ESAIM: COCV 31 (2025) 8
https://doi.org/10.1051/cocv/2024087
ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ϵ-NASH EQUILIBRIUM OF ANTICIPATIVE LARGE-POPULATION LQ GAME WITH PARTIAL OBSERVATIONS

Jize Li[1] and Yonghui Zhou[2,*]

Abstract. In this paper, we study a class of large-population linear-quadratic game problem that is driven by an anticipative signal-observation system with a correlation between the initial value of the signal and the observation noise. Firstly, we utilize a method of enlargement of filtration to transform the anticipative signal-observation system into a higher-dimensional non-anticipative one, and construct an extended equivalent large-population adapted game problem. Secondly, for each individual, by separation principle, filtering theory, and squared compensatory technology, we derive a closed-form decentralized equilibrium strategy for a limiting adapted version with a freezing term instead of average state, and obtain a consistency condition consisting of a forward-backward stochastic differential system with the coefficients affected by the correlation function. Finally, we prove for the extended equivalent large-population adapted game, the ϵ-Nash equilibrium properties of the decentralized strategy designed by means of locally observed information.

Mathematics Subject Classification. 60H10, 60H30, 93E20.

Received January 14, 2024. Accepted November 29, 2024.

1. Introduction

Large-population system arises in various fields, such as economics [1–3], population evolution [4], engineering, and social sciences [5]. Its most significant feature is that many negligible individuals are in competitive relationship with each other in the game system. A relatively complete theoretical framework for the multi-agent competition in non-cooperative games has been established by the pioneering work of [6, 7]. As a result, research on large-population games based on this setting has yielded rich results. Examples include a linear-quadratic-Gaussian game with a major player and a large number of minor players [8], a class of linear-quadratic-Gaussian control problems minimizing a social cost [9], a class of consensus games problem [10], a linear-quadratic mean-field (MF) game [11], a backward mean-field linear-quadratic-Gaussian game with weakly coupled large-population stochastic systems [12], partially observed stochastic dynamical systems of McKean-Vlasov type stochastic differential equations [13], and so on.

In the literature mentioned above, the initial state of a considered stochastic system is assumed to be independent of its noises. However, there are many situations that show some correlations between them, *e.g.*, insider trading in financial markets [14–16] and radar tracking problem [17], which means that the initial value

Keywords and phrases: Anticipative large-population LQ game, partial observation, enlargement of filtration, decentralized equilibrium strategy, filtering equation, ϵ-Nash equilibrium.

[1] School of Mathematical Sciences, Guizhou Normal University, Guiyang 550025, PR China.
[2] School of Big Data and Computer Science, Guizhou Normal University, Guiyang 550025, PR China.

[*] Corresponding author: yonghuizhou@gznu.edu.cn

of the state is *anticipative* (*i.e.*, not adapted). Recently, anticipative filtering problems have attracted attention that differ from the classical non-anticipative filtering theory described in [18–20]. It was Aase, Bjuland, and Øksendal [21] who first proposed an anticipative filtering problem and obtained a linear filtering equation by the expansion of filtration [22–25]. Inspired by the same idea, Tindel, Liu, and Lin [26] considered a nonlinear filtering problem for a multi-dimensional anticipative signal-observation system system, derived the corresponding filtering equation, and applied it to an anticipative radar tracking problem, which provides a much better estimation than the classical adapted Kalman filter.

Of course, large-population games may encounter some anticipative settings, such as finite datum, latent process, imperfect information, or access to some noisy observation of their own state. In this paper, we propose a model of an anticipative large-population LQ game with partial observations, where the initial value of the individual state is not independent of some observation noises.

Our main contributions are stated as follows.

(i) By using the method of expansion of filtration, the proposed anticipative large-population LQ game system with partial observations can be transformed into an equivalent adaptive one evolved by a higher dimensional signal-observation system. Under this system, an adapted large-population game model is introduced.

(ii) For the adapted extended system mentioned above, an adapted decentralized control problem is constructed by freezing a term instead of the average state, which may be thought of its limiting version. Using the separation principle, filtering theory, and squared compensatory technology, a closed-form decentralized equilibrium strategy is provided for each individual.

(iii) An ϵ-Nash equilibrium is obtained based on a profile of N decentralized strategies, along with a consistency condition that satisfies a forward-backward stochastic differential equation, which leads to the corresponding results in the independence setting discussed in [27].

This paper is structured as follows: Section 2 states problem formulation. In Section 3, an equivalent non-anticipative large-population game is established. In Section 4, for a limiting version of the above adapted game by freezing its average state, a decentralized equilibrium strategy with a consistency condition for each individual is solved. An ϵ-Nash equilibrium according to the profile of decentralized strategies is proved in Section 5. We provide a specific example in Section 6. And conclusions are drawn in the final Section.

2. Anticipative large-population LQ game model

In this paper, N, i, n_1, n_2, l, k are all natural numbers, and T is a fixed time horizon. Let $\mathbb{R}^{n_1}(\mathbb{R}^{n_1\times n_2})$ be the $n_1(n_1\times n_2)$-dimensional Euclidean space with the norm denoted by $|\cdot|$, and $\mathcal{S}^{n_1}$ be the set of $\mathbb{R}^{n_1\times n_1}$-valued symmetric matrices. $L^2(0,T;\mathbb{R}^{n_1})(L^2(0,T;\mathbb{R}^{n_1\times n_2}))$ represents the space of all deterministic functions x with values in $\mathbb{R}^{n_1}(\mathbb{R}^{n_1\times n_2})$ satisfying $\int_0^T|x(t)|^2\mathrm{d}t<+\infty$. Denote $L^\infty(0,T;\mathbb{R}^{n_1})(L^\infty(0,T;\mathbb{R}^{n_1\times n_2}), L^\infty(0,T;\mathcal{S}^{n_1}))$ the space of uniformly bounded functions with values in $\mathbb{R}^{n_1}(\mathbb{R}^{n_1\times n_2},\mathcal{S}^{n_1})$, and $C([0,T];\mathbb{R}^{n_1})$ $(C([0,T];\mathbb{R}^{n_1\times n_2}))$ the space of the continuous functions on $[0,T]$ with values in $\mathbb{R}^{n_1}(\mathbb{R}^{n_1\times n_2})$. For a given vector or matrix M, M^* stands for its transpose. Let $M\in\mathcal{S}^{n_1}$ be positive (semi)definite, denoted by $M>(\geqslant)0$. And if $M\in L^\infty(0,T;\mathcal{S}^{n_1})$ and $M(t)>(\geqslant)0$ for any $t\in[0,T]$, then M is also called positive (semi)definite and is aslo denoted by $M>(\geqslant)0$ without confusion.

We now consider a large-population system with N individual agents $\mathcal{A}_i, i=1,2,...,N$, on a finite horizon $[0,T]$. $(\Omega,\mathbb{F},\mathcal{F}_T,P)$ is a complete probability space on which a standard $(n_1\times 1\times N+l\times 1\times N+n_2\times 1)$-dimensional standard Brownain motion $(w_i,\nu_i,w;1\leqslant i\leqslant N)$ is defined, and $\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ with $\mathcal{F}_t\triangleq\sigma\{w_{is},\nu_{is},w_s;0\leqslant s\leqslant t,1\leqslant i\leqslant N\}$ and $\mathcal{F}_t^w=\sigma\{w_s;0\leqslant s\leqslant t\}$ for $t\in[0,T]$. For the agent $\mathcal{A}_i$, state x_i controlled by her/his action u_i is governed by the following linear dynamics for $t\in[0,T]$

$$\mathrm{d}x_{it}=[a(t)x_{it}+b(t)u_{it}+c(t)x_t^{(N)}+m(t)]\mathrm{d}t+\sigma(t)\mathrm{d}w_{it}+\widetilde{\sigma}(t)\mathrm{d}w_t,\ x_i(0)=x_{i0},\tag{2.1}$$

where x_{i0} is the initial state and $x_t^{(N)}\triangleq\frac{1}{N}\sum\limits_{i=1}^{N}x_{it}$ stands for the state-average of multi-agent in large-population system, and w is a common noise exactly as in [27, 28]. Although the agent $\mathcal{A}_i$ does not possess the information on the state x_i, but she/he can observe a process z_i satisfying the linear dynamics below

$$\mathrm{d}z_{it} = [h(t)x_{it} + f(t)x^{(N)} + \phi(t)]\mathrm{d}t + \mathrm{d}\nu_{it},\ z_i(0) = 0. \tag{2.2}$$

Then, the agent $\mathcal{A}_i$ acquires the available information up to time t as $\mathcal{Z}_t^i \triangleq \sigma\{z_{is}; 0 \leqslant s \leqslant t\} \vee \mathcal{F}_t^w$. In addition, we assume that there exists a correlation function ρ (not zero in general) between the initial state x_{i0} of the agent $\mathcal{A}_i$ and her/his observation noise ν_i.

In the study of large-population games[11, 27, 29], it is often assumed that the initial state x_{i0} of the underlying state process is independent of all noises in the agent's evolutionary system. However, the hypothesis of independence is not sufficient in a variety of realistic situations, such as insider trading [14] and filter problem for an anticipative signal-observation system [21, 26]. The anticipative system in [21, 26] is resulted in a correlation function, denoted ρ, between the initial state of signal and the observation noise. Thus, in general, the system (2.1)–(2.2) may be *anticipative*. For example, let $x_{i0} = \nu_{iT}$, then $x_{i0} \notin \mathcal{F}_t$ for any $t \in [0, T]$, and in this case, the evolutionary system $(x_{it}, z_{it}) \notin \mathcal{F}_t$ within the framework we have set up.

Suppose that an admissible set of the agent $\mathcal{A}_i$ based on the available information $\mathcal{Z}_t^i$ is

$$\mathcal{U}_i \triangleq \{u_i | u_i \in L^2_{\mathcal{Z}^i}(0, T; \mathbb{R}^k)\},$$

and the cost function is taken as the form

$$\mathcal{J}_i(u_i, u_{-i}) = E\left(\int_0^T (x_{it} - x_t^{(N)})^* Q(t)(x_{it} - x_t^{(N)}) + u_{it}^* R(t) u_{it}\mathrm{d}t + x_{iT}^* G x_{iT}\right), \tag{2.3}$$

for $u_i \in \mathcal{U}_i$ and $u_{-i} = (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N)$, where Q and R are the weight matrix functions of the state and the control in the running cost, respectively, and G is the terminal weight of the state.

To ensure that our game model is well defined, we specify some necessary assumptions below.

(A1). The coefficients of the signal-observation system system (x_i, z_i) for the agent $\mathcal{A}_i$ satisfy

$$\begin{cases} a(\cdot) \in L^\infty(0, T; \mathbb{R}^{n_1 \times n_1}), \quad b(\cdot) \in L^\infty(0, T; \mathbb{R}^{n_1 \times k}), \quad c(\cdot) \in L^\infty(0, T; \mathbb{R}^{n_1 \times n_1}), \\ m(\cdot) \in L^2(0, T; \mathbb{R}^{n_1}), \quad \sigma(\cdot) \in L^2(0, T; \mathbb{R}^{n_1 \times n_1}), \quad \widetilde{\sigma}(\cdot) \in L^2(0, T; \mathbb{R}^{n_1 \times n_2}), \\ h(\cdot) \in L^\infty(0, T; \mathbb{R}^{l \times n_1}), \quad f(\cdot) \in L^\infty(0, T; \mathbb{R}^{l \times n_1}), \quad \phi(\cdot) \in L^2(0, T; \mathbb{R}^l). \end{cases}$$

(A2). For $1 \leqslant i \leqslant N$, the family $(x_{i0}, \nu_{it}, t \in [0, T])$ is jointly centered Gaussian, independent of w_i and w, and the correlation function between x_{i0} and the observation noise ν_i is the same, defined by

$$\rho(t) = E[\nu_{it} x_{i0}^*],$$

where $\rho(\cdot) \in C^2([0, T]; \mathbb{R}^{l \times n_1})$. Moreover, $x_{i0}, i = 1, 2, ..., N$, are independent and identically normal distributed as $N(0, \Sigma)$.

(A3). The weight matrixes of the cost function satisfy

$$\begin{cases} Q(\cdot) \in L^\infty(0, T; \mathcal{S}^{n_1}),\ Q \geqslant 0, \quad R(\cdot) \in L^\infty(0, T; \mathcal{S}^k),\ R \geqslant \delta I,\ \textit{for some}\ \delta > 0, \\ G \in \mathcal{S}^{n_1},\ G \geqslant 0. \end{cases}$$

Problem (AP): Under (A1)–(A3), seek a strategy profile $\bar{u} = (\bar{u}_1, \cdots, \bar{u}_N)$ such that for all agents $i = 1, 2, ..., N$, $\bar{u}_i \in \mathcal{U}_i$ and

$$\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) = \inf_{u_i \in \mathcal{U}_i} \mathcal{J}_i(u_i, \bar{u}_{-i}),$$

where the state is governed by (2.1)–(2.2), and the agent's cost function is described by (2.3).

If such a strategy $\bar{u}$ exists, then it is called a Nash equilibrium. And due to the anticipativeness of the signal-observation system (2.1)–(2.2), Problem (AP) is then called *an anticipative large-population LQ game* when the number of players is large enough. Obviously, it is just an adaptive large-population LQ game in [27] if $\rho(t) \equiv 0$ for $t \in [0,T]$.

Since it is very difficult to directly search a Nash equilibrium for Problem (AP), we will only consider its ϵ-Nash equilibrium, similar to the solving of the non-anticipative large-population problems in [11, 27].

Definition 2.1. A profile $u = (u_1, \cdots, u_N)$ of controls $u_i \in \mathcal{U}_i, i = 1, 2, ..., N$, is called an ϵ-Nash equilibrium for Problem (AP), if there exists $\epsilon = \epsilon_N \geqslant 0$ with $\epsilon_N \to 0$ as $N \to \infty$ such that for all agents $i = 1, 2, ..., N$ and any strategy $u_i' \in \mathcal{U}_i$,

$$\mathcal{J}_i(u_i, u_{-i}) \leqslant \mathcal{J}_i(u_i', u_{-i}) + \epsilon.$$

In order to obtain an ϵ-Nash equilibrium for Problem (AP), we will break it into three steps: (i) an equivalent adapted game is obtained; (ii) a decentralized equilibrium strategy for each individual is given for a limit version of the equivalent adapted game; (iii) an ϵ-Nash equilibrium is deduced.

3. An equivalent adapted game model

Since Problem (AP) is anticipative, the methods used to solve the adaptive large-population LQ game presented in [27] cannot be directly applied. To overcome this difficulty, we will make use of the enlarged filtration in [23] or [26] to turn it into an adapted one, as an insider who knows the initial signal x_{i0} for each $i = 1, 2, ..., N$. The following lemma about enlarged filtration plays an important role in the solving of Problem (AP), and it is a synthesis version of Lemma 2.1 and Lemma 2.2 in [26] with its proof omitted.

Lemma 3.1. *Denote by, for $(s,t) \in [0,t] \times [0,T]$,*

$$\begin{cases} g(t) = \displaystyle\int_0^t \rho'(s)(\Sigma - \int_0^s \rho^{'*}(r)\rho^{'}(r)dr)^{-1}\mathrm{d}s, \\ p(t) = \rho''(t)^*, \;\; q(t) = -g(t)\rho''(t)^* - g'(t)\rho'(t)^*, \\ \lambda(t,s) = g(t)p(s) + q(s). \end{cases}$$

If $\rho'(T) \neq 0$, $g \in C([0,T];\mathbb{R}^{l\times n_1})$, $p \in C([0,T];\mathbb{R}^{n_1\times l})$, q and $\lambda \in C([0,T];\mathbb{R}^{l\times l})$, then the process $\widetilde{\nu}_{it}$, defined

$$\widetilde{\nu}_{it} = \nu_{it} - \int_0^t \lambda(t,s)\nu_{is}\mathrm{d}s - g(t)x_{i0}, \tag{3.1}$$

is a $\mathcal{G}_t$-Brownian motion, where $\mathcal{G}_t = \bigvee_{i=1}^N \mathcal{G}_{it}$ with $\mathcal{G}_{it} = \sigma\{x_{i0}, \nu_{is}, w_{is}; 0 \leqslant s \leqslant t\} \vee \mathcal{F}_t^w$ for $i = 1, 2, ..., N$.

