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EXISTENCE OF OPTIMAL CONTROLS FOR STOCHASTIC VOLTERRA EQUATIONS

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Abstract. We provide sufficient conditions that guarantee the existence of relaxed optimal controls in the weak formulation of control problems for stochastic Volterra equations (SVEs). Our study can be applied when the kernel appearing in the controlled SVE is singular at zero. The existence of relaxed optimal policies relies on the interaction between integrability hypotheses on the kernel and growth conditions on the running cost functional and the coefficients of the controlled SVEs. Under classical convexity assumptions, we can also deduce the existence of optimal strict controls.

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1. Introduction

There has been a rapidly growing interest in studying stochastic Volterra equations (SVEs) of convolution type since they provide suitable models for applications that benefit from the memory and the varying levels of regularity of their dynamics. Such applications include, among others, turbulence modeling in physics [1, 2], modeling of energy markets [3], and modeling of rough volatility in finance [4, 5].

In this paper, we consider finite-horizon control problems for SVEs of convolution type driven by a multi-dimensional Brownian motion with linear-growth coefficients and control policies with values on a metrizable topological space of Suslin type. We are particularly interested in fractional kernels proportional to $t^{H-\frac{1}{2}}$ with $H \in (0, 1)$. These kernels are important because they allow modeling trajectories with different levels of regularity compared to classical Brownian motion. They have been used, for instance, in financial models with rough volatility which reproduce features of time series of estimated spot volatility [4] and implied volatility surfaces [6, 7]. Fractional kernels also help incorporate dynamic memory effects with power-law fading, see *e.g.* [8, 9] and the references therein.

We propose to study the existence problem using so-called relaxed controls in a weak probabilistic setting. This approach compactifies the original control system by embedding it into a framework in which control policies are probability measures on the control set, and the probability space is also part of the class of admissible controls. Thus, in this setting, the unknown is no longer only the control-state process but rather an

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array consisting of the stochastic basis and the control-state pair solution to the relaxed version of the controlled SVE.

In the stochastic case, relaxed control of finite-dimensional stochastic systems goes back to [10] and was followed extensively by [11–15] to name a few. It has experienced a resurgence of interest in the past decade, especially within the context of stochastic and mean-field games [16–25] and optimal control of piece-wise deterministic Markov processes [26–30]. This renewed attention underscores the enduring relevance and potential of relaxed control methods in addressing contemporary challenges in stochastic optimal control.

To the authors' knowledge, no existing works specifically address optimal controls for singular Volterra integral equations using relaxation methods. The works by [31] and, more recently [32], employ relaxed controls for deterministic Volterra integral equations but without singular kernels. There are few recent studies on the optimal control of singular Volterra equations using the Pontryagin principle, see *e.g.* [33–35]. The recent paper [36] presents a novel formulation for the numerical solution of optimal control problems related to nonlinear Volterra fractional integral equations systems using a spectral approach based on Chelyshkov polynomials.

Several studies have investigated the optimal control of SVEs. [37] uses the maximum principle method to obtain optimality conditions in terms of an adjoint backward stochastic Volterra equation. [38] also uses the maximum principle and Malliavin calculus to obtain the adjoint equation as a standard backward SDE. Although the kernel considered in these papers is not restricted to convolution type, the required conditions do not allow the singularity of the kernel at zero. Recently, [39] derived an extended Bellman equation for the associated controlled Volterra equation, and [40] proved the existence of open-loop optimal controls for a class of forward stochastic Volterra integral equations (FSVIE) using type-II backward SVIEs. They also introduced the notion of causal state feedback representation and characterized the optimal solution using a path-dependent Riccati equation for an operator-valued function.

The particular case of linear-quadratic control problems for SVEs, with controlled drift and additive fractional noise with Hurst parameter $H > 1/2$, has been studied in [41]. Similarly, in [42] the authors consider a general Gaussian noise with an optimal control expressed as the sum of the well-known linear feedback control for the associated deterministic linear-quadratic control problem and the prediction of the response of a system to the future noise process. [43] investigated the linear-quadratic problem of stochastic Volterra equations by providing characterizations of optimal control in terms of a forward-backward system, but leaving aside its solvability, and under some assumptions on the coefficients that preclude (singular) fractional kernels of interest.

[44] studied control problems for linear SVEs with quadratic cost function and kernels that are the Laplace transforms of certain signed matrix measures that are not necessarily finite. They establish a correspondence between the initial problem and an infinite dimensional Markovian problem on a certain Banach space. Using a refined martingale verification argument combined with the completion of squares technique, they prove that the value function is of linear quadratic form in the new state variables, with a linear optimal feedback control, depending on nonstandard Banach space-valued Riccati equations. They also show that conventional finite dimensional Markovian linear-quadratic problems can approximate the value function of the stochastic Volterra optimization problem.

More recently, [45] investigated infinite horizon stochastic control problems for SVEs with singular coefficients, establishing both necessary and sufficient conditions for optimality using Pontryagin's maximum principle, where the adjoint equation is described as an infinite horizon backward SVE. [46] proved both necessary and sufficient maximum principles for infinite horizon discounted control problems of stochastic Volterra integral equations with finite delay and a convex control domain.

[47] and [48] studied causal-type feedback solutions to closed-loop linear−quadratic control problems for stochastic Volterra integral equations (SVIEs) with singular and non-convolution-type coefficients. In [47] they proved a duality principle and a representation formula for the quadratic functional of controlled SVIEs using type-II extended backward stochastic Volterra integral equation and a Lyapunov−Volterra equation, and in [48] they considered weighting matrices in the cost functional that are not necessarily non-negative definite.

We emphasize that our methods differ significantly from those employed in the recent works [44–48] as their primary focus is either Pontryagin's maximum principle or linear-quadratic problems. Our approach using relaxed controls offers a more general framework in certain aspects and has some clear advantages well-documented in the literature, see *e.g.* [49–51]. In particular, it does not require linearity in the coefficients with respect to the control process. This generality comes with additional restrictions in specific parts of our model. For instance, we do not cover cost functions with 'quadratic growth' in the control variable.

The main purpose of this paper is to provide a set of conditions that ensures the existence of optimal relaxed controls for SVEs, see Theorem 3.5. We use methods that are similar to the approach employed by [52] for stochastic PDEs. Our main contribution is that we allow singular kernels and coefficients that are not necessarily bounded in the control variable. Under one additional assumption on the coefficients and cost function, familiar in relaxed control theory since the work of [53], we prove that the optimal relaxed value is attained by strict policies on the original control set, see Theorem 3.9.

The paper is structured as follows. In Section 2 we establish some preliminary results on controlled stochastic Volterra equations (CSVEs). In Section 3 we describe the weak relaxed formulation of the control problem, state our main results, namely Theorems 3.5 and 3.9, and provide some examples. We sketch briefly an application to irreversible investment problems with power-law memory fading, in which firms base their production capacity decisions on past demand trends, and may adjust their strategies based on historical market performance. Here, the convolution kernel captures memory effects, which are essential in this setting because once a production capacity investment is made it cannot be easily reversed by the firm.

Section 4 contains the proofs of the main results. In Appendix A we recall an important measurability result needed for the existence of optimal strict controls. Appendix B contains an overview of the main results on relative compactness and limit theorems for Young measures that are used in the proofs of the main theorems.

2. Controlled stochastic Volterra equations (CSVEs)

Let $T > 0$ and $d, d' \in \mathbb{N}$ be fixed. We consider the control problem of minimizing the cost functional of the form

$$\mathbb{E}\left[\int_0^T l(t, X_t, u_t)\mathrm{d}t + G(X_T)\right] \tag{2.1}$$

subject to $X = (X_t)_{t\in[0,T]}$ being a $\mathbb{R}^d$-valued solution to the controlled stochastic Volterra equation (CSVE) of the form

$$X_t = x_0(t) + \int_0^t K(t-s)b(s, X_s, u_s)\,\mathrm{d}s + \int_0^t K(t-s)\sigma(s, X_s, u_s)\,\mathrm{d}W_s,\ \ t \in [0, T] \tag{2.2}$$

over a certain class of control processes $(u_t)_{t\in[0,T]}$ taking values in a measurable control set U. The function $K \in L^r_{\mathrm{loc}}(0, T; \mathbb{R}^{d\times d})$ with $r > 2$ is a given kernel, the initial condition x_0 is a deterministic $\mathbb{R}^d$-valued continuous function on $[0, T]$, and $(W_t)_{t\in[0,T]}$ is a d'-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. In our main existence results, we will consider solutions to (2.2) in a weak sense, see Definition 3.2.

Throughout, we will assume the following condition on the kernel K,

Assumption I. There exist $r \in (2, \infty)$ and $\gamma \in (0, 2]$ such that $K \in L^r_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^{d\times d})$ and

$$\int_0^h |K(t)|^2\,\mathrm{d}t = \mathcal{O}(h^\gamma), \quad \text{and} \quad \int_0^T |K(t+h) - K(t)|^2\,\mathrm{d}t = \mathcal{O}(h^\gamma).$$

The following are examples of kernels that satisfy Assumption I

1. Let K be locally Lipschitz. Then K satisfies Assumption I with $\gamma = 1$ and for any $r \in (2, \infty)$.
2. The fractional kernel $K(t) = t^{H-\frac{1}{2}}$ with $H \in (0, \frac{1}{2})$ satisfies Assumption I with $r \in (2, \frac{2}{1-2H})$ and $\gamma = 2H$.

We consider, for now, a control set U which is assumed to be a Hausdorff topological space endowed with the Borel $\sigma-$algebra $\mathcal{B}(U)$. We will assume later more specific conditions on U.

Assumption II.

1. The coefficients $b : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times d'}$ are continuous in $u \in U$, and in $(t,x) \in [0,T] \times \mathbb{R}^d$ uniformly with respect to u.
2. There exists a measurable function $\vartheta : [0,T] \times U \to [0,+\infty]$ and a constant $c_{\text{lin}} > 0$ such that

$$|b(t,x,u)| + |\sigma(t,x,u)| \le c_{\text{lin}}|x| + \vartheta(t,u), \quad (t,x,u) \in [0,T] \times \mathbb{R}^d \times U. \tag{2.3}$$

The following result extends the a-priori estimates of Lemma 3.1 of [54] to the case of CSVEs.

Theorem 2.1. *Suppose that Assumption II holds and that* $K \in L^r_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^{d\times d})$ *for some* $r > 2$*. Let* $(u_t)_{t\in[0,T]}$ *be a* U*-valued adapted control process such that*

$$\mathbb{E}\int_0^T \vartheta(t,u_t)^p \,\mathrm{d}t < \infty$$

for some p *satisfying* $\frac{1}{p} + \frac{1}{r} < \frac{1}{2}$*. Let* X *be a* $\mathbb{R}^d$*-valued solution to the controlled equation (2.2) with initial condition* $x_0 \in \mathcal{C}(0,T;\mathbb{R}^d)$*. Then, for all* $m > 2$ *satisfying* $\frac{m}{p} + \frac{2}{r} \le 1$ *we have*

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^m\right] \le c \tag{2.4}$$

where the constant c *depends on* $m, p, c_{\text{lin}}, T, C_B$[1]*,* $|x_0|_{\mathcal{C}(0,T;\mathbb{R}^d)}$*,* $K|_{[0,T]}$ *and* $\mathbb{E}\int_0^T \vartheta(t,u_t)^p \,\mathrm{d}t$*.*

Proof. For simplicity, but without loss of generality, we take $d = d' = 1$. Let $t \in [0,T]$ be fixed. Then, for any $m > 1$ we have

$$\begin{aligned} |X_t|^m &\le 3^{m-1}\left[|x_0|^m + \left|\int_0^t K(t-s)b(s,X_s,u_s)\,\mathrm{d}s\right|^m + \left|\int_0^t K(t-s)\sigma(s,X_s,u_s)\,\mathrm{d}W_s\right|^m\right] \\ &= 3^{m-1}\left[|x_0|^m + I + II\right]. \end{aligned}$$

Using Burkholder-Davis-Gundy inequality, and Jensen's inequality with the measure

$$\rho(\mathrm{d}s) := \frac{K(t-s)^2\,\mathrm{d}s}{\int_0^t |K(t-\tau)|^2\,\mathrm{d}\tau}$$

we have

$$\begin{aligned} \mathbb{E}[II] &\le C_B\mathbb{E}\left[\left|\int_0^t K(t-s)^2\sigma(s,X_s,u_s)^2\,\mathrm{d}s\right|^{m/2}\right] \\ &\le C_B\,\|K\|_{L^2}^{m-2}\int_0^t |\sigma(s,X_s,u_s)|^m\,|K(t-s)|^2\,\mathrm{d}s. \end{aligned}$$

[1] C_B is the constant in the Burkholder-Davis-Gundy inequality, see *e.g.* Section 4, Chapter IV in [55].

