\ x(n{+}1) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(n).
\end{aligned} \tag{2.8}$$

Each form above has its own advantage, especially in case that M, K, D have a symmetric or skew-symmetric structure. In this example, let us take the matrix coefficients

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.9}$$

Direct computation, for example [10], turns out that only in the fourth form, the index of the matrix pair (E, A) is three, while in the others, the index is four, which suggests a wrong prediction, that $x(n)$ depends also on $u(n+3)$, instead of only $u(n)$, $u(n+1)$, $u(n+2)$.

In control theory, classical design approaches usually require that the system is at least Y-controllable. Nevertheless, this is not always fulfilled as shown below.

Example 2.9. Consider the artificial descriptor system (1.1) with

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ K = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Clearly, this is in fact a first order system. We can directly check that this system is Y-controllable by verifying the rank condition **C2** in (2.6). Nevertheless, all the formulations in (2.8) are not.

For another input matrix $B = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, checking the rank condition **C1** yields that (1.1) is C-controllable, while all the formulations in (2.8) are not.

In view of all these difficulties, it is natural to seek for a suitable first order reformulation that is at least Y-controllable. This reformulation must also be beneficial to further study on other controllability properties of (1.1). This task will be done in the next section. Two auxiliary lemmata below are necessary for our analysis later.

Lemma 2.10. *([6], Lem. 4.1) Given some matrices $\breve{A}$, $\breve{B}$, $\breve{C}$ in $\mathbb{R}^{d,d}$ and $\breve{D}$ in $\mathbb{R}^{d,m}$. Then there exists an orthogonal matrix $\breve{U} \in \mathbb{R}^{d,d}$ such that*

$$\breve{U}\begin{bmatrix}\breve{A} & \breve{B} & \breve{C} & | & \breve{D}\end{bmatrix} = \left[\begin{array}{ccc|c} \breve{A}_1 & \breve{B}_1 & \breve{C}_1 & \breve{D}_1 \\ 0 & \breve{B}_2 & \breve{C}_2 & 0 \\ 0 & 0 & \breve{C}_3 & 0 \\ \hline 0 & \breve{B}_4 & \breve{C}_4 & \breve{D}_4 \\ 0 & 0 & \breve{C}_5 & \breve{D}_5 \\ 0 & 0 & 0 & 0 \end{array}\right], \tag{2.10}$$

where the matrices $\breve{A}_1$, $\breve{B}_2$, $\breve{B}_4$, $\breve{C}_3$, $\left[\begin{smallmatrix}\breve{D}_4 \\ \breve{D}_5\end{smallmatrix}\right]$ have full row rank.

Lemma 2.11. *Let $P = \left[\begin{smallmatrix}P_1 \\ P_2\end{smallmatrix}\right] \in \mathbb{R}^{\tilde{p},\tilde{d}}$, $Q = \left[\begin{smallmatrix}0 \\ Q_2\end{smallmatrix}\right] \in \mathbb{R}^{\tilde{q},\tilde{d}}$ be two matrices. Furthermore, assume that Q_2 has full row rank. Then there exist a matrix $F \in \mathbb{R}^{\tilde{d},\tilde{d}}$ such that $P + QF$ has full row rank if and only if P_1 also has full row rank.*

Proof. The necessary part is obtained directly from the observation that

$$P + QF = \begin{bmatrix}P_1 \\ P_2\end{bmatrix} + \begin{bmatrix}0 \\ Q_2\end{bmatrix} F = \begin{bmatrix}P_1 \\ P_2 + Q_2 F\end{bmatrix} = \begin{bmatrix}P_1 & 0 \\ P_2 & Q_2\end{bmatrix}\begin{bmatrix}I \\ F\end{bmatrix}.$$

For the sufficient part, including the construction of F, the readers can see [6], Lemma 4.2 (ii). □

2.1. On causality and strangeness-freeness

Now let us introduce the notions for solvability of the second order system (1.1).

Definition 2.12. (i) For a given input sequence $u = \{u(n)\}_{n \geqslant n_0}$, initial conditions $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$ is called *consistent w.r.t u* if the corresponding IVP (1.1) has at least one solution.
(ii) System (1.1) is called *regular* if for any input sequence $u = \{u(n)\}_{n \geqslant n_0}$ and any consistent initial conditions (x_0, x_1) w.r.t u, the corresponding IVP (1.1) has a unique solution.

A regular system (1.1) is called
(iii) *causal* if for each $n \geqslant n_0 + 2$, $x(n)$ does not depend on an input u at future time, *i.e.*, $u(n+1)$, $u(n+2)$,.... but depends only on present and past time, *i.e.*, $u(n)$, $u(n-1)$, ..., $u(n_0)$ and on initial conditions $x(n_0)$, $x(n_0+1)$.
(iv) *strictly causal* if for each $n \geqslant n_0 + 2$, $x(n)$ depends only on an input at past time, *i.e.*, $u(n-1)$, ..., $u(n_0)$ and on initial conditions $x(n_0)$, $x(n_0+1)$.

Definition 2.13. ([7]) System (1.1) is called *strangeness-free* if there exists a nonsingular matrix $P \in \mathbb{R}^{d,d}$ such that by multiplying system (1.1) with P from the left, we obtain a new system of the form

$$\begin{bmatrix}\hat{M}_1 \\ 0 \\ 0 \\ 0\end{bmatrix} x(n+2) + \begin{bmatrix}\hat{D}_1 \\ \hat{D}_2 \\ 0 \\ 0\end{bmatrix} x(n+1) + \begin{bmatrix}\hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \\ 0\end{bmatrix} x(n) = \begin{bmatrix}\hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \\ \hat{B}_4\end{bmatrix} u(n) \quad \text{for all } n \geqslant n_0, \tag{2.11}$$

where the matrix $\mathcal{M} := \begin{bmatrix}\hat{M}_1^T & \hat{D}_2^T & \hat{K}_3^T\end{bmatrix}^T$ has full row rank. Notice that, in the special case where $M = 0$, we obtain exactly the well-known concept *strangeness-free* for first order systems, see *e.g.* [3].

Let us denote by $\mathcal{P}(\lambda)$ the characteristic polynomial of system (1.1), *i.e.* $\mathcal{P}(\lambda) := \det(\lambda^2 M + \lambda D + K)$. The matrix triple (M, D, K) is called *regular* if $\mathcal{P}(\lambda)$ is not identically zero.

Remark 2.14. (i) Assume that system (1.1) is already in the strangeness-free from (2.11) then the regularity of the matrix triple (M, D, K) implies that the last row of system (2.11) is not present. Consequently, we obtained the consistency condition for $x(n_0)$ and $x(n_0 + 1)$ as follows:

$$\begin{cases} \hat{D}_2 x(n_0+1) + & \hat{K}_2 x(n_0) = \hat{B}_2 u(n_0), \\ & \hat{K}_3 x(n_0) = \hat{B}_3 u(n_0). \end{cases} \tag{2.12}$$

Furthermore, for any given input u and consistent x_0, x_1 w.r.t u, the solution to system (1.1) is exactly the unique solution to the second order difference system (see [7, 8])

$$x(n+2) = -\mathcal{M}^{-1}\mathcal{D}x(n+1) - \mathcal{M}^{-1}\mathcal{K}x(n) + \mathcal{M}^{-1}\mathcal{B}\begin{bmatrix} u(n) \\ u(n+1) \\ u(n+2) \end{bmatrix}, \tag{2.13}$$

where $\mathcal{M} := \begin{bmatrix} \hat{M}_1^T & \hat{D}_2^T & \hat{K}_3^T \end{bmatrix}^T$, $\mathcal{D} := \begin{bmatrix} \hat{D}_1^T & \hat{K}_2^T & 0 \end{bmatrix}^T$, $\mathcal{K} := \begin{bmatrix} \hat{K}_1^T & 0 & 0 \end{bmatrix}^T$ and $\mathcal{B} := \begin{bmatrix} \hat{B}_1 & 0 & 0 \\ 0 & \hat{B}_2 & 0 \\ 0 & 0 & \hat{B}_3 \end{bmatrix}$. This implies the regularity of the descriptor system (1.1).

(ii) In general, it is not always possible to bring system (1.1) to the strangeness-free form (2.11) by only scaling with some matrix P. This can be done by using the so-called *index reduction procedure*, see *e.g.* [3]. Furthermore, the regularity of the triple (M, D, K) and of system (1.1) are equivalent. For more details on the regularity and solvability of system (1.1), we refer the interested readers to [6], Section 3 and [7], Section 2.

The relations between causality, strict causality and strangeness-freeness are given in the following lemma.

