ESAIM: COCV 31 (2025) 17
https://doi.org/10.1051/cocv/2024084

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

APPROXIMATION OF FEEDBACK GAINS FOR ABSTRACT PARABOLIC SYSTEMS

Mehdi Badra* and Jean-Pierre Raymond

Abstract. We consider parabolic controlled systems represented by a pair (A, B), where $(A, \mathcal{D}(A))$ is the infinitesimal generator of an analytic semigroup on a Hilbert space Z and B is an unbounded control operator from a control space U into Z. We consider approximate controlled systems $(A_\varepsilon, B_\varepsilon)$, for $\varepsilon > 0$, where $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$ is the infinitesimal generator of an analytic semigroup on a Hilbert space Z_ε and B_ε is an unbounded control operator from the control space U into Z_ε. Since Z_ε is not included in Z, we are in the case of nonconforming approximations. We assume that both Z and Z_ε are Hilbert subspaces of another Hilbert space H, and that there exist projectors $P \in \mathcal{L}(H)$ and $P_\varepsilon \in \mathcal{L}(H)$ such that $Z = PH$ and $Z_\varepsilon = P_\varepsilon H$, and for which (A, B, P) and $(A_\varepsilon, B_\varepsilon, P_\varepsilon)$ satisfy suitable approximation assumptions. When the pair (A, B) is exponentially feedback stabilizable in Z, we first prove that the pair $(A_\varepsilon, B_\varepsilon)$ is exponentially feedback stabilizable in Z_ε, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 > 0$. We next prove that Riccati-based feedback laws stabilizing (A, B) in Z can be approximated by feedback laws stabilizing $(A_\varepsilon, B_\varepsilon)$ in Z_ε. This type of results has been established in the eighties and the nineties in the case of conforming approximation, that is when $Z_\varepsilon \subset Z$. To the best of our knowledge nothing is known in the case of nonconforming approximations. We also extend, to the case of nonconforming approximations, convergence rates obtained in the case of conforming approximations. Nonconforming approximations play a central role in fluid mechanics. In M. Badra and J.-P. Raymond, Approximation of feedback gains for the Oseen system. Preprint (2025). https://hal.science/hal-04880955, we have shown that the results proved in the present paper apply to the Oseen system (the Navier–Stokes equations linearized around a steady state) and its semidiscrete approximation by a Finite Element Method.

Mathematics Subject Classification. 93B52, 93B40, 76D55, 93C20, 93C95.

Received November 21, 2023. Accepted November 23, 2024.

1. Introduction

In this paper, we consider a controlled system, in a Hilbert space Z, of the form

$$z' = Az + Bu \quad \text{in } (0, \infty), \quad z(0) = z_0 \in Z. \tag{1.1}$$

In this setting, $(A, \mathcal{D}(A))$ is the infinitesimal generator of an analytic semigroup $(\mathrm{e}^{tA})_{t \geq 0}$ on Z, the control operator B is a bounded operator from a Hilbert space U into $(\mathcal{D}(A^*))'$, where $(A^*, \mathcal{D}(A^*))$ is the adjoint of $(A, \mathcal{D}(A))$.

Keywords and phrases: Feedback control, parabolic systems, stabilization, approximation of feedback laws.

Institut de Mathematiques de Toulouse, Universite Paul Sabatier & CNRS, 31062 Toulouse Cedex, France.

* Corresponding author: mehdi.badra@math.univ-toulouse.fr

We are going to consider nonconforming approximations of system (1.1), in a Hilbert space Z_ε, of the form

$$z_\varepsilon' = A_\varepsilon z_\varepsilon + B_\varepsilon u \quad \text{in } (0,\infty), \quad z_\varepsilon(0) = z_{\varepsilon,0} \in Z_\varepsilon, \tag{1.2}$$

where $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$ is the infinitesimal generator of an analytic semigroup $(\mathrm{e}^{tA_\varepsilon})_{t\geq 0}$ on Z_ε, the control operator B_ε is a bounded operator from U into $(\mathcal{D}(A_\varepsilon^*))'$, where $(A_\varepsilon^*, \mathcal{D}(A_\varepsilon^*))$ is the adjoint of $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$.

We speak of nonconforming approximation because $Z_\varepsilon \not\subset Z$, but Z and Z_ε are both Hilbert subspaces of a larger Hilbert space H.

The main results of the paper are the following:

- In Theorem 4.4(i), we prove that if the pair (A, B) is feedback stabilizable in Z, then the pair $(A_\varepsilon, B_\varepsilon)$ is feedback stabilizable in Z_ε, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, for $\varepsilon_0 > 0$ small enough.

- In Theorem 4.4(ii), we prove that if there exists a family of feedback operators $F_\varepsilon \in \mathcal{L}(Z_\varepsilon, U)$ for all $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 > 0$, stabilizing the pair $(A_\varepsilon, B_\varepsilon)$ in Z_ε, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, then $F_\varepsilon P_\varepsilon \in \mathcal{L}(Z, U)$ also stabilizes the pair (A, B) in Z.

- In Section 5, we study feedback operators defined in the LQR theory by the pair (A, B) and an output operator $\mathcal{C} \in \mathcal{L}(H, Y)$, where Y is another Hilbert space. We assume that the pair (A, B) is stabilizable and the pair $(A, \mathcal{C}|_Z)$ is detectable in Z. We denote by $\Pi \in \mathcal{L}(Z, Z')$ the solution to the algebraic Riccati equation associated with the triplet $(A, B, \mathcal{C}|_Z)$, and by $\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon, Z_\varepsilon')$ the solution to the algebraic Riccati equation associated with the triplet $(A_\varepsilon, B_\varepsilon, \mathcal{C}|_{Z_\varepsilon})$. In Proposition 5.12, we establish a convergence rate for $\|B^*\Pi P - B_\varepsilon^*\Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H,U)}$. Thus the assumptions of Theorem 4.4 are satisfied by the feedback laws $F = -B^*\Pi P$ and $F_\varepsilon = -B_\varepsilon^*\Pi_\varepsilon P_\varepsilon$, and the main result of Section 5 are collected in Theorem 5.13.

We would like to emphasize that many applications, for which assumptions $(H_1) - (H_5)$ stated in Section 2 are satisfied, fit with the abstract framework introduced here. In [1] (see also [2, 3]), we apply the results of Section 5 to the numerical approximation by a finite element method of feedback laws stabilizing the Oseen system in a bounded polyhedral domain of $\mathbb{R}^3$ (or a bounded polygonal domain of $\mathbb{R}^2$).

- The results of Section 5 can also be applied in the case where Z_ε is of infinite dimension. It is for example the case when the incompressibility condition in the Oseen system is approximated by the pseudo-compressibility method. For application of the results of Section 5 to that case, we refer to [4]. Let us notice that in [5], using results proved here in Section 3, we determine feedback laws for both the initial and the approximate system by using reduced order models based on spectral projections. We still obtain convergence rate for feedback laws in that case too.

As far as we know, the main results of the present paper – Theorems 4.4 and 5.13 – are the first ones of this type in the case of nonconforming approximations. But they can also be applied to the case of conforming approximations, that is when $Z = H$, $P = I$, and $Z_\varepsilon \subset Z$. Theorem 4.4 is new in the case of nonconforming approximations, and its proof relies on new tools introduced here for nonconforming approximations. We think that Theorem 5.13 is new, even in the case of conforming approximations, because both our assumptions and our method of proof are different from the existing results in the literature.

Let us make some comparisons with the existing literature. For bounded control operators and conforming approximations some results similar to those of Theorem 5.13 are established in [6, 7]. For unbounded control operators and conforming approximations the main contributions are due to Lasiecka and Triggiani in a series of papers [8–10] and in the book [11]. See also [12]. In Theorem 4.4, we have collected results which can be found in a weaker form in the case of conforming approximations in [11], Section 4.4.1, see also [10], Theorem 4.2 and [8], Theorem 2.3 (only convergence results are given in these references and not convergence rates. It is why these results are weaker than those stated in Thm. 4.4). Here, in order to extend the results stated in [11], Section 4.4.1, we have first to extend the notion of gap from an operator to another one, see [13], Chap. IV, par. 2.4, p.201, when these approximations are not defined in a subspace of Z (contrarily to the case of conforming approximations). This is done in Section 3. We also need a new resolvent identity adapted to the case of nonconforming approximations, which is stated and proved in Appendix A. But, results similar to those stated in (4.4) and (4.9) are neither given nor used in [8] or in [11]. The reason is that our method for proving Theorem 5.13 and that used in [10] or in [11] are different. This is what we explain at the end of Section 5.7.

As explained in Remark 5.14 at the end of Section 5.7, we think that some estimates are missing in [10] or in [11], Section 4.5.

Let us compare the assumptions used in [11], Section 4.5 and those we state in Section 2. The assumptions concerning the convergence rates stated in (2.4) and (2.6) are, for nonconforming approximations, similar to those in [11],(4.1.2.4), (4.1.2.8) for conforming approximations. Let us however notice that our choice (ε^s and ε^r in place of ε^s and $\varepsilon^{s(1-\gamma)}$) allows us to treat more general examples. The uniform bound stated in [11], (4.1.1.4) is similar to (2.5). The uniform bound (2.3) is not needed in [11] because the analogue of our projector P_ε is an orthogonal projector (denoted by Π_h in [11]). The main difference in the assumptions is in the uniform bound stated in (2.8). The corresponding assumptions in [11] are stated in [11], (4.1.2.6), (4.1.2.9). First notice that [11], (4.1.2.6) is an inverse inequality which can be satisfied only for finite dimensional approximations, while our assumption (2.8) can be used both for finite and infinite dimensional approximations. Moreover our method allows us to consider cases for which P and P_ε are not orthogonal projectors. It is for example the case when we define reduced order models based on spectral projections, see *e.g.* [5].

Let us finally mention that approximation results for infinite time horizon LQR problems, and eventually for associated Riccati operators, have been previously obtained *e.g.* in [14–16] or in [17], when the control operator B is bounded and the semigroup $(\mathrm{e}^{At})_{t\geq 0}$ is not necessarily analytic.

2. Assumptions and preliminary results

2.1. Notation

The inner product and the norm in H (resp. U) will be denoted by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ (resp. $(\cdot,\cdot)_U$ and $\|\cdot\|_U$) respectively. In Section 5, we will introduce another Hilbert space Y whose inner product and norm will be denoted by $(\cdot,\cdot)_Y$ and $\|\cdot\|_Y$ respectively.

The Hilbert spaces Z and Z_ε, which are continuously embedded in H, are equipped with the norm in H. Since $P\in\mathcal{L}(H)$ is a projector in H onto Z, if we set $Z_o=(I-P)H$, P is obviously the projector onto Z parallel to Z_o. We have $H=Z\oplus Z_o$. In what follows, we identify H with its dual H', U with U', and Y with Y'. Since H and H' are identified, if P is not an orthogonal projector in H, we cannot identify Z with Z', but the duality product $\langle\cdot,\cdot\rangle_{Z,Z'}$ is nothing but the inner product $(\cdot,\cdot)_H$.

We need to introduce $P^*\in\mathcal{L}(H)$, the adjoint of $P\in\mathcal{L}(H)$. We have $H=P^*H\oplus(I-P^*)H$. If P is not an orthogonal projector, we have $P\neq P^*$ and $Z\neq P^*H$. We can easily check that P^*H can be identified with Z', and $(I-P^*)H$ can be identified with Z_o'. With such identifications $H=Z'\oplus Z_o'$, and P^* is the projector in H onto Z' parallel to Z_o'.

Similarly, we set $Z_{o,\varepsilon}=(I-P_\varepsilon)H$. We introduce $P_\varepsilon^*\in\mathcal{L}(H)$, the adjoint of $P_\varepsilon\in\mathcal{L}(H)$. We will identify P_ε^*H with Z_ε', and $(I-P_\varepsilon^*)H$ with $Z_{o,\varepsilon}'$. We can identify Z_ε' with Z_ε, and $Z_{o,\varepsilon}'$ with $Z_{o,\varepsilon}$, only if P_ε is an orthogonal projector in H.

To shorten the notation, for $1\leq p\leq\infty$ and any Hilbert space X, the Bochner space $L^p(0,\infty;X)$ will be denoted by $L^p(X)$:

$$L^p(X)\stackrel{\text{def}}{=}L^p(0,\infty;X).$$

Throughout what follows, C denotes a generic constant which may vary from one occurence to another one, but is independent of the parameter ε and of $\lambda\in\mathbb{C}$. Sometimes, we emphasize the dependence of a constant on some other parameters θ, ℓ, or K, by writing C_θ, C_ℓ or C_K.

2.2. Assumptions

(H_1) There exists $(\omega_0,\delta)\in\mathbb{R}\times(0,\pi/2)$ such that:

$$\begin{aligned}&\{\omega_0\}+\mathbb{S}_{\pi/2+\delta}\subset\rho(A),\\ &\|(\lambda I-A)^{-1}\|_{\mathcal{L}(Z)}\leq\frac{C}{|\lambda-\omega_0|},\quad\text{for all }\lambda\in\{\omega_0\}+\mathbb{S}_{\pi/2+\delta},\end{aligned}\tag{2.1}$$

and

$$\begin{aligned} &\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A_\varepsilon), \quad \forall \varepsilon \in (0,1), \\ &\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le \frac{C}{|\lambda - \omega_0|}, \quad \forall \varepsilon \in (0,1), \quad \forall \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}, \end{aligned} \tag{2.2}$$

where, for $\delta \in (0, \pi/2)$, the subset $\mathbb{S}_{\pi/2+\delta} \subset \mathbb{C}$ denotes the sector $\{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \pi/2 + \delta\}$, and $\rho(A)$ and $\rho(A_\varepsilon)$ are the resolvent sets of A and A_ε respectively.

In what follows, $(A^*, \mathcal{D}(A^*))$, the adjoint of $(A, \mathcal{D}(A))$, is an unbounded operator in Z', and $(A_\varepsilon^*, \mathcal{D}(A_\varepsilon^*))$, the adjoint of $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$, is an unbounded operator in Z'_ε. We set

$$\widehat{A} \stackrel{\text{def}}{=} A - \lambda_0 I \quad \text{and} \quad \widehat{A}_\varepsilon \stackrel{\text{def}}{=} A_\varepsilon - \lambda_0 I, \quad \text{with } \lambda_0 > \omega_0.$$

We recall that $P \in \mathcal{L}(H)$ is a projector from H onto Z, and that $P_\varepsilon \in \mathcal{L}(H)$ is a projector from H onto Z_ε.

(H_2) The family of projectors $P_\varepsilon \in \mathcal{L}(H)$ satisfies the uniform bound

$$\sup_{\varepsilon \in (0,1)} \|P_\varepsilon\|_{\mathcal{L}(H)} < +\infty, \tag{2.3}$$

and the pair $(\widehat{A}, \widehat{A}_\varepsilon)$ satisfies the following approximation assumption

$$\|\widehat{A}^{-1} P - \widehat{A}_\varepsilon^{-1} P_\varepsilon\|_{\mathcal{L}(H)} \le C\varepsilon^s, \quad \forall \varepsilon \in (0,1), \quad \text{with } s > 0. \tag{2.4}$$

Remark 2.1. Note that (2.1) and (2.4) do not imply (2.2). To observe that, suppose that $Z = Z_\varepsilon = H$, let $(e_n)_{n \in \mathbb{N}}$ be an Hilbertian basis of H and let define A and A_ε by

$$Az = \sum_{n=1}^{+\infty} n(z, \varphi_n)_H \varphi_n \quad \text{and} \quad A_\varepsilon z = \sum_{n=1}^{N_\varepsilon - 1} n(z, \varphi_n)_H \varphi_n + \sum_{n=N_\varepsilon}^{+\infty} in(z, \varphi_n)_H \varphi_n$$

where N_ε is the integer satisfying $N_\varepsilon - 1 \le \frac{2}{\varepsilon} < N_\varepsilon$. It is easy to see that A satisfies (2.1) and that A_ε does not satisfies (2.2) since its spectrum is defined by $\sigma(A_\varepsilon) = \{n \in \mathbb{N}^* \mid n \le N_\varepsilon - 1\} \cup \{i\, n \mid n \in \mathbb{N}^*, n \ge N_\varepsilon\}$. But the following calculations show that (2.4) is satisfied with $s = 1$:

$$\left\|A^{-1} z - A_\varepsilon^{-1} z\right\|_H = \left\| \sum_{n=N_\varepsilon}^{+\infty} \frac{1}{n} (z, \varphi_n)_H \varphi_n - \sum_{n=N_\varepsilon}^{+\infty} \frac{1}{in} (z, \varphi_n)_H \varphi_n \right\|_H \le \frac{2}{N_\varepsilon} \|z\|_H \le \varepsilon \|z\|_H.$$

(H_3) The control operator B belongs to $\mathcal{L}(U, (\mathcal{D}(A^*))')$, and it satisfies

$$(-\widehat{A})^{-\gamma} B \in \mathcal{L}(U, Z) \quad \text{for some } \gamma \in [0,1). \tag{2.5}$$

(H_4) For all $\varepsilon \in (0,1)$, B_ε belongs to $\mathcal{L}(U, (\mathcal{D}(A_\varepsilon^*))')$. The pair (B, B_ε) satisfies the following approximation assumption

$$\|\widehat{A}^{-1} B - \widehat{A}_\varepsilon^{-1} B_\varepsilon\|_{\mathcal{L}(U,H)} \le C\varepsilon^r \quad \text{for all } \varepsilon \in (0,1), \quad \text{with } 0 < r \le s(1-\gamma). \tag{2.6}$$

In addition, when $\gamma > 0$ and $r = s(1-\gamma)$, we assume that

$$[Z, \mathcal{D}(A)]_\theta = \mathcal{D}(\widehat{A}^\theta), \quad \forall \theta \in (0,1), \tag{2.7}$$

where $[\cdot,\cdot]_\theta$ stands for the complex interpolation.

Remark 2.2. Assumption (2.7) is equivalent to the local boundedness of imaginary powers of $\widehat{A}$ (see [18], Thm. 1.15.3, p. 103). Assumption (2.7) is used in the proof of Proposition 2.8 to replace $[Z, \mathcal{D}(A)]_{1-\gamma}$ by $\mathcal{D}((\widehat{A})^{1-\gamma})$ (and next Prop. 2.8 is used to prove Props. 4.7 and 4.8). We do not need such an argument for $(Z_\varepsilon, A_\varepsilon)$, and therefore the analogue of Assumption (2.7) is not needed for $(Z_\varepsilon, A_\varepsilon)$. Assumption (2.7) is satisfied in usual cases, for instance when $\widehat{A}$ is maximal accretive (see [19], Prop. 6.1, p. 171). Let us notice that this type of condition is needed in [11], Chapter 4 even if this assumption is not explicitely stated there. For instance, it is required to derive [11], (4.3.3) and (4.3.6).

(H_5) The family of operators $(B_\varepsilon)_{\varepsilon\in(0,1)}$ satisfies $(-\widehat{A}_\varepsilon)^{-\gamma} B_\varepsilon \in \mathcal{L}(U, Z_\varepsilon)$, for all $\varepsilon \in (0,1)$, and the uniform bound

$$\|\mathrm{e}^{tA_\varepsilon} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \leq C\,\frac{\mathrm{e}^{\omega_0 t}}{t^{\overline{\gamma}}}, \quad \forall t \in (0,\overline{\varepsilon}) = (0, \varepsilon^{r/(1-\gamma)}),\ \forall \varepsilon \in (0,1), \tag{2.8}$$

for some $\overline{\gamma} \in [\gamma, 1)$.

Remark 2.3. When $\gamma = 0$, the operator B belongs to $\mathcal{L}(U, Z)$. And if in (2.8), $\overline{\gamma}$ is also equal to zero, this means that $B_\varepsilon \in \mathcal{L}(U, Z_\varepsilon)$, and that

$$\|B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \leq C, \quad \forall \varepsilon \in (0,1).$$

But (2.8) allows us to consider sequences of operators $(B_\varepsilon)_{\varepsilon\in(0,1)}$ which are not necessarily bounded, even if $B \in \mathcal{L}(U, Z)$.

Remark 2.4. From (H_3) and (H_4), it follows that

$$\sup_{\varepsilon\in(0,1)} \|(-\widehat{A}_\varepsilon)^{-1} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} < +\infty. \tag{2.9}$$

We can also notice that if

$$\sup_{\varepsilon\in(0,1)} \|(-\widehat{A}_\varepsilon)^{-\overline{\gamma}} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} < +\infty, \tag{2.10}$$

is satisfied, then Assumption (2.8) is verified as a consequence of (2.14) below. However, we will see in some applications that Assumption (2.8) can be verified while (2.10) is not necessarily satisfied (see [1]).

2.3. First error estimates for the semigroup $(\mathrm{e}^{tA})_{t>0}$

For all $\theta \in [0,1]$, and all $\varepsilon \in (0,1)$, we have the following estimates:

$$\|(-\widehat{A})^\theta (\lambda I - A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C_\theta}{|\lambda - \omega_0|^{1-\theta}},\ \forall \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}, \tag{2.11}$$

$$\|(-\widehat{A})^\theta \mathrm{e}^{At}\|_{\mathcal{L}(Z)} \leq C_\theta \frac{\mathrm{e}^{\omega_0 t}}{t^\theta}, \quad \forall t > 0, \tag{2.12}$$

$$\|(-\widehat{A}_\varepsilon)^\theta (\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq \frac{C_\theta}{|\lambda - \omega_0|^{1-\theta}},\ \forall \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}, \tag{2.13}$$

$$\|(-\widehat{A}_\varepsilon)^\theta \mathrm{e}^{A_\varepsilon t}\|_{\mathcal{L}(Z_\varepsilon)} \le C_\theta \frac{\mathrm{e}^{\omega_0 t}}{t^\theta}, \quad \forall t > 0. \tag{2.14}$$

Estimate (2.11) is a consequence of (2.1) and of the interpolation inequality stated in [20], (6.19) in Chapter 2, Theorem 6.10. With [19], Chapter II-1, (2.43) and (2.44), estimate (2.12) for $\theta = 1$ and $\theta = 0$ follows from (2.1). Estimate (2.12) for $0 < \theta < 1$ is obtained by interpolation. Similarly, (2.13) and (2.14) follow from (2.2).

Theorem 2.5. *Let us assume that* (2.1) *to* (2.4) *are satisfied. For all* $0 \le \theta \le 1$, *and all* $\varepsilon \in (0,1)$, *and* $t > 0$, *we have*

$$\|\mathrm{e}^{At}P - \mathrm{e}^{A_\varepsilon t}P_\varepsilon\|_{\mathcal{L}(H)} \le C \frac{\mathrm{e}^{\omega_0 t}}{t^\theta} \varepsilon^{s\theta}. \tag{2.15}$$

Moreover, we have

$$\begin{aligned} &\|\mathrm{e}^{\widehat{A}(\cdot)}P - \mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon\|_{L^1(\mathcal{L}(H))} \le C\varepsilon^s |\ln \varepsilon|, \\ &\text{and} \\ &\|\mathrm{e}^{\widehat{A}(\cdot)}P - \mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon\|_{L^p(\mathcal{L}(H))} \le C\varepsilon^{\frac{s}{p}}, \quad \text{for all } p \in (1,\infty). \end{aligned} \tag{2.16}$$

Proof. Step 1. Let us first prove that, for all $\theta \in [0,1]$, we have:

$$\|(\lambda I - A)^{-1}P - (\lambda I - A_\varepsilon)^{-1}P_\varepsilon\|_{\mathcal{L}(H)} \le \frac{C_{\theta,\rho}}{|\lambda - \omega_0|^{1-\theta}} \varepsilon^{s\theta}, \quad \forall \lambda \in \{\omega_0\} + \Gamma_{\rho,\delta}, \tag{2.17}$$

where $\Gamma_{\rho,\delta}$ is the path, oriented from $\infty \mathrm{e}^{-i\frac{\pi+\delta}{2}}$ to $\infty \mathrm{e}^{i\frac{\pi+\delta}{2}}$, defined by

$$\Gamma_{\rho,\delta} \overset{\text{def}}{=} \{r\mathrm{e}^{-i\frac{\pi+\delta}{2}}\}_{r>\rho} \cup \{\rho \mathrm{e}^{i\alpha}\}_{|\alpha| \le \frac{\pi}{2} + \frac{\delta}{2}} \cup \{r\mathrm{e}^{i\frac{\pi+\delta}{2}}\}_{r>\rho}, \quad \rho > 0. \tag{2.18}$$

For $\theta = 0$, (2.17) follows from (2.1) and (2.2).