With the help of the above lemma, we can get an extended adaptive signal-observation system containing the original state (2.1)–(2.2), as in the following theorem.

Theorem 3.2. *Under the assumptions in Lemma 3.1, the $\mathcal{G}_t$-adapted extended signal-observation system $(\mathbf{x}_i, z_i)$, which contains the primary (x_i, z_i) driven by (2.1)–(2.2), satisfies the following*

$$\begin{cases} \mathrm{d}\mathbf{x}_{it} = [\mathbf{a}(t)\mathbf{x}_{it} + \mathbf{b}(t)u_{it} + \mathbf{c}(t)\mathbf{x}^{(N)} + \mathbf{m}(t)]\mathrm{d}t + \sigma_1(t)\mathrm{d}w_{it} + \sigma_2(t)\mathrm{d}w_t + \sigma_0\mathrm{d}\widetilde{\nu}_{it}, \\ \mathrm{d}z_{it} = [\mathbf{h}(t)\mathbf{x}_{it} + \mathbf{f}(t)\mathbf{x}^{(N)} + \phi(t)]\mathrm{d}t + \mathrm{d}\widetilde{\nu}_{it}, \end{cases} \tag{3.2}$$

where the process $\mathbf{x}_i, \mathbf{x}^{(N)}$ *and* $x_i^\star$ *are denoted by* $\mathbf{x}_i = (x_i, x_i^\star, \nu_i)^*, \mathbf{x}^{(N)} = (x^{(N)}, x^{\star(N)}, \nu^{(N)})^*$ *and* $x_{it}^\star = x_{i0} + \int_0^t p(s)\nu_{is}\mathrm{d}s$, *respectively, and the coefficients* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{m}, \sigma_1, \sigma_2, \sigma_0, \mathbf{h}, \mathbf{f}$ *are as below*

$$\mathbf{a}(t) = \begin{bmatrix} a(t) & 0 & 0 \\ 0 & 0 & p(t) \\ 0 & g'(t) & \lambda(t,t) \end{bmatrix}, \mathbf{b}(t) = \begin{bmatrix} b(t) \\ 0 \\ 0 \end{bmatrix}, \mathbf{c}(t) = \begin{bmatrix} c(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{m}(t) = \begin{bmatrix} m(t) \\ 0 \\ 0 \end{bmatrix},$$
$$\sigma_1(t) = \begin{bmatrix} \sigma(t) \\ 0 \\ 0 \end{bmatrix}, \sigma_2(t) = \begin{bmatrix} \widetilde{\sigma}(t) \\ 0 \\ 0 \end{bmatrix}, \sigma_0 = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \mathbf{h}(t) = [h(t)\ g'(t)\ \lambda(t,t)], \mathbf{f}(t) = [f(t)\ 0\ 0].$$

Proof. If we bring the function $\lambda(t,s) = g(t)p(s) + q(s)$ into (3.1), then ν_i can be rewritten as

$$\nu_{it} = \widetilde{\nu}_{it} + \int_0^t q(s)\nu_{is}\mathrm{d}s + g(t)x_{it}^\star.$$

In addition, by the integral formula $\varphi_t\chi_t - \varphi_0\chi_0 = \int_0^t \varphi_s \mathrm{d}\chi_s + \int_0^t \chi_s \mathrm{d}\varphi_s$, if we set $\varphi = g$ and $\chi = x^\star$, then the process ν_i can be also expressed as the form

$$\nu_{it} = \int_0^t [g'(s)x_{is}^\star + \lambda(s,s)\nu_{is}]\mathrm{d}s + \widetilde{\nu}_{it}.$$

Therefore, the observation process z_i can be rewritten as

$$\mathrm{d}z_{it} = [h(t)x_{it} + f(t)x_t^{(N)} + g'(t)x_{it}^\star + \lambda(t,t)\nu_{it} + \phi(t)]\mathrm{d}t + \mathrm{d}\widetilde{\nu}_{it},\ z_{i0} = 0,$$

or as the vector form

$$\mathrm{d}z_{it} = [\mathbf{h}(t)\mathbf{x}_{it} + \mathbf{f}(t)\mathbf{x}_t^{(N)} + \phi(t)]\mathrm{d}t + \mathrm{d}\widetilde{\nu}_{it},\ z_{i0} = 0.$$

According to the explicit expressions for $x_i^\star$ and ν_i, we can directly assert that the extended $\mathcal{G}_t$-adapted state $(\mathbf{x}_i, z_i)$ is governed by (3.2) and contains the original state (x_i, z_i), which is driven by (2.1)–(2.2). Thus, the proof is complete. □

The above theorem tell us that for the agent $\mathcal{A}_i$ as an insider, the initial state $\mathbf{x}_{i0}$ is independent of all noises for the $\mathcal{G}_t$-adapted extended state $(\mathbf{x}_i, z_i)$. However, the extended state (3.2) of the agent $\mathcal{A}_i$ is incompatible with the original cost function (2.3). Therefore, we are going to design the following equivalent cost function to (2.3)

$$\bar{\mathcal{J}}_i(u_i, u_{-i}) = E\left(\int_0^T (\mathbf{x}_{it} - \mathbf{x}_t^{(N)})^* Q_1(t)(\mathbf{x}_{it} - \mathbf{x}_t^{(N)}) + u_{it}^* R(t) u_{it}\mathrm{d}t + \mathbf{x}_{iT}^* G_1 \mathbf{x}_{iT}\right), \tag{3.3}$$

where the weight coefficient matrices Q_1 and G_1 are given by

$$Q_1(t) = \begin{bmatrix} Q(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad G_1 = \begin{bmatrix} G & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

that is, for any $u_i \in \mathcal{U}_i, i = 1, 2, ..., N$,

$$\mathcal{J}_i(u_i, u_{-i}) = \bar{\mathcal{J}}_i(u_i, u_{-i}).$$

By the expansion of filtration presented in Lemma 3.1, a new game consists of a $\mathcal{G}_t$-adapted extended evolutionary state and a modified compatible cost function, in which each agent adopts the same strategy as in the original anticipative game. In order to solve the original Problem (AP), we now present the following equivalent non-anticipative problem.

Problem(NAP): Under (A1)–(A3), seek a profile of $\bar{u}_i \in \mathcal{U}_i$ for $i = 1, 2, ..., N$ such that

$$\bar{\mathcal{J}}_i(\bar{u}_i, \bar{u}_{-i}) = \inf_{u_i \in \mathcal{U}_i} \bar{\mathcal{J}}_i(u_i, \bar{u}_{-i}),$$

with the state and the cost function governed by (3.2) and (3.3), respectively.

Here, we obtain a higher-dimensional equivalent game problem through the expansion of filtration. Similarly, seeking Nash equilibrium of the Problem (NAP) is quite difficult as the number of agents increases. We then consider a class of approximate equilibria as the number of agents becomes sufficiently large, *i.e.*, ϵ-Nash equilibrium.

4. A decentralized strategy for a limiting non-anticipative version

To derive an ϵ-Nash equilibrium of the original anticipative problem, we make a simple transformation by using expansion of filtration to build a new adapted equivalent game Problem (NAP) from Problem (AP). Inspired by the idea in [27], to establish an ϵ-Nash equilibrium for Problem (NAP), we will first obtain a decentralized strategy for a class of limiting control problems, where we use a freezing term $\mathbf{x}^0$ instead of the state average $\mathbf{x}^{(N)}$ in (3.2) and (3.3)(in fact, $\mathbf{x}^0$ is assumed to be the limit of the average state $\mathbf{x}^{(N)}$ with the consistency condition in subsection 4.3).

For each agent $\mathcal{A}_i$, $i = 1, 2, ..., N$, consider below an auxiliary signal-observation system in terms of $(\bar{\mathbf{x}}_i, \bar{z}_i)$ controlled by u_i

$$\begin{cases} \mathrm{d}\bar{\mathbf{x}}_{it} = \{\mathbf{a}(t)\bar{\mathbf{x}}_{it} + \mathbf{b}(t)u_{it} + \mathbf{c}(t)\mathbf{x}^0_t + \mathbf{m}(t)\}\mathrm{d}t + \sigma_1(t)\mathrm{d}w_{it} + \sigma_2(t)\mathrm{d}w_t + \sigma_0\mathrm{d}\widetilde{\nu}_{it}, \\ \mathrm{d}\bar{z}_{it} = \{\mathbf{h}(t)\bar{\mathbf{x}}_{it} + \mathbf{f}(t)\mathbf{x}^0_t + \phi(t)\}\mathrm{d}t + \mathrm{d}\widetilde{\nu}_{it}, \\ \bar{\mathbf{x}}_{i0} = (x_{i0}, x_{i0}, 0)^*, \ \bar{z}_{i0} = 0, \end{cases} \tag{4.1}$$

with a freezing term $\mathbf{x}^0_t$ adapted to $\mathcal{F}^w_t$ and satisfying $E \sup_{0 \leqslant t \leqslant T} |\mathbf{x}^0_t|^2 < \infty$.

Next, we introduce the following two equations, one of which is a signal-observation process (α_i, β_i) below

$$\begin{cases} \mathrm{d}\alpha_{it} = [\mathbf{a}(t)\alpha_{it} + \mathbf{m}(t)]\mathrm{d}t + \sigma_1(t)\mathrm{d}w_{it} + \sigma_0\mathrm{d}\widetilde{\nu}_{it}, \ \alpha_{i0} = \bar{\mathbf{x}}_{i0}, \\ \mathrm{d}\beta_{it} = [\mathbf{h}(t)\alpha_{it} + \phi(t)]\mathrm{d}t + \mathrm{d}\widetilde{\nu}_{it}, \ \beta_{i0} = 0, \end{cases} \tag{4.2}$$

with excluding the control u_i and the aggregate term $\mathbf{x}^0$; and the other one contains $\mathbf{x}^0, u_i$ and satisfies the following dynamics

$$\begin{cases} \mathrm{d}\bar{\mathbf{x}}^1_{it} = [\mathbf{a}(t)\bar{\mathbf{x}}^1_{it} + \mathbf{b}(t)u_{it} + \mathbf{c}(t)\mathbf{x}^0_t]\mathrm{d}t + \sigma_2(t)\mathrm{d}w_t, \ \mathbf{x}^1_{i0} = 0, \\ \mathrm{d}z^1_{it} = [\mathbf{h}(t)\bar{\mathbf{x}}^1_{it} + \mathbf{f}(t)\mathbf{x}^0_t]\mathrm{d}t, \ z^1_{i0} = 0. \end{cases} \tag{4.3}$$

Here, for the systems (4.1), (4.2) and (4.3), we will call u_i an admissible control or decentralized strategy if the control u_i belongs to $\mathcal{U}_i^d$, defined by

$$\mathcal{U}_i^d = \{u_i \in L^2(0,T;\mathbb{R}^k)|\ u_i \text{ is adapted to } \mathcal{Z}_t^{\bar{z}_i,w} \text{ and } \mathcal{Z}_t^{\beta_i,w}\},$$

where $\mathcal{Z}_t^{\bar{z}_i,w} = \sigma\{\bar{z}_{is}; 0 \leqslant s \leqslant t\} \vee \mathcal{F}_t^w$ and $\mathcal{Z}_t^{\beta_i,w} = \sigma\{\beta_{is}; 0 \leqslant s \leqslant t\} \vee \mathcal{F}_t^w$.

And then, under (A1)–(A2), (4.2) exists a unique solution (α_i, β_i) by the standard argument of SDEs, and (4.3) admits a unique L^2-solution for any control $u_i \in \mathcal{U}_i^d$. Obviously, we can assert that the limiting process $(\bar{\mathbf{x}}_i, \bar{z}_i)$ can be rewritten as follows

$$\bar{\mathbf{x}}_{it} = \alpha_{it} + \bar{\mathbf{x}}_{it}^1, \quad \bar{z}_{it} = \beta_{it} + z_{it}^1. \tag{4.4}$$

Thus, the following non-anticipative problem is formulated for each player $\mathcal{A}_i$, $i = 1, 2, ..., N$.

Problem(LNAP): Under (A1)–(A3), seek a control $\hat{u}_i \in \mathcal{U}_i^d$ such that

$$J_i(\hat{u}_i) = \inf_{u_i \in \mathcal{U}_i^d} J_i(u_i),$$

where the auxiliary limiting cost function J_i satisfies

$$J_i(u_i) = E\left(\int_0^T (\bar{\mathbf{x}}_{it} - \mathbf{x}_t^0)^* Q_1(t)(\bar{\mathbf{x}}_{it} - \mathbf{x}_t^0) + u_{it}^* R(t) u_{it} \mathrm{d}t + \bar{\mathbf{x}}_{iT}^* G_1 \bar{\mathbf{x}}_{iT}\right) \tag{4.5}$$

for $u_i \in \mathcal{U}_i^d$, and the state $(\bar{\mathbf{x}}_i, \bar{z}_i)$ is governed by the system (4.1).

4.1. The filter of limiting system

Based on the separation principles outlined in [30, 31], we first present a key lemma before deriving the filtering equation for (4.1).

Lemma 4.1. *For $i = 1, 2, ..., N$, let $u_i \in \mathcal{U}_i^d$, then for $t \in [0,T]$*

$$\mathcal{Z}_t^{\bar{z}_i,w} = \mathcal{Z}_t^{\beta_i,w}. \tag{4.6}$$

Proof. Clearly, the assertion $\mathcal{Z}_t^{\bar{z}_i,w} \supseteq \mathcal{Z}_t^{\beta_i,w}$ holds, since (4.4). And now, we will check $\mathcal{Z}_t^{\bar{z}_i,w} \subseteq \mathcal{Z}_t^{\beta_i,w}$.

