

## A RATE-INDEPENDENT MODEL FOR GEOMATERIALS UNDER COMPRESSION COUPLING STRAIN GRADIENT PLASTICITY AND DAMAGE

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**Abstract.** We study a strain gradient-enhanced version of a model for geomaterials under compression by Marigo and Kazymyrenko (2019) coupling damage and small-strain associative plasticity. We prove that the jumps in time of the plastic variable may happen only along jumps of the damage variable. Moreover, we perform a vanishing-viscosity analysis showing existence of Balanced Viscosity quasistatic solutions *à la* Mielke–Rossi–Savaré.

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### 1. INTRODUCTION

In this paper we study a strain gradient-enhanced version of a model for geomaterials under compression by Marigo and Kazymyrenko [1] coupling damage and small-strain associative plasticity, proving a better regularity in time for the corresponding evolutions.

The model of [1] is based on a micromechanical analysis, accounting for microcracks inside the material as the main responsible of the progressive loss of rigidity and then for the macroscopic elasto-plastic coupled behavior.

Given a *reference domain*  $\Omega \subset \mathbb{R}^n$  and denoting by  $u: \Omega \rightarrow \mathbb{R}^n$  the *displacement field*, in the general framework of small-strain elasto-plasticity the total strain  $Eu = (\nabla u)^{\text{sym}} \in \mathbb{M}_{\text{sym}}^{n \times n}$  is additively decomposed into *elastic strain* and *plastic strain*:

$$Eu = e + p$$

and the *stress tensor*  $\sigma: \Omega \rightarrow \mathbb{M}_{\text{sym}}^{n \times n}$  depends linearly on  $e$  according to Hooke's law

$$\sigma = \mathbb{C}e,$$

where  $\mathbb{C}$  is the fourth order positive definite *Hooke's tensor*, with  $\sigma$  lying in a fixed closed and convex set  $K \subset \mathbb{M}_{\text{sym}}^{n \times n}$  (the *constraint set*) and satisfying equilibrium conditions involving the external loads.

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In order to describe the model in [1], consider the reference case of a triaxial test - a shear compression plus a compression normal to the shear plane - and the observed phenomenon of *dilatance*, *i.e.*, the irreversible increase of volume in compression. A micromechanical justification is that in the shear compression the volume may increase with that of the internal voids between the granuli, interpreted as microcracks, due to a less efficient particle organization; conversely, in the normal compression a part of these microcracks may be closed, preventing a free sliding and thus the free relaxation to the initial configuration in unloading.

Then a *damage variable*  $\alpha: [0, T] \times \Omega \rightarrow [0, 1]$  is considered in [1], related to the density of closed microcracks. Since the presence of such microcracks, that is of the damage, prevents the release of energy, the variable  $\alpha$  is related to irreversible deformations and thus appears into an internal blocked energy depending on plastic strain. Assuming Coulomb law for the sliding with friction between the lips of closed microcracks, plasticity follows a standard Drucker-Prager criterion with an associative flow rule, that is the plastic flow is normal to the constraint set  $K$ . In particular, the coupling between damage and plasticity only occurs through an hardening term. In the original model, see in particular [1], Section 3.2, this is justified by a micromechanical approach showing that, for a microcracked material with contact and friction between the lips of the cracks, only the hardening tensor depends on the crack state, while the elasticity tensor represents the stiffness of the sound material.

Moreover, damage is assumed to be an irreversible process (consequently  $\alpha$  is non-increasing in time) and the growth of the microcracks is modelled by the terms usually present in gradient damage models

$$D(\alpha(t)) + \ell_\alpha \|\nabla \alpha(t)\|_2^2, \quad D(\alpha) = \int_\Omega d(\alpha(x)) \, dx,$$

$\ell_\alpha > 0$ , *cf.* [2, 3], Section 4, [1], Remark 3.1.

In [4] the existence of quasistatic evolutions for such a model is proven, fulfilling the notion of *energetic solutions* by Mielke and Theil [5]: this is based on a *global stability condition*, which prescribes at each time the minimization, among the admissible configurations, of the total internal energy plus the dissipation potential, and on an *energy-dissipation balance* between the total variation in time of the internal energy, the total dissipated energy, and the work of the external loadings.

In [6] the model has been enriched by including, still employing an irreversible degradation function  $\alpha$ , also the tensile response. This accounts for the fact that the evolution of opening microcracks in tension leads to brittle behavior and mode I fracture. Conversely, the response in compression shares many features with [1], but plasticity is required to satisfy a non-associative flow rule.

The present contribution focuses on a different modification of the model in [1], obtained by formally including in the internal energy the  $L^2$  norm of the plastic strain gradient, times a dimensional (length) constant. Terms involving higher order derivatives are considered on the one hand by their capability to capture the correct scale size of shear bands, usually observed in failure in soils induced by shearing, and on the other hand to avoid mesh dependence in numerical simulations. Inspired by the work of Aifantis [7], several plastic strain gradient models have been proposed, for both crystalline and granular materials (*e.g.* [8–15]).

The framework of our strain gradient enhancement is that of the *explicit gradient theories*, in particular the term involving the plastic gradient does not depend on other variables, following the lines of [8, 11, 12].

Therefore, the stored energy at time  $t$  is the sum of the elastic energy of the sound material, of the kinematical hardening term (depending on  $\alpha$ ), and of the strain gradient term

$$\frac{1}{2} \int_\Omega \mathbb{C}e(t, x) : e(t, x) \, dx + \int_\Omega \mathbb{B}(\alpha(t, x))p(t, x) : p(t, x) \, dx + \ell_p \|\nabla p(t)\|_2^2,$$

for  $\mathbb{B}$  a fourth-order tensor positive semidefinite (usually positive definite except for  $\alpha = 0$ ). The gradient term, which enforces the plasticity to be in  $H^1$  (in space), introduces a characteristic internal length related to the

scale of the observed shear bands. In this setting, the plastic dissipation from a plastic strain  $q$  is the functional

$$\int_{\Omega} H(p(t, x) - q(x)) dx, \quad H(\xi) = \sup_{\sigma \in K} \sigma : \xi.$$

We observe that in the original model  $p(t)$  is in general not in  $L^2$  in the set  $\{\alpha = 0\}$ , so shear bands with infinitesimal thickness may therein display. Therefore, in [4] the dissipation from a plastic strain  $q$  is the relaxation of the functional above for  $p(t)$  a bounded Radon measure. Furthermore, we stress again that the damage and plasticity variable are explicitly coupled only through the hardening term, as in the original model.

In a strong formulation, our *energetic solution*  $(\alpha, u, e, p)$  satisfies for  $t \in [0, T]$  and  $x \in \Omega$ :

- (sf0) *irreversibility*:  $\dot{\alpha} \leq 0$ ,
- (sf1) *additive decomposition*:  $Eu = e + p$ ,
- (sf2) *constitutive equation*:  $\sigma = \mathbb{C}e$ ,
- (sf3) *equilibrium*:  $\operatorname{div} \sigma = 0$ ,
- (sf4) *stress constraint*:  $\sigma - 2\mathbb{B}(\alpha)p + 2\ell_p \Delta p \in K$ ,
- (sf5) *associative flow rule*:  $\dot{p} \in N_K(\sigma - 2\mathbb{B}(\alpha)p + 2\ell_p \Delta p)$ , where  $N_K(\xi)$  is the normal cone to the constraint set  $K$  at  $\xi \in K$ ,
- (sf6) *Kuhn-Tucker conditions*:

$$\dot{d}(\alpha) - 2\ell_\alpha \Delta \alpha + \mathbb{B}(\alpha)p : p \leq 0 \quad \text{and} \quad (\dot{d}(\alpha) - 2\ell_\alpha \Delta \alpha + \mathbb{B}(\alpha)p : p)\dot{\alpha} = 0,$$

- (sf7) *prescribed boundary displacement*:  $u = w$  on  $\partial\Omega$ .

In the present setting, the additional regularity of the plastic strain with respect to [4] makes the existence of energetic solutions much easier to prove; the core of the paper is the study of the regularity in time of the evolutions. In fact, with general nonconvex total energies, evolutions may display jumps in time. Basing on standard techniques we show in Lemma 3.9 that energetic solutions, whose existence is proven in Theorem 3.8, are strongly continuous outside an at most countable set of time instants.

Further, we develop a more precise analysis exploiting the particular structure of the model: we prove that the variations in time of the whole set of variables with respect to the target norms are controlled in terms of the variation in time of the damage in the  $L^1(\Omega)$ -norm, plus that of the external loading (*cf.* Thm. 3.15 for the precise statement). This ensures that, as soon as damage does not jump, the whole evolution, and in particular the plastic strain, has no time discontinuities. Our analysis is in the spirit of [16], which proves that small strain perfect plasticity is absolutely continuous in time.

The interplay between damage and plastic strain is a key point in the analysis of coupled damage-plasticity models. We consider, for instance, the uniaxial responses in [3], a reference work for the variational analysis of such coupling, and in [1]: in [3] it is shown that strain concentrates into ‘cohesive crack points’ where the space derivative of the damage is discontinuous; in [1], assuming solutions smooth in time and a linear  $D$ , it is observed that damage evolves only with plasticity.

By inspecting our proof it is apparent that the strain gradient regularization is crucial for the improved regularity. This could suggest, in analogous cases with gradient plasticity, to carefully verify if jumps in time of plasticity but not of damage are analytically ruled out and if this is in accordance with the expected experimental results.

In the final part of the paper we study the existence of quasistatic evolution obtained through a vanishing viscosity approximation, satisfying the notion of Balanced Viscosity (BV) solutions, see Definition 5.1.

BV solutions, which have been described in an ‘abstract’ context in [17] and the subsequent [18, 19], are obtained through the approximation of a rate-independent system by viscously perturbed system and provide a selection criterion for solutions with a mechanically feasible behavior at jumps, as well as a description of trajectories joining states before and after jumps. In fact, the notion of energetic solutions is not satisfactory from the physical point of view, whenever jumps develop in time: these jumps may appear ‘too long and too

early', being forced to overtake energy barriers between energy wells (see the characterization in [20] and the references therein).

The vanishing-viscosity technique has also been adopted in various concrete applications, ranging from plasticity (*cf.*, *e.g.*, [21–26]), to damage, fracture, and fatigue (see for instance [27–33]).

In a strong formulation, for a given viscosity parameter  $\varepsilon > 0$ , our *viscosity approximation*  $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$  satisfies for  $t \in [0, T]$  and  $x \in \Omega$ :

- (sf0) $_\varepsilon$  *irreversibility*:  $\dot{\alpha}_\varepsilon \leq 0$ ,
- (sf1) $_\varepsilon$  *additive decomposition*:  $\mathbb{E}u_\varepsilon = e_\varepsilon + p_\varepsilon$ ,
- (sf2) $_\varepsilon$  *constitutive equation*:  $\sigma_\varepsilon = \mathbb{C}e_\varepsilon$ ,
- (sf3) $_\varepsilon$  *equilibrium*:  $\operatorname{div} \sigma_\varepsilon = 0$ ,
- (sf4) $_\varepsilon$  *stress constraint*:  $\sigma_\varepsilon - 2\mathbb{B}(\alpha_\varepsilon)p_\varepsilon + 2\ell_p\Delta p_\varepsilon \in K$ ,
- (sf5) $_\varepsilon$  *associative flow rule*:  $\dot{p}_\varepsilon \in N_K(\sigma_\varepsilon - 2\mathbb{B}(\alpha_\varepsilon)p_\varepsilon + 2\ell_p\Delta p_\varepsilon)$ ,
- (sf6) $_\varepsilon$  *Kuhn-Tucker conditions*:

$$\dot{d}(\alpha_\varepsilon) - 2\ell_\alpha\Delta\alpha_\varepsilon + \dot{\mathbb{B}}(\alpha_\varepsilon)p_\varepsilon : p_\varepsilon + \varepsilon\dot{\alpha}_\varepsilon \leq 0 \quad \text{and} \quad (\dot{d}(\alpha_\varepsilon) - 2\ell_\alpha\Delta\alpha_\varepsilon + \dot{\mathbb{B}}(\alpha_\varepsilon)p_\varepsilon : p_\varepsilon + \varepsilon\dot{\alpha}_\varepsilon)\dot{\alpha}_\varepsilon = 0,$$

- (sf7) $_\varepsilon$  *prescribed boundary displacement*:  $u_\varepsilon = w$  on  $\partial\Omega$ .

We show existence of BV solutions (*cf.* Thm. 5.4), mostly taking advantage of the analysis carried out for the regularity result. Arguing similarly, we show that as soon as the damage variable is regularized in time with respect to the  $L^2$ -norm (in space), then all the evolution gains the same regularity in time, in the energy space. Moreover, using an argument from [29, 30], we get that the evolution is absolutely continuous in time uniformly respect to the viscosity parameter  $\varepsilon > 0$ . Actually, for fixed  $\varepsilon > 0$  any viscous evolution is  $H^1$  in time, and the  $W^{1,1}$  bound is uniform in  $\varepsilon > 0$ . Nevertheless, as usual we reparameterize in time to obtain a family of 1-Lipschitz evolutions from the time interval to the target spaces of the internal variables. Eventually, such regularity permits to follow the lines of [30] to conclude existence of Balanced Viscosity solutions.

The paper is organised as follows: In Section 2 we fix the notation used throughout the paper and we introduced the mathematical framework of our model. In Section 3 we first prove in Theorem 3.8 the existence of global quasistatic evolutions, and then in Theorem 3.15 we investigate their regularity in time. We continue with Section 4, where we introduced a  $\varepsilon$ -regularization in time with respect to the  $L^2$ -norm in space of the damage variable, and in Theorem 4.14 we prove the existence of  $\varepsilon$ -approximate viscous evolution. Finally, in Section 5 we prove the existence of BV solutions (see Thm. 5.4).

## 2. PRELIMINARIES

### 2.1. Notation

The space of  $n \times n$  symmetric matrices with real entries is denoted by  $\mathbb{M}_{\text{sym}}^{n \times n}$ . Given two matrices  $A_1, A_2 \in \mathbb{M}_{\text{sym}}^{n \times n}$ , their scalar product is denoted by  $A_1 : A_2$  and the induced norm of  $A \in \mathbb{M}_{\text{sym}}^{n \times n}$  by  $|A|$ . The identity matrix is denoted by  $I \in \mathbb{M}_{\text{sym}}^{n \times n}$ .

For a measurable set  $B \subset \mathbb{R}^n$  we use  $\mathcal{L}^n(B)$  to denote the  $n$ -dimensional Lebesgue measure of  $B$ . By  $\chi_B : \mathbb{R}^n \rightarrow \{0, 1\}$  we denote the corresponding characteristic function. Given an open subset  $\Omega$  of  $\mathbb{R}^n$ , the set of all distributions on  $\Omega$ , namely the continuous dual space of  $C_c^\infty(\Omega)$ , endowed with the strong dual topology, is denoted by  $\mathcal{D}'(\Omega)$ . We adopt standard notation for Lebesgue spaces on measurable subsets  $B \subset \mathbb{R}^n$  and Sobolev spaces on open subsets  $\Omega \subset \mathbb{R}^n$ . According to the context, we use  $\|\cdot\|_p$  to denote the norm in  $L^p(B)$  for all  $1 \leq p \leq \infty$ , and  $\|\cdot\|_{H^1}$  to denote the norm  $H^1(\Omega)$ . The boundary values of a Sobolev function are always intended in the sense of traces.

The partial derivatives with respect to the variable  $x_i$  are denoted by  $\partial_i$ . Given an open subset  $\Omega \subset \mathbb{R}^n$  and a function  $u : \Omega \rightarrow \mathbb{R}^d$ , we denote its Jacobian matrix by  $\nabla u$ , whose components are  $(\nabla u)_{ij} := \partial_j u_i$  for  $i = 1, \dots, d$  and  $j = 1, \dots, n$ . For a function  $u : \Omega \rightarrow \mathbb{R}^n$  we use  $\mathbb{E}u := (\nabla u)^{\text{sym}} = \frac{1}{2}(\nabla u + \nabla u^T) \in \mathbb{M}_{\text{sym}}^{n \times n}$  to

denote the symmetric part of the gradient. Given a tensor field  $F: \Omega \rightarrow \mathbb{M}_{\text{sym}}^{n \times n}$ , by  $\text{div } F$  we mean its divergence with respect to lines, namely  $(\text{div } F)_i := \sum_{j=1}^n \partial_j F_{ij}$  for  $i = 1, \dots, n$ .

## 2.2. Mathematical framework

We fix the mathematical framework of the model of complete damage and strain gradient plasticity presented in the introduction. For simplicity, all the physical constants are fixed to 1.

Throughout the paper the *reference configuration*  $\Omega$  is a bounded domain with Lipschitz boundary in  $\mathbb{R}^n$ , where  $n = 2, 3$ . The *displacement field* is represented by a function  $u \in H^1(\Omega; \mathbb{R}^n)$ , and we decompose its *total strain* as  $Eu = e + p$ , where  $e$  denotes the *elastic strain* and  $p$  the *plastic strain*. To simplify the notation, we set

$$\mathcal{A} := H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

Given a function  $w \in H^1(\Omega; \mathbb{R}^n)$ , the set of admissible displacements and strains for the boundary datum  $w$  on  $\partial\Omega$  is defined as

$$\mathcal{A}(w) := \{(u, e, p) \in \mathcal{A} : Eu = e + p \text{ in } \Omega, u = w \text{ on } \partial\Omega\}.$$

**Remark 2.1.** In this model we require a higher regularity in the plastic strain  $p$ , which usually is only a Radon measure. This property is ensured by the addition in the total energy (2.8) of the  $L^2$  norm of the plastic strain gradient term.

We fix  $T > 0$ . Let us consider a time-dependent boundary displacement  $w$  satisfying

$$w \in H^1((0, T); H^1(\Omega; \mathbb{R}^n)). \quad (2.1)$$

The damage variable is a function  $\alpha \in H^1(\Omega; [0, 1])$ . Given  $\alpha^0 \in H^1(\Omega; [0, 1])$  we set

$$\mathcal{D}(\alpha^0) := \{\alpha \in H^1(\Omega) : 0 \leq \alpha \leq \alpha^0\} \subset H^1(\Omega; [0, 1]),$$

so that

$$\mathcal{D}(\alpha^2) \subset \mathcal{D}(\alpha^1) \quad \text{for all } \alpha^2 \in \mathcal{D}(\alpha^1).$$

The *energy dissipated in damage growth* is the functional  $D: H^1(\Omega; [0, 1]) \rightarrow [0, \infty)$  given by

$$D(\alpha) := \int_{\Omega} d(\alpha(x)) \, dx \quad \text{for all } \alpha \in H^1(\Omega; [0, 1]),$$

where  $d: [0, 1] \rightarrow \mathbb{R}$  satisfies

$$d \in C^{1,1}([0, 1]; [0, \infty)), \quad \dot{d}(0) \leq 0 \quad (2.2)$$

(notice that the assumption  $\dot{d}(0) \leq 0$  is compatible with the ones proposed in [1]). The *elastic energy* is the functional  $\mathcal{Q}: L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \rightarrow [0, \infty)$  given by

$$\mathcal{Q}(e) := \frac{1}{2} \int_{\Omega} \mathbb{C}e(x) : e(x) \, dx \quad \text{for all } e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}),$$

where  $\mathbb{C} \in \text{Lin}(\mathbb{M}_{\text{sym}}^{n \times n}; \mathbb{M}_{\text{sym}}^{n \times n})$  satisfies

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl} \quad \text{for all } i, j, k, l = 1, \dots, n, \quad (2.3)$$

$$\gamma_1 |\xi|^2 \leq \mathbb{C}\xi : \xi \leq \gamma_2 |\xi|^2 \quad \text{for all } \xi \in \mathbb{M}_{\text{sym}}^{n \times n}, \quad (2.4)$$

with constants  $\gamma_1, \gamma_2 > 0$ . In particular, it holds

$$|\mathbb{C}\xi| \leq \gamma_2 |\xi| \quad \text{for all } \xi \in \mathbb{M}_{\text{sym}}^{n \times n}.$$

The *kinematical hardening term* is the functional  $\tilde{Q}: H^1(\Omega; [0, 1]) \times H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \rightarrow [0, \infty)$  given by

$$\tilde{Q}(\alpha, p) := \int_{\Omega} \mathbb{B}(\alpha(x))p(x) : p(x) \, dx \quad \text{for all } \alpha \in H^1(\Omega; [0, 1]) \text{ and } p \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}),$$

where  $\mathbb{B}: [0, 1] \rightarrow \text{Lin}(\mathbb{M}_{\text{sym}}^{n \times n}; \mathbb{M}_{\text{sym}}^{n \times n})$  satisfies

$$\mathbb{B} \in C^{1,1}([0, 1]; \text{Lin}(\mathbb{M}_{\text{sym}}^{n \times n}; \mathbb{M}_{\text{sym}}^{n \times n})), \quad \dot{\mathbb{B}}(0)\xi : \xi \leq 0 \quad \text{for all } \xi \in \mathbb{M}_{\text{sym}}^{n \times n}, \quad (2.5)$$

$$\mathbb{B}_{ijkl}(\alpha) = \mathbb{B}_{klij}(\alpha) = \mathbb{B}_{jikl}(\alpha) \quad \text{for all } i, j, k, l = 1, \dots, n \text{ and } \alpha \in [0, 1], \quad (2.6)$$

$$\mathbb{B}(\alpha)\xi : \xi \geq 0 \quad \text{for all } \alpha \in [0, 1] \text{ and } \xi \in \mathbb{M}_{\text{sym}}^{n \times n}. \quad (2.7)$$

**Remark 2.2.** Thanks to (2.6) and (2.7), we derive the Cauchy-Schwartz Inequality

$$|\mathbb{B}(\alpha)\xi_1 : \xi_2|^2 \leq (\mathbb{B}(\alpha)\xi_1 : \xi_1)(\mathbb{B}(\alpha)\xi_2 : \xi_2) \quad \text{for all } \alpha \in [0, 1] \text{ and } \xi_1, \xi_2 \in \mathbb{M}_{\text{sym}}^{n \times n}.$$

Define the *total energy*  $\mathcal{E}: H^1(\Omega; [0, 1]) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \rightarrow [0, \infty)$  as

$$\mathcal{E}(\alpha, e, p) := \mathcal{Q}(e) + D(\alpha) + \|\nabla \alpha\|_2^2 + \tilde{Q}(\alpha, p) + \|\nabla p\|_2^2. \quad (2.8)$$

**Remark 2.3.** By the Sobolev Embedding Theorem, for  $n = 2, 3$  and  $d \in \mathbb{N}$  we have that the embedding  $H^1(\Omega; \mathbb{R}^d) \hookrightarrow L^\theta(\Omega; \mathbb{R}^d)$  is continuous for all  $\theta \in [1, 6]$  and compact for all  $\theta \in [1, 6)$ . In particular, by (2.5) for all  $\alpha \in H^1(\Omega; [0, 1])$ ,  $\beta \in H^1(\Omega)$ , and  $p, q \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  we have:

$$\begin{aligned} \left| \int_{\Omega} \mathbb{B}(\alpha(x))p(x) : q(x) \, dx \right| &\leq \|\mathbb{B}\|_{\infty} \|p\|_2 \|q\|_2, \\ \left| \int_{\Omega} \dot{\mathbb{B}}(\alpha(x))\beta(x)p(x) : q(x) \, dx \right| &\leq \|\dot{\mathbb{B}}\|_{\infty} \|\beta\|_2 \|p\|_4 \|q\|_4, \\ \left| \int_{\Omega} [\mathbb{B}(\alpha_1(x)) - \mathbb{B}(\alpha_2(x))]p(x) : q(x) \, dx \right| &\leq \|\dot{\mathbb{B}}\|_{\infty} \|\alpha_1 - \alpha_2\|_2 \|p\|_4 \|q\|_4, \\ \left| \int_{\Omega} [\dot{\mathbb{B}}(\alpha_1(x)) - \dot{\mathbb{B}}(\alpha_2(x))]\beta(x)p(x) : q(x) \, dx \right| &\leq \|\ddot{\mathbb{B}}\|_{\infty} \|\alpha_1 - \alpha_2\|_4 \|\beta\|_4 \|p\|_4 \|q\|_4. \end{aligned}$$

The *constraint set* for the stress tensor  $\sigma$  is identified by a closed and convex set  $K \subset \mathbb{M}_{\text{sym}}^{n \times n}$  satisfying

$$\{\sigma \in \mathbb{M}_{\text{sym}}^{n \times n} : |\sigma| \leq r_H\} \subset K \quad (2.9)$$

for a radius  $r_H > 0$ .

**Remark 2.4.** Notice that  $K$  can be unbounded. In particular, as in [4], we can consider constraint sets of the form

$$K := \{\sigma \in \mathbb{M}_{\text{sym}}^{n \times n} : \tau \sigma_m + |\sigma_D| - \kappa \geq 0\}$$

for  $\tau, \kappa > 0$ , where  $\sigma_m := \frac{\text{tr} \sigma}{n} \in \mathbb{R}$  is the *mean stress* and  $\sigma_D := \sigma - \sigma_m I$  is the *deviatoric part* of  $\sigma$ .

Let us consider the support function  $H: \mathbb{M}_{\text{sym}}^{n \times n} \rightarrow [0, \infty]$  of  $K$ , which is defined as

$$H(\xi) := \sup_{\sigma \in K} \sigma : \xi \quad \text{for all } \xi \in \mathbb{M}_{\text{sym}}^{n \times n}.$$

Clearly  $H$  is convex, lower semicontinuous, and positively 1-homogeneous. Moreover

$$r_H |\xi| \leq H(\xi) \quad \text{for all } \xi \in \mathbb{M}_{\text{sym}}^{n \times n}. \quad (2.10)$$

The *plastic potential*  $\mathcal{H}: L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \rightarrow [0, \infty]$  is defined as

$$\mathcal{H}(p) := \int_{\Omega} H(p(x)) \, dx \quad \text{for all } p \in L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

**Remark 2.5.** It is easy to check that the functional  $\mathcal{H}$  is lower semicontinuous in  $L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ . Moreover, since  $\mathcal{H}: L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \rightarrow [0, \infty]$  is convex, we derive that  $\mathcal{H}$  is weakly (sequentially) lower semicontinuous in  $L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ . Similarly, also the functional  $p \mapsto \int_{t_1}^{t_2} \mathcal{H}(p(t)) \, dt$  is lower semicontinuous in  $L^1([t_1, t_2]; L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$  for all  $[t_1, t_2] \subset [0, T]$ .

Given  $[t_1, t_2] \subset [0, T]$  and  $p: [t_1, t_2] \rightarrow L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ , the  $\mathcal{H}$ -*dissipation* of  $p$  on  $[t_1, t_2]$  is defined as

$$\mathcal{V}_{\mathcal{H}}(p; t_1, t_2) := \sup \left\{ \sum_{j=1}^N \mathcal{H}(p(s_j) - p(s_{j-1})) : N \in \mathbb{N}, t_1 = s_0 < s_1 < \dots < s_{N-1} < s_N = t_2 \right\}.$$

We recall that this notion has been introduced in [16], Appendix.