To prove (2.17) for $\theta = 1$, we use the following resolvent identity (see Appendix A):

$$\begin{aligned} &(\lambda I - A_\varepsilon)^{-1}P_\varepsilon - (\lambda I - A)^{-1}P \\ &= (I - (\lambda - \lambda_0)(\lambda I - A)^{-1}P)(\widehat{A}^{-1}P - \widehat{A}_\varepsilon^{-1}P_\varepsilon)(I - (\lambda - \lambda_0)(\lambda I - A_\varepsilon)^{-1}P_\varepsilon), \end{aligned} \tag{2.19}$$

for $\lambda \in \{\omega_0\} + \Gamma_{\rho,\delta}$. From (2.1) and (2.2), it follows that

$$\|(\lambda - \lambda_0)(\lambda I - A_\varepsilon)^{-1}P_\varepsilon\|_{\mathcal{L}(H)} + \|(\lambda - \lambda_0)(\lambda I - A)^{-1}P\|_{\mathcal{L}(H)} \le \frac{C|\lambda - \lambda_0|}{|\lambda - \omega_0|} \le C_\rho,$$

for all $\lambda \in \{\omega_0\} + \Gamma_{\rho,\delta}$. Thus, with (2.19) and (2.4), we have:

$$\|(\lambda I - A)^{-1}P - (\lambda I - A_\varepsilon)^{-1}P_\varepsilon\|_{\mathcal{L}(H)} \le C_\rho \, \varepsilon^s.$$

For $0 < \theta < 1$, inequality (2.17) follows by interpolation from the estimates proved for $\theta = 0$ and for $\theta = 1$.

Step 2. Proof of (2.15). With the Dunford integral formula (see [19], Thm. 2.10, p. 109 or [20], formula (7.25), Chap. 1), we have:

$$\mathrm{e}^{At}P - \mathrm{e}^{A_\varepsilon t}P_\varepsilon = \frac{1}{2i\pi} \int_{\{\omega_0\} + \Gamma_{\rho,\delta}} \mathrm{e}^{\lambda t}((\lambda I - A)^{-1}P - (\lambda I - A_\varepsilon)^{-1}P_\varepsilon) \mathrm{d}\lambda.$$

Making the change of variable $\xi = t(\lambda-\omega_0)$, which maps $\{\omega_0\}+\Gamma_{\rho,\delta}$ to $t\Gamma_{\rho,\delta}$, we obtain

$$\int_{\{\omega_0\}+\Gamma_{\rho,\delta}} \mathrm{e}^{\lambda t}((\lambda I-A)^{-1}P-(\lambda I-A_\varepsilon)^{-1}P_\varepsilon)\mathrm{d}\lambda \\ = \int_{t\Gamma_{\rho,\delta}} \mathrm{e}^{\xi+t\omega_0}(((\xi/t+\omega_0)I-A)^{-1}P-((\xi/t+\omega_0)I-A_\varepsilon)^{-1}P_\varepsilon)\frac{\mathrm{d}\xi}{t}. \quad (2.20)$$

For $t>0$ the oriented path $t\Gamma_{\rho,\delta}$ is the sum of the oriented path $\Gamma_{\rho,\delta}$ and another one denoted by $\mathcal{C}_{t,\rho,\delta}$, which is a closed oriented path. Since $\mathcal{C}_{t,\rho,\delta}$ is closed and does not enclose 0, we have

$$\int_{\mathcal{C}_{t,\rho,\delta}} \mathrm{e}^{\xi+t\omega_0}(((\xi/t+\omega_0)I-A)^{-1}P-((\xi/t+\omega_0)I-A_\varepsilon)^{-1}P_\varepsilon)\frac{\mathrm{d}\xi}{t}=0.$$

The path $t\Gamma_{\rho,\delta}$ in (2.20) can be replaced by $\Gamma_{\rho,\delta}$, and, with (2.17), we obtain

$$\|\mathrm{e}^{At}P-\mathrm{e}^{A_\varepsilon t}P_\varepsilon\|_{\mathcal{L}(H)} \le \frac{C_{\theta,\rho}\varepsilon^{s\theta}}{2\pi t^\theta}\mathrm{e}^{\omega_0 t}\left|\int_{\Gamma_{\rho,\delta}}\frac{\mathrm{e}^{\xi}}{|\xi|^{1-\theta}}\mathrm{d}\xi\right| \le \frac{C\mathrm{e}^{\omega_0 t}\varepsilon^{s\theta}}{t^\theta}.$$

Step 3. Proof of (2.16). To prove (2.16) for $1\le p<\infty$, it is sufficient to write

$$\begin{aligned}\|\mathrm{e}^{\widehat{A}t}P-\mathrm{e}^{\widehat{A}_\varepsilon t}P_\varepsilon\|_{L^p(\mathcal{L}(H))} &\le C\Big(\int_0^{\varepsilon^s}\mathrm{d}t+\varepsilon^{sp}\int_{\varepsilon^s}^1\frac{\mathrm{d}t}{t^p}+\varepsilon^{sp}\int_1^\infty \mathrm{e}^{-p(\lambda_0-\omega_0)t}\mathrm{d}t\Big)^{1/p}\\ &\le C\begin{cases}(\varepsilon^s+\varepsilon^s|\ln(\varepsilon)|), & \text{if } p=1,\\ (\varepsilon^{s/p}+\varepsilon^s), & \text{if } 1<p<\infty,\end{cases}\end{aligned}$$

where we have used (2.15) with $\theta=0$ if $t\in[0,\varepsilon^s]$, and with $\theta=1$ if $t\in(\varepsilon^s,\infty)$. □

Remark 2.6. There is a link between the Trotter–Kato Theorem and Theorem 2.5. The Trotter–Kato Theorem mainly states that, under the stability property of the approximations of a stable semigroup, the convergence of the approximate resolvents is equivalent to the convergence of the approximate semigroups, see [21], Theorem 2.1. Here we derive not only convergence properties but also convergence rates.

Remark 2.7. A brief check of the proof of Theorem 2.5 shows that inequality (2.17) is true for $\lambda\in\{\omega_0\}+(\mathbb{S}_{\pi/2+\delta}\setminus\bar{\mathbb{B}}_\eta)$, where for $\eta>0$, $\bar{\mathbb{B}}_\eta=\{\lambda\in\mathbb{C}\mid|\lambda|\le\eta\}$. More precisely, for $\eta>0$, we have

$$\|(\lambda I-A_\varepsilon)^{-1}P_\varepsilon-(\lambda I-A)^{-1}P\|_{\mathcal{L}(H)}\le\frac{C_\eta}{|\lambda-\omega_0|^{1-\theta}}\varepsilon^{s\theta}, \quad (2.21)$$

for all $0\le\theta\le1$ and $\lambda\in\{\omega_0\}+(\mathbb{S}_{\pi/2+\delta}\setminus\bar{\mathbb{B}}_\eta)$.

Proposition 2.8. *Let $\eta>0$ and set $\bar{\mathbb{B}}_\eta=\{\lambda\in\mathbb{C}\mid|\lambda|\le\eta\}$. For all $\lambda\in\{\omega_0\}+(\mathbb{S}_{\pi/2+\delta}\setminus\bar{\mathbb{B}}_\eta)$ we have*

$$\|(\lambda I-A_\varepsilon)^{-1}P_\varepsilon-(\lambda I-A)^{-1}\|_{\mathcal{L}(\mathcal{D}((-\widehat{A})^{1-\gamma}),Z)}\le\frac{C_\eta}{|\lambda-\omega_0|}\varepsilon^r. \quad (2.22)$$

Proof. Let us prove that, for $\theta \in [0,1]$ and $\lambda \in \{\omega_0\} + (\mathbb{S}_{\pi/2+\delta} \setminus \bar{\mathbb{B}}_\eta)$, we have

$$\|(\lambda I - A)^{-1} - (\lambda I - A_\varepsilon)^{-1} P_\varepsilon\|_{\mathcal{L}([Z,\mathcal{D}(A)]_\theta, Z)} \leq \frac{C}{|\lambda - \omega_0|} \varepsilon^{s\theta}. \tag{2.23}$$

From the second statement of Lemma A.1 in Appendix A, we have

$$((\lambda I - A)^{-1} - (\lambda I - A_\varepsilon)^{-1} P_\varepsilon) P = (I - (\lambda - \lambda_0)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon)(\widehat{A}_\varepsilon^{-1} P_\varepsilon - \widehat{A}^{-1} P)(\lambda I - A)^{-1}(-\widehat{A})P.$$

From this identity and from (2.1), (2.2) and (2.4) we deduce (2.23) for $\theta = 1$. The case $\theta = 0$ is a direct consequence of (2.1) and (2.2). Thus, (2.23) follows by interpolation.

If $\gamma = 0$, then (2.22) is an immediate consequance of (2.23) with $\theta = 1$. If $\gamma > 0$ and $r = s(1-\gamma)$, then we deduce (2.22) from (2.23) and from assumption (2.7), both with $\theta = 1 - \gamma$.

If $\gamma > 0$ and $r < s(1-\gamma)$, then we deduce (2.22) from (2.23) with $\theta = r/s$ and from the continuous embedding $\mathcal{D}(\widehat{A}^{1-\gamma}) \hookrightarrow [Z, \mathcal{D}(A)]_{r/s}$. This last embedding is true since $r/s < 1 - \gamma$, see [18], Theorem 1.15.2 (d) p. 101, Theorem 1.3.3 (a) and (e) and para 1.18.10. Remark 3 (3), p. 143. □

3. Approximation of the semigroup and of the resolvent set

Here we suppose that Z_1 and Z_2 are two closed subspaces of H, that $P_1 : H \to Z_1$ and $P_2 : H \to Z_2$ are projection operators and that $\mathbb{A}_1$ and $\mathbb{A}_2$ are closed linear operators densely defined in Z_1 and Z_2 respectively.

The main goal of this section is to prove Theorem 3.2 and Corollary 3.3. This last will be used in the proof of Theorem 4.4−(i) with $\mathbb{A}_1 = A_\varepsilon + B_\varepsilon F_\varepsilon$ and $\mathbb{A}_2 = A + BF$, and in the proof of Theorem 4.4−(ii) with $\mathbb{A}_1 = A + BF^{(\varepsilon)}$ and $\mathbb{A}_2 = A_\varepsilon + B_\varepsilon F_\varepsilon$, where $F \in \mathcal{L}(Z, U)$, $F_\varepsilon \in \mathcal{L}(Z_\varepsilon, U)$ and $F^{(\varepsilon)} \in \mathcal{L}(Z, U)$ are feedback operators. To prove Theorem 3.2, we have to extend the notion of gap introduced in [13] to the case where $\mathbb{A}_1$ and $\mathbb{A}_2$ are not defined in the same space. For that, we are going to define the gap between the pairs $(\mathbb{A}_1, P_1)$ and $(\mathbb{A}_2, P_2)$. We first set

$$\delta((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2)) = \sup_{(z_1,\zeta) \in S(\mathbb{A}_1, P_1)} \inf_{z_2 \in \mathcal{D}(\mathbb{A}_2)} \Big\{ \|z_1 - z_2\|_H + \|P_2(\mathbb{A}_1 z_1 + (I - P_1)\zeta) - \mathbb{A}_2 z_2\|_H \Big\}, \tag{3.1}$$

where

$$S(\mathbb{A}_1, P_1) = \{(z_1, \zeta) \in \mathcal{D}(\mathbb{A}_1) \times H \mid \|z_1\|_H + \|\mathbb{A}_1 z_1 + (I - P_1)\zeta\|_H = 1\}.$$

The *gap between* $(\mathbb{A}_1, P_1)$ *and* $(\mathbb{A}_2, P_2)$ is defined by:

$$\widehat{\delta}((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2)) = \max\{\delta((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2)), \delta((\mathbb{A}_2, P_2), (\mathbb{A}_1, P_1))\}. \tag{3.2}$$

Note that when $Z_1 = Z_2 = H$ and $P_1 = P_2 = I$ is the identity in H, such a notion of gap coincides with that in [13], Chapter IV, para 2.4, p. 201. The arguments of the proof of Proposition 3.1 below are largely borrowed from [13].

Proposition 3.1. *Let* $\mathbb{A}_1$ *and* $\mathbb{A}_2$ *be closed linear operators densely defined on* Z_1 *and* Z_2 *respectively.*

(i) *If* $\mathbb{A}_1$ *and* $\mathbb{A}_2$ *are both boundedly invertible then*

$$\widehat{\delta}((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2)) \leq \|\mathbb{A}_1^{-1} P_1 - \mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H)}. \tag{3.3}$$

(ii) *If* $\mathbb{A}_2$ *has a bounded inverse on* Z_2 *and if the following inequality holds:*

$$\widehat{\delta}((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2)) < \frac{1}{2(1 + \|\mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H,Z_2)})}, \tag{3.4}$$

then $\mathbb{A}_1$ *admits a bounded inverse on* Z_1*. Moreover, the following inequality holds:*

$$\frac{\|\mathbb{A}_1^{-1} - \mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(Z_1)}}{2(1 + \|\mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H,Z_2)})^2} \le \widehat{\delta}((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2)). \tag{3.5}$$

(iii) *Let us define*

$$M(P_1, P_2) \stackrel{def}{=} \max\left\{\|P_1\|_{\mathcal{L}(H,Z_1)}, \|P_2\|_{\mathcal{L}(H,Z_2)}\right\}. \tag{3.6}$$

For all $\lambda \in \mathbb{C}$*, the following inequality holds:*

$$\widehat{\delta}((\mathbb{A}_1 - \lambda I, P_1), (\mathbb{A}_2 - \lambda I, P_2)) \le (1 + |\lambda| M(P_1, P_2))^2 \widehat{\delta}((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2)). \tag{3.7}$$

Proof. (i) It suffices to prove $\delta((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2)) \le \|\mathbb{A}_1^{-1} P_1 - \mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H)}$. The inequality $\delta((\mathbb{A}_2, P_2), (\mathbb{A}_1, P_1)) \le \|\mathbb{A}_1^{-1} P_1 - \mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H)}$ will next be deduced by reversing the role of $(\mathbb{A}_1, P_1)$ and $(\mathbb{A}_2, P_2)$. Since $\mathbb{A}_1$ and $\mathbb{A}_2$ are both boundedly invertible then, with $\xi = \mathbb{A}_1 z_1 + (I - P_1)\zeta$ and $\xi_2 = \mathbb{A}_2 z_2$ in (3.1), we deduce that

$$\delta((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2)) = \sup_{\xi \in H,\, \|\xi\|_H + \|\mathbb{A}_1^{-1} P_1 \xi\|_H = 1} \ \inf_{\xi_2 \in Z_2} \left\{ \|\mathbb{A}_1^{-1} P_1 \xi - \mathbb{A}_2^{-1} \xi_2\|_H + \|P_2 \xi - \xi_2\|_H \right\}. \tag{3.8}$$

For any given $\xi \in H$ such that $\|\xi\|_H + \|\mathbb{A}_1^{-1} P_1 \xi\|_H = 1$, we choose $\xi_2 = P_2 \xi$ to first get $\|\mathbb{A}_1^{-1} P_1 \xi - \mathbb{A}_2^{-1} P_2 \xi\|_H$ as an upper bound of the infimum in (3.8). Next, from $\|\xi\|_H \le \|\xi\|_H + \|\mathbb{A}_1^{-1} P_1 \xi\|_H = 1$ we deduce that $\|\mathbb{A}_1^{-1} P_1 \xi - \mathbb{A}_2^{-1} P_2 \xi\|_H \le \|\mathbb{A}_1^{-1} P_1 - \mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H)}$, and we conclude by taking the supremum on $\xi \in H$ such that $\|\xi\|_H + \|\mathbb{A}_1^{-1} P_1 \xi\|_H = 1$.

(ii) We first prove that $\mathbb{A}_1$ is one-to-one. Let us argue by contradiction. We assume that there exists $z_1 \in \mathcal{D}(\mathbb{A}_1)$ satisfying $\|z_1\|_H = 1$ and $\mathbb{A}_1 z_1 = 0$. Then $(z_1, 0) \in S(\mathbb{A}_1, P_1)$ and, according to the definition of $\delta((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2))$ in (3.1), we can choose $z_2 \in \mathcal{D}(\mathbb{A}_2)$ such that

$$\|z_1 - z_2\|_H + \|\mathbb{A}_2 z_2\|_H < (2 + 2\|\mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H,Z_2)})^{-1}.$$

It follows that $1 = \|z_1\|_H \le \|z_1 - z_2\|_H + \|\mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H,Z_2)} \|\mathbb{A}_2 z_2\|_H < 1/2$, which gives a contradiction. Thus $\mathbb{A}_1$ is one-to-one.

Let $\mathcal{R}(\mathbb{A}_1)$ be the range of $\mathbb{A}_1$. Since $\mathbb{A}_1$ is one-to-one it admits an inverse $\mathbb{A}_1^{-1}$ defined on $\mathcal{R}(\mathbb{A}_1)$.

Let us prove that $\mathbb{A}_1^{-1}$ is bounded on $\mathcal{R}(\mathbb{A}_1)$. Let $\xi_1 \in \mathcal{R}(\mathbb{A}_1)$ and set $r_1 = \|\xi_1\|_H + \|\mathbb{A}_1^{-1} \xi_1\|_H$. There exists $z_1 \in \mathcal{D}(\mathbb{A}_1)$ such that $\xi_1 = \mathbb{A}_1 z_1$ and moreover $r_1 = \|z_1\|_H + \|\mathbb{A}_1 z_1\|_H$. For all $z_2 \in \mathcal{D}(\mathbb{A}_2)$, we set $\xi_2 = \mathbb{A}_2 z_2$, and we have

$$\begin{aligned}
\left\|(\mathbb{A}_1^{-1} - \mathbb{A}_2^{-1} P_2) \frac{\xi_1}{r_1}\right\|_H &\le \left\|\mathbb{A}_1^{-1} \frac{\xi_1}{r_1} - \mathbb{A}_2^{-1} \xi_2\right\|_H + \left\|\mathbb{A}_2^{-1}\left(\xi_2 - P_2 \frac{\xi_1}{r_1}\right)\right\|_H \\
&\le (1 + \|\mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H,Z_2)}) \left(\left\|\mathbb{A}_1^{-1} \frac{\xi_1}{r_1} - \mathbb{A}_2^{-1} \xi_2\right\|_H + \left\|P_2 \frac{\xi_1}{r_1} - \xi_2\right\|_H\right) \\
&= (1 + \|\mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H,Z_2)}) \left(\left\|\frac{z_1}{r_1} - z_2\right\|_H + \left\|P_2 \mathbb{A}_1 \frac{z_1}{r_1} - \mathbb{A}_2 z_2\right\|_H\right).
\end{aligned}$$

By taking the infimum over $z_2 \in \mathcal{D}(\mathbb{A}_2)$, and by observing that $z_1 \in \mathcal{D}(\mathbb{A}_1)$ satisfies $\left\|\frac{z_1}{r_1}\right\|_H + \left\|\mathbb{A}_1\frac{z_1}{r_1}\right\|_H = 1$, we obtain

$$\|(\mathbb{A}_1^{-1} - \mathbb{A}_2^{-1}P_2)\xi_1\|_H \leq (1 + \|\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H,Z_2)})\delta((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2))(\|\mathbb{A}_1^{-1}\xi_1\|_H + \|\xi_1\|_H).$$

The above inequality, with

$$\begin{aligned}\|\mathbb{A}_1^{-1}\xi_1\|_H + \|\xi_1\|_H &\leq \|(\mathbb{A}_1^{-1} - \mathbb{A}_2^{-1}P_2)\xi_1\|_H + \|\mathbb{A}_2^{-1}P_2\xi_1\|_H + \|\xi_1\|_H \\ &\leq \|(\mathbb{A}_1^{-1} - \mathbb{A}_2^{-1}P_2)\xi_1\|_H + (1 + \|\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H,Z_2)})\|\xi_1\|_H,\end{aligned}$$

gives

$$\begin{aligned}\|(\mathbb{A}_1^{-1} - \mathbb{A}_2^{-1}P_2)\xi_1\|_H \leq (1 + \|\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H,Z_2)})\delta((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2))\|(\mathbb{A}_1^{-1} - \mathbb{A}_2^{-1}P_2)\xi_1\|_H \\ + (1 + \|\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H,Z_2)})^2\delta((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2))\|\xi_1\|_H. \quad (3.9)\end{aligned}$$

Estimates (3.9) and (3.4) give

$$\|(\mathbb{A}_1^{-1} - \mathbb{A}_2^{-1}P_2)\xi_1\|_H \leq 2(1 + \|\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H,Z_2)})^2\delta((\mathbb{A}_1, P_1), (\mathbb{A}_2, P_2))\|\xi_1\|_H, \quad (3.10)$$

for all $\xi_1 \in \mathcal{R}(\mathbb{A}_1)$. Thus $\mathbb{A}_1^{-1}$ is bounded on $\mathcal{R}(\mathbb{A}_1)$, or equivalently, there exists $c > 0$ such that $\|A_1 z\|_H \geq c\|z\|_H$ for all z in $\mathcal{D}(\mathbb{A}_1)$. By combining this with the fact that $\mathbb{A}_1$ is a closed linear operator on Z_1 we deduce that $\mathcal{R}(\mathbb{A}_1)$ is closed in Z_1 (see *e.g.* [22], Rem. 18, p. 47).

Hence, if we prove that $\mathcal{R}(\mathbb{A}_1)$ is dense in Z_1 then we will deduce that $\mathcal{R}(\mathbb{A}_1) = Z_1$ and (3.5) will follow from the inequality (3.10).