Lemma 2.15. *Consider system* (1.1) *and assume that the matrix triple* (M, D, K) *is regular. Then the following assertions hold true.*
(i) System (1.1) *is strangeness-free if and only if the following identity holds.*

$$\deg(\mathcal{P}(\lambda)) = \operatorname{rank}(\begin{bmatrix} M & D \end{bmatrix}) + \operatorname{rank}(M). \tag{2.14}$$

In the case $M = 0$, (2.14) *becomes the well-known condition for causality of first order system in [10].*
(ii) For system (1.1)*, the strangeness-freeness implies the causality.*
(iii) System (1.1) *is strangeness-free and strictly causal if and only if in addition to* (2.14)*, the following identity also hold true.*

$$\operatorname{rank}(\begin{bmatrix} M & D \end{bmatrix}) = \operatorname{rank}(\begin{bmatrix} M & D & K \end{bmatrix}). \tag{2.15}$$

Proof. (i) Firstly, since the degree of $\mathcal{P}(\lambda)$ is preserved under system scaling with any nonsingular matrix, without loss of generality, we can assume that system (1.1) is already in the form (2.11), where M_1, D_2, K_3 are of full row rank. Furthermore, the regularity of the matrix triple (M, D, K) implies that the last row of system (2.11) is not present. Thus, we have

$$\deg(\mathcal{P}(\lambda)) = \deg\left(\det\left(\lambda^2 \begin{bmatrix} M_1 \\ D_2 \\ K_3 \end{bmatrix} + \lambda \begin{bmatrix} D_1 \\ K_2 \\ 0 \end{bmatrix} + \begin{bmatrix} K_1 \\ 0 \\ 0 \end{bmatrix}\right)\right) - \operatorname{rank}(D_2) - 2\operatorname{rank}(K_3).$$

Hence,

$$\deg(\mathcal{P}(\lambda)) \leqslant 2\left(\operatorname{rank}(M_1)+\operatorname{rank}(D_2)+\operatorname{rank}(K_3)\right)-\operatorname{rank}(D_2)-2\operatorname{rank}(K_3)=2\operatorname{rank}(M_1)+\operatorname{rank}(D_2), \quad (2.16)$$

while the equality holds if and only if the matrix $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ is nonsingular. On the other hand, we see that

$$\operatorname{rank}(\begin{bmatrix} M & D \end{bmatrix})+\operatorname{rank}(M)=\operatorname{rank}\left(\begin{bmatrix} M_1 & D_1 \\ 0 & D_2 \\ 0 & 0 \end{bmatrix}\right)+\operatorname{rank}\left(\begin{bmatrix} M_1 \\ 0 \\ 0 \end{bmatrix}\right)=2\operatorname{rank}(M_1)+\operatorname{rank}(D_2). \quad (2.17)$$

From (2.16) and (2.17), we deduce that equality (2.14) holds true if and only if the matrix $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ is nonsingular, which means that system (1.1) is strangeness-free.
(ii) The second claim is straight forward from the previous part i), since the solution to system (2.11) is also a solution to the second order difference system (2.13), which is causal.
(iii) We notice, that $x(n)$ does not depend on $u(n)$ if only if in (2.11), the third block row equation is not present. This is equivalent to the condition (2.15). Hence, this completes the proof.

□

The following example concludes this section, by showing that causality does not implies strangeness-freeness.

Example 2.16. It is clear that system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n+1)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(n)+\begin{bmatrix} 1 \\ 0 \end{bmatrix} u(n)$$

has an index $\nu = 2$ and the characteristic polynomial $\mathcal{P}(\lambda) = -1$. The solution formula is $x(n) = \left[\begin{smallmatrix} 0 \\ -u(n) \end{smallmatrix}\right]$, which implies that the system is causal. Nevertheless, the system is not strangeness-free, since condition (2.14) is not satisfied.

3. Condensed forms and causal controllability

In this section, we will modify and develop an *algebraic method* presented in [12] to study the causal controllability (Y-controllability) of system (1.1). The main idea is to transform (1.1) directly, but not reformulate it as a first order one, into so-called *condensed forms*. Moreover, in comparison to [12], the main advantage of our method is two folds. First, our condensed forms are not only more concise but also can be computed in a stable way. Second, it is helpful to design a suitable feedback that makes the closed-loop system to be regular and causal (resp., impulse-free) in the discrete (resp., continuous) time case.

Definition 3.1. Two second order descriptor systems of the form (1.1) with system matrices (M, D, K, B), and $(\tilde{M}, \tilde{D}, \tilde{K}, \tilde{B})$ are called *left equivalent* if there exist nonsingular matrices $U \in \mathbb{R}^{d,d}$ and $V \in \mathbb{R}^{m,m}$ such that

$$\tilde{M} = UM,\ \tilde{D} = UD,\ \tilde{K} = UK,\ \tilde{B} = UBV,$$

For convenience, we write $(M, D, K, B) \overset{\ell}{\sim} (\tilde{M}, \tilde{D}, \tilde{K}, \tilde{B})$.

Recently, by using the left transformation $\overset{\ell}{\sim}$, condensed forms and the solvability analysis for system (1.1) has been discussed in [6]. In the following theorem, we present the first condensed form of system (1.1).

Lemma 3.2. *Consider the descriptor system (1.1). Then there exist two orthogonal matrices $\hat{U}$, $\hat{V}$ such that the following identities hold.*

$$\hat{U}\begin{bmatrix} M, & D, & K \end{bmatrix} = \left[\begin{array}{ccc} M_1 & D_1 & K_1 \\ 0 & D_2 & K_2 \\ 0 & 0 & K_3 \\ \hline 0 & D_4 & K_4 \\ 0 & 0 & K_5 \\ 0 & 0 & 0 \end{array}\right], \ \hat{U}B\hat{V} = \left[\begin{array}{ccc} B_{11} & B_{12} & B_{13} \\ 0 & 0 & B_{23} \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & B_{43} \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{array}\right], \ \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{3.1}$$

where sizes of the block rows are r_2, r_1, r_0, φ_1, φ_0, v. Here the matrices M_1, D_2, D_4, K_3 are of full row rank, and the matrices Σ_1, Σ_0 are nonsingular.

Proof. The proof is obtained directly from Lemma 2.10 by consecutively partitioning two matrices $\check{D}_5$ and $\check{D}_4$ in (2.10) *via* Singular Value Decomposition, [24]. □

Lemma 3.2 has one direct corollary below.

Corollary 3.3. *In the condensed form (3.1), the condition $r_0 = v = 0$ holds true if and only if the matrix $\begin{bmatrix} M, & D, & B \end{bmatrix}$ has full row rank.*

In order to study the Y-controllability of system (1.1), we need the concept of feedback transformations as follows.

Definition 3.4. Two systems $Mx(n+2) + Dx(n+1) + Kx(n) = Bu(n)$ and $\tilde{M}x(n+2)+\tilde{D}x(n+1)+\tilde{K}x(n) = \tilde{B}u(n)$ are called *equivalent under*

(i) *displacement feedback* if there exists a matrix $F_d \in \mathbb{R}^{m,d}$ such that $(M, D, K, B) \overset{\ell}{\sim} (\tilde{M}, \tilde{D}, \tilde{K} + F_d\tilde{B}, \tilde{B})$.
(ii) *velocity feedback* if there exists a matrix $F_v \in \mathbb{R}^{m,d}$ such that $(M, D, K, B) \overset{\ell}{\sim} (\tilde{M}, \tilde{D} + F_v\tilde{B}, \tilde{K}, \tilde{B})$.
(iii) *acceleration feedback* if there exists a matrix $F_a \in \mathbb{R}^{m,d}$ such that $(M, D, K, B) \overset{\ell}{\sim} (\tilde{M} + F_a\tilde{B}, \tilde{D}, \tilde{K}, \tilde{B})$.

Here F_d, F_v, F_a are called displacement, velocity, acceleration gain matrices, respectively.

We notice that this concept is equivalent to classical feedback concepts as in mechanics for continuous-time descriptor systems [5, 25]. In general, one can mimic three feedback types together, *i.e.*,

$$u(n) = -F_a x(n+2) - F_v x(n+1) - F_d x(n). \tag{3.2}$$

Consequently, the resulting closed-loop system is

$$(M + BF_a)x(n+2) + (D + BF_v)x(n+1) + (K + BF_d)x(n) = 0. \tag{3.3}$$

Definition 3.5. The descriptor system (1.1) is called *Y-controllable via displacement-velocity-acceleration feedback* if there exists a feedback of the form (3.2) such that the closed-loop system (3.3) is regular and strangeness-free.

If the feedback (3.2) has no acceleration part, velocity part or displacement part then the corresponding Y-controllability of (1.1) is defined similarly.

Lemma 3.6. *The Y-controllability is invariant under left equivalent transformations.*

Proof. Due to Definition 3.1, by choosing

$$u(n) = -V^{-1}F_a x(n+2) - V^{-1}F_v x(n+1) - V^{-1}F_d x(n)$$

the proof is straightforward. □

Remark 3.7. As pointed out in [12], Remark 2.7 the condensed form (2.3) there cannot be implemented in a numerically reliable way, since the considered system is transformed by using non-orthogonal transformations. In contrast, our condensed form (3.1) can be numerically stably computed due to the orthogonality of $\hat{U}$ and $\hat{V}$. This is an important advantage of our approach, in comparison to [12], in particular for the feedback design strategy and the computation of a minimal extension form presented below.

3.1. Causal controllability via displacement and velocity feedbacks

Now we are ready to present our first main result about the Y-controllability of (1.1). We emphasize, that the characterization for Y-controllability *via* displacement feedback is more strict than the one *via* velocity feedback.