Let $\bar{\mathbf{x}}_{it}^1 \in \mathcal{Z}_t^{\bar{z}_i,w}$. Then by the first equation of (4.3), $\bar{\mathbf{x}}_i^1$ can be expressed explicitly as

$$\bar{\mathbf{x}}_{it}^1 = \int_0^t \Phi(t,s)(\mathbf{b}(s)u_{is} + \mathbf{c}(s)\mathbf{x}_s^0)\mathrm{d}s + \int_0^t \Phi(t,s)\sigma_2(s)\mathrm{d}w_s,$$

where $\Phi(t,s) = \exp\{\int_s^t \mathbf{a}(r)dr\}$. Then $\bar{\mathbf{x}}_{it}^1 \in \mathcal{Z}_t^{\beta_i,w}$, since $u_i \in \mathcal{U}_i^d$ and $\mathbf{x}_t^0$ is $\mathcal{F}_t^w$-adapted. That is, $\mathcal{Z}_t^{\bar{z}_i,w} \subseteq \mathcal{Z}_t^{\beta_i,w}$. And the proof is complete. □

For convenience, denote for $t \in [0,T]$,

$$\hat{\bar{\mathbf{x}}}_{it} = E[\bar{\mathbf{x}}_{it}|\mathcal{Z}_t^{\bar{z}_i,w}], \quad \Sigma_i(t) = E[(\bar{\mathbf{x}}_{it} - \hat{\bar{\mathbf{x}}}_{it})(\bar{\mathbf{x}}_{it} - \hat{\bar{\mathbf{x}}}_{it})^*]. \tag{4.7}$$

Theorem 4.2. *Under (A1)–(A3), the process $\hat{\bar{\mathbf{x}}}_i$ satisfies*

$$\mathrm{d}\hat{\bar{\mathbf{x}}}_{it} = [\mathbf{a}(t)\hat{\bar{\mathbf{x}}}_{it} + \mathbf{b}(t)u_{it} + \mathbf{c}(t)\mathbf{x}_t^0 + \mathbf{m}(t)]\mathrm{d}t + [\Sigma_i(t)\mathbf{h}^*(t) + \sigma_0]dI_{it} + \sigma_2(t)\mathrm{d}w_t, \tag{4.8}$$

where Σ_i *satisfies*

$$\dot{\Sigma}_i(t) = \mathbf{a}(t)\Sigma_i(t) + \Sigma_i(t)\mathbf{a}^*(t) + (\sigma_1(t)\sigma_1^*(t) + \sigma_0\sigma_0^*) - [\sigma_0 + \Sigma_i(t)\mathbf{h}^*(t)] \times [\sigma_0 + \Sigma_i(t)\mathbf{h}^*(t)]^*, \tag{4.9}$$

and the innovation process I_i *is a* $\mathcal{Z}_t^{\bar{z},w}$*-Browian motion satisfying*

$$dI_{it} = \mathrm{d}\bar{z}_{it} - [\mathbf{h}(t)\hat{\bar{\mathbf{x}}}_{it} + \mathbf{f}(t)\mathbf{x}_t^0 + \phi(t)]\mathrm{d}t.$$

Proof. Obviously, the common noise w is independent of $w_i, \widetilde{\nu}_i$ and x_{i0} in (4.2), then w is independent of (α_i, β_i). Therefore, we can directly obtain the process $\hat{\alpha}_i$ satisfying

$$\hat{\alpha}_{it} = E[\alpha_{it}|\mathcal{Z}_t^{\beta_i,w}] = E[\alpha_{it}|\mathcal{F}_t^{\beta_i}].$$

According to Theorem 12.7 in [20], the process $\hat{\alpha}_i$ is evolved as

$$\mathrm{d}\hat{\alpha}_{it} = [\mathbf{a}(t)\hat{\alpha}_{it} + \mathbf{m}(t)]\mathrm{d}t + [\Sigma_i(t)\mathbf{h}^*(t) + \sigma_0]dI_{it}, \ \ \hat{\alpha}_{i0} = 0.$$

By (4.4), (4.6) and (4.7), we have

$$\hat{\bar{\mathbf{x}}}_{it} = \mathbf{x}_{it}^1 + E[\alpha_{it}|\mathcal{Z}_t^{\bar{z}_i,w}] = \mathbf{x}_{it}^1 + E[\alpha_{it}|\mathcal{Z}_t^{\beta_i,w}].$$

So, the process $\hat{\bar{\mathbf{x}}}_i$ satisfies

$$\mathrm{d}\hat{\bar{\mathbf{x}}}_{it} = [\mathbf{a}(t)\hat{\bar{\mathbf{x}}}_{it} + \mathbf{b}(t)u_{it} + \mathbf{c}(t)\mathbf{x}_t^0 + \mathbf{m}(t)]\mathrm{d}t + [\Sigma_i(t)\mathbf{h}^*(t) + \sigma_0]dI_{it} + \sigma_2(t)\mathrm{d}w_t.$$

If we define an error process $\widetilde{\mathbf{x}}_i \triangleq \bar{\mathbf{x}}_i - \hat{\bar{\mathbf{x}}}_i$, then it satisfies

$$\mathrm{d}\widetilde{\mathbf{x}}_{it} = \mathbf{a}(t)\widetilde{\mathbf{x}}_{it}\mathrm{d}t + \sigma_1(t)\mathrm{d}w_{it} - [\Sigma_i(t)\mathbf{h}^*(t) + \sigma_0]dI_{it}.$$

And since $\Sigma_i(t) = E[\widetilde{\mathbf{x}}_{it}\widetilde{\mathbf{x}}_{it}^*]$, then by Itô formula, we can get the ODE (4.9). □

4.2. Optimal control of problem (LNAP)

Note that the above process $\widetilde{\mathbf{x}}_i$ is independent of the decentralized strategy $u_i \in \mathcal{U}_i^d$. Moreover, it is orthogonal to $\hat{\bar{\mathbf{x}}}_i$, or,

$$E < \widetilde{\mathbf{x}}_{it}, \hat{\bar{\mathbf{x}}}_{it} >= 0, \quad E|\bar{\mathbf{x}}_{it}|^2 = E|\hat{\bar{\mathbf{x}}}_{it}|^2 + E|\widetilde{\mathbf{x}}_{it}|^2.$$

Thus, the cost function (4.5) can be split into two parts, *i.e.*,

$$J_i(u_i) = \hat{J}_i(u_i) + \widetilde{J}_i,$$

where

$$\hat{J}_i(u_i) = E\left(\int_0^T (\hat{\bar{\mathbf{x}}}_{it} - \mathbf{x}_t^0)^* Q_1(t)(\hat{\bar{\mathbf{x}}}_{it} - \mathbf{x}_t^0) + u_{it}^* R(t) u_{it}\mathrm{d}t + \hat{\bar{\mathbf{x}}}_{iT} G_1 \hat{\bar{\mathbf{x}}}_{iT}\right), \tag{4.10}$$

and

$$\widetilde{J}_i = \int_0^T tr[Q_1(t)\Sigma_i(t)]\mathrm{d}t + tr[G_1\Sigma_i(T)].$$

Clearly, $\widetilde{J}_i$ is a fixed constant, and it is independent of the decentralized strategy u_i.

We now introduce the following full information control problem, which is equivalent to Problem (LNAP).
Problem (FLNAP): Find a decentralized strategy $\hat{u}_i \in \mathcal{U}_i^d$ for each agent $\mathcal{A}_i$, $i = 1, 2, ..., N$, such that

$$\hat{J}_i(\hat{u}_i) = \inf_{u_i \in \mathcal{U}_i^d} \hat{J}_i(u_i),$$

where the state $\hat{\hat{\mathbf{x}}}_i$ and the cost function $\hat{J}_i$ are subject to (4.8) and (4.10), respectively.
To solve the above problem, we first give two equations in terms of P_i and (γ_i, Λ_i) below,

$$\dot{P}_i(t) + P_i(t)\mathbf{a}(t) + \mathbf{a}^*(t)P_i(t) + Q_1(t) - P_i(t)\mathbf{b}(t)R(t)^{-1}\mathbf{b}^*(t)P_i(t) = 0, \;\; P_i(T) = G_1, \tag{4.11}$$

and

$$\begin{cases} \mathrm{d}\gamma_{it} = -[P_i(t)(\mathbf{c}(t)\mathbf{x}_t^0 + \mathbf{m}(t)) + (\mathbf{a}^*(t) - P_i(t)\mathbf{b}(t)R(t)^{-1}\mathbf{b}^*(t))\gamma_{it} - Q_1(t)\mathbf{x}_t^0]\mathrm{d}t + \Lambda_{it}\mathrm{d}w_t, \\ \gamma_{iT} = 0, \end{cases} \tag{4.12}$$

respectively.
Under (A1)–(A3), (4.11) has a unique solution P_i by Theorem 7.2 in [32]. Since (4.12) is a linear BSDE with the coefficients depending on ρ, its has a unique solution (γ_i, Λ_i) by Theorem 3.1 in [34]. Note that the freezing term $\mathbf{x}^0$ is fixed at this stage, and the two solutions $P_i, (\gamma_i, \Lambda_i)$ do not depend on the choice of index i.
Note that the problem (FLNAP) is an LQ control problem, and the following theorem can be derived, inspired by [33].

Theorem 4.3. *Under (A1)–(A3), for Problem (LNAP), there exists a unique optimal control as*

$$\hat{u}_{it} = -R^{-1}(t)\mathbf{b}^*(t)(P_i(t)\hat{\hat{\mathbf{x}}}_{it} + \gamma_{it}), \;\; t \in [0, T], \tag{4.13}$$

where $P_i, (\gamma_i, \Lambda_i)$ satisfy (4.11) and (4.12), respectively.

Proof. Firstly, from the terminal linear structure of the cost function (4.10), we may construct an affine function p_i as

$$p_{it} = P_i(t)\hat{\hat{\mathbf{x}}}_{it} + 2\gamma_{it}, \;\; \forall t \in [0, T]. \tag{4.14}$$

We then get the following result by Itô formula

$$\begin{aligned} \mathrm{d}p_{it} &= d(P_i(t)\hat{\hat{\mathbf{x}}}_{it}) + 2\mathrm{d}\gamma_{it} = \dot{P}_i(t)\hat{\hat{\mathbf{x}}}_{it}\mathrm{d}t + P_i(t)\mathrm{d}\hat{\hat{\mathbf{x}}}_{it} + 2\mathrm{d}\gamma_{it} \\ &= \{-[\mathbf{a}^*(t)P_i(t) + Q_1(t) - P_i(t)\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P_i(t)]\hat{\hat{\mathbf{x}}}_{it} + P_i(t)\mathbf{b}(t)u_{it} \\ &\quad - 2(\mathbf{a}^*(t) - P_i(t)\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t))\gamma_{it} + 2Q_1(t)\mathbf{x}_t^0 - P_i(t)(\mathbf{c}(t)\mathbf{x}_t^0 + \mathbf{m}(t))\}\mathrm{d}t \\ &\quad + P_i(t)[\Sigma_i(t)\mathbf{h}^*(t) + \sigma_0]dI_{it} + [P_i(t)\sigma_2(t) + 2\Lambda_{it}]\mathrm{d}w_t. \end{aligned}$$

And applying Itô formula again to $t \to < p_{it}, \hat{\hat{\mathbf{x}}}_{it} >$, we have

$$\begin{aligned} E < p_{it}, \hat{\hat{\mathbf{x}}}_{it} >= & E\left(\int_0^t < \mathrm{d}p_{is}, \hat{\hat{\mathbf{x}}}_{is} > + \int_0^t < p_{is}, \mathrm{d}\hat{\hat{\mathbf{x}}}_{is} > + \int_0^t < \mathrm{d}p_{is}, \mathrm{d}\hat{\hat{\mathbf{x}}}_{is} >\right) \\ = & E\left(\int_0^t < -[Q_1(s) - P_i(s)\mathbf{b}(s)R^{-1}(s)\mathbf{b}^*(s)P_i(s)]\hat{\hat{\mathbf{x}}}_{is} + 2P_i(s)\mathbf{b}(s)u_{is}\right. \\ & + 2P_i(s)\mathbf{b}(s)R^{-1}(s)\mathbf{b}^*(s)\gamma_{is} + 2Q_1(s)\mathbf{x}_s^0, \hat{\hat{\mathbf{x}}}_{is} > \mathrm{d}s + \int_0^t 2 < \gamma_{is}, \mathbf{b}(s)u_{is} > \\ & + 2 < \gamma_{is}, \mathbf{c}(s)\mathbf{x}_s^0 + \mathbf{m}(s) > \mathrm{d}s + \int_0^t < P_i(s)[\Sigma_i(s)\mathbf{h}^*(s) + \sigma_0], \Sigma_i(s)\mathbf{h}^*(s) + \sigma_0 > \\ & \left. + < P_i(s)\sigma_2(s) + 2\Lambda_{is}, \sigma_2(s) > \mathrm{d}s\right). \end{aligned}$$

Together with (4.11), (4.12) and (4.14), the cost function (4.10) is rewritten as

$$\begin{aligned} \hat{J}_i(u_i) = & E\left(\int_0^T < Q_1(t)\hat{\hat{\mathbf{x}}}_{it}, \hat{\hat{\mathbf{x}}}_{it} > -2 < Q_1(t)\mathbf{x}_t^0, \hat{\hat{\mathbf{x}}}_{it} > + < Q_1(t)\mathbf{x}_t^0, \mathbf{x}_t^0 >\right. \\ & \left. + < R(t)u_{it}, u_{it} > \mathrm{d}t + < G_1\hat{\hat{\mathbf{x}}}_{iT}, \hat{\hat{\mathbf{x}}}_{iT} >\right) \\ = & E\left(\int_0^T |R^{\frac{1}{2}}(t)[u_{it} + R^{-1}(t)\mathbf{b}^*(t)(P_i(t)\hat{\hat{\mathbf{x}}}_{it} + \gamma_{it})]|^2 + 2 < \gamma_{it}, \mathbf{c}(s)\mathbf{x}_t^0 + \mathbf{m}(t) >\right. \\ & - < \mathbf{b}^*(t)\gamma_{it}, R^{-1}(t)\mathbf{b}^*(t)\gamma_{it} > + < Q_1(t)\mathbf{x}_t^0, \mathbf{x}_t^0 > \mathrm{d}t \\ & \left. + \int_0^T < P_i(t)[\Sigma_i(t)\mathbf{h}^*(t) + \sigma_0], \Sigma_i(t)\mathbf{h}^*(t) + \sigma_0 > + < P_i(t)\sigma_2(t) + 2\Lambda_{it}, \sigma_2(t) > \mathrm{d}t\right). \end{aligned}$$

Then under the assumption (A3), there exists a unique optimal control

$$\hat{u}_{it} = -R^{-1}(t)\mathbf{b}^*(t)(P_i(t)\hat{\hat{\mathbf{x}}}_{it} + \gamma_{it}).$$

□

4.3. The consistency condition

In this subsection, our goal is to derive a consistency condition that is satisfied by the decentralized strategies $\hat{u}_i$, *i.e.*, the sufficient condition for the existence of the limit term $\mathbf{x}^0$, which has been described above.