Let us prove that $\mathcal{R}(\mathbb{A}_1)$ is dense in Z_1. For that, we will use the following bound implied by (3.4):

$$\delta((\mathbb{A}_2, P_2), (\mathbb{A}_1, P_1)) \leq \frac{1}{2(1 + \|\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H,Z_2)})}.$$

From the above inequality we deduce that, for all $(z_2, \zeta) \in \mathcal{D}(\mathbb{A}_2) \times H$ such that $\|\mathbb{A}_2 z_2 + (I - P_2)\zeta\|_H + \|z_2\|_H = 1$, there exists $z_1 \in \mathcal{D}(\mathbb{A}_1)$ such that we have

$$\|P_1(\mathbb{A}_2 z_2 + (I - P_2)\zeta) - \mathbb{A}_1 z_1\|_H \leq \frac{1}{2(1 + \|\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H,Z_2)})}. \quad (3.11)$$

Let $\xi_1 \in Z_1$ such that $\|\xi_1\|_H = 1$, and set $r_1 = 1 + \|\mathbb{A}_2^{-1}P_2\xi_1\|_H$. We have $\|\frac{\xi_1}{r_1}\|_H + \|\mathbb{A}_2^{-1}P_2\frac{\xi_1}{r_1}\|_H = 1$. With $z_2 = \mathbb{A}_2^{-1}P_2\frac{\xi_1}{r_1}$ and $\zeta = \frac{\xi_1}{r_1}$ in (3.11) we deduce

$$\left\|\frac{\xi_1}{r_1} - \mathbb{A}_1 z_1\right\|_H \leq \frac{1}{2(1 + \|\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H,Z_2)})}.$$

Hence, with $r_1 \leq (1 + \|\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H,Z_2)})$, we deduce

$$\|\xi_1 - r_1\mathbb{A}_1 z_1\|_H \leq \frac{1}{2}.$$

This implies that $\mathrm{dist}(\xi_1, \mathcal{R}(\mathbb{A}_1)) \leq 1/2$. Since this last inequality holds for all ξ_1 in the unit sphere of Z_1, from [13], Lemma 1.12, p. 131, we deduce that $\mathcal{R}(\mathbb{A}_1)$ is dense in Z_1.

(iii) Let $(z_1,\zeta)\in\mathcal{D}(\mathbb{A}_1)\times H$ be such that $\|(\mathbb{A}_1-\lambda I)z_1+(I-P_1)\zeta\|_H+\|z_1\|_H=1$, and set $r_1=\|\mathbb{A}_1z_1+(I-P_1)\zeta\|_H+\|z_1\|_H$. For all $z_2\in\mathcal{D}(\mathbb{A}_2)$, we have

$$\begin{aligned}\|z_1-r_1z_2\|_H&+\|P_2((\mathbb{A}_1-\lambda I)z_1+(I-P_1)\zeta)-(\mathbb{A}_2-\lambda I)r_1z_2\|_H\\ &\le\left(\left\|\frac{z_1}{r_1}-z_2\right\|_H+\left\|P_2(\mathbb{A}_1\frac{z_1}{r_1}+(I-P_1)\frac{\zeta}{r_1})-\mathbb{A}_2z_2\right\|_H+|\lambda|\left\|P_2\frac{z_1}{r_1}-z_2\right\|_H\right)r_1\\ &\le(1+|\lambda|\|P_2\|_{\mathcal{L}(H,Z_2)})\left(\left\|\frac{z_1}{r_1}-z_2\right\|_H+\left\|P_2\left(\mathbb{A}_1\frac{z_1}{r_1}+(I-P_1)\frac{\zeta}{r_1}\right)-\mathbb{A}_2z_2\right\|_H\right)r_1.\end{aligned}$$

By taking the infimum for $z_2\in\mathcal{D}(\mathbb{A}_2)$, and by remarking that $(\frac{z_1}{r_1},\frac{\zeta}{r_1})$ obeys $\|\frac{z_1}{r_1}\|_H+\|\mathbb{A}_1\frac{z_1}{r_1}+(I-P_1)\frac{\zeta}{r_1}\|_H=1$, we obtain

$$\begin{aligned}\inf_{z_2\in\mathcal{D}(\mathbb{A}_2)}&\Big\{\|z_1-r_1z_2\|_H+\|P_2((\mathbb{A}_1-\lambda I)z_1+(I-P_1)\zeta)z_1-(\mathbb{A}_1-\lambda I)r_1z_2\|_H\Big\}\\ &\le(1+|\lambda|\|P_2\|_{\mathcal{L}(H,Z_2)})\delta((\mathbb{A}_1,P_1),(\mathbb{A}_2,P_2))r_1.\end{aligned}\tag{3.12}$$

Moreover, we have

$$r_1=\|z_1\|_H+\|\mathbb{A}_1z_1+(I-P_1)\zeta\|_H\le\|z_1\|_H+\|(\mathbb{A}_1-\lambda I)z_1+(I-P_1)\zeta\|_H+|\lambda|\|z_1\|_H\le 1+|\lambda|.$$

With $1\le\|P_2\|_{\mathcal{L}(H,Z_2)}$ we deduce $r_1\le 1+|\lambda|\|P_2\|_{\mathcal{L}(H,Z_2)}$, and (3.12) yields

$$\delta((\mathbb{A}_1-\lambda I,P_1),(\mathbb{A}_2-\lambda I,P_2))\le(1+|\lambda|\|P_2\|_{\mathcal{L}(H,Z_2)})^2\delta((\mathbb{A}_1,P_1),(\mathbb{A}_2,P_2)).$$

Finally, by reversing the role of $(\mathbb{A}_1,P_1)$ and $(\mathbb{A}_2,P_2)$, we also have

$$\delta((\mathbb{A}_2-\lambda I,P_2),(\mathbb{A}_1-\lambda I,P_1))\le(1+|\lambda|\|P_1\|_{\mathcal{L}(H,Z_1)})^2\delta((\mathbb{A}_2,P_2),(\mathbb{A}_1,P_1)),$$

and (3.7) is proved. □

Theorem 3.2. *Let $M(P_1,P_2)$ be defined in (3.6) and let $\mathbb{A}_1$ and $\mathbb{A}_2$ be closed linear operators densely defined on Z_1 and Z_2 respectively. If $\mathbb{A}_1$ and $\mathbb{A}_2$ both admit a bounded inverse in Z_1 and Z_2 respectively, if $\lambda\in\mathbb{C}$ belongs to the resolvent set of $\mathbb{A}_2$, and if*

$$\|\mathbb{A}_1^{-1}P_1-\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H)}<\frac{1}{2(1+|\lambda|M(P_1,P_2))^2(1+\|(\mathbb{A}_2-\lambda I)^{-1}P_2\|_{\mathcal{L}(H,Z_2)})},\tag{3.13}$$

then $\mathbb{A}_1-\lambda I$ admits a bounded inverse in Z_1, and

$$\|(\lambda I-\mathbb{A}_1)^{-1}\|_{\mathcal{L}(Z_1)}\le 1+2\|(\lambda I-\mathbb{A}_2)^{-1}P_2\|_{\mathcal{L}(H,Z_2)}.\tag{3.14}$$

Proof. By using (3.7) and (3.3), we first obtain

$$\begin{aligned}\widehat{\delta}((\mathbb{A}_1-\lambda I,P_1),(\mathbb{A}_2-\lambda I,P_2))&\le(1+|\lambda|M(P_1,P_2))^2\widehat{\delta}((\mathbb{A}_1,P_1),(\mathbb{A}_2,P_2))\\ &\le(1+|\lambda|M(P_1,P_2))^2\|\mathbb{A}_1^{-1}P_1-\mathbb{A}_2^{-1}P_2\|_{\mathcal{L}(H)}.\end{aligned}$$

Then, with (3.13), it yields

$$\widehat{\delta}((\mathbb{A}_1 - \lambda I, P_1), (\mathbb{A}_2 - \lambda I, P_2)) \leq \frac{1}{2(1 + \|(\mathbb{A}_2 - \lambda I)^{-1} P_2\|_{\mathcal{L}(H,Z_2)})}. \tag{3.15}$$

Which (by point *(ii)* of Prop. 3.1) ensures that $\mathbb{A}_1 - \lambda I$ admits a bounded inverse on Z_1. Finally, with (3.5), in which $\mathbb{A}_1$ and $\mathbb{A}_2$ are replaced by $\mathbb{A}_1 - \lambda I$ and $\mathbb{A}_2 - \lambda I$, we obtain

$$\frac{\|(\mathbb{A}_1 - \lambda I)^{-1} - (\mathbb{A}_2 - \lambda I)^{-1} P_2\|_{\mathcal{L}(Z_1)}}{2(1 + \|(\mathbb{A}_2 - \lambda I)^{-1} P_2\|_{\mathcal{L}(H,Z_2)})^2} \leq \widehat{\delta}((\mathbb{A}_1 - \lambda I, P_1), (\mathbb{A}_2 - \lambda I, P_2)).$$

With (3.15), we deduce that

$$\|(\mathbb{A}_1 - \lambda I)^{-1} - (\mathbb{A}_2 - \lambda I)^{-1} P_2\|_{\mathcal{L}(Z_1)} \leq (1 + \|(\mathbb{A}_2 - \lambda I)^{-1} P_2\|_{\mathcal{L}(H,Z_2)}).$$

Thus (3.14) follows from the triangle inequality. □

Corollary 3.3. *Let $M(P_1, P_2)$ be defined in (3.6) and let $\mathbb{A}_1$ and $\mathbb{A}_2$ be closed linear operators densely defined on Z_1 and Z_2 respectively. If $\widehat{\lambda} \in \mathbb{C}$ and if the operators $\mathbb{A}_1 - \widehat{\lambda} I$ and $\mathbb{A}_2 - \widehat{\lambda} I$ both admit a bounded inverse in Z_1 and Z_2 respectively, then if $\lambda \in \mathbb{C}$ belongs to the resolvent set of $\mathbb{A}_2$, and if*

$$\|(\mathbb{A}_1 - \widehat{\lambda} I)^{-1} P_1 - (\mathbb{A}_2 - \widehat{\lambda} I)^{-1} P_2\|_{\mathcal{L}(H)} < \frac{1}{2(1 + |\lambda - \widehat{\lambda}| M(P_1, P_2))^2 (1 + \|(\mathbb{A}_2 - \lambda I)^{-1} P_2\|_{\mathcal{L}(H,Z_2)})},$$

then $\mathbb{A}_1 - \lambda I$ admits a bounded inverse in Z_1 which satisfies (3.14).

Proof. It suffices to apply Theorem 3.2 to $\mathbb{A}_1 - \widehat{\lambda} I$ and $\mathbb{A}_2 - \widehat{\lambda} I$ with $\lambda - \widehat{\lambda}$ instead of λ in (3.13) and (3.14). □

4. Approximation of feedback gains

Throughout this section, we assume that (H_1) to (H_4) are satisfied.

Definition 4.1. 1. A strongly continuous semigroup $(\mathrm{e}^{t\mathbb{A}})_{t\geq 0}$ on Z is said to be exponentially stable if there exist $\alpha > 0$ and $C > 0$ such that

$$\|\mathrm{e}^{t\mathbb{A}}\|_{\mathcal{L}(Z)} \leq C\,\mathrm{e}^{-t\alpha}, \quad \forall t \geq 0.$$

2. A parameter dependent family $(\mathrm{e}^{t\mathbb{A}_\varepsilon})_{t\geq 0}$, with $\varepsilon \in (0, \varepsilon_0)$, of strongly continuous semigroups on Z_ε, is said to be exponentially stable, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, if there exist $\alpha > 0$ and $C > 0$ such that

$$\|\mathrm{e}^{t\mathbb{A}_\varepsilon}\|_{\mathcal{L}(Z_\varepsilon)} \leq C\,\mathrm{e}^{-t\alpha}, \quad \forall t \geq 0, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Definition 4.2.

1. The pair (A, B) is said to be *feedback stabilizable* in Z if there exists $F \in \mathcal{L}(Z, U)$ such that $(A + BF, \mathcal{D}(A + BF))$, with $\mathcal{D}(A + BF) = \{z \in Z \mid (A + BF)z \in Z\}$, is the infinitesimal generator of an exponentially stable strongly continuous semigroup on Z.

2. The parameter dependent pair $(A_\varepsilon, B_\varepsilon)$ is said to be feedback stabilizable in Z_ε, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, if there is a bounded family $(F_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ in $\mathcal{L}(Z_\varepsilon, U)$ such that $(A_\varepsilon + B_\varepsilon F_\varepsilon, \mathcal{D}(A_\varepsilon + B_\varepsilon F_\varepsilon))$, with $\mathcal{D}(A_\varepsilon + B_\varepsilon F_\varepsilon) = \{z \in Z_\varepsilon \mid (A_\varepsilon + B_\varepsilon F_\varepsilon)z \in Z_\varepsilon\}$, is the infinitesimal generator of an exponentially stable strongly continuous semigroup on Z_ε, uniformly in $\varepsilon \in (0, \varepsilon_0)$.

Remark 4.3. The operator A can be considered as a bounded operator from Z into $(\mathcal{D}((A)^*))'$ and BF too. Thus $A+BF$ is a bounded operator from Z into $(\mathcal{D}((A)^*))'$. Since $Z \hookrightarrow (\mathcal{D}((A)^*))'$, the definition of $\mathcal{D}(A+BF)$ is meaningful.

Theorem 4.4.

(i) *Let us assume that there exist $F \in \mathcal{L}(Z,U)$ and $\omega_F > 0$ such that $A+\omega_F I + BF$ is the infinitesimal generator of an exponentially stable strongly continuous semigroup on Z, and that $(F_\varepsilon)_{0<\varepsilon<1} \subset \mathcal{L}(Z_\varepsilon, U)$ is a family satisfying*

$$\|FP - F_\varepsilon\|_{\mathcal{L}(Z_\varepsilon,U)} \le \sigma(\varepsilon), \quad \forall \varepsilon \in (0,1), \tag{4.1}$$

where σ is a continuous function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $\sigma(0)=0$. We set $A_F \overset{def}{=} A+BF$ and $A_{\varepsilon,F_\varepsilon} \overset{def}{=} A_\varepsilon + B_\varepsilon F_\varepsilon$. Let $\delta \in (0,\pi/2)$ be the angle introduced in assumption (H_1).

Then, for all $\widetilde{\delta} \in (0,\delta)$, there exist $\varrho > 0$ and $\varepsilon_0 \in (0,1)$ such that $\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}} \subset \rho(A_{\varepsilon,F_\varepsilon})$, with $\omega_{F,\varepsilon} \overset{def}{=} \omega_F - \varrho(\varepsilon^r + \sigma(\varepsilon))$, and

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le \frac{C}{|\lambda + \omega_{F,\varepsilon}|}, \quad \forall \lambda \in \{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}}, \quad \forall \varepsilon \in (0,\varepsilon_0). \tag{4.2}$$

Moreover, we have

$$\|\mathrm{e}^{A_{\varepsilon,F_\varepsilon} t}\|_{\mathcal{L}(Z_\varepsilon)} \le C\mathrm{e}^{-\omega_{F,\varepsilon} t},\ \forall t \ge 0,\ \forall \varepsilon \in (0,\varepsilon_0), \tag{4.3}$$

$$\|\mathrm{e}^{A_F t}P - \mathrm{e}^{A_{\varepsilon,F_\varepsilon} t}P_\varepsilon\|_{\mathcal{L}(H)} \le C\,\mathrm{e}^{-\omega_{F,\varepsilon} t}\left(\frac{\varepsilon^r}{t^{r/s}} + \sigma(\varepsilon)\right), \quad \forall t \ge 0,\ \forall \varepsilon \in (0,\varepsilon_0). \tag{4.4}$$

In particular, (4.3) and (4.4) hold to be true for $F_\varepsilon = FP$, with $\sigma \equiv 0$.

(ii) *Let us assume that $(F_\varepsilon)_{0<\varepsilon<1}$ is a family in $\mathcal{L}(Z_\varepsilon, U)$, satisfying*

$$\|F_\varepsilon\, P_\varepsilon\|_{\mathcal{L}(Z,U)} \le C, \quad \forall \varepsilon \in (0,1), \tag{4.5}$$

and that $(F^{(\varepsilon)})_{0<\varepsilon<1}$ is a family in $\mathcal{L}(Z,U)$ satisfying

$$\|F_\varepsilon P_\varepsilon - F^{(\varepsilon)}\|_{\mathcal{L}(Z,U)} \le \sigma(\varepsilon), \quad \forall \varepsilon \in (0,1), \tag{4.6}$$

where σ is a continuous function from $\mathbb{R}^+$ into $\mathbb{R}^+$ satisfying $\sigma(0)=0$.

In addition, we assume that there exists $\omega_F > 0$ such that the family $((\mathrm{e}^{t(A_\varepsilon + \omega_F I + B_\varepsilon F_\varepsilon)})_{t\ge 0})_{0<\varepsilon<1}$, of strongly continuous semigroups on Z_ε, is exponentially stable, uniformly in $\varepsilon \in (0,1)$.

We set $A_{F^{(\varepsilon)}} \overset{def}{=} A + BF^{(\varepsilon)}$ and $A_{\varepsilon,F_\varepsilon} \overset{def}{=} A_\varepsilon + B_\varepsilon F_\varepsilon$. Let $\delta \in (0,\pi/2)$ be the angle introduced in assumption (H_1).

Then, for all $\widetilde{\delta} \in (0,\delta)$, there exist $\varrho > 0$ and $\varepsilon_0 \in (0,1)$ such that $\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}} \subset \rho(A_{F^{(\varepsilon)}})$, with $\omega_{F,\varepsilon} \overset{def}{=} \omega_F - \varrho(\varepsilon^r + \sigma(\varepsilon))$, and

$$\|(\lambda I - A_{F^{(\varepsilon)}})^{-1}\|_{\mathcal{L}(Z)} \le \frac{C}{|\lambda + \omega_{F,\varepsilon}|},\ \forall \lambda \in \{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}},\ \forall \varepsilon \in (0,\varepsilon_0). \tag{4.7}$$

Moreover, we have

$$\|\mathrm{e}^{A_{F^{(\varepsilon)}} t}\|_{\mathcal{L}(Z)} \le C\mathrm{e}^{-\omega_{F,\varepsilon} t},\ \forall t \ge 0,\ \forall \varepsilon \in (0,\varepsilon_0), \tag{4.8}$$

$$\|e^{A_{\varepsilon,F_\varepsilon} t}P - e^{A_{F^{(\varepsilon)}} t}P_\varepsilon\|_{\mathcal{L}(H)} \le C\, e^{-\omega_{F,\varepsilon} t}\left(\frac{\varepsilon^r}{t^{r/s}} + \sigma(\varepsilon)\right), \quad \forall t \ge 0,\ \forall \varepsilon \in (0,\varepsilon_0). \tag{4.9}$$

In particular, (4.8) *and* (4.9) *hold to be true for* $F^{(\varepsilon)} = F_\varepsilon P_\varepsilon$*, with* $\sigma \equiv 0$.

Remark 4.5. For simplicity, the assumptions of Theorem 4.4 are stated for $\varepsilon \in (0,1)$. But it obviously remains valid if the assumptions are stated for $\varepsilon \in (0,\tilde{\varepsilon}_0)$, for some $\tilde{\varepsilon}_0 \in (0,1)$. In the later case it suffices to set $A_\varepsilon = A_{\tilde{\varepsilon}_0}$, $B_\varepsilon = B_{\tilde{\varepsilon}_0}$, $P_\varepsilon = P_{\tilde{\varepsilon}_0}$, $F_\varepsilon = F_{\tilde{\varepsilon}_0}$ and $F^{(\varepsilon)} = F^{(\tilde{\varepsilon}_0)}$ for $\varepsilon \in [\tilde{\varepsilon}_0, 1)$.

Remark 4.6. In Theorem 4.4-(i), we state that if the pair (A,B) is feedback stabilizable, then there exists $\varepsilon_0 \in (0,1)$ such that the family $(A_\varepsilon, B_\varepsilon)$ is feedback stabilizable uniformly with respect to $\varepsilon \in (0,\varepsilon_0)$. From Theorem 4.4-(ii), we deduce that if the family $(A_\varepsilon, B_\varepsilon)$ is feedback stabilizable uniformly with respect to $\varepsilon \in (0,\varepsilon_0)$, for some $\varepsilon_0 \in (0,1)$, then the pair (A,B) is feedback stabilizable. In other words, the feedback stabilizability of the pair (A,B) is equivalent to the uniform feedback stabilizability of the family $(A_\varepsilon, B_\varepsilon)$ for $\varepsilon \in (0,\varepsilon_0)$, for some $\varepsilon_0 \in (0,1)$.

Before proving Theorem 4.4, we need the following proposition.

Proposition 4.7. *Assume that* $(H_1) - (H_4)$ *are satisfied. Let* $\eta > 0$ *and set* $\bar{\mathbb{B}}_\eta = \{\lambda \in \mathbb{C} \mid |\lambda| \le \eta\}$*. For all* $\lambda \in \{\omega_0\} + (\mathbb{S}_{\pi/2+\delta} \setminus \bar{\mathbb{B}}_\eta)$*, we have*

$$\|(\lambda I - A_\varepsilon)^{-1}B_\varepsilon - (\lambda I - A)^{-1}B\|_{\mathcal{L}(U,H)} \le C_\eta\, \varepsilon^r\,. \tag{4.10}$$

Proof. For all $\lambda \in \{\omega_0 - \lambda_0\} + \mathbb{S}_{\pi/2+\delta}$, we set

$$M(\lambda) = (\lambda I - \widehat{A}_\varepsilon)^{-1}B_\varepsilon - (\lambda I - \widehat{A})^{-1}B. \tag{4.11}$$

We have

$$\begin{aligned} M(\lambda) &= (\lambda I - \widehat{A}_\varepsilon)^{-1}\widehat{A}_\varepsilon\widehat{A}_\varepsilon^{-1}B_\varepsilon - (\lambda I - \widehat{A})^{-1}\widehat{A}\widehat{A}^{-1}B \\ &= (\lambda(\lambda I - \widehat{A}_\varepsilon)^{-1} - I)\widehat{A}_\varepsilon^{-1}B_\varepsilon - (\lambda(\lambda I - \widehat{A})^{-1} - I))\widehat{A}^{-1}B \\ &= \widehat{A}^{-1}B - \widehat{A}_\varepsilon^{-1}B_\varepsilon + \lambda(\lambda I - \widehat{A}_\varepsilon)^{-1}\widehat{A}_\varepsilon^{-1}B_\varepsilon - \lambda(\lambda I - \widehat{A})^{-1}\widehat{A}^{-1}B \\ &= \widehat{A}^{-1}B - \widehat{A}_\varepsilon^{-1}B_\varepsilon + \lambda(\lambda I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon(\widehat{A}_\varepsilon^{-1}B_\varepsilon - \widehat{A}^{-1}B) \\ &\quad +\lambda((\lambda I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon - (\lambda I - \widehat{A})^{-1}))\widehat{A}^{-1+\gamma}\widehat{A}^{-\gamma}B. \end{aligned} \tag{4.12}$$

To prove (4.10), we have to estimate $M(\lambda - \lambda_0) = (\lambda I - A_\varepsilon)^{-1}B_\varepsilon - (\lambda I - A)^{-1}B$. We estimate $\widehat{A}^{-1}B - \widehat{A}_\varepsilon^{-1}B_\varepsilon$ with (2.6). We estimate

$$(\lambda - \lambda_0)((\lambda - \lambda_0)I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon(\widehat{A}_\varepsilon^{-1}B_\varepsilon - \widehat{A}^{-1}B)$$

with (2.2), (2.3), and (2.6). We estimate

$$(\lambda - \lambda_0)(((\lambda - \lambda_0)I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon - ((\lambda - \lambda_0)I - \widehat{A})^{-1}))\widehat{A}^{-1}B$$

by using (2.22) and the fact that $\widehat{A}^{-\gamma}B \in \mathcal{L}(U,H)$. □

We also need the following variant of the above proposition.