**Remark 2.6.** By (2.10) we have

$$r_H \mathcal{V}(p; t_1, t_2) \leq \mathcal{V}_{\mathcal{H}}(p; t_1, t_2), \quad (2.11)$$

where  $\mathcal{V}(p; t_1, t_2)$  is the total variation of  $p: [t_1, t_2] \rightarrow L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ , which is defined as

$$\mathcal{V}(p; t_1, t_2) := \sup \left\{ \sum_{j=1}^N \|p(s_j) - p(s_{j-1})\|_1 : N \in \mathbb{N}, t_1 = s_0 < s_1 < \dots < s_{N-1} < s_N = t_2 \right\}.$$

Moreover, if  $p_k: [t_1, t_2] \rightarrow L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ ,  $k \in \mathbb{N}$ , satisfy

$$p_k(r) \rightarrow p(r) \quad \text{in } L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \text{ as } k \rightarrow \infty \text{ for all } r \in [t_1, t_2],$$

then, by Remark 2.5 we have

$$\mathcal{V}_{\mathcal{H}}(p; t_1, t_2) \leq \liminf_{k \rightarrow \infty} \mathcal{V}_{\mathcal{H}}(p_k; t_1, t_2). \quad (2.12)$$

We conclude this subsection with the following result, whose proof follows simply from the Gagliardo-Nirenberg Inequality.

**Lemma 2.7.** *For all  $\theta \in [2, 6)$  and  $\delta > 0$  there exists a constant  $C_{\theta, \delta} > 0$  such that*

$$\|u\|_{\theta}^2 \leq C_{\theta, \delta} \|u\|_1^2 + \delta \|\nabla u\|_2^2 \quad \text{for all } u \in H^1(\Omega; \mathbb{R}^d).$$

**Remark 2.8.** As a consequence of Lemma 2.7, there exists a constant  $C > 0$  such that

$$\|u\|_{H^1}^2 \leq C (\|u\|_1^2 + \|\nabla u\|_2^2) \quad \text{for all } u \in H^1(\Omega; \mathbb{R}^d). \quad (2.13)$$

### 3. GLOBAL QUASISTATIC EVOLUTIONS

In this section we prove Theorem 3.8, *i.e.*, the existence of global quasistatic evolutions (also referred to as energetic solutions) for the model of geomaterials coupling damage and small-strain associative plasticity introduced before. Moreover, in Theorem 3.15 we perform a detailed analysis of the regularity in time of these evolutions.

#### 3.1. Definition and preliminary results

We start by introducing the mathematical formulation of our model of geomaterials. Following [4] and the definition of energetic solution of [5], we give the following notion of global quasistatic evolutions.

**Definition 3.1.** Assume (2.1)–(2.7) and (2.9). A quadruple  $(\alpha, u, e, p)$  from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$  is a *global quasistatic evolution* with datum  $w$  if:

(qs0) for a.e.  $x \in \Omega$  the map

$$t \mapsto \alpha(t, x) \quad \text{is non-increasing on } [0, T];$$

(qs1)  $(u(t), e(t), p(t)) \in \mathcal{A}(w(t))$  for all  $t \in [0, T]$  and

$$\mathcal{E}(\alpha(t), e(t), p(t)) \leq \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p(t)) \quad \text{for all } (\beta, v, \eta, q) \in \mathcal{D}(\alpha(t)) \times \mathcal{A}(w(t));$$

(qs2)  $p: [0, T] \rightarrow H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  has bounded  $\mathcal{H}$ -variation and for all  $t \in [0, T]$

$$\mathcal{E}(\alpha(t), e(t), p(t)) + \mathcal{V}_{\mathcal{H}}(p; 0, t) = \mathcal{E}(\alpha(0), e(0), p(0)) + \int_0^t (\mathbb{C}e(s), E\dot{w}(s)) \, ds.$$

**Remark 3.2.** The first condition (qs0) models the *irreversibility* of the damage process, (qs1) is the *global stability condition*, and (qs2) is the *energy-dissipation balance*.

Let us check that the integral in (qs2) is well defined. To this aim, we first prove the following stability result.

**Lemma 3.3.** *Assume (2.2)–(2.7). Let  $w_k \in H^1(\Omega; \mathbb{R}^n)$ , and  $(\alpha_k, u_k, e_k, p_k) \in H^1(\Omega; [0, 1]) \times \mathcal{A}(w_k)$  for all  $k \in \mathbb{N}$ . Assume that as  $k \rightarrow \infty$*

$$\begin{aligned} u_k &\rightharpoonup u_{\infty} && \text{in } H^1(\Omega; \mathbb{R}^n), && \alpha_k &\rightharpoonup \alpha_{\infty} && \text{in } H^1(\Omega), \\ e_k &\rightharpoonup e_{\infty} && \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), && p_k &\rightharpoonup p_{\infty} && \text{in } H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \\ w_k &\rightharpoonup w_{\infty} && \text{in } H^1(\Omega; \mathbb{R}^n). \end{aligned}$$



Then  $(\alpha_\infty, e_\infty, p_\infty) \in \mathcal{A}(w_\infty)$ . If in addition

$$\mathcal{E}(\alpha_k, e_k, p_k) \leq \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p_k) \quad \text{for all } (\beta, v, \eta, q) \in \mathcal{D}(\alpha_k) \times \mathcal{A}(w_k), \quad (3.1)$$

then

$$\mathcal{E}(\alpha_\infty, e_\infty, p_\infty) \leq \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p_\infty) \quad \text{for all } (\beta, v, \eta, q) \in \mathcal{D}(\alpha_\infty) \times \mathcal{A}(w_\infty). \quad (3.2)$$

*Proof.* We argue as in [34], Theorem 3.6. Since

$$Eu_k = e_k + p_k \quad \text{in } \Omega \text{ for all } k \in \mathbb{N}, \quad u_k = w_k \quad \text{on } \partial\Omega \text{ for all } k \in \mathbb{N},$$

by passing to the limit as  $k \rightarrow \infty$  and using the continuity of the trace operator, we derive that

$$Eu_\infty = e_\infty + p_\infty \quad \text{in } \Omega, \quad u_\infty = w_\infty \quad \text{on } \partial\Omega.$$

Therefore,  $(u_\infty, e_\infty, p_\infty) \in \mathcal{A}(w_\infty)$ .

We fix  $(\beta, v, \eta, q) \in \mathcal{D}(\alpha_\infty) \times \mathcal{A}(w_\infty)$  and for all  $k \in \mathbb{N}$  we set

$$\beta_k := \beta \wedge \alpha_k, \quad v_k := v - u_\infty + u_k, \quad \eta_k := \eta - e_\infty + e_k, \quad q_k := q - p_\infty + p_k.$$

By construction  $(\beta_k, v_k, \eta_k, q_k) \in \mathcal{D}(\alpha_k) \times \mathcal{A}(w_k)$  for all  $k \in \mathbb{N}$ , and as  $k \rightarrow \infty$

$$v_k \rightharpoonup v \quad \text{in } H^1(\Omega; \mathbb{R}^n), \quad \beta_k \rightharpoonup \beta \quad \text{in } H^1(\Omega), \quad (3.3)$$

$$\eta_k \rightharpoonup \eta \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad q_k \rightharpoonup q \quad \text{in } H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}). \quad (3.4)$$

We use  $(\beta_k, v_k, \eta_k, q_k)$  as test function in (3.1), and from the identity

$$\|\nabla\beta\|_2^2 + \|\nabla\alpha_k\|_2^2 = \|\nabla(\beta \wedge \alpha_k)\|_2^2 + \|\nabla(\beta \vee \alpha_k)\|_2^2,$$

we derive

$$\begin{aligned} & D(\alpha_k) + \|\nabla(\beta \vee \alpha_k)\|_2^2 - \|\nabla\beta\|_2^2 + \tilde{Q}(\alpha_k, p_k) \\ &= D(\alpha_k) + \|\nabla\alpha_k\|_2^2 - \|\nabla\beta_k\|_2^2 + \tilde{Q}(\alpha_k, p_k) \\ &\leq \mathcal{Q}(\eta_k) - \mathcal{Q}(e_k) + D(\beta_k) + \|\nabla q_k\|_2^2 - \|\nabla p_k\|_2^2 + \tilde{Q}(\beta_k, q_k) + \mathcal{H}(q_k - p_k) \\ &= \frac{1}{2}(\mathbb{C}(\eta - e_\infty + 2e_k), \eta - e_\infty)_2 + D(\beta_k) + \tilde{Q}(\beta_k, q_k) \\ &\quad + (\nabla q - \nabla p_\infty + 2\nabla p_k, \nabla q - \nabla p_\infty)_2 + \mathcal{H}(q - p_\infty). \end{aligned} \quad (3.5)$$

By (3.3)–(3.4) and Remark 2.3, for all  $\theta \in [1, 6)$  we have as  $k \rightarrow \infty$

$$\begin{aligned} \alpha_k &\rightarrow \alpha_\infty \quad \text{in } L^\theta(\Omega), & \beta_k &\rightarrow \beta \quad \text{in } L^\theta(\Omega), \\ p_k &\rightarrow p_\infty \quad \text{in } L^\theta(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), & q_k &\rightarrow q \quad \text{in } L^\theta(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}). \end{aligned}$$

In particular, by using (2.2) and (2.5) and arguing as in Remark 2.3 we deduce

$$\lim_{k \rightarrow \infty} D(\alpha_k) = D(\alpha_\infty), \quad \lim_{k \rightarrow \infty} D(\beta_k) = D(\beta),$$

$$\lim_{k \rightarrow \infty} \tilde{Q}(\alpha_k, p_k) = \tilde{Q}(\alpha_\infty, p_\infty), \quad \lim_{k \rightarrow \infty} \tilde{Q}(\beta_k, q_k) = \tilde{Q}(\beta, q).$$

Moreover,  $\beta \vee \alpha_k \rightarrow \alpha_\infty$  in  $H^1(\Omega)$  as  $k \rightarrow \infty$ , which implies

$$\|\nabla \alpha_\infty\|_2^2 \leq \liminf_{k \rightarrow \infty} \|\nabla(\beta \vee \alpha_k)\|_2^2.$$

Hence, by passing to the limit as  $k \rightarrow \infty$  in (3.5) we get

$$\begin{aligned} & D(\alpha_\infty) + \|\nabla \alpha_\infty\|_2^2 - \|\nabla \beta\|_2^2 + \tilde{Q}(\alpha_\infty, p_\infty) \\ & \leq \frac{1}{2}(\mathbb{C}(\eta + e_\infty), \eta - e_\infty)_2 + D(\beta) + \tilde{Q}(\beta, q) + (\nabla q + \nabla p_\infty, \nabla q - \nabla p_\infty)_2 + \mathcal{H}(q - p_\infty) \\ & = \mathcal{Q}(\eta) - \mathcal{Q}(e_\infty) + D(\beta) + \tilde{Q}(\beta, q) + \|\nabla q\|_2^2 - \|\nabla p_\infty\|_2^2 + \mathcal{H}(q - p_\infty), \end{aligned}$$

which gives (3.2).  $\square$

Thanks to Lemma 3.3 we deduce the following regularity result.

**Lemma 3.4.** *Assume (2.1)–(2.7) and (2.9). Let  $(\alpha, u, e, p)$  from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$  be a quadruple satisfying (qs0) and (qs1), and such that  $p: [0, T] \rightarrow H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  has bounded  $\mathcal{H}$ -variation. Then there exists a constant  $C > 0$ , independent of  $t \in [0, T]$ , such that*

$$\|\alpha(t)\|_{H^1} + \|u(t)\|_{H^1} + \|e(t)\|_2 + \|p(t)\|_{H^1} \leq C \quad \text{for all } t \in [0, T]. \quad (3.6)$$

Moreover, there exists a countable set  $N \subset [0, T]$  such that the quadruple  $(\alpha, u, e, p)$  is weakly continuous from  $[0, T] \setminus N$  into  $H^1(\Omega) \times \mathcal{A}$ . In particular, the quadruple  $(\alpha, u, e, p)$  is strongly measurable from  $[0, T]$  into  $H^1(\Omega) \times \mathcal{A}$ .

*Proof.* Since  $p: [0, T] \rightarrow L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  has bounded  $\mathcal{H}$ -variation, by (2.11) we deduce that  $p$  has bounded total variation. In particular,  $\|p(t)\|_1$  is uniformly bounded in  $[0, T]$ , being

$$\|p(t)\|_1 \leq \|p(t) - p(0)\|_1 + \|p(0)\|_1 \leq \mathcal{V}(p; 0, T) + \|p(0)\|_1 \quad \text{for all } t \in [0, T]. \quad (3.7)$$

Since  $(0, u(T) - w(T) + w(t), e(T) - Ew(T) + Ew(t), p(T)) \in \mathcal{D}(\alpha(t)) \times \mathcal{A}(w(t))$ , by the global stability condition (qs1) we deduce

$$\begin{aligned} & \mathcal{E}(\alpha(t), e(t), p(t)) \\ & \leq \mathcal{Q}(e(T)) + \mathcal{Q}(Ew(T) - Ew(t)) + (\mathbb{C}e(T), Ew(T) - Ew(t))_2 + \tilde{Q}(0, p(T)) + \|\nabla p(T)\|_2^2 + \mathcal{H}(p(T) - p(t)) \\ & \leq \frac{\gamma_2}{2} \|e(T)\|_2^2 + \frac{\gamma_2}{2} \|Ew(T) - Ew(t)\|_2^2 + \gamma_2 \|e(T)\|_2 \|Ew(T) - Ew(t)\|_2 \\ & \quad + \|\mathbb{B}\|_\infty \|p(T)\|_2^2 + \|\nabla p(T)\|_2^2 + \mathcal{V}_{\mathcal{H}}(p, t, T) \\ & \leq \gamma_2 \|e(T)\|_2^2 + \gamma_2 \left( \int_0^T \|E\dot{w}(r)\|_2 \, dr \right)^2 + \|\mathbb{B}\|_\infty \|p(T)\|_2^2 + \|\nabla p(T)\|_2^2 + \mathcal{V}_{\mathcal{H}}(p; 0, T). \end{aligned}$$

Notice that the quantity on the right-hand side is independent of  $t \in [0, T]$ . Since

$$\mathcal{E}(\alpha(t), e(t), p(t)) \geq \frac{\gamma_1}{2} \|e(t)\|_2^2 + \|\nabla \alpha(t)\|_2^2 + \|\nabla p(t)\|_2^2,$$

we deduce that there exists a constant  $C_1 > 0$ , independent of  $t \in [0, T]$ , such that

$$\|\nabla\alpha(t)\|_2^2 + \|e(t)\|_2^2 + \|\nabla p(t)\|_2^2 \leq C_1 \quad \text{for all } t \in [0, T].$$

By using (2.13), (3.7), and the fact that  $0 \leq \alpha(t) \leq 1$  a.e. in  $\Omega$  for all  $t \in [0, T]$  we obtain a constant  $C_2 > 0$ , independent of  $t \in [0, T]$ , such that

$$\|\alpha(t)\|_{H^1}^2 + \|e(t)\|_2^2 + \|p(t)\|_{H^1}^2 \leq C_2 \quad \text{for all } t \in [0, T]. \quad (3.8)$$

Finally, since  $u(t) - w(t) \in H_0^1(\Omega; \mathbb{R}^n)$ , by Korn Inequality there exists a constant  $C_3 > 0$ , independent of  $t \in [0, T]$ , such that

$$\begin{aligned} \|u(t)\|_2 &\leq \|u(t) - w(t)\|_2 + \|w(t)\|_2 \\ &\leq C\|Eu(t) - Ew(t)\|_2 + \|w(t)\|_2 \leq C\|e(t)\|_2 + C\|p(t)\|_2 + C\|Ew(t)\|_2 + \|w(t)\|_2. \end{aligned}$$

Therefore, by (3.8) and Korn Inequality there exists a constant  $C_5 > 0$ , independent of  $t \in [0, T]$ , such that

$$\|u(t)\|_{H^1} \leq C.$$

This implies (3.6).

It remains to prove that there exists a countable set  $N \subset [0, T]$  such that the quadruple  $(\alpha, u, e, p)$  is weakly continuous from  $[0, T] \setminus N$  into  $H^1(\Omega) \times \mathcal{A}$ . Indeed, if this holds, then the quadruple  $(\alpha, u, e, p)$  is weakly measurable from  $[0, T]$  into  $H^1(\Omega) \times \mathcal{A}$ , and therefore strongly measurable, since  $H^1(\Omega) \times \mathcal{A}$  is a separable Banach space.

We start by observing that  $\alpha: [0, T] \rightarrow H^1(\Omega; [0, 1])$  satisfies

$$\|\alpha(t)\|_\infty \leq 1 \quad \text{for all } t \in [0, T], \quad \alpha(t_2) \leq \alpha(t_1) \quad \text{a.e. in } \Omega \text{ for all } 0 \leq t_1 \leq t_2 \leq T.$$

By [34], Lemma A.2 there exists a countable set  $N_1 \subset [0, T]$  such that the function  $\alpha: [0, T] \setminus N_1 \rightarrow L^2(\Omega)$  is (strongly) continuous. In particular, thanks to the uniform estimate (3.6) we derive that  $\alpha: [0, T] \setminus N_1 \rightarrow H^1(\Omega)$  is weakly continuous. Similarly, since  $p$  has bounded total variation in  $L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ , there exists a countable set  $N_2 \subset [0, T]$  such that the function  $p: [0, T] \setminus N_2 \rightarrow L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  is strongly continuous. Hence, the uniform estimate (3.6) yields that  $p: [0, T] \setminus N_2 \rightarrow H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  is weakly continuous.

Let us define  $N := N_1 \cup N_2$ . We claim that the pair  $(u, e)$  is weakly continuous from  $[0, T] \setminus N$  into  $H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ . Indeed, let us fix  $t_\infty \in [0, T] \setminus N$  and let us consider a sequence  $(t_k)_k \subset [0, T] \setminus N$  such that  $t_k \rightarrow t_\infty$  as  $k \rightarrow \infty$ . By the uniform estimate (3.6) there exists a constant  $C > 0$  independent of  $k \in \mathbb{N}$  such that

$$\|u(t_k)\|_{H^1} + \|e(t_k)\|_2 \leq C \quad \text{for all } k \in \mathbb{N}.$$

Therefore, there exists a subsequence  $(t_{k_j})_j \subset (t_k)_k$  and a pair  $(u_\infty, e_\infty) \in H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  such that as  $j \rightarrow \infty$

$$u(t_{k_j}) \rightharpoonup u_\infty \quad \text{in } H^1(\Omega; \mathbb{R}^n), \quad e(t_{k_j}) \rightharpoonup e_\infty \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

Moreover, as  $j \rightarrow \infty$  we have

$$\alpha(t_{k_j}) \rightharpoonup \alpha(t_\infty) \quad \text{in } H^1(\Omega), \quad p(t_{k_j}) \rightharpoonup p(t_\infty) \quad \text{in } H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad w(t_{k_j}) \rightarrow w(t_\infty) \quad \text{in } H^1(\Omega; \mathbb{R}^n).$$

Thus, we can apply Lemma 3.3 to deduce that  $(\alpha(t_\infty), u_\infty, e_\infty, p(t_\infty)) \in \mathcal{A}(w(t_\infty))$  and satisfies

$$\mathcal{E}(\alpha(t_\infty), e_\infty, p(t_\infty)) \leq \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p(t_\infty)) \quad \text{for all } (\beta, v, \eta, q) \in \mathcal{D}(\alpha(t_\infty)) \times \mathcal{A}(w(t_\infty)).$$

In particular, by choosing  $\beta = \alpha(t_\infty)$  and  $q = p(t_\infty)$ , we derive that the pair  $(u_\infty, e_\infty)$  minimizes the functional  $F: H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \rightarrow [0, \infty)$ , defined as

$$F(v, \eta) := \mathcal{Q}(\eta) \quad \text{for all } (v, \eta) \in H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}),$$

over the convex set

$$G := \{(v, \eta) \in H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) : (v, \eta, p(t_\infty)) \in \mathcal{A}(w(t_\infty))\}.$$

This implies that  $(u_\infty, e_\infty)$  is uniquely determined for all  $t \in [0, T]$  by the strict convexity of  $\mathcal{Q}$ . Therefore,  $u(t_\infty) = u_\infty$  and  $e(t_\infty) = e_\infty$ . In particular, we get that as  $k \rightarrow \infty$

$$u(t_k) \rightharpoonup u(t_\infty) \quad \text{in } H^1(\Omega; \mathbb{R}^n), \quad e(t_k) \rightharpoonup e(t_\infty) \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}),$$

*i.e.*, the pair  $(u, e)$  is weakly continuous from  $[0, T] \setminus N$  into  $H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ .  $\square$

**Remark 3.5.** By Lemma 3.4 the integral in (qs2) is well defined for all quadruples  $(\alpha, u, e, p)$  from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$  satisfying (qs0) and (qs1) and such that  $p: [0, T] \rightarrow H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  has bounded  $\mathcal{H}$ -variation.

**Remark 3.6.** If  $(\alpha, u, e, p)$  from  $[0, T]$  in  $H^1(\Omega; [0, 1]) \times \mathcal{A}$  is a global quasistatic evolution, then the constant  $C$  appearing in (3.6) can be chosen depending only on  $w, d, \mathbb{C}, \mathbb{B}, K$ , and the initial datum  $(\alpha(0), u(0), e(0), p(0))$ , and not on the quasistatic evolution  $(\alpha, u, e, p)$  itself. Indeed, by Lemma 3.4 we have that the function  $t \mapsto \|e(t)\|_2$  is bounded and measurable on  $[0, T]$ . In particular, if we define

$$M := \sup_{t \in [0, T]} \|e(t)\|_2,$$

by (qs2) we derive

$$\frac{\gamma_1}{2} M^2 \leq \mathcal{E}(\alpha(0), e(0), p(0)) + \gamma_2 M \int_0^T \|\text{E}\dot{w}(s)\|_2 \, ds.$$

This implies the existence of a constant  $C_1$ , which depends only on  $w, d, \mathbb{C}, \mathbb{B}$ , and on the initial datum  $(\alpha(0), e(0), p(0))$ , satisfying

$$M = \sup_{t \in [0, T]} \|e(t)\|_2 \leq C_1.$$

Therefore, by using again (qs2) we derive the existence of a further constant  $C_2$ , depending only on  $w, d, \mathbb{C}, \mathbb{B}, K$ , and on the initial datum  $(\alpha(0), u(0), e(0), p(0))$ , such that

$$\|\alpha(t)\|_{H^1} + \|u(t)\|_{H^1} + \|e(t)\|_2 + \|p(t)\|_{H^1} \leq C_2 \quad \text{for all } t \in [0, T].$$

### 3.2. Existence of a global quasistatic evolution

In this subsection we prove the existence of a global quasistatic evolution satisfying a fixed initial configuration. As in [4], the proof relies on the De Giorgi's Minimizing Movement approach to quasistatic evolutions,

where time-continuous evolutions are approximated by discrete-time ones, constructed by solving incremental minimization problems.

We consider an initial configuration

$$(\alpha^0, u^0, e^0, p^0) \in H^1(\Omega; [0, 1]) \times \mathcal{A}(w(0)),$$

which satisfies

$$\mathcal{E}(\alpha^0, e^0, p^0) \leq \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p^0) \quad \text{for all } (\beta, v, \eta, q) \in \mathcal{D}(\alpha^0) \times \mathcal{A}(w(0)). \quad (3.9)$$

For all  $k \in \mathbb{N}$  we define

$$\tau_k := \frac{T}{k}, \quad t_k^i := i\tau_k, \quad w_k^i := w(t_k^i) \quad \text{for } i = 0, \dots, k.$$

Starting from

$$(\alpha_k^0, u_k^0, e_k^0, p_k^0) := (\alpha^0, u^0, e^0, p^0) \in H^1(\Omega; [0, 1]) \times \mathcal{A}(w(0)),$$

for all  $i = 1, \dots, k$  we define

$$(\alpha_k^i, u_k^i, e_k^i, p_k^i) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i)$$

as the solution of

$$\min_{(\beta, v, \eta, q) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i)} \{ \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p_k^{i-1}) \}. \quad (3.10)$$

Notice that

$$(\alpha_k^{i-1}, w_k^i, \mathbf{E}w_k^i, 0) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i) \neq \emptyset \quad \text{for all } i = 1, \dots, k.$$

Moreover, for all  $i = 1, \dots, k$  and  $(\alpha, u, e, p) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i)$  we have

$$\mathcal{E}(\alpha, e, p) + \mathcal{H}(p - p_k^{i-1}) \geq \frac{\gamma_1}{2} \|e\|_2^2 + \|\nabla \alpha\|_2^2 + \|\nabla p\|_2^2 + r_H \|p\|_1 - r_H \|p_k^{i-1}\|_1$$

and the functional

$$(\alpha, u, e, p) \mapsto \mathcal{E}(\alpha, e, p) + \mathcal{H}(p - p_k^i)$$

is sequentially lower semicontinuous with respect to the weak convergences in the spaces  $H^1(\Omega)$ ,  $H^1(\Omega; \mathbb{R}^n)$ ,  $L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ , and  $H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ . Hence, by the Direct Method of Calculus of Variations, for all  $k \in \mathbb{N}$  and  $i = 1, \dots, k$  there exists a solution  $(\alpha_k^i, u_k^i, e_k^i, p_k^i) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i)$  to the minimum problem (3.10). In particular, since

$$\mathcal{H}(q - p_k^{i-1}) - \mathcal{H}(p_k^i - p_k^{i-1}) \leq \mathcal{H}(q - p_k^i),$$

the quadruple  $(\alpha_k^i, u_k^i, e_k^i, p_k^i)$  satisfies

$$\mathcal{E}(\alpha_k^i, e_k^i, p_k^i) \leq \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p_k^i) \quad \text{for all } (\beta, v, \eta, q) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i). \quad (3.11)$$

In the following lemma, we derive an energy estimate for the discrete evolution  $\{(\alpha_k^i, u_k^i, e_k^i, p_k^i)\}_{i=1}^k$ . First, for all  $k \in \mathbb{N}$  we define

$$\dot{w}_k^i := \frac{w_k^i - w_k^{i-1}}{\tau_k} \quad \text{for } i = 1, \dots, k.$$

**Lemma 3.7.** *Assume (2.2)–(2.7), (2.9), and (3.9). For all  $k \in \mathbb{N}$  and  $i = 1, \dots, k$  we have*

$$\mathcal{E}(\alpha_k^i, e_k^i, p_k^i) + \sum_{j=1}^i \mathcal{H}(p_k^j - p_k^{j-1}) \leq \mathcal{E}(\alpha^0, e^0, p^0) + \sum_{j=1}^i \tau_k (\mathbb{C}e_k^{j-1}, \mathbb{E}w_k^j)_2 + \delta_k, \quad (3.12)$$

where

$$\delta_k := \tau_k \frac{\gamma_2}{2} \int_0^T \|\mathbb{E}w(r)\|_2^2 dr \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (3.13)$$

In particular, there exists a constant  $C > 0$  independent of  $k$  and  $i$ , such that

$$\max_{i=0, \dots, k} \|e_k^i\|_2 + \max_{i=0, \dots, k} \|p_k^i\|_{H^1} + \max_{i=0, \dots, k} \|\alpha_k^i\|_{H^1} + \sum_{j=1}^k \mathcal{H}(p_k^j - p_k^{j-1}) \leq C.$$

*Proof.* We fix  $i = 1, \dots, k$  and for  $j = 1, \dots, i$  we consider

$$(\alpha_k^{j-1}, u_k^{j-1} - w_k^{j-1} + w_k^j, e_k^{j-1} - \mathbb{E}w_k^{j-1} + \mathbb{E}w_k^j, p_k^{j-1}) \in \mathcal{D}(\alpha_k^{j-1}) \times \mathcal{A}(w_k^j).$$

By the minimality condition (3.10) we get

$$\begin{aligned} \mathcal{E}(\alpha_k^j, e_k^j, p_k^j) + \mathcal{H}(p_k^j - p_k^{j-1}) &\leq \mathcal{E}(\alpha_k^{j-1}, e_k^{j-1} - \mathbb{E}w_k^{j-1} + \mathbb{E}w_k^j, p_k^{j-1}) \\ &\leq \mathcal{E}(\alpha_k^{j-1}, e_k^{j-1}, p_k^{j-1}) + \tau_k (\mathbb{C}e_k^{j-1}, \mathbb{E}w_k^j)_2 + \frac{\gamma_2}{2} \|\mathbb{E}w_k^j - \mathbb{E}w_k^{j-1}\|_2^2. \end{aligned}$$

We sum over  $j = 1, \dots, i$  and we use the inequality

$$\sum_{j=1}^k \|\mathbb{E}w_k^j - \mathbb{E}w_k^{j-1}\|_2^2 \leq \tau_k \int_0^T \|\mathbb{E}w(r)\|_2^2 dr$$

to obtain (3.12).

