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SEMI-GLOBAL STABILIZATION OF A NONLINEAR TRANSPORT EQUATION WITH NONLOCAL VELOCITY

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Abstract. In this paper, we establish the semi-global feedback stabilization for a class of nonlinear transport equations with nonlocal velocity, which models a highly re-entrant system encountered in semi-conductor manufacturing. We design two new time-varying feedback laws: one yields a semi-global exponential stability of the corresponding closed-loop system, the other one yields a local exponential stability result for the corresponding closed-loop system. The crucial assumption on the velocity function $\lambda(\cdot)$ and the target equilibrium $\overline{\rho}$ is strongly reduced compared to the previous results. Numerical simulations are also provided as illustration of the theoretical results.

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1. Introduction

In this paper, we study the stabilization of a nonlinear transport equation with nonlocal velocity

$$\partial_t \rho + \lambda(W(t))\partial_x \rho = 0, \quad (x,t) \in (0,1) \times (0,\infty), \tag{1.1}$$

in which ρ denotes the density and W denotes the total mass:

$$W(t) = \int_0^1 \rho(x,t)\ \mathrm{d}x, \quad t \in (0,\infty), \tag{1.2}$$

and $\lambda : \mathbb{R} \mapsto (0,\infty)$ is a smooth positive function. Here, $x = 0$ and $x = 1$ denote the entrance and the exit of the manufacturing system, respectively.

For problem (1.1)–(1.2) to be well-posed, since the velocity λ is positive, we further need to impose some boundary conditions at $x = 0$. Here, the natural boundary condition is on the influx:

$$\lambda(W(t))\rho(0,t) = u(t), \quad t \in (0,\infty), \tag{1.3}$$

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and $u : (0, \infty) \to \mathbb{R}$ will be a control function. Furthermore, our goal is to design the control u from measurement made on the outflux

$$y(t) = \lambda(W(t))\rho(1, t), \quad t \in (0, \infty). \tag{1.4}$$

To be more precise, given $\overline{\rho} \in \mathbb{R}$, our goal is to design a feedback control u in terms of y, of the form

$$u(t) = \mathcal{F}[y(\cdot)], \quad t \in (0, \infty), \tag{1.5}$$

where $\mathcal{F}$ is a suitable function of the measurement y, such that the closed-loop system (1.1)–(1.2)–(1.3)–(1.4)–(1.5) is exponentially stable towards the state $\overline{\rho}$.

The nonlinear transport equation model we study arose from the semiconductor manufacturing systems and was firstly introduced by Armbruster *et al.* in [1]. The performance of the wafer fabrication stage of integrated circuit (IC) production is closely related to the profits. These manufacture systems are characterized by their highly re-entrant feature with very high volume (number of parts manufactured per unit time) and very large number of consecutive production steps as well. The continuum model introduced in [1] has the advantage that it is more accurate than the models derived by Discrete Event Simulation method [2] and Queueing Networks method [3].

Control problems for conservation laws and general hyperbolic systems have been widely studied, see [4, 5] and the references therein. Regarding the controllability of nonlinear hyperbolic equations (or systems), one can refer to [6–8] for the context of classical solutions, and to [9, 10] for the context of entropy solutions. Asymptotic stability and stabilization of hyperbolic systems have also been studied in the literature, and three main strategies have been used. The first one relies on the analysis of the solution along the characteristic curves; see in particular [11, 12]. The second one is the Lyapunov function approach; see, in particular, [13–16]. The third one consists in designing boundary feedback controls through the Backstepping method, which has been used for instance to stabilize exponentially the inhomogeneous quasilinear hyperbolic system in H^2 norm (see [17, 18]). One can also refer to [19] for many successful examples about feedback stabilization with this approach.

Concerning the manufacturing model (1.1) itself, the well-posedness of the open-loop system for (1.1)–(1.3) was firstly established in [20]. An optimal control problem motivated by [1, 21] and related to the *Demand Tracking Problem* is also considered in [20]. The objective of that optimal control problem is to minimize, by choosing the influx u, the L^p-norm ($p \geqslant 1$) of the difference between the actual outflux y and a given demand forecast y_d over a fixed time period. Another related work [22] gave a necessary condition for the possible optimal controls. Based on the $W^{1,p}$-regularity results of solutions of the adjoint system, the article [23] exhibits the optimality conditions for a special type of cost functional, which is motivated by the L^2 tracking type cost functional from [1, 20, 21]. Let us also point out that these results have been generalized: the well-posedness of a coupled system of nonlocal conservation laws, modeling the highly re-entrant multi-commodity manufacturing network, for L^p-, BV- and $W^{1,p}$-data and the existence of minimizers that solve the corresponding optimal control problem are proved in [24]. One can also refer to [25] for the well-posedness of system 1.1)–(1.2)–(1.3)–(1.4)–(1.5) in the space of measures with discontinuous velocities. It is also worth mentioning that the exact controllability of the solution and the outflux for (1.1)–(1.3) were obtained in [26].

The output feedback stabilization (*i.e.* with a closed loop control) for the system (1.1) under a suitable feedback law is a natural and interesting problem. More precisely, the problem of asymptotical stabilization can be described as follows: For any given equilibrium $\overline{\rho} \in \mathbb{R}$ and any initial data ρ_0, can we find a feedback law (1.5) giving the control u in terms of y such that $\overline{\rho}$ is asymptotically stable for the corresponding closed-loop control system (1.1)–(1.2)–(1.3)–(1.4)–(1.5), namely, the weak solution ρ to the system (1.1)–(1.2)–(1.3)–(1.4)–(1.5) converges to $\overline{\rho}$ asymptotically when time t goes to ∞?

In order to achieve this goal, a natural way would be to propose a feedback law of the proportional form

$$u(t) - \overline{\rho}\lambda(\overline{\rho}) = k(y(t) - \overline{\rho}\lambda(\overline{\rho})), \quad t \in (0, \infty), \tag{1.6}$$

for some $k \in (-1,1)$. With this feedback law (1.6), the authors in [27] give a sufficient and necessary condition on the exponential stabilization for the corresponding linearized control system by spectral analysis. Then by a Lyapunov function approach and perturbation techniques, local exponential stabilization was also proved for the original closed-loop system in general nonlinear cases. In particular, the feedback law (1.6) with $k \in (-1,1)$ provides local exponential stability if and only if

$$d := \frac{\overline{\rho}\lambda'(\overline{\rho})}{\lambda(\overline{\rho})} \text{ satisfies } d > -1. \tag{1.7}$$

Note that this condition depends on the equilibrium $\overline{\rho}$ and the velocity function λ. It is also important to emphasize that if $\overline{\rho} > 0$, the stabilization result (see [27], Thm. 4.2) for the nonlinear system is local in the sense that the initial data, and consequently the solution, belong to a small neighborhood of the equilibrium $\overline{\rho} > 0$. When $\overline{\rho} = 0$, a global stabilization result in $L^2(0,1)$ has been proved [27], Theorem 4.1 where there is no smallness limitation for the initial data and the solution.

Later on in [28], the stabilization results of [27] have been generalized upon using a Lyapunov function approach. Firstly if $\overline{\rho} = 0$, the global stabilization result in $L^2(0,1)$ for the closed-loop system (1.1)–(1.2)–(1.3)–(1.4)–(1.6) with a general velocity function $\lambda(\cdot) \in C^1([0,\infty);(0,\infty))$ is generalized to $L^p(0,1)$ $(p \geqslant 1)$ data. Secondly if $\overline{\rho} > 0$, the global exponential stabilization result in $L^2(0,1)$ for the closed-loop system (1.1)–(1.2)–(1.3)–(1.4)–(1.6) with a typical class of velocity functions

$$\lambda(s) = \frac{A}{B+s}, \quad s \in [0,\infty) \quad A > 0,\ B > 0 \tag{1.8}$$

is obtained. That is, the smallness restriction on the initial data in [27], Theorem 4.2 is removed by using the special feature of the velocity function (1.8). Additionally, stabilization results for the discrete counterpart of system (1.1)–(1.2) are obtained by the eigenvalue decomposition method and by a Lyapunov function method, and numerical examples are provided to illustrate that if the velocity function $\lambda(\cdot)$ is not of the form (1.8), the solution of the closed-loop system might converge to a different equilibrium or even diverge as the time t goes to ∞. Let us also remark that when $\overline{\rho} \neq 0$, the global stabilization result in $L^p(0,1)$ $(p \geqslant 1, p \neq 2)$ remains an open problem even for systems satisfying (1.8). The situation for general systems with different velocity functions other than (1.8) is widely open.