Proposition 4.8. *Assume that $(H_1) - (H_4)$ are satisfied. For all $\lambda \in \mathbb{S}_{\pi/2+\delta}$, we have*

$$\|(\lambda I - \widehat{A}_\varepsilon)^{-1}B_\varepsilon - (\lambda I - \widehat{A})^{-1}B\|_{\mathcal{L}(U,H)} \le C\varepsilon^r, \tag{4.13}$$

$$\|(\lambda I - \widehat{A}_\varepsilon)^{-2}B_\varepsilon - (\lambda I - \widehat{A})^{-2}B\|_{\mathcal{L}(U,H)} \le C\frac{\varepsilon^r}{|\lambda|}. \tag{4.14}$$

Proof. Step 1. The estimate (4.13) follows from (4.10) with $\eta < (\lambda_0 - \omega_0)\cos\delta$.

Step 2. Proof of 4.14. By differentiating (4.12) with respect to λ, we get

$$\begin{aligned} M'(\lambda) = &((\lambda I - \widehat{A}_\varepsilon)^{-1} - \lambda(\lambda I - \widehat{A}_\varepsilon)^{-2})P_\varepsilon(\widehat{A}_\varepsilon^{-1}B_\varepsilon - \widehat{A}^{-1}B) \\ &+ ((\lambda I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon - (\lambda I - \widehat{A})^{-1}))\widehat{A}^{-1}B \\ &- \lambda((\lambda I - \widehat{A}_\varepsilon)^{-2}P_\varepsilon - (\lambda I - \widehat{A})^{-2}))\widehat{A}^{-1}B. \end{aligned} \tag{4.15}$$

Since we have

$$\begin{aligned} (\lambda I - \widehat{A}_\varepsilon)^{-2}P_\varepsilon - (\lambda I - \widehat{A})^{-2}P = &(\lambda I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon((\lambda I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon - (\lambda I - \widehat{A})^{-1}P) \\ &+ ((\lambda I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon - (\lambda I - \widehat{A})^{-1})(\lambda I - \widehat{A})^{-1}P, \end{aligned}$$

with (2.22), (2.1), and (2.2), we obtain

$$\|(\lambda I - \widehat{A}_\varepsilon)^{-2}P_\varepsilon - (\lambda I - \widehat{A})^{-2}P\|_{\mathcal{L}(\mathcal{D}((-\widehat{A})^{1-\gamma}),H)} \le C\frac{\varepsilon^r}{|\lambda|^2}. \tag{4.16}$$

From the definition of $M(\lambda)$ it also follows that $M'(\lambda) = (\lambda I - \widehat{A})^{-2}B - (\lambda I - \widehat{A}_\varepsilon)^{-2}B_\varepsilon$. Thus we can estimate $(\lambda I - \widehat{A})^{-2}B - (\lambda I - \widehat{A}_\varepsilon)^{-2}B_\varepsilon$ by using the expression of $M'(\lambda)$ obtained in (4.15).

From (2.2), we have $\|(\lambda I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon\|_{\mathcal{L}(H)} \le \frac{C}{|\lambda|}$ and $\|\lambda(\lambda I - \widehat{A}_\varepsilon)^{-2}P_\varepsilon\|_{\mathcal{L}(H)} \le \frac{C}{|\lambda|}$. Thus the first line in (4.15) is estimated with (2.6).

The second line in (4.15), namely $((\lambda I - \widehat{A}_\varepsilon)^{-1}P_\varepsilon - (\lambda I - \widehat{A})^{-1}))\widehat{A}^{-1}B$, was already estimated to prove (4.13).

The last line in (4.15), $-\lambda((\lambda I - \widehat{A}_\varepsilon)^{-2}P_\varepsilon - (\lambda I - \widehat{A})^{-2}))\widehat{A}^{-1+\gamma}\widehat{A}^{-\gamma}B$, is estimated by using (4.16). □

Let us notice that (2.5) and (2.11), with the identity $(\lambda I - A)^{-1}B = (-\widehat{A})^\gamma(\lambda - A)^{-1}(-\widehat{A})^{-\gamma}B$, give

$$\|(\lambda I - A)^{-1}B\|_{\mathcal{L}(U,Z)} \le \frac{C}{|\lambda - \omega_0|^{1-\gamma}}, \quad \forall \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}, \tag{4.17}$$

and that (2.12), with $\mathrm{e}^{At}B = (-\widehat{A})^\gamma \mathrm{e}^{At}(-\widehat{A})^{-\gamma}B$, gives

$$\|\mathrm{e}^{At}B\|_{\mathcal{L}(U,Z)} \le C\frac{\mathrm{e}^{\omega_0 t}}{t^\gamma}, \quad \forall t > 0. \tag{4.18}$$

4.1. Proof of Theorem 4.4-(i)

Proof. The proof of Theorem 4.4 is quite long and delicate and is divided in 5 steps. The principal point is the proof of estimate (4.2). It is the object of steps 2, 3 and 4. Estimate (4.2) is a resolvent estimate for $A_{\varepsilon,F_\varepsilon}$ in the sector $\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}}$ for $\varepsilon \in (0,\varepsilon_0)$ and $\widetilde{\delta} \in (0,\delta)$. To prove (4.2) in the sector $\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}}$, we are going to first prove the estimate in the sector $\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}$ for $\widehat{\omega}_0>-\omega_F$ suitablely chosen, and next prove the estimate in the bounded set $\{-\omega_{F,\varepsilon}\}+\mathbb{S}_{\pi/2+\widetilde{\delta}}\setminus\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}$.

Plan of the proof.

In Step 1, we prove a preliminary resolvent estimate for A_F in the sector $\{-\omega_F\}+\mathbb{S}_{\pi/2+\delta}$.

In Step 2, we choose $\widehat{\omega}_0>-\omega_F$ and we prove the estimate (4.2) in the sector $\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}$. It is deduced from different resolvent estimates and from majorizations in a classical manner.

In Step 3, we are able to estimate the difference $(\lambda I-A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon-(\lambda I-A_F)^{-1}P$ in the norm $\mathcal{L}(H)$ in terms of ε, for λ belonging to the sector $\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}$.

Next, in Step 4, using the estimate of Step 3, we are able to show that

$$\|(\widehat{\lambda}I-A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon-(\widehat{\lambda}I-A_F)^{-1}P\|_{\mathcal{L}(H)}\le\frac{1}{2(1+|\lambda-\widehat{\lambda}|\widehat{M})^2(1+\|(\lambda I-A_F)^{-1}P\|_{\mathcal{L}(H,Z)})},$$

for $\widehat{\lambda}\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}$ fixed and for all λ in $K_\varepsilon=\overline{\{-\omega_{F,\varepsilon}\}+\mathbb{S}_{\pi/2+\widetilde{\delta}}\setminus\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}}$. In the above inequality $\widehat{M}$ denotes the supremum of $\{\|P\|_{\mathcal{L}(H,Z)},\|P_\varepsilon\|_{\mathcal{L}(H,Z_\varepsilon)},\varepsilon\in(0,1)\}$. From that estimate and from Corollary 3.3, we prove the inequality in (4.2) in the bounded set $\{-\omega_{F,\varepsilon}\}+\mathbb{S}_{\pi/2+\widetilde{\delta}}\setminus\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}$ (see (4.35) and (4.37)).

Finally, in Step 5 we deduce (4.3) and (4.4) from (4.2).

Step 1. Let us prove a preliminary resolvent estimate for A_F. A perturbation argument ensures that A_F is the infinitesimal generator of an analytic semigroup on Z (see, *e.g.*, [11], p. 151). Since $A+\omega_F I+BF$ is the infinitesimal generator of an exponentially stable semigroup, we can choose $\delta_F>0$ such that the following resolvent estimate holds:

$$\|(\lambda I-A_F)^{-1}P\|_{\mathcal{L}(H,Z)}\le\frac{C_F}{|\lambda+\omega_F|},\quad\text{for all }\lambda\in\{-\omega_F\}+\mathbb{S}_{\pi/2+\delta_F}.\tag{4.19}$$

Without loss of generality, in what follows, we can assume that $\delta_F=\delta$, where $\delta\in]0,\pi/2[$ is the angle appearing in (2.1) and (2.2).

Step 2. Let us prove that there exists $\widehat{\omega}_0>-\omega_F$ and $\varepsilon_0\in(0,1)$ such that

$$\|(\lambda I-A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)}\le\frac{C}{|\lambda+\omega_F|},\quad\forall\lambda\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta},\quad\forall\varepsilon\in(0,\varepsilon_0).\tag{4.20}$$

For that, we set $T_\varepsilon(\lambda)=(\lambda I-A_\varepsilon)^{-1}B_\varepsilon F_\varepsilon P_\varepsilon\in\mathcal{L}(H)$. Due to (4.1), (2.3), (4.17) and (4.10) with $\eta<\lambda_0-\omega_0$, we have

$$\|T_\varepsilon(\lambda)\|_{\mathcal{L}(H)}\le C\|(\lambda I-A_\varepsilon)^{-1}B_\varepsilon\|_{\mathcal{L}(U,H)}\le C\varepsilon^r+\frac{C}{|\lambda-\omega_0|^{1-\gamma}},\quad\forall\varepsilon\in(0,1),\ \forall\lambda\in\{\lambda_0\}+\mathbb{S}_{\pi/2+\delta}.$$

Let c_0 belong to $(0,1)$. We choose $\varepsilon_0\in(0,1)$ and $\widehat{\omega}_0>\max(-\omega_F,\lambda_0)$ such that

$$\|T_\varepsilon(\lambda)\|_{\mathcal{L}(H)}\le 1-c_0,\quad\forall\lambda\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta},\ \forall\varepsilon\in(0,\varepsilon_0).$$

Therefore, for $\lambda\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}$, $I-T_\varepsilon(\lambda)$ is invertible in $\mathcal{L}(H)$ and obeys

$$\|(I-T_\varepsilon(\lambda))^{-1}\|_{\mathcal{L}(H)}\le(1-\|T_\varepsilon(\lambda)\|_{\mathcal{L}(H)})^{-1}\le c_0^{-1}.\tag{4.21}$$

Moreover, we verify that

$$(I-T_\varepsilon(\lambda))^{-1}(\lambda I-A_\varepsilon)^{-1}=(\lambda I-A_{\varepsilon,F_\varepsilon})^{-1},\quad\forall\lambda\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}.\tag{4.22}$$

With (4.22) and (4.21), we have

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le c_0^{-1}\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(Z_\varepsilon)}, \quad \forall\lambda\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}.$$

Hence, we obtain (4.20) from (2.2) and the fact that $\frac{|\lambda+\omega_F|}{|\lambda-\omega_0|}$ is bounded in the sector $\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}$.

Step 3. Let us now prove that

$$\begin{aligned}\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon - (\lambda I - A_F)^{-1}P\|_{\mathcal{L}(H)} \\ \le C\left(\frac{\varepsilon^r}{|\lambda+\omega_F|^{1-r/s}} + \frac{\sigma(\varepsilon)}{|\lambda+\omega_F|}\right), \quad \forall\lambda\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}, \quad \forall\varepsilon\in(0,\varepsilon_0).\end{aligned} \tag{4.23}$$

To prove (4.23), we set $T(\lambda)=(\lambda I - A)^{-1}BFP$ and, arguing as we did to prove (4.21), for $c_0\in(0,1)$, we can assume that $\widehat{\omega}_0$ is chosen so that

$$\|(I-T(\lambda))^{-1}\|_{\mathcal{L}(H)} \le c_0^{-1}, \quad \forall\lambda\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}. \tag{4.24}$$

In the following we use the simplified notation $T=T(\lambda)$ and $T_\varepsilon = T_\varepsilon(\lambda)$ and we consider $\lambda\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}$ and $\varepsilon\in(0,\varepsilon_0)$ so that $(I-T)$ and $(I-T_\varepsilon)$ are both boundedly invertible. We start by writing

$$\begin{aligned}(\lambda I - A_F)^{-1}P - (\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon &= (I-T)^{-1}(\lambda I - A)^{-1}P - (I-T_\varepsilon)^{-1}(\lambda I - A_\varepsilon)^{-1}P_\varepsilon \\ &= (I-T)^{-1}((\lambda I - A)^{-1}P - (\lambda I - A_\varepsilon)^{-1}P_\varepsilon) \\ &\quad +((I-T)^{-1}-(I-T_\varepsilon)^{-1})(\lambda I - A_\varepsilon)^{-1}P_\varepsilon.\end{aligned} \tag{4.25}$$

Thus, by using the identity

$$(I-T)^{-1}-(I-T_\varepsilon)^{-1} = (I-T)^{-1}(T-T_\varepsilon)(I-T_\varepsilon)^{-1},$$

and (4.22), we obtain

$$\begin{aligned}(\lambda I - A_F)^{-1}P - (\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon = (I-T)^{-1}((\lambda I - A)^{-1}P - (\lambda I - A_\varepsilon)^{-1}P_\varepsilon) \\ + (I-T)^{-1}(T-T_\varepsilon)(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon.\end{aligned} \tag{4.26}$$

From (4.24) and from (2.21) for $\theta = r/s$, we deduce that

$$\|(\lambda I - A_F)^{-1}P - (\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon\|_{\mathcal{L}(H)} \le C\left(\frac{\varepsilon^r}{|\lambda-\omega_0|^{1-r/s}} + \|(T-T_\varepsilon)(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon\|_{\mathcal{L}(H)}\right). \tag{4.27}$$

We have the identity

$$\begin{aligned}(T-T_\varepsilon)(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon = ((\lambda I - A)^{-1}B - (\lambda I - A_\varepsilon)^{-1}B_\varepsilon)F_\varepsilon(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon \\ + (\lambda I - A)^{-1}B(FP-F_\varepsilon)(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon.\end{aligned}$$

With the uniform boundedness of F_ε (which follows from (4.1)), and with (4.13), we obtain

$$\|((\lambda I - A)^{-1}B - (\lambda I - A_\varepsilon)^{-1}B_\varepsilon)F_\varepsilon(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon\|_{\mathcal{L}(H)} \le C\,\varepsilon^r\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon\|_{\mathcal{L}(H)},$$

because $\lambda \in \{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta} \subset \{\lambda_0\} + \mathbb{S}_{\pi/2+\delta}$. With (4.17), and (4.1), we obtain

$$\|(\lambda I - A)^{-1} B(FP - F_\varepsilon)(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1} P_\varepsilon\|_{\mathcal{L}(H)} \leq C\, \frac{\sigma(\varepsilon)}{|\lambda - \omega_0|^{1-\gamma}} \|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1} P_\varepsilon\|_{\mathcal{L}(H)}.$$

Thus, with (4.20) and the fact that $1/|\lambda - \omega_0| \leq C$ and $1/|\lambda + \omega_F| \leq C$ for $\lambda \in \{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta}$, we deduce

$$\|(T - T_\varepsilon)(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1} P_\varepsilon\|_{\mathcal{L}(H)} \leq C(\varepsilon^r + \sigma(\varepsilon))\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1} P_\varepsilon\|_{\mathcal{L}(H)} \leq C\left(\frac{\varepsilon^r}{|\lambda + \omega_F|^{1-r/s}} + \frac{\sigma(\varepsilon)}{|\lambda + \omega_F|}\right).$$

Thus, (4.23) follows from (4.27) and the fact that $\frac{|\lambda+\omega_F|}{|\lambda-\omega_0|}$ is bounded in $\{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta}$.

Step 4. Let us prove (4.2). We fix $\widehat{\lambda} \in \{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta}$ and let $\varepsilon \in (0, \varepsilon_0)$. From (4.23), we deduce that

$$\|(\widehat{\lambda} I - A_{\varepsilon,F_\varepsilon})^{-1} P_\varepsilon - (\widehat{\lambda} I - A_F)^{-1} P\|_{\mathcal{L}(H)} \leq C\left(\frac{\varepsilon^r}{|\widehat{\lambda} + \omega_F|^{1-r/s}} + \frac{\sigma(\varepsilon)}{|\widehat{\lambda} + \omega_F|}\right) = \widehat{C}\,(\varepsilon^r + \sigma(\varepsilon)). \tag{4.28}$$

For $\widetilde{\delta} \in (0, \delta)$ given fixed, we set

$$K = \overline{\{-\omega_F\} + \mathbb{S}_{\pi/2+\widetilde{\delta}} \setminus \{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta}},$$

$$\widehat{M} = \sup_{\varepsilon \in (0,1)} \left\{\|P\|_{\mathcal{L}(H,Z)}, \|P_\varepsilon\|_{\mathcal{L}(H,Z_\varepsilon)}\right\}$$

and

$$C_K = \sup_{\lambda \in K} \left\{2(1 + |\lambda - \widehat{\lambda}|\widehat{M})^2(|\lambda + \omega_F| + C_F)\right\}. \tag{4.29}$$

We choose

$$\varrho > \frac{\widehat{C}\, C_K}{\cos(\widetilde{\delta})}, \tag{4.30}$$

where $\widehat{C}$ is introduced in (4.28), and we set

$$\omega_{F,\varepsilon} = \omega_F - \varrho\,(\varepsilon^r + \sigma(\varepsilon))$$

and

$$K_\varepsilon = \overline{\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}} \setminus \{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta}}.$$

Since $\widehat{\omega}_0 > -\omega_F$ then, by choosing ε_0 smaller if necessary, we can assume that

$$-\omega_F < -\omega_{F,\varepsilon} < \widehat{\omega}_0 \quad \text{for all } \varepsilon \in (0, \varepsilon_0),$$

so that K_ε is not empty. From the above definitions we observe that

$$K_\varepsilon \subset K \qquad \text{and} \qquad K_\varepsilon \subset \{-\omega_F\} + \mathbb{S}_{\pi/2+\delta}. \tag{4.31}$$

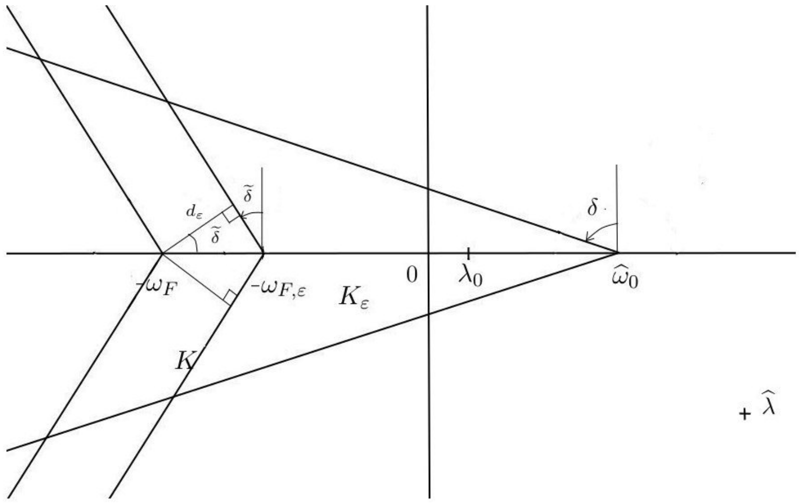

FIGURE 1. Representation of K, K_ε and of $d_\varepsilon = \mathrm{dist}(K_\varepsilon, -\omega_F)$.

Moreover, still by choosing ε_0 smaller if necessary, we can assume that

$$\mathrm{dist}(K_\varepsilon, -\omega_F) = \mathrm{dist}\left(\overline{\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}}}, -\omega_F\right) \quad \text{for all } \varepsilon \in (0, \varepsilon_0), \tag{4.32}$$

so that the following equality holds:

$$\mathrm{dist}(K_\varepsilon, -\omega_F) = (\omega_F - \omega_{F,\varepsilon}) \cos\widetilde{\delta} = \varrho \cos(\widetilde{\delta})\, (\varepsilon^r + \sigma(\varepsilon)). \tag{4.33}$$

The numbers ω_F, $\omega_{F,\varepsilon}$, $\mathrm{dist}(K_\varepsilon, -\omega_F)$ and the sets K, K_ε are represented in Figure 1.

From (4.28) and (4.30) we have

$$\|(\widehat{\lambda} I - A_{\varepsilon,F_\varepsilon})^{-1} P_\varepsilon - (\widehat{\lambda} I - A_F)^{-1} P\|_{\mathcal{L}(H)} \leq \widehat{C}\, (\varepsilon^r + \sigma(\varepsilon)) < \frac{\varrho \cos(\widetilde{\delta})\, (\varepsilon^r + \sigma(\varepsilon))}{C_K}.$$

Thus, from (4.33) we obtain

$$\|(\widehat{\lambda} I - A_{\varepsilon,F_\varepsilon})^{-1} P_\varepsilon - (\widehat{\lambda} I - A_F)^{-1} P\|_{\mathcal{L}(H)} \leq \frac{\mathrm{dist}(K_\varepsilon, -\omega_F)}{C_K} \leq \frac{|\lambda + \omega_F|}{C_K} \quad \forall \lambda \in K_\varepsilon,$$

and, by recalling the definition of C_K in (4.29), from (4.31) we deduce

$$\begin{aligned} &\|(\widehat{\lambda} I - A_{\varepsilon,F_\varepsilon})^{-1} P_\varepsilon - (\widehat{\lambda} I - A_F)^{-1} P\|_{\mathcal{L}(H)} \\ &\qquad \leq \frac{|\lambda + \omega_F|}{2(1 + |\lambda - \widehat{\lambda}|\widehat{M}\,)^2 (|\lambda + \omega_F| + C_F)} = \frac{1}{2(1 + |\lambda - \widehat{\lambda}|\widehat{M})^2 \left(1 + \frac{C_F}{|\lambda + \omega_F|}\right)} \quad \forall \lambda \in K_\varepsilon. \end{aligned}$$

Due to (4.19) (with $\delta_K = \delta$) and (4.31), we have

$$\|(A_F - \lambda I)^{-1} P\|_{\mathcal{L}(H,Z)} \leq \frac{C_F}{|\lambda + \omega_F|}, \quad \forall \lambda \in K_\varepsilon. \tag{4.34}$$

Thus, with (4.34) we obtain

$$\|(\widehat{\lambda}I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon - (\widehat{\lambda}I - A_F)^{-1}P\|_{\mathcal{L}(H)} \le \frac{1}{2(1+|\lambda-\widehat{\lambda}|\widehat{M})^2(1+\|(\lambda I - A_F)^{-1}P\|_{\mathcal{L}(H,Z)})} \quad \forall\lambda \in K_\varepsilon.$$

According to Corollary 3.3, with $(\mathbb{A}_2, P_2) = (A_F, P)$ and $(\mathbb{A}_1, P_1) = (A_{\varepsilon,F_\varepsilon}, P_\varepsilon)$, we have

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le 1 + 2\|(\lambda - A_F)^{-1}P\|_{\mathcal{L}(H,Z)}, \quad \forall\lambda \in K_\varepsilon.$$

Still with (4.34), we have

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le 1 + \frac{2C_F}{|\lambda+\omega_F|} \le \frac{\sup_{\lambda\in K}|\lambda+\omega_F| + 2C_F}{|\lambda+\omega_F|} \le \frac{C}{|\lambda+\omega_F|}, \quad \forall\lambda\in K_\varepsilon. \tag{4.35}$$

By combining (4.20) and (4.35), we obtain

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le \frac{C}{|\lambda+\omega_F|}, \quad \forall\lambda\in\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}}. \tag{4.36}$$

We notice that a rough majorization with (4.32) and (4.33) leads to

$$\frac{|\lambda+\omega_{F,\varepsilon}|}{|\lambda+\omega_F|} \le \frac{|\lambda+\omega_F|}{|\lambda+\omega_F|} + \frac{\varrho(\varepsilon^r+\sigma(\varepsilon))}{|\lambda+\omega_F|} \le 1 + \frac{\mathrm{dist}(K_\varepsilon, -\omega_F)}{\cos(\widetilde{\delta})|\lambda+\omega_F|} \le 1 + \frac{1}{\cos(\widetilde{\delta})}, \quad \forall\lambda\in\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}}. \tag{4.37}$$

Finally, (4.2) follows from (4.37) and (4.36).