Lemma 4.4. *Let for each $i = 1, 2, ..., N$, $\mathbf{x}_i$ be the solution to the system (3.2). If there exists a process $\mathbf{x}_t^0$ adapted to $\mathcal{F}_t^w$ and satisfying $E \sup\limits_{0\leqslant t\leqslant T} |\mathbf{x}_t^0|^2 < \infty$ such that*

$$\lim_{N\to\infty} \sup_{0\leqslant t\leqslant T} E|\mathbf{x}_t^{(N)} - \mathbf{x}_t^0|^2 = 0,$$

then under (A1)–(A3), $\mathbf{x}^0$ with (γ, Λ) satisfy the following forward-backward stochastic differential equation

$$\begin{cases} \mathrm{d}\mathbf{x}_t^0 = [\mathbb{A}(t)\mathbf{x}_t^0 + \mathbb{B}(t)\gamma_t + \mathbf{m}(t)]\mathrm{d}t + \sigma_2(t)\mathrm{d}w_t, \ \mathbf{x}_0^0 = 0, \\ \mathrm{d}\gamma_t = -[\mathbb{C}(t)\mathbf{x}_t^0 + \mathbb{D}(t)\gamma_t + \mathbb{G}(t)]\mathrm{d}t + \Lambda_t\mathrm{d}w_t, \ \gamma_T = 0, \end{cases} \tag{4.15}$$

where

$$\begin{cases}\mathbb{A}(t)=\mathbf{a}(t)+\mathbf{c}(t)-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t),\ \mathbb{B}(t)=-\mathbf{b}(t)R(t)^{-1}\mathbf{b}^*(t),\\ \mathbb{C}(t)=P(t)\mathbf{c}(t)-Q_1(t),\ \mathbb{D}(t)=\mathbf{a}^*(t)-P(t)\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t),\ \mathbb{G}(t)=P(t)\mathbf{m}(t),\end{cases}$$

and $P(t)=P_i(t)$ *is a solution to (4.11). Moreover,*

$$E\sup_{0\leqslant t\leqslant T}|\gamma_t|^2<\infty. \tag{4.16}$$

Proof. Similar to the proof of Lemma 2.3 in [27], for Problem (NAP), such a decentralized strategy can be designed as follows

$$\widetilde{u}_{it}=-R^{-1}(t)\mathbf{b}^*(t)(P_i(t)\hat{\mathbf{x}}_{it}+\gamma_{it}), \tag{4.17}$$

where the two controlled state processes $\mathbf{x}_i$ and $\hat{\mathbf{x}}_i$ based on the locally observed information satisfy a coupling system as

$$\begin{cases}\mathrm{d}\mathbf{x}_{it}=\{\mathbf{a}(t)\mathbf{x}_{it}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P_i(t)\hat{\mathbf{x}}_{it}+\gamma_{it})+\mathbf{c}(t)\mathbf{x}_t^{(N)}+\mathbf{m}(t)\}\mathrm{d}t+\sigma_1(t)\mathrm{d}w_{it}\\ \qquad\qquad+\sigma_2(t)\mathrm{d}w_t+\sigma_0\mathrm{d}\widetilde{\nu}_{it},\\ \mathrm{d}\hat{\mathbf{x}}_{it}=\{\mathbf{a}(t)\hat{\mathbf{x}}_{it}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P_i(t)\hat{\mathbf{x}}_{it}+\gamma_{it})+\mathbf{c}(t)\mathbf{x}_t^0+\mathbf{m}(t)\}\mathrm{d}t\\ \qquad\qquad+\sigma_2(t)\mathrm{d}w_t+[\Sigma_i(t)\mathbf{h}(t)+\sigma_0][(\mathbf{h}(t)(\mathbf{x}_{it}-\hat{\mathbf{x}}_{it})+\mathbf{f}(t)(\mathbf{x}_t^{(N)}-\mathbf{x}_t^0))\mathrm{d}t+\mathrm{d}\widetilde{\nu}_{it}],\\ \mathbf{x}_{i0}=(x_{i0},x_{i0}^\star,0)^*,\ \hat{\mathbf{x}}_{i0}=(0,0,0)^*.\end{cases}$$

For the sake of notion simplicity, we denote $\bar{\Sigma}_i(t)\triangleq\Sigma_i(t)\mathbf{h}^*(t)+\sigma_0$. Then $\hat{\mathbf{x}}_i$ is governed by the dynamics below

$$\begin{aligned}\mathrm{d}\hat{\mathbf{x}}_{it}=&\{[\mathbf{a}(t)-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P_i(t)-\bar{\Sigma}_i(t)\mathbf{h}(t)]\hat{\mathbf{x}}_{it}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)\gamma_{it}+\mathbf{c}(t)\mathbf{x}_t^0\\ &+\mathbf{m}(t)+\bar{\Sigma}_i(t)\mathbf{h}(t)\mathbf{x}_{it}+\bar{\Sigma}_i(t)\mathbf{f}(t)(\mathbf{x}_t^{(N)}-\mathbf{x}_t^0)\}\mathrm{d}t+\bar{\Sigma}_i(t)\mathrm{d}\widetilde{\nu}_{it}+\sigma_2(t)\mathrm{d}w_t.\end{aligned}$$

Meanwhile, we set $\hat{\mathbf{x}}_t^0\triangleq\lim\limits_{N\to\infty}\frac{1}{N}\sum\limits_{i=1}^N\hat{\mathbf{x}}_{it}$, then it satisfies

$$\begin{aligned}\mathrm{d}\hat{\mathbf{x}}_t^0=&\{[\mathbf{a}(t)-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P_i(t)-\bar{\Sigma}_i(t)\mathbf{h}(t)]\hat{\mathbf{x}}_t^0+(\bar{\Sigma}_i(t)\mathbf{h}(t)+\mathbf{c}(t))\mathbf{x}_t^0\\ &-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)\gamma_{it}+\mathbf{m}(t)\}\mathrm{d}t+\sigma_2(t)\mathrm{d}w_t,\ \hat{\mathbf{x}}_{i0}=(0,0,0)^*.\end{aligned}$$

Moreover, since $\mathbf{x}_t^0\triangleq\lim\limits_{N\to\infty}\frac{1}{N}\sum\limits_{i=1}^N\mathbf{x}_{it}$, then we have

$$\begin{aligned}\mathrm{d}\mathbf{x}_t^0=&\{(\mathbf{a}(t)+\mathbf{c}(t))\mathbf{x}_t^0-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P_i(t)\hat{\mathbf{x}}_t^0-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)\gamma_{it}\\ &+\mathbf{m}(t)\}\mathrm{d}t+\sigma_2(t)\mathrm{d}w_t,\mathbf{x}_0^0=(0,0,0)^*.\end{aligned}$$

Now define a process $\Psi\triangleq\mathbf{x}^0-\hat{\mathbf{x}}^0$, then it satisfies

$$\Psi_t=\int_0^t(\mathbf{a}(s)-\bar{\Sigma}_i(s)\mathbf{h}(s))(\mathbf{x}_s^0-\hat{\mathbf{x}}_s^0)\mathrm{d}s=\int_0^t(\mathbf{a}(s)-\bar{\Sigma}_i(s)\mathbf{h}(s))\Psi_s\mathrm{d}s,$$

and there exists a constant $\ell > 0$ by (A1)–(A3) such that

$$E|\Psi_t|^2 \leqslant \ell \int_0^t E|\Psi_s|^2 \mathrm{d}s.$$

So, by Gronwall's inequality, we can get $\Psi_t = 0$ a.s., that is to say $\mathbf{x}^0 = \hat{\mathbf{x}}^0$.

Therefore, we have

$$\begin{cases} \mathrm{d}\mathbf{x}_t^0 = \{[\mathbf{a}(t) + \mathbf{c}(t) - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P_i(t)]\mathbf{x}_t^0 - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)\gamma_{it} + \mathbf{m}(t)\}\mathrm{d}t + \sigma_2(t)\mathrm{d}w_t, \\ \mathrm{d}\gamma_{it} = -\{[P_i(t)\mathbf{c}(t) - Q_1(t)]\mathbf{x}_t^0 + [\mathbf{a}^*(t) - P_i(t)\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)]\gamma_{it} + P_i(t)\mathbf{m}(t)\}\mathrm{d}t + \Lambda_{it}\mathrm{d}w_t. \end{cases}$$

Clearly, if we denote $(\gamma, \Lambda) = (\gamma_i, \Lambda_i)$, and thus (4.15) holds. By a standard estimation arguments of SDEs and BSDEs, the inequality (4.16) holds; more certified details can be seen the proof of Lemma 2.4 in [27]. □

From the above lemma, we know that the consistency condition is made up of an FBSDE as (4.15). Next, we focus on discussing the well-posedness of this FBSDE. If we suppose that $\gamma_t = \mathbf{K}(t)\mathbf{x}_t^0 + \mathbf{\Phi}(t)$, then the equation (4.15) is decoupled with $\mathbf{K}(t)$ and $\mathbf{\Phi}(t)$ satisfying

$$\begin{cases} \dot{\mathbf{K}}(t) + \mathbf{K}(t)\mathbb{A}(t) + \mathbb{D}(t)\mathbf{K}(t) + \mathbf{K}(t)\mathbb{B}(t)\mathbf{K}(t) + \mathbb{C}(t) = 0, \ \mathbf{K}(T) = 0, \\ \dot{\mathbf{\Phi}}(t) + (\mathbb{D}(t) + \mathbf{K}(t)\mathbb{B}(t))\mathbf{\Phi}(t) + \mathbb{G}(t) + \mathbf{K}(t)\mathbf{m}(t) = 0, \ \mathbf{\Phi}(T) = 0. \end{cases} \tag{4.18}$$

Then, if the above equation has a unique solution, then (4.15) has a unique solution $(\mathbf{x}^0, \gamma_i, \Lambda_i)$. By applying the Banach fixed point theorem, we have the following existence result from [11, 27], and the proof is omitted.

Lemma 4.5. *Under (A1)–(A3), there exists a unique solution of (4.18) if*

$$\mathbb{L} < 1,$$

where

$$\mathbb{L} = T||\mathbb{C}||_T||\mathbb{B}||_T \cdot \exp\{[2||\mathbb{A}||_T + 2||\mathbb{D}||_T + ||\mathbb{B}||_T + ||\mathbb{C}||_T]T\},$$

and $||\cdot||_T$ *denotes the super-norm of matrix-valued function on* $[0, T]$.

Lemma 4.6. *Under (A1)–(A3), suppose that* $\mathbb{C}$ *is invertible and* $\phi(t, s)$ *is a fundamental solution to* $\mathbb{D}$. *Then, there exists a unique solution to (4.18) if*

$$\sqrt{T}||\phi||_T||\mathbb{A} - \mathbb{D}||_T < 1,$$

where $||\phi||_T = \sup\limits_{0\leqslant t\leqslant T} \sqrt{\int_t^T ||\phi'(s,t)\mathbb{C}_s^{\frac{1}{2}}||^2 \mathrm{d}s}$ *and* $||\mathbb{A} - \mathbb{D}||_T = \sup\limits_{0\leqslant t\leqslant T} ||(\mathbb{A} - \mathbb{D})_t\mathbb{C}_t^{-\frac{1}{2}}||$.

5. ϵ-Nash equilibrium

Our main goal is to assert ϵ-Nash equilibrium properties of the decentralized strategy profile $(\widetilde{u}_1, \cdots, \widetilde{u}_N)$ as (4.17) for Problem (NAP).

Theorem 5.1. *Under (A1)–(A3), assume that each agent* $\mathcal{A}_i$ *follows the decentralized strategy as (4.17) with* $(\mathbf{x}^0, \gamma, \Lambda)$ *driven by (4.15). Then* $(\widetilde{u}_1, \cdots, \widetilde{u}_N)$ *is an* ϵ*-Nash equilibrium for Problem (NAP) with* $\epsilon = O(\frac{1}{\sqrt{N}})$ *as*

$N\to\infty$, i.e. for $i=1,2,...,N$ and any $u_i'\in\mathcal{U}_i$,

$$|\bar{\mathcal{J}}_i(\widetilde{u}_i,\widetilde{u}_{-i})-\bar{\mathcal{J}}_i(u_i',\widetilde{u}_{-i})|=O(\frac{1}{\sqrt{N}}). \tag{5.1}$$

In what follows, we mainly focus on providing sufficient evidence to show that the equality (5.1) holds. Note that if the decentralized strategy $\hat{u}_i$ defined in (4.13) is taken, then for any $u_i\in\mathcal{U}_i$, we have

$$|\bar{\mathcal{J}}_i(\widetilde{u}_i,\widetilde{u}_{-i})-\bar{\mathcal{J}}_i(u_i,\widetilde{u}_{-i})|\leqslant|\bar{\mathcal{J}}_i(\widetilde{u}_i,\widetilde{u}_{-i})-J_i(\hat{u}_i)|+|\bar{\mathcal{J}}_i(u_i,\widetilde{u}_{-i})-J_i(u_i)|.$$

And then, we only need to prove the two propositions below

$$\bar{\mathcal{J}}_i(\widetilde{u}_i,\widetilde{u}_{-i})=J_i(\hat{u}_i)+O(\frac{1}{\sqrt{N}}), \tag{5.2}$$

and

$$\bar{\mathcal{J}}_i(u_i,\widetilde{u}_{-i})=J_i(u_i)+O(\frac{1}{\sqrt{N}}). \tag{5.3}$$

Next, we will divide the discussion into the following two subsections to fully demonstrate the above two propositions. Note that $\bar{\Sigma}_i(t)\triangleq\Sigma_i(t)\mathbf{h}^*(t)+\sigma_0$ in the following.

5.1. Estimation of $|\bar{\mathcal{J}}_i(\widetilde{u}_i,\widetilde{u}_{-i})-J_i(\hat{u}_i)|$

Before proceeding the derivation, we need to define the following two systems corresponding to the decentralized strategies $\widetilde{u}_i$ as (4.17) and $\hat{u}_i$ as (4.13), respectively,

$$\begin{cases}\mathrm{d}\mathbf{x}_{it}=[\mathbf{a}(t)\mathbf{x}_{it}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{x}}_{it}+\gamma_t)+\mathbf{c}(t)\mathbf{x}_t^{(N)}+\mathbf{m}(t)]\mathrm{d}t+\sigma_1(t)\mathrm{d}w_{it}\\ \qquad\quad+\sigma_2(t)\mathrm{d}w_t+\sigma_0\mathrm{d}\widetilde{\nu}_{it},\\ \mathrm{d}\hat{\mathbf{x}}_{it}=[\mathbf{a}(t)\hat{\mathbf{x}}_{it}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{x}}_{it}+\gamma_t)+\mathbf{c}(t)\mathbf{x}_t^0+\mathbf{m}(t)]\mathrm{d}t+\sigma_2(t)\mathrm{d}w_t\\ \qquad\quad+\bar{\Sigma}_i(t)[\mathrm{d}z_{it}-(\mathbf{h}(t)\hat{\mathbf{x}}_{it}+\mathbf{f}(t)\mathbf{x}_t^0+\phi(t))\mathrm{d}t],\\ \mathrm{d}z_{it}=[\mathbf{h}(t)\mathbf{x}_{it}+\mathbf{f}(t)\mathbf{x}_t^N+\phi(t)]\mathrm{d}t+\mathrm{d}\widetilde{\nu}_{it},\end{cases} \tag{5.4}$$

and

$$\begin{cases}\mathrm{d}\bar{\mathbf{x}}_{it}=[\mathbf{a}(t)\bar{\mathbf{x}}_{it}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\bar{\mathbf{x}}}_{it}+\gamma_t)+\mathbf{c}(t)\mathbf{x}_t^0+\mathbf{m}(t)]\mathrm{d}t+\sigma_1(t)\mathrm{d}w_{it}\\ \qquad\quad+\sigma_2(t)\mathrm{d}w_t+\sigma_0\mathrm{d}\widetilde{\nu}_{it},\\ \mathrm{d}\hat{\bar{\mathbf{x}}}_{it}=[\mathbf{a}(t)\hat{\bar{\mathbf{x}}}_{it}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\bar{\mathbf{x}}}_{it}+\gamma_t)+\mathbf{c}(t)\mathbf{x}_t^0+\mathbf{m}(t)]\mathrm{d}t+\sigma_2(t)\mathrm{d}w_t\\ \qquad\quad+\bar{\Sigma}_i(t)[\mathrm{d}\bar{z}_{it}-(\mathbf{h}(t)\hat{\bar{\mathbf{x}}}_{it}+\mathbf{f}(t)\mathbf{x}_t^0+\phi(t))\mathrm{d}t],\\ \mathrm{d}\bar{z}_{it}=[\mathbf{h}(t)\bar{\mathbf{x}}_{it}+\mathbf{f}(t)\mathbf{x}_t^0+\phi(t)]\mathrm{d}t+\mathrm{d}\widetilde{\nu}_{it}.\end{cases} \tag{5.5}$$

Define the following two average processes

$$\mathbf{x}^{(N)}\triangleq\frac{1}{N}\sum_{i=1}^N\mathbf{x}_i,\ \ \hat{\mathbf{x}}^{(N)}\triangleq\frac{1}{N}\sum_{i=1}^N\hat{\mathbf{x}}_i.$$

The following two lemmas holds.