Theorem 3.8. *Consider the second order descriptor system* (1.1) *and the condensed form* (3.1)*. Then we have that:*
(i) System (1.1) *is Y-controllable via displacement-velocity feedback if and only if* $v = 0$ *and the matrix* $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ *has full row rank.*
(ii) System (1.1) *is Y-controllable via displacement feedback if and only if* $v = 0$ *and the matrix* $\begin{bmatrix} M_1^T & D_2^T & K_3^T & D_4^T \end{bmatrix}^T$ *has full row rank.*
(iii) System (1.1) *is Y-controllable via velocity feedback if and only if* $v = 0$ *and the matrix* $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ *has full row rank.*

Proof. Since the proofs of these three claims are essentially the same, for the sake of brevity we will present only the detailed arguments for part i).
Necessity: Due to (3.1) we see that

$$\begin{bmatrix} M & D & K & | & B \end{bmatrix} \overset{\ell}{\sim} \left[\begin{array}{ccc|ccc} M_1 & D_1 & K_1 & B_{11} & B_{12} & B_{13} \\ 0 & D_2 & K_2 & 0 & 0 & B_{23} \\ 0 & 0 & K_3 & 0 & 0 & 0 \\ \hline 0 & D_4 & K_4 & 0 & \Sigma_1 & B_{43} \\ 0 & 0 & K_5 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \varphi_1 \\ \varphi_0 \\ v \end{array}.$$

Thus, by using Gaussian elimination to consecutively eliminate the matrices B_{43}, B_{23}, B_{13}, B_{12}, we obtain

$$\begin{bmatrix} M & D & K & | & B \end{bmatrix} \overset{\ell}{\sim} \left[\begin{array}{ccc|ccc} M_1 & D_1^{new} & K_1^{new} & B_{11} & 0 & 0 \\ 0 & D_2 & K_2^{new} & 0 & 0 & 0 \\ 0 & 0 & K_3 & 0 & 0 & 0 \\ \hline 0 & D_4 & K_4^{new} & 0 & \Sigma_1 & 0 \\ 0 & 0 & K_5 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{3.4}$$

where by the super script *new* we indicate a (possibly) new matrix at the same block position. This form implies that no matter which feedback has been applied, it will not affect the strangeness property of the upper part of the corresponding system. Thus, system (1.1) is Y-controllable only if the matrix $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ has full row rank. Finally, notice that system (1.1) is of square size, so it is regular only if $v = 0$. This completes the necessity part.
Sufficiency: Applying Lemma 2.11 for the matrices $P_1 = \begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$, $P_2 = \begin{bmatrix} D_4^T & K_5^T \end{bmatrix}^T$, and $Q_2 = \begin{bmatrix} 0 & \Sigma_1 & B_{43} \\ 0 & 0 & \Sigma_0 \end{bmatrix}$,

we see that there exist two matrices $F_d,\ F_v$ such that the matrix

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} 0 \\ Q_2 \end{bmatrix} \begin{bmatrix} F_d & F_v \end{bmatrix} = \left[\begin{array}{c} M_1 \\ D_2 \\ K_3 \\ \hline D_4 + \begin{bmatrix} 0 & \Sigma_1 & B_{43} \end{bmatrix} F_v \\ K_5 + \begin{bmatrix} 0 & 0 & \Sigma_0 \end{bmatrix} F_d \end{array}\right]$$

has full row rank. Consequently, for the displacement–velocity feedback

$$u(n) = -F_v x(n+1)(t) - F_d x(n) \text{ for all } n \geqslant n_0,$$

the closed loop system

$$Mx(n+2) + (D + BF_v)x(n+1) + (K + BF_d)x(n) = 0 \tag{3.5}$$

is strangeness-free. This, together with the fact $v = 0$, imply that the closed-loop system (3.5) is regular. This finishes the proof. □

We illustrate the application of Theorem 3.8 in the following example.

Example 3.9. Consider system (1.1) where

$$M = \begin{bmatrix} 1&0&0&0&0 \\ 2&0&0&0&0 \\ 2&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix},\ D = \begin{bmatrix} 0&0&0&0&0 \\ 1&0&0&1&0 \\ 2&0&0&2&0 \\ 0&0&0&0&0 \\ 2&0&0&2&0 \end{bmatrix},\ K = \begin{bmatrix} 1&1&0&0&0 \\ 0&0&1&0&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&0&1&0 \end{bmatrix},\ B = \begin{bmatrix} 0&0 \\ 1&1 \\ 1&1 \\ 1&0 \\ 0&0 \end{bmatrix}.$$

The resulting system (3.1) has the matrix coefficients

$$M = \left[\begin{array}{ccccc} -3&0&0&0&0 \\ \hline 0&0&0&0&0 \\ 0&0&0&0&0 \\ \hline 0&0&0&0&0 \\ 0&0&0&0&0 \end{array}\right],\ D = \left[\begin{array}{ccccc} -2&0&0&-2&0 \\ \hline -2.1213&0&0&-2.1213&0 \\ 0&0&0&0&0 \\ \hline 0.70711&0&0&0.70711&0 \\ 0&0&0&0&0 \end{array}\right],$$

$$K = \left[\begin{array}{ccccc} -1&-1&-1.3333&-0.66667&-1.3333 \\ \hline -1.1785&-1.1785&0&-1.1785&0 \\ -0.33333&-0.33333&0&-0.33333&0 \\ \hline -0.70711&-0.70711&0.4714&0.2357&0.4714 \\ -1&-1&-1&-1&-1 \end{array}\right],\ B = \left[\begin{array}{cc} -1.3333&1.3333 \\ \hline 0&0 \\ 0&0 \\ \hline 0.4714&-0.4714 \\ 0&1 \end{array}\right] \begin{array}{c} r_2 \\ \hline r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \end{array}.$$

Here the number of rows are $r_2 = r_1 = r_0 = \varphi_1 = \varphi_0 = 1$, $v = 0$. Since

$$\begin{bmatrix} M_1 \\ D_2 \\ K3 \end{bmatrix} = \begin{bmatrix} -3&0&0&0&0 \\ -2.1213&0&0&-2.1213&0 \\ -0.33333&-0.33333&0&-0.33333&0 \end{bmatrix}$$

has full row rank, the system is Y-controllable by displacement-velocity feedback, and velocity feedback as well. Furthermore, since

$$\begin{bmatrix} M_1 \\ D_2 \\ K3 \\ D_4 \end{bmatrix} = \begin{bmatrix} -3&0&0&0&0 \\ -2.1213&0&0&-2.1213&0 \\ -0.33333&-0.33333&0&-0.33333&0 \\ 0.70711&0&0&0.70711&0 \end{bmatrix}$$

does not have full row rank, the system is not Y-controllable *via* only displacement feedback.

Based on the condensed form (3.1), another condition can be directly deduced from Theorem 3.8 to directly verify the Y-controllability (but without any feedback design strategy) as below.

Corollary 3.10. *Consider the second order descriptor system* (1.1) *and the condensed form* (3.1)*. Then system* (1.1) *is Y-controllable via displacement-velocity feedback if and only if*

$$\operatorname{rank}\begin{bmatrix}M, & DS^1_\infty, & KS^2_\infty, & B\end{bmatrix} = d, \tag{3.6}$$

where columns of S^1_∞ form a basis of kernel M*, columns of S^2_∞ form the basis of*

$$\operatorname{span}\left\{\operatorname{kernel}\begin{bmatrix} M \\ Z_1^T D\end{bmatrix} \setminus \operatorname{kernel}\begin{bmatrix} M \\ Z_1^T D \\ Z_3^T K\end{bmatrix}\right\},$$

and columns of Z_1 and of Z_3 span the left null spaces of M and $[M \;\; D]$, respectively.

Making use of (3.1), we can rewrite system (1.1) as follows

$$\left[\begin{array}{ccc} M_1 & D_1 & K_1 \\ 0 & D_2 & K_2 \\ 0 & 0 & K_3 \\ \hline 0 & D_4 & K_4 \\ 0 & 0 & K_5 \\ 0 & 0 & 0 \end{array}\right]\begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n)\end{bmatrix} = \left[\begin{array}{ccc} B_{11} & B_{12} & B_{13} \\ 0 & 0 & B_{23} \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & B_{43} \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{array}\right] v(n), \qquad \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{3.7}$$

where $u(n) = Vv(n)$ for all $n \geqslant n_0$. Let $z(n) := M_1 x(n+1)$ then we can introduce a new variable $\xi(n) = \left[\begin{smallmatrix} z(n) \\ x(n)\end{smallmatrix}\right] \in \mathbb{R}^{r_2+d}$ and rewrite system (3.7) as the first order system

$$\underbrace{\left[\begin{array}{cc} I_{r_2} & D_1 \\ 0 & M_1 \\ 0 & D_2 \\ 0 & 0 \\ \hline 0 & D_4 \\ 0 & 0 \\ 0 & 0 \end{array}\right]}_{\tilde{E}} \xi(n+1) + \underbrace{\left[\begin{array}{cc} 0 & K_1 \\ -I_{r_2} & 0 \\ 0 & K_2 \\ 0 & K_3 \\ \hline 0 & K_4 \\ 0 & K_5 \\ 0 & 0 \end{array}\right]}_{-\tilde{A}} \xi(n) = \underbrace{\left[\begin{array}{ccc} B_{11} & B_{12} & B_{13} \\ 0 & 0 & 0 \\ 0 & 0 & B_{23} \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & B_{43} \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{array}\right]}_{\tilde{B}} v(n). \qquad \begin{array}{c} r_2 \\ r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{3.8}$$