We also refer to [29] for a result on designing PI control to stabilize the linearized system of the nonlinear model (1.1) by spectral analysis as in [27], or to the more recent work [30] for the design of event-triggered boundary controls to stabilize this model. We also refer to [31] for a nonlocal model which has both local and nonlocal nature, and to [32] for a nonlocal traffic flow model, which also shares the non-local feature of the system (1.1) under consideration.

In this article, to design a suitable feedback law for (1.1)–(1.2)–(1.3)–(1.4), our key observations are the following:

- If we know the value of W for all time $t > 0$, the natural feedback law

$$u(t) - \overline{\rho}\lambda(W(t)) = k(y(t) - \overline{\rho}\lambda(W(t))), \quad t \in (0,\infty), \tag{1.9}$$

with $k \in (-1,1)$ will work, since it is equivalent to

$$\rho(0,t) - \overline{\rho} = k(\rho(1,t) - \overline{\rho}), \quad k \in (-1,1), \tag{1.10}$$

which obviously stabilizes the system.

• If we do not know the function W but only an approximation $\widehat{W}$ on $(0,\infty)$, the feedback control

$$u(t) - \overline{\rho}\lambda(\widehat{W}(t)) = k(y(t) - \overline{\rho}\lambda(\widehat{W}(t))), \quad t \in (0,\infty), \tag{1.11}$$

for $k \in (-1,1)$ may work as well, at least if $\widehat{W}$ converges to W as $t \to \infty$.

• The dynamics of the non-local term W is completely determined by the outflux and the influx, since

$$\frac{\mathrm{d}W}{\mathrm{d}t} = u - y \tag{1.12}$$

in the sense of distribution.

• If one wants to estimate W precisely, it is convenient to set $u = y$, so that the equation (1.1)–(1.2)–(1.3)–(1.4) can be seen as an equation on the torus with constant speed, which thus satisfies several nice properties, such as time periodicity for instance.

Based on these ideas and elements, we show a stabilization result by alternating two types of phases:

- Acquisition steps, in which we simply choose $u = y$ and try to improve the approximation $\widehat{W}$ of W;
- Stabilization steps, in which we choose the feedback law (1.11) for $k \in (-1,1)$, with $\widehat{W}$ solving

$$\frac{\mathrm{d}\widehat{W}}{\mathrm{d}t} = u - y \tag{1.13}$$

  during these steps.

We therefore propose two new closed-loop systems with time varying feedback laws for which we can establish the local exponential stability of $\overline{\rho}$ under the weaker condition

$$s \mapsto g(s) := s\lambda(s) \text{ is a local } C^1 \text{ diffeomorphism around } \overline{\rho}, \tag{1.14}$$

which is in fact guaranteed if and only if $g'(\overline{\rho}) \neq 0$, *i.e.*

$$d = \frac{\overline{\rho}\lambda'(\overline{\rho})}{\lambda(\overline{\rho})} \neq -1. \tag{1.15}$$

Theorem 1.1. *Let $\overline{\rho} \in \mathbb{R}$, $\lambda \in C^1(\mathbb{R};(0,\infty))$, and $k \in (-1,1)$. Let $R_0 > r_0 > 0$ be such that the function g defined by $g(s) = s\lambda(s)$ is a C^1 diffeomorphism of $[\overline{\rho} - R_0, \overline{\rho} + R_0]$ on its image. Assume that $\rho_0 \in L^p(0,1)$ for some $p \in (1,\infty)$ satisfies*

$$\|\rho_0 - \overline{\rho}\|_{L^p(0,1)} \leqslant r_0. \tag{1.16}$$

Then there exists $\mathcal{T}_1 = \mathcal{T}_1(r_0, R_0)$ and $\mathcal{T}_2 = \mathcal{T}_2(r_0, R_0)$ (see Rem. 3.5) such that, introducing the sequence of times

$$T_0 = 0 < T_1 < T_2 < \cdots < T_i < T_{i+1} < \cdots \longrightarrow +\infty, \qquad \text{with } T_{2i+1} - T_{2i} = \mathcal{T}_1 \text{ and } T_{2i+2} - T_{2i+1} = \mathcal{T}_2, \tag{1.17}$$

the feedback

$$u(t) = y(t) \text{ for } t \in (T_{2i}, T_{2i+1}), i \in \mathbb{N} \tag{1.18}$$

and

$$
\begin{cases}
u(t) - \overline{\rho}\lambda(\widehat{W}(t)) = k(y(t) - \overline{\rho}\lambda(\widehat{W}(t)), & \text{for } t \in (T_{2i+1}, T_{2i+2}), i \in \mathbb{N}, \\
\widehat{W}(T_{2i+1}) = g^{-1}\left(\dfrac{1}{T_{2i+1} - T_{2i}} \displaystyle\int_{T_{2i}}^{T_{2i+1}} y(\tau)\,\mathrm{d}\tau\right), \\
\widehat{W}' = (k-1)y(t) + (1-k)\overline{\rho}\lambda(\widehat{W}(t)) & \text{for } t \in (T_{2i+1}, T_{2i+2}), i \in \mathbb{N},
\end{cases}
\tag{1.19}
$$

is an exponentially stable feedback law around $\overline{\rho}$ for (1.1)–(1.2)–(1.3)–(1.4). More precisely, for $p \in (1, \infty)$, there exist $C_p = C_p(r_0, R_0) > 0$ and $\alpha = \alpha(r_0, R_0) > 0$ such that, for any $\rho_0 \in L^p(0,1)$ satisfying (1.16), the solution ρ of (1.1)–(1.2)–(1.3)–(1.4) with initial datum ρ_0, with the feedback law given by (1.18) for $t \in (T_{2i}, T_{2i+1})$ and by (1.19) for $t \in (T_{2i+1}, T_{2i+2})$ satisfies:

$$
\|\rho(t) - \overline{\rho}\|_{L^p(0,1)} \leqslant C_p \exp(-\alpha t) \|\rho_0 - \overline{\rho}\|_{L^p(0,1)}, \quad \forall t \in [0, \infty). \tag{1.20}
$$

Remark 1.2. Strictly speaking, the feedback law proposed in Theorem 1.1 is not of the form (1.5), although the control u can be computed only from the knowledge of y.

First, we alternate between acquisition steps, corresponding to the time intervals (T_{2i}, T_{2i+1}), $i \in \mathbb{N}$, and stabilization steps, corresponding to the intervals (T_{2i+1}, T_{2i+2}), $i \in \mathbb{N}$. Thus, our feedback law is *time-dependent*, alternating between two different laws.

Second, it is important to notice that, for $i \in \mathbb{N}$, $\widehat{W}(T_{2i+1})$ depends on y on (T_{2i}, T_{2i+1}). Therefore, to compute the control function u at time $t = T_{2i+1}^+$, one needs to know y at time $t = T_{2i+1}^+$ *and* $y|_{(T_{2i}, T_{2i+1})}$.

Third, let us remark that on time intervals of the form (T_{2i+1}, T_{2i+2}), $i \in \mathbb{N}$, the state of the system becomes augmented with the extra unknown $\widehat{W}$, which is introduced to track W on the time interval.

Remark 1.3. The stabilization result in Theorem 1.1 is semiglobal, since R_0 and r_0 are not supposed to be small.

For instance, the proof of Theorem 1.1 gives that, if $\lambda(s) = A/(B+s)$ for some A and B positive constants, then for every $\overline{\rho} \geqslant 0$, for every non-negative initial state ρ_0, the feedback law proposed in 1.1 applies and the corresponding solution ρ converges exponentially, as $t \to \infty$, to $\overline{\rho}$.