By the discrete energy inequality (3.12) we get

$$\frac{\gamma_1}{2} \|e_k^i\|_2^2 \leq \mathcal{E}(\alpha^0, e^0, p^0) + \gamma_2 \sum_{j=1}^i \tau_k \|\mathbb{E}w_k^j\|_2 \|e_k^{j-1}\|_2 + \delta_k. \quad (3.14)$$

If we define

$$M_k := \max_{i=0, \dots, k} \|e_k^i\|_2 \quad \text{for } k \in \mathbb{N},$$

then by (3.14) we derive

$$\frac{\gamma_1}{2} M_k^2 \leq \mathcal{E}(\alpha^0, e^0, p^0) + \gamma_2 \left( \int_0^T \|\mathbb{E}\dot{w}(r)\|_2 \, dr \right) M_k + \delta_k.$$

Since  $\delta_k \rightarrow 0$  as  $k \rightarrow \infty$ , there exists a constant  $C_1 > 0$ , independent of  $k$ , such that

$$M_k = \max_{i=0, \dots, k} \|e_k^i\|_2 \leq C_1.$$

Therefore, by (3.12) we can find another constant  $C_2 > 0$ , independent of  $k$  and  $i$ , such that

$$\max_{i=0, \dots, k} \|\nabla \alpha_k^i\|_2^2 + \max_{i=0, \dots, k} \|\nabla p_k^i\|_2^2 + \sum_{i=1}^k \mathcal{H}(p_k^j - p_k^{j-1}) \leq C_2.$$

By (2.13) and the inequality

$$\|p_k^i\|_1 \leq \|p^0\|_1 + \sum_{j=1}^k \|p_k^j - p_k^{j-1}\|_1 \leq \|p^0\|_1 + \frac{1}{r_H} \sum_{j=1}^k \mathcal{H}(p_k^j - p_k^{j-1}),$$

we deduce the existence a constant  $C_3 > 0$ , independent of  $k$  and  $i$ , such that

$$\max_{i=0, \dots, k} \|p_k^i\|_{H^1} \leq C_3.$$

Since  $0 \leq \alpha_k^i \leq 1$  in  $\Omega$ , there exists a constant  $C_4 > 0$  independent of  $k$  and  $i$  such that

$$\max_{i=0, \dots, k} \|\alpha_k^i\|_{H^1} \leq \sqrt{|\Omega|} + \max_{i=0, \dots, k} \|\nabla \alpha_k^i\|_2 \leq C_4.$$

This concludes the proof.  $\square$

We introduce the *piecewise constant interpolant*  $\bar{\alpha}_k: [0, T] \rightarrow H^1(\Omega; [0, 1])$  by

$$\bar{\alpha}_k(t) := \alpha_k^i \quad \text{for } t \in (t_k^{i-1}, t_k^i], \quad i = 1, \dots, k, \quad \bar{\alpha}_k(0) = \alpha_k^0, \quad (3.15)$$

and similarly we consider  $\bar{u}_k: [0, T] \rightarrow H^1(\Omega; \mathbb{R}^n)$ ,  $\bar{e}_k: [0, T] \rightarrow L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ ,  $\bar{p}_k: [0, T] \rightarrow H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ , and  $\bar{w}_k: [0, T] \rightarrow H^1(\Omega; \mathbb{R}^n)$ . Moreover, we define  $\bar{\tau}_k: [0, T] \rightarrow [0, T]$  by

$$\bar{\tau}_k(t) := t_k^i \quad \text{for } t \in (t_k^{i-1}, t_k^i], \quad i = 1, \dots, k, \quad \bar{\tau}_k(0) = 0. \quad (3.16)$$

By definition, we have

$$\sum_{j=1}^k \mathcal{H}(p_k^j - p_k^{j-1}) = \mathcal{V}_{\mathcal{H}}(\bar{p}_k; 0, T),$$

and thanks to Lemma 3.7 we deduce that the piecewise constant interpolants satisfy the following estimate: there is a constant  $C > 0$  independent of  $k \in \mathbb{N}$  and  $t \in [0, T]$  such that

$$\|\bar{e}_k(t)\|_2 + \|\bar{p}_k(t)\|_{H^1} + \|\bar{\alpha}_k(t)\|_{H^1} + \mathcal{V}_{\mathcal{H}}(\bar{p}_k; 0, T) \leq C \quad \text{for all } t \in [0, T]. \quad (3.17)$$

We can now prove our existence result for global quasistatic evolution.

**Theorem 3.8.** *Assume (2.1)–(2.7), (2.9), and (3.9). Then there exists a global quasistatic evolution  $(\alpha, u, e, p)$  from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$  satisfying the initial condition*

$$(\alpha(0), u(0), e(0), p(0)) = (\alpha^0, u^0, e^0, p^0). \quad (3.18)$$

*Proof. Step 1.* Starting from  $(\alpha^0, u^0, e^0, p^0)$  we consider the incremental problem (3.10) for all  $k \in \mathbb{N}$  and  $i = 1, \dots, k$ , and we obtain a sequence of approximate solutions  $(\bar{\alpha}_k, \bar{u}_k, \bar{e}_k, \bar{p}_k)$  from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$ . Since all functions  $\bar{\alpha}_k$  are non-increasing in time, we can apply a generalized version of Helly Theorem (for example, [16], Lemma 7.2 with  $X$  equal to the space of bounded Radon measures on  $\Omega$ ) to conclude that there exist a subsequence, still denoted by  $\bar{\alpha}_k$ , and a function  $\alpha: [0, T] \rightarrow H^1(\Omega; [0, 1])$  non-increasing in time, such that for all  $t \in [0, T]$  as  $k \rightarrow \infty$

$$\bar{\alpha}_k(t) \rightharpoonup \alpha(t) \quad \text{in } H^1(\Omega).$$

In particular, thanks to Remark 2.3, for all  $t \in [0, T]$  and  $\theta \in [1, 6)$  we have as  $k \rightarrow \infty$

$$\bar{\alpha}_k(t) \rightarrow \alpha(t) \quad \text{in } L^\theta(\Omega).$$

Moreover, by (3.17) there exists a constant  $C > 0$ , independent of  $k$ , such that

$$r_H \mathcal{V}(\bar{p}_k; 0, T) \leq \mathcal{V}_{\mathcal{H}}(\bar{p}_k; 0, T) = \sum_{j=1}^k \mathcal{H}(p_k^j - p_k^{j-1}) \leq C, \quad \|\bar{p}_k(0)\|_{H^1} = \|p^0\|_{H^1} \leq C.$$

Hence, by using the same generalized version of Helly Theorem there exist a further subsequence, still denoted by  $\bar{p}_k$ , and a function  $p: [0, T] \rightarrow H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  with bounded variation in  $L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  such that for all  $t \in [0, T]$  as  $k \rightarrow \infty$

$$\bar{p}_k(t) \rightharpoonup p(t) \quad \text{in } H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

In particular, by (2.12) we get

$$\mathcal{V}_{\mathcal{H}}(p; 0, T) \leq \liminf_{k \rightarrow \infty} \mathcal{V}_{\mathcal{H}}(\bar{p}_k; 0, T) \leq C,$$

which yields that  $p$  has bounded  $\mathcal{H}$ -variation. Furthermore, for all  $t \in [0, T]$  and  $\theta \in [1, 6)$  we have as  $k \rightarrow \infty$

$$\bar{p}_k(t) \rightarrow p(t) \quad \text{in } L^\theta(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

**Step 2.** For all fixed  $t \in [0, T]$  there exist a subsequence  $k_j$  and functions  $e^* \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  and  $u^* \in H^1(\Omega; \mathbb{R}^n)$ , depending on  $t \in [0, T]$ , such that as  $j \rightarrow \infty$

$$\bar{e}_{k_j}(t) \rightharpoonup e^* \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad \bar{u}_{k_j}(t) \rightarrow u^* \quad \text{in } H^1(\Omega; \mathbb{R}^n).$$

By construction  $(\bar{\alpha}_k(t), \bar{u}_k(t), \bar{e}_k(t), \bar{p}_k(t)) \in H^1(\Omega; [0, 1]) \times \mathcal{A}(\bar{w}_k(t))$ , and in view of (3.11)

$$\mathcal{E}(\bar{\alpha}_k(t), \bar{e}_k(t), \bar{p}_k(t)) \leq \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - \bar{p}_k(t)) \quad \text{for all } (\beta, v, \eta, q) \in \mathcal{D}(\bar{\alpha}_k(t)) \times \mathcal{A}(\bar{w}_k(t)).$$



By writing the inequality for  $k_j$  and using Lemma 3.3 we deduce

$$\mathcal{E}(\alpha(t), e^*, p(t)) \leq \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p(t)) \quad \text{for all } (\beta, v, \eta, q) \in \mathcal{D}(\alpha(t)) \times \mathcal{A}(w(t)).$$

Therefore, we can argue as in Lemma 3.4 to derive that  $(u^*, e^*)$  is uniquely determined for all  $t \in [0, T]$ . In particular, by defining  $u(t) := u^*$  and  $e(t) := e^*$  we deduce that as  $k \rightarrow \infty$

$$\bar{e}_k(t) \rightharpoonup e(t) \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad \bar{u}_k(t) \rightarrow u(t) \quad \text{in } H^1(\Omega; \mathbb{R}^n),$$

and the quadruple  $(\alpha, u, e, p)$  satisfies (qs0), (qs1), and (3.18).

**Step 3.** By Lemma 3.4, we deduce that  $(\alpha, u, e, p)$  are strongly measurable from  $[0, T]$  into  $H^1(\Omega) \times \mathcal{A}$ . We define  $\underline{e}_k: [0, T] \rightarrow L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  as

$$\underline{e}_k(t) := e_k^{i-1} \quad \text{for } t \in [t_k^{i-1}, t_k^i], \quad i = 1, \dots, k, \quad \underline{e}_k(T) = e_k^k.$$

By the discrete energy inequality (3.12), for all  $k \in \mathbb{N}$ ,  $t \in (t_k^{i-1}, t_k^i]$ , and  $i = 1, \dots, k$  we have

$$\mathcal{E}(\bar{\alpha}_k(t), \bar{e}_k(t), \bar{p}_k(t)) + \mathcal{V}_{\mathcal{H}}(\bar{p}_k; 0, t_k^i) \leq \mathcal{E}(\alpha^0, e^0, p^0) + \int_0^{\bar{\tau}_k(t)} (\mathbb{C}\underline{e}_k(r), \text{E}\dot{w}(r))_2 \, dr + \delta_k. \quad (3.19)$$

Since  $\underline{e}_k(t) = \bar{e}_k(t - \tau_k)$  for a.e.  $t \in [\tau_k, T]$ , we derive that as  $k \rightarrow \infty$

$$\underline{e}_k \rightharpoonup e \quad \text{in } L^2((0, T); L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})).$$

Hence, we can use the convergences of Steps 1 and 2 to pass to the limit as  $k \rightarrow \infty$  in (3.19), and to obtain that the quadruple  $(\alpha, u, e, p)$  satisfies for all  $t \in [0, T]$

$$\mathcal{E}(\alpha(t), e(t), p(t)) + \mathcal{V}_{\mathcal{H}}(p; 0, t) \leq \mathcal{E}(\alpha^0, e^0, p^0) + \int_0^t (\mathbb{C}e(r), \text{E}\dot{w}(r))_2 \, dr,$$

since all the terms on the left-hand side are sequentially lower semicontinuous with respect to the convergences of  $\bar{\alpha}_k, \bar{e}_k, \bar{p}_k$ . It remains to prove the opposite inequality.

We fix  $t \in (0, T]$  and for all  $k \in \mathbb{N}$  we consider a collection of points  $\{s_k^i\}_{i=0}^k$  in  $[0, t]$  satisfying

$$0 = s_k^0 < s_k^1 < \dots < s_k^{k-1} < s_k^k = t, \quad \lim_{k \rightarrow \infty} \max_{i=1, \dots, k} |s_k^i - s_k^{i-1}| \rightarrow 0.$$

For all  $i = 1, \dots, k$  we have

$$(\alpha(s_k^i), u(s_k^i) - w(s_k^i) + w(s_k^{i-1}), e(s_k^i) - \text{E}w(s_k^i) + \text{E}w(s_k^{i-1}), p(s_k^i)) \in \mathcal{D}(\alpha(s_k^{i-1})) \times \mathcal{A}(w(s_k^{i-1})).$$

By using (qs1) in  $t = s_k^{i-1}$  for all  $k \in \mathbb{N}$  and  $i = 1, \dots, k$  we deduce

$$\begin{aligned} \mathcal{E}(\alpha(s_k^{i-1}), e(s_k^{i-1}), p(s_k^{i-1})) &\leq \mathcal{E}(\alpha(s_k^i), e(s_k^i) - \text{E}w(s_k^i) + \text{E}w(s_k^{i-1}), p(s_k^i)) + \mathcal{H}(p(s_k^i) - p(s_k^{i-1})) \\ &= \mathcal{E}(\alpha(s_k^i), e(s_k^i), p(s_k^i)) + \mathcal{H}(p(s_k^i) - p(s_k^{i-1})) \\ &\quad - (\mathbb{C}e(s_k^i), \text{E}w(s_k^i) - \text{E}w(s_k^{i-1}))_2 + \mathcal{Q}(\text{E}w(s_k^i) - \text{E}w(s_k^{i-1})) \\ &\leq \mathcal{E}(\alpha(s_k^i), e(s_k^i), p(s_k^i)) + \mathcal{H}(p(s_k^i) - p(s_k^{i-1})) - \int_{s_k^{i-1}}^{s_k^i} (\mathbb{C}e(s_k^i), \text{E}\dot{w}(r))_2 \, dr \end{aligned}$$

$$+ \max_{i=1,\dots,k} |s_k^i - s_k^{i-1}| \frac{\gamma_2}{2} \int_0^t \|\mathbb{E}\dot{w}(r)\|_2^2 dr.$$

By summing over  $i = 1, \dots, k$  we get

$$\begin{aligned} \mathcal{E}(\alpha^0, e^0, p^0) &\leq \mathcal{E}(\alpha(t), e(t), p(t)) + \mathcal{V}_{\mathcal{H}}(p; 0, t) - \int_0^t (\mathbb{C}\tilde{e}_k(r), \mathbb{E}\dot{w}(r))_2 dr \\ &+ \max_{i=1,\dots,k} |s_k^i - s_k^{i-1}| \frac{\gamma_2}{2} \int_0^t \|\mathbb{E}\dot{w}(r)\|_2^2 dr, \end{aligned} \quad (3.20)$$

where  $\tilde{e}_k: [0, T] \rightarrow L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  is defined as

$$\tilde{e}_k(t) := e(s_k^i) \quad \text{for all } t \in [s_k^{i-1}, s_k^i), \quad \tilde{e}_k(T) := e(T).$$

Since  $\tilde{e}_k(r) \rightarrow e(r)$  in  $L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  for a.e.  $r \in [0, t]$  as  $k \rightarrow \infty$ , we can use the Dominated Convergence Theorem to derive

$$\lim_{k \rightarrow \infty} \int_0^t (\mathbb{C}\tilde{e}_k(r), \mathbb{E}\dot{w}(r))_2 dr = \int_0^t (\mathbb{C}e(r), \mathbb{E}\dot{w}(r))_2 dr.$$

By passing to the limit as  $k \rightarrow \infty$  in (3.20) we get the other opposite energy inequality. Hence,  $(\alpha, u, e, p)$  satisfies (qs2) and it is a global quasistatic evolution from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$ .  $\square$

### 3.3. Continuity in time

In this subsection we investigate the regularity in time of global quasistatic evolutions  $(\alpha, u, e, p)$  in the sense of Definition 3.1. More precisely, we show that every evolution is continuous at every continuity point  $t$  of  $\alpha: [0, T] \rightarrow L^1(\Omega)$ . We point out that such a property improves the continuity result of Lemma 3.4.

We start by proving several preliminary results. In the first one we show that every global quasistatic evolution is strongly continuous up to a countable set.

**Lemma 3.9.** *Assume (2.2)–(2.7) and (2.9). Let  $(\alpha, u, e, p)$  be a global quasistatic evolution from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$ . There exists a countable set  $N \subset [0, T]$  such that the quadruple  $(\alpha, u, e, p)$  is strongly continuous from  $[0, T] \setminus N$  into  $H^1(\Omega) \times \mathcal{A}$ .*

*Proof.* From Lemma 3.4 we know that there exists a countable set  $N_1 \subset [0, T]$  such that every global quasistatic evolution  $(\alpha, u, e, p)$  is weakly continuous from  $[0, T] \setminus N_1$  into  $H^1(\Omega) \times \mathcal{A}$ . Moreover, since the map  $t \mapsto \mathcal{V}_{\mathcal{H}}(p; 0, t)$  from  $[0, T]$  into  $\mathbb{R}$  is increasing, there exists a countable set  $N_2 \subset [0, T]$  such that  $t \mapsto \mathcal{V}_{\mathcal{H}}(p; 0, t)$  is continuous from  $[0, T] \setminus N_2$  into  $\mathbb{R}$ . In particular, the energy equality (qs2) yields that the map  $t \mapsto \mathcal{E}(\alpha(t), e(t), p(t))$  is continuous from  $[0, T] \setminus N_2$  into  $\mathbb{R}$ .

Let us define  $N := N_1 \cup N_2$ . We fix  $t_\infty \in [0, T] \setminus N$  and a sequence  $(t_k)_k \subset [0, T] \setminus N$  such that  $t_k \rightarrow t_\infty$  as  $k \rightarrow \infty$ . Thanks to Lemma 3.4 we have

$$\begin{aligned} \alpha(t_k) &\rightharpoonup \alpha(t_\infty) \quad \text{in } H^1(\Omega) \text{ as } k \rightarrow \infty, & u(t_k) &\rightharpoonup u(t_\infty) \quad \text{in } H^1(\Omega; \mathbb{R}^n) \text{ as } k \rightarrow \infty, \\ e(t_k) &\rightharpoonup e(t_\infty) \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \text{ as } k \rightarrow \infty, & p(t_k) &\rightharpoonup p(t_\infty) \quad \text{in } H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \text{ as } k \rightarrow \infty, \end{aligned}$$

and by the Sobolev Embedding Theorem (see also Rem. 2.3), for all  $\theta \in [1, 6)$  we get

$$\begin{aligned} \alpha(t_k) &\rightarrow \alpha(t_\infty) \quad \text{in } L^\theta(\Omega) \text{ as } k \rightarrow \infty, & u(t_k) &\rightarrow u(t_\infty) \quad \text{in } L^\theta(\Omega; \mathbb{R}^n) \text{ as } k \rightarrow \infty, \\ p(t_k) &\rightarrow p(t_\infty) \quad \text{in } L^\theta(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \text{ as } k \rightarrow \infty, & w(t_k) &\rightarrow w(t_\infty) \quad \text{in } H^1(\Omega; \mathbb{R}^n) \text{ as } k \rightarrow \infty. \end{aligned}$$

In particular, we derive that

$$\begin{aligned} & \mathcal{E}(\alpha(t_\infty), e(t_\infty), p(t_\infty)) \\ & \leq \liminf_{k \rightarrow \infty} \mathcal{Q}(e(t_k)) + \lim_{k \rightarrow \infty} D(\alpha(t_k)) + \liminf_{k \rightarrow \infty} \|\nabla \alpha(t_k)\|_2^2 + \lim_{k \rightarrow \infty} \tilde{\mathcal{Q}}(\alpha(t_k), p(t_k)) + \liminf_{k \rightarrow \infty} \|\nabla p(t_k)\|_2^2 \\ & \leq \lim_{k \rightarrow \infty} \mathcal{E}(\alpha(t_k), e(t_k), p(t_k)) = \mathcal{E}(\alpha(t_\infty), e(t_\infty), p(t_\infty)). \end{aligned}$$

This implies that

$$\mathcal{Q}(e(t_\infty)) = \liminf_{k \rightarrow \infty} \mathcal{Q}(e(t_k)), \quad \|\nabla \alpha(t_\infty)\|_2^2 = \liminf_{k \rightarrow \infty} \|\nabla \alpha(t_k)\|_2^2, \quad \|\nabla p(t_\infty)\|_2^2 = \liminf_{k \rightarrow \infty} \|\nabla p(t_k)\|_2^2.$$

If we use also that

$$\liminf_{k \rightarrow \infty} (a_k + b_k) \leq \limsup_{k \rightarrow \infty} a_k + \liminf_{k \rightarrow \infty} b_k \leq \limsup_{k \rightarrow \infty} (a_k + b_k) \quad \text{for all two sequences } (a_k)_k, (b_k)_k \subset \mathbb{R},$$

we conclude that

$$\mathcal{Q}(e(t_\infty)) = \lim_{k \rightarrow \infty} \mathcal{Q}(e(t_k)), \quad \|\nabla \alpha(t_\infty)\|_2^2 = \lim_{k \rightarrow \infty} \|\nabla \alpha(t_k)\|_2^2, \quad \|\nabla p(t_\infty)\|_2^2 = \lim_{k \rightarrow \infty} \|\nabla p(t_k)\|_2^2.$$

Hence, thanks to (2.4) we obtain the following strong convergences:

$$\begin{aligned} \alpha(t_k) & \rightarrow \alpha(t_\infty) \quad \text{in } H^1(\Omega) \text{ as } k \rightarrow \infty, & e(t_k) & \rightarrow e(t_\infty) \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \text{ as } k \rightarrow \infty, \\ p(t_k) & \rightarrow p(t_\infty) \quad \text{in } H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \text{ as } k \rightarrow \infty. \end{aligned}$$

Finally, by Korn Inequality, we derive

$$\|\nabla u(t_k) - \nabla u(t_\infty)\|_2 \leq C (\|u(t_k) - u(t_\infty)\|_2 + \|e(t_k) - e(t_\infty)\|_2 + \|p(t_k) - p(t_\infty)\|_2) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Therefore, the quadruple  $(\alpha, u, e, p)$  is strongly continuous from  $[0, T] \setminus N$  into  $H^1(\Omega) \times \mathcal{A}$ .  $\square$

Moreover, for all  $\alpha \in H^1(\Omega; [0, 1])$ ,  $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ ,  $p \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ , and  $\beta \in H^1(\Omega)$  we define

$$\partial_\alpha \mathcal{E}(\alpha, e, p)[\beta] := \int_\Omega \dot{d}(\alpha(x)) \beta(x) \, dx + 2 \int_\Omega \nabla \alpha(x) \cdot \nabla \beta(x) \, dx + \int_\Omega \mathbb{B}(\alpha(x)) \beta(x) p(x) : p(x) \, dx.$$

By Remark 2.3 we deduce that  $\partial_\alpha \mathcal{E}(\alpha, e, p)[\beta]$  is well defined, and we can identify  $\partial_\alpha \mathcal{E}(\alpha, e, p)$  as an element of  $(H^1(\Omega))'$  by setting

$$\langle \partial_\alpha \mathcal{E}(\alpha, e, p), \beta \rangle := \partial_\alpha \mathcal{E}(\alpha, e, p)[\beta] \quad \text{for all } \beta \in H^1(\Omega).$$

**Remark 3.10.** If we extend  $d$  and  $\mathbb{B}$  to the whole  $\mathbb{R}$  as

$$d(\alpha) := d(0) + \alpha \dot{d}(0), \quad \mathbb{B}(\alpha) := \mathbb{B}(0) + \alpha \dot{\mathbb{B}}(0) \quad \text{for all } \alpha \in (-\infty, 0), \quad (3.21)$$

$$d(\alpha) := d(1) + (\alpha - 1) \dot{d}(1), \quad \mathbb{B}(\alpha) := \mathbb{B}(1) + (\alpha - 1) \dot{\mathbb{B}}(1) \quad \text{for all } \alpha \in (1, \infty), \quad (3.22)$$

so that

$$d \in C^1(\mathbb{R}; [0, \infty)) \cap C^{0,1}(\mathbb{R}; [0, \infty)), \quad \mathbb{B} \in C^1(\mathbb{R}; \text{Lin}(\mathbb{M}_{\text{sym}}^{n \times n}; \mathbb{M}_{\text{sym}}^{n \times n})) \cap C^{0,1}(\mathbb{R}; \text{Lin}(\mathbb{M}_{\text{sym}}^{n \times n}; \mathbb{M}_{\text{sym}}^{n \times n})),$$

then by the Dominated Convergence Theorem we derive that  $\partial_\alpha \mathcal{E}(\alpha, e, p)$  is nothing else than the Gateaux derivative of  $\mathcal{E}$  at  $\alpha \in H^1(\Omega; [0, 1])$  in the direction  $\beta \in H^1(\Omega)$ .

In the next lemma, following the ideas of [16], Theorem 3.6 and [35], Lemmas 3.2, we show that every global quasistatic evolution satisfies some variational inequalities.