Theorem 3.11. *Consider the descriptor system* (1.1) *and the condensed form* (3.1)*. Furthermore, assume that $v = 0$ and the matrix $\begin{bmatrix} M_1^T & D_2^T & K_3^T\end{bmatrix}^T$ has full row rank. Then the first order system* (3.8) *is Y-controllable.*

Proof. In order to prove the desired claim we will verify the rank condition **C2** in (2.6). Let $S_\infty(\tilde{E})$ be a full column rank matrix whose columns form an orthogonal basis of kernel$(\tilde{E})$. Partition $S_\infty(\tilde{E}) = \left[\begin{smallmatrix} U_1 \\ V_1\end{smallmatrix}\right] \in \mathbb{R}^{r_2+d, r_2+d}$ correspondingly to (3.8), we see that

$$D_2 V_1 = 0, \;\; M_1 V_1 = 0.$$

Now we will prove that $K_3 V_1$ has full row rank. To do it first we perform an SVD for the full row rank matrix $\left[\begin{smallmatrix} M_1 \\ D_2\end{smallmatrix}\right]$ to obtain

$$U_2^T \left[\begin{smallmatrix} M_1 \\ D_2\end{smallmatrix}\right] V_2 = \begin{bmatrix}\Sigma & 0\end{bmatrix},$$

where Σ is a nonsingular, diagonal matrix. Hence, $V_1 = V_2 \left[\begin{smallmatrix} 0 \\ I\end{smallmatrix}\right]$. Partitioning $U_2^T K_3 V_2$ correspondingly, we have $U_2^T K_3 V_2 = \begin{bmatrix} K_{31} & K_{32}\end{bmatrix}$. Notice that since the matrix $\begin{bmatrix} M_1^T & D_2^T & K_3^T\end{bmatrix}^T$ has full row rank, K_{32} has full row rank. Thus,

$$K_3V_1 = U_2 \begin{bmatrix} K_{31} & K_{32} \end{bmatrix} V_2^T \; V_2 \begin{bmatrix} 0 \\ I \end{bmatrix} = U_2 \; K_{32},$$

which has full row rank. Therefore, we see that

$$\begin{bmatrix} \tilde{E} & \tilde{A}S_\infty(\tilde{E}) & \widetilde{B} \end{bmatrix} = \left[\begin{array}{cc|c|ccc} I & D_1 & K_1V_1 & B_{11} & B_{12} & B_{13} \\ 0 & M_1 & U_1 & 0 & 0 & 0 \\ 0 & D_2 & K_2V_1 & 0 & 0 & B_{23} \\ 0 & 0 & K_3V_1 & 0 & 0 & 0 \\ \hline 0 & D_4 & K_5V_1 & 0 & \Sigma_1 & B_{43} \\ 0 & 0 & 0 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{c} r_2 \\ r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array}$$

has full row rank if and only if $v = 0$. This completes the proof. □

Remark 3.12. From Theorems 3.8, 3.11 above, we see that one can interpret the upper part of system (3.7) as a *causal uncontrollable part*, while the lower part is the *causal controllable part*. Furthermore, the key point for constructing a suitable first order reformulation to (1.1) (and also for feedback design strategies) is to bring system (1.1) to the condensed form (3.1), where the upper part must be *strangeness-free, i.e.*, $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ has full row rank. In other words, the index reduction procedure has been performed only for the causal uncontrollable part. Recently, this task has been finished in both theoretical and numerical ways. To keep the brevity of this paper, we will omit the details and refer the interested readers to [6], Section 4. Below we recall one important result taken from this research.

Proposition 3.13. *([6], Thm. 4.7) Consider the descriptor system (1.1). Then it has exactly the same solution set as the strangeness-free descriptor system*

$$\underbrace{\left[\begin{array}{c} \hat{M}_1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]}_{\hat{M}} x(n+2) + \underbrace{\left[\begin{array}{c} \hat{D}_1 \\ \hat{D}_2 \\ 0 \\ \hline \hat{D}_4 \\ 0 \\ 0 \end{array}\right]}_{\hat{D}} x(n+1) + \underbrace{\left[\begin{array}{c} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \\ \hline \hat{K}_4 \\ \hat{K}_5 \\ 0 \end{array}\right]}_{\hat{K}} x(n) = \left[\begin{array}{ccc} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ 0 & 0 & \hat{B}_{23} \\ 0 & 0 & 0 \\ \hline 0 & \hat{\Sigma}_1 & \hat{B}_{43} \\ 0 & 0 & \hat{\Sigma}_0 \\ 0 & 0 & 0 \end{array}\right] v(n), \qquad \begin{array}{c} \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hline \hat{\varphi}_1 \\ \hat{\varphi}_0 \\ \hat{v} \end{array} \tag{3.9}$$

for all $n \geqslant n_0$*, where* $\begin{bmatrix} \hat{M}_1^T & \hat{D}_2^T & \hat{K}_3^T \end{bmatrix}^T$ *has full row rank,* $\hat{\Sigma}_1$ *and* $\hat{\Sigma}_0$ *are nonsingular, and* $u(n) = Vv(n)$ *for all* $n \geqslant n_0$*, where* V *is nonsingular. Furthermore, if system (1.1) is regular then* $\hat{v} = 0$*. The process that brings system (1.1) into the form (3.9) is called an* index-reduction procedure*.*

Making use of Theorem 3.11 and Proposition 3.13, we can fully analyze the Y-controllability of (1.1) as follows. First, we perform an index reduction procedure in Proposition 3.13 to obtain the strangeness-free form (3.9), and then utilizing Theorem 3.8 to verify the Y-controllability of (1.1) and to obtain the so-called *minimal* *extension form*

$$\left[\begin{array}{cc} I_{\hat{r}_2} & \hat{D}_1 \\ 0 & \hat{M}_1 \\ 0 & \hat{D}_2 \\ 0 & 0 \\ \hline 0 & \hat{D}_4 \\ 0 & 0 \\ 0 & 0 \end{array}\right] \xi(n+1) + \left[\begin{array}{cc} 0 & \hat{K}_1 \\ -I_{\hat{r}_2} & 0 \\ 0 & \hat{K}_2 \\ 0 & \hat{K}_3 \\ \hline 0 & \hat{K}_4 \\ 0 & \hat{K}_5 \\ 0 & 0 \end{array}\right] \xi(n) = \left[\begin{array}{ccc} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & \hat{B}_{23} \\ 0 & 0 & 0 \\ \hline 0 & \hat{\Sigma}_1 & \hat{B}_{43} \\ 0 & 0 & \hat{\Sigma}_0 \\ 0 & 0 & 0 \end{array}\right] v(n), \quad \begin{array}{c} \hat{r}_2 \\ \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{3.10}$$

where $\xi(n) = \begin{bmatrix} \hat{M}_1 x(n+1) \\ x(n) \end{bmatrix} \in \mathbb{R}^{r_2+d}$.

Remark 3.14. (i) We notice one property of the index-reduction procedure (see [6]) is that it reduces the degree $\deg(P(\lambda))$ of the characteristic polynomial $P(\lambda) = \det(\lambda^2 M + \lambda D + K)$ until becomes stationary. Consequently, we have that $\deg\det(P(\lambda)) \geqslant \deg\det(\hat{P}(\lambda))$, where $\hat{P}(\lambda) := \det(\lambda^2 \hat{M} + \lambda \hat{D} + \hat{K})$.
(ii) System (3.10) is minimal in the sense that the dimension of this first order reformulation is smallest, which is $d + \hat{r}_2$. Here we only need to introduce $\hat{r}_2$ new variables. Another advantage of this system is that it is always Y-controllable. As we have seen before all the first order reformulation in Example 2.8 are of dimension $2d$ and they are not always Y-controllable.

3.2. Causal controllability via acceleration feedback

For second order systems, one can consider different types of feedback (acceleration/velocity/displacement) separately or mimic them together. For the continuous-time case, in the pioneering work [12], Loose and Mehrmann considered three feedback types: displacement, velocity, and displacement-velocity; while recently Abdelaziz ([26]) considered displacement-acceleration feedback, and Zhu and Zhang ([27]) considered the most general form (3.2). Now we will study the effectiveness of acceleration feedback. Clearly, to incooperate another feedback type, we need a new condensed form instead of (3.1). This is given in the following lemma.

Lemma 3.15. *Consider the descriptor system (1.1). Then there exist two orthogonal matrices $\tilde{U}$, $\tilde{V}$ such that the following identities hold.*

$$\tilde{U}\left[M,\ D,\ K\right] = \left[\begin{array}{ccc} \tilde{M}_1 & \tilde{D}_1 & \tilde{K}_1 \\ 0 & \tilde{D}_2 & \tilde{K}_2 \\ 0 & 0 & \tilde{K}_3 \\ \hline \tilde{M}_4 & \tilde{D}_4 & \tilde{K}_4 \\ 0 & \tilde{D}_5 & \tilde{K}_5 \\ 0 & 0 & \tilde{K}_6 \\ \hline 0 & 0 & 0 \end{array}\right], \quad \tilde{U}B\tilde{V} = \left[\begin{array}{cccc} 0 & 0 & \tilde{B}_{13} & \tilde{B}_{14} \\ 0 & 0 & 0 & \tilde{B}_{24} \\ 0 & 0 & 0 & 0 \\ \hline 0 & \tilde{\Sigma}_2 & \tilde{B}_{43} & \tilde{B}_{44} \\ 0 & 0 & \tilde{\Sigma}_1 & \tilde{B}_{54} \\ 0 & 0 & 0 & \tilde{\Sigma}_0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \quad \begin{array}{c} \tilde{r}_2 \\ r_1 \\ r_0 \\ \hline \tilde{\varphi}_2 \\ \varphi_1 \\ \varphi_0 \\ \hline v \end{array} \tag{3.11}$$

where sizes of the block rows are $\tilde{r}_2,\ r_1,\ r_0,\ \tilde{\varphi}_2,\ \varphi_1,\ \varphi_0,\ v$. Here the matrices $\begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_4 \end{bmatrix}$, $\begin{bmatrix} \tilde{D}_2 \\ \tilde{D}_5 \end{bmatrix}$, $\tilde{K}_3$ are of full row rank, and the matrices $\tilde{\Sigma}_2$, $\tilde{\Sigma}_1$, $\tilde{\Sigma}_0$ are nonsingular. Furthermore, compare to the condensed form (3.1) we have $r_2 = \tilde{r}_2 + \tilde{\varphi}_2$.