In fact, in addition to the proof of Theorem 1.1, one should only check that the function $s \mapsto g(s) := s\lambda(s)$ is a global C^1 diffeomorphism of $(0, \infty)$ on $(0, A)$.

Remark 1.4. It is interesting to notice that, if the condition (1.14) holds, then Theorem 1.1 holds for R_0 and r_0 small enough. Accordingly, in this case, it provides a local stabilization result in the neighborhood of $\overline{\rho}$.

Thus, the feedback law proposed here allows to consider situations in which the condition $d > -1$ (recall the definition of d in (1.7)) is not satisfied, which is the necessary and sufficient condition for the feedback law of the proportional form (1.6) to work see [27].

Also note that Theorem 1.1 generalizes the stabilization result of [27] in space $L^2(0,1)$ to $L^p(0,1)$ spaces $(p \in (1, \infty))$.

One of the drawback of the stabilization law proposed in Theorem 1.1 is that $\mathcal{T}_1$ and $\mathcal{T}_2$ should be large enough and are not very explicit (see Rem. 3.5). This might seem not very effective when considering the acquisition steps (the time intervals of the form $[T_{2i}, T_{2i+1})$), in which there is no stability process involved. Therefore, we propose below another approach for the acquisition step, in which the time for the acquisition step is reduced, but which is effective only locally.

Theorem 1.5. *Let $\overline{\rho} \in \mathbb{R}$, $\lambda \in C^2(\mathbb{R}; (0, \infty))$, $p \in (1, \infty)$ and $k \in (-1, 1)$.*

Set

$$d = \frac{\overline{\rho}\lambda'(\overline{\rho})}{\lambda(\overline{\rho})}, \tag{1.21}$$

and assume that $d \neq -1$.

Then there exists $\mathcal{T}_2 > 0$ *such that, introducing the sequence of times*

$$T_0 = 0 < T_1 < T_2 < \cdots < T_i < T_{i+1} < \cdots \longrightarrow +\infty, \quad \text{with } T_{2i+2} - T_{2i+1} = \mathcal{T}_2, \tag{1.22}$$

the feedback

$$u(t) = y(t) \text{ for } t \in (T_{2i}, T_{2i+1}), i \in \mathbb{N} \tag{1.23}$$

with

$$T_{2i+1} = T_{2i} + \frac{1}{\lambda(\widehat{W}(T_{2i}))}, i \in \mathbb{N} \tag{1.24}$$

and

$$\begin{cases} u(t) - \overline{\rho}\lambda(\widehat{W}(t)) = k(y(t) - \overline{\rho}\lambda(\widehat{W}(t)), & \text{for } t \in (T_{2i+1}, T_{2i+2}), i \in \mathbb{N}, \\ \widehat{W}(T_{2i+1}) = \widehat{W}(T_{2i}) - \dfrac{1}{1+d}\left(\widehat{W}(T_{2i}) - \displaystyle\int_{T_{2i}}^{T_{2i+1}} y(s)ds\right), & \\ \widehat{W}' = (k-1)y(t) + (1-k)\overline{\rho}\lambda(\widehat{W}(t)) & \text{for } t \in (T_{2i+1}, T_{2i+2}), i \in \mathbb{N}, \end{cases} \tag{1.25}$$

is a locally exponentially stable feedback law around $\overline{\rho}$ *for* (1.1)–(1.2)–(1.3)–(1.4). *More precisely, for any* $\mathcal{T}_2$ *such that* $\mathcal{T}_2 > 1/\lambda(\overline{\rho})$, *there exist* $r_0 > 0$, $C_p > 0$ *and* $\alpha > 0$ *such that for all* $\rho_0 \in L^p(0,1)$ *and* $\widehat{W}_0$ *satisfying*

$$\|\rho_0 - \overline{\rho}\|_{L^p(0,1)} + |\widehat{W}_0 - W_0| \leqslant r_0, \tag{1.26}$$

the solution $(\rho, \widehat{W})$ *of* (1.1)–(1.2)–(1.3)–(1.4) *with initial datum* $(\rho_0, \widehat{W}_0)$, *with the feedback law given by* (1.23) *for* $t \in (T_{2i}, T_{2i+1})$ *and by* (1.19) *for* $t \in (T_{2i+1}, T_{2i+2})$ *satisfies*[1]

$$\|\rho(t) - \overline{\rho}\|_{L^p(0,1)} + |\widehat{W}(t) - W(t)| \leqslant C_p \exp(-\alpha t)\left(\|\rho_0 - \overline{\rho}\|_{L^p(0,1)} + |\widehat{W}_0 - W_0|\right), \quad \forall t \in [0, \infty). \tag{1.27}$$

Of course, in practice, one can alternate the acquisition steps proposed in Theorem 1.1 and Theorem 1.5, keeping in mind that the one proposed in Theorem 1.5 is in principle effective only locally around the stationary state $\overline{\rho}$.

The organization of this paper is as follows: In Section 2, we prove the well-posedness of the closed-loop system. Then in Section 3, we give the proof of Theorem 1.1. In Section 4, we focus on the proof of Theorem 1.5. Finally, some numerical simulations are shown in Section 5.

[1]Properly speaking, $\widehat{W}$ is defined only on the time intervals $[T_{2i+1}, T_{2i+2}]$, and for convenience of notation, we simply set $\widehat{W}(t) = \widehat{W}(T_{2i})$ for $t \in (T_{2i}, T_{2i+1})$.

2. Wellposedness of the closed-loop system

In this section, we prove the well-posedness of the closed-loop system presented in Theorem 1.1.

The boundary condition $y = u$ is classical and we refer to [27], Lemma 2.1 for the following well-posedness result:

Proposition 2.1. *Let $T > 0$, $\overline{\rho} \in \mathbb{R}$, $\lambda \in C^1(\mathbb{R}; (0, \infty))$. Then for all $\rho_0 \in L^1(0,1)$, there exists a unique solution $\rho \in C^0([0,T]; L^1(0,1))$ of*

$$\begin{cases} \partial_t \rho + \lambda(W(t)) \partial_x \rho = 0, & (x,t) \in (0,1) \times (0,T), \\ u(t) = y(t), & t \in (0,T), \\ \rho(x,0) = \rho_0(x), & x \in (0,1), \end{cases} \tag{2.1}$$

where

$$\begin{cases} W(t) = \displaystyle\int_0^1 \rho(x,t) \mathrm{d}x, & t \in (0,T), \\ u(t) = \lambda(W(t)) \rho(0,t), & t \in (0,T), \\ y(t) = \lambda(W(t)) \rho(1,t), & t \in (0,T), \end{cases} \tag{2.2}$$

in the following sense: $\rho \in C^0([0,T]; L^1(0,1))$, $(W, u, y) \in W^{1,1}(0,T) \times L^1(0,T) \times L^1(0,T)$, and for every $s \in (0,T]$, and every $\varphi \in C^1([0,1] \times [0,s])$ satisfying $\varphi(x,s) = 0$, for all $x \in [0,1]$, one has

$$\int_0^s \int_0^1 \rho(x,t)(\partial_t \varphi(x,t) + \lambda(W(t)) \partial_x \varphi(x,t)) \mathrm{d}x \mathrm{d}t \\ - \int_0^s y(t) \varphi(1,t) \mathrm{d}t + \int_0^s u(t) \varphi(0,t) \mathrm{d}t + \int_0^1 \rho_0(x) \varphi(x,0) \mathrm{d}x = 0. \tag{2.3}$$

Furthermore, if $\rho_0 \in L^p(0,1)$ for some $p \in [1, \infty)$, then the solution ρ of (2.1) belongs to $C^0([0,T]; L^p(0,1))$, and $(W, u, y) \in W^{1,p}(0,T) \times L^p(0,T) \times L^p(0,T)$.

We then focus on the feedback law (1.19) for the system (1.1)–(1.2)–(1.3)–(1.4), and check that this corresponds to a well-posed system. We first introduce the corresponding definition of weak solution.