Step 5. Let us prove (4.3) and (4.4). According to [19], Chapter II-1, (2.43) we have that (4.3) is a consequence of (4.2). To prove (4.4) we first state the following resolvent identity that can be proved as in the Appendix:

$$\begin{aligned}&(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon - (\lambda I - A_F)^{-1}P\\ &= \Big(I - (\lambda-\widehat{\lambda})(\lambda I - A_F)^{-1}P\Big)\Big((A_F - \widehat{\lambda}I)^{-1}P - (A_{\varepsilon,F_\varepsilon} - \widehat{\lambda}I)^{-1}P_\varepsilon\Big)\Big(I - (\lambda-\widehat{\lambda})(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon\Big).\end{aligned} \tag{4.38}$$

Hence, with (4.38), (4.28), (4.19) and (4.36) we deduce that for all $\varepsilon\in(0,\varepsilon_0)$,

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon - (\lambda I - A_F)^{-1}P\|_{\mathcal{L}(H)} \le C\left(\varepsilon^r + \sigma(\varepsilon)\right), \quad \forall\lambda\in K_\varepsilon.$$

With the fact that $|\lambda+\omega_F| \le C$ for all $\lambda\in K$, and with $K_\varepsilon\subset K$, we deduce that for all $\varepsilon\in(0,\varepsilon_0)$,

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon - (\lambda I - A_F)^{-1}P\|_{\mathcal{L}(H)} \le C\left(\frac{\varepsilon^r}{|\lambda+\omega_F|^{1-r/s}} + \frac{\sigma(\varepsilon)}{|\lambda+\omega_F|}\right), \quad \forall\lambda\in\Big(\{-\omega_{F,\varepsilon}\} + \Gamma_{\rho,\widetilde{\delta}}\Big)\cap K_\varepsilon,$$

where $\Gamma_{\rho,\widetilde{\delta}}$ is defined in (2.18).

Next, from (4.23) we deduce that for all $\varepsilon\in(0,\varepsilon_0)$,

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon - (\lambda I - A_F)^{-1}P\|_{\mathcal{L}(H)} \le C\left(\frac{\varepsilon^r}{|\lambda+\omega_F|^{1-r/s}} + \frac{\sigma(\varepsilon)}{|\lambda+\omega_F|}\right), \quad \forall\lambda\in\Big(\{-\omega_{F,\varepsilon}\} + \Gamma_{\rho,\widetilde{\delta}}\Big)\backslash K_\varepsilon.$$

Finally, by combining the two above inequalities, with (4.37) we deduce

$$\begin{aligned} &\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon - (\lambda I - A_F)^{-1}P\|_{\mathcal{L}(H)} \\ &\qquad \le C\left(\frac{\varepsilon^r}{|\lambda+\omega_{F,\varepsilon}|^{1-r/s}} + \frac{\sigma(\varepsilon)}{|\lambda+\omega_{F,\varepsilon}|}\right), \quad \forall\lambda\in\{-\omega_{F,\varepsilon}\}+\Gamma_{\rho,\widetilde{\delta}}, \quad \forall\varepsilon\in(0,\varepsilon_0), \end{aligned} \tag{4.39}$$

and (4.4) can be deduced by arguing as in the proof of Theorem 2.5. □

4.2. Proof of Theorem 4.4-(ii)

Proof. We set $A_{\varepsilon,F_\varepsilon} \overset{\text{def}}{=} A_\varepsilon + B_\varepsilon F_\varepsilon$. First, we would like to prove that $A_{\varepsilon,F_\varepsilon}$ is the infinitesimal generator of an analytic semigroup on Z_ε. Here, we cannot use a perturbation argument as in Step 1 of the proof of Theorem 4.4-(i). Indeed, it is used in [11], p. 152 that $B^*(-\widehat{A}^*)^{-\gamma}$ belongs to $\mathcal{L}(Z',U)$. Here, we know that $B_\varepsilon^*(-\widehat{A}_\varepsilon^*)^{-\gamma}$ belongs to $\mathcal{L}(Z_\varepsilon',U)$. Thus, as in Step 1 of the proof of Theorem 4.4-(i), we can prove that, for all $\varepsilon\in(0,1)$, $A_{\varepsilon,F_\varepsilon}$ is the infinitesimal generator of an analytic semigroup on Z_ε, and that

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le \frac{C}{|\lambda+\omega_F|}, \quad \forall\lambda\in\{-\omega_F\}+\mathbb{S}_{\pi/2+\delta_\varepsilon},\ \forall\varepsilon\in(0,1),$$

for some $\delta_\varepsilon\in(0,\pi/2)$. But the family $(B_\varepsilon^*(-\widehat{A}_\varepsilon)^{-\gamma})_{\varepsilon\in(0,1)}$ is not uniformly bounded in $\mathcal{L}(Z_\varepsilon',U)$, and therefore we cannot prove that $\inf_{\varepsilon\in(0,1)}\delta_\varepsilon>0$.

We proceed differently. As in Step 2 of the proof of Theorem 4.4-(i), we can show that, there exist $\varepsilon_0\in(0,1)$ and $\widehat{\omega}_0>\max(-\omega_F,\omega_0)$, such that

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le \frac{C}{|\lambda+\omega_F|}, \quad \forall\lambda\in\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta},\ \forall\varepsilon\in(0,\varepsilon_0). \tag{4.40}$$

Thus, for all $\varepsilon\in(0,\varepsilon_0)$, $A_{\varepsilon,F_\varepsilon}$ is the infinitesimal generator of an analytic semigroup on Z_ε.

Since, we have

$$\|\mathrm{e}^{t(A_{\varepsilon,F_\varepsilon}+\omega_F I)}\|_{\mathcal{L}(Z_\varepsilon)} \le C\,\mathrm{e}^{-t\alpha}, \quad \forall\varepsilon\in(0,1),$$

for some $\alpha>0$, we deduce that

$$\rho(A_{\varepsilon,F_\varepsilon}) \subset \{\lambda\in\mathbb{C} \mid \mathrm{Re}\lambda > -\omega_F-\alpha/2\}, \quad \forall\varepsilon\in(0,1),$$

and

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le C, \quad \forall\lambda\in\{\lambda\in\mathbb{C} \mid \mathrm{Re}\lambda > -\omega_F-\alpha/2\}, \quad \forall\varepsilon\in(0,1).$$

Therefore, combining (4.40) and the above estimate in

$$\{\lambda\in\mathbb{C} \mid \mathrm{Re}\lambda > -\omega_F-\alpha/2\}\setminus\left(\{\widehat{\omega}_0\}+\mathbb{S}_{\pi/2+\delta}\right)$$

we deduce there exists $\widetilde{\delta}\in(0,\pi/2)$ such that we have

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le \frac{C}{|\lambda+\omega_F|}, \quad \forall\lambda\in\{-\omega_F\}+\mathbb{S}_{\pi/2+\widetilde{\delta}}, \quad \forall\varepsilon\in(0,\varepsilon_0). \tag{4.41}$$

Thus, we can rewrite the proof of Theorem 4.4-(i) in which we replace F_ε by $F^{(\varepsilon)}$, F by F_ε, and by reversing the role of (A,B,P,F) and $(A_\varepsilon,B_\varepsilon,P_\varepsilon,F_\varepsilon)$. □

4.3. Properties of the open-loop system

Let $\mathcal{K}\in\mathcal{L}(L^2(U),L^2(H))$ and $\mathcal{K}_\varepsilon\in\mathcal{L}(L^2(U),L^2(H))$ be defined by

$$\mathcal{K}(u)(t)=\int_0^t \mathrm{e}^{\widehat{A}(t-\tau)}Bu(\tau)\mathrm{d}\tau,\quad \mathcal{K}_\varepsilon(u)(t)=\int_0^t \mathrm{e}^{\widehat{A}_\varepsilon(t-\tau)}B_\varepsilon u(\tau)\mathrm{d}\tau. \tag{4.42}$$

We recall that $L^2(U)$ and $L^2(H)$ stand for $L^2(0,\infty;U)$ and $L^2(0,\infty;H)$ respectively (see Sect. 2.1). The fact that $\mathcal{K}$ belongs to $\mathcal{L}(L^2(U),L^2(H))$ follows from (4.18). Next, the fact that $\mathcal{K}_\varepsilon$ belongs to $\mathcal{L}(L^2(U),L^2(H))$ follows from Corollary 4.10 stated below.

Proposition 4.9. *The following estimate holds:*

$$\|\mathrm{e}^{At}B-\mathrm{e}^{A_\varepsilon t}B_\varepsilon\|_{\mathcal{L}(U,H)}\le C\frac{\varepsilon^r\mathrm{e}^{\omega_0 t}}{t},\quad \forall\varepsilon\in(0,1),\ \forall t>0. \tag{4.43}$$

Proof. We have

$$\mathrm{e}^{At}B-\mathrm{e}^{A_\varepsilon t}B_\varepsilon=\frac{1}{2i\pi}\int_{\{\omega_0\}+\Gamma_{\rho,\delta}}\mathrm{e}^{\lambda t}\left((\lambda I-A_\varepsilon)^{-1}B_\varepsilon-(\lambda I-A)^{-1}B\right)\mathrm{d}\lambda,$$

where $\Gamma_{\rho,\delta}$ is defined by (2.18). The estimate (4.43) follows from the above identity and from (4.13), by arguing as in the proof of Theorem 2.5. □

Corollary 4.10. *Assume that* (H_1) *to* (H_5) *are satisfied. The operators* $\mathcal{K}$ *and* $\mathcal{K}_\varepsilon$*, defined in* (4.42)*, satisfy the following estimate:*

$$\|(\mathcal{K}_\varepsilon-\mathcal{K})u\|_{L^p(H)}\le C\varepsilon^r\,|\ln\varepsilon|\,\|u\|_{L^p(U)},\quad\forall\varepsilon\in(0,1/2), \tag{4.44}$$

for all $p\in[1,\infty]$.

Proof. With Young's inequality, we have

$$\|(\mathcal{K}_\varepsilon-\mathcal{K})u\|_{L^p(H)}\le C\|\mathrm{e}^{\widehat{A}(\cdot)}B-\mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}B_\varepsilon\|_{L^1(\mathcal{L}(U,H)))}\|u\|_{L^p(U)}.$$

Let us recall that $\overline{\varepsilon}=\varepsilon^{r/(1-\gamma)}$. We estimate $\int_0^{\overline{\varepsilon}^\alpha}\|\mathrm{e}^{\widehat{A}t}B-\mathrm{e}^{\widehat{A}_\varepsilon t}B_\varepsilon\|_{\mathcal{L}(U,H)}dt$ by using (4.18) and (2.8), with $\alpha=\frac{1-\gamma}{1-\overline{\gamma}}$. We estimate $\int_{\overline{\varepsilon}^\alpha}^1\|\mathrm{e}^{\widehat{A}t}B-\mathrm{e}^{\widehat{A}_\varepsilon t}B_\varepsilon\|_{\mathcal{L}(U,H)}dt$ by using (4.43). We estimate $\int_1^\infty\|\mathrm{e}^{\widehat{A}t}B-\mathrm{e}^{\widehat{A}_\varepsilon t}B_\varepsilon\|_{\mathcal{L}(U,H)}dt$ by using

$$\|\mathrm{e}^{At}B-\mathrm{e}^{A_\varepsilon t}B_\varepsilon\|_{\mathcal{L}(U,H)}\le C\varepsilon^r\mathrm{e}^{\omega_0 t},\quad\forall t\ge 1.$$

The proof is complete. □

Below, in Corollary 4.12, we improve the estimate proved in Corollary 4.10 in the case when $p\in(1,\infty)$. For that we need the following theorem.

Theorem 4.11 ([23], Thm. 6.1.6, p. 135)**.** *Let X and Y be two Hilbert spaces. Let $\mathcal{M} \in C^1(\mathbb{R}\backslash\{0\}; \mathcal{L}(X,Y))$ satisfy*

$$\|\mathcal{M}(\xi)\|_{\mathcal{L}(X,Y)} + |\xi| \|\mathcal{M}'(\xi)\|_{\mathcal{L}(X,Y)} \le C_{\mathcal{M}}, \quad \forall \xi \in \mathbb{R}^*.$$

Then, for all $p \in]1,\infty[$, the operator $\mathcal{T}_{\mathcal{M}}$ defined by

$$(\mathcal{T}_{\mathcal{M}} f)(t) = \int_{\mathbb{R}} \mathrm{e}^{it\xi} \mathcal{M}(\xi) \widehat{f}(\xi) d\xi \quad \forall f \in L^p(\mathbb{R}; X),$$

where $\widehat{f}$ is the Fourier transform of f, belongs to $\mathcal{L}(L^p(\mathbb{R};X), L^p(\mathbb{R};Y))$ and satisfies

$$\|\mathcal{T}_{\mathcal{M}} f\|_{L^p(\mathbb{R};Y)} \le C_p C_{\mathcal{M}} \|f\|_{L^p(\mathbb{R};X)}.$$

Corollary 4.12. *For all $p \in (1,\infty)$, the following estimate holds:*

$$\|(\mathcal{K}_\varepsilon - \mathcal{K})u\|_{L^p(H)} \le C\varepsilon^r \|u\|_{L^p(U)}, \quad \forall \varepsilon \in (0,1). \tag{4.45}$$

Proof. For $u \in L^p(H)$, we have $(\mathcal{K}_\varepsilon - \mathcal{K})u = \mathcal{T}_{\mathcal{M}} u$, where u is extended by zero to $\mathbb{R}^-$, $\mathcal{M}(\xi) = M(\imath\xi)$, and M is defined in (4.11). Thus, (4.45) is a direct consequence of (4.13), (4.14) and of Theorem 4.11. □

We end this section by stating estimates helpul in the proof of convergence rates for Riccati-based feedback laws (see Sect. 5.6).

Corollary 4.13. *Let us recall that $\overline{\varepsilon} = \varepsilon^{r/(1-\gamma)}$. We have the following estimates over the intervals $(0,\overline{\varepsilon})$ and $(\overline{\varepsilon},\infty)$:*

$$\|\mathrm{e}^{A_\varepsilon t} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le C\,\varepsilon^r \frac{\mathrm{e}^{\omega_0 t}}{t}, \quad \forall t \in (0,\overline{\varepsilon}), \tag{4.46}$$

and

$$\|\mathrm{e}^{A_\varepsilon t} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le C \frac{\mathrm{e}^{\omega_0 t}}{t^\gamma}, \quad \forall t \in (\overline{\varepsilon},\infty). \tag{4.47}$$

Proof. Step 1. From (4.43) and (4.18), it follows that

$$\|\mathrm{e}^{A_\varepsilon t} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le C\,\varepsilon^r \frac{\mathrm{e}^{\omega_0 t}}{t} + C\,\frac{\mathrm{e}^{\omega_0 t}}{t^\gamma}, \quad \forall t > 0. \tag{4.48}$$

For $0 < t < \overline{\varepsilon}$, we have $t^{1-\gamma} < \varepsilon^r$ and

$$\|\mathrm{e}^{A_\varepsilon t} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le C\,\varepsilon^r \frac{\mathrm{e}^{\omega_0 t}}{t} + C\,\frac{\mathrm{e}^{\omega_0 t}}{t^\gamma} \le C\,\varepsilon^r \frac{\mathrm{e}^{\omega_0 t}}{t}, \quad \forall t \in (0,\overline{\varepsilon}).$$

Step 2. For $t > \overline{\varepsilon}$, we have $\frac{\varepsilon^r}{t^{1-\gamma}} < 1$. Thus, with (4.48), we have

$$\|\mathrm{e}^{A_\varepsilon t} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le C \frac{\mathrm{e}^{\omega_0 t}}{t^\gamma}, \quad \forall t \in (\overline{\varepsilon},\infty). \tag{4.49}$$

This completes the proof. □

5. Approximate Riccati feedback law

We assume that the triplets (A, B, P) and $(A_\varepsilon, B_\varepsilon, P_\varepsilon)$ satisfy the assumptions (H_1) to (H_5), and that

$$\text{The pair } (A, B) \text{ is stabilizable in } Z. \tag{5.1}$$

Let $\mathcal{C}$ belong to $\mathcal{L}(H, Y)$, where Y is a Hilbert space. We denote by $\mathcal{C}|_Z$ the restriction of $\mathcal{C}$ to Z and by $\mathcal{C}|_{Z_\varepsilon}$ the restriction of $\mathcal{C}$ to Z_ε.

We assume that

$$\text{The pair } (A, \mathcal{C}|_Z) \text{ is detectable in } Z. \tag{5.2}$$

Assumption (5.2) means that there exists $L \in \mathcal{L}(Y, Z)$ such that $(A + L\mathcal{C}|_Z, \mathcal{D}(A))$ is the infinitesimal generator of an exponentially stable strongly continuous semigroup on Z. An equivalent definition is to say that the pair $(A^*, (\mathcal{C}|_Z)^*)$ is stabilizable in Z, see [19], Definition V-1-3.1, p. 488.

In this section, we are going to construct a feedback law $F \in \mathcal{L}(Z, U)$, determined via a Riccati equation associated to the triplet $(A, B, \mathcal{C}|_Z)$, stabilizing the pair (A, B) in Z, and a feedback law $F_\varepsilon \in \mathcal{L}(Z_\varepsilon, U)$, determined via a Riccati equation associated to the triplet $(A_\varepsilon, B_\varepsilon, \mathcal{C}|_{Z_\varepsilon})$, stabilizing the pair $(A_\varepsilon, B_\varepsilon)$ in Z_ε. We want to show that, in that case, the pair (F, F_ε) satisfies the condition (4.1). Thus the results of Theorem 4.4-(i) may apply.

5.1. Feedback stabilization of the pair (A, B)

Let us set

$$\mathcal{I}(z, u) \overset{\text{def}}{=} \frac{1}{2} \int_0^\infty \|\mathcal{C}z(t)\|_Y^2 \, \mathrm{d}t + \frac{1}{2} \int_0^\infty \|u(t)\|_U^2 \, \mathrm{d}t,$$

and consider the evolution equation

$$z' = Az + Bu, \quad z(0) = Pz_0 \in Z, \tag{5.3}$$

where $z_0 \in H$. Let us consider the optimal control problem

$$(\mathcal{P}) \qquad \inf\{\mathcal{I}(z, u) \mid (z, u) \in L^2(Z) \times L^2(U) \text{ obeys } (5.3)\}.$$

Since (A, B) is stabilizable in Z, and the pair $(A, \mathcal{C}|_Z)$ is detectable in Z, problem $(\mathcal{P})$ admits a unique solution and the optimal pair $(\widehat{z}, \widehat{u})$ satisfies the feedback relation

$$\widehat{u}(t) = -B^*\Pi\widehat{z}(t) \quad \text{for all } t > 0, \tag{5.4}$$

where $\Pi \in \mathcal{L}(Z, Z')$ is the unique solution to the algebraic Riccati equation

$$\begin{aligned} &\Pi \in \mathcal{L}(Z, Z'), \quad \Pi = \Pi^* \geq 0, \quad B^*\Pi \in \mathcal{L}(Z, U), \\ &\Pi A + A^*\Pi - \Pi BB^*\Pi + (\mathcal{C}|_Z)^*\mathcal{C}|_Z = 0, \end{aligned} \tag{5.5}$$

where $(\mathcal{C}|_Z)^*$ is the adjoint of $\mathcal{C}|_Z \in \mathcal{L}(Z, Y)$. Let us notice that since Z' is identified with P^*H (see Sect. 2.1) and Y' is identified with Y then we have $(\mathcal{C}|_Z)^* = P^*\mathcal{C} \in \mathcal{L}(Y, Z')$ and

$$(\mathcal{C}|_Z)^*\mathcal{C}|_Z = P^*\mathcal{C}^*\mathcal{C}|_Z. \tag{5.6}$$

The existence and uniqueness of solution to (5.5) is established *e.g.* in [11], Theorem 2.2.1, Chapter 2, p. 125 if B is an unbounded operator, or in [24], Theorem 4.3, p. 240 if B is bounded.

Moreover, the semigroup generated by $A_\Pi \overset{\text{def}}{=} A - BB^*\Pi$ is analytic and exponentially stable on Z (see [11], Chap. 2, Thm. 2.2.2), and we have:

$$(\mathrm{e}^{t(A_\Pi+\omega_\Pi I)})_{t\geq 0} \quad \text{is exponentially stable on } Z, \tag{5.7}$$

for some $\omega_\Pi > 0$.

The optimal state is defined by $\widehat{z}(t) = \mathrm{e}^{A_\Pi t}Pz_0$. It is also well-known that the optimal pair $(\widehat{z},\widehat{u})$ obeys

$$\mathcal{I}(\widehat{z},\widehat{u}) = \frac{1}{2}(\Pi P z_0, P z_0)_H, \tag{5.8}$$

and that Π satisfies the following integral identity

$$\Pi = \int_0^\infty \mathrm{e}^{\widehat{A}^*\tau}((\mathcal{C}|_Z)^*\mathcal{C}|_Z + 2\lambda_0\Pi)\mathrm{e}^{\widehat{A}_\Pi\tau}\mathrm{d}\tau = \int_0^\infty \mathrm{e}^{\widehat{A}^*\tau}P^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0\Pi)\,\mathrm{e}^{\widehat{A}_\Pi\tau}\mathrm{d}\tau, \tag{5.9}$$

where $\widehat{A}_\Pi \overset{\text{def}}{=} A_\Pi - \lambda_0 I$ (see *e.g.* [11], Thm. 2.2.1). Note that the above second equality is justified by (5.6).

Proposition 5.1. *We have*

$$\mathcal{K}^*\left[(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P^*\Pi P)\mathrm{e}^{\widehat{A}_\Pi(\cdot)}Pz_0\right](t) = B^*\Pi\mathrm{e}^{\widehat{A}_\Pi t}Pz_0, \quad \forall t\geq 0, \tag{5.10}$$

and all $z_0\in H$, where $\mathcal{K}$ is defined in (4.42) and

$$(\mathcal{K}^*z)(t) = B^*\int_t^\infty \mathrm{e}^{\widehat{A}^*(\tau-t)}P^*z(\tau)\,d\tau.$$

In addition, the optimal trajectory $\widehat{z}$ of Problem $(\mathcal{P})$ satisfies

$$\left[\big(I + \mathcal{K}\mathcal{K}^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0\Pi)\big)\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z}\right](t) = \mathrm{e}^{\widehat{A}t}Pz_0. \tag{5.11}$$

Proof. With (5.9), we can write

$$\begin{aligned}
B^*\Pi\mathrm{e}^{\widehat{A}_\Pi t}Pz_0 &= B^*\int_0^\infty \mathrm{e}^{\widehat{A}^*\tau}P^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0\Pi)\mathrm{e}^{\widehat{A}_\Pi(\tau+t)}Pz_0\mathrm{d}\tau,\\
&= B^*\int_t^\infty \mathrm{e}^{\widehat{A}^*(\tau-t)}P^*\big((\mathcal{C}^*\mathcal{C} + 2\lambda_0\Pi)\mathrm{e}^{\widehat{A}_\Pi\tau}Pz_0\big)\mathrm{d}\tau,\\
&= \mathcal{K}^*\left[(\mathcal{C}^*\mathcal{C} + 2\lambda_0\Pi)\mathrm{e}^{\widehat{A}_\Pi(\cdot)}Pz_0\right](t), \quad \forall t\geq 0.
\end{aligned}$$

Thus, (5.10) is proved.