**Lemma 3.11.** *Assume (2.2)–(2.7) and (2.9). Let  $(\alpha, u, e, p)$  be a global quasistatic evolution from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$ .*

(a) *For all  $t \in [0, T]$  we have*

$$\partial_\alpha \mathcal{E}(\alpha(t), e(t), p(t))[\beta] \geq 0 \quad \text{for all } \beta \in H^1(\Omega) \text{ with } \beta \leq 0. \quad (3.23)$$

(b) *For all  $t \in [0, T]$  and  $(v, \eta, q) \in \mathcal{A}(0)$  we have*

$$-\mathcal{H}(q) \leq (\mathbb{C}e(t), \eta)_2 + 2(\mathbb{B}(\alpha(t))p(t), q)_2 + 2(\nabla p(t), \nabla q)_2 \leq \mathcal{H}(-q). \quad (3.24)$$

(c) *For all  $t \in [0, T]$  and  $(v, \eta, q) \in \mathcal{A}(w(t))$*

$$\begin{aligned} & \mathcal{Q}(e(t)) + \mathcal{Q}(\eta - e(t)) + \tilde{\mathcal{Q}}(\alpha(t), p(t)) + \tilde{\mathcal{Q}}(\alpha(t), q - p(t)) + \|\nabla p(t)\|_2^2 + \|\nabla q - \nabla p(t)\|_2^2 \\ & \leq \mathcal{Q}(\eta) + \tilde{\mathcal{Q}}(\alpha(t), q) + \|\nabla q\|_2^2 + \mathcal{H}(q - p(t)). \end{aligned} \quad (3.25)$$

*Proof.* (a) We first extend  $d$  and  $\mathbb{B}$  to the whole  $\mathbb{R}$  as in (3.21) and (3.22). Hence, as observed in Remark 3.10, for all  $\alpha \in H^1(\Omega; [0, 1])$ ,  $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ ,  $p \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ , and  $\beta \in H^1(\Omega)$  we have

$$\lim_{\lambda \rightarrow 0^+} \frac{\mathcal{E}(\alpha + \lambda\beta, e, p) - \mathcal{E}(\alpha, e, p)}{\lambda} = \partial_\alpha \mathcal{E}(\alpha, e, p)[\beta]. \quad (3.26)$$

We fix  $t \in [0, T]$ . Let  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  and let  $\lambda > 0$ . Since  $(\alpha(t) + \lambda\beta)^+ \in \mathcal{D}(\alpha(t))$ , by (qs1) we get

$$\mathcal{E}(\alpha(t), e(t), p(t)) \leq \mathcal{E}((\alpha(t) + \lambda\beta)^+, e(t), p(t)).$$

Moreover, by (2.2), (2.5), (3.21), (3.22), and the fact that  $\alpha(t) \geq 0$  we have

$$\begin{aligned} D((\alpha(t) + \lambda\beta)^+) & \leq D(\alpha(t) + \lambda\beta), & \|\nabla((\alpha(t) + \lambda\beta)^+)\|_2^2 & \leq \|\nabla(\alpha(t) + \lambda\beta)\|_2^2, \\ \tilde{\mathcal{Q}}((\alpha(t) + \lambda\beta)^+, p(t)) & \leq \tilde{\mathcal{Q}}(\alpha(t) + \lambda\beta, p(t)). \end{aligned}$$

Hence, we derive

$$\mathcal{E}(\alpha(t), e(t), p(t)) \leq \mathcal{E}(\alpha(t) + \lambda\beta, e(t), p(t)).$$

By dividing for  $\lambda > 0$ , sending  $\lambda \rightarrow 0^+$ , and using (3.26) we get (3.23).

(b) For all  $(v, \eta, q) \in \mathcal{A}(0)$  and  $\lambda > 0$  we test (qs1) with

$$(\alpha(t), u(t) + \lambda v, e(t) + \lambda \eta, p(t) + \lambda q) \in \mathcal{D}(\alpha(t)) \times \mathcal{A}(w(t)),$$

and we get

$$\mathcal{Q}(e(t)) + \tilde{\mathcal{Q}}(\alpha(t), p(t)) + \|\nabla p(t)\|_2^2 \leq \mathcal{Q}(e(t) + \lambda \eta) + \tilde{\mathcal{Q}}(\alpha(t), p(t) + \lambda q) + \|\nabla(p(t) + \lambda q)\|_2^2 + \lambda \mathcal{H}(q).$$

By dividing for  $\lambda > 0$  and sending  $\lambda \rightarrow 0^+$  we get one inequality of (3.24). To get the other inequality, it is enough to replace  $(v, \eta, q) \in \mathcal{A}(0)$  with  $(-v, -\eta, -q) \in \mathcal{A}(0)$ .

(c) By (3.24) for all  $(v, \eta, q) \in \mathcal{A}(w(t))$  we get

$$\begin{aligned} & \mathcal{Q}(e(t)) + \mathcal{Q}(\eta - e(t)) - \mathcal{Q}(\eta) + \tilde{\mathcal{Q}}(\alpha(t), p(t)) + \tilde{\mathcal{Q}}(\alpha(t), q - p(t)) - \tilde{\mathcal{Q}}(\alpha(t), q) \\ & \quad + \|\nabla p(t)\|_2^2 + \|\nabla q - \nabla p(t)\|_2^2 - \|\nabla q\|_2^2 \\ & = (\mathbb{C}e(t), e(t) - \eta)_2 + 2(\mathbb{B}(\alpha(t))p(t), p(t) - q)_2 + 2(\nabla p(t), \nabla p(t) - \nabla q)_2 \leq \mathcal{H}(q - p(t)), \end{aligned}$$

which is (3.25).  $\square$

**Remark 3.12.** The assumptions  $\dot{d}(0) \leq 0$  and  $\dot{\mathbb{B}}(0)\xi : \xi \leq 0$  for all  $\xi \in \mathbb{M}_{\text{sym}}^{n \times n}$  in (2.2) and (2.5) are needed to deduce (3.23) for all  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$ .

By (3.24) we deduce that every global quasistatic evolution  $(\alpha, u, e, p)$  satisfies

$$\operatorname{div} \mathbb{C}e(t) = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n) \text{ for all } t \in [0, T]. \quad (3.27)$$

Indeed, since  $(\phi, E\phi, 0) \in \mathcal{A}(0)$  for all  $\phi \in C_c^\infty(\Omega; \mathbb{R}^n)$  and  $\mathcal{H}(0) = 0$ , by (3.24) we deduce

$$(\mathbb{C}e(t), E\phi)_2 = 0 \quad \text{for all } \phi \in C_c^\infty(\Omega; \mathbb{R}^n),$$

which is (3.27). Let us recall also the following integration by parts formula.

**Lemma 3.13.** *Let  $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  satisfy*

$$\operatorname{div} \mathbb{C}e = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n). \quad (3.28)$$

*Then, for all  $w \in H^1(\Omega; \mathbb{R}^n)$  and  $(v, \eta, q) \in \mathcal{A}(w)$  we have*

$$(\mathbb{C}e, q)_2 = -(\mathbb{C}e, \eta - Ew)_2. \quad (3.29)$$

*Proof.* Since  $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ , by using a density argument in (3.28) we derive

$$(\mathbb{C}e, E\psi)_2 = 0 \quad \text{for all } \psi \in H_0^1(\Omega; \mathbb{R}^n).$$

In particular, for all  $w \in H^1(\Omega; \mathbb{R}^n)$  and  $(v, \eta, q) \in \mathcal{A}(w)$ , we get

$$(\mathbb{C}e, q)_2 = (\mathbb{C}e, Eu - \eta)_2 = -(\mathbb{C}e, \eta - Ew)_2 + (\mathbb{C}e, E(u - w))_2 = -(\mathbb{C}e, \eta - Ew)_2,$$

which is (3.29).  $\square$

Moreover, we need the following version of Gronwall Inequality, whose proof is analogous to the one of [16], Lemma 5.3.

**Lemma 3.14.** *Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $f : [a, b] \rightarrow [0, \infty)$  be a bounded measurable function, let  $g : [a, b] \rightarrow [0, \infty)$  be a non-decreasing function, and let  $h : [a, b] \rightarrow [0, \infty)$  be an integrable function. Suppose that*

$$f(t)^2 \leq g(t)^2 + \int_a^t f(r)h(r) \, dr + \left( \int_a^t h(r) \, dr \right)^2 \quad \text{for all } t \in [a, b]. \quad (3.30)$$

Then,

$$f(t) \leq g(t) + \frac{3}{2} \int_a^t h(r) dr \quad \text{for all } t \in [a, b]. \quad (3.31)$$

*Proof.* Let  $t_0 \in [a, b]$  be fixed. Define

$$\gamma_0 := g(t_0)^2 + \left( \int_a^{t_0} h(r) dr \right)^2. \quad (3.32)$$

If  $\gamma_0 = 0$ , then  $g(t_0) = 0$  and  $h = 0$  a.e. on  $[0, t_0]$ , which implies that  $f(t_0) = 0$  by (3.30), and so (3.31) is satisfied. Therefore, we may assume that  $\gamma_0 > 0$  and we define the function

$$V(t) := \int_a^t f(r)h(r) dr \quad \text{for all } r \in [a, t_0].$$

By assumptions  $V \in AC([a, t_0])$  and  $\dot{V}(t) = f(t)h(t)$  for a.e.  $t \in [a, t_0]$ . Therefore, thanks to (3.30) we derive that  $\dot{V}(t) \leq h(t)\sqrt{\gamma_0 + V(t)}$  for a.e.  $t \in [a, t_0]$ . Integrating between  $a$  and  $t_0$ , and using (3.32) we get

$$2\sqrt{V(t_0) + \gamma_0} \leq 2\sqrt{\gamma_0} + \int_a^{t_0} h(r) dr \leq 2g(t_0) + 3 \int_a^{t_0} h(r) dr.$$

On the other hand, by (3.30) we derive  $f(t_0) \leq \sqrt{V(t_0) + \gamma_0}$ , and so

$$2f(t_0) \leq 2g(t_0) + 3 \int_a^{t_0} h(r) dr,$$

which gives (3.31). □

We can finally prove the strong continuity of a global quasistatic evolution  $(\alpha, u, e, p)$  from  $[0, T]$  into  $H^1(\Omega) \times \mathcal{A}$  at every continuity point  $t$  of  $\alpha: [0, T] \rightarrow L^1(\Omega)$ .

**Theorem 3.15.** *Assume (2.2)–(2.7) and (2.9). Let  $(\alpha, u, e, p)$  be a global quasistatic evolution from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$ . Then, there exists a constant  $C > 0$  such that for all  $t_1, t_2 \in [0, T]$  with  $t_1 < t_2$  we have*

$$\begin{aligned} & \|\alpha(t_2) - \alpha(t_1)\|_{H^1} + \|e(t_2) - e(t_1)\|_2 + \|p(t_2) - p(t_1)\|_{H^1} \\ & \leq C \left( \|\alpha(t_2) - \alpha(t_1)\|_1 + \int_{t_1}^{t_2} \|\mathbb{E}\dot{w}(r)\|_2 dr \right), \end{aligned} \quad (3.33)$$

$$\|u(t_2) - u(t_1)\|_{H^1} \leq C \left( \|\alpha(t_2) - \alpha(t_1)\|_1 + \int_{t_1}^{t_2} \|\dot{w}(r)\|_{H^1} dr \right). \quad (3.34)$$

**Remark 3.16.** Therefore, every global quasistatic evolution  $(\alpha, u, e, p)$  is (strongly) continuous from  $[0, T]$  into  $H^1(\Omega) \times \mathcal{A}$ , except for a countable subset of  $[0, T]$ , which is the set of discontinuity points of  $\alpha$  with respect to the convergence in  $L^1(\Omega)$ . Moreover, if  $\alpha \in AC([0, T]; L^1(\Omega))$ , then  $(\alpha, u, e, p)$  are absolutely continuous from  $[0, T]$  into  $H^1(\Omega) \times \mathcal{A}$ .

*Proof.* We fix  $t_1, t_2 \in [0, T]$  with  $t_1 < t_2$ . We test (3.23) in  $t_1$  with  $\beta := \alpha(t_2) - \alpha(t_1)$  and we use the identity

$$\|\nabla\alpha(t_2)\|_2^2 - \|\nabla\alpha(t_2) - \nabla\alpha(t_1)\|_2^2 - \|\nabla\alpha(t_1)\|_2^2 = 2(\nabla\alpha(t_1), \nabla\alpha(t_2) - \nabla\alpha(t_1))_2,$$

to derive that

$$\begin{aligned} & \|\nabla\alpha(t_2) - \nabla\alpha(t_1)\|_2^2 \\ & \leq \|\nabla\alpha(t_2)\|_2^2 - \|\nabla\alpha(t_1)\|_2^2 + (\dot{d}(\alpha(t_1)), \alpha(t_2) - \alpha(t_1))_2 + (\mathring{\mathbb{B}}(\alpha(t_1))(\alpha(t_2) - \alpha(t_1))p(t_1), p(t_1))_2. \end{aligned} \quad (3.35)$$

We now test (3.25) in  $t_1$  with

$$(v, \eta, q) := (u(t_2) - w(t_2) + w(t_1), e(t_2) - \mathbb{E}w(t_2) + \mathbb{E}w(t_1), p(t_2)) \in \mathcal{A}(w(t_1)),$$

and we get

$$\begin{aligned} & \mathcal{Q}(e(t_1)) + \mathcal{Q}(e(t_2) - e(t_1) - \mathbb{E}(w(t_2) - w(t_1))) \\ & \quad + \tilde{Q}(\alpha(t_1), p(t_1)) + \tilde{Q}(\alpha(t_1), p(t_2) - p(t_1)) + \|\nabla p(t_1)\|_2^2 + \|\nabla p(t_2) - \nabla p(t_1)\|_2^2 \\ & \leq \mathcal{Q}(e(t_2) - \mathbb{E}(w(t_2) - w(t_1))) + \tilde{Q}(\alpha(t_1), p(t_2)) + \|\nabla p(t_2)\|_2^2 + \mathcal{H}(p(t_2) - p(t_1)). \end{aligned}$$

In particular

$$\begin{aligned} & \mathcal{Q}(e(t_2) - e(t_1)) + \tilde{Q}(\alpha(t_1), p(t_2) - p(t_1)) + \|\nabla p(t_2) - \nabla p(t_1)\|_2^2 \\ & \leq \mathcal{Q}(e(t_2)) - \mathcal{Q}(e(t_1)) - (\mathbb{C}e(t_1), \mathbb{E}(w(t_2) - w(t_1)))_2 \\ & \quad + \tilde{Q}(\alpha(t_1), p(t_2)) - \tilde{Q}(\alpha(t_1), p(t_1)) + \|\nabla p(t_2)\|_2^2 - \|\nabla p(t_1)\|_2^2 + \mathcal{H}(p(t_2) - p(t_1)). \end{aligned} \quad (3.36)$$

Moreover, by (qs2) evaluated in  $t_1$  and  $t_2$ , we have that

$$\begin{aligned} \mathcal{H}(p(t_2) - p(t_1)) & \leq \mathcal{V}_{\mathcal{H}}(p; t_1, t_2) \\ & = \mathcal{E}(\alpha(t_1), e(t_1), p(t_1)) - \mathcal{E}(\alpha(t_2), e(t_2), p(t_2)) + \int_{t_1}^{t_2} (\mathbb{C}e(s), \mathbb{E}\dot{w}(s))_2 ds. \end{aligned} \quad (3.37)$$

If we sum (3.35) and (3.36) and we use (3.37) we deduce that

$$\begin{aligned} & \mathcal{Q}(e(t_2) - e(t_1)) + \tilde{Q}(\alpha(t_1), p(t_2) - p(t_1)) + \|\nabla p(t_2) - \nabla p(t_1)\|_2^2 + \|\nabla\alpha(t_2) - \nabla\alpha(t_1)\|_2^2 \\ & \leq \int_{t_1}^{t_2} (\mathbb{C}(e(s) - e(t_1)), \mathbb{E}\dot{w}(s))_2 ds + D(\alpha(t_1)) - D(\alpha(t_2)) + (\dot{d}(\alpha(t_1)), \alpha(t_2) - \alpha(t_1))_2 \\ & \quad + \tilde{Q}(\alpha(t_1), p(t_2)) - \tilde{Q}(\alpha(t_2), p(t_2)) + (\mathring{\mathbb{B}}(\alpha(t_1))(\alpha(t_2) - \alpha(t_1))p(t_1), p(t_1))_2. \end{aligned}$$

By (2.2) and (2.5), for all  $\alpha_1, \alpha \in [0, 1]$  and  $\xi \in \mathbb{M}_{\text{sym}}^{n \times n}$  we have

$$\begin{aligned} |d(\alpha_1) - d(\alpha_2) - \dot{d}(\alpha_1)(\alpha_1 - \alpha_2)| & \leq \|\ddot{d}\|_{\infty} |\alpha_1 - \alpha_2|^2, \\ |\mathbb{B}(\alpha_1)\xi : \xi - \mathbb{B}(\alpha_2)\xi : \xi - \mathring{\mathbb{B}}(\alpha_1)(\alpha_1 - \alpha_2)\xi : \xi| & \leq \|\ddot{\mathbb{B}}\|_{\infty} |\alpha_1 - \alpha_2|^2 |\xi|^2. \end{aligned}$$

Therefore, thanks to (2.4) and the uniform estimates (3.6), we have

$$\begin{aligned} & \frac{\gamma_2}{2} \|e(t_2) - e(t_1)\|_2^2 + \tilde{Q}(\alpha(t_1), p(t_2) - p(t_1)) + \|\nabla p(t_2) - \nabla p(t_1)\|_2^2 + \|\nabla\alpha(t_2) - \nabla\alpha(t_1)\|_2^2 \\ & \leq \int_{t_1}^{t_2} (\mathbb{C}(e(s) - e(t_1)), \mathbb{E}\dot{w}(s))_2 ds + \|\ddot{d}\|_{\infty} \|\alpha(t_2) - \alpha(t_1)\|_2^2 + \|\ddot{\mathbb{B}}\|_{\infty} \|\alpha(t_2) - \alpha(t_1)\|_4^2 \|p(t_1)\|_4^2 \\ & \quad + \tilde{Q}(\alpha(t_1), p(t_2)) - \tilde{Q}(\alpha(t_2), p(t_2)) - \tilde{Q}(\alpha(t_1), p(t_1)) + \tilde{Q}(\alpha(t_2), p(t_1)), \end{aligned}$$

$$\begin{aligned}
&= \int_{t_1}^{t_2} (\mathbb{C}(e(s) - e(t_1)), \mathbb{E}\dot{w}(s))_2 ds + \|\ddot{d}\|_\infty \|\alpha(t_2) - \alpha(t_1)\|_2^2 + \|\ddot{\mathbb{B}}\|_\infty \|\alpha(t_2) - \alpha(t_1)\|_4^2 \|p(t_1)\|_4^2 \\
&\quad + ((\mathbb{B}(\alpha(t_1)) - \mathbb{B}(\alpha(t_2)))(p(t_2) - p(t_1)), p(t_2) + p(t_1))_2 \\
&\leq C_1 \|\alpha(t_2) - \alpha(t_1)\|_4^2 + C_1 \|\alpha(t_1) - \alpha(t_2)\|_4 \|p(t_2) - p(t_1)\|_2 + C_1 \int_{t_1}^{t_2} \|e(s) - e(t_1)\|_2 \|\mathbb{E}\dot{w}(s)\|_2 ds \quad (3.38)
\end{aligned}$$

for a constant  $C_1 > 0$ .

By Remark 2.2, the uniform estimates (3.6), and (2.10), (3.37), there exists a constant  $C_2 > 0$  such that

$$\begin{aligned}
r_H \|p(t_2) - p(t_1)\|_1 &\leq \mathcal{H}(p(t_2) - p(t_1)) \\
&\leq \mathcal{E}(\alpha(t_1), e(t_1), p(t_1)) - \mathcal{E}(\alpha(t_2), e(t_2), p(t_2)) + \int_{t_1}^{t_2} (\mathbb{C}e(s), \mathbb{E}\dot{w}(s))_2 ds \\
&\leq \mathcal{Q}(e(t_1)) - \mathcal{Q}(e(t_2)) + D(\alpha(t_1)) - D(\alpha(t_2)) + \|\nabla\alpha(t_1)\|_2^2 - \|\nabla\alpha(t_2)\|_2^2 \\
&\quad + \tilde{\mathcal{Q}}(\alpha(t_1), p(t_1)) - \tilde{\mathcal{Q}}(\alpha(t_1), p(t_2)) + \tilde{\mathcal{Q}}(\alpha(t_1), p(t_2)) - \tilde{\mathcal{Q}}(\alpha(t_2), p(t_2)) \\
&\quad + \|\nabla p(t_1)\|_2^2 - \|\nabla p(t_2)\|_2^2 + \int_{t_1}^{t_2} (\mathbb{C}e(s), \mathbb{E}\dot{w}(s))_2 ds \\
&\leq C_2 (\|e(t_2) - e(t_1)\|_2 + \|\alpha(t_2) - \alpha(t_1)\|_4 + \|\nabla\alpha(t_2) - \nabla\alpha(t_1)\|_2) \\
&\quad + C_2 \left( \sqrt{\tilde{\mathcal{Q}}(\alpha(t_1), p(t_2) - p(t_1))} + \|\nabla p(t_2) - \nabla p(t_1)\|_2 + \int_{t_1}^{t_2} \|\mathbb{E}\dot{w}(s)\|_2 ds \right).
\end{aligned}$$

Therefore, if we use also (3.38) we deduce the existence of a constant  $C_3 > 0$  such that

$$\begin{aligned}
&r_H^2 \|p(t_2) - p(t_1)\|_1^2 \\
&\leq 3C_2^2 (\|e(t_2) - e(t_1)\|_2^2 + \|\alpha(t_2) - \alpha(t_1)\|_4^2 + \|\nabla\alpha(t_2) - \nabla\alpha(t_1)\|_2^2) \\
&\quad + 3C_2^2 \left( \tilde{\mathcal{Q}}(\alpha(t_1), p(t_2) - p(t_1)) + \|\nabla p(t_2) - \nabla p(t_1)\|_2^2 + \left( \int_{t_1}^{t_2} \|\mathbb{E}\dot{w}(s)\|_2 ds \right)^2 \right) \\
&\leq C_3 \|\alpha(t_2) - \alpha(t_1)\|_4^2 + C_3 \|\alpha(t_1) - \alpha(t_2)\|_4 \|p(t_2) - p(t_1)\|_2 \\
&\quad + C_3 \int_{t_1}^{t_2} \|e(s) - e(t_1)\|_2 \|\mathbb{E}\dot{w}(s)\|_2 ds + C_3 \left( \int_{t_1}^{t_2} \|\mathbb{E}\dot{w}(s)\|_2 ds \right)^2. \quad (3.39)
\end{aligned}$$

By summing (3.38) and (3.39) and using also (2.13), we can find a constant  $C_4 > 0$  such that

$$\begin{aligned}
&\|e(t_2) - e(t_1)\|_2^2 + \|p(t_2) - p(t_1)\|_{H^1}^2 + \|\alpha(t_2) - \alpha(t_1)\|_{H^1}^2 \\
&\leq C_4 \|\alpha(t_2) - \alpha(t_1)\|_4^2 + C_4 \|\alpha(t_1) - \alpha(t_2)\|_4 \|p(t_2) - p(t_1)\|_2 \\
&\quad + C_4 \int_{t_1}^{t_2} \|e(s) - e(t_1)\|_2 \|\mathbb{E}\dot{w}(s)\|_2 ds + C_4 \left( \int_{t_1}^{t_2} \|\mathbb{E}\dot{w}(s)\|_2 ds \right)^2.
\end{aligned}$$

Firstly, we use Young Inequality

$$\|\alpha(t_1) - \alpha(t_2)\|_4 \|p(t_2) - p(t_1)\|_2 \leq C_\epsilon \|\alpha(t_1) - \alpha(t_2)\|_4^2 + \epsilon \|p(t_2) - p(t_1)\|_{H^1}^2 \quad \text{for all } \epsilon > 0,$$



and then Lemma 2.7 with  $\theta = 4$  to derive that

$$\begin{aligned} & \|e(t_2) - e(t_1)\|_2^2 + \|p(t_2) - p(t_1)\|_{H^1}^2 + \|\alpha(t_2) - \alpha(t_1)\|_{H^1}^2 \\ & \leq C_5 \|\alpha(t_2) - \alpha(t_1)\|_1^2 + C_5 \int_{t_1}^{t_2} \|e(s) - e(t_1)\|_2 \|\mathbf{E}\dot{w}(s)\|_2 \, ds + C_5 \left( \int_{t_1}^{t_2} \|\mathbf{E}\dot{w}(s)\|_2 \, ds \right)^2 \end{aligned}$$

for a constant  $C_5 > 0$ . In particular, there exist a constant  $C > 0$  such that

$$\begin{aligned} & (\|e(t_2) - e(t_1)\|_2 + \|p(t_2) - p(t_1)\|_{H^1} + \|\alpha(t_2) - \alpha(t_1)\|_{H^1})^2 \\ & \leq C^2 \|\alpha(t_2) - \alpha(t_1)\|_1^2 + C \int_{t_1}^{t_2} \|e(s) - e(t_1)\|_2 \|\mathbf{E}\dot{w}(s)\|_2 \, ds + \left( C \int_{t_1}^{t_2} \|\mathbf{E}\dot{w}(s)\|_2 \, ds \right)^2. \end{aligned} \quad (3.40)$$

Let us define the functions

$$\begin{aligned} f(t) & := \|e(t) - e(t_1)\|_2 + \|p(t) - p(t_1)\|_{H^1} + \|\alpha(t) - \alpha(t_1)\|_{H^1} \quad \text{for all } t \in [t_1, T], \\ g(t) & := C \|\alpha(t) - \alpha(t_1)\|_1 \quad \text{for all } t \in [t_1, T], \quad h(t) := C \|\mathbf{E}\dot{w}(t)\|_2 \quad \text{for a.e. } t \in [t_1, T]. \end{aligned}$$

Therefore, by (3.40) we deduce

$$f(t)^2 \leq g(t)^2 + \int_{t_1}^t f(s)h(s) \, ds + \left( \int_{t_1}^t h(s) \, ds \right)^2 \quad \text{for all } t \in [t_1, T].$$

We can apply Lemma 3.14 to deduce (3.33). Finally, by Korn Inequality we have

$$\|u(t_2) - u(t_1)\|_2 \leq C_6 \|e(t_2) - e(t_1)\|_2 + C_6 \|p(t_2) - p(t_1)\|_2 + C_6 \|w(t_2) - w(t_1)\|_{H^1}$$

for a constant  $C_6 > 0$ . Hence, by (3.33) we derive (3.34).  $\square$

#### 4. APPROXIMATE VISCOUS EVOLUTIONS

The present section is devoted to prove the existence of a quasistatic evolution for a viscous regularization of our elastoplastic-damage model, driven by a small parameter  $\varepsilon \in (0, 1)$  (see Def. 4.10). As in the previous section, this is done by using a time discretization scheme.