Definition 2.2. Let $T > 0$, $\overline{\rho} \in \mathbb{R}$, $k \in \mathbb{R}$, $\rho_0 \in L^1(0,1)$ and $w_0 \in \mathbb{R}$. A pair of functions $(\rho, w) \in C^0([0,T]; L^1(0,1)) \times W^{1,1}(0,T)$ is a weak solution of the Cauchy problem

$$\begin{cases} \partial_t \rho + \lambda(W(t)) \partial_x \rho = 0, & (x,t) \in (0,1) \times (0,T), \\ u(t) = k y(t) + (1-k) \overline{\rho} \lambda(w(t)), & t \in (0,T), \\ w'(t) = u(t) - y(t), & t \in (0,T), \\ \rho(x,0) = \rho_0(x), & x \in (0,1), \\ w(0) = w_0, \end{cases} \tag{2.4}$$

where (W, u, y) are as in (2.2), if the following properties hold: $(\rho, w) \in C^0([0,T]; L^1(0,1)) \times W^{1,1}(0,T)$, $(W, u, y) \in W^{1,1}(0,T) \times L^1(0,T) \times L^1(0,T)$, and for every $s \in (0,T]$, and every $\varphi \in C^1([0,1] \times [0,s])$ satisfying

$$\forall x \in [0,1], \quad \varphi(x,s) = 0,$$

one has

$$\int_0^s \int_0^1 \rho(x,t)(\partial_t\varphi(x,t) + \lambda(W(t))\partial_x\varphi(x,t))\mathrm{d}x\mathrm{d}t \\ - \int_0^s y(t)\varphi(1,t)\mathrm{d}t + \int_0^s u(t)\varphi(0,t)\mathrm{d}t + \int_0^1 \rho_0(x)\varphi(x,0)\mathrm{d}x = 0 \quad (2.5)$$

and

$$w' = u - y \quad \text{in } \mathscr{D}'(0,T). \quad (2.6)$$

Remark 2.3. In the previous work [27], equation (3.26), it has been observed that, when ρ satisfies the transport equation $(2.4)_{(1)}$ the total mass $W(t)$ of ρ, defined by $(2.2)_{(1)}$, satisfies

$$W' = u - y, \quad \text{in } \mathscr{D}'(0,T).$$

where $(W,u,y) \in W^{1,1}(0,T) \times L^1(0,T) \times L^1(0,T)$ are defined in system (2.2). Accordingly, a solution (ρ,w) of (2.4) should satisfy that $w - W$ is a constant function. This motivates the choice done in $(1.19)_{(1)}$.

Remark 2.4. Note that the problem (2.4) is of course completely equivalent to the Cauchy problem

$$\begin{cases} \partial_t\rho + \lambda(W(t))\partial_x\rho = 0, & (x,t) \in (0,1)\times(0,T), \\ u(t) = ky(t) + (1-k)\overline{\rho}\lambda(w(t)), & t \in (0,T), \\ w'(t) = (k-1)y(t) + (1-k)\overline{\rho}\lambda(w(t)), & t \in (0,T), \\ \rho(x,0) = \rho_0(x), & x \in (0,1), \\ w(0) = w_0, \end{cases} \quad (2.7)$$

where (W,u,y) are as in (2.2), which is of course completely similar to the feedback law given in (1.19) for the system (1.1)–(1.2)–(1.3)–(1.4).

We now give a well-posedness result for the system (2.4).

Theorem 2.5. *Let $\overline{\rho} \in \mathbb{R}$ and $k \in [-1,1]$ be given. For any initial data $(\rho_0,w_0) \in L^1(0,1)\times\mathbb{R}$, there exists a time $T^* > 0$ such that the Cauchy problem (2.4) has a unique weak solution $(\rho,w) \in C^0([0,T^*);L^1(0,1)) \times W^{1,1}_{loc}([0,T^*))$ with $(W,u,y) \in W^{1,1}_{loc}([0,T^*)) \times L^1_{loc}([0,T^*)) \times L^1_{loc}([0,T^*))$.*

Moreover, for $p \in (1,\infty)$, and $\rho_0 \in L^p(0,1)$, the solution (ρ,w) of the Cauchy problem (2.4) belongs to the set $C^0([0,T^*);L^p(0,1)) \times W^{1,p}_{loc}([0,T^*))$ with $(W,u,y) \in W^{1,p}_{loc}([0,T^*)) \times L^p_{loc}([0,T^*)) \times L^p_{loc}([0,T^*))$. Besides, if we denote by T^* the maximal time of existence in this class, T^* is finite if and only if $\lim_{t\to T^*} \|\rho(t)\|_{L^p(0,1)} = \infty$.

Sketch of the proof of Theorem 2.5. We first sketch the proof of the local result of Theorem 2.5, and show that there exists $\delta > 0$ depending on (ρ_0,w_0), such that the Cauchy problem (2.4) has a unique weak solution (ρ,w) $\in C^0([0,\delta];L^1(0,1)) \times W^{1,1}(0,\delta)$ with $(W,u,y) \in W^{1,1}(0,\delta)\times L^1(0,\delta)\times L^1(0,\delta)$.

Clearly, since we are looking for $W \in W^{1,1}(0,\delta)$ and since λ takes values in $(0,\infty)$, the system (2.4) can be rewritten as

$$\begin{cases} \partial_t\rho + \lambda(W(t))\partial_x\rho = 0, & (x,t) \in (0,1)\times(0,\delta), \\ \rho(0,t) = k\rho(1,t) + (1-k)\overline{\rho}\,\dfrac{\lambda(w(t))}{\lambda(W(t))}, & t \in (0,\delta), \\ w'(t) = u(t) - y(t), & t \in (0,\delta), \\ \rho(x,0) = \rho_0(x), & x \in (0,1), \\ w(0) = w_0. \end{cases} \quad (2.8)$$

with

$$W(t) = \int_0^1 \rho(x,t)\mathrm{d}x, \quad t \in (0,\delta). \tag{2.9}$$

As mentioned in Remark 2.3, this implies that

$$\forall t \in (0,\delta), \quad w(t) = W(t) - W_0 + w_0, \quad \text{with } W_0 = \int_0^1 \rho_0(x)\mathrm{d}x. \tag{2.10}$$

Therefore, for parameters $\delta > 0$, and $M > 0$, that will be chosen later on, we define the following convex and closed subset of $L^1(0,\delta)$:

$$\Sigma_{\delta,M} = \{W \in L^\infty(0,\delta) \text{ with } \|W\|_{L^\infty(0,\delta)} \leqslant M\}. \tag{2.11}$$

We then introduce the mapping $\mathbb{F}_\delta : W \in \Sigma_{\delta,M} \mapsto \widetilde{W}$ where $\widetilde{W}$ is obtained by solving the linear system

$$\begin{cases} \partial_t \widetilde{\rho} + \lambda(W(t))\partial_x \widetilde{\rho} = 0, & (x,t) \in (0,1)\times(0,\delta), \\ \widetilde{\rho}(0,t) = k\widetilde{\rho}(1,t) + (1-k)\overline{\rho}\dfrac{\lambda(W(t)-W_0+w_0)}{\lambda(W(t))}, & t \in (0,\delta), \\ \widetilde{\rho}(x,0) = \rho_0(x), & x \in (0,1), \end{cases} \tag{2.12}$$

and setting

$$\widetilde{W}(t) = \int_0^1 \widetilde{\rho}(x,t)\mathrm{d}x, \qquad t \in (0,\delta). \tag{2.13}$$

Note that the equation (2.12) is a classical Cauchy problem for a transport equation, and is thus easily seen to be well-posed.

Similarly as in [20] for a similar open-loop system and [27] for a similar closed-loop system, we can prove the following properties:
1. For all $M > |W_0|$, there exists $\delta(M) > 0$ such that for all $\delta \in (0, \delta(M)]$, the mapping $\mathbb{F}_\delta$ maps $\Sigma_{\delta,M}$ into itself.
2. There exists $\delta(M) > 0$ such that for all $\delta \in (0,\delta(M)]$, the following contraction type estimate holds: For W_1 and W_2 in $\Sigma_{\delta,M}$,

$$\int_0^\delta |\mathbb{F}_\delta(W_1(t)) - \mathbb{F}_\delta(W_2(t))|\,\mathrm{d}t \leqslant C_M \int_0^\delta |W_1(t)-W_2(t)|\,\mathrm{d}t \int_{1-S(M)\delta}^1 |\rho_0(x)|\,\mathrm{d}x + C_M\delta \int_0^\delta |W_1(t)-W_2(t)|\,\mathrm{d}t, \tag{2.14}$$

with $S(M) = \sup_{[-M,M]} \lambda$, and C_M a positive constant dependent on M only.