By condition (2.3)

$$\begin{aligned}\mathbb{E}[II] &\leq C_B 2^{m-1} c_{\text{lin}}^m \|K\|_{L^2}^{m-2} \left(\int_0^t \mathbb{E}\,|X_s|^m K(t-s)^2 \,\mathrm{d}s + c_{\text{lin}}^{-m} \mathbb{E} \int_0^t \vartheta(s,u_s)^m K(t-s)^2 \,\mathrm{d}s \right) \\ &= k_1 \left(\int_0^t \mathbb{E}\,|X_s|^m |K(t-s)|^2 \, ds + k_2 \right). \end{aligned} \tag{2.5}$$

Note that k_2 is finite since by Hölder's inequality we have

$$k_2 = c_{\text{lin}}^{-m} \mathbb{E} \int_0^t \vartheta(s,u_s)^m K(t-s)^2 \,\mathrm{d}s \leq c_{\text{lin}}^{-m} T^{1-\frac{m}{p}-\frac{2}{r}} \left[\mathbb{E} \int_0^T \vartheta(s,u_s)^p \,\mathrm{d}s \right]^{m/p} \|K\|_{L^r}^2 .$$

A similar argument for the first term I yields

$$\begin{aligned}\mathbb{E}[I] &\leq t^{m/2} 2^{m-1} c_{\text{lin}}^m \|K\|_{L^2}^{m-2} \left(\int_0^t \mathbb{E}\,|X_s|^m K(t-s)^2 \,\mathrm{d}s + c_{\text{lin}}^{-m} \mathbb{E} \int_0^t \vartheta(s,u_s)^m K(t-s)^2 \,\mathrm{d}s \right) \\ &= t^{m/2} C_B^{-1} k_1 \left(\int_0^t \mathbb{E}\,|X_s|^m |K(t-s)|^2 \, ds + k_2 \right). \end{aligned} \tag{2.6}$$

For each $n \in \mathbb{N}$ set $\tau_n = \inf\{t \geq 0 : |X_t| \geq n\} \wedge T$. By the Corollary of Theorem II.18 in [55] we have that,

$$|X_t|^m \mathbb{1}_{\{t<\tau_n\}} \leq \left| x_0 + \int_0^t K(t-s)(b(s, X_s \mathbb{1}_{\{s<\tau_n\}}, u_s)\,\mathrm{d}s) + \sigma(s, X_s \mathbb{1}_{\{s<\tau_n\}}, u_s)\,\mathrm{d}W_s) \right|^m .$$

Let $f_n(t) = \mathbb{E}\,|X_t|^m \mathbb{1}_{\{t<\tau_n\}}$. Then, by (2.6) and (2.5) we have

$$f_n \leq |x_0|_{\mathcal{C}(0,T;\mathbb{R}^d)} + \bar{k} k_2 + \bar{k}\,|K|^2 * f_n$$

where $\bar{k} = k_1(1 + T^{m/2} C_B^{-1})$. The same argument in the proof of Lemma 3.1 of [54] yields the desired result (2.4).

□

Corollary 2.2. *Under the same Assumptions of Theorem 2.1, suppose further Assumption I also holds with γ satisfying $\gamma > \frac{2}{m}$, where $\frac{m}{p} + \frac{2}{r} = 1$. Then X admits a version with paths in $\mathcal{C}^\alpha(0,T;\mathbb{R}^d)$ for any $\alpha \in \left[0, \frac{\gamma}{2} - \frac{1}{m}\right)$. For this version, denoted again with X, we have the following:*

$$\mathbb{E}\left[|X - x_0|_{\mathcal{C}^\alpha(0,T;\mathbb{R}^d)}^m \right] \leq c, \tag{2.7}$$

with c depending on $m, p, c_{\text{lin}}, T, C_B, |x_0|_{\mathcal{C}(0,T;\mathbb{R}^d)}, K|_{[0,T]}$ and $\mathbb{E}\int_0^T \vartheta(t,u_t)^p \,\mathrm{d}t$.

Proof. Follows directly from the estimate (2.4) and Lemma 2.4 in [54]. □

We will also frequently use the following result in the proof of the main existence result of relaxed controls. This alternative formulation of stochastic Volterra equations, by considering the integrated process $\int_0^\cdot X_s \,\mathrm{d}s$, is inspired by the martingale problem approach in [56] and facilitates the justification of convergence arguments that will be useful in our setting.

Lemma 2.3. *Suppose that Assumption II holds, $K \in L^2_{\rm loc}(\mathbb{R}_+;\mathbb{R}^{d\times d})$ and*

$$\mathbb{E}\int_0^T \vartheta(t,u_t)^2\,\mathrm{d}t < \infty.$$

Let X be a solution to the CSVE (2.2) and let Z be the controlled process $Z_t = \int_0^t b(s,X_s,u_s)\,\mathrm{d}s + \int_0^t \sigma(s,X_s,u_s)\,\mathrm{d}W_s$. If X has paths in $L^2_{\rm loc}$ then

$$\int_0^t X_s\,\mathrm{d}s = \int_0^t x_0(s)\,\mathrm{d}s + \int_0^t K(t-s)Z_s\,\mathrm{d}s, \quad t\in[0,T]. \tag{2.8}$$

Conversely, if X satisfies (2.8) with paths in $L^2_{\rm loc}$ then it solves the CSVE (2.2).

Proof. Follows from Lemma 3.2 of [56]. □

Remark 2.4.
- A key challenge in addressing non-convolution kernels arises from the tightness arguments required in the proof of Theorem 3.5. In the non-convolution setup, the integral formulation of the CSVE in Lemma 2.3 is unavailable because the stochastic Fubini theorem cannot be applied. This complicates the limiting arguments in law. For this reason, most recent studies on the weak existence of solutions for stochastic Volterra equations either focus on the convolution case, *e.g.* [56], or impose strong regularity conditions on non-convolution kernels to apply tightness arguments analogous to the convolution scenario, *e.g.* [57]. Given these difficulties, we have restricted our analysis to the convolution case.
- When the convolution kernel exhibits sufficient regularity, following the argument in [58], Lemma 1, the CSVE can be expressed as a semi-martingale with a path-dependent drift. The path-dependence introduces complexities absent in classical SDEs. In our specific framework, the semi-martingale formulation of the CSVE is not advantageous for the tightness arguments employed in Theorem 3.5. Instead, the formulation of the CSVE in Lemma 2.3 facilitates the limiting arguments.

3. Relaxed control formulation

The use of stochastic relaxed controls is inspired by the works of [12] and [13]. In what follows, $\mathcal{P}(U)$ denotes the set of all probability measures on $\mathcal{B}(U)$ endowed with the $\sigma-$algebra generated by the projection maps

$$\begin{aligned}\theta_C : \mathcal{P}(U) &\mapsto [0,1]\\ \pi &\mapsto \pi(C), \quad C\in\mathcal{B}(U).\end{aligned}$$

We associate a relaxed control system to the original control problem (2.1)–(2.2) as follows. First, we extend the definition of coefficients and cost functionals with the convention

$$\bar{F}(t,x,\pi) = \int_U F(t,x,u)\,\pi(\mathrm{d}u)$$

provided that for each $t\in[0,T]$ and $x\in\mathbb{R}^d$ the map $F(t,x,\cdot)$ is integrable with respect to $\pi\in\mathcal{P}(U)$.

Definition 3.1. A stochastic process $\pi = (\pi_t)_{t\in[0,T]}$ with values in $\mathcal{P}(U)$ is called a *stochastic relaxed control* (or relaxed control process) on U if the map

$$\begin{aligned}[0,T]\times\Omega &\mapsto \mathcal{P}(U)\\ (t,\omega) &\mapsto \pi_t(\omega,\cdot)\end{aligned}$$

is predictable. In other words, a stochastic relaxed control on U is a predictable process with values in $\mathcal{P}(U)$.

Given a relaxed control process $(\pi_t)_{t\in[0,T]}$, the associated relaxed controlled equation now reads

$$X_t = x_0(t) + \int_0^t K(t-s)\bar{b}(s, X_s, \pi_s)\mathrm{d}s + \int_0^t K(t-s)\bar{\sigma}(s, X_s, \pi_s)\,\mathrm{d}W_s,\ \ t \in [0,T], \tag{3.1}$$

where $\bar{\sigma}$ is defined, with a slight abuse of notation, so that the following holds:

$$[\bar{\sigma}\bar{\sigma}^\top](t,x,\pi) = \int_U [\sigma\sigma^\top](t,x,u)\,\pi(\mathrm{d}u), \quad t \in [0,T],\, x \in \mathbb{R}^d,\, \pi \in \mathcal{P}(U).$$

For the existence of $\bar{\sigma}$ see e.g, Theorem 2.5-a in [12]. The relaxed cost functional is defined as

$$\mathcal{J}(X,\pi) = \mathbb{E}\left[\int_0^T \bar{l}(t, X_t, \pi_t)\,\mathrm{d}t + G\left(X_T\right)\right].$$

Notice that the original system (2.1)–(2.2) controlled by a U-valued process $u = (u_t)_{t\in[0,T]}$ coincides with the relaxed system controlled by the Dirac measures $\pi_t = \delta_{u_t}$, $t \in [0,T]$. Moreover, since relaxed controls are just usual (strict) controls with control set $\mathcal{P}(U)$, the results for strict controls in the previous section also hold for relaxed controls, with the control system defined in terms of the relaxed versions of coefficients, running cost and $\bar{\vartheta}(t,\pi)$.

3.1. Weak formulation of optimal control problem

We investigate the existence of an optimal control for the stochastic relaxed control system within the following formulation in weak (probabilistic) sense.

Definition 3.2. Let $T > 0$ and $x_0 \in \mathcal{C}(0,T;\mathbb{R}^d)$ be fixed. A weak admissible relaxed control for (K, b, σ) is a system

$$\Theta = (\Omega, \mathcal{F}, \mathbf{P}, \mathbb{F}, W, X, \pi) \tag{3.2}$$

such that the following hold:

1. $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, satisfying the usual conditions,
2. $W = (W_t)_{t\in[0,T]}$ is a d'-dimensional Brownian motion with respect to $\mathbb{F}$,
3. $\pi = (\pi_t)_{t\in[0,T]}$ is a $\mathbb{F}$-predictable process with values in $\mathcal{P}(U)$,
4. $X = (X_t)_{t\in[0,T]}$ is a $\mathbb{F}$-adapted solution to the relaxed controlled equation (3.1).
5. The map $[0,T] \times \Omega \ni (t,\omega) \mapsto \bar{l}(t, X_t(\omega), \pi_t(\omega)) \in \mathbb{R}$ belongs to $L^1([0,T]\times\Omega;\mathbb{R})$ and $G(X_T) \in L^1(\Omega;\mathbb{R})$.