Proof. The proof can be obtained directly by performing one SVD for the matrix B_{11} in (3.1). To keep the brevity of this paper we will omit the detail. □

The following corollaries are direct consequences of Lemmas 2.11 and 3.15.

Corollary 3.16. *Consider the descriptor system (1.1) and the condensed form (3.11). Then for any kind of feedback that involves acceleration (d-v-a, d-a, v-a, a), system (1.1) is Y-controllable via that feedback type if and only if* $v = 0$ *and the matrix* $\begin{bmatrix} \tilde{M}_1^T & \tilde{D}_2^T & \tilde{K}_3^T \end{bmatrix}^T$ *is of full row rank.*

Corollary 3.17. *Consider the descriptor system (1.1) and the condensed form (3.11). Then the following assertions hold true.*
(i) System (1.1) is Y-controllable via only displacement feedback if and only if in (3.1), we have $v = 0$ *and the matrix* $\begin{bmatrix} \tilde{M}_1^T & \tilde{D}_2^T & \tilde{K}_3^T & \tilde{M}_4^T & \tilde{D}_5^T \end{bmatrix}^T$ *is of full row rank.*
(ii) System (1.1) is Y-controllable via displacement-velocity feedback (or velocity feedback) if and only if in (3.1), $v = 0$ *and the matrix* $\begin{bmatrix} \tilde{M}_1^T & \tilde{D}_2^T & \tilde{K}_3^T & \tilde{M}_4^T \end{bmatrix}^T$ *is of full row rank.*

Example 3.18. To illustrate the effectiveness of an acceleration feedback, we consider the discrete-time version of a non-gyroscopic system (*e.g.* [28])

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(n). \tag{3.12}$$

Here we have that $\tilde{M}_4 = \tilde{K}_3 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\tilde{M}_1 = \tilde{D}_2 = \tilde{D}_4 = \tilde{D}_5 = \tilde{K}_6 = [\ ]$. Due to Corollary 3.16i) this system is Y-controllable by acceleration feedback. Nevertheless, it is not possible to achieve the Y-controllability by using only displacement and velocity feedbacks, since all the rank conditions in Corollary 3.17 fail.

4. Other controllability concepts and their characterizations

In this section, using the forms (3.1), (3.10) proposed above, we will discuss other controllability concepts and their characterizations. We will also point out the difference between discrete- and continuous-time systems and a new feature of second order system as well.

Now let us introduce some rank conditions for second order systems.

$$\begin{aligned} &\mathbf{C21}: \quad \operatorname{rank}\begin{bmatrix} \lambda^2 M + \lambda D + K, & B \end{bmatrix} = d \quad \text{for all } \lambda \in \mathbb{C}, \\ &\mathbf{C22}: \quad \operatorname{rank}\begin{bmatrix} M, & D, & B \end{bmatrix} = d, \\ &\mathbf{C23}: \quad \operatorname{rank}\begin{bmatrix} M, & B \end{bmatrix} = d. \end{aligned} \tag{4.1}$$

Definition 4.1. Consider the descriptor system (1.1) and $(x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d$.
(i) A set $\mathcal{R}(x_0, x_1) \subseteq \mathbb{R}^d$ is called a *reachable set* starting from (x_0, x_1) if for every $x_0^f \in \mathcal{R}(x_0, x_1)$ there exists an input sequence u that transfers the system in finite time from $x(n_0) = x_0$, $x(n_0+1) = x_1$ to $x(n_f) = x_0^f$.
(ii) A set $\mathcal{R}_2(x_0, x_1) \subseteq \mathbb{R}^d \times \mathbb{R}^d$ is called a *reachable set* starting from the pair (x_0, x_1) if for every $(x_0^f, x_1^f) \in \mathcal{R}_2(x_0, x_1)$ there exists an input sequence u that transfers the system in finite time from $x(n_0) = x_0$, $x(n_0+1) = x_1$ to $x(n_f) = x_0^f$, $x(n_f+1) = x_1^f$.

We define

$$\begin{aligned} \mathcal{R} &:= \bigcup \{\mathcal{R}(x_0, x_1) \mid (x_0, x_1) \text{ is consistent w.r.t some } u\}, \\ \mathcal{R}_2 &:= \bigcup \{\mathcal{R}_2(x_0, x_1) \mid (x_0, x_1) \text{ is consistent w.r.t some } u\}. \end{aligned}$$

(iii) The system is called *R-controllable* if any state $x_0^f \in \mathcal{R}$ can be reached from any consistent pair (x_0, x_1) in finite time.
(iv) The system is called *R2-controllable* if any pair $(x_0^f, x_1^f) \in \mathcal{R}_2$ can be reached from any consistent pair (x_0, x_1) in finite time.

(v) The system is called *C-controllable* if for any consistent pair $(x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d$ and any $x_0^f \in \mathbb{R}^d$ there exist a finite time n_f and an input sequence u such that $x(n_f) = x_0^f$, $x(n_0) = x_0$, $x(n_1) = x_1$.
(vi) The system is called *C2-controllable* if for any consistent pair $(x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d$ and any pair $(x_0^f, x_1^f) \in \mathbb{R}^d \times \mathbb{R}^d$ there exist a finite time n_f and an input sequence u such that $x(n_f) = x_0^f$, $x(n_f+1) = x_1^f$, $x(n_0) = x_0$, $x(n_1) = x_1$.

It is straightforward to see that all these controllability concepts are invariant under left equivalent transformation. In the following theorem, we give a characterization for the C2- and R2-controllability.

Theorem 4.2. *Consider the descriptor system* (1.1) *and its first order companion form* (2.8)*. Then the following assertions hold true.*
(i) System (1.1) *is R2-controllable if and only if the system matrix coefficients satisfy condition* **C21***.*
(ii) Besides that, system (1.1) *is C2-controllable if and only if the system matrix coefficients satisfy both conditions* **C21** *and* **C23***.*

Proof. Following directly from Definition 4.1, we see that system (1.1) is C2-controllable (resp., R2-controllable) if and only if its first order companion form (2.8) is C-controllable (resp., R-controllable). Thus, the proof is straightforward by checking the rank criteria in Proposition 2.5. □

Making use of Theorem 4.2 and Proposition 2.7, we deduce the following corollary, which analyze the amount of time needed to reach any desired pair $(x_0^f, x_1^f) \in \mathbb{R}^d \times \mathbb{R}^d$.

Corollary 4.3. *If a regular system* (1.1) *is* $C2$*-controllable then it can reach any desired pair* $(x_0^f, x_1^f) \in \mathbb{R}^d \times \mathbb{R}^d$ *in* $\deg\det(\lambda^2 M + \lambda D + K)$ *steps.*

Proof. Since the definition of $C2$-controllability is equivalent to the C-controllability of the companion form (2.8), the proof can be derived directly from Theorem 4.2 and Proposition 2.7. □

Remark 4.4. The $R2$-controllability has a closed relation to the *behavioral controllability* consider by Rocha and Willems in [29], which leads to the same condition $\mathbf{C_{21}}$. Nevertheless, in this manuscript we are also interested in other controllability concepts, which require more detailed analysis on the structure of the system's coefficients. Here we also refer to [30, 31] about the behavior approach for discrete time systems.

Now let us come back to the strangeness-free form (3.9) and the minimal extension form (3.10). The following example demonstrates, that sometimes, it is reasonable to control $x(n)$ and only the part $M_1 x(n+1)$ but not the whole $x(n+1)$. This fact motivates the concept of weakly $C2$-controllability.

Example 4.5. We consider the simple system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix} x(n) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} u(n). \tag{4.2}$$

We can see directly, that $\operatorname{rank}(\begin{bmatrix} M, B \end{bmatrix}) \neq 3$, so the system is not $C2$-controllable. Nevertheless, by introducing a new variable $z(n) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(n+1)$ and $\xi(n) = \begin{bmatrix} z(n) \\ x(n) \end{bmatrix}$, the minimal extension form reads

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xi(n+1) + \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 0 & 0 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xi(n) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} u(n), \tag{4.3}$$

which is C-controllable.

Definition 4.6. Consider the descriptor system (1.1) and assume that it is already in the strangeness-free form (3.9). Then system (1.1) is called *weakly C2-controllable* if the minimal extension form (3.10) is C-controllable.