Lemma 5.2. *Under (A1)–(A3),*

$$\sup_{0\leqslant t\leqslant T} E|\mathbf{x}_t^{(N)}-\mathbf{x}_t^0|^2=O(\frac{1}{N}),\quad \sup_{0\leqslant t\leqslant T} E|\hat{\mathbf{x}}_t^{(N)}-\mathbf{x}_t^0|^2=O(\frac{1}{N}).$$

Proof. By (5.4), the average processes $\mathbf{x}^{(N)}$ and $\hat{\mathbf{x}}^{(N)}$ satisfy the coupling system

$$\begin{cases} \mathrm{d}\mathbf{x}_t^{(N)}=[\mathbf{a}(t)\mathbf{x}_t^{(N)}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{x}}_t^{(N)}+\gamma_t)+\mathbf{c}(t)\mathbf{x}_t^{(N)}+\mathbf{m}(t)]\mathrm{d}t \\ \qquad +\frac{1}{N}\sum_{i=1}^N\sigma_1(t)\mathrm{d}w_{it}+\sigma_2(t)\mathrm{d}w_t+\frac{1}{N}\sum_{i=1}^N\sigma_0\mathrm{d}\widetilde{\nu}_{it}, \\ \mathrm{d}\hat{\mathbf{x}}_t^{(N)}=[\mathbf{a}(t)\hat{\mathbf{x}}_t^{(N)}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{x}}_t^{(N)}+\gamma_t)+\mathbf{c}(t)\mathbf{x}_t^0+\mathbf{m}(t)]\mathrm{d}t+\sigma_2(t)\mathrm{d}w_t \\ \qquad +\bar{\Sigma}_i(t)[\mathbf{h}(t)(\mathbf{x}_t^{(N)}-\hat{\mathbf{x}}_t^{(N)})+\mathbf{f}(t)(\mathbf{x}_t^{(N)}-\mathbf{x}_t^0)]\mathrm{d}t+\frac{1}{N}\sum_{i=1}^N\bar{\Sigma}_i(t)\mathrm{d}\widetilde{\nu}_{it}. \end{cases}$$

Obviously, by the standard arguments for estimation of SDE, the Gronwall's inequality and (4.16), we have

$$E|\mathbf{x}_t^{(N)}|^2=O(\frac{1}{N}),E|\hat{\mathbf{x}}_t^{(N)}|^2=O(\frac{1}{N}).$$

Then, together with the first equation in (4.15), the processes $\mathbf{x}^{(N)}-\mathbf{x}^0$ and $\hat{\mathbf{x}}^{(N)}-\mathbf{x}^0$ satisfy the following system

$$\begin{cases} d(\mathbf{x}_t^{(N)}-\mathbf{x}_t^0)=\{(\mathbf{a}(t)+\mathbf{c}(t))(\mathbf{x}_t^{(N)}-\mathbf{x}_t^0)-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)(\hat{\mathbf{x}}_t^{(N)}-\mathbf{x}_t^0)\}\mathrm{d}t \\ \qquad \frac{1}{N}\sum_{i=1}^N\sigma_1(t)\mathrm{d}w_{it}+\frac{1}{N}\sum_{i=1}^N\sigma_0\mathrm{d}\widetilde{\nu}_{it}, \\ d(\hat{\mathbf{x}}_t^{(N)}-\mathbf{x}_t^0)=\{[\mathbf{a}(t)-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)](\hat{\mathbf{x}}_t^{(N)}-\mathbf{x}_t^0)+\bar{\Sigma}_i(t)[\mathbf{h}(t)(\mathbf{x}_t^{(N)}-\hat{\mathbf{x}}_t^N) \\ \qquad +\mathbf{f}(t)(\mathbf{x}^{(N)}-\mathbf{x}_t^0)]\}\mathrm{d}t+\frac{1}{N}\sum_{i=1}^N\bar{\Sigma}_i(t)\mathrm{d}\widetilde{\nu}_{it}. \end{cases}$$

It follows from the Gronwall's inequality and the assumptions (A1)–(A3),

$$\begin{aligned} E|\mathbf{x}_t^N-\mathbf{x}_t^0|^2 &\leqslant \ell E\int_0^t|\mathbf{x}_s^{(N)}-\mathbf{x}_s^0|^2+|\hat{\mathbf{x}}_s^{(N)}-\mathbf{x}_s^0|^2\mathrm{d}s+\frac{\ell}{N^2}E\left(\left|\int_0^t\sum_{i=1}^N\sigma_1(s)\mathrm{d}w_{is}\right|^2\right. \\ &\qquad \left.+\left|\int_0^t\sum_{i=1}^N\sigma_0\mathrm{d}\widetilde{\nu}_{is}\right|^2\right)+\frac{2E|x_{i0}|^2}{N} \\ &\leqslant \hat{\ell}\left(E\int_0^t|\hat{\mathbf{x}}_s^N-\mathbf{x}_s^0|^2\mathrm{d}s+O(\frac{1}{N})\right), \end{aligned}$$

where $\hat{\ell} = \hat{\ell}(\ell, T)$ is denoted by a constant only depending on ℓ and T, since

$$\frac{1}{N^2} E\left(\left|\int_0^t \sum_{i=1}^N \sigma_1(s)\mathrm{d}w_{is}\right|^2 + \left|\int_0^t \sum_{i=1}^N \sigma_0 \mathrm{d}\widetilde{\nu}_{is}\right|^2\right) + \frac{2E|x_{i0}|^2}{N} = O(\frac{1}{N}).$$

Moreover,

$$\begin{aligned} E|\hat{\mathbf{x}}_t^{(N)} - \mathbf{x}_t^0|^2 &\leqslant \ell E\int_0^t |\hat{\mathbf{x}}_s^N - \mathbf{x}_s^0|^2 + |\mathbf{x}_s^N - \mathbf{x}_s^0|^2 \mathrm{d}s + \frac{\ell}{N^2} E\left|\int_0^t \sum_{i=1}^N \bar{\Sigma}_i(s)\mathrm{d}\widetilde{\nu}_{is}\right|^2 \\ &\leqslant \hat{\ell}\left(E\int_0^t |\mathbf{x}_s^N - \mathbf{x}_s^0|^2 \mathrm{d}s + O(\frac{1}{N})\right), \end{aligned}$$

since

$$\frac{1}{N^2} E\left|\int_0^t \sum_{i=1}^N \bar{\Sigma}_i(s)\mathrm{d}\widetilde{\nu}_{is}\right|^2 = O(\frac{1}{N}).$$

□

Lemma 5.3. *Under (A1)–(A3),*

$$\sup_{0\leqslant t\leqslant T} E|\mathbf{x}_{it} - \bar{\mathbf{x}}_{it}|^2 = O(\frac{1}{N}), \quad \sup_{0\leqslant t\leqslant T} E|\hat{\mathbf{x}}_{it} - \hat{\bar{\mathbf{x}}}_{it}|^2 = O(\frac{1}{N}), \quad \sup_{0\leqslant t\leqslant T} E|z_{it} - \bar{z}_{it}|^2 = O(\frac{1}{N}).$$

Proof. According to (5.4) and (5.5), the processes $\mathbf{x}_i - \bar{\mathbf{x}}_i, \hat{\mathbf{x}}_i - \hat{\bar{\mathbf{x}}}_i$ and $z_i - \bar{z}_i$ are governed by the three equations below,

$$\begin{cases} d(\mathbf{x}_{it} - \bar{\mathbf{x}}_{it}) = \{\mathbf{a}(t)(\mathbf{x}_{it} - \bar{\mathbf{x}}_{it}) - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)(\hat{\mathbf{x}}_{it} - \hat{\bar{\mathbf{x}}}_{it}) + \mathbf{c}(t)(\mathbf{x}_t^{(N)} - \mathbf{x}_t^0)\}\mathrm{d}t, \\ d(\hat{\mathbf{x}}_{it} - \hat{\bar{\mathbf{x}}}_{it}) = \{[\mathbf{a}(t) - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t) - \bar{\Sigma}_i(t)\mathbf{h}(t)](\hat{\mathbf{x}}_{it} - \hat{\bar{\mathbf{x}}}_{it}) + \bar{\Sigma}_i(t)\mathbf{h}(t)(\mathbf{x}_{it} - \bar{\mathbf{x}}_{it}) \\ \qquad\qquad + \bar{\Sigma}_i(t)\mathbf{f}(t)(\mathbf{x}_t^{(N)} - \mathbf{x}_t^0)\}\mathrm{d}t, \\ d(z_{it} - \bar{z}_{it}) = [\mathbf{h}(t)(\mathbf{x}_{it} - \bar{\mathbf{x}}_{it}) + \mathbf{f}(t)(\mathbf{x}_t^{(N)} - \mathbf{x}_t^0)]\mathrm{d}t. \end{cases}$$

As a result, by (A1)–(A3), there exists a constant ℓ such that

$$\begin{cases} E|\mathbf{x}_{it} - \bar{\mathbf{x}}_{it}|^2 \leqslant \ell E\left(\int_0^t |\mathbf{x}_{is} - \bar{\mathbf{x}}_{is}|^2 + |\hat{\mathbf{x}}_{is} - \hat{\bar{\mathbf{x}}}_{is}|^2 + |\mathbf{x}_s^{(N)} - \mathbf{x}_s^0|^2 \mathrm{d}s\right), \\ E|\hat{\mathbf{x}}_{it} - \hat{\bar{\mathbf{x}}}_{it}|^2 \leqslant \ell E\left(\int_0^t |\mathbf{x}_{is} - \bar{\mathbf{x}}_{is}|^2 + |\hat{\mathbf{x}}_{is} - \hat{\bar{\mathbf{x}}}_{is}|^2 + |\mathbf{x}_s^{(N)} - \mathbf{x}_s^0|^2 \mathrm{d}s\right), \\ E|z_{it} - \bar{z}_{it}|^2 \leqslant \ell E\left(\int_0^t |\mathbf{x}_{is} - \bar{\mathbf{x}}_{is}|^2 + |\mathbf{x}_s^{(N)} - \mathbf{x}_s^0|^2 \mathrm{d}s\right). \end{cases}$$

Meanwhile, we obtain the following result by the Gronwall's inequality

$$\sup_{0\leqslant s\leqslant t} E|\mathbf{x}_{is}-\bar{\mathbf{x}}_{is}|^2 \leqslant \hat{\ell}\left(\int_0^t \sup_{0\leqslant s\leqslant t} E|\mathbf{x}_s^{(N)}-\mathbf{x}_s^0|^2 \mathrm{d}s\right),$$
$$\sup_{0\leqslant s\leqslant t} E|\hat{\mathbf{x}}_{is}-\hat{\bar{\mathbf{x}}}_{is}|^2 \leqslant \hat{\ell}\left(\int_0^t \sup_{0\leqslant s\leqslant t} E|\mathbf{x}_s^{(N)}-\mathbf{x}_s^0|^2 \mathrm{d}s\right),$$
$$\sup_{0\leqslant s\leqslant t} E|z_{is}-\bar{z}_{is}|^2 \leqslant \ell\left(\int_0^t \sup_{0\leqslant s\leqslant t} E|\mathbf{x}_{is}-\bar{\mathbf{x}}_{is}|^2 + \sup_{0\leqslant s\leqslant t} E|\mathbf{x}_s^{(N)}-\mathbf{x}_s^0|^2 \mathrm{d}s\right).$$

The following results can be given directly by Lemma 5.2,

$$\sup_{0\leqslant t\leqslant T} E|\mathbf{x}_{it}-\bar{\mathbf{x}}_{it}|^2 = O(\frac{1}{N}),\ \ \sup_{0\leqslant t\leqslant T} E|\hat{\mathbf{x}}_{it}-\hat{\bar{\mathbf{x}}}_{it}|^2 = O(\frac{1}{N}).$$

According to the above results and Lemma 5.2,

$$\sup_{0\leqslant t\leqslant T} E|z_{it}-\bar{z}_{it}|^2 = O(\frac{1}{N}).$$

The proof is complete. □

Proposition 5.4. *Under (A1)–(A3),*

$$\bar{\mathcal{J}}_i(\widetilde{u}_i,\widetilde{u}_{-i}) = J_i(\hat{u}_i) + O(\frac{1}{\sqrt{N}}).$$

Proof. Denote $\delta\mathcal{J} \triangleq \bar{\mathcal{J}}_i(\widetilde{u}_i,\widetilde{u}_i) - J_i(\hat{u}_i)$, we have

$$\delta\mathcal{J} = E\left(\int_0^T (\hbar_{it}+\hbar_{2t})\mathrm{d}t + \hbar_3\right)$$

with $\hbar_1, \hbar_2$ and $\hbar_3$ satisfying

$$\begin{cases}\hbar_{1t} = (\mathbf{x}_{it}-\mathbf{x}_t^{(N)})^* Q_1(t)(\mathbf{x}_{it}-\mathbf{x}_t^{(N)}) - (\bar{\mathbf{x}}_{it}-\mathbf{x}_t^0)^* Q_1(t)(\bar{\mathbf{x}}_{it}-\mathbf{x}_t^0)\\ \hbar_{2t} = \widetilde{u}_{it}^* R(t)\widetilde{u}_{it} - \hat{u}_{it}^* R(t)\hat{u}_{it},\ \hbar_3 = \mathbf{x}_{iT}^* G_1 \mathbf{x}_{iT} - \bar{\mathbf{x}}_{iT}^* G_1 \bar{\mathbf{x}}_{iT}.\end{cases}$$

Because of $a^*Qa - b^*Qb = (a-b)^*Q(a-b) + 2(a-b)^*Qb$, $\hbar_1, \hbar_2$ and $\hbar_3$ can be rewritten as

$$\begin{cases}\hbar_{1t} = [(\mathbf{x}_{it}-\mathbf{x}_t^{(N)}) - (\bar{\mathbf{x}}_{it}-\mathbf{x}_t^0)]^* Q_1(t)[(\mathbf{x}_{it}-\mathbf{x}_t^{(N)}) - (\bar{\mathbf{x}}_{it}-\mathbf{x}_t^0)]\\ \qquad\quad + 2[(\mathbf{x}_{it}-\mathbf{x}_t^{(N)}) - (\bar{\mathbf{x}}_{it}-\mathbf{x}_t^0)]^* Q_1(t)(\bar{\mathbf{x}}_{it}-\mathbf{x}_t^0),\\ \hbar_{2t} = (\widetilde{u}_{it}-\hat{u}_i)^* R(t)(\widetilde{u}_{it}-\hat{u}_i) + 2(\widetilde{u}_{it}-\hat{u}_i)^* R(t)\hat{u}_{it},\\ \hbar_3 = (\mathbf{x}_{iT}-\bar{\mathbf{x}}_{iT})^* G_1(\mathbf{x}_{iT}-\bar{\mathbf{x}}_{iT}) + 2(\mathbf{x}_{iT}-\bar{\mathbf{x}}_{iT})^* G_1 \bar{\mathbf{x}}_{iT}.\end{cases}$$

Moreover, by Lemma 5.2, Lemma 5.3 and the Hölder's inequality, we get

$$\begin{aligned}&\sqrt{\sup_{0\leqslant t\leqslant T} E|\mathbf{x}_{it}-\bar{\mathbf{x}}_{it}|^2+\sup_{0\leqslant t\leqslant T} E|\mathbf{x}_t^{(N)}-\mathbf{x}_t^0|^2}\sqrt{\sup_{0\leqslant t\leqslant T} E|\bar{\mathbf{x}}_{it}-\mathbf{x}_t^0|^2}=O(\frac{1}{\sqrt{N}}),\\&\sqrt{\sup_{0\leqslant t\leqslant T} E|\widetilde{u}_{it}-\hat{u}_{it}|^2}\sqrt{\sup_{0\leqslant t\leqslant T} E|\hat{u}_{it}|^2}=O(\frac{1}{\sqrt{N}}).\end{aligned}$$