The set of weak admissible relaxed control systems with time horizon $[0,T]$ and initial value x_0 will be denoted by $\bar{\mathcal{U}}(x_0,T)$. Under this weak formulation, the relaxed cost functional is defined as

$$\bar{\mathcal{J}}(\Theta) = \mathbb{E}^{\mathbf{P}}\left[\int_0^T \bar{l}(s, X_s, \pi_s)\,\mathrm{d}s + G\left(X_T\right)\right], \quad \Theta = (\Omega, \mathcal{F}, \mathbf{P}, \mathbb{F}, W, X, \pi) \in \bar{\mathcal{U}}(x_0,T). \tag{3.3}$$

The relaxed control problem **(RCP)** consists in minimizing $\bar{\mathcal{J}}$ over $\bar{\mathcal{U}}(x_0,T)$. Namely, we seek $\widetilde{\Theta}\in\bar{\mathcal{U}}(x_0,T)$ such that

$$\bar{\mathcal{J}}(\tilde{\Theta}) = \inf_{\Theta\in\bar{\mathcal{U}}(x_0,T)} \bar{\mathcal{J}}(\Theta). \tag{3.4}$$

Definition 3.3. A weak admissible strict control for (K,b,σ) is a system $\Theta=(\Omega,\mathcal{F},\mathbf{P},\mathbb{F},W,X,\pi)$ satisfying the same conditions of Definition 3.2 and $\pi_t=\delta_{u(t)}$ $\mathbf{P}-$almost surely for all $t\in[0,T]$ for a $\mathbb{F}$-progressively measurable U-valued process $\{u_t\}_{t\in[0,T]}$.

3.2. Main existence result

To finalize the set of assumptions for the primary existence result, we require the following definition:

Definition 3.4. A function $\vartheta: U\to[0,+\infty]$ is called *inf-compact* if the level set $\{\vartheta\le R\}=\{u\in U:\vartheta(u)\le R\}$ is compact for every $R\ge 0$.

Observe that, since U is Hausdorff, for every inf-compact function ϑ the level sets $\{\vartheta\le R\}$ are closed. Therefore, every inf-compact function is lower semi-continuous and hence Borel-measurable. If U is compact, the converse holds too, *i.e.* every lower semi-continuous function is inf-compact. We will denote by $IC(0,T;U)$ the class of measurable functions $\vartheta:[0,T]\times U\to[0,+\infty]$ such that for all $t\in[0,T]$ the map $\vartheta(t,\cdot)$ is inf-compact.

Assumption III.
1. The **control set** U is a metrizable **Suslin** space *i.e.* there exists a Polish space S and a continuous mapping $\phi: S\to U$ such that $\phi(S)=U$.
2. The **running cost** function $l:[0,T]\times\mathbb{R}^d\times U\to(-\infty,+\infty]$ is measurable in $t\in[0,T]$ and lower semi-continuous with respect to $(x,u)\in\mathbb{R}^d\times U$.
3. There exist $\vartheta\in IC(0,T;U)$ and constants $C_1\in\mathbb{R},\ C_2>0$ such that l satisfies the following **coercivity** condition:

$$\vartheta(t,u)^p\le C_1+C_2 l(t,x,u),\quad (t,x,u)\in[0,T]\times\mathbb{R}^d\times U \tag{3.5}$$

   for some $p\ge 1$.
4. The **final cost** function $G:\mathbb{R}^d\to\mathbb{R}$ is continuous.

The following is the main result of this paper.

Theorem 3.5 (Existence of optimal relaxed controls)**.** *Let $T>0$ and $x_0\in\mathcal{C}(0,T;\mathbb{R}^d)$ be fixed. Suppose that Assumptions I, II and III hold with $r>2$, $\gamma\in(0,2]$ and p sufficiently large so that $\frac{1}{p}<\left(\frac{1}{2}-\frac{1}{r}\right)\min\{1,\gamma\}$. If there exists $\Theta\in\bar{\mathcal{U}}(x_0,T)$ such that $\bar{\mathcal{J}}(\Theta)<+\infty$, then* **(RCP)** *admits a weak optimal relaxed control.*

Example 3.6 (Fractional kernel)**.** For simplicity, we fix $d=d'=1$, and consider the fractional kernel $K(t)=t^{H-\frac{1}{2}}$ with $H\in\left(0,\frac{1}{2}\right)$. Suppose the coefficients and running cost function have the form

$$\begin{aligned} b(t,x,u) &= b_0(t,u)+b_1(t,u)x\\ \sigma(t,x,u) &= \sigma_0(t,u)+\sigma_1(t,u)x\\ l(t,x,u) &= l_0(t,u)+l_1(x)\end{aligned}$$

with

- b_i,σ_i measurable and continuous in $u\in U$, uniformly with respect to $t\in[0,T]$, for $i=0,1$,
- b_1,σ_1 uniformly bounded in (t,u),
- $l_0\in IC(0,T;U)$ and l_1 LSC and bounded from below.

Suppose further that $|f(t,u)|^p \leq Cl_0(t,u)$ for both $f = b_0, \sigma_0$, some constant $C > 0$ and p sufficiently large satisfying $p > \frac{1}{2H^2}$. Then, there exists $r > 2$ such that

$$1 - 2H < \frac{2}{r} < 1 - \frac{1}{pH}$$

so that Assumption I holds for this choice of r and $\gamma = 2H$. Assumptions II, III hold with $\vartheta = (Cl_0)^{1/p}$, $C_1 = -C \inf l_1$ and $C_2 = C$. Then, the existence of an optimal relaxed control follows from Theorem 3.5.

Remark 3.7. Recently, [44, 47, 48] proved existence results for Linear-Quadratic control problems for linear Volterra equations, and obtained linear feedback characterizations. Unlike those works, we do not assume linearity in the coefficients with respect to the control variable, which allows for broader applicability across various scenarios. However, we acknowledge that this generality comes with additional restrictions in specific parts of the model. These restrictions are necessary to ensure existence within the scope of our work. For instance, in the previous example, we do not cover cost functions with 'quadratic growth' in the control variable, since we are forced to have p sufficiently large so that $p > 1/2H^2$.

Example 3.8 (Irreversible investment with power-law memory effect)**.** Irreversible investment problems have been studied widely in the economic literature, see *e.g.* [59, 60] and the references therein. In these models, firms represent the economy's productive sector and make decisions regarding capital investment strategies.

Here we consider a firm that produces a single kind of perishable consumption good and chooses at each time t an investment plan u_t to increase its production capacity X_t. The dynamics of X_t is assumed to evolve according to the controlled linear SVE

$$X_t = x_0(t) + \int_0^t K(t-s)[b_0(t,u) + b_1(t,u)X_s]\,\mathrm{d}s + \int_0^t K(t-s)\sigma_1 X_s\,\mathrm{d}W_s,\ t \in [0,T] \tag{3.6}$$

where $b_1(t,u)$ represents the appreciation or depreciation rate coefficient, dependent on the investment rate, and $b_0(t,u)$ is a conversion factor in the sense that each unit of new investment u_t is converted into $b_0(t,u_t)$ units of capacity, including the cost of raising new equity. The 'volatility' σ_1 could also depend on u but for simplicity, we assume it is constant.

The coefficient b_1 is assumed to depend -possibly non-linearly- on the investment rate because investing in new technology can enhance production efficiency, increasing production capacity and reducing long-term costs. Regular maintenance investments can also decelerate the depreciation of production assets. Conversely, technological advancements can make older equipment obsolete, potentially accelerating depreciation. $b_1(t,u)$ can be used to model adjustment costs associated with changing investment levels in production capacities, including installation, setup, training, and adaptation expenses. This justifies the assumption that $b_1(t,u)$ depends on the investment rate.

The kernel captures the influence of past investments and technological changes on current output or production capacity. Here, the memory effects are essential because once an investment is made, it cannot be easily reversed. Firms may base their production capacity decisions on past demand trends, and investors may adjust their strategies based on historical market performance.

Furthermore, convolution with fractional kernels helps incorporate dynamic memory effects with power-law fading, see *e.g.* [8, 9, 61] and the references therein. Finally, the firm seeks to minimize

$$l(t,x,u) = l_0(t,u) - l_1(x)$$

where $l_0(t,u)$ is the production cost and $l_1(x)$ is the profit function. For example, if the kernel is fractional $K(t) = t^{H-\frac{1}{2}}$ with $H \in (0, \frac{1}{2})$, the control set is $U = \mathbb{R}_+$, $b_1(t,u)$ is bounded, $b_0(t,u) = B_0 u$ with $B_0 > 0$, the cost function is given by $l_0(t,u) = L_0 |u|^p$ with $L_0 > 0$ and $p > 1/2H^2$, and the profit function $l_1(x)$ is bounded above, by the result in Example 3.6 we can ensure existence of an optimal relaxed control.

3.3. Existence of strict controls

To establish the existence of optimal strict controls, our primary result requires an additional assumption, well-known in relaxed control theory since Filippov's work [53].

Assumption IV. 1. U is a closed subset of an Euclidean space.
2. For each $(t,x) \in [0,T] \times \mathbb{R}^d$, the set

$$\Gamma(t,x) = \left\{ \left([\sigma\sigma^\top](t,x,u), b(t,x,u), z \right) : u \in U,\ z \geq l(t,x,u) \right\} \tag{3.7}$$

is a convex and closed subset of $\mathcal{S}^d \times \mathbb{R}^d \times \mathbb{R}$.

Theorem 3.9 (Existence of optimal strict controls)**.** *Suppose that Assumption IV holds. Then, for each* $\Theta = (\Omega, \mathcal{F}, \mathbf{P}, \mathbb{F}, W, X, \pi) \in \bar{\mathcal{U}}(x_0,T)$ *there exists a* U*-valued* $\mathbb{F}$*-predictable control process* $u = (u_t)_{t\in[0,T]}$ *on the same probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *such that*

1. *X satisfies the Volterra equation (2.2) controlled by the strict control process* $u = (u_t)_{t\in[0,T]}$.
2. $\int_0^T \bar{l}(s,X_s,\pi_s)\,\mathrm{d}s \geq \int_0^T l(s,X_s,u_s)\,\mathrm{d}s$, $\mathbf{P}$*-a.s.*

In particular, if the Assumptions of Theorem 3.5 also hold, there exists a weak optimal strict control for (2.1)–(2.2).

Example 3.10. Let $U = \mathbb{R}$ or $U = [-\zeta,\zeta]$, with $0 < \zeta < \infty$, $d = d' = 1$ and $K(t) = t^{H-\frac{1}{2}}$ with $H \in \left(0, \frac{1}{2}\right)$. Suppose the coefficients have the form

$$\begin{aligned} b(t,x,u) &= b_0(t,x) + b_1(t,x)u^2 \\ \sigma(t,x,u) &= \sigma_1(t,x)u \\ l(t,x,u) &= l_0(t,u^2) + l_1(x), \end{aligned}$$

where

- b_i, σ_1 measurable and continuous in $x \in \mathbb{R}$, uniformly respect to $t \in [0,T]$, for $i = 0,1$,
- b_1, σ_1 uniformly bounded in (t,x) and $|b_0(t,x)| \leq c_0 |x|$, with $c_0 > 0$,
- $l_0(t,.)$ is a function convex on $\mathbb{R}^+$, for each $t \in [0,T]$, $l_0 \in IC(0,T;U^2)$ and l_1 LSC bounded from below.