##### 4.1. Discretization in time

Let  $\varepsilon \in (0, 1)$  be fixed. As in Section 3 we consider an initial configuration  $(\alpha^0, u^0, e^0, p^0) \in H^1(\Omega; [0, 1]) \times \mathcal{A}(w(0))$  which satisfies (3.9).

For all  $k \in \mathbb{N}$  we define

$$\tau_k := \frac{T}{k}, \quad t_k^i := i\tau_k, \quad w_k^i := w(t_k^i) \quad \text{for } i = 0, \dots, k.$$

Starting from

$$(\alpha_k^0, u_k^0, e_k^0, p_k^0) := (\alpha^0, u^0, e^0, p^0) \in H^1(\Omega; [0, 1]) \times \mathcal{A}(w(0)),$$

for all  $i = 1, \dots, k$  we define

$$(\alpha_k^i, u_k^i, e_k^i, p_k^i) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i)$$

as the solution of the minimum problem

$$\min_{(\beta, v, \eta, q) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i)} \left\{ \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p_k^{i-1}) + \frac{\varepsilon}{2\tau_k} \|\beta - \alpha_k^{i-1}\|_2^2 \right\}. \quad (4.1)$$

As in Section 3, the minimum problem (4.1) admits a solution  $(\alpha_k^i, u_k^i, e_k^i, p_k^i) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i)$  for all  $i = 1, \dots, k$ . Moreover, it satisfies the following properties.

**Lemma 4.1.** *Assume (2.2)–(2.7), (2.9), and (3.9). The following hold for all  $k \in \mathbb{N}$  and  $i = 0, \dots, k$ .*

(a) For all  $(v, \eta, q) \in \mathcal{A}(w_k^i)$

$$\mathcal{E}(\alpha_k^i, e_k^i, p_k^i) \leq \mathcal{E}(\alpha_k^i, \eta, q) + \mathcal{H}(q - p_k^i).$$

(b) For all  $(v, \eta, q) \in \mathcal{A}(0)$

$$-\mathcal{H}(q) \leq (\mathbb{C}e_k^i, \eta)_2 + 2(\mathbb{B}(\alpha_k^i)p_k^i, q)_2 + 2(\nabla p_k^i, \nabla q)_2 \leq \mathcal{H}(-q).$$

In particular

$$\operatorname{div} \mathbb{C}e_k^i = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n). \quad (4.2)$$

(c) For all  $(v, \eta, q) \in \mathcal{A}(w_k^i)$

$$\begin{aligned} & \mathcal{Q}(e_k^i) + \mathcal{Q}(\eta - e_k^i) + \tilde{\mathcal{Q}}(\alpha_k^i, p_k^i) + \tilde{\mathcal{Q}}(\alpha_k^i, q - p_k^i) + \|\nabla p_k^i\|_2^2 + \|\nabla q - \nabla p_k^i\|_2^2 \\ & \leq \mathcal{Q}(\eta) + \tilde{\mathcal{Q}}(\alpha_k^i, q) + \|\nabla q\|_2^2 + \mathcal{H}(q - p_k^i). \end{aligned}$$

*Proof.* (a) For  $i = 0$  this is true by (3.9), while for  $i = 1, \dots, k$  and  $(v, \eta, q) \in \mathcal{A}(w_k^i)$  it is enough to use (4.1) together with

$$\mathcal{H}(q - p_k^{i-1}) - \mathcal{H}(p_k^i - p_k^{i-1}) \leq \mathcal{H}(q - p_k^i).$$

(b) For all  $(v, \eta, q) \in \mathcal{A}(0)$  and  $\lambda > 0$  we text in (a) with  $(u_k^i + \lambda v, e_k^i + \lambda \eta, p_k^i + \lambda q) \in \mathcal{A}(w_k^i)$ , and we get

$$\mathcal{Q}(e_k^i) + \tilde{\mathcal{Q}}(\alpha_k^i, p_k^i) + \|\nabla p_k^i\|_2^2 \leq \mathcal{Q}(e_k^i - \lambda \eta) + \tilde{\mathcal{Q}}(\alpha_k^i, p_k^i - \lambda q) + \|\nabla(p_k^i - \lambda q)\|_2^2 + \lambda \mathcal{H}(-q).$$

By dividing for  $\lambda > 0$  and sending  $\lambda \rightarrow 0^+$  we get one inequality of (b). To get the other inequality, it is enough to replace  $(v, \eta, q) \in \mathcal{A}(0)$  with  $(-v, -\eta, -q) \in \mathcal{A}(0)$ .

Let now  $\phi \in C_c^\infty(\Omega)$ . Then  $(\phi, \mathbb{E}\phi, 0) \in \mathcal{A}(0)$  and by using (b) we get

$$(\mathbb{C}e_k^i, \mathbb{E}\phi)_2 = 0 \quad \text{for all } \phi \in C_c^\infty(\Omega; \mathbb{R}^n),$$

which gives (4.2).

(c) Thanks to (b), for all  $(v, \eta, q) \in \mathcal{A}(w_k^i)$  we get

$$\begin{aligned} & \mathcal{Q}(e_k^i) + \mathcal{Q}(\eta - e_k^i) - \mathcal{Q}(\eta) + \tilde{\mathcal{Q}}(\alpha_k^i, p_k^i) + \tilde{\mathcal{Q}}(\alpha_k^i, q - p_k^i) - \tilde{\mathcal{Q}}(\alpha_k^i, q) \\ & \quad + \|\nabla p_k^i\|_2^2 + \|\nabla q - \nabla p_k^i\|_2^2 - \|\nabla q\|_2^2 \\ & = (\mathbb{C}e_k^i, e_k^i - \eta)_2 + 2(\mathbb{B}(\alpha_k^i)p_k^i, p_k^i - q)_2 + 2(\nabla p_k^i, \nabla p_k^i - \nabla q)_2 \leq \mathcal{H}(q - p_k^i), \end{aligned}$$

which implies the desired inequality.  $\square$

For all  $k \in \mathbb{N}$  and  $i = 1, \dots, k$  we define

$$\dot{\alpha}_k^i := \frac{\alpha_k^i - \alpha_k^{i-1}}{\tau_k}, \quad \dot{u}_k^i := \frac{u_k^i - u_k^{i-1}}{\tau_k}, \quad \dot{e}_k^i := \frac{e_k^i - e_k^{i-1}}{\tau_k}, \quad \dot{p}_k^i := \frac{p_k^i - p_k^{i-1}}{\tau_k}, \quad \dot{w}_k^i := \frac{w_k^i - w_k^{i-1}}{\tau_k}.$$

By arguing as in Lemma 3.7, we can derive the following discrete energy estimates.

**Lemma 4.2.** *Assume (2.2)–(2.7), (2.9), and (3.9). For all  $k \in \mathbb{N}$  and  $i = 1, \dots, k$  we have*

$$\mathcal{E}(\alpha_k^i, e_k^i, p_k^i) + \sum_{j=1}^i \tau_k \mathcal{H}(\dot{p}_k^j) + \frac{\varepsilon}{2} \sum_{j=1}^i \tau_k \|\dot{\alpha}_k^j\|_2^2 \leq \mathcal{E}(\alpha^0, e^0, p^0) + \sum_{j=1}^i \tau_k (\mathbb{C}e_k^{j-1}, \text{E}u_k^j)_2 + \delta_k,$$

where  $\delta_k \rightarrow 0$  as  $k \rightarrow \infty$  is defined as in (3.13). In particular, there exists a constant  $C > 0$  independent of  $\varepsilon, k, i$  such that

$$\max_{i=0, \dots, k} \|e_k^i\|_2 + \max_{i=0, \dots, k} \|p_k^i\|_{H^1} + \max_{i=0, \dots, k} \|\alpha_k^i\|_{H^1} + \sum_{i=1}^k \tau_k \mathcal{H}(\dot{p}_k^i) + \varepsilon \sum_{i=1}^k \tau_k \|\dot{\alpha}_k^i\|_2^2 \leq C. \quad (4.3)$$

*Proof.* It is enough to proceed as in the proof of Lemma 3.7.  $\square$

We now show a variational inequality for the quadruple  $(\alpha_k^i, u_k^i, e_k^i, p_k^i)$ , similarly to Lemma 3.11-(a).

**Lemma 4.3.** *Assume (2.2)–(2.7), (2.9), and (3.9). For all  $i = 1, \dots, k$  we have*

$$\partial_\alpha \mathcal{E}(\alpha_k^i, e_k^i, p_k^i)[\beta] + \varepsilon (\dot{\alpha}_k^i, \beta)_2 \geq 0 \quad \text{for all } \beta \in H^1(\Omega) \text{ with } \beta \leq 0. \quad (4.4)$$

Moreover

$$\partial_\alpha \mathcal{E}(\alpha_k^i, e_k^i, p_k^i)[\dot{\alpha}_k^i] + \varepsilon \|\dot{\alpha}_k^i\|_2^2 = 0. \quad (4.5)$$

*Proof.* We extend  $d$  and  $\mathbb{B}$  to the whole  $\mathbb{R}$  as in (3.21) and (3.22). We fix  $k \in \mathbb{N}$  and  $i = 1, \dots, k$ . Let  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  and let  $\lambda > 0$ . If we consider  $(\alpha_k^i + \lambda\beta)^+ \in \mathcal{D}(\alpha_k^{i-1})$ , by (4.1) we get

$$\mathcal{E}(\alpha_k^i, e_k^i, p_k^i) + \frac{\varepsilon}{2\tau_k} \|\alpha_k^i - \alpha_k^{i-1}\|_2^2 \leq \mathcal{E}((\alpha_k^i + \lambda\beta)^+, e_k^i, p_k^i) + \frac{\varepsilon}{2\tau_k} \|(\alpha_k^i + \lambda\beta)^+ - \alpha_k^{i-1}\|_2^2.$$

Moreover, by (2.2), (2.5), and the fact that  $\alpha_k^{i-1} \geq 0$  we have

$$\begin{aligned} D((\alpha_k^i + \lambda\beta)^+) &\leq D(\alpha_k^i + \lambda\beta) & \|\nabla((\alpha_k^i + \lambda\beta)^+)\|_2^2 &\leq \|\nabla(\alpha_k^i + \lambda\beta)\|_2^2, \\ \tilde{Q}((\alpha_k^i + \lambda\beta)^+, p_k^i) &\leq \tilde{Q}(\alpha_k^i + \lambda\beta, p_k^i) & \|(\alpha_k^i + \lambda\beta)^+ - \alpha_k^{i-1}\|_2^2 &\leq \|\alpha_k^i + \lambda\beta - \alpha_k^{i-1}\|_2^2. \end{aligned}$$

Hence

$$\mathcal{E}(\alpha_k^i, e_k^i, p_k^i) + \frac{\varepsilon}{2\tau_k} \|\alpha_k^i - \alpha_k^{i-1}\|_2^2 \leq \mathcal{E}(\alpha_k^i + \lambda\beta, e_k^i, p_k^i) + \frac{\varepsilon}{2\tau_k} \|\alpha_k^i + \lambda\beta - \alpha_k^{i-1}\|_2^2.$$

By dividing for  $\lambda > 0$ , sending  $\lambda \rightarrow 0^+$ , and using (3.26) we get (4.4).

If we use  $\beta = \dot{\alpha}_k^i$  in (4.4) we get the first inequality of (4.5). To get the other inequality, we fix  $\lambda \in (0, \tau_k)$  and we consider  $\alpha_k^i - \lambda \dot{\alpha}_k^i \in \mathcal{D}(\alpha_k^{i-1})$ . Hence, thanks to (4.1) we have

$$\mathcal{E}(\alpha_k^i, e_k^i, p_k^i) + \frac{\varepsilon}{2\tau_k} \|\alpha_k^i - \alpha_k^{i-1}\|_2^2 \leq \mathcal{E}(\alpha_k^i - \lambda \dot{\alpha}_k^i, e_k^i, p_k^i) + \frac{\varepsilon}{2\tau_k} \|\alpha_k^i - \alpha_k^{i-1} - \lambda \dot{\alpha}_k^i\|_2^2.$$

By dividing for  $\lambda > 0$ , sending  $\lambda \rightarrow 0^+$ , and using (3.26) we derive (4.5).  $\square$

**Remark 4.4.** As in Section 3, the assumptions  $\dot{d}(0) \leq 0$  and  $\mathbb{B}(0)\xi : \xi \leq 0$  for all  $\xi \in \mathbb{M}_{\text{sym}}^{n \times n}$  in (2.2) and (2.5) are needed to prove that (4.4) holds. Indeed, another way to prove Lemma 4.3 is to observe that, in view of (2.2) and (2.5), the quadruple  $(\alpha_k^i, u_k^i, e_k^i, p_k^i) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i)$  satisfies the following minimum problem

$$\min_{(\beta, v, \eta, q) \in H^1(\Omega) \times \mathcal{A}(w_k^i)} \left\{ \mathcal{E}(\beta, \eta, q) + \mathcal{H}(q - p_k^{i-1}) + \frac{\varepsilon}{2\tau_k} \|\beta - \alpha_k^{i-1}\|_2^2 + I_{(-\infty, 0]}(\beta - \alpha_k^{i-1}) \right\},$$

where  $I_{(-\infty, 0]} : H^1(\Omega) \rightarrow [0, \infty]$  is defined as

$$I_{(-\infty, 0]}(\alpha) := \begin{cases} 0 & \text{if } \alpha \leq 0, \\ \infty & \text{otherwise} \end{cases} \quad \text{for all } \alpha \in H^1(\Omega).$$

Therefore, it follows that

$$-\partial_\alpha \mathcal{E}(\alpha_k^i, e_k^i, p_k^i) - \frac{\varepsilon}{\tau_k} (\alpha_k^i - \alpha_k^{i-1}) \in \partial I_{(-\infty, 0]}(\alpha_k^i - \alpha_k^{i-1}), \quad (4.6)$$

where  $\partial I_{(-\infty, 0]}(\alpha) \subset (H^1(\Omega))'$  denotes the subdifferential of  $I_{(-\infty, 0]}$  in  $\alpha$ , defined as

$$\partial I_{(-\infty, 0]}(\alpha) := \{\eta \in (H^1(\Omega))' : I_{(-\infty, 0]}(\beta) \geq I_{(-\infty, 0]}(\alpha) + \langle \eta, \beta - \alpha \rangle \text{ for all } \beta \in H^1(\Omega)\}.$$

Since

$$\eta \in \partial I_{(-\infty, 0]}(\alpha) \iff \begin{cases} \langle \eta, \beta \rangle \leq 0 & \text{for all } \beta \in H^1(\Omega) \text{ with } \beta \leq 0, \\ \langle \eta, \alpha \rangle = 0, \end{cases}$$

from (4.6) we immediately derive (4.4) and (4.5).

**Remark 4.5.** Arguing as before, by (3.9) for all  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  we deduce

$$\partial_\alpha \mathcal{E}(\alpha^0, e^0, p^0)[\beta] \geq 0. \quad (4.7)$$

In the same spirit of Theorem 3.15, in the following lemma we show that we can estimate the norms of  $\dot{u}_k$ ,  $\dot{e}_k$ ,  $\dot{p}_k$  by the norm of  $\dot{\alpha}_k$  and  $\dot{w}_k$  times a constant independent of  $k$  and  $\varepsilon$ . The proof follows the one of [30], Lemma 3.6.

**Lemma 4.6.** *Assume (2.2)–(2.7), (2.9), and (3.9). Define for all  $k \in \mathbb{N}$  and  $i = 1, \dots, k$*

$$\omega_k^i := \|\alpha_k^i - \alpha_k^{i-1}\|_2 + \|Ew_k^i - Ew_k^{i-1}\|_2.$$

*There exists a constant  $C > 0$ , independent of  $\varepsilon, k, i$ , such that for all  $i = 1, \dots, k$  and  $k \in \mathbb{N}$*

$$\|e_k^i - e_k^{i-1}\|_2 + \|p_k^i - p_k^{i-1}\|_{H^1} \leq C\omega_k^i. \quad (4.8)$$

In particular, for all  $i = 1, \dots, k$  and  $k \in \mathbb{N}$  we have

$$\|u_k^i - u_k^{i-1}\|_{H^1} \leq C(\|w_k^i - w_k^{i-1}\|_2 + \omega_k^i). \quad (4.9)$$

Moreover, for all  $i = 1, \dots, k$  and  $k \in \mathbb{N}$

$$\mathcal{H}(\dot{p}_k) \leq (\mathbb{C}e_k^i, \dot{p}_k)_2 - 2(\mathbb{B}(\alpha_k^i)p_k^i, \dot{p}_k^i) - 2(\nabla p_k^i, \nabla \dot{p}_k^i)_2 + C\tau_k (\|\dot{\alpha}_k^i\|_2^2 + \|\mathbb{E}w_k^i\|_2^2). \quad (4.10)$$

*Proof.* We fix  $k \in \mathbb{N}$  and  $i = 1, \dots, k$ . If we test Lemma 4.1-(c) in  $i - 1$  with

$$(u_k^i - (w_k^i - w_k^{i-1}), e_k^i - \mathbb{E}(w_k^i - w_k^{i-1}), p_k^i) \in \mathcal{A}(w_k^{i-1}),$$

we get

$$\begin{aligned} & \mathcal{Q}(e_k^{i-1}) + \mathcal{Q}(e_k^i - e_k^{i-1} - \mathbb{E}(w_k^i - w_k^{i-1})) \\ & \quad + \tilde{Q}(\alpha_k^{i-1}, p_k^{i-1}) + \tilde{Q}(\alpha_k^{i-1}, p_k^i - p_k^{i-1}) + \|\nabla p_k^{i-1}\|_2^2 + \|\nabla p_k^i - \nabla p_k^{i-1}\|_2^2 \\ & \leq \mathcal{Q}(e_k^i - \mathbb{E}(w_k^i - w_k^{i-1})) + \tilde{Q}(\alpha_k^{i-1}, p_k^i) + \|\nabla p_k^i\|_2^2 + \mathcal{H}(p_k^i - p_k^{i-1}). \end{aligned}$$

In particular

$$\begin{aligned} & \mathcal{Q}(e_k^i - e_k^{i-1}) + \tilde{Q}(\alpha_k^{i-1}, p_k^i - p_k^{i-1}) + \|\nabla p_k^i - \nabla p_k^{i-1}\|_2^2 \\ & \leq \mathcal{Q}(e_k^i) - \mathcal{Q}(e_k^{i-1}) - (\mathbb{C}e_k^{i-1}, \mathbb{E}(w_k^i - w_k^{i-1})) \\ & \quad + \tilde{Q}(\alpha_k^{i-1}, p_k^i) - \tilde{Q}(\alpha_k^{i-1}, p_k^{i-1}) + \|\nabla p_k^i\|_2^2 - \|\nabla p_k^{i-1}\|_2^2 + \mathcal{H}(p_k^i - p_k^{i-1}). \end{aligned} \quad (4.11)$$

We now use the minimality of  $(\alpha_k^i, u_k^i, e_k^i, p_k^i) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i)$  with

$$(\alpha_k^i, u_k^{i-1} + (w_k^i - w_k^{i-1}), e_k^{i-1} + \mathbb{E}(w_k^i - w_k^{i-1}), p_k^{i-1}) \in \mathcal{D}(\alpha_k^{i-1}) \times \mathcal{A}(w_k^i),$$

and we get

$$\begin{aligned} & \mathcal{Q}(e_k^i) + \tilde{Q}(\alpha_k^i, p_k^i) + \|\nabla p_k^i\|_2^2 + \mathcal{H}(p_k^i - p_k^{i-1}) \\ & \leq \mathcal{Q}(e_k^{i-1} + \mathbb{E}(w_k^i - w_k^{i-1})) + \tilde{Q}(\alpha_k^i, p_k^{i-1}) + \|\nabla p_k^{i-1}\|_2^2. \end{aligned}$$

Hence

$$\begin{aligned} & \mathcal{H}(p_k^i - p_k^{i-1}) \\ & \leq \mathcal{Q}(e_k^{i-1} + \mathbb{E}(w_k^i - w_k^{i-1})) - \mathcal{Q}(e_k^i) + \tilde{Q}(\alpha_k^i, p_k^{i-1}) - \tilde{Q}(\alpha_k^i, p_k^i) + \|\nabla p_k^{i-1}\|_2^2 - \|\nabla p_k^i\|_2^2. \end{aligned} \quad (4.12)$$

By combining together (4.11) and (4.12), we deduce the existence of a constant  $C_1 > 0$ , independent of  $\varepsilon, k, i$ , such that

$$\begin{aligned} & \mathcal{Q}(e_k^i - e_k^{i-1}) + \tilde{Q}(\alpha_k^{i-1}, p_k^i - p_k^{i-1}) + \|\nabla p_k^i - \nabla p_k^{i-1}\|_2^2 \\ & \leq \mathcal{Q}(\mathbb{E}(w_k^i - w_k^{i-1})) + \tilde{Q}(\alpha_k^{i-1}, p_k^i) - \tilde{Q}(\alpha_k^i, p_k^i) - \tilde{Q}(\alpha_k^{i-1}, p_k^{i-1}) + \tilde{Q}(\alpha_k^i, p_k^{i-1}) \\ & \leq \frac{\gamma^2}{2} \|\mathbb{E}w_k^i - \mathbb{E}w_k^{i-1}\|_2^2 - ([\mathbb{B}(\alpha_k^i) - \mathbb{B}(\alpha_k^{i-1})](p_k^i - p_k^{i-1}), p_k^i + p_k^{i-1})_2 \\ & \leq C_1 (\|\mathbb{E}w_k^i - \mathbb{E}w_k^{i-1}\|_2^2 + \|\alpha_k^i - \alpha_k^{i-1}\|_2 \|p_k^i - p_k^{i-1}\|_{H^1}). \end{aligned} \quad (4.13)$$

Moreover, by Remarks 2.2 and 2.3 and the uniform estimate (4.3), there is a constant  $C_2 > 0$ , independent of  $\varepsilon, k, i$ , such that

$$\begin{aligned}
r_H \|p_k^i - p_k^{i-1}\|_1 &\leq \mathcal{H}(p_k^i - p_k^{i-1}) \\
&\leq \mathcal{Q}(e_k^i) - \mathcal{Q}(e_k^{i-1}) + \mathcal{Q}(\mathbb{E}(w_k^i - w_k^{i-1})) + (\mathbb{C}e_k^{i-1}, \mathbb{E}(w_k^i - w_k^{i-1}))_2 \\
&\quad + \tilde{\mathcal{Q}}(\alpha_k^i, p_k^{i-1}) - \tilde{\mathcal{Q}}(\alpha_k^{i-1}, p_k^{i-1}) + \tilde{\mathcal{Q}}(\alpha_k^{i-1}, p_k^i) - \tilde{\mathcal{Q}}(\alpha_k^i, p_k^i) \\
&\quad + \tilde{\mathcal{Q}}(\alpha_k^{i-1}, p_k^{i-1}) - \tilde{\mathcal{Q}}(\alpha_k^{i-1}, p_k^i) + \|\nabla p_k^{i-1}\|_2^2 - \|\nabla p_k^i\|_2^2 \\
&\leq C_2 \left( \sqrt{\mathcal{Q}(e_k^i - e_k^{i-1})} + \sqrt{\tilde{\mathcal{Q}}(\alpha_k^{i-1}, p_k^i - p_k^{i-1})} + \|\nabla p_k^i - \nabla p_k^{i-1}\|_2 \right) \\
&\quad + C_2 (\|\mathbb{E}(w_k^i - w_k^{i-1})\|_2 + \|\alpha_k^i - \alpha_k^{i-1}\|_2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\|p_k^i - p_k^{i-1}\|_1^2 \\
&\leq C_3 \left( \mathcal{Q}(e_k^i - e_k^{i-1}) + \tilde{\mathcal{Q}}(\alpha_k^{i-1}, p_k^i - p_k^{i-1}) + \|\nabla p_k^i - \nabla p_k^{i-1}\|_2^2 + \|\mathbb{E}(w_k^i - w_k^{i-1})\|_2^2 + \|\alpha_k^i - \alpha_k^{i-1}\|_2^2 \right) \\
&\leq C_4 (\|\alpha_k^i - \alpha_k^{i-1}\|_2 \|p_k^i - p_k^{i-1}\|_{H^1} + \|\mathbb{E}w_k^i - \mathbb{E}w_k^{i-1}\|_2^2 + \|\alpha_k^i - \alpha_k^{i-1}\|_2^2), \tag{4.14}
\end{aligned}$$

for two constants  $C_3, C_4 > 0$ , independent of  $\varepsilon, k, i$ . If we sum (4.13) and (4.14), and we use (2.4) and (2.13), we derive that

$$\begin{aligned}
\|e_k^i - e_k^{i-1}\|_2^2 + \|p_k^i - p_k^{i-1}\|_{H^1}^2 &\leq C_5 (\|\alpha_k^i - \alpha_k^{i-1}\|_2 \|p_k^i - p_k^{i-1}\|_{H^1} + \|\mathbb{E}w_k^i - \mathbb{E}w_k^{i-1}\|_2^2 + \|\alpha_k^i - \alpha_k^{i-1}\|_2^2) \\
&\leq \frac{1}{2} \|p_k^i - p_k^{i-1}\|_{H^1}^2 + C_6 (\|\mathbb{E}w_k^i - \mathbb{E}w_k^{i-1}\|_2^2 + \|\alpha_k^i - \alpha_k^{i-1}\|_2^2),
\end{aligned}$$

for two constants  $C_5, C_6 > 0$ , independent of  $\varepsilon, k, i$ . This gives (4.8). Moreover, to obtain (4.9) it is enough to combine (4.8) with Korn Inequality.