Then, the estimation (5.2) holds. □

5.2. Estimation of $|\bar{\mathcal{J}}_i(u_i,\widetilde{u}_{-i})-J_i(u_i)|$

Let $u_i\in\mathcal{U}_i$ be a perturbed control for the agent $\mathcal{A}_i$, with the corresponding extended signal-observation system $(\mathbf{e}_i,z_i^{\mathbf{e}})$ satisfying

$$\begin{cases}\mathrm{d}\mathbf{e}_{it}=[\mathbf{a}(t)\mathbf{e}_{it}+\mathbf{b}(t)u_{it}+\mathbf{c}(t)\mathbf{e}_t^{(N)}+\mathbf{m}(t)]\mathrm{d}t+\sigma_1(t)\mathrm{d}w_{it}+\sigma_2(t)\mathrm{d}w_t+\sigma_0\mathrm{d}\widetilde{\nu}_{it},\ \mathbf{e}_{i0}=\mathbf{x}_{i0},\\ \mathrm{d}z_{it}^{\mathbf{e}}=[\mathbf{h}(t)\mathbf{e}_{it}+\mathbf{f}(t)\mathbf{e}_t^{(N)}+\phi(t)]\mathrm{d}t+\mathrm{d}\widetilde{\nu}_{it},\ z_{i0}^{\mathbf{e}}=0,\end{cases}$$

while other agents $\mathcal{A}_j, j\neq i$, still maintain the control $\widetilde{u}_j$, with the corresponding extended state satisfying

$$\begin{cases}\mathrm{d}\mathbf{e}_{jt}=[\mathbf{a}(t)\mathbf{e}_{jt}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{e}}_{jt}+\gamma_t)+\mathbf{c}(t)\mathbf{e}_t^{(N)}+\mathbf{m}(t)]\mathrm{d}t\\ \qquad\quad+\sigma_1(t)\mathrm{d}w_{jt}+\sigma_2(t)\mathrm{d}w_t+\sigma_0\mathrm{d}\widetilde{\nu}_{jt},\\ \mathrm{d}\hat{\mathbf{e}}_{jt}=[\mathbf{a}(t)\hat{\mathbf{e}}_{jt}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{e}}_{jt}+\gamma_t)+\mathbf{c}(t)\mathbf{x}_t^0+\mathbf{m}(t)]\mathrm{d}t\\ \qquad\quad+\sigma_2(t)\mathrm{d}w_t+\bar{\Sigma}_j(t)\{\mathrm{d}z_{jt}^{\mathbf{e}}-[\mathbf{h}(t)\hat{\mathbf{e}}_{jt}+\mathbf{f}(t)\mathbf{x}_t^0+\phi(t)]\mathrm{d}t\},\\ \mathrm{d}z_{jt}^{\mathbf{e}}=[\mathbf{h}(t)\mathbf{e}_{jt}+\mathbf{f}(t)\mathbf{e}_t^{(N)}+\phi(t)]\mathrm{d}t+\mathrm{d}\widetilde{\nu}_{jt},\end{cases}\tag{5.6}$$

where $\mathbf{e}_t^{(N)}=\frac{1}{N}\sum_{i=1}^N\mathbf{e}_{it}$. According to the definition of ϵ-Nash equilibrium, we only need to consider the perturbed control $u_i\in\mathcal{U}_i$ such that $\bar{\mathcal{J}}_i(u_i,\widetilde{u}_{-i})\leqslant\bar{\mathcal{J}}_i(\widetilde{u}_i,\widetilde{u}_{-i})$, which implies

$$\frac{1}{2}E\int_0^T u_{it}^*R(t)u_{it}\mathrm{d}t\leqslant\bar{\mathcal{J}}_i(u_i,\widetilde{u}_{-i})\leqslant\bar{\mathcal{J}}_i(\widetilde{u}_i,\widetilde{u}_{-i})=J_i(\hat{u}_i)+O(\frac{1}{\sqrt{N}}),$$

then,

$$E\int_0^T|u_{it}|^2\mathrm{d}t\leqslant c_1,$$

where c_1 is a positive constant independent of N.

Correspondingly, the extended limit signal-observation process (4.1) for the agent $\mathcal{A}_i$, with the selection of the disturbed control u_i for Problem (LNAP), is as follows

$$\begin{cases}\mathrm{d}\bar{\mathbf{e}}_{it}=[\mathbf{a}(t)\bar{\mathbf{e}}_{it}+\mathbf{b}(t)u_{it}+\mathbf{c}(t)\mathbf{x}_t^0+\mathbf{m}(t)]\mathrm{d}t+\sigma_1(t)\mathrm{d}w_{it}+\sigma_2(t)\mathrm{d}w_t+\sigma_0\mathrm{d}\widetilde{\nu}_{it},\ \bar{\mathbf{e}}_{i0}=\mathbf{x}_{i0},\\ \mathrm{d}z_{it}^{\bar{\mathbf{e}}}=[\mathbf{h}(t)\bar{\mathbf{e}}_{it}+\mathbf{f}(t)\mathbf{x}_t^0+\phi(t)]\mathrm{d}t+\mathrm{d}\widetilde{\nu}_{it},\ z_{i0}^{\bar{\mathbf{e}}}=0,\end{cases}$$

and for the agent $\mathcal{A}_j,\ j\neq i$,

$$\begin{cases}\mathrm{d}\bar{\mathbf{e}}_{jt}=[\mathbf{a}(t)\bar{\mathbf{e}}_{jt}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\bar{\mathbf{e}}}_{jt}+\gamma_t)+\mathbf{c}(t)\mathbf{x}_t^0+\mathbf{m}(t)]\mathrm{d}t+\sigma_2(t)\mathrm{d}w_t\\ \qquad\quad +\sigma_1(t)\mathrm{d}w_{jt}+\sigma_0\mathrm{d}\widetilde{\nu}_{jt},\ \bar{\mathbf{e}}_{j0}=\mathbf{x}_{j0},\\ \mathrm{d}\hat{\bar{\mathbf{e}}}_{jt}=[\mathbf{a}(t)\hat{\bar{\mathbf{e}}}_{jt}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\bar{\mathbf{e}}}_{jt}+\gamma_t)+\mathbf{c}(t)\mathbf{x}_t^0+\mathbf{m}(t)]\mathrm{d}t+\sigma_2(t)\mathrm{d}w_t\\ \qquad\quad +\bar{\Sigma}_j(t)[\mathrm{d}z_{jt}^{\bar{\mathbf{e}}}-(\mathbf{h}(t)\hat{\bar{\mathbf{e}}}_{jt}+\mathbf{f}(t)\mathbf{x}_t^0+\phi(t))\mathrm{d}t],\ \hat{\bar{\mathbf{e}}}_{j0}=0,\\ \mathrm{d}z_{jt}^{\bar{\mathbf{e}}}=[\mathbf{h}(t)\bar{\mathbf{e}}_{jt}+\mathbf{f}(t)\mathbf{x}_t^0+\phi(t)]\mathrm{d}t+\mathrm{d}\widetilde{\nu}_{jt},\ z_{j0}^{\bar{\mathbf{e}}}=0.\end{cases}$$

Lemma 5.5. *Under (A1)–(A3), there exists a constant $\hat{\ell}$, independent of N and j, such that*

$$\sup_{0\leqslant t\leqslant T}E\left[|\mathbf{e}_{jt}|^2+|\hat{\mathbf{e}}_{jt}|^2+|z_{jt}^{\mathbf{e}}|^2\right]\leqslant\hat{\ell}.$$

Proof. By (5.6), the processes $\mathbf{e}_j$ and $\hat{\mathbf{e}}_j$ can be rewritten as

$$\begin{cases}\mathbf{e}_{jt}=\mathbf{x}_{j0}+\displaystyle\int_0^t[\mathbf{a}(s)\mathbf{e}_{js}-\mathbf{b}(s)R^{-1}(s)\mathbf{b}^*(s)(P(s)\hat{\mathbf{e}}_{js}+\gamma_s)+\mathbf{c}(s)\mathbf{e}_s^{(N)}+\mathbf{m}(s)]\mathrm{d}s\\ \qquad\quad +\displaystyle\int_0^t\sigma_1(s)\mathrm{d}w_{js}+\int_0^t\sigma_2(s)\mathrm{d}w_s+\int_0^t\sigma_0\mathrm{d}\widetilde{\nu}_{js},\\ \hat{\mathbf{e}}_{jt}=\displaystyle\int_0^t\{[\mathbf{a}(s)-\mathbf{b}(s)R^{-1}(s)\mathbf{b}^*(s)P(s)-\bar{\Sigma}_j(s)\mathbf{h}(s)]\hat{\mathbf{e}}_{js}+\bar{\Sigma}_j(s)\mathbf{h}(s)\mathbf{e}_{js}+\bar{\Sigma}_j(s)\mathbf{f}(s)\mathbf{e}_s^{(N)}\\ \qquad\quad +[\mathbf{c}(s)-\bar{\Sigma}_j(s)\mathbf{f}(s)]\mathbf{x}_s^0-\mathbf{b}(s)R^{-1}(s)\mathbf{b}^*(s)\gamma_s+\mathbf{m}(s)\}\mathrm{d}s+\displaystyle\int_0^t\sigma_2(s)\mathrm{d}w_s+\int_0^t\bar{\Sigma}_j(s)\mathrm{d}\widetilde{\nu}_{js}.\end{cases}$$

Under (A1)–(A3), there exists a constant ℓ such that

$$\begin{cases}|\mathbf{e}_{jt}|^2\leqslant\ell\left(|\mathbf{x}_{j0}|^2+\displaystyle\int_0^t|\mathbf{e}_{js}|^2+|\hat{\mathbf{e}}_{js}|^2+\frac{1}{N}\sum_{i=1}^N|\mathbf{e}_{is}|^2+|\gamma_s|^2+|\mathbf{m}(s)|^2\mathrm{d}s\right.\\ \qquad\qquad\left.+\left|\displaystyle\int_0^t\sigma_1(s)\mathrm{d}w_{js}\right|^2+\left|\int_0^t\sigma_2(s)\mathrm{d}w_s\right|^2+\left|\int_0^t\sigma_0\mathrm{d}\widetilde{\nu}_{js}\right|^2\right),\\ |\hat{\mathbf{e}}_{jt}|^2\leqslant\ell\left(\displaystyle\int_0^t(|\hat{\mathbf{e}}_{js}|^2+|\mathbf{e}_{js}|^2+\frac{1}{N}\sum_{i=1}^N|\mathbf{e}_{is}|^2+|\gamma_s|^2+|\mathbf{x}_s^0|^2+|\mathbf{m}(s)|^2)\mathrm{d}s+\left|\int_0^t\sigma_2(s)\mathrm{d}w_s\right|^2\right.\\ \qquad\qquad\left.+\left|\displaystyle\int_0^t\bar{\Sigma}_j(s)\mathrm{d}\widetilde{\nu}_{js}\right|^2\right).\end{cases}$$

By the BDG's inequality, we can obtain

$$\begin{cases}E|\mathbf{e}_{jt}|^2\leqslant\ell E\left(|\mathbf{x}_{i0}|^2+\displaystyle\int_0^t(|\mathbf{e}_{js}|^2+|\hat{\mathbf{e}}_{js}|^2+\frac{1}{N}\sum_{i=1}^N|\mathbf{e}_{is}|^2)\mathrm{d}s+\int_0^t(|\gamma_s|^2+k_1(s))\mathrm{d}s\right),\\ E|\hat{\mathbf{e}}_{jt}|^2\leqslant\ell E\left(\displaystyle\int_0^t(|\hat{\mathbf{e}}_{js}|^2+|\mathbf{e}_{js}|^2+\frac{1}{N}\sum_{i=1}^N|\mathbf{e}_{is}|^2)\mathrm{d}s+\int_0^t(|\gamma_s|^2+|\mathbf{x}_s^0|^2+k_2(s))\mathrm{d}s\right),\end{cases}$$

where $k_1(t)\triangleq|\mathbf{m}(t)|^2+|\sigma_1(t)|^2+|\sigma_2(t)|^2+|\sigma_0|^2$ and $k_2(t)\triangleq|\mathbf{m}(t)|^2+|\sigma_2(t)|^2+|\bar{\Sigma}_j(t)|^2$.

By directly calculating, we can obtain the following results

$$\sum_{j=1}^{N} E|\mathbf{e}_{jt}|^2 \leqslant \ell E\left(N|\mathbf{x}_{i0}|^2 + \int_0^t 2\sum_{j=1}^{N}|\mathbf{e}_{js}|^2 + \sum_{j=1}^{N}|\hat{\mathbf{e}}_{js}|^2 + N(|u_{is}|^2 + |\gamma_s|^2 + k_1(s))\mathrm{d}s\right),$$

$$\sum_{j=1}^{N} E|\hat{\mathbf{e}}_{jt}|^2 \leqslant \ell E\left(\int_0^t \sum_{j=1}^{N}|\hat{\mathbf{e}}_{js}|^2 + 2\sum_{j=1}^{N}|\mathbf{e}_{js}|^2 + N(|u_{is}|^2 + |\gamma_s|^2 + |\mathbf{x}_s^0|^2 + k_2(s))\mathrm{d}s\right),$$

and

$$\sum_{j=1}^{N} E|\mathbf{e}_{jt}|^2 + \sum_{j=1}^{N} E|\hat{\mathbf{e}}_{jt}|^2 \leqslant 4\ell\int_0^t \sum_{j=1}^{N} E|\mathbf{e}_{js}|^2 + \sum_{j=1}^{N} E|\hat{\mathbf{e}}_{js}|^2\mathrm{d}s + 2\ell N E\left(|\mathbf{x}_{i0}|^2 + \int_0^t |u_{is}|^2 + |\gamma_s|^2 + |\mathbf{x}_s^0|^2 + k_1(s) + k_2(s)\mathrm{d}s\right).$$

So, by the Gronwall's inequality, we can get

$$\sum_{j=1}^{N} E|\mathbf{e}_{jt}|^2 + \sum_{j=1}^{N} E|\hat{\mathbf{e}}_{jt}|^2 \leqslant 2N\ell \mathrm{e}^{4\ell T}\left(|\mathbf{x}_{i0}|^2 + \int_0^t |u_{is}|^2 + |\gamma_s|^2 + |\mathbf{x}_s^0|^2 + k_1(s) + k_2(s)\mathrm{d}s\right),$$

or,

$$\frac{1}{N}\sum_{j=1}^{N} E|\mathbf{e}_{jt}|^2 + \frac{1}{N}\sum_{j=1}^{N} E|\hat{\mathbf{e}}_{jt}|^2 \leqslant 2\ell \mathrm{e}^{4\ell T} E\left(|\mathbf{x}_{i0}|^2 + \int_0^T |u_{is}|^2 + |\gamma_s|^2 + |\mathbf{x}_s^0|^2 + k_1(s) + k_2(s)\mathrm{d}s\right).$$

Since u_i is L^2-bounded, then by the assumption (A1) and (4.16), there exists a constant $\hat{\ell}$, independent of j and N, such that the following result holds

$$\frac{1}{N}\sum_{j=1}^{N} E|\mathbf{e}_{jt}|^2 + \frac{1}{N}\sum_{j=1}^{N} E|\hat{\mathbf{e}}_{jt}|^2 \leqslant \hat{\ell}. \tag{5.7}$$

Since $z_j^{\mathbf{e}}$ satisfies the equation below,

$$z_{jt}^{\mathbf{e}} = \int_0^t \mathbf{h}(s)\mathbf{e}_{js} + \mathbf{f}(s)\mathbf{e}_s^{(N)} + \phi(s)\mathrm{d}s + \widetilde{\nu}_{jt},$$

then,

$$|z_{jt}^{\mathbf{e}}|^2 \leqslant \ell\left(\int_0^t |\mathbf{e}_{js}|^2 + \frac{1}{N}\sum_{j=1}^{N}|\mathbf{e}_{js}|^2\mathrm{d}s\right) + \left(\int_0^t |\phi(s)|^2\mathrm{d}s + |\widetilde{\nu}_{jt}|^2\right).$$