Suppose further that $|\phi(t,u)|^p \leq C l_0(t,u)$, with $\phi(t,u) = \max\{|u|^2, |u|\}$ for all $t \in [0,T]$, and p satisfying $\frac{1}{p} < 2H^2$. As in the Example 3.6 there exists $r > 2$ such that Assumptions II, III hold with $\vartheta = (Cl_0)^{1/p}$, $C_1 = -C \inf l_1$ and $C_2 = C$. By Theorem 3.5 there is an optimal relaxed control. Let $\Gamma^1(t,x) = \left\{ (\widetilde{u}, z) : \widetilde{u} \in U^2, z \geq l_0(t,\widetilde{u}) + l_1(x) \right\}$. Then Γ in (3.7) can be written as an affine transformation of Γ^1. More precisely, $\Gamma(t,x) = \mathbf{b}(t,x) + \mathbf{A}(t,x)\Gamma^1(t,x)$ where

$$\mathbf{b}(t,x) = \begin{bmatrix} 0 \\ b_0(t,x) \\ 0 \end{bmatrix}, \quad \mathbf{A}(t,x) = \begin{bmatrix} \sigma_1^2(t,x) & 0 \\ b_1(t,x) & 0 \\ 0 & 1 \end{bmatrix}, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

Since l_0 is a function convex on $\mathbb{R}_+$ the epigraph Γ^1 is a convex set, then Γ is a convex set and by Theorem 3.9 there is an optimal strict control.

Remark 3.11. If σ does not depend on $u \in U$, Assumption IV holds if the set $\{(b(t,x,u), z) : u \in U, z \geq l(t,x,u)\}$ is convex and closed in $\mathbb{R}^d \times \mathbb{R}$. This is the case, for instance, if the drift coefficient is affine in u, *i.e.* it has the form $b(t,x,u) = b_0(t,x) + b_1(t,x)u$ and if $l(t,x,\cdot)$ is lower semi-continuous and convex.

4. Proofs of the main theorems

4.1. Relaxed controls and Young measures

Definition 4.1. Let $\mathcal{L}(\mathrm{d}t)$ denote the Lebesgue measure on $[0,T]$ and μ be a bounded non-negative $\sigma-$additive measure on $\mathcal{B}(U \times [0,T])$. We say that μ is a *Young measure* on U if and only if μ satisfies

$$\mu(U \times D) = \mathcal{L}(D), \quad D \in \mathcal{B}([0,T]), \tag{4.1}$$

i.e. the marginal of μ on $\mathcal{B}([0,T])$ is equal to the Lebesgue measure. We denote by $\mathcal{Y}(0,T;U)$ the set of Young measures on U. We endow $\mathcal{Y}(0,T;U)$ with the *stable topology* defined as the weakest topology for which the mappings

$$\mathcal{Y}(0,T;U) \ni \mu \mapsto \int_{U \times D} f(u)\, \mu(\mathrm{d}u, \mathrm{d}t) \in \mathbb{R}$$

are continuous, for every $D \in \mathcal{B}([0,T])$ and $f \in \mathcal{C}_b(U)$.

The following result connects random Young measures with predictable relaxed controls. For the proof, see *e.g.* Section 3.3 of [62] or Section 2.4 of [21].

Lemma 4.2 (Predictable disintegration of random Young measures). *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let U be a Radon space. Let $\mu : \Omega \to \mathcal{Y}(0,T;U)$ be such that, for every $J \in \mathcal{B}(U \times [0,T])$, the mapping*

$$\Omega \ni \omega \mapsto \mu(\omega)(J) = \mu(\omega, J) \in [0,T]$$

is measurable. Then there exists a stochastic relaxed control $(\pi_t)_{t \in [0,T]}$ on U such that for $\mathbf{P}-$a.e. $\omega \in \Omega$ we have

$$\mu(\omega, C \times D) = \int_D \pi_t(\omega, C)\, \mathrm{d}t, \quad C \in \mathcal{B}(U),\ D \in \mathcal{B}([0,T]). \tag{4.2}$$

Moreover, if $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a given filtration that satisfies the usual conditions and $\mu([0,\cdot) \times C)$ is $\mathbb{F}-$adapted, for all $C \in \mathcal{B}(U)$, then π is a $\mathbb{F}-$predictable process.

Remark 4.3. We will denote the disintegration formula (4.2) by $\mu(\mathrm{d}u, \mathrm{d}t) = \pi_t(\mathrm{d}u)\, \mathrm{d}t$. Note that $\pi_t(C)$ can be seen as the time-derivative of $\mu([0,t) \times C)$ that exists for almost every $t \in [0,T]$, for all $C \in \mathcal{B}(U)$.

Remark 4.4. It can be proved (see *e.g.* Rem. 3.20 [63]) that if U is a separable and metrisable topological space, then $\mu : \Omega \to \mathcal{Y}(0,T;U)$ is measurable with respect to the Borel $\sigma-$algebra generated by the stable topology if and only if for every $J \in \mathcal{B}(U \times [0,T])$ the mapping

$$\Omega \ni \omega \mapsto \mu(w)(J) \in [0,T]$$

is measurable. This justifies referring to the maps considered in Lemma 4.2 as random Young measures.

For the two following Lemmas, E denotes a Euclidean space with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$.

Lemma 4.5. *Let $f : [0,T] \times \mathbb{R}^d \times U \to E$ be a Borel-measurable function, continuous in $u \in U$, and continuous in $x \in \mathbb{R}^d$ uniformly with respect to $u \in U$, satisfying the growth condition*

$$|f(t,x,u)|_E \le c_{\text{lin}} |x|^\delta + \vartheta(t,u) \tag{4.3}$$

with $\vartheta \in IC(0,T;U)$, for some $\delta \ge 1$. For $\beta \ge 1$ fixed, we denote

$$\mathcal{Y}^\beta(0,T;U) := \left\{ \mu \in \mathcal{Y}(0,T;U) : \vartheta \in L^\beta(\mu) \right\}.$$

Then, for each $t \in [0,T]$, the mapping $\Sigma_t : \mathcal{C}(0,T;\mathbb{R}^d) \times \mathcal{Y}^\beta(0,T;U) \to \mathbb{R}^d$ defined by

$$\Sigma_t(x,\mu) = \int_{U\times[0,t]} f(s,x(s),u)\, \mu(\mathrm{d}u,\mathrm{d}s), \tag{4.4}$$

is Borel-measurable.

Proof. We fix $t \in [0,T]$. For each $N \in \mathbb{N}$ and $i \in \{1,\dots,d\}$ define

$$\phi^i_N(\mu) = \int_{U\times[0,t]} \min\{N, f^i(s,x(s),u)\}\, \mu(\mathrm{d}u,\mathrm{d}s), \quad \mu \in \mathcal{Y}^\beta(0,T;U).$$

The integrand in the above expression is bounded and continuous with respect to $u \in U$. Therefore, by Theorem B.6 ϕ_N is continuous for each $N \in \mathbb{N}$, and by dominated convergence, $\phi^i_N(\mu) \to \Sigma^i_t(x,\mu)$ as $N \to \infty$ for all $\mu \in \mathcal{Y}^\beta(0,T;U)$. Hence, $\Sigma_t(x,\cdot)$ is measurable. Now, we prove that for $\mu \in \mathcal{Y}^\beta(0,T;U)$ fixed, the map $\Sigma_t(\cdot,\mu)$ is continuous. Let $x_n \to x$ in $\mathcal{C}\left(0,T;\mathbb{R}^d\right)$. Then, by assumption we have

$$\left| f(s,x(s),u) - f(s,x_n(s),u) \right| \to 0 \text{ as } n \to \infty, \quad (s,u) \in [0,t] \times U.$$

Moreover, since (x_n) converges in $\mathcal{C}\left(0,T;\mathbb{R}^d\right)$, it is bounded and there exists $\bar\rho > 0$ such that

$$\sup_{s\in[0,T]} \left| x_n(s) - x(s) \right| < \bar\rho, \quad \forall n \in \mathbb{N}.$$

Therefore by (4.3), we have

$$\left| f(s,x(s),u) - f(s,x_n(s),u) \right| \le c_{\text{lin}} \left\{ \bar\rho + 2\, |x|_{\mathcal{C}(0,T;\mathbb{R}^d)} \right\} + \vartheta(s,u)$$

As ϑ belongs to $L^1([0,T]\times U;\mu)$, so does the right side of the above inequality. Therefore, by Lebesgue's dominated convergence theorem we have

$$|\Sigma_t(x,\mu) - \Sigma_t(x_n,\mu)| \le \int_{U\times[0,t]} |f(s,x(s),u) - f(s,x_n(s),u)|\, \mu(\mathrm{d}u,\mathrm{d}s) \to 0$$

as $n \to \infty$, that is, $\Sigma_t(\cdot,\mu)$ is continuous. Since $\mathcal{Y}^\beta(0,T;U)$ is separable and metrizable, by Lemma 1.2.3 in [64] it follows that Σ_t is jointly measurable. □

Recall that $\phi^n \rightharpoonup \phi$ weakly in $L^1([0,T]\times\Omega;E)$ if

$$\mathbb{E}\int_0^T \langle \phi^n(t), \psi(t)\rangle \,\mathrm{d}t \to \mathbb{E}\int_0^T \langle \phi(t), \psi(t)\rangle \,\mathrm{d}t, \quad \forall \psi \in L^\infty([0,T]\times\Omega;E).$$

We have the following result.

Lemma 4.6. *Let $f : [0,T] \times \mathbb{R}^d \times U \to E$ be a Borel-measurable function, continuous in $x \in \mathbb{R}^d$ uniformly with respect to $u \in U$, satisfying the growth condition (4.3) for some $\delta \geq 1$, with $\vartheta \in IC(0,T;U)$. Let $(X^n)_{n\in\mathbb{N}}$ a sequence of $\mathbb{R}^d$-valued processes, and $\mu^n(\mathrm{d}u, \mathrm{d}t) = \pi^n_t(\mathrm{d}u)\,\mathrm{d}t$ a sequence of stochastic relaxed controls defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that*

- *$X^n \to X$ point-wise $\mathbf{P}$-a.s. and in $L^{\beta\delta}(\Omega \times [0,T]; \mathbb{R}^d)$ for some $\beta > 1$, and*
- *$\mu^n \to \mu$ in the stable topology $\mathbf{P}$-a.s., with $\mu(\mathrm{d}u, \mathrm{d}t) = \pi_t(\mathrm{d}u)\,\mathrm{d}t$.*

Suppose further

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbf{P}} \int_{U\times[0,T]} \vartheta(t,u)^\beta \,\mu^n(\mathrm{d}u,\mathrm{d}t) < \infty. \tag{4.5}$$

For each $n \in \mathbb{N}$, set $f^n_t = \bar{f}(t, X^n_t, \pi^n_t)$, $\hat{f}^n_t = \bar{f}(t, X_t, \pi^n_t)$, $t \in [0,T]$. Then

1. *$f^n - \hat{f}^n \to 0$, (strongly) in $L^1([0,T] \times \Omega; E)$*
2. *$\hat{f}^n \rightharpoonup f$, weakly in $L^1([0,T] \times \Omega; E)$*

with $f_t = \bar{f}(t, X_t, \pi_t)$, $t \in [0,T]$.