Finally, by combining (4.12) and (3.29), with (4.8) and the identity

$$|a|^2 - |b|^2 = -2b \cdot (b - a) + |b - a|^2 \quad \text{for all } a, b \in \mathbb{R}^d,$$

we obtain

$$\begin{aligned}
&\mathcal{H}(p_k^i - p_k^{i-1}) \\
&\leq (\mathbb{C}e_k^i, \mathbb{E}(w_k^i - w_k^{i-1}) - (e_k^i - e_k^{i-1})) + \mathcal{Q}(\mathbb{E}(w_k^i - w_k^{i-1})) \\
&\quad - (\mathbb{C}(e_k^i - e_k^{i-1}), \mathbb{E}(w_k^i - w_k^{i-1}))_2 + \mathcal{Q}(e_k^i - e_k^{i-1}) \\
&\quad - 2(\mathbb{B}(\alpha_k^i)p_k^i, p_k^i - p_k^{i-1})_2 + \tilde{\mathcal{Q}}(p_k^i - p_k^{i-1}) - 2(\nabla p_k^i, \nabla p_k^i - \nabla p_k^{i-1})_2 + \|\nabla p_k^i - \nabla p_k^{i-1}\|_2^2 \\
&\leq (\mathbb{C}e_k^i, p_k^i - p_k^{i-1})_2 - 2(\mathbb{B}(\alpha_k^i)p_k^i, p_k^i - p_k^{i-1})_2 - 2(\nabla p_k^i, \nabla p_k^i - \nabla p_k^{i-1})_2 \\
&\quad + C (\|\alpha_k^i - \alpha_k^{i-1}\|_2^2 + \|\mathbb{E}w_k^i - \mathbb{E}w_k^{i-1}\|_2^2),
\end{aligned}$$

which gives (4.10).  $\square$

**Remark 4.7.** We can rephrase Lemma 4.6 in the following way: there is a constant  $C$ , independent of  $\varepsilon, k, i$ , such that for all  $k \in \mathbb{N}$  and  $i = 1, \dots, k$

$$\|\dot{e}_k^i\|_2 + \|\dot{p}_k^i\|_{H^1} \leq C (\|\dot{\alpha}_k^i\|_{H^1} + \|\mathbb{E}\dot{w}_k^i\|_2), \quad \|\dot{u}_k^i\|_{H^1} \leq C (\|\dot{\alpha}_k^i\|_{H^1} + \|\dot{w}_k^i\|_{H^1}).$$

We now estimate, by means of the following two lemmas, the regularity in time of the viscously regularized evolutions, taking value in the target spaces of the internal variables, with respect to the  $H^1(0, T)$  norm (for fixed  $\varepsilon \in (0, 1)$ ) and the  $W^{1,1}(0, T)$  norm (uniform in  $\varepsilon \in (0, 1)$ ).

**Lemma 4.8.** *Assume (2.2)–(2.7), (2.9), and (3.9). There exists a constant  $C > 0$ , independent of  $\varepsilon, k, i$ , such that*

$$\|\dot{\alpha}_k^i\|_2^2 + \sum_{j=1}^i \tau_k \|\dot{\alpha}_k^j\|_{H^1}^2 \leq \frac{C}{\varepsilon} e^{\frac{C}{\varepsilon} t_k} \quad \text{for all } k \geq \frac{C}{\varepsilon} \text{ and } i = 1, \dots, k. \quad (4.15)$$

*Proof.* We fix  $k \in \mathbb{N}$ . By the uniform estimate (4.3), the identity (4.5) with  $i = 1$ , the variational inequality (4.7) with  $\beta = \dot{\alpha}_k^1$ , the (4.8), and Remark 2.3 we have

$$\begin{aligned} \varepsilon \|\dot{\alpha}_k^1\|_2^2 + 2\tau_k \|\dot{\alpha}_k^1\|_{H^1}^2 &\leq \partial_\alpha \mathcal{E}(\alpha_k^0, e_k^0, p_k^0)[\dot{\alpha}_k^1] - \partial_\alpha \mathcal{E}(\alpha_k^1, e_k^1, p_k^1)[\dot{\alpha}_k^1] + 2\tau_k \|\dot{\alpha}_k^1\|_{H^1}^2 \\ &= \int_\Omega [d(\alpha_k^0(x)) - d(\alpha_k^1(x))] \dot{\alpha}_k^1(x) \, dx + 2\tau_k \|\dot{\alpha}_k^1\|_2^2 \\ &\quad + 2 \int_\Omega [\mathbb{B}(\alpha_k^0(x)) - \mathbb{B}(\alpha_k^1(x))] \dot{\alpha}_k^1(x) p_k^0(x) : p_k^0(x) \, dx \\ &\quad - 2 \int_\Omega \mathbb{B}(\alpha_k^1(x)) \dot{\alpha}_k^1(x) (p_k^1(x) - p_k^0(x)) : (p_k^1(x) + p_k^0(x)) \, dx \\ &\leq C_1 \tau_k (\|\dot{\alpha}_k^1\|_2^2 + \|\dot{\alpha}_k^1\|_4^2 + \|\dot{\alpha}_k^1\|_2 \|p_k^1 - p_k^0\|_{H^1}) \\ &\leq C_2 \tau_k (\|\dot{\alpha}_k^1\|_4^2 + \|\mathbb{E}w_k^1\|_2^2), \end{aligned} \quad (4.16)$$

for two constants  $C_1, C_2 > 0$ , independent of  $\varepsilon, k$ . Similarly, for  $j = 2, \dots, k$  we use (4.5) in  $j$  and (4.4) in  $j - 1$  with  $\beta = \dot{\alpha}_k^j$ , and we get

$$\begin{aligned} \varepsilon (\dot{\alpha}_k^j, \dot{\alpha}_k^j - \dot{\alpha}_k^{j-1})_2 + 2\tau_k \|\dot{\alpha}_k^j\|_{H^1}^2 &\leq \partial_\alpha \mathcal{E}(\alpha_k^{j-1}, e_k^{j-1}, p_k^{j-1})[\dot{\alpha}_k^j] - \partial_\alpha \mathcal{E}(\alpha_k^j, e_k^j, p_k^j)[\dot{\alpha}_k^j] + 2\tau_k \|\dot{\alpha}_k^j\|_{H^1}^2 \\ &= \int_\Omega [d(\alpha_k^{j-1}(x)) - d(\alpha_k^j(x))] \dot{\alpha}_k^j(x) \, dx + 2\tau_k \|\dot{\alpha}_k^j\|_2^2 \\ &\quad + 2 \int_\Omega [\mathbb{B}(\alpha_k^{j-1}(x)) - \mathbb{B}(\alpha_k^j(x))] \dot{\alpha}_k^j(x) p_k^{j-1}(x) : p_k^{j-1}(x) \, dx \\ &\quad - 2 \int_\Omega \mathbb{B}(\alpha_k^j(x)) \dot{\alpha}_k^j(x) (p_k^j(x) - p_k^{j-1}(x)) : (p_k^j(x) + p_k^{j-1}(x)) \, dx \\ &\leq C_3 \tau_k (\|\dot{\alpha}_k^j\|_2^2 + \|\dot{\alpha}_k^j\|_4^2 + \|\dot{\alpha}_k^j\|_2 \|p_k^j - p_k^{j-1}\|_{H^1}) \\ &\leq C_4 \tau_k (\|\dot{\alpha}_k^j\|_4^2 + \|\mathbb{E}w_k^j\|_2^2), \end{aligned} \quad (4.17)$$

for two constants  $C_3, C_4 > 0$ , independent of  $\varepsilon, k, j$ . We now use the estimate

$$\varepsilon (\dot{\alpha}_k^j, \dot{\alpha}_k^j - \dot{\alpha}_k^{j-1})_2 \geq \frac{\varepsilon}{2} \|\dot{\alpha}_k^j\|_2^2 - \frac{\varepsilon}{2} \|\dot{\alpha}_k^{j-1}\|_2^2 \quad \text{for all } i = 2, \dots, k,$$

and we sum the two above inequalities over  $j = 1, \dots, i$  to derive

$$\varepsilon \|\dot{\alpha}_k^i\|_2^2 + \sum_{j=1}^i \tau_k \|\dot{\alpha}_k^j\|_{H^1}^2 \leq C_5 \left( 1 + \sum_{j=1}^i \tau_k \|\dot{\alpha}_k^j\|_4^2 \right) \quad \text{for all } i = 1 \dots, k,$$

for a constant  $C_5 > 0$ , independent of  $\varepsilon, k, i$ . By Lemma 2.7 with  $\theta = 4$  for all  $\lambda > 0$  there exists a constant  $C_\lambda > 0$  such that

$$\|\alpha\|_4^2 \leq \lambda \|\nabla \alpha\|_2^2 + C_\lambda \|\alpha\|_2^2 \quad \text{for all } \alpha \in H^1(\Omega).$$

Hence, there exists a constant  $C_6 > 0$ , independent of  $\varepsilon, k, i$ , such that

$$\varepsilon \|\dot{\alpha}_k^i\|_2^2 + \sum_{j=1}^i \tau_k \|\dot{\alpha}_k^j\|_{H^1}^2 \leq C_6 \left( 1 + \sum_{j=1}^i \tau_k \|\dot{\alpha}_k^j\|_2^2 \right). \quad (4.18)$$

Therefore, by applying the discrete version of Gronwall Inequality of [36], Lemma 3.2.4, we derive that

$$\|\dot{\alpha}_k^i\|_2^2 \leq \frac{2C_6}{\varepsilon} e^{\frac{2C_6}{\varepsilon} t_k^i} \quad \text{for all } k > \frac{2TC_6}{\varepsilon} \text{ and } i = 1, \dots, k.$$

Finally, if we combine the above inequality with (4.18) we obtain (4.15).  $\square$

**Lemma 4.9.** *Assume (2.2)–(2.7), (2.9), and (3.9). There exists a constant  $C > 0$ , independent of  $\varepsilon, k, i$ , such that*

$$\sum_{i=1}^k \tau_k \|\dot{\alpha}_k^i\|_{H^1} \leq C \quad \text{for all } k \in \mathbb{N}. \quad (4.19)$$

*Proof.* We fix  $k \in \mathbb{N}$ . If we set  $\dot{\alpha}_k^0 := 0$ , by (4.16), (4.17), and the inequality

$$\varepsilon (\dot{\alpha}_k^j, \dot{\alpha}_k^j - \dot{\alpha}_k^{j-1})_2 \geq \varepsilon \|\dot{\alpha}_k^j\|_2 (\|\dot{\alpha}_k^j\|_2 - \|\dot{\alpha}_k^{j-1}\|_2) \quad \text{for all } j = 1, \dots, k,$$

we can find a constant  $C_1 > 0$ , independent of  $\varepsilon, k, j$ , such that

$$\varepsilon \|\dot{\alpha}_k^j\|_2 (\|\dot{\alpha}_k^j\|_2 - \|\dot{\alpha}_k^{j-1}\|_2) + 2\tau_k \|\dot{\alpha}_k^j\|_{H^1}^2 \leq C_1 \tau_k (\|\dot{\alpha}_k^j\|_4^2 + \|\mathbf{E}w_k^j\|_2^2) \quad \text{for all } j = 1, \dots, k.$$

By Lemma 2.7 with  $\theta = 4$ , for all  $\lambda > 0$  there exist  $C_\lambda, \tilde{C}_\lambda > 0$  such that

$$\|\alpha\|_4^2 \leq \lambda \|\nabla \alpha\|_2^2 + C_\lambda \|\alpha\|_1^2 \leq \lambda \|\nabla \alpha\|_2^2 + \tilde{C}_\lambda \|\alpha\|_1 \|\alpha\|_2 \quad \text{for all } \alpha \in H^1(\Omega).$$

Hence, we deduce that

$$\varepsilon \|\dot{\alpha}_k^j\|_2 (\|\dot{\alpha}_k^j\|_2 - \|\dot{\alpha}_k^{j-1}\|_2) + \tau_k \|\dot{\alpha}_k^j\|_{H^1}^2 \leq C_2 \tau_k (\|\dot{\alpha}_k^j\|_1 \|\dot{\alpha}_k^j\|_2 + \|\mathbf{E}w_k^j\|_2^2) \quad \text{for all } j = 1, \dots, k,$$

for a constant  $C_2 > 0$ , independent of  $\varepsilon, k, j$ . By multiplying the above inequality by  $\frac{2}{\varepsilon}$  and taking into account that  $\|\dot{\alpha}_k^j\|_{H^1} \geq \|\dot{\alpha}_k^j\|_2$ , we derive that

$$2\|\dot{\alpha}_k^j\|_2 (\|\dot{\alpha}_k^j\|_2 - \|\dot{\alpha}_k^{j-1}\|_2) + \frac{\tau_k}{\varepsilon} \|\dot{\alpha}_k^j\|_2^2 + \frac{\tau_k}{\varepsilon} \|\dot{\alpha}_k^j\|_{H^1}^2 \leq \frac{2C_2\tau_k}{\varepsilon} (\|\dot{\alpha}_k^j\|_1 \|\dot{\alpha}_k^j\|_2 + \|\mathbf{E}w_k^j\|_2^2) \quad \text{for all } j = 1, \dots, k.$$

From now on, the proof follows closely the proof of [30], Proposition 3.8, employing in particular the discrete Gronwall Inequality with weights from [29], Lemma 4.1. We detail all the passages below, for reader's convenience.



Let us set  $a_0 := \|\dot{\alpha}_k^0\|_2 = 0$  and for  $j = 1, \dots, k$

$$\zeta := \frac{\tau_k}{2\varepsilon}, \quad a_j := \|\dot{\alpha}_k^j\|_2, \quad b_j := \sqrt{\frac{\tau_k}{\varepsilon}} \|\dot{\alpha}_k^j\|_{H^1}, \quad c_j := \sqrt{\frac{2C_2\tau_k}{\varepsilon}} \|\mathbb{E}\dot{w}_k^j\|_2, \quad d_j := \frac{C_2\tau_k}{\varepsilon} \|\dot{\alpha}_k^j\|_1.$$

Therefore, we can rephrase the above inequality as

$$2a_j(a_j - a_{j-1}) + 2\zeta a_j^2 + b_j^2 \leq c_j^2 + 2a_j d_j \quad \text{for all } j = 1, \dots, k.$$

Hence, we can apply [29], Lemma 4.1 and we use that  $a_0 = 0$  to deduce

$$\sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} b_h^2 \leq 2 \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} c_h^2 + 4 \left( \sum_{h=1}^j (1 + \zeta)^{h-j-1} d_h \right)^2 \quad \text{for all } j = 1, \dots, k.$$

In particular, for all  $j = 1, \dots, k$  we derive

$$2\zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} \|\dot{\alpha}_k^h\|_{H^1}^2 \leq 8C_2\zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} \|\mathbb{E}w_k^h\|_2^2 + \left( 4C_2\zeta \sum_{h=1}^j (1 + \zeta)^{h-j-1} \|\dot{\alpha}_k^h\|_1 \right)^2.$$

We now use the inequality

$$a^2 + b^2 \leq (a + b)^2 \leq (1 + a^2 + b^2) \quad \text{for all } a, b \geq 0,$$

to estimate from above the right-hand side by

$$\left( 1 + 8C_2\zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} \|\mathbb{E}w_k^h\|_2^2 + 4C_2\zeta \sum_{h=1}^j (1 + \zeta)^{h-j-1} \|\dot{\alpha}_k^h\|_1 \right)^2.$$

Moreover, for all  $j = 1, \dots, k$  we have

$$\begin{aligned} \zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} \|\dot{\alpha}_k^h\|_{H^1} &\leq \left( \zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} \right)^{\frac{1}{2}} \left( \zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} \|\dot{\alpha}_k^h\|_{H^1}^2 \right)^{\frac{1}{2}} \\ &\leq \left( \zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} \|\dot{\alpha}_k^h\|_{H^1}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

since

$$\zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} = \frac{\zeta}{1 + \zeta} \sum_{h=0}^{j-1} (1 + \zeta)^{-2h} = \frac{\zeta(1 + \zeta)(1 - (1 + \zeta)^{-2j})}{(1 + \zeta)^2 - 1} \leq \frac{1 + \zeta}{2 + \zeta} \leq 1. \quad (4.20)$$

Therefore, by combining the previous inequalities we deduce that for  $j = 1, \dots, k$

$$\zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} \|\dot{\alpha}_k^h\|_{H^1} \leq C_3 \left( 1 + \zeta \sum_{h=1}^j (1 + \zeta)^{2(h-j)-1} \|\mathbb{E}w_k^h\|_2^2 + \zeta \sum_{h=1}^j (1 + \zeta)^{h-j-1} \|\dot{\alpha}_k^h\|_1 \right) \quad (4.21)$$

for a constant  $C_3 > 0$  independent of  $\varepsilon, k, j$ . We now multiply both sides by  $\tau_k$  and we sum over  $j = 1, \dots, k$ , to obtain

$$\begin{aligned} & \zeta \sum_{j=1}^k \sum_{h=1}^j \tau_k (1 + \zeta)^{2(h-j)-1} \|\dot{\alpha}_k^h\|_{H^1} \\ & \leq C_3 \left( T + \zeta \sum_{j=1}^k \sum_{h=1}^j \tau_k (1 + \zeta)^{2(h-j)-1} \|\mathbb{E} \dot{w}_k^h\|_2^2 + \zeta \sum_{j=1}^k \sum_{h=1}^j \tau_k (1 + \zeta)^{h-j-1} \|\dot{\alpha}_k^h\|_1 \right). \end{aligned} \quad (4.22)$$

If we change the order of the sum and we argue as in (4.20), we derive

$$\zeta \sum_{j=1}^k \sum_{h=1}^j \tau_k (1 + \zeta)^{2(h-j)-1} \|\mathbb{E} \dot{w}_k^h\|_2^2 \leq \frac{1 + \zeta}{2 + \zeta} \sum_{h=1}^k \tau_k \|\mathbb{E} \dot{w}_k^h\|_2^2 \leq \sum_{h=1}^k \tau_k \|\mathbb{E} \dot{w}_k^h\|_2^2, \quad (4.23)$$

$$\zeta \sum_{j=1}^k \sum_{h=1}^j \tau_k (1 + \zeta)^{h-j-1} \|\dot{\alpha}_k^h\|_1 \leq \frac{1}{1 + \zeta} \sum_{h=1}^k \tau_k \|\dot{\alpha}_k^h\|_1 \leq \sum_{h=1}^k \tau_k \|\dot{\alpha}_k^h\|_1, \quad (4.24)$$

$$\zeta \sum_{j=1}^k \sum_{h=1}^j \tau_k (1 + \zeta)^{2(h-j)-1} \|\dot{\alpha}_k^h\|_{H^1} = \frac{1 + \zeta}{2 + \zeta} \sum_{h=1}^k \tau_k (1 - (1 + \zeta)^{2(h-k-1)}) \|\dot{\alpha}_k^h\|_{H^1}. \quad (4.25)$$

Hence, by combining (4.22)–(4.25) we derive

$$\frac{1}{2} \sum_{h=1}^k \tau_k \|\dot{\alpha}_k^h\|_{H^1} \leq C_3 \left( T + \sum_{h=1}^k \tau_k \|\mathbb{E} \dot{w}_k^h\|_2^2 + \sum_{h=1}^k \tau_k \|\dot{\alpha}_k^h\|_1 \right) + \sum_{h=1}^k \tau_k (1 + \zeta)^{2(h-k-1)} \|\dot{\alpha}_k^h\|_{H^1}.$$

Since  $\tau_k = 2\varepsilon\zeta$ , by (4.21) we have

$$\begin{aligned} \sum_{h=1}^k \tau_k (1 + \zeta)^{2(h-k-1)} \|\dot{\alpha}_k^h\|_{H^1} &= \frac{2\varepsilon}{1 + \zeta} \zeta \sum_{h=1}^k (1 + \zeta)^{2(h-k)-1} \|\dot{\alpha}_k^h\|_{H^1} \\ &\leq C_3 \left( 2\varepsilon + \sum_{h=1}^k \tau_k \|\mathbb{E} \dot{w}_k^h\|_2^2 + \sum_{h=1}^k \tau_k \|\dot{\alpha}_k^h\|_1 \right), \end{aligned}$$

which gives

$$\sum_{h=1}^k \tau_k \|\dot{\alpha}_k^h\|_{H^1} \leq C_4 \left( 1 + \sum_{h=1}^k \tau_k \|\mathbb{E} \dot{w}_k^h\|_2^2 + \sum_{h=1}^k \tau_k \|\dot{\alpha}_k^h\|_1 \right)$$

for a constant  $C_4 > 0$  independent of  $\varepsilon, i, k$ , since  $\varepsilon \in (0, 1)$ . Finally, we observe that

$$\sum_{h=1}^k \tau_k \|\dot{\alpha}_k^h\|_1 = \sum_{h=1}^k \int_{\Omega} (\alpha_k^{h-1}(x) - \alpha_k^h(x)) \, dx = \int_{\Omega} (\alpha_k^0(x) - \alpha_k^k(x)) \, dx \leq \mathcal{L}^n(\Omega).$$

Hence, we obtain (4.19).  $\square$

As done in the previous section in (3.15) and (3.16), we introduce the *piecewise constant interpolants*  $\bar{\alpha}_k: [0, T] \rightarrow H^1(\Omega; [0, 1])$ ,  $\bar{u}_k: [0, T] \rightarrow H^1(\Omega; \mathbb{R}^n)$ ,  $\bar{e}_k: [0, T] \rightarrow L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ ,  $\bar{p}_k: [0, T] \rightarrow H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ ,  $\bar{w}_k: [0, T] \rightarrow H^1(\Omega; \mathbb{R}^n)$ , and  $\bar{\tau}_k: [0, T] \rightarrow [0, T]$ . Moreover, we define the *piecewise affine interpolant*  $\alpha_k \in H^1((0, T); H^1(\Omega; [0, 1]))$  as

$$\alpha_k(t) := \alpha_k^{i-1} + (t - t_k^{i-1})\dot{\alpha}_k^i \quad \text{for } t \in [t_k^{i-1}, t_k^i], \quad i = 1, \dots, k,$$

and similarly  $u_k \in H^1((0, T); H^1(\Omega; \mathbb{R}^n))$ ,  $e_k \in H^1((0, T); L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ , and  $p_k \in H^1((0, T); H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ .

By using the piecewise affine interpolants, we can rephrase Lemmas 4.8 and 4.9 in the following way: there exists a constant  $C$ , independent of  $k$  and  $\varepsilon$ , such that

$$\int_0^T (\|\dot{\alpha}_k(r)\|_{H^1}^2 + \|\dot{u}_k(r)\|_{H^1}^2 + \|\dot{e}_k(r)\|_2^2 + \|\dot{p}_k(r)\|_{H^1}^2) \, dr \leq \frac{C}{\varepsilon} e^{\frac{C}{\varepsilon}} \quad \text{for all } k \geq \frac{C}{\varepsilon}, \quad (4.26)$$

$$\int_0^T (\|\dot{\alpha}_k(r)\|_{H^1} + \|\dot{u}_k(r)\|_{H^1} + \|\dot{e}_k(r)\|_2 + \|\dot{p}_k(r)\|_{H^1}) \, dr \leq C \quad \text{for all } k \in \mathbb{N}. \quad (4.27)$$

## 4.2. Passage to the limit

In this section we fix  $\varepsilon \in (0, 1)$  and we shall send  $k \rightarrow \infty$  to find an  $\varepsilon$ -approximate viscous evolution according to the definition below.

**Definition 4.10.** Assume (2.2)–(2.7) and (2.9). Let  $\varepsilon \in (0, 1)$  be fixed. A quadruple  $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$  from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$  is an  $\varepsilon$ -approximate viscous evolution with datum  $w$  if

$$\alpha_\varepsilon \in H^1(0, T; H^1(\Omega; [0, 1])), \quad u_\varepsilon \in H^1(0, T; H^1(\Omega; \mathbb{R}^n)), \quad (4.28)$$

$$e_\varepsilon \in H^1(0, T; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})), \quad p_\varepsilon \in H^1(0, T; H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})), \quad (4.29)$$

and the following conditions are satisfied:

(ev0) $_\varepsilon$  *irreversibility*: for a.e.  $x \in \Omega$  the map

$$t \mapsto \alpha_\varepsilon(t, x) \quad \text{is non-increasing on } [0, T];$$

(ev1) $_\varepsilon$  *kinematic condition and equilibrium*: for all  $t \in [0, T]$

$$(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in \mathcal{A}(w(t)) \quad \text{and} \quad \text{div } \mathbb{C}e_\varepsilon(t) = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n);$$

(ev2) $_\varepsilon$  *stress constraint*: for all  $t \in [0, T]$

$$\mathcal{H}(q) \geq (\mathbb{C}e_\varepsilon(t), q)_2 - 2(\mathbb{B}(\alpha_\varepsilon(t))p_\varepsilon(t), q)_2 - 2(\nabla p_\varepsilon(t), \nabla q)_2 \quad \text{for all } q \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n});$$

(ev3) $_\varepsilon$  *Kuhn–Tucker Inequality*: for a.e.  $t \in [0, T]$

$$\partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))[\beta] + \varepsilon(\dot{\alpha}_\varepsilon(t), \beta)_2 \geq 0 \quad \text{for all } \beta \in H^1(\Omega) \text{ with } \beta \leq 0;$$

(ev4) $_\varepsilon$  *energy balance*: for all  $t \in [0, T]$

$$\mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) + \int_0^t \mathcal{H}(\dot{p}_\varepsilon(r)) \, dr + \varepsilon \int_0^t \|\dot{\alpha}_\varepsilon(r)\|_2^2 \, dr = \mathcal{E}(\alpha_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) + \int_0^t (\mathbb{C}e_\varepsilon(r), E\dot{w}(r))_2 \, dr.$$

Condition  $(ev2)_\varepsilon$  is a generalization of the standard stress constraint in plasticity. Indeed, we have the following result.

**Lemma 4.11.** *Assume (2.2)–(2.7) and (2.9). Let  $(\alpha, u, e, p) \in H^1(\Omega; [0, 1]) \times \mathcal{A}$  be satisfying*

$$\mathcal{H}(q) \geq (\mathbb{C}e, q)_2 - 2(\mathbb{B}(\alpha)p, q)_2 - 2(\nabla p, \nabla q)_2 \quad \text{for all } q \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}). \quad (4.30)$$

*If  $p \in H^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$  and  $K$  is compact, then  $(\alpha, u, e, p) \in H^1(\Omega; [0, 1]) \times \mathcal{A}$  satisfies*

$$\mathbb{C}e(x) - 2\mathbb{B}(\alpha(x))p(x) + 2\Delta p(x) \in K \quad \text{for a.e. } x \in \Omega.$$

*Proof.* Let  $q \in C_c^\infty(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ . By integrating by parts in (4.30) we get

$$\mathcal{H}(q) \geq (\mathbb{C}e - 2\mathbb{B}(\alpha)p + 2\Delta p, q)_2 \quad \text{for all } q \in C_c^\infty(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

Since  $K$  is bounded, there exists  $R > 0$  such that  $K \subset B_R(0)$ . Then

$$\mathcal{H}(q) \leq R\|q\|_1 \quad \text{for all } q \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}),$$

which implies that  $\mathcal{H}$  is continuous on  $L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ . Therefore, by a density argument we derive

$$\mathcal{H}(q) \geq (\mathbb{C}e - 2\mathbb{B}(\alpha)p + 2\Delta p, q)_2 \quad \text{for all } q \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

Let us fix  $\xi \in \mathbb{M}_{\text{sym}}^{n \times n}$ . Since for all measurable set  $B \subset \Omega$  we can take  $q := \chi_B \xi \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ , we derive

$$H(\xi) \geq [\mathbb{C}e(x) - 2\mathbb{B}(\alpha(x))p(x) + 2\Delta p(x)] : \xi \quad \text{for all } \xi \in \mathbb{M}_{\text{sym}}^{n \times n} \text{ and for a.e. } x \in \Omega.$$

Therefore,  $\mathbb{C}e(x) - 2\mathbb{B}(\alpha(x))p(x) + 2\Delta p(x) \in \partial H(0) = K$  for a.e.  $x \in \Omega$ .  $\square$

The following proposition is instrumental to prove existence of  $\varepsilon$ -approximate viscous evolutions.