Accordingly, there exists a unique fixed point W of $\mathbb{F}_\delta$ in $\Sigma_{\delta,M}$. By construction, this fixed point corresponds to a solution (ρ,w) of (2.4), and we easily check that (ρ,w) belongs to $C^0([0,\delta);L^1(0,1)) \times W^{1,1}(0,\delta)$ with $(W,u,y) \in W^{1,1}(0,\delta)\times L^1(0,\delta)\times L^1(0,\delta)$.

Also note, for later use, that if $\rho_0 \in L^p(0,1)$ for some $p > 1$, M can be chosen as $M = 2\|\rho_0\|_{L^1(0,1)}$, and the contractivity of $\mathbb{F}_\delta$ is guaranteed by taking $\delta > 0$ such that

$$C_M\|\rho_0\|_{L^p(0,1)}\delta^{1-1/p} + C_M\delta = \frac{1}{2},$$

where C_M is a constant depending on M only but not on δ. Accordingly, the time of local existence is bounded from below uniformly for ρ_0 in bounded balls of $L^p(0,1)$.

The fact that this solution is unique comes from the fact that M can be taken arbitrarily large in the above process.

When $\rho_0 \in L^p(0,1)$ for some $p>1$, of course the strategy above applies and yields the existence and uniqueness of a solution (ρ, w) of (2.4) in $C^0([0,T^*); L^1(0,1)) \times W^{1,1}_{loc}([0,T^*))$ with $(W,u,y) \in W^{1,1}_{loc}([0,T^*)) \times L^1_{loc}([0,T^*)) \times L^1_{loc}([0,T^*))$ for some $T^*>0$.

It is easy to check in the above strategy that $\rho_0 \in L^p(0,1)$ implies that, for $W \in \Sigma_{\delta,M}$, the solution $(\widetilde{\rho}, \widetilde{W})$ of (2.12)–(2.13) satisfies: $\widetilde{\rho}(1,\cdot) \in L^p(0,\delta)$, $\widetilde{\rho}(0,\cdot) \in L^p(0,\delta)$, $\widetilde{\rho} \in C^0([0,\delta]; L^p(0,1))$, and $\widetilde{W} \in W^{1,p}(0,\delta)$. Accordingly, the solution (ρ, W) of (2.4) constructed above as the fixed point of $\mathbb{F}_\delta$ belongs to $C^0([0,\delta]; L^p(0,1)) \times W^{1,p}_{loc}(0,\delta)$ with $(W,u,y) \in W^{1,p}(0,\delta) \times L^p(0,\delta) \times L^p(0,\delta)$.

Finally, if we denote by T^* the maximal time of existence of the solution (ρ, W) of (2.4), if T^* is finite, then necessarily $\lim_{T\to T^*} \|\rho(T)\|_{L^p(0,1)} = \infty$. This is indeed a straightforward consequence of the fact, already proved, that the time of local existence is bounded from below uniformly for ρ_0 in bounded balls of $L^p(0,1)$. □

Remark 2.6. Note that in Theorem 2.5, we have stated that $T^* < \infty$ if and only if $\lim_{T\to T^*} \|\rho(T)\|_{L^p(0,1)} = \infty$ for some $p \in (1,\infty)$.

The case L^1 is more delicate since the contractivity of the map $\mathbb{F}_\delta$ in the above proof is not quantified in terms of the L^1 norm of ρ_0, recall (2.14).

However, in the specific case $k=1$, which corresponds to $u=y$ in (2.4), the situation is very degenerate, as W and w do not depend on time. In this case, it is easy to check that the problem (2.4) is globally well-posed in L^1: the solution (ρ, w) belongs to $C^0([0,\infty); L^1(0,1)) \times W^{1,1}_{loc}([0,\infty))$ with $(W,u,y) \in W^{1,1}_{loc}([0,\infty)) \times L^1_{loc}([0,\infty)) \times L^1_{loc}([0,\infty))$.

For later use, we also give the following well-posedness result, which can be proved in a similar (and in fact, easier) way as the previous result, whose proof is thus left to the reader:

Theorem 2.7. *Let $k \in \mathbb{R}$. For $\rho_0 \in L^1(0,1)$ and $h \in L^\infty(0,\infty)$, there exists a time T^* and a unique weak solution (ρ, W) of*

$$\begin{cases} \partial_t \rho + \lambda(W(t)) \partial_x \rho = 0, & (x,t) \in (0,1) \times (0,T), \\ u(t) = k y(t) + h(t) \lambda(W(t)), & t \in (0,T), \\ \rho(x,0) = \rho_0(x), & x \in (0,1), \end{cases} \tag{2.15}$$

where (W,u,y) are as in (2.2), with $(\rho, W) \in C^0([0,T^*); L^1(0,1)) \times W^{1,1}_{loc}([0,T^*))$, and $(u,y) \in L^1_{loc}([0,T^*)) \times L^1_{loc}([0,T^*))$.

Moreover, for $p \in (1,\infty)$, and $\rho_0 \in L^p(0,1)$, the solution (ρ, w) of the Cauchy problem (2.4) belongs to the set $C^0([0,T^*); L^p(0,1)) \times W^{1,p}_{loc}([0,T^*))$ with $(W,u,y) \in W^{1,p}_{loc}([0,T^*)) \times L^p_{loc}([0,T^*)) \times L^p_{loc}([0,T^*))$. In this case, T^* is finite if and only if $\lim_{t\to T^*} \|\rho(t)\|_{L^p(0,1)} = \infty$.

3. Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. As pointed in the introduction, the knowledge of W is critical to design a suitable exponentially stable feedback law, and this is why the algorithm we propose alternate between two phases, one corresponding to an acquisition step in which we try to estimate W at best, and another one corresponding to the natural feedback law that one would use if W were known.

To make the proof of Theorem 1.1 easier to follow, we split this section in several parts: Section 3.1 studies the acquisition step, Section 3.2 focuses on the stabilizing step, and Section 3.3 puts together the results obtained in each section to give the complete proof of Theorem 1.1.

3.1. The acquisition step in Theorem 1.1

Here we focus on the acquisition steps in which the feedback law is given by $u = y$, recall (1.18). For convenience, we study this case in an infinite time horizon.

We therefore focus on the equation

$$\begin{cases} \partial_t \rho + \lambda(W(t))\partial_x \rho = 0, & (x,t) \in (0,1)\times(0,\infty),\\ u(t) = y(t), & t \in (0,\infty),\\ \rho(x,0) = \rho_0(x), & x \in (0,1),\\ \text{where} \quad W(t) = \displaystyle\int_0^1 \rho(x,t)\mathrm{d}x, & t \in (0,\infty),\\ \qquad\qquad u(t) = \lambda(W(t))\rho(0,t), & t \in (0,\infty),\\ \qquad\qquad y(t) = \lambda(W(t))\rho(1,t), & t \in (0,\infty). \end{cases} \tag{3.1}$$

Note that the boundary condition $(3.1)_{(2)}$ also reads

$$\rho(0,t) = \rho(1,t), \quad t \in (0,\infty). \tag{3.2}$$

That way, we identify $x = 0$ and $x = 1$, and the equation (3.1) can now be thought as taking place in the torus $\mathbb{T}$. We then easily get the following lemma, whose proof is left to the reader:

Lemma 3.1. *The solution ρ of (3.1) with initial datum $\rho_0 \in L^1(0,1)$ is global in time and satisfies:*

- *for all $t \geqslant 0$,*

$$W(t) = W_0 = \int_0^1 \rho_0 \,\mathrm{d}x \text{ and } \lambda(W(t)) = \lambda(W_0). \tag{3.3}$$

- *if $\rho_0 \in L^p(0,1)$ for some $p \in [1,\infty)$, ρ belongs to $C^0(\mathbb{R}_+; L^p(0,1))$ and for all constant $\overline{\rho} \in \mathbb{R}$, for all $t \geqslant 0$,*

$$\|\rho(t) - \overline{\rho}\|_{L^p(0,1)} = \|\rho_0 - \overline{\rho}\|_{L^p(0,1)}. \tag{3.4}$$

Once we have noticed that the velocity $t \mapsto \lambda(W(t))$ does not depend of time according to (3.3), the solution ρ of (3.1) with initial datum $\rho_0 \in L^1(0,1)$ is periodic in time of period $1/\lambda(W_0)$, and $y(t) = \lambda(W(t))\rho(1,t)$ has the same period.