To prove (5.11), we notice that

$$\begin{aligned}
\mathrm{e}^{-\lambda_0 t}\widehat{z}(t) &= \mathrm{e}^{\widehat{A}_\Pi t}Pz_0 = \mathrm{e}^{(\widehat{A}-BB^*\Pi)t}Pz_0,\\
&= \mathrm{e}^{\widehat{A}t}Pz_0 - \int_0^t \mathrm{e}^{\widehat{A}(t-\tau)}B\big(B^*\Pi\mathrm{e}^{\widehat{A}_\Pi\tau}Pz_0\big)\mathrm{d}\tau,\\
&= \mathrm{e}^{\widehat{A}t}Pz_0 - \left[\mathcal{K}\big(B^*\Pi\mathrm{e}^{\widehat{A}_\Pi(\cdot)}Pz_0\big)\right](t), \quad \forall t\geq 0.
\end{aligned}$$

Next, we substitute $B^*\Pi e^{\widehat{A}_\Pi t}Pz_0$ by $\mathcal{K}^*\Big[(\mathcal{C}^*\mathcal{C}+2\lambda_0\Pi)e^{\widehat{A}_\Pi(\cdot)}Pz_0\Big](t)$, and we use the equality $e^{\widehat{A}_\Pi t}Pz_0 = e^{-\lambda_0 t}\widehat{z}(t)$, in the above identity. □

5.2. Detectability of the pair $(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon})$

For $\varepsilon_0 \in (0,1)$ let us recall that, by definition, the family $(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon})_{0<\varepsilon<\varepsilon_0}$ is detectable in Z_ε, uniformly with respect to $\varepsilon \in (0,\varepsilon_0)$, if and only if $(A_\varepsilon^*, (\mathcal{C}|_{Z_\varepsilon})^*)_{0<\varepsilon<\varepsilon_0} = (A_\varepsilon^*, P_\varepsilon^*\mathcal{C}^*)_{0<\varepsilon<\varepsilon_0}$ is stabilizable in Z'_ε, uniformly with respect to $\varepsilon \in (0,\varepsilon_0)$.

In the following lemma we deduce from (5.2) that $(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon})_{0<\varepsilon<\varepsilon_0}$ is detectable in Z_ε, uniformly with respect to $\varepsilon \in (0,\varepsilon_0)$, for some $\varepsilon_0 \in (0,1)$.

Lemma 5.2. *There exists $\varepsilon_0 \in (0,1)$ such that the family of pairs of operators $(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon})_{0<\varepsilon<\varepsilon_0}$ is detectable in Z_ε, uniformly with respect to $\varepsilon \in (0,\varepsilon_0)$.*

Proof. According to (5.2) the pair $(A, \mathcal{C}|_Z)$ is detectable in Z which is equivalent to the fact that $(A^*, (\mathcal{C}|_Z)^*) = (A^*, P^*\mathcal{C}^*)$ is stabilizable in Z'. Thus, there exists $K^* \in \mathcal{L}(Z', Y)$ such that the semigroup generated by $A^* + P^*\mathcal{C}^*K^*$ is exponentially stable on Z'. Let P_ε^* be the adjoint of $P_\varepsilon \in \mathcal{L}(H)$. We can easily check that P_ε^* is a projector in H onto Z'_ε (see Sect. 2.1). We are going to use Theorem 4.4-(i) in which we replace A, A_ε, P, P_ε, U, B, B_ε, F, and F_ε by A^*, A_ε^*, P^*, P_ε^*, Y, $P^*\mathcal{C}^*$, $P_\varepsilon^*\mathcal{C}^*$, K^*, and K^*P^*, respectively. In that case, it is clear that (H_1) is satisfied. Assumption (H_3) and (H_5) are satisfied with $\gamma = \overline{\gamma} = 0$. Assumption (H_2) and Assumption (H_4), with $r = s$ and $\gamma = 0$, are satisfied because we have

$$\begin{aligned}&\|\widehat{A}^{-*}P^* - \widehat{A}_\varepsilon^{-*}P_\varepsilon^*\|_{\mathcal{L}(H)} = \|P^*\widehat{A}^{-*}P^* - P_\varepsilon^*\widehat{A}_\varepsilon^{-*}P_\varepsilon^*\|_{\mathcal{L}(H)}\\&= \|(P\widehat{A}^{-1}P - P_\varepsilon\widehat{A}_\varepsilon^{-1}P_\varepsilon)^*\|_{\mathcal{L}(H)} = \|(\widehat{A}^{-1}P - \widehat{A}_\varepsilon^{-1}P_\varepsilon)^*\|_{\mathcal{L}(H)} \le C\varepsilon^s.\end{aligned}$$

Thus, according to Theorem 4.4-(i), there exists $\varepsilon_0 \in (0,1)$ such that the semigroup generated by $A_\varepsilon^* + P_\varepsilon^*\mathcal{C}^*K^*P^*$ is analytic and exponentially stable on Z'_ε, uniformly in $\varepsilon \in (0,\varepsilon_0)$. Thus, since $(A_\varepsilon^* + P_\varepsilon^*\mathcal{C}^*K^*P^*)^* = A_\varepsilon + P_\varepsilon K\mathcal{C}|_{Z_\varepsilon}$ we deduce that the semigroup generated by $A_\varepsilon + P_\varepsilon K\mathcal{C}|_{Z_\varepsilon}$ is analytic and exponentially stable, uniformly in $\varepsilon \in (0,\varepsilon_0)$. Then $(A_\varepsilon, \mathcal{C}_{|Z_\varepsilon})_{0<\varepsilon<\varepsilon_0}$ is detectable in Z_ε, uniformly in $\varepsilon \in (0,\varepsilon_0)$. □

Lemma 5.3. *Let ε_0 be the parameter introduced in Lemma 5.2. There exists $K \in \mathcal{L}(H,Z)$ such that $A_{\varepsilon,K} \stackrel{def}{=} A_\varepsilon + P_\varepsilon K\mathcal{C}|_{Z_\varepsilon}$ generates an analytic and exponentially stable semigroup on Z_ε, uniformly in $\varepsilon \in (0,\varepsilon_0)$.*

Moreover, the linear operator $\mathcal{K}_{\varepsilon,K}$ defined by

$$\mathcal{K}_{\varepsilon,K}(u)(t) = \int_0^t e^{A_{\varepsilon,K}(t-\tau)}B_\varepsilon u(\tau)d\tau, \tag{5.12}$$

is bounded from $L^2(U)$ into $L^2(H)$, uniformly in $\varepsilon \in (0,\varepsilon_0)$.

Proof. The existence of $K \in \mathcal{L}(H,Z)$ satisfying the first part of the Lemma has been proved at the end of the proof of Lemma 5.2. It implies that there exist $\omega_K > 0$ and $\delta_K > 0$ such that

$$\|(\lambda I - A_{\varepsilon,K})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le \frac{C}{|\lambda + \omega_K|}, \quad \text{for all } \lambda \in \{-\omega_K\} + \mathbb{S}_{\pi/2+\delta_K}.$$

In particular, the above inequality is satisfied for all $\lambda \in \mathbb{S}_{\pi/2+\delta_K}$. Hence, since for all $\lambda \in \mathbb{S}_{\pi/2+\delta_K}$ we have $|\lambda|/|\lambda + \omega_K| \le C$, we deduce that

$$|\lambda|\|(\lambda I - A_{\varepsilon,K})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \le C, \quad \text{for all } \lambda \in \mathbb{S}_{\pi/2+\delta_K}. \tag{5.13}$$

Thus, from (5.13) with $(\lambda I - A_{\varepsilon,K})^{-1}A_{\varepsilon,K} = -I + \lambda(\lambda I - A_{\varepsilon,K})^{-1}$ we deduce that

$$\|(\lambda I - A_{\varepsilon,K})^{-1}A_{\varepsilon,K}\|_{\mathcal{L}(Z_\varepsilon)} \le C, \quad \text{for all } \lambda \in \mathbb{S}_{\pi/2+\delta_K}. \tag{5.14}$$

Let $\mu_0 > \lambda_0$ large enough so that

$$\|(\mu_0 I - A_\varepsilon)^{-1}P_\varepsilon K\mathcal{C}\|_{\mathcal{L}(Z_\varepsilon)} < 1/2. \tag{5.15}$$

Then $I - (\mu_0 I - A_\varepsilon)^{-1}P_\varepsilon K\mathcal{C}$ is boundedly invertible in H and

$$(\mu_0 I - A_{\varepsilon,K})^{-1}B_\varepsilon = (I - (\mu_0 I - A_\varepsilon)^{-1}P_\varepsilon K\mathcal{C})^{-1}(\mu_0 I - A_\varepsilon)^{-1}B_\varepsilon.$$

The above equality with (5.15) and (2.6) yields

$$\|(\mu_0 I - A_{\varepsilon,K})^{-1}B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le 2\|(\mu_0 I - A_\varepsilon)^{-1}B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le C. \tag{5.16}$$

Hence, from (5.16) with (5.13) and (5.14) we deduce that for all $\lambda \in \mathbb{S}_{\pi/2+\delta_K}$,

$$\|(\lambda I - A_{\varepsilon,K})^{-1}B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le \|(\lambda I - A_{\varepsilon,K})^{-1}(\mu_0 I - A_{\varepsilon,K})\|_{\mathcal{L}(Z_\varepsilon)}\|(\mu_0 I - A_{\varepsilon,K})^{-1}B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le C$$

and

$$\|(\lambda I - A_{\varepsilon,K})^{-2}B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le \|(\lambda I - A_{\varepsilon,K})^{-1}\|_{\mathcal{L}(Z_\varepsilon)}\|(\lambda I - A_{\varepsilon,K})^{-1}B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \le \frac{C}{|\lambda|}.$$

Finally, the conclusion follows from Theorem 4.11. Indeed with the notation there we have $\mathcal{K}_{\varepsilon,K}u = \mathcal{T}_{\mathcal{M}}u$, where u is extended by zero to $\mathbb{R}^-$, and $\mathcal{M}(\xi) = (\imath\xi - A_{\varepsilon,K})^{-1}B_\varepsilon$ and $\mathcal{M}'(\xi) = \imath(\imath\xi - A_{\varepsilon,K})^{-2}B_\varepsilon$. □

5.3. Feedback stabilization of the pair $(A_\varepsilon, B_\varepsilon)$

Let us consider the following evolution equation in Z_ε:

$$z_\varepsilon' = A_\varepsilon z_\varepsilon + B_\varepsilon u, \quad z_\varepsilon(0) = P_\varepsilon z_0 \in Z_\varepsilon. \tag{5.17}$$

And let us consider the optimal control problem

$$(\mathcal{P}_\varepsilon) \qquad \inf\{\mathcal{I}(z_\varepsilon, u) \mid (z_\varepsilon, u) \in L^2(Z_\varepsilon) \times L^2(U) \text{ obeys } (5.17)\}.$$

Here, we propose to determine an approximation of the feedback control law $-B^*\Pi$ introduced in (5.4) by looking for the solution to the optimal control problem $(\mathcal{P}_\varepsilon)$.

Since (A, B) is stabilizable, from Theorem 4.4-(i) with $\sigma \equiv 0$, it follows that we can choose $\varepsilon_0 \in (0, 1)$ so that the family $(A_\varepsilon, B_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is stabilizable uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$. More precisely, for all $\varepsilon \in (0, \varepsilon_0)$, the following functions

$$\widetilde{z}_\varepsilon \stackrel{\text{def}}{=} \mathrm{e}^{(A_\varepsilon - B_\varepsilon(B^*\Pi P))(\cdot)}P_\varepsilon z_0 \quad \text{and} \quad \widetilde{u}_\varepsilon \stackrel{\text{def}}{=} -(B^*\Pi P)\widetilde{z}_\varepsilon \tag{5.18}$$

satisfy

$$\widetilde{z}_\varepsilon' = A_\varepsilon \widetilde{z}_\varepsilon + B_\varepsilon \widetilde{u}_\varepsilon, \quad \widetilde{z}_\varepsilon(0) = P_\varepsilon z_0 \in Z_\varepsilon, \tag{5.19}$$

and there exists $\varrho > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, we have

$$\|\widetilde{z}_\varepsilon(t)\|_{Z_\varepsilon} \le C\mathrm{e}^{-\omega_{\Pi,\varepsilon} t}\|z_0\|_H, \tag{5.20}$$

$$\|\widehat{z}(t) - \widetilde{z}_\varepsilon(t)\|_H \le C\frac{\mathrm{e}^{-\omega_{\Pi,\varepsilon} t}}{t^{r/s}}\varepsilon^r\|z_0\|_H, \tag{5.21}$$

where $\omega_{\Pi,\varepsilon} \stackrel{\text{def}}{=} \omega_\Pi - \varrho\varepsilon^r$.

This means that $(\widetilde{z}_\varepsilon, \widetilde{u}_\varepsilon)$ is an admissible pair for the approximate problem $(\mathcal{P}_\varepsilon)$. Thus $(\mathcal{P}_\varepsilon)$ admits a unique solution, and the optimal pair $(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)$ satisfies the feedback relation

$$\widehat{u}_\varepsilon(t) = -B_\varepsilon^*\Pi_\varepsilon\widehat{z}_\varepsilon(t), \quad \text{for all } t > 0, \tag{5.22}$$

where $\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon, Z_\varepsilon')$ is the unique solution to the algebraic Riccati equation

$$\begin{aligned} &\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon, Z_\varepsilon'), \quad \Pi_\varepsilon = \Pi_\varepsilon^* \ge 0, \quad B_\varepsilon^*\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon, U), \\ &\Pi_\varepsilon A_\varepsilon + A_\varepsilon^*\Pi_\varepsilon - \Pi_\varepsilon B_\varepsilon B_\varepsilon^*\Pi_\varepsilon + (\mathcal{C}|_{Z_\varepsilon})^*\mathcal{C}|_{Z_\varepsilon} = 0. \end{aligned} \tag{5.23}$$

Moreover, the semigroup generated by $A_{\varepsilon,\Pi_\varepsilon} \stackrel{\text{def}}{=} A_\varepsilon - B_\varepsilon B_\varepsilon^*\Pi_\varepsilon$ is analytic and exponentially stable on Z_ε, and the associated cost functional is defined by

$$\mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon) = \frac{1}{2}(\Pi_\varepsilon P_\varepsilon z_0, P_\varepsilon z_0)_H. \tag{5.24}$$

The Riccati operator Π_ε satisfies the integral relation

$$\Pi_\varepsilon = \int_0^\infty \mathrm{e}^{\widehat{A}_\varepsilon^*\tau}((\mathcal{C}|_{Z_\varepsilon})^*\mathcal{C}|_{Z_\varepsilon} + 2\lambda_0\Pi_\varepsilon)\mathrm{e}^{\widehat{A}_{\varepsilon,\Pi_\varepsilon}\tau}\mathrm{d}\tau = \int_0^\infty \mathrm{e}^{\widehat{A}_\varepsilon^*\tau}P_\varepsilon^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0\Pi_\varepsilon)\mathrm{e}^{\widehat{A}_{\varepsilon,\Pi_\varepsilon}\tau}\mathrm{d}\tau, \tag{5.25}$$

where $\widehat{A}_{\varepsilon,\Pi_\varepsilon} \stackrel{\text{def}}{=} A_{\varepsilon,\Pi_\varepsilon} - \lambda_0 I$. Moreover, $\widehat{z}_\varepsilon$ satisfies:

$$\big(I + \mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0\Pi_\varepsilon)\big)(\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z}_\varepsilon) = \mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon z_0, \tag{5.26}$$

where $\mathcal{K}_\varepsilon \in \mathcal{L}(L^2(U), L^2(H))$ is defined in (4.42) and $\mathcal{K}_\varepsilon^* \in \mathcal{L}(L^2(H), L^2(U))$ is defined by

$$(\mathcal{K}_\varepsilon^* z)(t) = B_\varepsilon^*\int_t^\infty \mathrm{e}^{\widehat{A}_\varepsilon^*(\tau - t)}P_\varepsilon^* z(\tau)\,\mathrm{d}\tau.$$

Proposition 5.4. *We have*

$$\sup_{0<\varepsilon<\varepsilon_0}\|\mathrm{e}^{A_{\varepsilon,\Pi_\varepsilon}(\cdot)}P_\varepsilon\|_{\mathcal{L}(H,L^2(H))} < +\infty. \tag{5.27}$$

Proof. The optimality of $(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)$ and (5.24) give

$$(\Pi_\varepsilon P_\varepsilon z_0, P_\varepsilon z_0)_H \le 2\mathcal{I}(\widetilde{z}_\varepsilon, \widetilde{u}_\varepsilon).$$

From (5.20), the definition of $\widetilde{u}_\varepsilon$ in (5.18), and the definition of $\omega_{\Pi,\varepsilon}$ introduced after (5.21), it follows that

$$(\Pi_\varepsilon P_\varepsilon z_0, P_\varepsilon z_0)_H \le 2\mathcal{I}(\widetilde{z}_\varepsilon, \widetilde{u}_\varepsilon) \le C\|z_0\|_H^2. \tag{5.28}$$

Since $P_\varepsilon^*\Pi_\varepsilon P_\varepsilon$ is self-adjoint, by taking the supremum over $\{z \in H \mid \|z_0\|_H = 1\}$, we obtain:

$$\sup_{0<\varepsilon<\varepsilon_0} \|P_\varepsilon^*\Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)} < +\infty. \tag{5.29}$$

Let $K \in \mathcal{L}(Y, Z)$ be given in Lemma 5.3 such that $A_{\varepsilon,K} \stackrel{\text{def}}{=} A_\varepsilon + P_\varepsilon K\mathcal{C}|_{Z_\varepsilon}$ generates an analytic and exponentially stable semigroup, uniformly in $\varepsilon \in (0, \varepsilon_0)$. From $A_{\varepsilon,\Pi_\varepsilon} = A_{\varepsilon,K} - P_\varepsilon K\mathcal{C}|_{Z_\varepsilon} - B_\varepsilon B_\varepsilon^*\Pi_\varepsilon$ we deduce

$$\mathrm{e}^{A_{\varepsilon,\Pi_\varepsilon} t} P_\varepsilon z_0 = \mathrm{e}^{A_{\varepsilon,K} t} P_\varepsilon z_0 - \int_0^t \mathrm{e}^{A_{\varepsilon,K}(t-\tau)} P_\varepsilon K\mathcal{C}\widehat{z}_\varepsilon(\tau)\mathrm{d}\tau - \int_0^t \mathrm{e}^{A_{\varepsilon,K}(t-\tau)} B_\varepsilon \widehat{u}_\varepsilon(\tau)\mathrm{d}\tau$$

Hence, with the uniform exponential stability of $(\mathrm{e}^{A_{\varepsilon,K} t})_{t\geq 0}$, the uniform boundedness of $P_\varepsilon K\mathcal{C} \in \mathcal{L}(H)$ and of $\mathcal{K}_{\varepsilon,K} \in \mathcal{L}(L^2(U), L^2(H))$ (defined by (5.12)) we deduce

$$\|\mathrm{e}^{A_{\varepsilon,\Pi_\varepsilon}(\cdot)} P_\varepsilon z_0\|_{L^2(H)}^2 \leq C(\|z_0\|_H^2 + \mathcal{I}(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)).$$

Finally, from (5.24) and (5.28) we obtain $\|\mathrm{e}^{A_{\varepsilon,\Pi_\varepsilon}(\cdot)} P_\varepsilon z_0\|_{L^2(H)}^2 \leq C\|z_0\|_H^2$. □

5.4. Uniform stability of the family of approximate semigroups

Throughout this section ε_0 is the parameter introduced in Sections 5.2 and 5.3.

The goal of this section is to prove the uniform exponential stability of the semigroup $(\mathrm{e}^{A_{\varepsilon,\Pi_\varepsilon} t})_{t>0}$. This result is established in Theorem 5.8. It is based on the generalization of Datko's theorem in the case of a parameter dependent semigroup [25], which, in addition to (5.27), requires a bound of the form

$$\sup_{0<\varepsilon<\varepsilon_0} \|\mathrm{e}^{A_{\varepsilon,\Pi_\varepsilon} t} P_\varepsilon\|_{\mathcal{L}(H)} \leq C\mathrm{e}^{at}, \quad \forall t \geq 0, \tag{5.30}$$

for some $a > 0$.

Lemma 5.5. *If the parameter $\overline{\gamma}$ introduced in (2.8) obeys $\overline{\gamma} \in (0, 1)$, and if $p \in [1, \infty]$, the operator $\mathcal{K}_\varepsilon$ defined in (4.42) satisfies*

$$\|\mathcal{K}_\varepsilon\|_{\mathcal{L}(L^p(U), L^q(H))} \leq C_{p,q}, \quad \forall \varepsilon \in (0, \varepsilon_0), \tag{5.31}$$

where $p \leq q < \frac{p}{(1-(1-\overline{\gamma})p)}$ if $p \in]1, \frac{1}{1-\overline{\gamma}}[$, where $\frac{1}{1-\overline{\gamma}} \leq q < \infty$ if $p = \frac{1}{1-\overline{\gamma}}$, and where $p \leq q \leq \infty$ if $p \in]\frac{1}{1-\overline{\gamma}}, \infty]$.

If $\gamma = \overline{\gamma} = 0$, then (5.31) holds for all $q \in [p, \infty]$ and all $p \in [1, \infty]$.

Proof. Let us first assume that $\gamma \in (0, 1)$. Due to Assumption (H_5) and Corollary 4.13, we have

$$\begin{aligned}
&\|\mathrm{e}^{\widehat{A}_\varepsilon t} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \leq C\frac{\mathrm{e}^{-(\lambda_0-\omega_0)t}}{t^{\overline{\gamma}}}, \quad \text{if } 0 < t < \varepsilon^{r/(1-\gamma)} = \overline{\varepsilon}, \\
&\|\mathrm{e}^{\widehat{A}_\varepsilon t} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \leq C\frac{\mathrm{e}^{-(\lambda_0-\omega_0)t}}{t^{\gamma}} \leq C\frac{\mathrm{e}^{-(\lambda_0-\omega_0)t}}{t^{\overline{\gamma}}}, \quad \text{if } \overline{\varepsilon} \leq t \leq 1, \\
&\|\mathrm{e}^{\widehat{A}_\varepsilon t} B_\varepsilon\|_{\mathcal{L}(U,Z_\varepsilon)} \leq C\mathrm{e}^{-(\lambda_0-\omega_0)t}, \quad \text{if } 1 \leq t.
\end{aligned}$$

Thus, we have

$$\|(\mathcal{K}_\varepsilon u)(t)\|_H \leq C\int_0^t k_\varepsilon(t-\tau)\|u(\tau)\|_U \,\mathrm{d}\tau, \quad \forall t \in (0, \infty), \tag{5.32}$$

with

$$k_\varepsilon(t) = \frac{\mathrm{e}^{-(\lambda_0-\omega_0)t}}{t^{\overline{\gamma}}}\,\chi_{[0,1]}(t) + \mathrm{e}^{-(\lambda_0-\omega_0)t}\,\chi_{(1,\infty)}(t),$$

where $\chi_{[0,1]}$ is the characteristic function of $[0,1]$, and $\chi_{(1,\infty)}$ is the characteristic function of $(1,\infty)$. To prove (5.31) it is sufficient to apply Young's inequality to (5.32).