**Proposition 4.12.** *Assume (2.2)–(2.7) and (2.9). Let  $\varepsilon \in (0, 1)$  be fixed and let  $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$  be satisfying (4.28)–(4.29),  $(ev0)_\varepsilon$ ,  $(ev1)_\varepsilon$ ,  $(ev2)_\varepsilon$ , and  $(ev3)_\varepsilon$ . Then  $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$  is an  $\varepsilon$ -approximate viscous evolution, i.e., it satisfies the energy balance  $(ev4)_\varepsilon$ , if and only if any of the conditions below holds true:*

$(ev4')_\varepsilon$  for a.e.  $t \in [0, T]$  the following hold:

*Kuhn–Tucker equality:*

$$\partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))[\dot{\alpha}_\varepsilon(t)] + \varepsilon \|\dot{\alpha}_\varepsilon(t)\|_2^2 = 0;$$

*Hill's maximum plastic work principle:*

$$\mathcal{H}(\dot{p}_\varepsilon(t)) = (\mathbb{C}e_\varepsilon(t), \dot{p}_\varepsilon(t))_2 - 2(\mathbb{B}(\alpha_\varepsilon(t))p_\varepsilon(t), \dot{p}_\varepsilon(t))_2 - 2(\nabla p_\varepsilon(t), \nabla \dot{p}_\varepsilon(t))_2;$$

$(ev4'')_\varepsilon$  energy inequality: for all  $t \in [0, T]$

$$\mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) + \int_0^t \mathcal{H}(\dot{p}_\varepsilon(r)) \, dr + \varepsilon \int_0^t \|\dot{\alpha}_\varepsilon(r)\|_2^2 \, dr \leq \mathcal{E}(\alpha_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) + \int_0^t (\mathbb{C}e_\varepsilon(r), E\dot{w}(r))_2 \, dr.$$

*Proof.*  $(ev4)_\varepsilon \iff (ev4')_\varepsilon$ . Since  $\alpha_\varepsilon$ ,  $e_\varepsilon$ , and  $p_\varepsilon$  satisfy (4.28) and (4.29), then  $t \mapsto \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))$  is absolutely continuous on  $[0, T]$ , and for a.e.  $t \in [0, T]$  we have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) &= \partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))[\dot{\alpha}_\varepsilon(t)] + (\mathbb{C}e_\varepsilon(t), \dot{e}_\varepsilon(t))_2 \\ &\quad + 2(\nabla p_\varepsilon(t), \nabla \dot{p}_\varepsilon(t))_2 + 2(\mathbb{B}(\alpha_\varepsilon(t))p_\varepsilon(t), \dot{p}_\varepsilon(t))_2. \end{aligned}$$

By  $(ev1)_\varepsilon$  we deduce that  $(\dot{u}_\varepsilon(t), \dot{e}_\varepsilon(t), \dot{p}_\varepsilon(t)) \in \mathcal{A}(\dot{w}(t))$  and

$$(\mathbb{C}e_\varepsilon(t), \dot{e}_\varepsilon(t)) = (\mathbb{C}e_\varepsilon(t), E\dot{w}(t) - \dot{p}_\varepsilon(t))_2 \quad \text{for a.e. } t \in [0, T].$$

Hence, we derive that for a.e.  $t \in [0, T]$

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) &= \partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))[\dot{\alpha}_\varepsilon(t)] + (\mathbb{C}e_\varepsilon(t), E\dot{w}(t))_2 \\ &\quad - (\mathbb{C}e_\varepsilon(t), \dot{p}_\varepsilon(t))_2 + 2(\nabla p_\varepsilon(t), \nabla \dot{p}_\varepsilon(t))_2 + 2(\mathbb{B}(\alpha_\varepsilon(t))p_\varepsilon(t), \dot{p}_\varepsilon(t))_2. \end{aligned} \quad (4.31)$$

Notice that  $(ev4)_\varepsilon$  holds if and only if for a.e.  $t \in [0, T]$

$$\frac{d}{dt} \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) = -\mathcal{H}(\dot{p}_\varepsilon(t)) - \varepsilon \|\dot{\alpha}_\varepsilon(t)\|_2^2 + (\mathbb{C}e_\varepsilon(t), E\dot{w}(t))_2.$$

Therefore,  $(ev4)_\varepsilon$  is equivalent to the following identity: for a.e.  $t \in [0, T]$

$$\begin{aligned} &\mathcal{H}(\dot{p}_\varepsilon(t)) + \varepsilon \|\dot{\alpha}_\varepsilon(t)\|_2^2 \\ &= -\partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))[\dot{\alpha}_\varepsilon(t)] + (\mathbb{C}e_\varepsilon(t), \dot{p}_\varepsilon(t))_2 - 2(\nabla p_\varepsilon(t), \nabla \dot{p}_\varepsilon(t))_2 - 2(\mathbb{B}(\alpha_\varepsilon(t))p_\varepsilon(t), \dot{p}_\varepsilon(t))_2. \end{aligned}$$

By  $(ev2)_\varepsilon$  and  $(ev3)_\varepsilon$  we have

$$\mathcal{H}(\dot{p}_\varepsilon(t)) \geq (\mathbb{C}e_\varepsilon(t), \dot{p}_\varepsilon(t))_2 - 2(\nabla p_\varepsilon(t), \nabla \dot{p}_\varepsilon(t))_2 - 2(\mathbb{B}(\alpha_\varepsilon(t))p_\varepsilon(t), \dot{p}_\varepsilon(t))_2 \quad \text{for a.e. } t \in [0, T], \quad (4.32)$$

$$\varepsilon \|\dot{\alpha}_\varepsilon(t)\|_2^2 \geq -\partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))[\dot{\alpha}_\varepsilon(t)] \quad \text{for a.e. } t \in [0, T]. \quad (4.33)$$

Hence, we derive that  $(ev4)_\varepsilon$  is equivalent to  $(ev4')_\varepsilon$ .

$(ev4)_\varepsilon \iff (ev4'')_\varepsilon$ . Clearly  $(ev4)_\varepsilon$  implies  $(ev4'')_\varepsilon$ . Let us prove the other implication. By (4.31), (4.32), and (4.33) we deduce for a.e.  $t \in [0, T]$

$$\frac{d}{dt} \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \geq (\mathbb{C}e_\varepsilon(t), E\dot{w}(t))_2 - \mathcal{H}(\dot{p}_\varepsilon(t)) - \varepsilon \|\dot{\alpha}_\varepsilon(t)\|_2^2.$$

Integrating over the interval  $[0, t]$  for all  $t \in [0, T]$  we get

$$\mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) + \int_0^t \mathcal{H}(\dot{p}_\varepsilon(r)) dr + \varepsilon \int_0^t \|\dot{\alpha}_\varepsilon(r)\|_2^2 dr \geq \mathcal{E}(\alpha_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) + \int_0^t (\mathbb{C}e_\varepsilon(r), E\dot{w}(r))_2 dr,$$

which implies  $(ev4)_\varepsilon$  by  $(ev4'')_\varepsilon$ .  $\square$

By  $(ev3)_\varepsilon$  and  $(ev4')_\varepsilon$  it follows that

$$\varepsilon \|\dot{\alpha}_\varepsilon(t)\|_2 = \sup_{\beta \in F} \langle -\partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)), \beta \rangle \quad \text{for a.e. } t \in [0, T],$$

where

$$F := \{\beta \in H^1(\Omega) : \beta \leq 0, \|\beta\|_2 = 1\}.$$

If we define

$$\Phi(f) := \sup_{\beta \in F} \langle -f, \beta \rangle \quad \text{for all } f \in (H^1(\Omega))', \quad (4.34)$$

$$\Psi(\alpha, e, p) := \Phi(\partial_\alpha \mathcal{E}(\alpha, e, p)) \quad \text{for all } (\alpha, e, p) \in H^1(\Omega; [0, 1]) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad (4.35)$$

then we deduce

$$\Psi(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) = \varepsilon \|\alpha_\varepsilon(t)\|_2. \quad (4.36)$$

Hence, we can rewrite the energy balance  $(ev4)_\varepsilon$  as: for all  $t \in [0, T]$ .

$$\begin{aligned} & \mathcal{E}(\alpha_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) + \int_0^t \mathcal{H}(\dot{p}_\varepsilon(r)) \, dr + \int_0^t \|\dot{\alpha}_\varepsilon(r)\|_2 \Psi(\alpha_\varepsilon(r), e_\varepsilon(r), p_\varepsilon(r)) \, dr \\ &= \mathcal{E}(\alpha_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) + \int_0^t (\mathbb{C}e_\varepsilon(r), \text{E}\dot{w}(r))_2 \, dr. \end{aligned}$$

By arguing as in [30], Lemma 4.4 we can give another characterization for the operator  $\Phi$  introduced in (4.34).

**Lemma 4.13.** *Define*

$$G := \{h \in (H^1(\Omega))' : \langle h, \beta \rangle \geq 0 \text{ for all } \beta \in H^1(\Omega) \text{ with } \beta \leq 0\}$$

and

$$d_2(f, G) := \min\{\|g\|_2 : f + g \in G\}.$$

Then

$$\Phi(f) = d_2(f, G) \quad \text{for all } f \in (H^1(\Omega))'.$$

*Proof.* The proof follows the same lines of the one of Lemma 4.4 of [30]. It is enough to replace  $H^m(\Omega)$  with  $H^1(\Omega)$  and to use the density of  $C^1(\bar{\Omega})$  in  $H^1(\Omega)$ .  $\square$

We are now ready to prove the existence result of  $\varepsilon$ -approximate viscous evolution for fixed  $\varepsilon \in (0, 1)$ .

**Theorem 4.14.** *Assume (2.2)–(2.7), (2.9), and (3.9). For all  $\varepsilon \in (0, 1)$  there exists an  $\varepsilon$ -approximate viscous evolution  $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$  satisfying the initial condition*

$$(\alpha_\varepsilon(0), u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) = (\alpha^0, u^0, e^0, p^0), \quad (4.37)$$

and the uniform estimates

$$\int_0^T \|\dot{\alpha}_\varepsilon(r)\|_{H^1} \, dr + \int_0^T \|\dot{u}_\varepsilon(r)\|_{H^1} \, dr + \int_0^T \|\dot{e}_\varepsilon(r)\|_2 \, dr + \int_0^T \|\dot{p}_\varepsilon(r)\|_{H^1} \, dr \leq C, \quad (4.38)$$

for a constant  $C > 0$  independent of  $\varepsilon \in (0, 1)$ .

*Proof. Step 1.* Let us fix  $\varepsilon \in (0, 1)$ . Starting from  $(\alpha^0, u^0, e^0, p^0)$  we consider the incremental problem (4.1) and we obtain a sequence of piecewise affine interpolants  $(\alpha_k, u_k, e_k, p_k)$  from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$ . By the uniform bounds (4.26) and Remark 2.3 there exist a subsequence (not relabeled) and a quadruple  $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$  of functions from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$  satisfying as  $k \rightarrow \infty$

$$\alpha_k \rightharpoonup \alpha_\varepsilon \quad \text{in } H^1((0, T); H^1(\Omega)), \quad \alpha_k \rightarrow \alpha_\varepsilon \quad \text{in } C^0([0, T]; L^\theta(\Omega)), \quad (4.39)$$

$$p_k \rightharpoonup p_\varepsilon \quad \text{in } H^1((0, T); H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})), \quad p_k \rightarrow p_\varepsilon \quad \text{in } C^0([0, T]; L^\theta(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})), \quad (4.40)$$

$$e_k \rightharpoonup e \quad \text{in } H^1((0, T); L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})), \quad u_k \rightharpoonup u \quad \text{in } H^1((0, T); H^1(\Omega; \mathbb{R}^n)) \quad (4.41)$$

for all  $\theta \in [1, 6)$ . Moreover, for all  $t \in [0, T]$  as  $k \rightarrow \infty$

$$\bar{\alpha}_k(t) \rightharpoonup \alpha_\varepsilon(t) \quad \text{in } H^1(\Omega), \quad \bar{p}_k(t) \rightharpoonup p_\varepsilon(t) \quad \text{in } H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad (4.42)$$

$$\bar{e}_k(t) \rightharpoonup e(t) \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad \bar{u}_k(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega; \mathbb{R}^n). \quad (4.43)$$

In particular, the initial conditions (4.37) are satisfied, and by passing to the limit as  $k \rightarrow \infty$  in (4.27), we derive the uniform estimates (4.38). Moreover, the weak convergences (4.42) and (4.43) imply that  $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in \mathcal{A}(w(t))$  and  $\alpha_\varepsilon(t) \in H^1(\Omega; [0, 1])$  for all  $t \in [0, T]$ , and that for a.e.  $x \in \Omega$

$$t \mapsto \alpha_\varepsilon(t, x) \quad \text{is non-increasing on } [0, T].$$

By (4.2) and (4.42) we conclude that for all  $t \in [0, T]$

$$\operatorname{div} \mathbb{C}e_\varepsilon(t) = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n).$$

Furthermore, by using Lemma 4.1 together with (4.42)–(4.43), we deduce that for all  $t \in [0, T]$

$$\mathcal{H}(q) \geq (\mathbb{C}e_\varepsilon(t), q)_2 - 2(\mathbb{B}(\alpha_\varepsilon(t))p_\varepsilon(t), q)_2 - 2(\nabla p_\varepsilon(t), \nabla q)_2 \quad \text{for all } q \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

To conclude, it remains to prove  $(ev3)_\varepsilon$  and  $(ev4'')_\varepsilon$ .

**Step 2.** Since the functions  $\alpha_k$ ,  $e_k$ , and  $p_k$  are absolutely continuous, we derive that the function  $t \mapsto \mathcal{E}(\alpha_k(t), e_k(t), p_k(t))$  is absolutely continuous and for a.e.  $t \in [0, T]$

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\alpha_k(t), e_k(t), p_k(t)) &= (\mathbb{C}e_k(t), \dot{e}_k(t))_2 + 2(\mathbb{B}(\alpha_k(t))p_k(t), \dot{p}_k(t))_2 + 2(\nabla p_k(t), \nabla \dot{p}_k(t))_2 \\ &\quad + \partial_\alpha \mathcal{E}(\alpha_k(t), e_k(t), p_k(t))[\dot{\alpha}_k(t)]. \end{aligned}$$

Hence, by using also (4.5) we conclude that for a.e.  $t \in [0, T]$

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\alpha_k(t), e_k(t), p_k(t)) \\ = (\mathbb{C}e_k(t), \dot{e}_k(t))_2 + 2(\mathbb{B}(\alpha_k(t))p_k(t), \dot{p}_k(t))_2 + 2(\nabla p_k(t), \nabla \dot{p}_k(t))_2 - \varepsilon \|\dot{\alpha}_k(t)\|_2^2. \end{aligned} \quad (4.44)$$

Notice that

$$(\mathbb{C}e_k(t), \dot{e}_k(t))_2 = (\mathbb{C}\bar{e}_k(t), \dot{e}_k(t))_2 + (\mathbb{C}(e_k(t) - \bar{e}_k(t)), \dot{e}_k(t))_2, \quad (4.45)$$

$$\begin{aligned} (\mathbb{B}(\alpha_k(t))p_k(t), \dot{p}_k(t))_2 &= (\mathbb{B}(\bar{\alpha}_k(t))\bar{p}_k(t), \dot{p}_k(t))_2 + (\mathbb{B}(\bar{\alpha}_k(t))[p_k(t) - \bar{p}_k(t)], \dot{p}_k(t))_2 \\ &\quad + ([\mathbb{B}(\alpha_k(t)) - \mathbb{B}(\bar{\alpha}_k(t))]p_k(t), \dot{p}_k(t))_2, \end{aligned} \quad (4.46)$$

$$(\nabla p_k(t), \nabla \dot{p}_k(t))_2 = (\nabla \bar{p}_k(t), \nabla \dot{p}_k(t))_2 + (\nabla \bar{p}_k(t) - \nabla p_k(t), \nabla \dot{p}_k(t))_2. \quad (4.47)$$

Thanks to (3.29), since  $(\dot{u}_k(t), \dot{e}_k(t), \dot{p}_k(t)) \in \mathcal{A}(\dot{w}_k(t))$  for a.e.  $t \in [0, T]$ , we get

$$(\mathbb{C}\bar{e}_k(t), \dot{e}_k(t))_2 = (\mathbb{C}\bar{e}_k(t), \mathbb{E}\dot{w}_k(t))_2 - (\mathbb{C}\bar{e}_k(t), \dot{p}_k(t))_2 \quad \text{for a.e. } t \in [0, T]. \quad (4.48)$$

By (4.10), for a.e.  $t \in [0, T]$  we get

$$\begin{aligned} & - (\mathbb{C}\bar{e}_k(t), \dot{p}_k(t))_2 + 2(\mathbb{B}(\bar{\alpha}_k(t))\bar{p}_k(t), \dot{p}_k(t))_2 + 2(\nabla\bar{p}_k(t), \nabla\dot{p}_k(t))_2 \\ & \leq -\mathcal{H}(\dot{p}_k(t)) + C\tau_k (\|\dot{\alpha}_k(t)\|_2^2 + \|\mathbb{E}\dot{w}_k(t)\|_2^2). \end{aligned} \quad (4.49)$$

By integrating (4.44) over the interval  $[0, t]$  for all  $t \in [0, T]$ , and using (4.45)–(4.49), (4.26), and Remark 2.3 we deduce

$$\begin{aligned} & \mathcal{E}(\alpha_k(t), e_k(t), p_k(t)) - \mathcal{E}(\alpha^0, e^0, p^0) \\ & = \int_0^t ((\mathbb{C}e_k(r), \dot{e}_k(r))_2 + 2(\mathbb{B}(\alpha_k(r))p_k(r), \dot{p}_k(r))_2 + 2(\nabla p_k(r), \nabla\dot{p}_k(r))_2 - \varepsilon\|\dot{\alpha}_k(r)\|_2^2) \, dr \\ & \leq \int_0^t (\mathbb{C}\bar{e}_k(r), \mathbb{E}\dot{w}_k(r))_2 \, dr - \int_0^t \mathcal{H}(\dot{p}_k(r)) \, dr - \varepsilon \int_0^t \|\dot{\alpha}_k(r)\|_2^2 \, dr \\ & \quad + C \int_0^t (\|e_k(r) - \bar{e}_k(r)\|_2 \|\dot{e}_k(r)\|_2 + \|p_k(r) - \bar{p}_k(r)\|_2 \|\dot{p}_k(r)\|_2) \, dr \\ & \quad + C \int_0^t (\|\alpha_k(r) - \bar{\alpha}_k(r)\|_2 \|p_k(r)\|_{H^1} \|\dot{p}_k(r)\|_{H^1} + \|\nabla\bar{p}_k(r) - \nabla p_k(r)\|_2 \|\nabla\dot{p}_k(r)\|_2) \, dr \\ & \quad + C\tau_k \int_0^t (\|\dot{\alpha}_k(r)\|_2^2 + \|\mathbb{E}\dot{w}_k(r)\|_2^2) \, dr \\ & \leq \int_0^t (\mathbb{C}\bar{e}_k(r), \mathbb{E}\dot{w}_k(r))_2 \, dr - \int_0^t \mathcal{H}(\dot{p}_k(r)) \, dr - \varepsilon \int_0^t \|\dot{\alpha}_k(r)\|_2^2 \, dr \\ & \quad + C\tau_k \int_0^t (\|\dot{e}_k(r)\|_2^2 + \|\dot{p}_k(r)\|_{H^1}^2 + \|\dot{\alpha}_k(r)\|_2^2 + \|\mathbb{E}\dot{w}_k(r)\|_2^2) \, dr \\ & \leq \int_0^t (\mathbb{C}\bar{e}_k(r), \mathbb{E}\dot{w}_k(r))_2 \, dr - \int_0^t \mathcal{H}(\dot{p}_k(r)) \, dr - \varepsilon \int_0^t \|\dot{\alpha}_k(r)\|_2^2 \, dr + C\tau_k \end{aligned}$$

for a constant  $C > 0$  independent of  $k$ . Hence, for every  $t \in [0, T]$  we have

$$\mathcal{E}(\alpha_k(t), e_k(t), p_k(t)) + \int_0^t \mathcal{H}(\dot{p}_k(r)) \, dr + \varepsilon \int_0^t \|\dot{\alpha}_k(r)\|_2^2 \, dr \leq \mathcal{E}(\alpha^0, e^0, p^0) + \int_0^t (\mathbb{C}\bar{e}_k(r), \mathbb{E}\dot{w}_k(r))_2 \, dr + C\tau_k.$$

By sending  $k \rightarrow \infty$  and using the weak convergences (4.39)–(4.43) together with the lower semicontinuity of the left-hand side (see Rem. 2.5), we get  $(ev_4'')_\varepsilon$ .

**Step 3.** By (4.4), for all  $\beta \in L^\infty((0, T); H^1(\Omega))$  with  $\beta \leq 0$  we have

$$\begin{aligned} 0 & \leq (\dot{d}(\alpha_\varepsilon(t)), \beta(t))_2 + 2(\nabla\bar{\alpha}_k(t), \nabla\beta(t))_2 + (\mathbb{B}(\alpha_\varepsilon(t))\beta(t)\bar{p}_k(t), \bar{p}_k(t))_2 + \varepsilon(\dot{\alpha}_k(t), \beta(t))_2 \\ & \quad + (\dot{d}(\bar{\alpha}_k(t)) - \dot{d}(\alpha_\varepsilon(t)), \beta(t))_2 + ([\mathbb{B}(\bar{\alpha}_k(t)) - \mathbb{B}(\alpha_\varepsilon(t))]\beta(t)\bar{p}_k(t), \bar{p}_k(t))_2. \end{aligned}$$

By Remark 2.3, as  $k \rightarrow \infty$

$$\begin{aligned} |(\dot{d}(\bar{\alpha}_k(t)) - \dot{d}(\alpha_\varepsilon(t)), \beta(t))_2| & \leq \|\dot{d}\|_\infty \|\bar{\alpha}_k(t) - \alpha_\varepsilon(t)\|_2 \|\beta(t)\|_2 \rightarrow 0, \\ |([\mathbb{B}(\bar{\alpha}_k(t)) - \mathbb{B}(\alpha_\varepsilon(t))]\beta(t)\bar{p}_k(t), \bar{p}_k(t))_2| & \leq \|\mathbb{B}\|_\infty \|\bar{\alpha}_k(t) - \alpha_\varepsilon(t)\|_4 \|\beta(t)\|_4 \|\bar{p}_k(t)\|_4^2 \rightarrow 0 \end{aligned}$$



in view of the strong convergences in (4.39), see also (4.42). Moreover, thanks to the strong convergences in (4.40) we get

$$(\dot{\mathbb{B}}(\alpha_\varepsilon(t))\beta(t)p_\varepsilon(t), p_\varepsilon(t))_2 = \lim_{k \rightarrow \infty} (\dot{\mathbb{B}}(\alpha_\varepsilon(t))\beta(t)\bar{p}_k(t), \bar{p}_k(t))_2.$$

Hence, we can apply the Dominated Convergence Theorem to derive that

$$\begin{aligned} 0 &\leq \lim_{k \rightarrow \infty} \int_0^T [(\dot{d}(\alpha_\varepsilon(r)), \beta(r))_2 + 2(\nabla \bar{\alpha}_k(r), \nabla \beta(r))_2 + (\dot{\mathbb{B}}(\alpha_\varepsilon(r))\beta(r)\bar{p}_k(r), \bar{p}_k(r))_2 + \varepsilon(\dot{\alpha}_k(r), \beta(r))_2] dr \\ &= \int_0^T [(\dot{d}(\alpha_\varepsilon(r)), \beta(r))_2 + 2(\nabla \alpha_\varepsilon(r), \nabla \beta(r))_2 + (\dot{\mathbb{B}}(\alpha_\varepsilon(r))\beta(r)p_\varepsilon(r), p_\varepsilon(r))_2 + \varepsilon(\dot{\alpha}_\varepsilon(r), \beta(r))_2] dr. \end{aligned}$$

If we fix  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  a.e. in  $\Omega$ , by choosing  $\beta(t) = \beta \chi_A(t)$  with  $A \subset [0, T]$  arbitrary set, we get

$$(\dot{d}(\alpha_\varepsilon(t)), \beta)_2 + (\dot{\mathbb{B}}(\alpha_\varepsilon(t))\beta p_\varepsilon(t), p_\varepsilon(t))_2 + 2(\nabla \alpha_\varepsilon(t), \nabla \beta)_2 + \varepsilon(\dot{\alpha}_\varepsilon(t), \beta)_2 \geq 0 \quad \text{for all } t \in [0, T] \setminus E_\beta,$$

where  $E_\beta$  is a negligible set depending on  $\beta$ . Since  $\{\alpha \in H^1(\Omega) : \alpha \leq 0\}$  is separable, we get  $(ev_3)_\varepsilon$ .  $\square$

## 5. BALANCED VISCOSITY (BV) SOLUTIONS

In the previous section for all  $\varepsilon \in (0, 1)$  we found an  $\varepsilon$ -approximate viscous evolution  $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$  from  $[0, T]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$  which satisfies (4.38). We now introduce a “slow” time scale  $s$  and we pass to the limit as  $\varepsilon \rightarrow 0$ , to obtain a Balanced Viscosity quasistatic evolution in the spirit of [30], see also [17, 18, 23]. We first introduce the following definition.

**Definition 5.1.** Assume (2.2)–(2.7) and (2.9). A quintuplet of Lipschitz functions  $(\alpha_0, u_0, e_0, p_0, t_0)$  from  $[0, S]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A} \times [0, T]$  is *Balanced Viscosity quasistatic evolution* in the time interval  $[0, S]$  with datum  $w$ , if setting for all  $s \in [0, S]$

$$w_0(s) := w(t_0(s)) \quad \text{and} \quad U_0 := \{s \in [0, S] : t_0 \text{ is constant in a neighborhood of } s\},$$

the following conditions are satisfied:

(ev0) *irreversibility*:  $t_0$  is non-decreasing and surjective, and for a.e.  $x \in \Omega$  the map

$$s \mapsto \alpha_0(s, x) \quad \text{is non-increasing on } [0, S];$$

(ev1) *kinematic condition and equilibrium*: for all  $s \in [0, S]$

$$(u_0(s), e_0(s), p_0(s)) \in \mathcal{A}(w_0(s)) \quad \text{and} \quad \operatorname{div} \mathbb{C}e_0(s) = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n);$$

(ev2) *stress constraint*: for all  $s \in [0, S]$

$$\mathcal{H}(q) \geq (\mathbb{C}e_0(s), q)_2 - 2(\mathbb{B}(\alpha_0(s))p_0(s), q)_2 - 2(\nabla p_0(s), \nabla q)_2 \quad \text{for all } q \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n});$$

(ev3) *Kuhn–Tucker Inequality in*  $[0, S] \setminus U_0$ : for all  $s \in [0, S] \setminus U_0$

$$\partial_\alpha \mathcal{E}(\alpha_0(s), e_0(s), p_0(s))[\beta] \geq 0 \quad \text{for all } \beta \in H^1(\Omega) \text{ with } \beta \leq 0;$$

(ev4) *energy balance*: for all  $s \in [0, S]$

$$\begin{aligned} & \mathcal{E}(\alpha_0(s), e_0(s), p_0(s)) + \int_0^s \mathcal{H}(\dot{p}_0(r)) \, dr + \int_0^s \|\dot{\alpha}_0(r)\|_2 \Psi(\alpha_0(r), e_0(r), p_0(r)) \, dr \\ &= \mathcal{E}(\alpha_0(0), e_0(0), p_0(0)) + \int_0^s (\mathbb{C}e_0(r), E\dot{w}_0(r))_2 \, dr, \end{aligned}$$

where  $\Psi$  is the function defined in (4.35), with the convention  $0 \cdot \infty = 0$ .