Our claim is that we can then recover $\lambda(W_0)W_0$ in the following way:

Proposition 3.2. *Let $\rho_0 \in L^1(0,1)$, and ρ the corresponding solution of (3.1).*
Then y is $1/\lambda(W_0)$ periodic and we have the following identity:

$$\lambda(W_0)\int_0^{1/\lambda(W_0)} y(\tau)\,\mathrm{d}\tau = \lambda(W_0)W_0. \tag{3.5}$$

Besides, setting, for all $t > 0$,

$$F(t) = \frac{1}{t}\int_0^t y(\tau)\,\mathrm{d}\tau, \tag{3.6}$$

we get the following estimate:

$$\forall t > 0, \qquad |F(t) - \lambda(W_0)W_0| \leqslant \frac{1}{t}\int_0^1 |\rho_0(x) - W_0|\,\mathrm{d}x. \tag{3.7}$$

Before going into the proof of Proposition 3.2, let us point out that this simple result is in fact crucial to our approach, as it states that F in (3.6) is an estimator of $g(W_0)$ (recall that g is defined by $g(s) = s\lambda(s)$). In particular, if g is injective, one can recover W_0 from F.

Proof. The proof of (3.5) follows from the fact that the velocity $\lambda(W_0)$ does not depend on time, so that $y(t) = \lambda(W_0)\rho(t,1)$ and, from the characteristic formula,

$$y(t) = \lambda(W_0)\rho_0(1 - t\lambda(W_0)) \text{ for all } t \in [0, 1/\lambda(W_0)].$$

We therefore have

$$\int_0^{1/\lambda(W_0)} y(\tau)\,\mathrm{d}\tau = \lambda(W_0)\int_0^{1/\lambda(W_0)} \rho_0(1 - t\lambda(W_0))\,\mathrm{d}t = \int_0^1 \rho_0(x)\,\mathrm{d}x = W_0,$$

which implies (3.5).

Now, with F as in (3.6), using (3.5) we have

$$\begin{aligned} |F(t) - \lambda(W_0)W_0| &= \frac{1}{t}\left|\int_0^t (y(\tau) - \lambda(W_0)W_0)\,\mathrm{d}\tau\right| \\ &\leqslant \frac{1}{t}\sup_{\widetilde{t}\in[0,1/\lambda(W_0)]}\left|\int_0^{\widetilde{t}} (y(\tau) - \lambda(W_0)W_0)\,\mathrm{d}\tau\right| \\ &\leqslant \frac{1}{t}\int_0^{1/\lambda(W_0)} |y(\tau) - \lambda(W_0)W_0|\,\mathrm{d}\tau \\ &\leqslant \frac{1}{t}\int_0^1 |\rho_0(x) - W_0|\,\mathrm{d}x, \end{aligned}$$

where the last identity comes from the formula $y(t) = \lambda(W_0)\rho_0(1 - t\lambda(W_0))$ and the change of variable $t \to 1 - t\lambda(W_0)$. This completes the proof of (3.7). □

Remark 3.3. During the acquisition step, we get an estimator of $g(W_0)$ (recall the definition of g in (1.14)) and thus g has to be at least injective to recover W_0 from the knowledge of $g(W_0)$. This is in fact a necessary and sufficient condition to identify W_0 when considering the system equation (3.1)–(3.2). Indeed, if g is not injective, we can find distinct $W_{0,a}$, $W_{0,b}$ such that $g(W_{0,a}) = g(W_{0,b})$, and one easily checks that the constant functions ρ_a, ρ_b defined by $\rho_a(x,t) = W_{0,a}$ and $\rho_b(x,t) = W_{0,b}$ for all $x \in (0,1)$ and $t \geqslant 0$ are solutions of (3.1) for which the corresponding outputs $y_a(t) = \lambda(W_{0,a})\rho_a(1,t) = g(W_a)$ and $y_b(t) = \lambda(W_{0,b})\rho_b(1,t) = g(W_b)$ are the same.

3.2. The stabilizing step

In this subsection, we focus on the equations (1.1)–(1.2) completed with boundary conditions of the form

$$\rho(0,t) = k\rho(1,t) + h(t), \quad t \in (0,\infty), \tag{3.8}$$

for some $k \in (-1,1)$ and in which we assume

$$h \in L^\infty(0,\infty). \tag{3.9}$$

Clearly, (3.8) is equivalent to

$$u(t) = ky(t) + h(t)\lambda(W(t)), \quad t \in (0,\infty), \tag{3.10}$$

which thus corresponds to the Cauchy problem (2.15).

Our goal is to prove the following result:

Proposition 3.4. *Let $k \in (-1,1)$, $h \in L^\infty(0,\infty)$ and $\rho_0 \in L^p(0,1)$ for some $p \in (1,\infty)$. Then the solution ρ of (2.15) with initial datum ρ_0 belongs to $C^0([0,\infty); L^p(0,1))$. Besides, taking $\beta > 0$, $\varepsilon \in (0,1)$ such that*

$$e^{-\beta} > |k|(1+\varepsilon), \tag{3.11}$$

and introducing, for all $t \geqslant 0$,

$$\lambda_{\min}(t) = \inf_{\tau\in[0,t]}\{\lambda(W(\tau))\}, \quad \lambda_{\max}(t) = \sup_{\tau\in[0,t]}\{\lambda(W(\tau))\}, \tag{3.12}$$

we get the following estimates: for all $t \geqslant 0$,

$$\|\rho(t)\|_{L^p(0,1)} \leqslant e^{\beta(1-\lambda_{\min}(t)t)}\|\rho_0\|_{L^p(0,1)} + (C_p(\varepsilon))^{1/p}e^{\beta}\left(1+\frac{1}{\beta}\right)\|h\|_{L^\infty(0,t)}, \tag{3.13}$$

where $C_p(\varepsilon)$ is the constant introduced in Lemma A.1.

Proof. For $p \in (1,\infty)$, let T^* be the maximal time of existence of the solution of (2.15). We have, in $\mathscr{D}'(0,T^*)$,

$$\begin{aligned}
&\frac{d}{d\tau}\left(\int_0^1 e^{-p\beta x}|\rho(x,\tau)|^p\,dx\right) = p\int_0^1 e^{-p\beta x}|\rho(x,\tau)|^{p-2}\rho(x,\tau)\partial_\tau\rho(\tau,x)\,dx\\
&= -\int_0^1 e^{-p\beta x}\lambda(W(\tau))\partial_x(|\rho(x,\tau)|^p)\,dx\\
&= -e^{-p\beta}\lambda(W(\tau))^{1-p}|y(\tau)|^p + \lambda(W(\tau))^{1-p}|u(\tau)|^p - p\beta\lambda(W(\tau))\left(\int_0^1 e^{-p\beta x}|\rho(\tau,x)|^p\,dx\right).
\end{aligned}$$

In particular, using (3.10) and Lemma A.1, for all $t \in (0,T^*)$ and $\tau \in (0,t)$, we obtain

$$\begin{aligned}
&\frac{d}{d\tau}\left(\int_0^1 e^{-p\beta x}|\rho(x,\tau)|^p\,dx\right) + p\beta\lambda(W(\tau))\left(\int_0^1 e^{-p\beta x}|\rho(x,\tau)|^p\,dx\right)\\
&\qquad\leqslant \lambda(W(\tau))^{1-p}\left(-e^{-p\beta}|y(\tau)|^p + (1+\varepsilon)^p k^p|y(\tau)|^p + C_p(\varepsilon)\lambda(W(\tau))^p|h(\tau)|^p\right).
\end{aligned}$$