The case where $\gamma = \overline{\gamma} = 0$ is easy and left to the reader. □

Theorem 5.6. *For all $\varepsilon \in (0,\varepsilon_0)$, the operator $I + \mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)$ is an automorphism in $L^2(H)$, and, for all $p \in [2,\infty]$, $(I + \mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^*\Pi_\varepsilon P_\varepsilon))^{-1}$ is bounded in $\mathcal{L}(L^2(H)\cap L^p(H), L^p(H))$, uniformly in $\varepsilon \in (0,\varepsilon_0)$.*

Proof. We notice that $\mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*$ and $(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)$ are both nonnegative and self-adjoint operators in $L^2(H)$. Thus, from [11], Lemma 2A.1, it follows that $I + \mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)$ is an isomorphism of $L^2(H)$ and that

$$\|(I + \mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^*\Pi_\varepsilon P_\varepsilon))^{-1}\|_{\mathcal{L}(L^2(H))} \le 1 + \|\mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*\|_{\mathcal{L}(L^2(H))}\|(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)\|_{\mathcal{L}(H)}.$$

Hence, with (5.31) and (5.29), the theorem is proved for $p = 2$.

If $p > 2$, let us set $\mathcal{T}_\varepsilon = \mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)$ for readability convenience. For all $(x,y) \in H\times H$ such that $x + \mathcal{T}_\varepsilon x = y$, an easy recurrence argument gives

$$x = (-1)^n\mathcal{T}_\varepsilon^{(n)}(I+\mathcal{T}_\varepsilon)^{-1}y + \sum_{k=0}^{n-1}(-1)^k\mathcal{T}_\varepsilon^{(k)}y, \quad \forall n \in \mathbb{N}^*. \tag{5.33}$$

Let $p \in [2,\infty]$ be given fixed. Due to Lemma 5.5, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{T}_\varepsilon^{(n_0)} \in \mathcal{L}(L^2(H), L^p(H))$, with operator norm uniform in ε. We have proved that $(I+\mathcal{T}_\varepsilon)^{-1}$ belongs to $\mathcal{L}(L^2(H))$, with operator norm uniform in ε. Due to Lemma 5.5, $\mathcal{T}_\varepsilon^{(n_0)}$ belongs to $\mathcal{L}(L^p(H))$ and $\mathcal{T}_\varepsilon^{(k)}$ belongs to $\mathcal{L}(L^p(H))$, for all $k = 1,\dots n_0-1$, with operator norm uniform in ε. Thus, the identity (5.33) with $n = n_0$ gives $\|x\|_{L^p(H)} \le C(\|y\|_{L^2(H)} + \|y\|_{L^p(H)})$. The proof of the theorem is complete. □

Remark 5.7. Proceeding as in the proof of Theorem 5.6, we can show, as in the above proof, that the operator $I + \mathcal{K}\mathcal{K}^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P^*\Pi P)$ is an automorphism in $L^2(H)$.

Theorem 5.8. *There exist $\omega_\Pi^* > 0$ and $C > 0$ such that*

$$\sup_{0<\varepsilon<\varepsilon_0}\|\mathrm{e}^{A_{\varepsilon,\Pi_\varepsilon}t}\|_{\mathcal{L}(Z_\varepsilon)} \le C\mathrm{e}^{-\omega_\Pi^* t}, \quad \forall t \ge 0. \tag{5.34}$$

Moreover, the following uniform bound holds:

$$\sup_{0<\varepsilon<\varepsilon_0}\|\widehat{A}_\varepsilon^{*\theta}\Pi_\varepsilon\|_{\mathcal{L}(Z_\varepsilon)} < +\infty, \quad \forall\theta \in [0,1[. \tag{5.35}$$

Proof. With (5.26), Theorem 5.6 for $p=\infty$, and the bound $\|\mathrm{e}^{\widehat{A}_\varepsilon t}P_\varepsilon z_0\|_H \le C\mathrm{e}^{-(\lambda_0-\omega_0)t}\|z_0\|_H$ give

$$\|\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z}_\varepsilon\|_{L^\infty(Z_\varepsilon)} \le C(\|\mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon z_0\|_{L^\infty(Z_\varepsilon)} + \|\mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon z_0\|_{L^2(Z_\varepsilon)}) \le C\|z_0\|_H, \tag{5.36}$$

for all $\varepsilon \in (0,\varepsilon_0)$. Thus, we obtain (5.30) for $a = \lambda_0$. Due to the generalization of Datko's theorem stated in [25], (5.34) follows from (5.27) and (5.30).

Finally, (5.25) gives

$$(-\widehat{A}_\varepsilon^*)^\theta \Pi_\varepsilon = \int_0^\infty (-\widehat{A}_\varepsilon^*)^\theta \mathrm{e}^{\widehat{A}_\varepsilon^* t} P_\varepsilon^* (\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^* \Pi_\varepsilon P_\varepsilon) \mathrm{e}^{\widehat{A}_{\varepsilon,\Pi_\varepsilon} t} \mathrm{d}t.$$

Then, (5.35) follows from (2.3), (5.34), (2.14) and (5.29). □

Lemma 5.9. *We have*

$$\sup_{0<\varepsilon<\varepsilon_0} \|B_\varepsilon^* \Pi_\varepsilon\|_{\mathcal{L}(U)} < +\infty. \tag{5.37}$$

Proof. We have

$$B_\varepsilon^* \Pi_\varepsilon = \int_0^\infty B_\varepsilon^* \mathrm{e}^{\widehat{A}_\varepsilon^* t} P_\varepsilon^* (\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^* \Pi_\varepsilon P_\varepsilon) \mathrm{e}^{\widehat{A}_{\varepsilon,\Pi_\varepsilon} t} \mathrm{d}t.$$

With (2.3), (2.8), (5.29), and (5.34), we have

$$\left\| \int_0^{\overline{\varepsilon}} B_\varepsilon^* \mathrm{e}^{\widehat{A}_\varepsilon^* t} P_\varepsilon^* (\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^* \Pi_\varepsilon P_\varepsilon) \mathrm{e}^{\widehat{A}_{\varepsilon,\Pi_\varepsilon} t} \mathrm{d}t \right\| \le C \int_0^{\overline{\varepsilon}} \frac{\mathrm{d}t}{t^{\overline{\gamma}}} < \infty.$$

With (2.3), (5.29), (5.34) and (4.47), we have

$$\left\| \int_{\overline{\varepsilon}}^\infty B_\varepsilon^* \mathrm{e}^{\widehat{A}_\varepsilon^* t} P_\varepsilon^* (\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^* \Pi_\varepsilon P_\varepsilon) \mathrm{e}^{\widehat{A}_{\varepsilon,\Pi_\varepsilon} t} \mathrm{d}t \right\| \le C \int_{\overline{\varepsilon}}^\infty \frac{\mathrm{e}^{(\omega_0 - \lambda_0 - \omega_\Pi^*)t}}{t^\gamma} \, \mathrm{d}t < \infty.$$

The proof of (5.37) is complete. □

Finally, by setting

$$\widetilde{z} \stackrel{\text{def}}{=} \mathrm{e}^{(A - B(B_\varepsilon^* \Pi_\varepsilon P_\varepsilon))(\cdot)} P z_0 \quad \text{ and } \quad \widetilde{u} \stackrel{\text{def}}{=} -(B_\varepsilon^* \Pi_\varepsilon P_\varepsilon)\widetilde{z}, \tag{5.38}$$

we have

$$\widetilde{z}' = A\widetilde{z} + B\widetilde{u}, \quad \widetilde{z}(0) = P z_0 \in Z. \tag{5.39}$$

From Theorem 4.4-(ii) with $\sigma \equiv 0$ (see Rem. 4.5), it follows that $\varepsilon_0 > 0$ can be chosen such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\|\widetilde{z}(t)\|_Z \le C \mathrm{e}^{(-\omega_\Pi^*/2)t} \|z_0\|_H, \quad \forall t \in (0,\infty), \tag{5.40}$$

$$\|\widehat{z}_\varepsilon(t) - \widetilde{z}(t)\|_H \le C \frac{\mathrm{e}^{(-\omega_\Pi^*/2)t}}{t^{r/s}} \varepsilon^r \|z_0\|_H, \quad \forall t \in (0,\infty). \tag{5.41}$$

5.5. Rate of convergence of the solutions to Riccati equations

Throughout this section ε_0 is the parameter introduced at the end of Section 5.4.

Theorem 5.10. *Let Π be the solution of (5.5) and Π_ε be the solution of (5.23).*
If $0 < r < s$, where r and s are the rates of convergence appearing in (2.4) and (2.6), then we have

$$\|P^* \Pi P - P_\varepsilon^* \Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)} \le C \varepsilon^r, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{5.42}$$

If $r = s$, we have

$$\|P^*\Pi P - P_\varepsilon^*\Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)} \le C\varepsilon^s\,|\ln(\varepsilon)|, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{5.43}$$

Proof. Step 1. First, for $z_0 \in H$, (5.8) and (5.24) give

$$\frac{1}{2}\big|\big((P^*\Pi P - P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)z_0, z_0\big)_H\big| = |\mathcal{I}(\widehat{z}, \widehat{u}) - \mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)|.$$

Thus, the optimality of $(\widehat{z}, \widehat{u})$ and that of $(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)$ give $\mathcal{I}(\widehat{z}, \widehat{u}) \le \mathcal{I}(\widetilde{z}, \widetilde{u})$ and $\mathcal{I}(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon) \le \mathcal{I}(\widetilde{z}_\varepsilon, \widetilde{u}_\varepsilon)$ (see the definitions given in (5.18), (5.38)). Therefore we have

$$\mathcal{I}(\widehat{z}, \widehat{u}) - \mathcal{I}_\varepsilon(\widetilde{z}_\varepsilon, \widetilde{u}_\varepsilon) \le \mathcal{I}(\widehat{z}, \widehat{u}) - \mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon) \le \mathcal{I}(\widetilde{z}, \widetilde{u}) - \mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon).$$

We deduce that $|\mathcal{I}(\widehat{z}, \widehat{u}) - \mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)| \le |\mathcal{I}(\widehat{z}, \widehat{u}) - \mathcal{I}_\varepsilon(\widetilde{z}_\varepsilon, \widetilde{u}_\varepsilon)| + |\mathcal{I}(\widetilde{z}, \widetilde{u}) - \mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)|$. As a consequence, we have:

$$\frac{1}{2}\big|\big((P\Pi P - P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)z_0, z_0\big)_H\big| \le |\mathcal{I}(\widetilde{z}, \widetilde{u}) - \mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)| + |\mathcal{I}_\varepsilon(\widetilde{z}_\varepsilon, \widetilde{u}_\varepsilon) - \mathcal{I}(\widehat{z}, \widehat{u})| \le (a) + (b), \tag{5.44}$$

where

$$\begin{aligned}(a) &= \big|\|\mathcal{C}\widetilde{z}\|^2_{L^2(Y)} - \|\mathcal{C}\widehat{z}_\varepsilon\|^2_{L^2(Y)}\big| + \big|\|\mathcal{C}\widetilde{z}_\varepsilon\|^2_{L^2(Y)} - \|\mathcal{C}\widehat{z}\|^2_{L^2(Y)}\big|,\\ (b) &= \big|\|\widetilde{u}\|^2_{L^2(U)} - \|\widehat{u}_\varepsilon\|^2_{L^2(U)}\big| + \big|\|\widetilde{u}_\varepsilon\|^2_{L^2(U)} - \|\widehat{u}\|^2_{L^2(U)}\big|.\end{aligned}$$

Step 2. We now assume that $0 < r/s < 1$. With the identity $\|\mathcal{C}x\|_Y^2 - \|\mathcal{C}y\|_Y^2 = \big(\mathcal{C}x - \mathcal{C}y, \mathcal{C}x\big)_Y + \big(\mathcal{C}y, \mathcal{C}x - \mathcal{C}y\big)_Y$, and with Hölder's inequality, for $p \in (1, s/r)$, we get:

$$(a) \le C\,\|\widetilde{z} - \widehat{z}_\varepsilon\|_{L^p(H)}\left(\|\widetilde{z}\|_{L^{\frac{p}{p-1}}(H)} + \|\widehat{z}_\varepsilon\|_{L^{\frac{p}{p-1}}(H)}\right) + \|\widetilde{z}_\varepsilon - \widehat{z}\|_{L^p(H)}\left(\|\widetilde{z}_\varepsilon\|_{L^{\frac{p}{p-1}}(H)} + \|\widehat{z}\|_{L^{\frac{p}{p-1}}(H)}\right).$$

From the exponential stability of $\widetilde{z}$, $\widehat{z}_\varepsilon$, $\widetilde{z}_\varepsilon$, and $\widehat{z}$ (see (5.40), (5.34), (5.20), and (5.7)), it follows that $\|\widetilde{z}\|_{L^{\frac{p}{p-1}}(H)} + \|\widehat{z}_\varepsilon\|_{L^{\frac{p}{p-1}}(H)}$ and $\|\widetilde{z}_\varepsilon\|_{L^{\frac{p}{p-1}}(H)} + \|\widehat{z}\|_{L^{\frac{p}{p-1}}(H)}$ are bounded, uniformly in ε. Thus, (5.21) and (5.41) give $(a) \le C\varepsilon^r$.

Moreover, (5.37) allows to bound (b) analogously.

From (5.44), it follows that

$$\big|\big((P^*\Pi P - P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)z_0, z_0\big)_H\big| \le C\varepsilon^r.$$

Since $P_\varepsilon^*\Pi_\varepsilon P_\varepsilon - P^*\Pi P$ is self-adjoint, we obtain (5.42).

Step 3. We now assume that $r = s$. We proceed as in Step 2 with $p = 1$. We already know that

$$\|\widetilde{z}\|_{L^\infty(H)} + \|\widehat{z}_\varepsilon\|_{L^\infty(H)} + \|\widetilde{z}_\varepsilon\|_{L^\infty(H)} + \|\widehat{z}\|_{L^\infty(H)} \le C.$$

The estimates of $\widetilde{z} - \widehat{z}_\varepsilon$ and $\widetilde{z}_\varepsilon - \widehat{z}$ in $L^1(H)$ can be obtained as we did to prove (2.16), by using (5.21) and (5.41) over the time interval (ε, ∞), and by using the bounds in $L^\infty(H)$ of $\widetilde{z}$, $\widehat{z}_\varepsilon$, $\widetilde{z}_\varepsilon$, and $\widehat{z}$ over the time interval $(0, \varepsilon)$. ☐

5.6. Rate of convergence of the feedback gains

Throughout this section ε_0 is the parameter introduced at the end of Section 5.4.

Proposition 5.11. *Let $(\widehat{z},\widehat{u})$ be the solution of $(\mathcal{P})$ and $(\widehat{z}_\varepsilon,\widehat{u}_\varepsilon)$ be the solution of $(\mathcal{P}_\varepsilon)$. For all $\varepsilon\in(0,\varepsilon_0)$, we have*

$$\begin{aligned}&\|\mathrm{e}^{-\lambda_0(\cdot)}(\widehat{z}-\widehat{z}_\varepsilon)\|_{L^{s/r}(H)}\le C\varepsilon^r\ \|z_0\|_H, \quad \text{if } r<s,\\ &\|\mathrm{e}^{-\lambda_0(\cdot)}(\widehat{z}-\widehat{z}_\varepsilon)\|_{L^{1}(H)}\le C\varepsilon^r\,|\ln\,\varepsilon|\,\|z_0\|_H, \qquad \text{if } r=s.\end{aligned} \tag{5.45}$$

Proof. Step 1. Let us recall that $\widehat{A}_\Pi \overset{\text{def}}{=} A_\Pi-\lambda_0 I$, $\widehat{A}_{\varepsilon,\Pi_\varepsilon}\overset{\text{def}}{=} A_{\varepsilon,\Pi_\varepsilon}-\lambda_0 I$, and let us set

$$\mathcal{T}(\cdot)\overset{\text{def}}{=}\mathcal{K}\mathcal{K}^*[(\mathcal{C}^*\mathcal{C}+2\lambda_0P^*\Pi P)(\cdot)]\in\mathcal{L}(L^2(H))$$

and

$$\mathcal{T}_\varepsilon(\cdot)\overset{\text{def}}{=}\mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*[(\mathcal{C}^*\mathcal{C}+2\lambda_0P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)(\cdot)]\in\mathcal{L}(L^2(H)).$$

From Theorem 5.6 and Remark 5.7, we know that both $I+\mathcal{T}$ and $I+\mathcal{T}_\varepsilon$ are automorphisms in $L^2(H)$. Starting from (5.11) and (5.26), we make the following calculations

$$\begin{aligned}\mathrm{e}^{\widehat{A}_\Pi(\cdot)}P-\mathrm{e}^{\widehat{A}_{\varepsilon,\Pi_\varepsilon}(\cdot)}P_\varepsilon &= (I+\mathcal{T})^{-1}(\mathrm{e}^{\widehat{A}(\cdot)}P)-(I+\mathcal{T}_\varepsilon)^{-1}(\mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon)\\ &=((I+\mathcal{T})^{-1}-(I+\mathcal{T}_\varepsilon)^{-1})(\mathrm{e}^{\widehat{A}(\cdot)}P)+(I+\mathcal{T}_\varepsilon)^{-1}(\mathrm{e}^{\widehat{A}(\cdot)}P-\mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon)\\ &=(I+\mathcal{T}_\varepsilon)^{-1}(\mathcal{T}_\varepsilon-\mathcal{T})(I+\mathcal{T})^{-1}(\mathrm{e}^{\widehat{A}(\cdot)}P)+(I+\mathcal{T}_\varepsilon)^{-1}(\mathrm{e}^{\widehat{A}(\cdot)}P-\mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon)\\ &=(I+\mathcal{T}_\varepsilon)^{-1}(\mathcal{T}_\varepsilon-\mathcal{T})(\mathrm{e}^{\widehat{A}_\Pi(\cdot)}P)+(I+\mathcal{T}_\varepsilon)^{-1}(\mathrm{e}^{\widehat{A}(\cdot)}P-\mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon).\end{aligned}$$

Thus, by writing

$$\begin{aligned}\mathcal{T}_\varepsilon(\cdot)-\mathcal{T}(\cdot)=(\mathcal{K}_\varepsilon-\mathcal{K})\mathcal{K}^*((\mathcal{C}^*\mathcal{C}+2\lambda_0P^*\Pi P)(\cdot))+\mathcal{K}_\varepsilon(\mathcal{K}_\varepsilon^*-\mathcal{K}^*)((\mathcal{C}^*\mathcal{C}+2\lambda_0P^*\Pi P)(\cdot))\\ +2\lambda_0\mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*((P_\varepsilon^*\Pi_\varepsilon P_\varepsilon-P^*\Pi P)(\cdot)),\end{aligned}$$

with $\mathcal{K}^*[(\mathcal{C}^*\mathcal{C}+2\lambda_0P^*\Pi P)\mathrm{e}^{\widehat{A}_\Pi(\cdot)}P]=B^*\Pi\mathrm{e}^{\widehat{A}_\Pi(\cdot)}P$, we obtain

$$\mathrm{e}^{\widehat{A}_\Pi(\cdot)}P-\mathrm{e}^{\widehat{A}_{\varepsilon,\Pi_\varepsilon}(\cdot)}P_\varepsilon=(I+\mathcal{T}_\varepsilon)^{-1}((a)+(b)+(c)+(d)), \tag{5.46}$$

where

$$\begin{aligned}&(a)=(\mathcal{K}_\varepsilon-\mathcal{K})(B^*\Pi\mathrm{e}^{\widehat{A}_\Pi(\cdot)}P),\\ &(b)=\mathcal{K}_\varepsilon(\mathcal{K}_\varepsilon^*-\mathcal{K}^*)(\mathcal{C}^*\mathcal{C}+2\lambda_0P^*\Pi P)(\mathrm{e}^{\widehat{A}_\Pi(\cdot)}P),\\ &(c)=2\lambda_0\mathcal{K}_\varepsilon\mathcal{K}_\varepsilon^*(P_\varepsilon^*\Pi_\varepsilon P_\varepsilon-P^*\Pi P)(\mathrm{e}^{\widehat{A}_\Pi(\cdot)}P),\\ &(d)=\mathrm{e}^{\widehat{A}(\cdot)}P-\mathrm{e}^{\widehat{A}_\varepsilon(\cdot)}P_\varepsilon.\end{aligned}$$

Step 2. We first assume that $r/s\in(0,1)$. In that case, (5.45) follows from (5.46) with (4.45), (5.31), (5.42), and (2.16) with $1<p=s/r$. In that case, (5.45) follows from (5.46) with (4.45), (5.31), (5.42), and (2.16) with $1<p=s/r$.