Proof. We first prove

$$f^n - \hat{f}^n \to 0, \qquad \text{(strongly) in } L^1([0,T]\times\Omega; E). \tag{4.6}$$

By uniform continuity with respect to $u \in U$, for each $n \in \mathbb{N}$ we have

$$I^n_t = \int_U \big|f(t, X^n_t, u) - f(t, X_t, u)\big|\,\pi^n_t(\mathrm{d}u) \leq \sup_{u\in U}\big|f(t, X^n_t, u) - f(t, X_t, u)\big| \to 0$$

as $n \to \infty$ for $t \in [0,T]$, $\mathbf{P}-$a.s. From (4.3) and (4.5), we get

$$\sup_{n\in\mathbb{N}} \mathbb{E} \int_0^T |I^n_t|^\beta \,\mathrm{d}t < +\infty.$$

Hence, $\{I^n\}_{n\in\mathbb{N}}$ is uniformly integrable on $\Omega \times [0,T]$. Lemma 4.11 [65] implies that

$$\mathbb{E}\int_0^T \big|f^n_t - \hat{f}^n_t\big|\,\mathrm{d}t \leq \mathbb{E}\int_0^T I^n_t \,\mathrm{d}t \to 0, \qquad \text{as } \; n \to \infty,$$

and (4.6) follows. Now, we will prove that

$$\hat{f}^n \rightharpoonup f, \qquad \text{weakly in } \; L^1([0,T]\times\Omega; E). \tag{4.7}$$

Let $\psi \in L^\infty([0,T] \times \Omega; E)$ be fixed. We denote $g(t,u) = \langle f(t, X_t, u), \psi_t\rangle$. Then,

$$\mathbb{E}\int_0^T \langle \hat{f}^n_t, \psi_t\rangle \,\mathrm{d}t = \mathbb{E}\int_0^T \left\langle \int_U f(t, X_t, u)\,\pi^n_t(\mathrm{d}u), \psi_t \right\rangle \mathrm{d}t = \mathbb{E}\int_{U\times[0,T]} g(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t)$$

for each $n \in \mathbb{N}$. Let $\varepsilon \in (0,1)$ be fixed and take $C_\varepsilon > \max\{\frac{R}{\varepsilon}, 1\}$ with R defined as the supremum in (4.5), and let $\mathcal{A}_\varepsilon = \{(t,u) \in [0,T] \times U : \vartheta(t,u)^{\beta-1} > C_\varepsilon\}$. Then, for this choice of C_ε, we have

$$\mathbb{E}\left[\mu^n(\mathcal{A}_\varepsilon)\right] = \mathbb{E}\int_{\mathcal{A}_\varepsilon} \mu^n(\mathrm{d}u, \mathrm{d}t) \leq \frac{1}{C_\varepsilon}\mathbb{E}\int_{\mathcal{A}_\varepsilon} \vartheta(t,u)^{\beta-1}\,\mu^n(\mathrm{d}u,\mathrm{d}t) < \varepsilon.$$

We write

$$\mathbb{E}\int_{[0,T]\times U} g(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t) = \mathbb{E}\int_{\mathcal{A}_\varepsilon^c} g(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t) + \mathbb{E}\int_{\mathcal{A}_\varepsilon} g(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t)$$

and observe first that by Theorem B.6 we have $\mathbf{P}-$a.s.

$$\int_{\mathcal{A}_\varepsilon^c} g(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t) \to \int_{\mathcal{A}_\varepsilon^c} g(t,u)\,\mu(\mathrm{d}u,\mathrm{d}t)$$

as $n \to \infty$ and, by (4.3),

$$\int_{\mathcal{A}_\varepsilon^c} g(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t) \leq \left[c_{\mathrm{lin}}\,|X|^\delta_{L^1(0,T;\mathbb{R}^d)} + C_\varepsilon^{1/(\beta-1)}\right] |\psi|_{L^\infty(0,T;E)}\,, \quad \mathbf{P}-\text{a.s.}$$

The right side of the last inequality has finite expectation by the hypothesis about ψ and the Cauchy-Schwarz' inequality. Thus, using Lebesgue's dominated convergence theorem we get

$$\mathbb{E}\int_{\mathcal{A}_\varepsilon^c} g(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t) \to \mathbb{E}\int_{\mathcal{A}_\varepsilon^c} g(t,u)\,\mu(\mathrm{d}u,\mathrm{d}t)$$

as $n \to \infty$. Now, for each $n \in \mathbb{N}$, define the measure $\kappa_n(\mathrm{d}u,\mathrm{d}t,\mathrm{d}\omega) = \mu^n(\omega)(\mathrm{d}u,\mathrm{d}t)\mathbf{P}(\mathrm{d}\omega)$ on $\mathcal{B}(U) \otimes \mathcal{B}([0,T]) \otimes \mathbb{F}$, so we have

$$\mathbb{E}\int_{\mathcal{A}_\varepsilon} \left|g(t,u)\right| \mu^n(\mathrm{d}u,\mathrm{d}t) \leq \int_{\Omega\times\mathcal{A}_\varepsilon} \varphi(t)\,\kappa_n(\mathrm{d}u,\mathrm{d}t,\mathrm{d}\omega) + \int_{\Omega\times\mathcal{A}_\varepsilon} \vartheta(t,u)\,|\psi_t|_E\;\kappa_n(\mathrm{d}u,\mathrm{d}t,\mathrm{d}\omega)$$

with $\varphi = c_{\mathrm{lin}}\left|X\right|^\delta |\psi|_E \in L^\beta([0,T]\times\Omega)$, since $\left|X\right|^\delta \in L^\beta([0,T]\times\Omega)$ and $\psi \in L^\infty([0,T]\times\Omega)$. Using Hölder's inequality we get

$$\begin{aligned}\int_{\Omega\times\mathcal{A}_\varepsilon} \varphi(t)\,\kappa_n(\mathrm{d}u,\mathrm{d}t,\mathrm{d}\omega) &\leq \left[\int_{\Omega\times[0,T]\times U} \varphi(t)^\beta\,\kappa_n(\mathrm{d}u,\mathrm{d}t,\mathrm{d}\omega)\right]^{1/\beta} \cdot \left(\mathbb{E}\left[\mu^n(\mathcal{A}_\varepsilon)\right]\right)^{1-1/\beta} \\ &< \|\varphi\|_{L^\beta([0,T]\times\Omega)}\,\varepsilon^{1-1/\beta}.\end{aligned}$$

Moreover,

$$\int_{\Omega\times\mathcal{A}_\varepsilon} \vartheta(t,u)\,\kappa_n(\mathrm{d}u,\mathrm{d}t,\mathrm{d}\omega) = \mathbb{E}\int_{\mathcal{A}_\varepsilon} \frac{\vartheta(t,u)^\beta}{\vartheta(t,u)^{\beta-1}}\,\mu^n(\mathrm{d}u,\mathrm{d}t) \leq \frac{1}{C_\varepsilon}\mathbb{E}\int_{\mathcal{A}_\varepsilon} \vartheta(t,u)^\beta\,\mu^n(\mathrm{d}u,\mathrm{d}t) \leq \frac{R}{C_\varepsilon} < \varepsilon$$

Then, we have

$$\int_{\Omega\times\mathcal{A}_\varepsilon} \vartheta(t,u)\,|\psi(t)|_E\;\kappa_n(\mathrm{d}u,\mathrm{d}t,\mathrm{d}\omega) < \|\psi\|_{L^\infty([0,T]\times\Omega;E)}\,\varepsilon$$

which holds uniformly with respect to $n \in \mathbb{N}$. Since $\vartheta(t, \cdot)$ is lower semi-continuous for all $t \in [0,T]$, by Lemma B.5 and Fatou's lemma we have

$$\mathbb{E}\int_{U\times[0,T]} \vartheta(t,u)^{\beta}\,\mu(\mathrm{d}u,\mathrm{d}t) \leq \liminf_{n\to\infty} \mathbb{E}\int_{U\times[0,T]} \vartheta(t,u)^{\beta}\,\mu^n(\mathrm{d}u,\mathrm{d}t) \leq R.$$

Therefore, the same estimates hold for μ, that is,

$$\mathbb{E}\int_{\mathcal{A}_\varepsilon} \left|g(t,u)\right| \mu(\mathrm{d}u,\mathrm{d}t) \leq \|\varphi\|_{L^{\beta}([0,T]\times\Omega)}\,\varepsilon^{1-1/\beta} + \|\psi\|_{L^{\infty}([0,T]\times\Omega;E)}\,\varepsilon$$

and since $\varepsilon \in (0,1)$ is arbitrary, we conclude that

$$\mathbb{E}\int_{U\times[0,T]} g(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t) \to \mathbb{E}\int_{U\times[0,T]} g(t,u)\,\mu(\mathrm{d}u,\mathrm{d}t)$$

as $n \to \infty$, and (4.7) follows. □

4.2. Proof of Theorem 3.5

Let $\Theta^n = (\Omega^n, \mathbb{F}^n, \mathbf{P}^n, W^n, \pi^n, X^n)$, $n \in \mathbb{N}$ be a minimizing sequence of weak admissible relaxed controls, that is,

$$\lim_{n\to\infty} \bar{\mathcal{J}}(\Theta^n) = \inf_{\theta\in\bar{\mathcal{U}}(x_0,T)} \bar{\mathcal{J}}(\Theta).$$

From this and Assumption III it follows that there exists $\mathcal{R} > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}^n\int_{U\times[0,T]} \vartheta(t,u)^p\,\pi_t^n(\mathrm{d}u)\,\mathrm{d}t \leq C_1 + C_2\mathbb{E}^n\int_{U\times[0,T]} l(t,X_n(t),u)\,\pi_t^n(\mathrm{d}u)\,\mathrm{d}t \leq \mathcal{R} \tag{4.8}$$

where $\mathbb{E}^n$ denotes expectation with respect to $\mathbf{P}^n$. We will divide the proof in several steps.

STEP 1. Define $m := p\left(1 - \frac{2}{r}\right)$ and let $\alpha \in \left[0, \frac{\gamma}{2} - \frac{1}{m}\right)$ be fixed. By Corollary 2.2 and (4.8) the processes X^n admit versions, which we also denote with X^n, with paths in $\mathcal{C}^{\alpha}(0,T;\mathbb{R}^d)$ satisfying

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n\left[|X^n - x_0|^m_{\mathcal{C}^{\alpha}(0,T;\mathbb{R}^d)}\right] < \infty.$$

Since $C^{\alpha}(0,T;\mathbb{R}^d)$ is compactly embedded in $\mathcal{C}(0,T;\mathbb{R}^d)$, by Chebyshev's inequality it follows that the family of laws of $\{X^n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{C}(0,T;\mathbb{R}^d)$. Using Lemma 2.3, for each $n \in \mathbb{N}$ the process X^n satisfies

$$\int_0^t X_s^n\,\mathrm{d}s = \int_0^t x_0(s)\mathrm{d}s + \int_0^t K(t-s)\,(\zeta_s^n + Y_s^n)\,\mathrm{d}s, \quad t \in [0,T],$$

where

$$\zeta_t^n = \int_0^t \bar{b}(s, X_s^n, \pi_s^n)\,\mathrm{d}s \quad \text{and} \quad Y_t^n = \int_0^t \bar{\sigma}(s, X_s^n, \pi_s^n)\,\mathrm{d}W_s^n.$$

A similar argument as in the proof of Theorem 2.1 and Corollary 2.2 with K replaced by the identity matrix of size d ensures that $\{\zeta^n\}_{n\in\mathbb{N}}$ and $\{Y^n\}_{n\in\mathbb{N}}$ are also tight in $C([0,T],\mathbb{R}^d)$. For each $n \in \mathbb{N}$ we define the random