Consequently, using (3.11), for all $t \in (0,T^*)$, we have, in $\mathscr{D}'(0,t)$,

$$\frac{d}{d\tau}\left(\int_0^1 e^{-p\beta x}|\rho(x,\tau)|^p\,dx\right) + p\beta\lambda(W(\tau))\left(\int_0^1 e^{-p\beta x}|\rho(x,\tau)|^p\,dx\right) \leqslant \lambda(W(\tau))C_p(\varepsilon)\|h\|^p_{L^\infty(0,t)}.$$

This inequality yields, for all $t \in (0,T^*)$,

$$\begin{aligned}
\int_0^1 e^{-p\beta x}|\rho(x,t)|^p\,dx &\leqslant e^{-p\beta\int_0^t\lambda(W(\tau))d\tau}\int_0^1 e^{-p\beta x}|\rho_0(x)|^p\,dx + \frac{1}{p\beta}C_p(\varepsilon)\|h\|^p_{L^\infty(0,t)}\\
&\leqslant e^{-p\beta\lambda_{\min}(t)}\int_0^1 e^{-p\beta x}|\rho_0(x)|^p\,dx + \frac{1}{p\beta}C_p(\varepsilon)\|h\|^p_{L^\infty(0,t)},
\end{aligned}$$

where we used

$$\int_0^t \exp\left(-p\beta \int_\tau^t \lambda(W(s))\,\mathrm{d}s\right) \lambda(W(\tau))\,\mathrm{d}\tau \leqslant \frac{1}{p\beta}.$$

This proves in particular that for all $t \in (0, T^*)$,

$$\left\|\mathrm{e}^{-\beta\cdot}\rho(t)\right\|_{L^p(0,1)} \leqslant \mathrm{e}^{-\beta\lambda_{\min}(t)t} \left\|\mathrm{e}^{-\beta\cdot}\rho_0\right\|_{L^p(0,1)} + (C_p(\varepsilon))^{1/p}\left(1+\frac{1}{\beta}\right)\|h\|_{L^\infty(0,t)}, \tag{3.14}$$

where we used that, as $p \in (1, \infty)$,

$$\left(\frac{1}{p\beta}\right)^{1/p} \leqslant \left(\frac{1}{\beta}\right)^{1/p} \leqslant 1 + \frac{1}{\beta}.$$

According to Theorem 2.5, this implies $T^* = \infty$. Besides, bounds on $\mathrm{e}^{-\beta\cdot}$ in $(0,1)$ yields (3.13). This concludes the proof of Proposition 3.4. □

3.3. Proof of Theorem 1.1

We assume the setting of Theorem 1.1, with $\rho_0 \in L^p(0,1)$ for some $p \in (1,\infty)$ and satisfying (1.16) for some r_0.

For later use, we introduce the notations

$$\lambda_{\min} = \inf_{s\in[\overline{\rho}-R_0,\overline{\rho}+R_0]}\{\lambda(s)\}, \tag{3.15}$$

$$\lambda_{\max} = \sup_{s\in[\overline{\rho}-R_0,\overline{\rho}+R_0]}\{\lambda(s)\}, \tag{3.16}$$

$$\gamma_g = \min\{|g(\overline{\rho}+R_0) - g(\overline{\rho}+r_0)|, |g(\overline{\rho}-R_0) - g(\overline{\rho}-r_0)|\}, \tag{3.17}$$

$$\|\lambda'\|_\infty = \|\lambda'\|_{L^\infty([\overline{\rho}-R_0,\overline{\rho}+R_0])}, \tag{3.18}$$

$$\left\|(g^{-1})'\right\|_\infty = \left\|(g^{-1})'\right\|_{L^\infty(g([\overline{\rho}-R_0,\overline{\rho}+R_0]))}. \tag{3.19}$$

We then set for all $i \in \mathbb{N}$,

$$\delta_p(i) = \|\rho(T_i) - \overline{\rho}\|_{L^p(0,1)}, \tag{3.20}$$

and, for $i \in \mathbb{N}$, we denote by $e(2i+1)$ the error between $W(T_{2i+1})$ and $\widehat{W}(T_{2i+1})$ given by

$$e(2i+1) = \left| W(T_{2i+1}) - g^{-1}\left(\frac{1}{T_{2i+1}-T_{2i}}\int_{T_{2i}}^{T_{2i+1}} y(\tau)\,\mathrm{d}\tau\right)\right| \quad \left(= |W(T_{2i+1} - \widehat{W}(T_{2i+1})|\right). \tag{3.21}$$

We will construct the iteration such that for all $i \in \mathbb{N}$,

$$\delta_p(2i) \leqslant r_0. \tag{3.22}$$

(Recall the assumption (1.16), which can be rewritten as $\delta_p(0) \leqslant r_0$.)

In order to show that our construction makes sense, we fix some $i \in \mathbb{N}$ such that our solution ρ has been constructed on $[0, T_{2i}]$ and

$$\delta_p(2i) \leqslant r_0. \tag{3.23}$$

This in particular implies that

$$|W(T_{2i}) - \overline{\rho}| \leqslant r_0,$$

and we show that for a suitable choice of $\mathcal{T}_1 = T_{2i+1} - T_{2i}$ and $\mathcal{T}_2 = T_{2i+2} - T_{2i+1}$, we get $\delta_p(2i+2) \leqslant r_0$.

3.3.1. Iteration: The acquisition step

According to Section 3.1, we have

$$\|\rho(t) - \overline{\rho}\|_{L^p(0,1)} = \delta_p(2i) \quad \text{for all} \ \ t \in [T_{2i}, T_{2i+1}], \tag{3.24}$$

and in particular,

$$\delta_p(2i+1) = \delta_p(2i). \tag{3.25}$$

Recall then that $W(t)$ is constant on $[T_{2i}, T_{2i+1}]$ from Lemma 3.1. We thus set

$$W_{2i} = W(T_{2i}) \text{ and } \lambda_{2i} = \lambda(W(T_{2i})).$$

Besides, for $t > T_{2i}$, we define

$$F_{2i}(t) = \frac{1}{t - T_{2i}} \int_{T_{2i}}^{t} y(\tau)\,\mathrm{d}\tau,$$

Proposition 3.2 then implies that for $t \in (T_{2i}, T_{2i+1})$,

$$\begin{aligned}
|F_{2i}(t) - g(W_{2i})| &\leqslant \frac{1}{t - T_{2i}} \int_0^1 |\rho(x, T_{2i}) - W_{2i}|\,\mathrm{d}x \\
&\leqslant \frac{1}{t - T_{2i}} \left(\int_0^1 |\rho(x, T_{2i}) - \overline{\rho}|\,\mathrm{d}x + |\overline{\rho} - W_{2i}| \right) \\
&\leqslant \frac{2}{t - T_{2i}} \delta_p(2i).
\end{aligned}$$

Taking

$$T_{2i+1} - T_{2i} \geqslant \frac{2r_0}{\gamma_g}, \tag{3.26}$$

where γ_g is defined in (3.17), and noting (3.23), we obtain

$$\frac{2}{T_{2i+1} - T_{2i}} \delta_p(2i) \leqslant \gamma_g.$$

and $F(T_{2i+1}) \in g([\overline{\rho} - R_0, \overline{\rho} + R_0])$. Thus

$$\widehat{W}(T_{2i+1}) = g^{-1}\left(\frac{1}{T_{2i+1} - T_{2i}} \int_{T_{2i}}^{T_{2i+1}} y(\tau)\,\mathrm{d}\tau\right)$$

is well defined and takes values in $[\overline{\rho} - R_0, \overline{\rho} + R_0]$.

Moreover, using (3.23), we obtain

$$e(2i+1) \leqslant \frac{2}{T_{2i+1} - T_{2i}} \left\|(g^{-1})'\right\|_\infty \delta_p(2i), \tag{3.27}$$

where $\left\|(g^{-1})'\right\|_\infty$ is defined in (3.19).