Therefore,

$$\sum_{j=1}^{N} E|z_{jt}^{\mathbf{e}}|^2 \leqslant 2\ell\int_0^t \sum_{j=1}^{N} E|\mathbf{e}_{js}|^2\mathrm{d}s + NE\int_0^t (|\phi(s)|^2 + 1)\mathrm{d}s.$$

According to (5.7), we get

$$\frac{1}{N}\sum_{j=1}^{N} E|z_{jt}^{\mathbf{e}}|^2 \leqslant \mathrm{e}^{2\ell T} E\left(\int_0^T (|\phi(s)|^2 + 1)ds\right).$$

According to the assumption (A2), (5.7) and the above inequality, there exists a constant $\hat{\ell}$, independent of j, N and t, such that the result holds in this lemma. The proof is complete. □

To adequately demonstrate ϵ-Nash equilibrium properties of the decentralization strategy $\widetilde{u}_i$, we must provide some necessary estimates for the states controlled by a perturbed control u_i and their corresponding filters. We introduce an intermediate state $(\mathbf{n}_i, z_i^{\mathbf{n}})$ controlled by u_i for $\mathcal{A}_i$ as

$$\begin{cases} \mathrm{d}\mathbf{n}_{it} = [\mathbf{a}(t)\mathbf{n}_{it} + \mathbf{b}(t)u_{it} + \mathbf{c}(t)\dfrac{N-1}{N}\mathbf{n}_t^{(N-1)} + \mathbf{m}(t)]\mathrm{d}t + \sigma_1(t)\mathrm{d}w_{it} + \sigma_1(t)\mathrm{d}w_t + \sigma_0\mathrm{d}\widetilde{\nu}_{it}, \\ \mathrm{d}z_{it}^{\mathbf{n}} = [\mathbf{h}(t)\mathbf{n}_{it} + \mathbf{f}(t)\dfrac{N-1}{N}\mathbf{n}_t^{(N-1)} + \phi(t)]\mathrm{d}t + \mathrm{d}\widetilde{\nu}_{it}, \\ \mathbf{n}_{i0} = \mathbf{x}_{i0},\ z_{i0}^{\mathbf{n}} = 0, \end{cases}$$

and for each agent $\mathcal{A}_j, j \neq i$, the corresponding state $(\mathbf{n}_j, z_j^{\mathbf{n}}, \hat{\mathbf{n}}_j)$ with selecting $\widetilde{u}_j$ satisfies

$$\begin{cases} \mathrm{d}\mathbf{n}_{jt} = [\mathbf{a}(t)\mathbf{n}_{jt} - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{n}}_{jt} + \gamma_t) + \mathbf{c}(t)\dfrac{N-1}{N}\mathbf{n}_t^{(N-1)} + \mathbf{m}(t)]\mathrm{d}t \\ \qquad\qquad + \sigma_1(t)\mathrm{d}w_{jt} + \sigma_2(t)\mathrm{d}w_t + \sigma_0\mathrm{d}\widetilde{\nu}_{jt}, \\ \mathrm{d}\hat{\mathbf{n}}_{jt} = [\mathbf{a}(t)\hat{\mathbf{n}}_{jt} - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{n}}_{jt} + \gamma_t) + \mathbf{x}_t^0 + \mathbf{m}(t)]\mathrm{d}t \\ \qquad\qquad + \sigma_2(t)\mathrm{d}w_t + \bar{\Sigma}_j(t)[\mathrm{d}z_{jt}^{\mathbf{n}} - (\mathbf{h}(t)\hat{\mathbf{n}}_{jt} + \mathbf{f}(t)\mathbf{x}_t^0 + \phi(t))\mathrm{d}t], \\ \mathrm{d}z_{jt}^{\mathbf{n}} = [\mathbf{h}(t)\mathbf{n}_{jt} + \mathbf{f}(t)\dfrac{N-1}{N}\mathbf{n}_t^{(N-1)} + \phi(t)]\mathrm{d}t + \mathrm{d}\widetilde{\nu}_{jt}, \\ \mathbf{n}_{j0} = \mathbf{x}_{j0},\ \hat{\mathbf{n}}_{j0} = 0,\ z_{j0}^{\mathbf{n}} = 0. \end{cases}$$

For any $t \in [0, T]$, define

$$\mathbf{n}^{(N-1)} \triangleq \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \mathbf{n}_{jt}, \quad \hat{\mathbf{n}}^{(N-1)} \triangleq \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \hat{\mathbf{n}}_{jt},$$
$$z_{\mathbf{e}}^{(N-1)} \triangleq \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} z_{jt}^{\mathbf{e}}, \quad z_{\mathbf{n}}^{(N-1)} \triangleq \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} z_{jt}^{\mathbf{n}}.$$

We then can get the following results

$$\begin{cases} \mathrm{d}\mathbf{n}^{(N-1)} = \{[\mathbf{a}(t) + \mathbf{c}(t)\dfrac{N-1}{N}]\mathbf{n}^{(N-1)} - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{n}}^{(N-1)} + \gamma_t) + \mathbf{m}(t)\}\mathrm{d}t \\ \qquad\qquad + \dfrac{1}{N-1}\displaystyle\sum_{j=1, j\neq i}^{N} \sigma_1(t)\mathrm{d}w_{jt} + \sigma_2(t)\mathrm{d}w_t + \dfrac{1}{N-1}\displaystyle\sum_{j=1, j\neq i}^{N} \sigma_0\mathrm{d}\widetilde{\nu}_{jt}, \\ \mathrm{d}\hat{\mathbf{n}}^{(N-1)} = \{[\mathbf{a}(t) - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)]\hat{\mathbf{n}}^{(N-1)} - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)\gamma_t + \mathbf{c}(t)\mathbf{x}_t^0 + \mathbf{m}(t)\}\mathrm{d}t \\ \qquad\qquad + \sigma_2(t)\mathrm{d}w_t + \bar{\Sigma}_j(t)[\mathrm{d}z_{\mathbf{n}}^{(N-1)} - (\mathbf{h}(t)\hat{\mathbf{n}}^{(N-1)} + \mathbf{f}(t)\mathbf{x}_t^0 + \phi(t))\mathrm{d}t], \\ \mathrm{d}z_{\mathbf{n}}^{(N-1)} = \{[\mathbf{h}(t) + \dfrac{N-1}{N}\mathbf{f}(t)]\mathbf{n}^{(N-1)} + \phi(t)\}\mathrm{d}t + \dfrac{1}{N-1}\displaystyle\sum_{j=1, j\neq i}^{N} \mathrm{d}\widetilde{\nu}_{jt}, \\ \mathbf{n}_0^{N-1} = \dfrac{1}{N-1}\displaystyle\sum_{j=1, j\neq i}^{N} \mathbf{x}_{j0},\ \hat{\mathbf{n}}_0^{(N-1)} = 0,\ z_{\mathbf{n}0}^{(N-1)} = 0. \end{cases} \tag{5.8}$$

And by (5.6), we obtain

$$\begin{cases} \mathrm{d}\mathbf{e}^{(N-1)} = \{[\mathbf{a}(t)+\mathbf{c}(t)\frac{N-1}{N}]\mathbf{e}^{(N-1)} - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)(P(t)\hat{\mathbf{e}}^{(N-1)}+\gamma_t) + \frac{\mathbf{c}(t)}{N}\mathbf{e}_{it} \\ \qquad +\mathbf{m}(t)\}\mathrm{d}t + \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\sigma_1(t)\mathrm{d}w_{jt} + \sigma_2(t)\mathrm{d}w_t + \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\sigma_0\mathrm{d}\widetilde{\nu}_{jt}, \\ \mathrm{d}\hat{\mathbf{e}}^{(N-1)} = \{[\mathbf{a}(t)-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)]\hat{\mathbf{e}}^{(N-1)} - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)\gamma_t + \mathbf{c}(t)\mathbf{x}_t^0 + \mathbf{m}(t)\}\mathrm{d}t \\ \qquad + \sigma_2(t)\mathrm{d}w_t + \bar{\Sigma}_j(t)[\mathrm{d}z_{\mathbf{e}}^{(N-1)} - (\mathbf{h}(t)\hat{\mathbf{e}}^{(N-1)} + \mathbf{f}(t)\mathbf{x}_t^0 + \phi(t))\mathrm{d}t], \\ \mathrm{d}z_{\mathbf{e}}^{(N-1)} = \{[\mathbf{h}(t)+\frac{N-1}{N}\mathbf{f}(t)]\mathbf{e}^{(N-1)} + \frac{\mathbf{f}(t)}{N}\mathbf{e}_{it} + \phi(t)\}\mathrm{d}t + \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\mathrm{d}\widetilde{\nu}_{jt}, \end{cases} \tag{5.9}$$

where $\mathbf{e}^{(N-1)} \triangleq \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\mathbf{e}_{jt}$, $\hat{\mathbf{e}}^{(N-1)} \triangleq \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\hat{\mathbf{e}}_{jt}$ and $z_{\mathbf{e}}^{(N-1)} \triangleq \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}z_{jt}^{\mathbf{e}}$.

Lemma 5.6. *If the assumptions (A1)–(A3) hold, then*

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T} E|\mathbf{e}^{(N-1)} - \mathbf{n}^{(N-1)}|^2 = O(\frac{1}{N}), \quad \sup_{0\leqslant t\leqslant T} E|\mathbf{e}^{(N)} - \mathbf{e}^{(N-1)}|^2 = O(\frac{1}{N}), \\ &\sup_{0\leqslant t\leqslant T} E|\hat{\mathbf{e}}^{(N-1)} - \hat{\mathbf{n}}^{(N-1)}|^2 = O(\frac{1}{N}), \quad \sup_{0\leqslant t\leqslant T} E|\hat{\mathbf{n}}^{(N-1)} - \hat{\mathbf{x}}^0|^2 = O(\frac{1}{N}), \\ &\sup_{0\leqslant t\leqslant T} E|z_{\mathbf{e}}^{(N-1)} - z_{\mathbf{n}}^{(N-1)}|^2 = O(\frac{1}{N}), \quad \sup_{0\leqslant t\leqslant T} E|\mathbf{n}^{(N-1)} - \mathbf{x}^0|^2 = O(\frac{1}{N}). \end{aligned}$$

Proof. According to (5.8) and (5.9), we can derive the following results

$$\begin{cases} d[\mathbf{e}^{(N-1)} - \mathbf{n}^{(N-1)}] = \{[\mathbf{a}(t)+\mathbf{c}(t)\frac{N-1}{N}][\mathbf{e}^{(N-1)} - \mathbf{e}^{(N-1)}] + \frac{\mathbf{c}(t)}{N}\mathbf{e}_{it} \\ \qquad -\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)[\hat{\mathbf{e}}^{(N-1)} - \hat{\mathbf{n}}^{(N-1)}]\}\mathrm{d}t, \\ d[\hat{\mathbf{e}}^{(N-1)} - \hat{\mathbf{n}}^{(N-1)}] = \{[\mathbf{a}(t) - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t) - \bar{\Sigma}_j(t)\mathbf{h}(t)][\hat{\mathbf{e}}^{(N-1)} - \hat{\mathbf{n}}^{(N-1)}]\}\mathrm{d}t \\ \qquad + \bar{\Sigma}_j(t)d[z_{\mathbf{e}}^{(N-1)} - z_{\mathbf{n}}^{(N-1)}], \\ d[z_{\mathbf{e}}^{(N-1)} - z_{\mathbf{n}}^{(N-1)}] = \{[\mathbf{h}(t)+\frac{N-1}{N}\mathbf{f}(t)][\mathbf{e}^{(N-1)} - \mathbf{n}^{(N-1)}] + \frac{\mathbf{f}(t)}{N}\mathbf{e}_{it}\}\mathrm{d}t, \end{cases}$$

i.e.,

$$\begin{cases} d[\mathbf{e}^{(N-1)} - \mathbf{n}^{(N-1)}] = \{[\mathbf{a}(t)+\mathbf{c}(t)\frac{N-1}{N}][\mathbf{e}^{(N-1)} - \mathbf{e}^{(N-1)}] + \frac{\mathbf{c}(t)}{N}\mathbf{e}_{it} \\ \qquad -\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)[\hat{\mathbf{e}}^{(N-1)} - \hat{\mathbf{n}}^{(N-1)}]\}\mathrm{d}t, \\ d[\hat{\mathbf{e}}^{(N-1)} - \hat{\mathbf{n}}^{(N-1)}] = \{[\mathbf{a}(t) - \mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t) - \bar{\Sigma}_j(t)\mathbf{h}(t)][\hat{\mathbf{e}}^{(N-1)} - \hat{\mathbf{n}}^{(N-1)}]\}\mathrm{d}t \\ \qquad + \bar{\Sigma}_j(t)\{[\mathbf{h}(t)+\frac{N-1}{N}\mathbf{f}(t)][\mathbf{e}^{(N-1)} - \mathbf{n}^{(N-1)}] + \frac{\mathbf{f}(t)}{N}\mathbf{e}_{it}\}\mathrm{d}t. \end{cases}$$

So,

$$\begin{cases} E|\mathbf{e}^{(N-1)}-\mathbf{n}^{(N-1)}|^2 \leqslant \ell E\displaystyle\int_0^t |\mathbf{e}^{(N-1)}-\mathbf{e}^{(N-1)}|^2+|\hat{\mathbf{e}}^{(N-1)}-\hat{\mathbf{n}}^{(N-1)}|^2+\frac{|\mathbf{e}_{is}|^2}{N^2}\mathrm{d}s,\\ E|\hat{\mathbf{e}}^{(N-1)}-\hat{\mathbf{n}}^{(N-1)}|^2 \leqslant \ell E\displaystyle\int_0^t |\hat{\mathbf{e}}^{(N-1)}-\hat{\mathbf{n}}^{(N-1)}|^2+|\mathbf{e}^{(N-1)}-\mathbf{n}^{(N-1)}|^2+\frac{|\mathbf{e}_{is}|^2}{N^2}|\mathrm{d}s, \end{cases}$$

and so

$$E\left[|\mathbf{e}^{(N-1)}-\mathbf{n}^{(N-1)}|^2+|\hat{\mathbf{e}}^{(N-1)}-\hat{\mathbf{n}}^{(N-1)}|^2\right] \leqslant 2\ell E\int_0^t |\mathbf{e}^{(N-1)}-\mathbf{n}^{(N-1)}|^2+|\hat{\mathbf{e}}^{(N-1)}-\hat{\mathbf{n}}^{(N-1)}|^2 + \frac{|\mathbf{e}_{is}|^2}{N^2}\mathrm{d}s.$$

By the Gronwall's inequality and Lemma 5.5, we can get

$$E|\mathbf{e}^{(N-1)}-\mathbf{n}^{(N-1)}|^2+E|\hat{\mathbf{e}}^{(N-1)}-\hat{\mathbf{n}}^{(N-1)}|^2 \leqslant \mathrm{e}^{2\ell T}E\int_0^T \frac{|\mathbf{e}_{is}|^2}{N^2}\mathrm{d}s < \frac{\hat{\ell}}{N}.$$

Similarly, according to the above inequality, we can obtain

$$E|z_{\mathbf{e}}^{(N-1)}-z_{\mathbf{n}}^{(N-1)}|^2 \leqslant \ell E\int_0^t |\mathbf{e}^{(N-1)}-\mathbf{n}^{(N-1)}|^2+\frac{|\mathbf{e}_{is}|^2}{N^2}\mathrm{d}s < \frac{\hat{\ell}}{N}.$$

We now know that the average process $\mathbf{e}^{(N)}$ and $\mathbf{e}^{(N-1)}$ are given by the following dynamics

$$\begin{aligned} \mathrm{d}\mathbf{e}^{(N)} = &\{[\mathbf{a}(t)+\mathbf{c}(t)]\mathbf{e}^{(N)}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)[P(t)\frac{N-1}{N}\hat{\mathbf{e}}^{(N-1)}+\gamma_t]+\frac{\mathbf{b}(t)}{N}u_{it}+\mathbf{m}(t)\}\mathrm{d}t\\ &+\frac{1}{N}\sum_{j=1}^{N}\sigma_1(t)\mathrm{d}w_{jt}+\sigma_2(t)\mathrm{d}w_t+\frac{1}{N}\sum_{j=1}^{N}\sigma_0\mathrm{d}\widetilde{\nu}_{jt}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}\mathbf{e}^{(N-1)} = &\{\mathbf{a}(t)\mathbf{e}^{(N-1)}+\mathbf{c}(t)\mathbf{e}^{(N)}-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)[P(t)\hat{\mathbf{e}}^{(N-1)}+\gamma_t]+\mathbf{m}(t)\}\mathrm{d}t\\ &+\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\sigma_1(t)\mathrm{d}w_{jt}+\sigma_2(t)\mathrm{d}w_t+\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\sigma_0\mathrm{d}\widetilde{\nu}_{jt}, \end{aligned}$$

respectively.