Young measure

$$\mu^n(\mathrm{d}u, \mathrm{d}t) = \pi_t^n(\mathrm{d}u)\,\mathrm{d}t. \tag{4.9}$$

We also claim that the family of laws of $\{\mu^n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{Y}(0,T;U)$. Indeed, for each $\varepsilon > 0$ define the set

$$D_\varepsilon = \left\{\mu \in \mathcal{Y}(0,T;U) : \int_{U\times[0,T]} \vartheta(t,u)^p\,\mu(\mathrm{d}u,\mathrm{d}t) \le \frac{\mathcal{R}}{\varepsilon}\right\}.$$

By Theorems B.3 and B.4, D_ε is relatively compact in the stable topology of $\mathcal{Y}(0,T;U)$, and by Chebyshev's inequality we have

$$\mathbf{P}^n\left(\mu^n \in \mathcal{Y}(0,T;U)\setminus \bar{D}_\varepsilon\right) \le \mathbf{P}^n\left(\mu^n \in \mathcal{Y}(0,T;U)\setminus D_\varepsilon\right) \le \frac{\varepsilon}{\mathcal{R}}\,\mathbb{E}^n \int_{U\times[0,T]} \vartheta(t,u)^p\,\mu^n(\mathrm{d}u,\mathrm{d}t) \le \varepsilon$$

and the tightness of the laws of $\{\mu^n\}_{n\in\mathbb{N}}$ follows. We use now Prohorov's theorem to ensure existence of a probability measure Ψ on $C\left([0,T],\mathbb{R}^d\right)^3 \times \mathcal{Y}(0,T;U)$ and a subsequence of $\{X^n,\zeta^n,Y^n,\mu^n\}_{n\in\mathbb{N}}$, which we denote using the same index $n\in\mathbb{N}$, such that

$$\text{law}\,(X^n,\zeta^n,Y^n,\mu^n) \to \Psi, \quad n\to\infty. \tag{4.10}$$

Step 2. Dudley's generalization of Skorohod's representation theorem (see Thm. 4.30 in [65]) ensures existence of a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and a sequence of random variables $\{\tilde X^n,\tilde\zeta^n,\tilde Y^n,\tilde\mu^n\}_{n\in\mathbb{N}}$ with values in $\mathcal{C}([0,T];\mathbb{R}^d)^3\times\mathcal{Y}(0,T;U)$, defined on $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, such that

$$(\tilde X^n,\tilde\zeta^n,\tilde Y^n,\tilde\mu^n) \stackrel{d}{=} (X^n,\zeta^n,Y^n,\mu^n), \quad n\in\mathbb{N}, \tag{4.11}$$

and, on the same stochastic basis $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, a random variable $\left(\tilde X,\tilde\zeta,\tilde Y,\tilde\mu\right)$ with values in $\mathcal{C}([0,T];\mathbb{R}^d)^3\times\mathcal{Y}(0,T;U)$ such that

$$(\tilde X_n,\tilde\zeta_n,\tilde Y_n) \to (\tilde X,\tilde\zeta,\tilde Y), \quad \text{in } \ \mathcal{C}([0,T];\mathbb{R}^d)^3, \quad \tilde{\mathbb{P}}-\text{a.s.} \tag{4.12}$$

and

$$\tilde\mu^n \to \tilde\mu, \quad \text{stably in } \ \mathcal{Y}(0,T;U), \quad \tilde{\mathbb{P}}-\text{a.s.} \tag{4.13}$$

Step 3. For each $t\in[0,T]$ let φ_t denote the evaluation map $\mathcal{C}([0,T];\mathbb{R}^d)\ni\zeta\mapsto\zeta(t)\in\mathbb{R}^d$, and let $\Gamma_t : C([0,T];\mathbb{R}^d)^2\times\mathcal{Y}^p(0,T;U)\to\mathbb{R}^d$ be defined as

$$\Gamma_t(x,\zeta,\mu) = \Sigma_t(x,\mu) - \varphi_t(\zeta),\ (x,\zeta)\in C([0,T];\mathbb{R}^d)^2,\ \mu\in\mathcal{Y}^p(0,T;U)$$

with Σ_t as in (4.4) with $f=b$. Using Lemma 4.5 with $\vartheta=\vartheta 1$, it follows that Γ_t is measurable. Hence by (4.11) and the definition of ζ^n, for each $t\in[0,T]$ and $n\in\mathbb{N}$ we have

$$\tilde\zeta_t^n = \int_{U\times[0,t]} b(s,\tilde X_s^n,u)\,\tilde\mu^n(\mathrm{d}u,\mathrm{d}s).$$

By Theorem 6.1 in [66], the map $Z \mapsto \int_0^t K(t-s)Z_s \,\mathrm{d}s$ is continuous from $\mathcal{C}(0,T;\mathbb{R}^d)$ to itself. In particular, it is measurable, so we also have

$$\int_0^t \tilde{X}_s^n \,\mathrm{d}s = \int_0^t x_0(s)\,\mathrm{d}s + \int_0^t K(t-s)\left[\tilde{\zeta}_s^n + \tilde{Y}_s^n\right]\mathrm{d}s. \tag{4.14}$$

Since U is a Suslin space, it also separable and Radon, see *e.g.* Ch. II in [67]. In particular, Lemma 4.2 applies, so there exists a relaxed control process $(\tilde{\pi}_t^n)_{t\in[0,T]}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$\tilde{\mu}^n(\mathrm{d}u,\mathrm{d}t) = \tilde{\pi}_t^n(\mathrm{d}u)\,\mathrm{d}t, \quad \tilde{\mathbb{P}}-\text{a.s.}$$

Now, Y^n is a $\mathbb{F}^n$-martingale with quadratic variation

$$\langle Y^n\rangle_t = \int_0^t (\bar{\sigma}\bar{\sigma}^\top)(s, X_s^n, \pi_s^n)\,\mathrm{d}s, \quad t\in[0,T]$$

and $(X_n, \pi^n) \overset{d}{=} (\tilde{X}_n, \tilde{\pi}^n)$. Then, using once again Lemma 4.5, now with $f = \sigma\sigma^\top$ and $\vartheta = \vartheta^2$, it follows that $\tilde{Y}^n$ is also a martingale with respect to the filtration

$$\tilde{\mathcal{F}}_t^n = \sigma\left\{(\tilde{X}_s^n, \tilde{\pi}_s^n) : s\in[0,t]\right\}, \quad t\in[0,T]$$

and quadratic variation $\left\langle \tilde{Y}^n\right\rangle_t = \int_0^t (\bar{\sigma}\bar{\sigma}^\top)(s, \tilde{X}_s^n, \tilde{\pi}_s^n)\,\mathrm{d}s$. Again, using continuity of the map $Z \mapsto \int_0^t K(t-s)Z_s\,\mathrm{d}s$ from $\mathcal{C}(0,T;\mathbb{R}^d)$ to itself, we obtain

$$\int_0^t \tilde{X}_s \,\mathrm{d}s = \int_0^t x_0(s)\,\mathrm{d}s + \int_0^t K(t-s)\left[\tilde{\zeta}_s + \tilde{Y}_s\right]\mathrm{d}s.$$

We use Lemma 4.2 one last time to ensure the existence of a relaxed control process $(\tilde{\pi}_t)_{t\in[0,T]}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$\tilde{\mu}(\mathrm{d}u,\mathrm{d}t) = \tilde{\pi}_t(\mathrm{d}u)\,\mathrm{d}t, \quad \tilde{\mathbb{P}}-\text{a.s.} \tag{4.15}$$

The filtration $\tilde{\mathbb{F}} = \left\{\tilde{\mathcal{F}}_t\right\}_{t\in[0,T]}$ is defined by

$$\tilde{\mathcal{F}}_t = \sigma\{(\tilde{X}_s, \tilde{\pi}_s) : s\in[0,t]\}, \qquad t\in[0,T].$$

We now claim that $\tilde{Y}$ is a $\tilde{\mathbb{F}}$-martingale. Indeed, From (4.12) we have

$$\sup_{t\in[0,T]} \left|\tilde{X}_t^n - \tilde{X}_t\right|^2 \to 0, \quad \text{as } n\to\infty, \quad \tilde{\mathbb{P}}-\text{a.s.} \tag{4.16}$$

By Theorem 2.1, Corollary 2.2 and (4.11), it follows that

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]} \left|\tilde{X}_t\right|^m\right] < \infty. \tag{4.17}$$

Also by Theorem 2.1, and Chebyshev's inequality, the random variables in (4.16) are uniformly integrable. Then, by Lemma 4.11 in [65] we have

$$\tilde{\mathbb{E}}\Big[\sup_{t\in[0,T]}\big|\tilde{X}^n_t-\tilde{X}_t\big|^2\Big]\to 0,\quad \text{as } n\to\infty. \tag{4.18}$$

Similarly, we have

$$\tilde{\mathbb{E}}\Big[\sup_{t\in[0,T]}\big|\tilde{Y}^n_t-\tilde{Y}_t\big|^2\Big]\to 0,\quad \text{as } n\to\infty. \tag{4.19}$$

This, in conjunction with the martingale property of $\tilde{Y}^n$, implies that for all $0<s<t\leq T$ and for all

$$\phi\in\mathcal{C}_b\left(\mathcal{C}(0,s;\mathbb{R}^d)\times\mathcal{Y}(0,s;U)\right)$$

we have that as $n\to\infty$

$$0=\tilde{\mathbb{E}}\left[\big(\tilde{Y}^n_t-\tilde{Y}^n_s\big)\phi(\tilde{X}^n,\tilde{\mu}^n)\right]\to\tilde{\mathbb{E}}\left[\big(\tilde{Y}_t-\tilde{Y}_s\big)\phi(\tilde{X},\tilde{\mu})\right],$$

which implies that $\tilde{Y}$ is a $\tilde{\mathbb{F}}-$martingale.