**Remark 5.2.** A similar definition for a different model with incomplete damage can be found in [30], where it is called rescaled quasistatic viscosity evolution. By Lemma 4.13 and (ev3), for every  $s \in [0, S] \setminus U_0$  we have

$$0 \leq d(\partial_\alpha \mathcal{E}(\alpha_0(s), e_0(s), p_0(s)), G) = \Psi(\alpha_0(s), e_0(s), p_0(s)) = \sup_{\beta \in F} \langle -\partial_\alpha \mathcal{E}(\alpha_0(s), e_0(s), p_0(s)), \beta \rangle \leq 0,$$

which gives

$$\Psi(\alpha_0(s), e_0(s), p_0(s)) = 0 \quad \text{for all } s \in [0, S] \setminus U_0.$$

Similarly to Proposition 4.12, the following is a technical proposition instrumental to prove existence of Balanced Viscosity quasistatic solutions.

**Proposition 5.3.** *Assume (2.2)–(2.7) and (2.9). Let  $(\alpha_0, u_0, e_0, p_0, t_0)$  be a quintuplet of Lipschitz functions from  $[0, S]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A} \times [0, T]$  satisfying (ev0), (ev1), (ev2), and (ev3). Then  $(\alpha_0, u_0, e_0, p_0, t_0)$  is a Balanced Viscosity quasistatic evolution, i.e., it satisfies the energy balance (ev4), if and only if any of the conditions below holds true:*

(ev4') *for a.e.  $s \in [0, S]$  the following hold:*  
*generalized Kuhn–Tucker equality:*

$$-\partial_\alpha \mathcal{E}(\alpha_0(s), e_0(s), p_0(s))[\dot{\alpha}_0(s)] = \|\dot{\alpha}_0(s)\|_2 \Psi(\alpha_0(s), e_0(s), p_0(s));$$

*Hill's maximum plastic work principle:*

$$\mathcal{H}(\dot{p}_0(s)) = (\mathbb{C}e_0(t), \dot{p}_0(s))_2 - 2(\mathbb{B}(\alpha_0(s))p_0(s), \dot{p}_0(s))_2 - 2(\nabla p_0(s), \nabla \dot{p}_0(s))_2;$$

(ev4'') *energy inequality*: for all  $s \in [0, S]$

$$\begin{aligned} & \mathcal{E}(\alpha_0(s), e_0(s), p_0(s)) + \int_0^s \mathcal{H}(\dot{p}_0(r)) \, dr + \int_0^s \|\dot{\alpha}_0(r)\|_2 \Psi(\alpha_0(r), e_0(r), p_0(r)) \, dr \\ & \leq \mathcal{E}(\alpha_0(0), e_0(0), p_0(0)) + \int_0^s (\mathbb{C}e_0(r), E\dot{w}_0(r))_2 \, dr. \end{aligned}$$

*Proof.* The proof is analogous to the one of Proposition 4.12. □

We eventually prove existence of Balanced Viscosity quasistatic solutions.

**Theorem 5.4.** *Assume (2.2)–(2.7), (2.9), and (3.9). Then there exist  $S > 0$  and a Balanced Viscosity quasistatic solution  $(\alpha_0, u_0, e_0, p_0, t_0)$  in the time interval  $[0, S]$  satisfying the initial condition*

$$(\alpha_0(0), u_0(0), e_0(0), p_0(0), t_0(0)) = (\alpha^0, u^0, e^0, p^0, 0). \tag{5.1}$$

*Proof. Step 1.* Let  $\varepsilon \in (0, 1)$  be fixed and let  $(\alpha_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$  be the  $\varepsilon$ -approximate viscous evolution given by Theorem 4.14. For all  $t \in [0, T]$  we define the function

$$s_\varepsilon^0(t) := t + \int_0^t \|\dot{\alpha}_\varepsilon(r)\|_{H^1} dr + \int_0^t \|\dot{e}_\varepsilon(r)\|_2 dr + \int_0^t \|\dot{p}_\varepsilon(r)\|_{H^1} dr.$$

The function  $s_\varepsilon^0$  is absolutely continuous, increasing, and bijective on  $[0, T]$  and we have

$$s_\varepsilon^0(t_2) - s_\varepsilon^0(t_1) \geq t_2 - t_1 \quad \text{for all } 0 \leq t_1 \leq t_2 \leq T. \quad (5.2)$$

Let  $S_\varepsilon := s_\varepsilon^0(T) \geq T$  and let  $t_\varepsilon^0: [0, S_\varepsilon] \rightarrow [0, T]$  be the inverse of  $s_\varepsilon^0$ . By (4.38) we can find a constant  $C > 0$  independent of  $\varepsilon$  such that  $S_\varepsilon \leq C$  for all  $\varepsilon \in (0, 1)$ . Hence, up to a subsequence,  $S_\varepsilon \rightarrow S$  as  $\varepsilon \rightarrow 0$ , with  $S \geq T$ .

We set  $\bar{S} = \sup_{\varepsilon \in (0, 1)} S_\varepsilon \geq S$  and we extend every function  $t_\varepsilon^0$  to the interval  $[0, \bar{S}]$  by defining  $t_\varepsilon^0(s) := T$  for all  $s \in [S_\varepsilon, \bar{S}]$ . For all  $\varepsilon \in (0, 1)$  we define the following rescaled functions on  $[0, \bar{S}]$ : for all  $s \in [0, \bar{S}]$

$$\begin{aligned} \alpha_\varepsilon^0(s) &:= \alpha_\varepsilon(t_\varepsilon^0(s)), & w_\varepsilon^0(s) &:= w(t_\varepsilon^0(s)), & u_\varepsilon^0(s) &:= u_\varepsilon(t_\varepsilon^0(s)), \\ e_\varepsilon^0(s) &:= e_\varepsilon(t_\varepsilon^0(s)), & p_\varepsilon^0(s) &:= p_\varepsilon(t_\varepsilon^0(s)). \end{aligned}$$

Notice that the functions  $t_\varepsilon^0: [0, \bar{S}] \rightarrow [0, T]$  are uniformly Lipschitz in  $\varepsilon \in (0, 1)$  by (5.2), since

$$0 \leq t_\varepsilon^0(s_2) - t_\varepsilon^0(s_1) \leq s_2 - s_1 \quad \text{for all } 0 \leq s_1 \leq s_2 \leq \bar{S}.$$

Hence, there exists a subsequence and a Lipschitz map  $t_0: [0, \bar{S}] \rightarrow [0, T]$  such that  $t_\varepsilon^0 \rightarrow t_0$  uniformly on  $[0, \bar{S}]$  and weakly\* in  $W^{1, \infty}((0, \bar{S}); (0, T))$  as  $\varepsilon \rightarrow 0$ . In particular  $t_0(0) = 0$ ,  $t_0(s) = T$  for all  $s \in [S, \bar{S}]$ , and

$$0 \leq t_0(s_2) - t_0(s_1) \leq s_2 - s_1 \quad \text{for all } 0 \leq s_1 \leq s_2 \leq \bar{S}.$$

In particular, the map  $t_0: [0, S] \rightarrow [0, T]$  is non-decreasing and surjective. Moreover, as  $\varepsilon \rightarrow 0$  we derive

$$w_\varepsilon^0(s) \rightarrow w_0(s) := w(t_0(s)) \quad \text{in } H^1(\Omega; \mathbb{R}^n) \text{ for all } s \in [0, S].$$

By using the definitions of  $s_\varepsilon^0$  and  $t_\varepsilon^0$  we obtain

$$\|\alpha_\varepsilon^0(s_2) - \alpha_\varepsilon^0(s_1)\|_{H^1} + \|e_\varepsilon^0(s_2) - e_\varepsilon^0(s_1)\|_2 + \|p_\varepsilon^0(s_2) - p_\varepsilon^0(s_1)\|_{H^1} \leq s_2 - s_1 \quad \text{for all } 0 \leq s_1 \leq s_2 \leq S. \quad (5.3)$$

By Ascoli-Arzelà Theorem there exists a quadruple  $(\alpha_0, u_0, e_0, p_0)$  of Lipschitz functions from  $[0, S]$  into  $H^1(\Omega; [0, 1]) \times \mathcal{A}$  satisfying: for all  $s \in [0, S]$  as  $\varepsilon \rightarrow 0$

$$\alpha_\varepsilon^0(s) \rightharpoonup \alpha_0(s) \quad \text{in } H^1(\Omega), \quad u_\varepsilon^0(s) \rightharpoonup u_0(s) \quad \text{in } H^1(\Omega; \mathbb{R}^n), \quad (5.4)$$

$$e_\varepsilon^0(s) \rightharpoonup e_0(s) \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad p_\varepsilon^0(s) \rightharpoonup p_0(s) \quad \text{in } H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}). \quad (5.5)$$

In particular, by Remark 2.3, for all  $s \in [0, S]$  as  $\varepsilon \rightarrow 0$  we have

$$\alpha_\varepsilon^0 \rightarrow \alpha_0 \quad \text{in } C^0([0, S]; L^\theta(\Omega)), \quad p_\varepsilon^0 \rightarrow p_0 \quad \text{in } C^0([0, S]; L^\theta(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})) \quad (5.6)$$

for all  $\theta \in [1, 6)$ . Furthermore, if we combine the uniform Lipschitz estimate (5.3) with the weak convergences (5.4)–(5.5), we get

$$\|\alpha_0(s_2) - \alpha_0(s_1)\|_{H^1} + \|e_0(s_2) - e_0(s_1)\|_2 + \|p_0(s_2) - p_0(s_1)\|_{H^1} \leq s_2 - s_1 \quad \text{for all } 0 \leq s_1 \leq s_2 \leq S,$$

which yields

$$\|\dot{\alpha}_0(s)\|_{H^1} + \|\dot{e}_0(s)\|_2 + \|\dot{p}_0(s)\|_{H^1} \leq 1 \quad \text{for a.e. } s \in [0, S].$$

Moreover, the initial conditions (5.1) hold by construction.

**Step 2.** Let us prove that  $(\alpha_0, u_0, e_0, p_0, t_0)$  is a Balanced Viscosity quasistatic solution on  $[0, S]$ . By (5.4) and (5.5) we derive that  $(u_0(s), e_0(s), p_0(s)) \in \mathcal{A}(w_0(s))$  and  $\operatorname{div} \mathbb{C}e_0(s) = 0$  in  $\mathcal{D}'(\Omega; \mathbb{R}^n)$  for all  $s \in [0, S]$ , and that for a.e.  $x \in \Omega$  the map  $s \mapsto \alpha_0(s, x)$  is non-increasing on  $[0, S]$ . Hence  $(\alpha_0, u_0, e_0, p_0, t_0)$  satisfies (ev0) and (ev1). Let us fix  $s \in [0, S]$  and  $q \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ . Then  $t_\varepsilon(s) \in [0, T]$  for all  $s \in [0, S]$  and  $\varepsilon \in (0, 1)$ . Hence, thanks to (ev2) $_\varepsilon$ , for all  $s \in [0, S]$  we get

$$\mathcal{H}(q) \geq (\mathbb{C}e_\varepsilon^0(s), q)_2 - 2(\mathbb{B}(\alpha_\varepsilon^0(s))p_\varepsilon^0(s), q)_2 - 2(\nabla p_\varepsilon^0(s), \nabla q)_2 \quad \text{for all } q \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

By sending  $\varepsilon \rightarrow 0$  and exploiting the convergences (5.4)–(5.6), for all  $s \in [0, S]$  we derive

$$\mathcal{H}(q) \geq (\mathbb{C}e_0(s), q)_2 - 2(\mathbb{B}(\alpha_0(s))p_0(s), q)_2 - 2(\nabla p_0(s), \nabla q)_2 \quad \text{for all } q \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

Therefore, also condition (ev2) is satisfied.

**Step 3.** We introduce the functions  $s_0^+, s_0^- : [0, T] \rightarrow [0, S]$ , defined as

$$\begin{aligned} s_0^+(t) &:= \inf\{s \in [0, S] : t_0(s) > t\} \quad \text{for } t \in [0, T), & s_0^+(T) &= S, \\ s_0^-(t) &:= \sup\{s \in [0, S] : t_0(s) < t\} \quad \text{for } t \in (0, T], & s_0^-(0) &= 0. \end{aligned}$$

Then

$$s_0^-(t) \leq \liminf_{\varepsilon \rightarrow 0} s_\varepsilon^0(t) \leq \limsup_{\varepsilon \rightarrow 0} s_\varepsilon^0(t) \leq s_0^+(t) \quad \text{for all } t \in [0, T], \quad (5.7)$$

and

$$t_0(s_0^-(t)) = t = t_0(s_0^+(t)) \quad \text{for all } t \in [0, T], \quad s_0^-(t_0(s)) \leq s \leq s_0^+(t_0(s)) \quad \text{for all } s \in [0, S].$$

In particular, the set

$$S_0 := \{t \in [0, T] : s_0^-(t) < s_0^+(t)\}$$

is at most countable and

$$U_0 := \{s \in [0, S] : t_0 \text{ is constant in a neighborhood of } s\} = \bigcup_{t \in S_0} (s_0^-(t), s_0^+(t)).$$

Thanks to (5.7), for all  $t \in [0, T] \setminus S_0$  we have that  $s_\varepsilon^0(t) \rightarrow s_0^-(t) = s_0^+(t)$  as  $\varepsilon \rightarrow 0$ . Hence, by exploiting (5.3)–(5.5) we deduce that for all  $t \in [0, T] \setminus S_0$  as  $\varepsilon \rightarrow 0$

$$\alpha_\varepsilon(t) \rightarrow \alpha_0(s_0^-(t)) \quad \text{in } H^1(\Omega), \quad u_\varepsilon(t) \rightarrow u_0(s_0^-(t)) \quad \text{in } H^1(\Omega; \mathbb{R}^n), \quad (5.8)$$

$$e_\varepsilon(t) \rightarrow e_0(s_0^-(t)) \quad \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad p_\varepsilon(t) \rightarrow p_0(s_0^-(t)) \quad \text{in } H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad (5.9)$$

and by Remark 2.3 for all  $t \in [0, T] \setminus S_0$  as  $\varepsilon \rightarrow 0$

$$\alpha_\varepsilon(t) \rightarrow \alpha_0(s_0^-(t)) \quad \text{in } L^\theta(\Omega), \quad p_\varepsilon(t) \rightarrow p_0(s_0^-(t)) \quad \text{in } L^\theta(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$$

for all  $\theta \in [1, 6)$ .

We want to show (ev3) and we define

$$A_0 := \{s \in [0, S] : \Psi(\alpha_0(s), e_0(s), p_0(s)) > 0\}.$$

We notice that it is enough to prove that

$$A_0 \subset U_0. \quad (5.10)$$

Indeed, if (5.10) holds, then by (4.34)–(4.35) for all  $s \in [0, T] \setminus U_0$  and for all  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  and  $\|\beta\|_2 = 1$  we have

$$\langle -\partial_\alpha \mathcal{E}(\alpha_0(s), e_0(s), p_0(s)), \beta \rangle \leq \Psi(\alpha_0(s), e_0(s), p_0(s)) \leq 0,$$

which implies (ev3).

Since for all  $\beta \in H^1(\Omega)$  the map  $s \mapsto \langle -\partial_\alpha \mathcal{E}(\alpha_0(s), e_0(s), p_0(s)), \beta \rangle$  is continuous on  $[0, S]$ , we derive that the map  $s \mapsto \Psi(\alpha_0(s), e_0(s), p_0(s))$  is lower semicontinuous on  $[0, T]$ . In particular, the set  $A_0$  is open on  $[0, T]$ . Moreover, for all  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  we have

$$\langle -\partial_\alpha \mathcal{E}(\alpha_0(s), e_0(s), p_0(s)), \beta \rangle \leq \liminf_{\varepsilon \rightarrow 0} \langle -\partial_\alpha \mathcal{E}(\alpha_\varepsilon^0(s), e_\varepsilon^0(s), p_\varepsilon^0(s)), \beta \rangle \leq \liminf_{\varepsilon \rightarrow 0} \Psi(\alpha_\varepsilon^0(s), e_\varepsilon^0(s), p_\varepsilon^0(s)),$$

which gives

$$\Psi(\alpha_0(s), e_0(s), p_0(s)) \leq \liminf_{\varepsilon \rightarrow 0} \Psi(\alpha_\varepsilon^0(s), e_\varepsilon^0(s), p_\varepsilon^0(s)). \quad (5.11)$$

Let us define

$$D_0 := \{s \in (0, S) : \dot{t}_0(s) = 0\},$$

and let us show that

$$\limsup_{\varepsilon \rightarrow 0} \dot{t}_\varepsilon^0(s) > 0 \quad \text{for a.e. } s \in [0, S] \setminus D_0. \quad (5.12)$$

On the contrary, since  $\dot{t}_\varepsilon^0$  is non-decreasing, we could find a measurable set  $A \subset (0, S) \setminus D_0$  with positive measure such that

$$\lim_{\varepsilon \rightarrow 0} \dot{t}_\varepsilon^0(s) = 0 \quad \text{for all } s \in A.$$

Since every function  $\dot{t}_\varepsilon^0$  is 1-Lipschitz on  $[0, S]$ , by applying the Dominated Convergence Theorem we conclude that

$$\lim_{\varepsilon \rightarrow 0} \int_A \dot{t}_\varepsilon^0(r) \, dr = 0.$$

On the other hand, we have that  $\dot{t}_\varepsilon^0 \rightharpoonup \dot{t}_0$  weakly\* in  $W^{1,\infty}((0, S); (0, T))$ , which gives

$$\lim_{\varepsilon \rightarrow 0} \int_A \dot{t}_\varepsilon^0(r) \, dr = \int_A \dot{t}_0(r) \, dr > 0,$$

since  $\dot{t}_0(s) > 0$  for a.e.  $s \in [0, S] \setminus D_0$ . This leads to a contradiction and proves (5.12).

For a.e.  $s \in [0, S] \setminus D_0$  we derive

$$0 \leq \Psi(\alpha_0(s), e_0(s), p_0(s)) \leq \liminf_{\varepsilon \rightarrow 0} \Psi(\alpha_\varepsilon^0(s), e_\varepsilon^0(s), p_\varepsilon^0(s)) = \liminf_{\varepsilon \rightarrow 0} \varepsilon \|\dot{\alpha}_\varepsilon(t_\varepsilon^0(s))\|_2 = \liminf_{\varepsilon \rightarrow 0} \varepsilon \frac{\|\dot{\alpha}_\varepsilon^0(s)\|_2}{\dot{t}_\varepsilon^0(s)} = 0$$

by (4.36), (5.3), and (5.12). Therefore, we have

$$\Psi(\alpha_0(s), e_0(s), p_0(s)) = 0 \quad \text{for a.e. } s \in [0, S] \setminus D_0,$$

which gives that  $A_0 \subset D_0$ , *i.e.*,

$$\dot{t}_0(s) = 0 \quad \text{for a.e. } s \in A_0.$$

Since  $A_0$  is open on  $[0, T]$ , every  $s \in A_0$  has an open neighborhood where  $\dot{t}_0 = 0$ . Since  $t_0$  is Lipschitz, we conclude that (5.10) holds. This gives (ev3).

**Step 4.** By Proposition 5.3, in order to show that  $(\alpha_0, u_0, e_0, p_0, t_0)$  is a Balanced Viscosity quasistatic evolution, it is enough to prove the energy inequality (ev4''). By using (ev4) $_\varepsilon$  in  $t = t_\varepsilon^0(s)$  together with (4.35) and the change of variable formula, for all  $\varepsilon \in (0, 1)$  and  $s \in [0, S]$  we get

$$\begin{aligned} & \mathcal{E}(\alpha_\varepsilon^0(s), e_\varepsilon^0(s), p_\varepsilon^0(s)) + \int_0^s \mathcal{H}(\dot{p}_\varepsilon^0(r)) \, dr + \int_0^s \|\dot{\alpha}_\varepsilon^0(r)\|_2 \Psi(\alpha_\varepsilon^0(r), e_\varepsilon^0(r), p_\varepsilon^0(r)) \, dr \\ &= \mathcal{E}(\alpha^0, e^0, p^0) + \int_0^{t_\varepsilon^0(s)} (\mathbb{C}e_\varepsilon(r), E\dot{w}(r))_2 \, dr. \end{aligned} \quad (5.13)$$

By the weak convergences in (5.4) and (5.5) we get

$$\mathcal{E}(\alpha_0(s), e_0(s), p_0(s)) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{E}(\alpha_\varepsilon^0(s), e_\varepsilon^0(s), p_\varepsilon^0(s)).$$

Moreover, by Remark 2.5 and the fact that  $\dot{p}_\varepsilon^0 \rightharpoonup \dot{p}_0$  weakly\* in  $L^\infty((0, S); H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$  as  $\varepsilon \rightarrow 0$ , we have

$$\int_0^s \mathcal{H}(\dot{p}_0(r)) \, dr \leq \liminf_{\varepsilon \rightarrow 0} \int_0^s \mathcal{H}(\dot{p}_\varepsilon^0(r)) \, dr,$$

We want to show that

$$\int_{A_0} \|\dot{\alpha}_0(r)\|_2 \Psi(\alpha_0(r), e_0(r), p_0(r)) \, dr \leq \liminf_{\varepsilon \rightarrow 0} \int_{A_0} \|\dot{\alpha}_\varepsilon^0(r)\|_2 \Psi(\alpha_\varepsilon^0(r), e_\varepsilon^0(r), p_\varepsilon^0(r)) \, dr. \quad (5.14)$$

Let  $C \subset A_0$  be a compact set and let  $\psi: C \rightarrow [0, \infty)$  be a continuous function such that

$$\Psi(\alpha_0(r), e_0(r), p_0(r)) > \psi(r) \quad \text{for all } r \in C.$$

By (5.11) and the compactness of  $C$ , we deduce that for all  $\varepsilon > 0$  small enough

$$\Psi(\alpha_\varepsilon^0(r), e_\varepsilon^0(r), p_\varepsilon^0(r)) > \psi(r) \quad \text{for all } r \in C.$$

By a standard approximation argument from below, since  $s \mapsto \Psi(\alpha_0(r), e_0(r), p_0(r))$  is lower semicontinuous, in order to prove (5.14) it suffices to show that

$$\int_C \|\dot{\alpha}_0(r)\|_2 \psi(r) \, dr \leq \liminf_{\varepsilon \rightarrow 0} \int_C \|\dot{\alpha}_\varepsilon^0(r)\|_2 \psi(r) \, dr \quad (5.15)$$

for all compact set  $C \subset A_0$  and every continuous function  $\psi: C \rightarrow [0, \infty)$ . To this end, let  $(\varphi_i)_i \subset C_c^\infty(\Omega)$  be a dense sequence in the unit ball of  $L^2(\Omega)$ . Since

$$\|\dot{\alpha}_\varepsilon^0(r)\|_2 = \sup_i (\varphi_i, \dot{\alpha}_\varepsilon^0(r))_2 \quad \text{for all } r \in C,$$

by the Localization Lemma (see, *e.g.*, [37], Lem. 2.3.2) we have

$$\int_C \|\dot{\alpha}_\varepsilon^0(r)\|_2 \psi(r) \, dr = \sup \sum_{i=1}^k \int_{C_i} (\varphi_i, \dot{\alpha}_\varepsilon^0(r))_2 \psi(r) \, dr,$$

where the supremum is taken over all integers  $k$  and over all finite Borel partitions  $C_1, \dots, C_k$  of  $C$ . For all  $i$  the real-valued functions  $r \mapsto (\varphi_i, \dot{\alpha}_\varepsilon^0(r))_2$  are equi-Lipschitz on  $[0, S]$  and converge to  $r \mapsto (\varphi_i, \dot{\alpha}_0(r))_2$  for all  $r \in [0, S]$  as  $\varepsilon \rightarrow 0$ . Hence the functions  $r \mapsto (\varphi_i, \dot{\alpha}_\varepsilon^0(r))_2$  converge to  $r \mapsto (\varphi_i, \dot{\alpha}_0(r))_2$  weakly\* in  $L^\infty(0, S)$ . It follows that

$$\sum_{i=1}^k \int_{C_i} (\varphi_i, \dot{\alpha}_0(r))_2 \psi(r) \, dr = \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^k \int_{C_i} (\varphi_i, \dot{\alpha}_\varepsilon^0(r))_2 \psi(r) \, dr \leq \liminf_{\varepsilon \rightarrow 0} \int_C \|\dot{\alpha}_\varepsilon^0(r)\|_2 \psi(r) \, dr.$$

This implies (5.15), which gives (5.14).

It remains to study the limit as  $\varepsilon \rightarrow 0$  of the last term in the right hand side of (5.13). By the Dominated Convergence Theorem and the weak convergences of (5.8) and (5.9) we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^{t_\varepsilon^0(s)} (\mathbb{C}e_\varepsilon(r), E\dot{w}(r))_2 \, dr = \int_0^{t_0(s)} (\mathbb{C}e_0(s_0^-(r)), E\dot{w}(r))_2 \, dr.$$

Since  $t_0$  is Lipschitz and  $w$  satisfies (2.1), we get that  $w_0 \in AC([0, S]; H^1(\Omega; \mathbb{R}^n))$  and

$$E\dot{w}_0(r) = E\dot{w}(t_0(r))\dot{t}_0(r) \quad \text{for a.e. } r \in [0, S].$$

Hence, by the area formula we derive

$$\int_0^{t_0(s)} (\mathbb{C}e_0(s_0^-(r)), E\dot{w}(r))_2 \, dr = \int_0^s (\mathbb{C}e_0(s_0^-(t_0(r))), E\dot{w}(t_0(r)))_2 \dot{t}_0(r) \, dr = \int_0^s (\mathbb{C}e_0(r), E\dot{w}_0(r))_2 \, dr,$$

because  $s_0^-(t_0(r)) = r$  for all  $r \in [0, S] \setminus U_0$  and  $\dot{t}_0(r) = 0$  for all  $r \in U_0$ . This implies the energy inequality (ev4''), which concludes the proof.  $\square$

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