Thus, the following result is obtained by direct calculation

$$\begin{aligned} d(\mathbf{e}^{(N)}-\mathbf{e}^{(N-1)}) = &[\mathbf{a}(t)(\mathbf{e}^{(N)}-\mathbf{e}^{(N-1)})+\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)\frac{1}{N}\hat{\mathbf{e}}^{(N-1)}+\frac{\mathbf{b}(t)}{N}u_{it}]\mathrm{d}t\\ &-\frac{1}{N(N-1)}\sum_{j=1,j\neq i}^{N}\sigma_1(t)\mathrm{d}w_{jt}-\frac{1}{N(N-1)}\sum_{j=1,j\neq i}^{N}\sigma_0\mathrm{d}\widetilde{\nu}_{jt}+\frac{1}{N}[\sigma_1(t)\mathrm{d}w_{it}+\sigma_0\mathrm{d}\widetilde{\nu}_{it}], \end{aligned}$$

and due to the BDG's inequality and the assumptions (A1)–(A3), there exists a constant ℓ such that

$$\begin{aligned}E|\mathbf{e}^{(N)}-\mathbf{e}^{(N-1)}|^2\leqslant{}&\ell E\int_0^t|\mathbf{e}^{(N)}-\mathbf{e}^{(N-1)}|^2\mathrm{d}s\\&+\frac{\ell}{N}E\left(|\mathbf{x}_{i0}|^2+\int_0^t|\hat{\mathbf{e}}^{(N-1)}|^2+|u_{is}|^2+|\sigma_1(s)|^2+|\sigma_0|^2\mathrm{d}s\right).\end{aligned}$$

Since u_i is L^2-bounded, we get by the Gronwall's inequality and Lemma 5.5

$$E|\mathbf{e}^{(N)}-\mathbf{e}^{(N-1)}|^2\leqslant\frac{\hat{\ell}}{N}E\left(|\mathbf{x}_{i0}|^2+\int_0^T(|\sigma_1(s)|^2+|\sigma_0|^2+1)\mathrm{d}s\right),$$

i.e.,

$$\sup_{0\leqslant t\leqslant T}E|\mathbf{e}^{(N)}-\mathbf{e}^{(N-1)}|^2=O(\frac{1}{N}).$$

Because the processes $\hat{\mathbf{n}}^{(N-1)}-\hat{\mathbf{x}}^0$ and $\mathbf{n}^{(N-1)}-\mathbf{x}^0$ satisfy

$$\begin{cases}d(\hat{\mathbf{n}}^{(N-1)}-\hat{\mathbf{x}}^0_t)=\{[\mathbf{a}(t)-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)-\bar{\Sigma}_j(t)\mathbf{h}(t)][\hat{\mathbf{n}}^{(N-1)}-\hat{\mathbf{x}}^0_t]\\\qquad+\bar{\Sigma}_j(t)[\mathbf{h}(t)+\frac{N-1}{N}\mathbf{f}(t)][\mathbf{n}^{(N-1)}-\mathbf{x}^0_t]-\frac{1}{N}\bar{\Sigma}_j(t)\mathbf{f}(t)\mathbf{x}^0_t\}\mathrm{d}t\\\qquad+\frac{1}{N-1}\sum_{j=1,j\neq i}^N\bar{\Sigma}_j(t)\mathrm{d}\widetilde{\nu}_{jt},\\d(\mathbf{n}^{(N-1)}-\mathbf{x}^0_t)=\{\mathbf{a}(t)[\mathbf{n}^{(N-1)}-\mathbf{x}^0_t]+\mathbf{c}(t)\frac{N-1}{N}[\mathbf{n}^{(N-1)}-\mathbf{x}^0_t]-\frac{\mathbf{c}(t)}{N}\mathbf{x}^0_t\\\qquad-\mathbf{b}(t)R^{-1}(t)\mathbf{b}^*(t)P(t)[\hat{\mathbf{n}}^{(N-1)}-\hat{\mathbf{x}}^0_t]\}\mathrm{d}t\\\qquad+\frac{1}{N-1}\sum_{j=1,j\neq i}^N\sigma_1(t)\mathrm{d}w_{jt}+\frac{1}{N-1}\sum_{j=1,j\neq i}^N\sigma_0\mathrm{d}\widetilde{\nu}_{jt},\end{cases}$$

we obtain

$$\begin{cases}E|\hat{\mathbf{n}}^{(N-1)}-\hat{\mathbf{x}}^0_t|^2\leqslant\ell E\int_0^t|\hat{\mathbf{n}}^{(N-1)}-\hat{\mathbf{x}}^0_s|^2+|\mathbf{n}^{(N-1)}-\mathbf{x}^0_s|^2\mathrm{d}s+\frac{\ell}{N}E(|\mathbf{x}_{i0}|^2+T),\\E|\mathbf{n}^{(N-1)}-\mathbf{x}^0_t|^2\leqslant\ell E\int_0^t|\mathbf{n}^{(N-1)}-\mathbf{x}^0_s|^2+|\hat{\mathbf{n}}^{(N-1)}-\hat{\mathbf{x}}^0_s|^2\mathrm{d}s\\\qquad+\frac{\ell}{N}E\left(|\mathbf{x}_{i0}|^2+\int_0^t(|\sigma_1(s)|^2+|\sigma_0|^2+1)\mathrm{d}s\right).\end{cases}$$

By the Gronwall's inequality, we get the following result

$$E[|\hat{\mathbf{n}}^{(N-1)}-\hat{\mathbf{x}}^0_t|^2+|\mathbf{n}^{(N-1)}-\mathbf{x}^0_t|^2]\leqslant\frac{\hat{\ell}}{N}E\left(|\mathbf{x}_{i0}|^2+\int_0^T(|\sigma_1(t)|^2+|\sigma_0|^2+1)\mathrm{d}t\right),$$

i.e.,

$$\sup_{0\leqslant t\leqslant T} E[|\hat{\mathbf{n}}^{(N-1)}-\hat{\mathbf{x}}_t^0|^2+|\mathbf{n}^{(N-1)}-\mathbf{x}_t^0|^2]=O(\frac{1}{N}).$$

The proof is complete. □

Lemma 5.7. *Under (A1)–(A3),*

$$\sup_{0\leqslant t\leqslant T} E|\mathbf{e}^{(N)}-\mathbf{x}^0|^2=O(\frac{1}{N}),\ \sup_{0\leqslant t\leqslant T} E|\mathbf{e}_i-\bar{\mathbf{e}}_i|^2=O(\frac{1}{N}).$$

Proposition 5.8. *Under (A1)–(A3),*

$$\bar{\mathcal{J}}_i(u_i,\widetilde{u}_{-i})=J_i(u_i)+O(\frac{1}{N}).$$

Proof. Note that

$$\begin{aligned}\bar{\mathcal{J}}_i(u_i,\widetilde{u}_{-i})-J_i(u_i)=E\Bigg(&\int_0^T(\mathbf{e}_{it}-\mathbf{e}^{(N)})^*Q_1(t)(\mathbf{e}_{it}-\mathbf{e}^{(N)})-(\bar{\mathbf{e}}_{it}-\mathbf{x}_t^0)^*Q_1(t)(\bar{\mathbf{e}}_{it}-\mathbf{x}_t^0)\mathrm{d}t\\&+\mathbf{e}_{iT}^*G_1\mathbf{e}_{iT}-\bar{\mathbf{e}}_{iT}^*G_1\bar{\mathbf{e}}_{iT}\Bigg).\end{aligned}$$

Since $a^*Qa-b^*Qb=(a-b)^*Q(a-b)+2(a-b)^*Qb$, we can get

$$\begin{aligned}&(\mathbf{e}_{it}-\mathbf{e}^{(N)})^*Q_1(t)(\mathbf{e}_{it}-\mathbf{e}^{(N)})-(\bar{\mathbf{e}}_{it}-\mathbf{x}_t^0)^*Q_1(t)(\bar{\mathbf{e}}_{it}-\mathbf{x}_t^0)\\&=[(\mathbf{e}^{(N)}-\mathbf{x}_t^0)-(\mathbf{e}_{it}-\bar{\mathbf{e}}_{it})]^*Q_1(t)[(\mathbf{e}^{(N)}-\mathbf{x}_t^0)-(\mathbf{e}_{it}-\bar{\mathbf{e}}_{it})]\\&+2[(\mathbf{e}^{(N)}-\mathbf{x}_t^0)-(\mathbf{e}_{it}-\bar{\mathbf{e}}_{it})]^*Q_1(t)(\bar{\mathbf{e}}_{it}-\mathbf{x}_t^0),\end{aligned}$$

and

$$\mathbf{e}_{iT}^*G_1\mathbf{e}_{iT}-\bar{\mathbf{e}}_{iT}^*G_1\bar{\mathbf{e}}_{iT}=(\mathbf{e}_{iT}-\bar{\mathbf{e}}_{iT})^*G_1(\mathbf{e}_{iT}-\bar{\mathbf{e}}_{iT})+2(\mathbf{e}_{iT}-\bar{\mathbf{e}}_{iT})^*G_1\bar{\mathbf{e}}_{iT}.$$

□

According to Lemmas 5.5–5.7, we have

$$\sup_{0\leqslant t\leqslant T} E[|\mathbf{e}^{(N)}-\mathbf{x}_t^0|^2+|\mathbf{e}_{it}-\bar{\mathbf{e}}_{it}|^2]=O(\frac{1}{N})$$

and

$$\sqrt{\sup_{0\leqslant t\leqslant T} E[|\mathbf{e}_{it}-\bar{\mathbf{e}}_{it}|^2+|\mathbf{e}^{(N)}-\mathbf{x}_t^0|^2]}\sqrt{\sup_{0\leqslant t\leqslant T} E|\bar{\mathbf{e}}_{it}-\mathbf{x}_t^0|^2}=O(\frac{1}{\sqrt{N}}).$$

So, the result holds, or (5.3) holds.

Proof of Theorem 5.1: It follows from Proposition 5.4 and Proposition 5.8.

6. A specific example

We consider a one-dimensional system with a fixed time horizon $[0,T]$. The signal-observation system (x_i, z_i) of the agent $\mathcal{A}_i$ for $i = 1, \cdots, N$ satisfies the following equation with constant coefficients

$$\begin{cases} \mathrm{d}x_{it} = [ax_{it} + bu_{it} + cx_t^N + m]\mathrm{d}t + \sigma \mathrm{d}w_{it} + \widetilde{\sigma}\mathrm{d}w_t, \ x_i(0) = x_{i0}, \\ \mathrm{d}z_{it} = [hx_{it} + fx_t^{(N)} + \phi]\mathrm{d}t + \mathrm{d}\nu_{it}, \ z_i(0) = 0, \end{cases} \tag{6.1}$$

where

(i) the initial value x_{i0} is a central Gaussian random variable, denoted by $x_{i0} \triangleq \xi_i + \eta_i$. Here, ξ_i is a random variable subject to the standard Gaussian distribution, and there exists a correlation function ρ between the variable η_i and the observation noise ν_i;

(ii) if we denote $\eta_i \triangleq \kappa\nu_{iT}$, then the correlation function ρ satisfies

$$\rho(t) = E[\nu_{it}x_{i0}] = \kappa t, \ t \in [0,T],$$

and κ is a positive constant that measures the strength of the correlations.

Correspondingly, by Theorem 3.2, the coefficients of the extended system $(\mathbf{x}_i, z_i)$ for the agent $\mathcal{A}_i$ consist of the matrices below

$$\mathbf{a} = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \hat{k}(t) & -\kappa\hat{k}(t) \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{m} = \begin{bmatrix} m \\ 0 \\ 0 \end{bmatrix}, \sigma_1 = \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix},$$

$$\sigma_2 = \begin{bmatrix} \widetilde{\sigma} \\ 0 \\ 0 \end{bmatrix}, \sigma_0 = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \mathbf{h}(t) = [h \ \hat{k}(t) \ \ -\kappa\hat{k}(t)], \mathbf{f}(t) = [f \ 0 \ 0],$$

where $\hat{k}(t) = \frac{\kappa}{1+\kappa^2(T-t)}$

Set $T = 0.1, a = 0.4, b = 3.87, c = 2.3, Q = 0.75, R = \frac{103}{3}, G = 0.55$. According to Lemma 4.5, we can obtain the following numerical results

$$\begin{aligned} &\kappa = 0, \ \mathbb{L} \approx 0.0424; \ \kappa = 0.5, \ \ \mathbb{L} \approx 0.0455; \ \ \kappa = 1, \ \mathbb{L} \approx 0.0556; \\ &\kappa = 1.5, \ \ \mathbb{L} \approx 0.0728; \ \ \kappa = 2, \ \ \mathbb{L} \approx 0.1480. \end{aligned}$$

The above results show that the equation (4.15) has a unique solution $(\mathbf{x}^0, \gamma)$, since $0 < \mathbb{L} < 1$ in Lemma 4.5. Therefore, the solution we provide is feasible, which means that our proposed anticipative large population game model has practical implications.

7. Conclusion

In this paper, a new large-population LQ game with anticipative partial observation is proposed, in which there is a correlation function between the initial value x_{i0} of the signal for each agent $\mathcal{A}_i$ and the self observation noise, meaning that the initial signal is not adapted to the original filtration in general.

Our main result is that for an equivalent extended adapted large population game to the original one, a decentralized strategy profile is obtained that satisfies the ϵ-Nash equilibrium property. Firstly, we turn the original anticipative LQ game into an adapted version. Secondly, for each individual, we introduce a corresponding adapted control problem with an average state limit freezing term and solve a decentralized equilibrium strategy that satisfies a consistency condition, governed by a forward-backward stochastic differential equation. Finally, an ϵ-Nash equilibrium of the equivalent adapted large-population LQ game is derived from the profile of

N decentralized strategies. It is shown that all the above results are partially influenced by the given correlation function.

Funding

Supported by the Natural Science Foundation of China (11861025), the QKZYD of Guizhou [2022]4055, Guizhou Province's New Generation Artificial Intelligence and Modern Service Technology Innovation Talent Team Development(Qian Science and Platform Talent-BQW[2024]015), Natural Science Research Project of Guizhou Provincial Department of Education (No. QJJ [2023] 011).

Conflicts of interest

The authors declare that they have no conflicts of interest.

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