Lemma 4.7. *Consider the descriptor system* (1.1)*, the strangeness-free form* (3.9) *and the minimal extension form* (3.10)*. Then we have that:*
(i) System (3.10) *is R-controllable if and only if system* (3.9) *satisfies condition* **C21**.
(ii) System (1.1) *is weakly $C2$-controllable if and only if system* (3.9) *satisfies both conditions* **C21** *and* **C22**.
(iii) The constant rank condition **C21** *is preserved under the index-reduction procedure, which transforms* (1.1) *to* (3.9).

Proof. For notational convenience, within this proof, we will omit the superscript $\hat{}$ in the strangeness-free form (3.9). Due to Definition 2.5, system (3.10) is R-controllable (resp. C-controllable) if and only if the matrix coefficients $\tilde{E}$, $\tilde{A}$, $\widetilde{B}$ satisfy the constant rank **C1** (resp., **C0**).
(i) Condition **C1** applied to system (3.10) reads

$$\operatorname{rank}\left[\begin{array}{cc|ccc} \lambda I_{r_2} & \lambda D_1 + K_1 & B_{11} & B_{12} & B_{13} \\ -I_{r_2} & \lambda M_1 & 0 & 0 & 0 \\ 0 & \lambda D_2 + K_2 & 0 & 0 & B_{23} \\ 0 & K_3 & 0 & 0 & 0 \\ \hline 0 & \lambda D_4 + K_4 & 0 & \Sigma_1 & B_{43} \\ 0 & K_5 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = d + r_2 \quad \text{for all } \lambda \in \mathbb{C}. \tag{4.4}$$

Using matrix row manipulation in order to eliminate λI_{r_2} in the first row, we see that (4.4) is equivalent to the condition

$$\operatorname{rank}\left[\begin{array}{cc|ccc} 0 & \lambda^2 M_1 + \lambda D_1 + K_1 & B_{11} & B_{12} & B_{13} \\ -I_{r_2} & \lambda M_1 & 0 & 0 & 0 \\ 0 & \lambda D_2 + K_2 & 0 & 0 & B_{23} \\ 0 & K_3 & 0 & 0 & 0 \\ \hline 0 & \lambda D_4 + K_4 & 0 & \Sigma_1 & B_{43} \\ 0 & K_5 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = d + r_2 \quad \text{for all } \lambda \in \mathbb{C}. \tag{4.5}$$

Clearly, this holds true if and only if $\operatorname{rank}\left[\lambda^2 M + \lambda D + K,\ B\right] = d$, which is exactly the rank condition **C21**.
(ii) Due to Definition 2.5, we see that **C0** = **C1** + **C3**, so we need to prove that condition **C3** is equivalent to condition **C22**. Now let us look at condition **C3**, which means that the matrix

$$\begin{bmatrix} M & D & B \end{bmatrix} = \left[\begin{array}{cc|ccc} I_{r_2} & D_1 & B_{11} & B_{12} & B_{13} \\ 0 & M_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 & B_{23} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & D_4 & 0 & \Sigma_1 & B_{43} \\ 0 & 0 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} r_2 \\ r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array}$$

has full row rank $(d + r_2)$. Recall that in the strangeness-free form (3.9) the matrix $\begin{bmatrix} \hat{M}_1 \\ \hat{D}_2 \end{bmatrix}$ has full row rank. Therefore, condition **C3** holds true if and only if $r_0 = v = 0$.

Moreover, condition **C22** means that the matrix

$$\begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \left[\begin{array}{cc|ccc} M_1 & D_1 & B_{11} & B_{12} & B_{13} \\ 0 & D_2 & 0 & 0 & B_{23} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & D_4 & 0 & \Sigma_1 & B_{43} \\ 0 & 0 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

has full row rank. This is fulfilled only when $r_0 = v = 0$. Thus, two conditions **C3** and **C22** are equivalent. This completes the proof of this part.
(iii) In order to prove that condition **C21** is preserved under the index-reduction procedure we only need to prove that it is preserved under one index reduction step. For any two left equivalent tuples $(M, D, K, B) \overset{\ell}{\sim} (\bar{M}, \bar{D}, \bar{K}, \bar{B})$ we have that

$$\left[\lambda^2 \bar{M} + \lambda \bar{D} + \bar{K},\ \bar{B}\right] = U\ \left[\lambda^2 M + \lambda D + K,\ B\right] \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}.$$

Thus, rank $\left[\lambda^2 M + \lambda D + K,\ B\right]$ is invariant under left equivalent relation. Consequently, we may assume that (M, D, K, B) takes the form in the right hand side of (3.4). Thus, our system reads

$$\left[\begin{array}{c} M_1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] x(n+2) + \left[\begin{array}{c} D_1 \\ D_2 \\ 0 \\ \hline D_4 \\ 0 \\ 0 \end{array}\right] x(n+1) + \left[\begin{array}{c} K_1 \\ K_2 \\ K_3 \\ \hline K_4 \\ K_5 \\ 0 \end{array}\right] x(n) = \left[\begin{array}{ccc} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & 0 \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{array}\right] v(n), \quad \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{4.6}$$

where M_1, D_2, K_3 have full row rank, and the matrices Σ_0, Σ_1 are diagonal and nonsingular. We recall, that due to [6], Lemma 4.4, typically one index-reduction step is to transform (4.6) into the new form which reads

$$\underbrace{\left[\begin{array}{c} S^{(2)}M_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]}_{\tilde{M}} x(n+2) + \underbrace{\left[\begin{array}{c} S^{(2)}D_1 \\ Z^{(2)}D_1 + Z^{(4)}K_2 \\ S^{(1)}D_2 \\ 0 \\ 0 \\ \hline D_4 \\ 0 \\ 0 \end{array}\right]}_{\widetilde{D}} x(n+1) + \underbrace{\left[\begin{array}{c} S^{(2)}K_1 \\ Z^{(2)}K_1 \\ S^{(1)}K_2 \\ Z^{(1)}K_2 \\ K_3 \\ \hline K_4 \\ K_5 \\ 0 \end{array}\right]}_{\widetilde{K}} x(n) = \underbrace{\left[\begin{array}{ccc} S^{(2)}B_{11} & 0 & 0 \\ Z^{(2)}B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & 0 \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{array}\right]}_{\tilde{B}} v(n). \quad \begin{array}{c} d_2 \\ s_2 \\ d_1 \\ s_1 \\ r_0 \\ \hline \varphi_2 \\ \varphi_1 \\ v \end{array} \tag{4.7}$$

Here, the matrices $S^{(i)}$ and $Z^{(j)}$ $(i = 1, 2,\ j = 1, ..., 5)$ satisfy the following conditions.

(i) For $i = 1, 2$, the matrices $\begin{bmatrix} S^{(i)} \\ Z^{(i)} \end{bmatrix} \in \mathbb{R}^{r_i, r_i}$ are orthogonal, and $r_i = d_i + s_i$.
(ii) The following identities hold true.

$$\begin{aligned} Z^{(1)}D_2 + Z^{(3)}K_3 &= 0, \\ Z^{(2)}M_1 + Z^{(4)}D_2 + Z^{(5)}K_3 &= 0. \end{aligned}$$

Utilizing the nonsingular matrix U_λ defined as

$$U_\lambda := \begin{bmatrix} \begin{bmatrix} S^{(2)} \\ Z^{(2)} \end{bmatrix} & \begin{bmatrix} 0 \\ \lambda Z^{(4)} \end{bmatrix} & \begin{bmatrix} 0 \\ \lambda^2 Z^{(5)} \end{bmatrix} & 0 & 0 & 0 \\ 0 & \begin{bmatrix} S^{(1)} \\ Z^{(1)} \end{bmatrix} & \begin{bmatrix} 0 \\ \lambda Z^{(3)} \end{bmatrix} & 0 & 0 & 0 \\ 0 & 0 & I_{r_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\varphi_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\varphi_0} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_v \end{bmatrix},$$

we see that

$$\left[\lambda^2 \tilde{M} + \lambda \tilde{D} + \tilde{K},\ \tilde{B}\right] = U_\lambda \left[\lambda^2 M + \lambda D + K,\ B\right].$$

Therefore, we have

$$\operatorname{rank}\left[\lambda^2 \tilde{M} + \lambda \tilde{D} + \tilde{K},\ \tilde{B}\right] = \operatorname{rank}\left[\lambda^2 M + \lambda D + K,\ B\right] \quad \text{for all } \lambda \in \mathbb{C},$$

and hence, condition **C21** is preserved under one index reduction step. This finishes our proof. □

In addition to the Y-controllability as shown in Theorem 3.11, the advantage of the minimal extension form (3.10) is enhanced in the following theorem.

Theorem 4.8. *Consider the descriptor system* (1.1)*, its the strangeness-free form* (3.9) *and the minimal extension form* (3.10)*. Then the following assertions are equivalent.*
(i) System (1.1) *is R2-controllable.*
(ii) System (3.9) *is R2-controllable.*
(iii) System (3.10) *is R-controllable.*

Proof. As a direct consequence of Theorem 4.2 (i) and Lemma 4.7 (iii), we see that the R2-controllability is preserved under the index-reduction procedure. Therefore, the equivalence of (i) and (ii) is straight forward. On the other hand, the equivalence of (ii) and (iii) can be established based on Theorem 4.2 (i) and Lemma 4.7 (i). This completes the proof. □

Theorem 4.9. *Consider the descriptor system* (1.1) *and its the strangeness-free form* (3.9)*. Then system* (1.1) *is weakly C2-controllable if and only if the following conditions are satisfied.*
(i) The matrix coefficients of system (1.1) *satisfies condition* **C21***.*
(ii) The matrix coefficients of the strangeness-free system (3.9) *satisfy condition* **C22***.*

Proof. The proof is followed directly from Definition 4.6 and Lemma 4.7. □

The following example shows that condition **C22** is not invariant under the index-reduction procedure.