STEP 4. We now pass to the limit to identify the process $(\tilde{X}_t)_{t\in[0,T]}$ as a solution of the equation controlled by $(\tilde{\pi}_t)_{t\in[0,T]}$. Using Lemma 4.6 with $E=\mathbb{R}^d$, $f=b$, $\beta=p>1$ and $\delta=1$, we obtain

$$\tilde{b}^n\rightharpoonup\tilde{b},\quad \text{weakly in } L^1([0,T]\times\tilde{\Omega};\mathbb{R}^d) \tag{4.20}$$

with $\tilde{b}_t=\bar{b}(t,\tilde{X}_t,\tilde{\pi}_t),\ t\in[0,T]$. We claim that the process $\tilde{Y}$ satisfies

$$\int_0^t\tilde{X}_s\mathrm{d}s=\int_0^t x_0(s)\mathrm{d}s+\int_0^t K(t-s)\left(\int_0^s\tilde{b}_\tau\,\mathrm{d}\tau+\tilde{Y}_s\right)\mathrm{d}s,\quad \tilde{\mathbb{P}}-\text{a.s.} \tag{4.21}$$

By (4.18) and (4.19), for any $\varepsilon>0$ there exists an integer $\bar{m}=\bar{m}(\varepsilon)\geq 1$ for which

$$\tilde{\mathbb{E}}\Big[\sup_{t\in[0,T]}\big|\tilde{X}^n_t-\tilde{X}_t\big|+\big|\tilde{Y}^n_t-\tilde{Y}_t\big|\Big]<\varepsilon,\quad \forall n\geq\bar{m}. \tag{4.22}$$

From (4.20) we have

$$\tilde{b}\in\overline{\{\tilde{b}^{\bar{m}},\tilde{b}^{\bar{m}+1},\dots\}}^w\subset\overline{\mathrm{co}\{\tilde{b}^{\bar{m}},\tilde{b}^{\bar{m}+1},\dots\}}^w$$

where $\mathrm{co}(\cdot)$ and $\overline{\cdot}^w$ denote the convex hull and weak-closure in $L^1([0,T]\times\tilde{\Omega};\mathbb{R}^d)$ respectively. By Mazur's, see for example Theorem 2.5.16 in [68]

$$\overline{\mathrm{co}\{\tilde{b}^{\bar{m}},\tilde{b}^{\bar{m}+1},\dots\}}^w=\overline{\mathrm{co}\{\tilde{b}^{\bar{m}},\tilde{b}^{\bar{m}+1},\dots\}}.$$

Therefore, there exist an integer $\bar{N}\geq 1$ and $\{\alpha_1,\dots,\alpha_{\bar{N}}\}$ with $\alpha_i\geq 0$, $\sum_{i=1}^{\bar{N}}\alpha_i=1$, such that

$$\Big\|\sum_{i=1}^{\bar{N}}\alpha_i\tilde{b}^{\bar{m}+i}-\tilde{b}\Big\|_{L^1([0,T]\times\tilde{\Omega};\mathbb{R}^d)}<\varepsilon. \tag{4.23}$$

Let $t \in [0,T]$ be fixed. Using the α_i's and (4.14) we can write

$$\int_0^t x_0(s)\mathrm{d}s = \sum_{i=1}^{\bar{N}} \alpha_i \left\{ \int_0^t \tilde{X}_s^{\bar{m}+i}\mathrm{d}s - \int_0^t K(t-s)\left(\int_0^s \tilde{b}_v^{\bar{m}+i}\,\mathrm{d}v + \tilde{Y}_s^{\bar{m}+i}\right)\mathrm{d}s\right\}.$$

Thus, we have

$$\begin{aligned}
I &= \Big|\int_0^t x_0(s)\mathrm{d}s + \int_0^t K(t-s)\left(\int_0^s \tilde{b}_v\,\mathrm{d}v + \tilde{Y}_s\right)\mathrm{d}s - \int_0^t \tilde{X}_s\mathrm{d}s\Big| \\
&= \Big|\sum_{i=1}^{\bar{N}} \alpha_i \left\{ \int_0^t \tilde{X}_s^{\bar{m}+i}\mathrm{d}s - \int_0^t K(t-s)\left(\int_0^s \tilde{b}_v^{\bar{m}+i}\,\mathrm{d}v + \tilde{Y}_s^{\bar{m}+i}\right)\mathrm{d}s\right\} \\
&+ \int_0^t K(t-s)\left(\int_0^s \tilde{b}_v\,\mathrm{d}v + \tilde{Y}_s\right)\mathrm{d}s - \int_0^t \tilde{X}_s\mathrm{d}s\Big| \\
&\le \Big|\sum_{i=1}^{\bar{N}} \alpha^i \int_0^t \tilde{X}_s^{\bar{m}+i}\mathrm{d}s - \int_0^t \tilde{X}_s\mathrm{d}s\Big| + \Big|\sum_{i=1}^{\bar{N}} \alpha^i \int_0^t K(t-s)\tilde{Y}_s^{\bar{m}+i}\mathrm{d}s - \int_0^t K(t-s)\tilde{Y}\mathrm{d}s\Big| \\
&+ \Big|\sum_{i=1}^{\bar{N}} \alpha^i \int_0^t K(t-s)\int_0^s \tilde{b}_v^{\bar{m}+i}\,\mathrm{d}v\,\mathrm{d}s - \int_0^t K(t-s)\int_0^s \tilde{b}_v\,\mathrm{d}v\,\mathrm{d}s\Big| = II + III + IV.
\end{aligned}$$

Then, by (4.22), we have

$$\begin{aligned}
\tilde{\mathbb{E}}(II) &= \tilde{\mathbb{E}}\left[\Big|\sum_{i=1}^{\bar{N}} \alpha^i \int_0^t \tilde{X}_s^{\bar{m}+i}\mathrm{d}s - \int_0^t \tilde{X}_s\mathrm{d}s\Big|\right] \le \sum_{i=1}^{\bar{N}} \alpha^i \tilde{\mathbb{E}}\left[\Big|\int_0^t (\tilde{X}_s^{\bar{m}+i} - \tilde{X}_s)\mathrm{d}s\Big|\right] \\
&\le \sum_{i=1}^{\bar{N}} \alpha^i \tilde{\mathbb{E}}\left[\int_0^t \sup_{s\in[0,T]} \big|\tilde{X}_s^{\bar{m}+i} - \tilde{X}_s\big|\,\mathrm{d}s\right] \le \varepsilon T.
\end{aligned}$$

By Fubini's theorem and (4.22), it follows

$$\begin{aligned}
\tilde{\mathbb{E}}(III) &= \tilde{\mathbb{E}}\left[\Big|\sum_{i=1}^{\bar{N}} \alpha^i \int_0^t K(t-s)\tilde{Y}_s^{\bar{m}+i}\,\mathrm{d}s - \int_0^t K(t-s)\tilde{Y}_s\,\mathrm{d}s\Big|\right] \\
&\le \sum_{i=1}^{\bar{N}} \alpha^i \tilde{\mathbb{E}}\left[\Big|\int_0^t K(t-s)\left(\tilde{Y}_s^{\bar{m}+i} - \tilde{Y}_s\right)\,\mathrm{d}s\Big|\right] \\
&\le \sum_{i=1}^{\bar{N}} \alpha^i \tilde{\mathbb{E}}\left[\Big|\int_0^t K(t-s)\left(\tilde{Y}_s^{\bar{m}+i} - \tilde{Y}_s\right)\,\mathrm{d}s\Big|\right] \le \varepsilon\,\|K\|_{L^1(0,T)}.
\end{aligned}$$

Using twice Jensen's inequality and (4.23),

$$\tilde{\mathbb{E}}(IV) = \tilde{\mathbb{E}}\left[\Big|\sum_{i=1}^{\bar{N}} \alpha^i \int_0^t K(t-s)\int_0^s \tilde{b}_v^{\bar{m}+i}\,\mathrm{d}v\,\mathrm{d}s - \int_0^t K(t-s)\int_0^s \tilde{b}_v\,\mathrm{d}v\,\mathrm{d}s\Big|\right]$$

$$
\begin{aligned}
&= \tilde{\mathbb{E}}\left[\left|\int_0^t K(t-s)\int_0^s\Big(\sum_{i=1}^{\bar N}\alpha^i\tilde b_v^{\bar m+i}-\tilde b_v\Big)\,\mathrm{d}v\,\mathrm{d}s\right|\right]\\
&\le \tilde{\mathbb{E}}\left[t\int_0^t |K(t-s)|\int_0^s\Big|\sum_{i=1}^{\bar N}\alpha^i\tilde b_v^{\bar m+i}-\tilde b_v\Big|\,\mathrm{d}v\,\mathrm{d}s\right]\le \varepsilon T\,\|K\|_{L^1(0,T)}\,.
\end{aligned}
$$

Then, $\tilde{\mathbb{E}}(I)\le [T+\|K\|_{L^1(0,T)}(T+1)]\varepsilon$. Since $\varepsilon>0$ is arbitrary, (4.21) follows.

STEP 5. Set $\tilde\sigma_t=\bar\sigma(t,\tilde X_t,\tilde\pi_t)$ and $\tilde\sigma_t^n=\bar\sigma(t,\tilde X_t^n,\tilde\pi_t^n)$, $t\in[0,T]$. Using Lemma 4.6 with $E=\mathbb{S}^d$, $f=\sigma\sigma^\top$, $\delta=2$ and $\beta=p/2>1$, we obtain

$$
\tilde\sigma^n\tilde\sigma^{n,\top}\rightharpoonup\tilde\sigma\tilde\sigma^\top,\qquad \text{weakly in}\ \ L^1([0,T]\times\tilde\Omega;\mathbb{S}^d).
$$

Let $t\in[0,T]$ be fixed. By (4.19) and the Burkholder-Davis-Gundy inequality we have $\langle\tilde Y^n\rangle_t\to\langle\tilde Y\rangle_t$ in $L^2(\tilde\Omega)$. Then, for any $\varepsilon>0$ there exists an integer $\bar m=\bar m(\varepsilon)\ge 1$ such that

$$
\tilde{\mathbb{E}}\Big[\big|\langle\tilde Y^n\rangle_t-\langle\tilde Y\rangle_t\big|\Big]<\varepsilon,\qquad\forall n\ge\bar m.
$$

As in the proof of Step 4, there also exists an integer $\bar N\ge1$ and $\{\alpha_1,\dots,\alpha_{\bar N}\}$ with $\alpha_i\ge0$, $\sum_{i=1}^{\bar N}\alpha_i=1$, such that

$$
\Big\|\sum_{i=1}^{\bar N}\alpha_i\tilde\sigma^{\bar m+i}\tilde\sigma^{(\bar m+i),\top}-\tilde\sigma\tilde\sigma^\top\Big\|_{L^1([0,T]\times\tilde\Omega;\mathbb{S}^m)}<\varepsilon.
$$

Thus, we have

$$
\begin{aligned}
\Big|\langle\tilde Y\rangle_t-\int_0^t\tilde\sigma_s\tilde\sigma_s^\top\,\mathrm{d}s\Big| &= \Big|\langle\tilde Y\rangle_t-\sum_{i=1}^{\bar N}\alpha^i\langle\tilde Y^{\bar m+i}\rangle_t+\sum_{i=1}^{\bar N}\alpha^i\int_0^t\tilde\sigma_s^{\bar m+i}\tilde\sigma_s^{(\bar m+i),\top}\,\mathrm{d}s-\int_0^t\tilde\sigma_s\tilde\sigma_s^\top\,\mathrm{d}s\Big|\\
&\le\Big|\langle\tilde Y\rangle_t-\sum_{i=1}^{\bar N}\alpha^i\langle\tilde Y^{\bar m+i}\rangle_t\Big|+\Big|\sum_{i=1}^{\bar N}\alpha^i\int_0^t\tilde\sigma_s^{\bar m+i}\tilde\sigma_s^{(\bar m+i),\top}\,\mathrm{d}s-\int_0^t\tilde\sigma_s\tilde\sigma_s^\top\,\mathrm{d}s\Big|.
\end{aligned}
$$

As in Step 4, we have

$$
\tilde{\mathbb{E}}\Big[\Big|\langle\tilde Y\rangle_t-\int_0^t\tilde\sigma_s\tilde\sigma_s^\top\,\mathrm{d}s\Big|\Big]<(1+T)\varepsilon.
$$

Since $\varepsilon>0$ and $t\in[0,T]$ are arbitrary, it follows $\langle\tilde Y\rangle_t=\int_0^t\tilde\sigma_s\tilde\sigma_s^\top\,\mathrm{d}s$ $\tilde{\mathbb{P}}$-a.s. for all $t\in[0,T]$. By the martingale representation theorem (see *e.g.* Thm. 4.2 in Chapter 3.4 [69]) there exist an extension of the probability space $(\tilde\Omega,\tilde{\mathcal F},\tilde{\mathbb P})$, which we also denote $(\tilde\Omega,\tilde{\mathcal F},\tilde{\mathbb P})$, and a d'-dimensional Brownian motion $(\tilde W_t)_{t\ge0}$ defined on $(\tilde\Omega,\tilde{\mathcal F},\tilde{\mathbb P})$, such that

$$
\tilde Y_t=\int_0^t\bar\sigma(s,\tilde X_s,\tilde\pi_s)\,\mathrm{d}\tilde W_s,\quad\tilde{\mathbb P}-\text{a.s.},\quad t\in[0,T].
$$