Step 3. Now, we assume that $r = s$. In that case, (5.45) follows from (5.46) with (4.44), (5.31), (5.43), and (2.16) for $p = 1$. □

Proposition 5.12. *Let* Π *be the solution of* (5.5) *and* Π_ε *be the solution of* (5.23)*. We have*

$$\|B^*\Pi P - B^*_\varepsilon \Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H,U)} \leq C\varepsilon^r |\ln \varepsilon|, \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{5.47}$$

Proof. We have

$$\begin{aligned} B^*\Pi P z_0 - B^*_\varepsilon \Pi_\varepsilon P_\varepsilon z_0 = & \left[(\mathcal{K}^* - \mathcal{K}^*_\varepsilon)((\mathcal{C}^*\mathcal{C} + 2\lambda_0 P^*\Pi P)\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z})\right](0) \\ & + 2\lambda_0 \left[\mathcal{K}^*_\varepsilon \left((P^*\Pi P - P^*_\varepsilon \Pi_\varepsilon P_\varepsilon)\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z}\right)\right](0) \\ & + \left[\mathcal{K}^*_\varepsilon \left((\mathcal{C}^*\mathcal{C} + 2\lambda_0 P^*_\varepsilon \Pi_\varepsilon P_\varepsilon)(\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z} - \mathrm{e}^{-\lambda_0(\cdot)}\widehat{z}_\varepsilon)\right)\right](0). \end{aligned} \tag{5.48}$$

Step 1. From (4.44), it follows that

$$\|\mathcal{K}^* - \mathcal{K}^*_\varepsilon\|_{\mathcal{L}(L^\infty(H),L^\infty(U))} \leq C\,\varepsilon^r |\ln\,\varepsilon|.$$

Thus, with (5.7), we have

$$\begin{aligned} \left\|\left[(\mathcal{K}^* - \mathcal{K}^*_\varepsilon)((\mathcal{C}^*\mathcal{C} + 2\lambda_0 P^*\Pi P)\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z})\right](0)\right\|_U & \leq \|\mathcal{K}^* - \mathcal{K}^*_\varepsilon\|_{\mathcal{L}(L^\infty(H),L^\infty(U))} \left\|(\mathcal{C}^*\mathcal{C} + 2\lambda_0 P^*\Pi P)\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z}\right\|_{L^\infty(H)} \\ & \leq C\,\varepsilon^r |\ln\,\varepsilon|\,\|z_0\|_H. \end{aligned}$$

Step 2. With (5.7), (5.31), and (5.42) (or (5.43) if $r = s$), we have

$$\begin{aligned} \|2\lambda_0 \left[\mathcal{K}^*_\varepsilon \left((P^*\Pi P - P^*_\varepsilon \Pi_\varepsilon P_\varepsilon)\widehat{z}\right)\right](0)\|_U & \leq C\|\mathcal{K}^*_\varepsilon\|_{\mathcal{L}(L^\infty(H),L^\infty(U))}\|P^*\Pi P - P^*_\varepsilon \Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)}\,\|\widehat{z}\|_{L^\infty(H)} \\ & \leq C\,\varepsilon^r\,|\ln\,\varepsilon|\,\|z_0\|_H. \end{aligned}$$

Step 3. From the definition of $\mathcal{K}^*_\varepsilon$, it follows that

$$\begin{aligned} & \left[\mathcal{K}^*_\varepsilon \left((\mathcal{C}^*\mathcal{C} + 2\lambda_0 P^*_\varepsilon \Pi_\varepsilon P_\varepsilon)(\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z} - \mathrm{e}^{-\lambda_0(\cdot)}\widehat{z}_\varepsilon)\right)\right](0) \\ & = \int_0^\infty B^*_\varepsilon \mathrm{e}^{\widehat{A}^*_\varepsilon \tau} P^*_\varepsilon (\mathcal{C}^*\mathcal{C} + 2\lambda_0 P^*_\varepsilon \Pi_\varepsilon P_\varepsilon)(\mathrm{e}^{-\lambda_0(\tau)}\widehat{z}(\tau) - \mathrm{e}^{-\lambda_0(\tau)}\widehat{z}_\varepsilon(\tau))\mathrm{d}\tau = \int_0^{\overline{\varepsilon}^\alpha} + \int_{\overline{\varepsilon}^\alpha}^{\overline{\varepsilon}} + \int_{\overline{\varepsilon}}^1 + \int_1^\infty, \quad \text{with } \overline{\varepsilon} = \varepsilon^{r/(1-\gamma)}. \end{aligned}$$

We are going to use (H_5). If we choose $\alpha = \frac{1-\gamma}{1-\overline{\gamma}} \geq 1$, where $\overline{\gamma} \in [\gamma, 1)$ is the exponent appearing in (2.8), it follows that

$$\begin{aligned} & \int_0^{\overline{\varepsilon}^\alpha} \left\|B^*_\varepsilon \mathrm{e}^{\widehat{A}^*_\varepsilon \tau} P^*_\varepsilon (\mathcal{C}^*\mathcal{C} + 2\lambda_0 P^*_\varepsilon \Pi_\varepsilon P_\varepsilon)(\mathrm{e}^{-\lambda_0(\tau)}\widehat{z}(\tau) - \mathrm{e}^{-\lambda_0(\tau)}\widehat{z}_\varepsilon(\tau))\right\|_U \mathrm{d}\tau \\ & \qquad \leq C \int_0^{\overline{\varepsilon}^\alpha} \frac{1}{\tau^{\overline{\gamma}}}\,\mathrm{d}\tau \times \left(\|\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z}\|_{L^\infty(H)} + \|\mathrm{e}^{-\lambda_0(\cdot)}\widehat{z}_\varepsilon\|_{L^\infty(H)}\right) \leq C\,\overline{\varepsilon}^{\alpha(1-\overline{\gamma})}\,\|z_0\|_H \leq C\,\varepsilon^r\,\|z_0\|_H. \end{aligned}$$

For the second term, with (4.46), we have

$$\int_{\overline{\varepsilon}^\alpha}^{\overline{\varepsilon}} \left\| B_\varepsilon^* e^{\widehat{A}_\varepsilon^* \tau} P_\varepsilon^* (\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^* \Pi_\varepsilon P_\varepsilon)(e^{-\lambda_0(\tau)} \widehat{z}(\tau) - e^{-\lambda_0(\tau)} \widehat{z}_\varepsilon(\tau)) \right\|_U d\tau$$
$$\leq C\, \varepsilon^r \int_{\overline{\varepsilon}^\alpha}^{\overline{\varepsilon}} \frac{1}{\tau} \left\| e^{-\lambda_0(\tau)} \widehat{z}(\tau) - e^{-\lambda_0(\tau)} \widehat{z}_\varepsilon(\tau) \right\|_H d\tau \leq C\, \varepsilon^r \,|\ln \varepsilon| \| e^{-\lambda_0(\cdot)} \widehat{z}(\cdot) - e^{-\lambda_0(\cdot)} \widehat{z}_\varepsilon(\cdot) \|_{L^\infty(H)}$$
$$\leq C\, \varepsilon^r \,|\ln\, \varepsilon| \|z_0\|_H.$$

For the third term, with (4.47) and (5.45), we obtain

$$\int_{\overline{\varepsilon}}^{1} \left\| B_\varepsilon^* e^{\widehat{A}_\varepsilon^* \tau} P_\varepsilon^* (\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^* \Pi_\varepsilon P_\varepsilon)(e^{-\lambda_0(\tau)} \widehat{z}(\tau) - e^{-\lambda_0(\tau)} \widehat{z}_\varepsilon(\tau)) \right\|_U d\tau$$
$$\leq C \int_{\overline{\varepsilon}}^{1} \frac{1}{\tau^\gamma} \left\| e^{-\lambda_0(\tau)} \widehat{z}(\tau) - e^{-\lambda_0(\tau)} \widehat{z}_\varepsilon(\tau) \right\|_H d\tau$$
$$\leq C \begin{cases} |\ln(\varepsilon)|^\gamma \, \| e^{-\lambda_0(\cdot)} \widehat{z}(\cdot) - e^{-\lambda_0(\cdot)} \widehat{z}_\varepsilon(\cdot) \|_{L^{s/r}(H)} & \text{if } r = s(1-\gamma), \\ \| e^{-\lambda_0(\cdot)} \widehat{z}(\cdot) - e^{-\lambda_0(\cdot)} \widehat{z}_\varepsilon(\cdot) \|_{L^{s/r}(H)} & \text{if } r < s(1-\gamma), \end{cases}$$
$$\leq C \begin{cases} \varepsilon^r \,|\ln(\varepsilon)| \|z_0\|_H & \text{if } r = s(1-\gamma) \text{ and } \gamma = 0, \\ \varepsilon^r \,|\ln(\varepsilon)|^\gamma \|z_0\|_H & \text{if } r = s(1-\gamma) \text{ and } \gamma > 0, \\ \varepsilon^r \, \|z_0\|_H & \text{if } r < s(1-\gamma). \end{cases}$$

The fourth term can be estimated with (5.45) and Hölder's inequality as follows

$$\int_{1}^{\infty} \left\| B_\varepsilon^* e^{\widehat{A}_\varepsilon^* \tau} P_\varepsilon^* (\mathcal{C}^*\mathcal{C} + 2\lambda_0 P_\varepsilon^* \Pi_\varepsilon P_\varepsilon)(e^{-\lambda_0(\tau)} \widehat{z}(\tau) - e^{-\lambda_0(\tau)} \widehat{z}_\varepsilon(\tau)) \right\|_U d\tau$$
$$\leq \int_{1}^{\infty} C e^{(\omega_0 - \lambda_0)\tau} \left\| e^{-\lambda_0(\tau)} \widehat{z}(\tau) - e^{-\lambda_0(\tau)} \widehat{z}_\varepsilon(\tau) \right\|_H d\tau$$
$$\leq C \left\| e^{-\lambda_0(\cdot)} \widehat{z} - e^{-\lambda_0(\cdot)} \widehat{z}_\varepsilon \right\|_{L^{s/r}(H)} \leq C \begin{cases} \varepsilon^r \, \|z_0\|_H & \text{if } s < r \\ \varepsilon^r \,|\ln(\varepsilon)| \|z_0\|_H & \text{if } s = r. \end{cases}$$

The proof is complete. □

5.7. Final results and final remarks

We can summarize the results of Section 5.6 in the following theorem.

Theorem 5.13. *We assume that the pair (A, B) is feedback stabilizable, and that the pair $(A, \mathcal{C})$ satisfies the condition (5.2).*

Let Π be the solution of (5.5) and Π_ε be the solution of (5.23).

There exist $\varrho > 0$ and $\varepsilon_0 \in (0, 1)$ such that

$$\| e^{(A_\varepsilon - B_\varepsilon B_\varepsilon^* \Pi_\varepsilon)t} \|_{\mathcal{L}(Z_\varepsilon)} \leq C e^{(-\omega_\Pi + \varrho \varepsilon^r |\ln \varepsilon|)t},$$
$$\| e^{(A_\varepsilon - B_\varepsilon B_\varepsilon^* \Pi_\varepsilon)t} P_\varepsilon - e^{(A - BB^*\Pi)t} P \|_{\mathcal{L}(H)} \leq C \frac{e^{(-\omega_\Pi + \varrho \varepsilon^r |\ln \varepsilon|)t}}{t^{r/s}} \varepsilon^r |\ln \varepsilon|,$$

for all $\varepsilon \in (0, \varepsilon_0)$ and all $t > 0$. (In the above estimates, ω_Π is the exponential decay introduced in (5.7)).

Proof. Since (5.47) is satisfied, we choose $F_\varepsilon = -B_\varepsilon^* \Pi_\varepsilon P_\varepsilon$, and the theorem is a direct consequence of Theorem 4.4-(i) (see Rem. 4.5) with $\sigma(\varepsilon) = C\,\varepsilon^r |\ln \varepsilon|$. □

Let us end this section by comparing our method of proof to establish the results of that section with similar ones proved in the case of conforming approximations.

Remark 5.14. The main difference between [11], Section 4.5 and our results in the present section is in the estimate of $\|\widehat{z}(t) - \widehat{z}_\varepsilon(t)\|_H$, where $\widehat{z}(t) = \mathrm{e}^{t(A+BK)} P z_0$ is the solution of the initial closed-loop system, and $\widehat{z}_\varepsilon(t) = \mathrm{e}^{t(A_\varepsilon + B_\varepsilon K_\varepsilon)} P_\varepsilon z_0$ is the solution of the approximate closed-loop system.

In [11], Section 4.5, the proof of the estimate of $\|\widehat{z}(t) - \widehat{z}_\varepsilon(t)\|_H$ is based on the estimate [11], (4.5.1.20), p. 474, and on the estimate of its right hand side, $\|(\lambda_0 I - A - BK)^{-1}P - (\lambda_0 I - A_\varepsilon - B_\varepsilon K_\varepsilon)^{-1} P_\varepsilon\|_{\mathcal{L}(H)}$ (with our notation), which is, according to [11], p. 475, estimated with [11], (4.5.1.6), p. 472. With our notation [11], (4.5.1.6), p. 472 corresponds to an estimate of $\|(\lambda_0 I - A - BK)^{-1}P - (\lambda_0 I - A_\varepsilon - B_\varepsilon KP)^{-1} P_\varepsilon\|_{\mathcal{L}(H)}$. Actually, using the triangle inequality

$$\begin{aligned}
&\|(\lambda_0 I - A - BK)^{-1}P - (\lambda_0 I - A_\varepsilon - B_\varepsilon K_\varepsilon)^{-1} P_\varepsilon\|_{\mathcal{L}(H)} \\
&\qquad \le \|(\lambda_0 I - A - BK)^{-1}P - (\lambda_0 I - A_\varepsilon - B_\varepsilon KP)^{-1} P_\varepsilon\|_{\mathcal{L}(H)} \\
&\qquad\qquad + \|(\lambda_0 I - A_\varepsilon - B_\varepsilon KP)^{-1}P - (\lambda_0 I - A_\varepsilon - B_\varepsilon K_\varepsilon)^{-1} P_\varepsilon\|_{\mathcal{L}(H)},
\end{aligned}$$

an estimate of $\|(\lambda_0 I - A_\varepsilon - B_\varepsilon KP)^{-1}P - (\lambda_0 I - A_\varepsilon - B_\varepsilon K_\varepsilon)^{-1} P_\varepsilon\|_{\mathcal{L}(H)}$, uniform in ε, would be needed. But this estimate is not given in [11], Section 4.5. This is why this point in the proof of [11], Section 4.5 is not clear for us. In our proof, we first establish convergence rates for the feedback laws in Proposition 5.12. The convergence rate for $\|\widehat{z}(t) - \widehat{z}_\varepsilon(t)\|_H$ is obtained in Theorem 5.13 as a consequence of Proposition 5.12 and of Theorem 4.4. In [11], Section 4.5, the convergence rate of feedback laws is obtained by using the convergence rate for $\|\widehat{z}(t) - \widehat{z}_\varepsilon(t)\|_H$. Since the estimate of $\|(\lambda_0 I - A_\varepsilon - B_\varepsilon KP)^{-1}P - (\lambda_0 I - A_\varepsilon - B_\varepsilon K_\varepsilon)^{-1} P_\varepsilon\|_{\mathcal{L}(H)}$ is not proved in [11], Section 4.5, we think that some arguments are missing there.

Data availability statement

No new data/codes were created or analyzed in this study.

References

[1] M. Badra and J.-P. Raymond, Approximation of feedback gains for the Oseen system. *Preprint* (2025). https://hal.science/hal-04880955.

[2] M. Badra and J.-P. Raymond, Numerical approximation of the Oseen system in polyhedral or polygonal domains. *Preprint* (2025). https://hal.science/hal-04880966.

[3] M. Badra and J.-P. Raymond, The Oseen system with nonhomogeneous boundary conditions in polyhedral or polygonal domains. *Preprint* (2025). https://hal.science/hal-04880963.

[4] M. Badra and J.-P. Raymond, Feedback stabilization of the Oseen system approximated by the pseudo-compressibility method, special issue of Mathematical Reports dedicated to Marius Tucsnak, Vol. 24 (2022) 39–57.

[5] M. Badra and J.-P. Raymond, Approximation of feedback gains using spectral projections: application to the Oseen system. *SIAM J. Control Optim.* **62** (2024) 2910–2935.

[6] H.T. Banks and K. Kunisch, The linear regulator problem for parabolic systems. *SIAM J. Control Optim.* **22** (1984) 684–698.

[7] M. Kroller and K. Kunisch, Convergence rates for the feedback operators arising in the linear quadratic regulator problem governed by parabolic equations. *SIAM J. Numer. Anal.* **28** (1991) 1350–1385.

[8] I. Lasiecka, Convergence rates for the approximations of the solutions to algebraic Riccati equations with unbounded coefficients: case of analytic semigroups. *Numer. Math.* **63** (1992) 357–390.

[9] I. Lasiecka and R. Triggiani, The regulator problem for parabolic equations with Dirichlet boundary control. II. Galerkin approximation. *Appl. Math. Optim.* **16** (1987) 187–216.

[10] I. Lasiecka and R. Triggiani, Numerical approximations of algebraic Riccati equations for abstract systems modelled by analytic semigroups, and applications. *Math. Comp.* **57** (1991) 639–662, S13–S37.

[11] I. Lasiecka and R. Triggiani, Control theory for partial differential equations: continuous and approximation theories. I. Abstract parabolic systems, Vol. 74 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (2000).

[12] H.T. Banks and K. Ito, Approximation in LQR problems for infinite-dimensional systems with unbounded input operators. *J. Math. Syst. Estim. Control* **7** (1997) 1–34.

[13] T. Kato, Perturbation Theory for Linear Operators, Classics in Mathematics. Springer-Verlag, Berlin (1995).

[14] J.S. Gibson, An analysis of optimal modal regulation: convergence and stability. *SIAM J. Control Optim.* **19** (1981) 686–707.

[15] J.S. Gibson and A. Adamian, Approximation Theory for Linear-Quadratic-Gaussian Optimal Control of Flexible Structures. *SIAM J. Control Optim.* **29** (1991) 1–37.

[16] F. Kappel and D. Salamon, An approximation theorem for the algebraic Riccati equation. *SIAM J. Control Optim.* **28** (1990) 1136–1147.

[17] K. Ramdani, T. Takahashi and M. Tucsnak, Uniformly exponentially stable approximations for a class of second order evolution equations—application to LQR problems. *ESAIM Control Optim. Calc. Var.* **13** (2007) 503–527.

[18] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, 2nd edn. Johann Ambrosius Barth, Heidelberg (1995).

[19] A. Bensoussan, G. Da Prato, M.C. Delfour and S.K. Mitter, Representation and control of infinite dimensional systems, Systems & Control: Foundations & Applications, 2nd edn. Birkhäuser Boston Inc., Boston, MA (2007).

[20] A. Pazy, Semigroups of linear operators and applications to partial differential equations. Vol. 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York (1983).

[21] K. Ito and F. Kappel, The Trotter-Kato theorem and approximation of PDEs. *Math. Comp.* **67** (1998) 21–44.

[22] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext. Springer, New York (2011).

[23] J. Bergh and J. Löfström, Interpolation Spaces. An Introduction. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin-New York (1976).

[24] J. Zabczyk, Mathematical Control Theory. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA (2008).

[25] R. Triggiani, A sharp result on the exponential operator-norm decay of a family $T_h(t)$ of strongly continuous semigroups uniformly in h, in Optimal control of differential equations (Athens, OH, 1993). Vol. 160 of *Lecture Notes in Pure and Applied Mathematics*. Dekker, New York (1994) 325–335.

Appendix A

The goal of this section is to prove the resolvent identity stated in the following lemma.

Lemma A.1. *For all* $\lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}$ *and* $\lambda_0 > \omega_0$*, we have*

$$(\lambda I - A_\varepsilon)^{-1} P_\varepsilon - (\lambda I - A)^{-1} P = (I - (\lambda - \lambda_0)(\lambda I - A)^{-1} P)(\widehat{A}^{-1} P - \widehat{A}_\varepsilon^{-1} P_\varepsilon) \left(I - (\lambda - \lambda_0)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon\right),$$

and

$$((\lambda I - A)^{-1} - (\lambda I - A_\varepsilon)^{-1} P_\varepsilon) P = (I - (\lambda - \lambda_0)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon)(\widehat{A}_\varepsilon^{-1} P_\varepsilon - \widehat{A}^{-1} P)(-\widehat{A})(\lambda I - A)^{-1} P.$$

Proof. We have

$$I - (\lambda - \lambda_0)(\lambda I - A)^{-1} P = I - P - \widehat{A}(\lambda I - A)^{-1} P$$

and

$$I - (\lambda - \lambda_0)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon = I - P_\varepsilon - \widehat{A}_\varepsilon(\lambda I - A_\varepsilon)^{-1} P_\varepsilon.$$

Next, we write

$$\begin{aligned}
\Delta &= \Big(I - P - \widehat{A}(\lambda I - A)^{-1} P\Big)\Big(\widehat{A}^{-1} P - \widehat{A}_\varepsilon^{-1} P_\varepsilon\Big)\Big(I - P_\varepsilon - \widehat{A}_\varepsilon(\lambda I - A_\varepsilon)^{-1} P_\varepsilon\Big) \\
&= (I - P)\Big(\widehat{A}^{-1} P - \widehat{A}_\varepsilon^{-1} P_\varepsilon\Big)\Big(I - P_\varepsilon - \widehat{A}_\varepsilon(\lambda I - A_\varepsilon)^{-1} P_\varepsilon\Big) \\
&\quad -\widehat{A}(\lambda I - A)^{-1} P\Big(\widehat{A}^{-1} P - \widehat{A}_\varepsilon^{-1} P_\varepsilon\Big)\Big(I - P_\varepsilon - \widehat{A}_\varepsilon(\lambda I - A_\varepsilon)^{-1} P_\varepsilon\Big) \\
&= -(I - P)\,\widehat{A}_\varepsilon^{-1} P_\varepsilon\Big(I - P_\varepsilon - \widehat{A}_\varepsilon(\lambda I - A_\varepsilon)^{-1} P_\varepsilon\Big) \\
&\quad -(\lambda I - A)^{-1} P\Big(P - \widehat{A} P \widehat{A}_\varepsilon^{-1} P_\varepsilon\Big)(I - P_\varepsilon) \\
&\quad +(\lambda I - A)^{-1} P\Big(P - \widehat{A} P \widehat{A}_\varepsilon^{-1} P_\varepsilon\Big)\widehat{A}_\varepsilon(\lambda I - A_\varepsilon)^{-1} P_\varepsilon \\
&= (I - P)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon - (\lambda I - A)^{-1} P (I - P_\varepsilon) \\
&\quad +(\lambda I - A)^{-1} P\Big(P - \widehat{A} P \widehat{A}_\varepsilon^{-1} P_\varepsilon\Big)\widehat{A}_\varepsilon(\lambda I - A_\varepsilon)^{-1} P_\varepsilon \\
&= (I - P)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon - (\lambda I - A)^{-1} P (I - P_\varepsilon) \\
&\quad +(\lambda I - A)^{-1} P \widehat{A}_\varepsilon(\lambda I - A_\varepsilon)^{-1} P_\varepsilon - (\lambda I - A)^{-1} \widehat{A} P (\lambda I - A_\varepsilon)^{-1} P_\varepsilon
\end{aligned}$$

We have

$$\widehat{A}_\varepsilon(\lambda I - A_\varepsilon)^{-1} P_\varepsilon = -P_\varepsilon + (\lambda - \lambda_0)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon$$

and

$$(\lambda I - A)^{-1} \widehat{A} P = -P + (\lambda - \lambda_0)(\lambda I - A)^{-1} P.$$

Thus, we finally obtain

$$\begin{aligned}
\Delta &= (I - P)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon - (\lambda I - A)^{-1} P (I - P_\varepsilon) \\
&\quad +(\lambda I - A)^{-1} P\big(-P_\varepsilon + (\lambda - \lambda_0)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon\big) \\
&\quad +\big(P - (\lambda - \lambda_0)(\lambda I - A)^{-1} P\big)(\lambda I - A_\varepsilon)^{-1} P_\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&= (I-P)\,(\lambda I - A_\varepsilon)^{-1}\,P_\varepsilon - (\lambda I - A)^{-1}\,P\,(I-P_\varepsilon) \\
&\quad -(\lambda I - A)^{-1}\,PP_\varepsilon + (\lambda I - A)^{-1}\,P\,\big((\lambda-\lambda_0)(\lambda I - A_\varepsilon)^{-1}\,P_\varepsilon\big) \\
&\quad +P(\lambda I - A_\varepsilon)^{-1}\,P_\varepsilon - (\lambda-\lambda_0)(\lambda I - A)^{-1}P(\lambda I - A_\varepsilon)^{-1}\,P_\varepsilon \\
&= (I-P)\,(\lambda I - A_\varepsilon)^{-1}\,P_\varepsilon - (\lambda I - A)^{-1}\,P\,(I-P_\varepsilon) \\
&\quad -(\lambda I - A)^{-1}\,PP_\varepsilon + P(\lambda I - A_\varepsilon)^{-1}\,P_\varepsilon \\
&= (\lambda I - A_\varepsilon)^{-1}\,P_\varepsilon - (\lambda I - A)^{-1}\,P.
\end{aligned}
$$

The proof of the first identity is complete.

By reversing the role of A and A_ε we also obtain

$$
(\lambda I - A)^{-1}P - (\lambda I - A_\varepsilon)^{-1}P_\varepsilon = (I - (\lambda-\lambda_0)(\lambda I - A_\varepsilon)^{-1}P_\varepsilon)(\widehat{A}_\varepsilon^{-1}P_\varepsilon - \widehat{A}^{-1}P)\left(I - (\lambda-\lambda_0)(\lambda I - A)^{-1}\,P\right).
$$

The second equality is a direct consequence of the above equality. □