Example 4.10. Consider the following system

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{M} x(n+2) + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} x(n+1) + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{K} x(n) = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u(n) \quad \text{for all } n \geqslant n_0. \tag{4.9}$$

In order to perform an index-reduction procedure in [6], first we shift the second row equation to obtain

$$\begin{bmatrix}1 & 0 & 0\end{bmatrix} x(n+2) + \begin{bmatrix}0 & 1 & 0\end{bmatrix} x(n+1) = 0.$$

Removing the left hand side part from the first equation of system (4.9), we obtain $\begin{bmatrix}1 & 0 & 0\end{bmatrix} x(n) = 0$. Therefore, we arrive at the new system

$$\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} x(n+1) + \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} x(n) = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} u(n).$$

Analogously, subtracting the shifted version of the first row equation from the second equation yields the strangeness-free formulation (3.9), which reads

$$\underbrace{\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\hat{M}} x(n+2) + \underbrace{\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\hat{D}} x(n+1) + \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\hat{K}} x(n) = \underbrace{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}}_{\hat{B}} u(n). \tag{4.10}$$

Clearly, $\operatorname{rank}\begin{bmatrix}M, & D, & B\end{bmatrix} = 3 > 1 = \operatorname{rank}\begin{bmatrix}\hat{M}, & \hat{D}, & \hat{B}\end{bmatrix}$. This means that condition **C22** is not invariant under the index-reduction procedure, and system (4.9) is not $C2$-controllable. Clearly, from (4.10) we deduce that system (4.9) is not C-controllable. On the other hand, examining condition **C21**

$$\operatorname{rank}\left[\lambda^2 M + \lambda D + K \quad \middle| \quad B\right] = \operatorname{rank}\left[\begin{array}{ccc|c}\lambda^2+1 & \lambda & 0 & 0\\ \lambda & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{array}\right] = 3$$

we see that system (4.9) is $R2$-controllable.

Remark 4.11. Another important difference between discrete and continuous time descriptor systems is, that in order to achieve the weakly $C2$-controllability, condition **C22** must be required for the strangeness-free system (3.9) instead of for the original system (1.1). In details, [12], Corollary 3.11 (ii) and Theorem 3.18 (iv) imply that the continuous-time version of system (4.9) is $C2$-controllable (resp. C-controllable).

Naturally, one may ask whether one can verify the weakly $C2$-controllability of system (1.1) without performing an index reduction procedure. In fact, the positive answer is given in the following theorem.

Theorem 4.12. *Consider the descriptor system (1.1) and its condensed form (3.1). Then system (1.1) is weakly $C2$-controllable if and only if two following conditions are satisfied.*
(i) The matrix coefficients of system (1.1) satisfies condition **C21**.
(ii) In the condensed form (3.1), $r_0 = v = 0$ and the matrix $\left[\begin{smallmatrix}M_1\\ D_2\end{smallmatrix}\right]$ has full row rank.
Finally, condition ii) is equivalent to the requirement that $\operatorname{rank}\begin{bmatrix}M, & D, & B\end{bmatrix} = d$ *and the matrix $\left[\begin{smallmatrix}M_1\\ D_2\end{smallmatrix}\right]$ has full row rank.*

Proof. Due to Definition 4.6 system (1.1) is weakly $C2$-controllable if and only if the minimal extension form (3.10) is C-controllable. From Definition 2.5 and Lemma 4.7 iii, we see that $\mathbf{C0} = \mathbf{C1} + \mathbf{C3}$ and **C1** is equivalent to condition **C21**.

Hence, we only need to prove that condition **C3** is equivalent to the claim ii). Now let us look at condition **C3**, which means that the matrix

$$\begin{array}{c} r_2 \\ r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \left[\begin{array}{cc|ccc} I_{r_2} & D_1 & B_{11} & B_{12} & B_{13} \\ 0 & M_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 & B_{23} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & D_4 & 0 & \Sigma_1 & B_{43} \\ 0 & 0 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

has full row rank. This condition is fulfilled if and only if $\left[\begin{smallmatrix} M_1 \\ D_2 \end{smallmatrix}\right]$ has full row rank and $r_0 = v = 0$, which is nothing else than the claim ii). Finally, the last claim follows from Corollary 3.3. This completes the proof. □

Making use of Proposition 2.7, we deduce the following result, which analyze the amount of time needed for reachability *via* the concept weakly $C2$-controllability.

Corollary 4.13. *Consider a regular system* (1.1) *and the strangeness-free system* (3.9)*. If system* (1.1) *is weakly* $C2$*-controllable then it can reach any desired pair* $(x_0^f, \hat{M}x_1^f) \in \mathbb{R}^d \times \mathbb{R}^{r_2}$ *in* $\deg\det(\lambda^2\hat{M} + \lambda\hat{D} + \hat{K}) = \operatorname{rank}(\begin{bmatrix} \hat{M} & \hat{D} \end{bmatrix}) + \operatorname{rank}(\hat{M})$ *steps.*

Proof. By direct computation we see that

$$\det\left(\lambda\left[\begin{array}{cc} I_{\hat{r}_2} & \hat{D}_1 \\ 0 & \hat{M}_1 \\ 0 & \hat{D}_2 \\ 0 & 0 \\ \hline 0 & \hat{D}_4 \\ 0 & 0 \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} 0 & \hat{K}_1 \\ -I_{\hat{r}_2} & 0 \\ 0 & \hat{K}_2 \\ 0 & \hat{K}_3 \\ \hline 0 & \hat{K}_4 \\ 0 & \hat{K}_5 \end{array}\right]\right) = \det\left(\left[\begin{array}{cc} \lambda I_{\hat{r}_2} & \lambda\hat{D}_1 + \hat{K}_1 \\ -I_{\hat{r}_2} & \lambda\hat{M}_1 \\ 0 & \lambda\hat{D}_2 + \hat{K}_2 \\ 0 & \hat{K}_3 \\ \hline 0 & \lambda\hat{D}_4 + \hat{K}_4 \\ 0 & \hat{K}_5 \end{array}\right]\right) = \det\left(\left[\begin{array}{cc} \lambda I_{\hat{r}_2} & \lambda^2\hat{M}_1 + \lambda\hat{D}_1 + \hat{K}_1 \\ -I_{\hat{r}_2} & \lambda\hat{M}_1 \\ 0 & \lambda\hat{D}_2 + \hat{K}_2 \\ 0 & \hat{K}_3 \\ 0 & \lambda\hat{D}_4 + \hat{K}_4 \\ \hline 0 & \hat{K}_5 \end{array}\right]\right)$$

$$= \det\left(\lambda^2\left[\begin{array}{c} \hat{M}_1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] + \lambda\left[\begin{array}{c} \hat{D}_1 \\ \hat{D}_2 \\ 0 \\ \hline \hat{D}_4 \\ 0 \end{array}\right] + \left[\begin{array}{c} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \\ \hline \hat{K}_4 \\ \hat{K}_5 \end{array}\right]\right) = \det(\lambda^2\hat{M} + \lambda\hat{D} + \hat{K}).$$

Thus, Proposition 2.7 and Lemma 2.15 applied to the strangeness-free system (3.9) completes the proof. □

Now let us discuss the C-controllability of system (1.1). In the following example we illustrate that for second order systems, C-controllability does not always imply Y-controllability.

Example 4.14. Consider the following system

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}_{M} x(n+2) + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{D} x(n+1) + \underbrace{\begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}}_{K} x(n) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(n) \quad \text{for all } n \geqslant n_0. \tag{4.11}$$

Clearly, the structure of the pair (M,D) implies that system (4.11) is not Y-controllable. By adding the shifted version of the second row equation to the first row, we can transform (4.11) to the first order system

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(n+1) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(n) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(n),$$

which can be directly verified that is C-controllable. Thus, C-controllability does not imply Y-controllability.

Example 4.14 suggests, that we should discuss the C-controllability of the strangeness-free formulation (3.9), which is always Y-controllable, instead of the original system (1.1). The characterizations of C-controllability for system (1.1) are given in the following theorem.

Theorem 4.15. *Consider system (1.1) and assume that it is already in the strangeness-free form (3.9). Let $\mathcal{R}_{ext}$ be the reachable set of the minimal extension form (3.10). Let $E_0 = \operatorname{diag}(0_{\hat{r}_2}, I_d)$. Then the following assertions are equivalent.*
(i) System (1.1) is C-controllable.
(ii) System (1.1) is R-controllable and $\mathrm{Im} E_0 \subseteq \mathcal{R}_{ext}$.
(iii) System (1.1) is R-controllable and $\operatorname{rank}\begin{bmatrix} M, & D, & B \end{bmatrix} = d$.