By (4.21), it follows that

$$\int_0^t \tilde{X}_s \mathrm{d}s = \int_0^t x_0(s)\mathrm{d}s + \int_0^t K(t-s)\left(\int_0^s \tilde{b}_v\,\mathrm{d}v + \tilde{Y}_s\right)\mathrm{d}s,\ \tilde{\mathbb{P}}-\text{a.s.}$$

for each $t \in [0,T]$. By Lemma 2.3 this is equivalent to $\tilde{X}$ being a solution to the stochastic Volterra equation controlled by $\tilde{\pi}$. In other words, $\tilde{\Theta} = \left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\mathbb{F}}, \tilde{W}, \tilde{X}, \tilde{\pi}\right)$ is a weak admissible relaxed control. By the Fiber Product Lemma B.7 we have

$$\underline{\delta}_{\tilde{X}^n} \otimes \mu^n \to \underline{\delta}_{\tilde{X}} \otimes \mu, \quad \text{stably in } \mathcal{Y}(0,T;\mathbb{R}\times U), \quad \tilde{\mathbb{P}}-\text{a.s.}$$

Since $\mathbb{R}\times U$ is also a metrisable Suslin space, using Lemma B.5 and Fatou's Lemma we get

$$\tilde{\mathbb{E}}\int_{U\times[0,T]} l(t,\tilde{X}_t,u)\,\tilde{\mu}(\mathrm{d}u,\mathrm{d}t) \le \liminf_{n\to\infty}\tilde{\mathbb{E}}\int_{U\times[0,T]} l(t,\tilde{X}^n_t,u)\,\tilde{\mu}^n(\mathrm{d}u,\mathrm{d}t)$$

and since $(\tilde{X}^n,\tilde{\mu}^n) \overset{d}{=} (X^n,\mu^n)$ it follows that

$$\begin{aligned}
\bar{\mathcal{J}}(\tilde{\Theta}) &= \tilde{\mathbb{E}}\int_{U\times[0,T]} l(t,\tilde{X}_t,u)\,\tilde{\mu}(\mathrm{d}u,\mathrm{d}t) + \tilde{\mathbb{E}}G(\tilde{X}_T)\\
&\le \liminf_{n\to\infty}\mathbb{E}^n\int_{U\times[0,T]} l(t,X^n_t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t) + \liminf_{n\to\infty}\mathbb{E}^n G(X^n_T)\\
&\le \liminf_{n\to\infty}\left[\mathbb{E}^n\int_{U\times[0,T]} l(t,X^n_t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t) + \mathbb{E}^n G(X^n_T)\right] = \inf_{\Theta\in\mathcal{U}(x_0)}\bar{\mathcal{J}}(\Theta),
\end{aligned}$$

that is, $\tilde{\Theta}$ is a weak optimal relaxed control for **(RCP)**, and this concludes the proof of Theorem 3.5.

4.3. Proof of Theorem 3.9

Proof. Let $\Theta = (\Omega,\mathcal{F},\mathbf{P},\mathbb{F},X,W,\pi) \in \bar{\mathcal{U}}(x_0,T)$. Define $\mathcal{E} : [0,T]\times\Omega \to \mathbb{R}^{d\times d}\times\mathbb{R}^d\times\mathbb{R}$ as

$$\mathcal{E}(t,\omega) := (\bar{\sigma}\bar{\sigma}^\top,\bar{b},\bar{h})(t,X_t(\omega),\pi_t(\omega)).$$

By Assumption IV, we have that $\mathcal{E}(t,\omega) \in \Gamma(t,X_t(\omega))$, as defined in (3.7), for all $(t,\omega)\in[0,T]\times\Omega$. We also define

$$c^1(t,\omega) := (\bar{\sigma}\bar{\sigma}^\top,\bar{b})(t,X_t(\omega),\pi_t(\omega)), \quad c^2(t,\omega) := \bar{l}(t,X_t(\omega),\pi_t(\omega)).$$

By Lemma 4.2, c^1 and c^2 are measurable with respect to the predictable σ-algebra $\mathcal{G}$ on $Y = [0,T]\times\Omega$. Using Theorem A.1 we conclude the existence of a function $u : [0,T]\times\Omega \to U$ measurable with respect to $\mathcal{G}$ such that

$$c^1(t,\omega) = (\sigma\sigma^\top,b)(t,X_t(\omega),u_t(\omega)), \quad c^2(t,\omega) \ge l(t,X_t(\omega),u_t(\omega)), \quad (t,\omega)\in Y \tag{4.24}$$

and the desired result follows. ☐

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Data availability statement

No new data/codes were created or analyzed in this study.

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Appendix A. Auxiliary results

Let $(Y, \mathcal{G}, \mu)$ be a measure space, k, m be natural numbers, and U a closed subset of an Euclidean space. Let

$$c^1 : Y \to \mathbb{R}^k, \quad c^2 : Y \to \mathbb{R}^m, \quad \phi : Y \times U \to \mathbb{R}^k, \quad \psi : Y \times U \to \mathbb{R}^m_+,$$

be given measurable functions with $u \to \phi(y, u)$ continuous and $u \to \psi_i(y, u)$ lower semi-continuous, for each $y \in Y$ and $i = 1, 2, ...m$. Define

$$\Gamma(y, U) = \Big\{(\phi(y, u), z) \in \mathbb{R}^k \times \mathbb{R}^m : u \in U, z_i \geq \psi_i(y, u) \text{ for } i = 1, \ldots, m\Big\}.$$

Theorem A.1. *If $(c^1(y), c^2(y)) \in \Gamma(y, U)$ for all $y \in Y$, then there exists a measurable function $u : Y \to U$ such that $c^1(y) = \phi(y, u(y))$ and $c^2_i(y) \geq \psi_i(y, u(y))$, $i = 1, \ldots, m$.*

Proof. [13], Theorem A.9. □

Appendix B. Relative compactness and limit theorems for Young measures

Young measures on metrizable Suslin control sets have been studied by [70] and [71]. We refer to the book [64] for more details.

Proposition B.1. *Let U be metrisable (resp. metrisable Suslin). Then the space $\mathcal{Y}(0, T; U)$ endowed with the stable topology is also metrizable (resp. metrizable Suslin).*

Proof. Propositions 2.3.1 and 2.3.3 in [64]. □

The notion of tightness for Young measures that we use was introduced by [72]. See also the book [63]. Recall that a set-valued function $[0, T] \ni t \mapsto D_t \subset U$ is said to be *measurable* if and only if for every open set $\tilde{U} \subset U$,

$$\Big\{t \in [0, T] : D_t \cap \tilde{U} \neq \varnothing\Big\} \in \mathcal{B}([0, T]).$$

Definition B.2. We say that a set $\mathfrak{J} \subset \mathcal{Y}(0, T; U)$ is *flexibly tight* if, for each $\varepsilon > 0$, there exists a measurable set-valued mapping $[0, T] \ni t \mapsto D_t \subset U$ such that D_t is compact for all $t \in [0, T]$ and

$$\sup_{\mu \in \mathfrak{J}} \int_{U \times [0,T]} \mathbf{1}_{D_t^c}(u)\, \mu(\mathrm{d}u, \mathrm{d}t) < \varepsilon.$$

Theorem B.3 (**Equivalence theorem for flexible tightness**)**.** *For any $\mathfrak{J} \subset \mathcal{Y}(0, T; U)$ the two following conditions are equivalent:*

1. *$\mathfrak{J}$ is flexibly tight*
2. *There exists $\vartheta \in IC(0, T; U)$ such that*

$$\sup_{\mu \in \mathfrak{J}} \int_{U \times [0,T]} \vartheta(t, u)\, \mu(\mathrm{d}u, \mathrm{d}t) < +\infty.$$

Proof. See the equivalence assertion in [73], Definition 3.3. □

Theorem B.4 (**Prohorov criterion for relative compactness**,)**.** *Let U be a metrisable Suslin space. Then every flexibly tight subset of $\mathcal{Y}(0, T; U)$ is sequentially relatively compact in the stable topology.*

Proof. [64], Theorem 4.3.5 □

Lemma B.5. *Let U be a metrisable Suslin space and $G \in L^1(0,T;\mathbb{R})$. Let us assume that*

$$l : [0,T] \times U \to [-\infty, +\infty]$$

is a measurable function such that $l(t,\cdot)$ is lower semi-continuous for every $t \in [0,T]$ and satisfies one of the two following conditions:

1. $|l(t,u)| \leq G(t)$, *a.e.* $t \in [0,T]$,
2. $l \geq 0$.

If $\mu^n \to \mu$ stably in $\mathcal{Y}(0,T;U)$, then

$$\int_{U\times[0,T]} l(t,u)\,\mu(\mathrm{d}u,\mathrm{d}t) \leq \liminf_{n\to\infty} \int_{U\times[0,T]} l(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t).$$

Proof. [52], Lemma 2.15 □

It is worth mentioning that these last two results are, in fact, the main reasons why it suffices for the control set U to be only metrizable and Suslin, in contrast with the existing literature on stochastic relaxed controls. Indeed, Theorem B.4 is key to obtain tightness of the laws of random Young measures in the proof of the main existence result, and Lemma B.5 is used to prove the lower semi-continuity of the relaxed cost functionals as well as Theorem B.6 below.

Theorem B.6. *Let U be a metrisable Suslin space. If $\mu^n \to \mu$ stably in $\mathcal{Y}(0,T;U)$, then for every $f \in L^1(0,T;\mathcal{C}_b(U))$ we have*

$$\lim_{n\to\infty} \int_{U\times[0,T]} f(t,u)\,\mu^n(\mathrm{d}u,\mathrm{d}t) = \int_{U\times[0,T]} f(t,u)\,\mu(\mathrm{d}t,\mathrm{d}u).$$

Proof. Use Lemma B.5 with f and $-f$. □

We will need the following version of the so-called Fiber Product Lemma. The proof can be found in [52], Lemma 2.17. For a measurable map $y : [0,T] \to U$, we denote by $\underline{\delta}_{y(\cdot)}(\cdot)$ the *degenerate Young measure* defined as $\underline{\delta}_{y(\cdot)}(\mathrm{d}u,\mathrm{d}t) = \delta_{y(t)}(\mathrm{d}u)\,\mathrm{d}t$.

Lemma B.7 (Fiber Product Lemma). *Let $\mathcal{S}$ and U be separable metric spaces and let $y_n : [0,T] \to \mathcal{S}$ be a sequence of measurable mappings which converge pointwise to a mapping $y : [0,T] \to \mathcal{S}$. Let $\mu^n \to \mu$ stably in $\mathcal{Y}(0,T;U)$ and consider the following sequence of Young measures on $\mathcal{S} \times U$:*

$$(\underline{\delta}_{y_n} \otimes \mu^n)(\mathrm{d}x,\mathrm{d}u,\mathrm{d}t) = \delta_{y_n(t)}(\mathrm{d}x)\,\mu^n(\mathrm{d}u,\mathrm{d}t), \quad n \in \mathbb{N},$$

and

$$(\underline{\delta}_{y} \otimes \mu)(\mathrm{d}x,\mathrm{d}u,\mathrm{d}t) = \delta_{y(t)}(\mathrm{d}x)\,\mu(\mathrm{d}u,\mathrm{d}t).$$

Then $\underline{\delta}_{y_n} \otimes \mu^n \to \underline{\delta}_{y} \otimes \mu$ stably in $\mathcal{Y}(0,T;S \times U)$.