Proof. Notice that in system (3.10) $\xi_n = \left[\begin{smallmatrix} z_n \\ x_n \end{smallmatrix}\right] \in \mathbb{R}^{\hat{r}_2+d}$, so the equivalence between i) and ii) is straightforward. From the definition of C-controllability, we see that $\hat{r}_0 = 0$. Furthermore, since system (1.1) is square, we also have $\hat{v} = 0$. Corollary 3.3, therefore, implies that $\operatorname{rank}\begin{bmatrix} M, & D, & B \end{bmatrix} = d$. Hence, we have proved that $i) \Rightarrow iii)$. Now we prove that $iii) \Rightarrow ii)$.

Due to Corollary 3.3, we see that $\hat{r}_0 = \hat{v} = 0$, and hence the 3rd and 6th rows are not present in the form (3.10). In analogous to the sufficiency part of Theorem 3.8 i), there exist two matrices $\hat{F}_d$, $\hat{F}_v$ such that the matrix $\begin{bmatrix} \hat{M}_1^T & D_2^T & K_3^T & \widetilde{D}_4^T & \widetilde{K}_5^T \end{bmatrix}^T$ has full row rank, where

$$\widetilde{D}_4 := \hat{D}_4 + \begin{bmatrix} 0 & \hat{\Sigma}_1 & \hat{B}_{43} \end{bmatrix} \hat{F}_v, \ \widetilde{K}_5 := \hat{K}_5 + \begin{bmatrix} 0 & 0 & \hat{\Sigma}_0 \end{bmatrix} \hat{F}_d.$$

Consequently, by introducing a new input function $w = \{w(n)\}$ such that

$$u(n) = -\hat{F}_v x(n+1)(t) - \hat{F}_d x(n) + w(n) \ \text{ for all } n \geqslant n_0,$$

we can transform the minimal extension form (3.10) to the closed loop system

$$\left[\begin{array}{cc} I_{\hat{r}_2} & \hat{D}_1 \\ 0 & \hat{M}_1 \\ 0 & \hat{D}_2 \\ 0 & \widetilde{D}_4 \\ \hline 0 & 0 \end{array}\right] \xi(n+1) + \left[\begin{array}{cc} 0 & \hat{K}_1 \\ -I_{\hat{r}_2} & 0 \\ 0 & \hat{K}_2 \\ 0 & \hat{K}_4 \\ \hline 0 & \widetilde{K}_5 \end{array}\right] \xi(n) = \left[\begin{array}{ccc} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & \hat{B}_{23} \\ 0 & \hat{\Sigma}_1 & \hat{B}_{43} \\ \hline 0 & 0 & \hat{\Sigma}_0 \end{array}\right] w(n), \quad \begin{array}{c} \hat{r}_2 \\ \hat{r}_2 \\ \hat{r}_1 \\ \hat{\varphi}_1 \\ \hat{\varphi}_0 \end{array}. \tag{4.12}$$

Notice that, since $w(n)$ can be freely chosen like $u(n)$, we neither change the R-controllability nor change the reachable set $\mathcal{R}$ of system (1.1). Since the matrix $\begin{bmatrix} \hat{M}_1^T & D_2^T & \widetilde{D}_4^T & \widetilde{K}_5^T \end{bmatrix}^T$ has full row rank, the matrix

$$\left[\begin{array}{cc} I_{\hat{r}_2} & \hat{D}_1 \\ 0 & \hat{M}_1 \\ 0 & \hat{D}_2 \\ 0 & \widetilde{D}_4 \\ \hline 0 & \widetilde{K}_5 \end{array}\right]$$

also has full row rank, and hence, system (4.12) is regular and strangeness-free. Corollary 2.2 applied to system (4.12) implies that the reachable subspace of (4.12) is $\mathcal{R}_{ext} = \mathbb{R}^{\hat{r}_2+d}$ and hence, $\mathrm{Im}E_0 \subseteq \mathcal{R}_{ext}$. This completes the proof. □

Now we discuss the R-controllability of system (1.1). Assume that it is regular and already in the strangeness-free form (3.9), then $\hat{v} = 0$. Similar to Theorem 4.15, we can transform the minimal extension form (3.10) to

$$\underbrace{\left[\begin{array}{cc} I_{\hat{r}_2} & \hat{D}_1 \\ 0 & \hat{M}_1 \\ 0 & \hat{D}_2 \\ 0 & 0 \\ \hline 0 & \widetilde{D}_4 \\ 0 & 0 \end{array}\right]}_{\hat{E}} \xi(n+1) + \underbrace{\left[\begin{array}{cc} 0 & \hat{K}_1 \\ -I_{\hat{r}_2} & 0 \\ 0 & \hat{K}_2 \\ 0 & \hat{K}_3 \\ \hline 0 & \widetilde{K}_4 \\ 0 & \widetilde{K}_5 \end{array}\right]}_{-\hat{A}} \xi(n) = \underbrace{\left[\begin{array}{ccc} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & \hat{B}_{23} \\ 0 & 0 & 0 \\ \hline 0 & \hat{\Sigma}_1 & \hat{B}_{43} \\ 0 & 0 & \hat{\Sigma}_0 \end{array}\right]}_{\hat{B}} v(n), \quad \begin{array}{c} \hat{r}_2 \\ \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hline \hat{\varphi}_1 \\ \hat{\varphi}_0 \end{array}. \tag{4.13}$$

with the matrix

$$\left[\begin{array}{cc} I_{\hat{r}_2} & \hat{D}_1 \\ 0 & \hat{M}_1 \\ 0 & \hat{D}_2 \\ 0 & \hat{K}_3 \\ \hline 0 & \widetilde{D}_4 \\ 0 & \widetilde{K}_5 \end{array}\right] \tag{4.14}$$

has full row rank.

For standard linear control systems, one key difference between the continuous and discrete time case is the equivalence between the (zero-/null-) controllability and the reachability (from zero) which holds in continuous time but not in discrete time. Analogously, for first order singular systems, the null-controllability does not imply the reachability from zero. We further notice, see *e.g.* [10], Chapter 2, that the reachability from zero is equivalent to the reachability from any consistent initial condition. Making use of the minimal extension form, we characterize the reachability from zero in the next theorem.

Theorem 4.16. *System (1.1) is R-controllable if and only if for the corresponding first order system (4.13), the matrix product* $\begin{bmatrix} 0 & I_d \end{bmatrix} \begin{bmatrix} \Phi_0\hat{B} & \Phi_1\hat{B} & \dots & \Phi_{\hat{n}-1}\hat{B} \end{bmatrix} \in \mathbb{R}^{d,\hat{n}m}$ *has full row rank. Here* $\hat{n} := \deg\det(\lambda\hat{E} - \hat{A})$ *and* $\{\Phi_i\}$ *is the fundamental solution matrix of (4.13).*

Proof. Since matrix (4.14) has full row rank, system (4.13) has index 1. From (2.5), we see that the first order system (4.13) has the reachable set from zero is

$$\mathcal{R}(0) = \left(\sum_{i=0}^{\hat{n}-1} \mathrm{Im}(\Phi_i\hat{B})\right) \oplus \mathrm{Im}(\Phi_{-1}\hat{B})$$

and the reachable set $\mathcal{R} = \mathbb{R}^{2r_2+r_1+\varphi_1} \oplus \mathrm{Im}(\Phi_{-1}\hat{B})$. Therefore, (4.13) is R-controllable if and only if $\sum_{i=0}^{\hat{n}-1} \mathrm{Im}(\Phi_i\hat{B}) = \mathbb{R}^{2\hat{r}_2+\hat{r}_1+\hat{\varphi}_1}$. Furthermore, notice that the first $\hat{r}_2$ variables of (4.13) come from the transformation of second order system (3.9) to the first order system (4.13) and are not relevant to consider for R-controllability of (1.1). This completes the proof. □

Remark 4.17. (i) Motivated from [11] we see that, for the R-controllability, a desired state can be reached in $\hat{n}$ steps.
(ii) Furthermore, also from [11] and the minimal extension form (4.13) above, the null-controllability can be characterized *via* the following condition.

$$-\begin{bmatrix}0 & I_d\end{bmatrix}(\Phi_0\hat{A})^{\hat{n}}\xi(0) \in \operatorname{Im}\begin{bmatrix}0 & I_d\end{bmatrix}\begin{bmatrix}\Phi_0\hat{B} & \Phi_1\hat{B} & \dots & \Phi_{\hat{n}-1}\hat{B}\end{bmatrix}.$$

Clearly this condition is weaker than the condition in Theorem 4.16.

5. Conclusion and outlook

In this paper, we have conducted a theoretical analysis of the controllability of linear, second-order descriptor systems in discrete-time settings. By modifying and extending the algebraic method proposed in previous works [12, 32], we have developed more concise and reliable condensed forms for practical purposes. Additionally, we introduced a feedback design strategy to ensure causal behavior in closed-loop systems. Our characterization of controllability relies on rank conditions that are numerically verifiable, allowing us to explore the fundamental concepts of controllability thoroughly. We highlighted key differences between continuous-time and discrete-time systems, notably in Remark 4.11, and demonstrating that C-controllability does not necessarily imply Y-controllability for second-order systems. Moreover, we addressed the minimal time required to achieve a desired state in Corollaries 4.3 and 4.13. Looking ahead, future research will focus on generalizing our approach to higher-order descriptor systems and investigating the preservation of controllability properties during discretization or sampling processes.

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Data availability statement

The research data associated with this article are included in the article.

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