In the following, we let

$$\frac{2}{T_{2i+1} - T_{2i}} \left\|(g^{-1})'\right\|_\infty \leqslant \frac{R_0 - r_0}{4r_0}, \tag{3.28}$$

so that we have

$$|W(T_{2i+1}) - \overline{\rho}| \leqslant \delta_p(2i) \leqslant r_0$$
$$\text{and} \quad |W(T_{2i+1}) - \widehat{W}(T_{2i+1})| \leqslant \frac{R_0 - r_0}{4} \quad \text{and} \quad |\widehat{W}(T_{2i+1}) - \overline{\rho}| \leqslant \frac{R_0 + 3r_0}{4}. \tag{3.29}$$

In the following, we choose $T_{2i+1} - T_{2i} = \mathcal{T}_1$ so that conditions (3.26) and (3.28) are satisfied, *i.e.*

$$\mathcal{T}_1 \geqslant \max\left\{\frac{2r_0}{\gamma_g}, \frac{8r_0\|(g^{-1})'\|_\infty}{R_0 - r_0}\right\}. \tag{3.30}$$

3.3.2. Iteration: The stabilization step

Setting

$$\widetilde{\rho}(x,t) = \rho(x,t) - \overline{\rho}, \qquad \text{for all } x \in (0,1) \text{ and } t \in (T_{2i+1}, T_{2i+2}),$$
$$\widetilde{\lambda}(s) = \lambda(s + \overline{\rho}), \qquad \text{for } s \in \mathbb{R},$$

we easily check that $\widetilde{\rho}$ solves

$$\partial_t \widetilde{\rho} + \widetilde{\lambda}(\widetilde{W}) \partial_x \widetilde{\rho} = 0 \qquad \text{for } (x,t) \in (0,1) \times (T_{2i+1}, T_{2i+2}), \tag{3.31}$$

in which $\widetilde{W}$ denotes the total mass of $\widetilde{\rho}$:

$$\widetilde{W}(t) = \int_0^1 \widetilde{\rho}(x,t)\,\mathrm{d}x \qquad \text{for } t \in (T_{2i+1}, T_{2i+2}), \tag{3.32}$$

with boundary conditions

$$\widetilde{\rho}(0,t) = k\widetilde{\rho}(1,t) + h(t) \qquad \text{for } t \in (T_{2i+1}, T_{2i+2}), \tag{3.33}$$

where

$$h(t) = (1-k)\overline{\rho}\left(\frac{\lambda(\widehat{W}(t))}{\lambda(W(t))} - 1\right). \tag{3.34}$$

and $\widehat{W}(t)$ is given by (1.19).

We then remark that, as long as the solution is well-defined, integrating the equation (1.1), we also have, for all $t \in (T_{2i+1}, T_{2i+2})$,

$$W'(t) = u(t) - y(t) = (k-1)y(t) + (1-k)\overline{\rho}\lambda(\widehat{W}(t)) = (\widehat{W})'(t).$$

Since $|\widehat{W}(T_{2i+1}) - W(T_{2i+1})| = e(2i+1)$, we therefore get

$$\forall t \in (T_{2i+1}, T_{2i+2}), \quad |\widehat{W}(t) - W(t)| = e(2i+1). \tag{3.35}$$

• **The solution is well-defined on** (T_{2i+1}, T_{2i+2})**.** W **and** $\widehat{W}$ **take value in** $[\overline{\rho} - R_0, \overline{\rho} + R_0]$**.**

Let us denote by T^*_{2i+2} the largest time smaller than T_{2i+2} such that

$$|\widehat{W}(t) - \overline{\rho}| \leqslant R_0 \text{ and } |W(t) - \overline{\rho}| \leqslant R_0 \text{ for } t \in (T_{2i+1}, T^*_{2i+2}) \text{ and } \|\widetilde{\rho}(t)\|_{L^p(0,1)} < \infty. \tag{3.36}$$

Our goal is to show that for suitable choices of $\mathcal{T}_1$ and $\mathcal{T}_2$, $T^*_{2i+2} = T_{2i+2}$.

Then using (3.34) and (3.20), (3.27) and (3.35),

$$\|h\|_{L^\infty(T_{2i+1}, T^*_{2i+2})} \leqslant \frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}} e(2i+1) \leqslant \frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}} \frac{2\left\|(g^{-1})'\right\|_\infty}{T_{2i+1} - T_{2i}} \delta_p(2i), \tag{3.37}$$

where we used the notations (3.15)–(3.18)–(3.19).

For $\beta > 0$ and $\varepsilon > 0$ small enough to satisfy $\mathrm{e}^{-\beta} > |k|(1+\varepsilon)$, applying Proposition 3.4 for (3.31)–(3.32)–(3.33)–(3.34), we get for all $t \in (T_{2i+1}, T^*_{2i+2})$ that

$$\|\widetilde{\rho}(t)\|_{L^p(0,1)} \leqslant \mathrm{e}^{\beta(1-\lambda_{\min}(t-T_{2i+1}))}\delta_p(2i) + (C_p(\varepsilon))^{1/p}\mathrm{e}^\beta\left(1+\frac{1}{\beta}\right)\frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}}\frac{2\left\|(g^{-1})'\right\|_\infty}{T_{2i+1}-T_{2i}}\delta_p(2i),$$

that is

$$\begin{aligned}\|\rho(t) - \overline{\rho}\|_{L^p(0,1)} \leqslant{}& \mathrm{e}^{\beta(1-\lambda_{\min}(t-T_{2i+1}))}\delta_p(2i) \\ &+ (C_p(\varepsilon))^{1/p}\mathrm{e}^\beta\left(1+\frac{1}{\beta}\right)\frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}}\frac{2\left\|(g^{-1})'\right\|_\infty}{T_{2i+1}-T_{2i}}\delta_p(2i).\end{aligned} \tag{3.38}$$

Therefore for all $t \in (T_{2i+1}, T^*_{2i+2})$,

$$|W(t) - \overline{\rho}| \leqslant \|\rho(t) - \overline{\rho}\|_{L^p(0,1)} \leqslant \mathrm{e}^\beta \delta_p(2i) + \frac{\mathcal{M}_\varepsilon}{T_{2i+1} - T_{2i}}\delta_p(2i), \tag{3.39}$$

where

$$\mathcal{M}_\varepsilon := (C_p(\varepsilon))^{1/p}\mathrm{e}^\beta\left(1+\frac{1}{\beta}\right)\frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}} 2\left\|(g^{-1})'\right\|_\infty. \tag{3.40}$$

According to the estimates (3.38)-(3.39) and Theorem 2.5, the solution $\widetilde{\rho}$ is well-defined on the whole interval (T_{2i+1}, T_{2i+2}), and T^*_{2i+2} necessarily equals to T_{2i+2} if we manage to show that the images of $\widehat{W}$ and W on (T_{2i+1}, T_{2i+2}) are contained in $[\overline{\rho} - R_0, \overline{\rho} + R_0]$. This is the goal of the analysis below.

Now for fixed $R_0 > r_0 > 0$, we take $\beta > 0$ small enough such that

$$\mathrm{e}^{\beta} < \min\left\{1 + \frac{R_0 - r_0}{3r_0}, \frac{1}{|k|}\right\}, \tag{3.41}$$

$\varepsilon > 0$ such that (3.11) holds, and, in accordance with (3.30), we choose $\mathcal{T}_1 = T_{2i+1} - T_{2i}$ such that

$$\mathcal{T}_1 \geqslant \max\left\{ \frac{2r_0}{\gamma_g}, \frac{8r_0\|(g^{-1})'\|_\infty}{R_0 - r_0}, \frac{3\mathcal{M}_\varepsilon r_0}{R_0 - r_0}\right\}, \tag{3.42}$$

where $\mathcal{M}_\varepsilon$ is given by (3.40).

Then it follows from (3.39) that for $t \in (T_{2i+1}, T^*_{2i+2})$,

$$|W(t) - \overline{\rho}| \leqslant \delta_p(2i)\left(1 + \frac{2(R_0 - r_0)}{3r_0}\right) \leqslant \frac{2R_0 + r_0}{3} < R_0.$$

On the other hand, we have from (3.29) that for all $t \in (T_{2i+1}, T^*_{2i+2})$,

$$\begin{aligned}|\widehat{W}(t) - \overline{\rho}| &\leqslant |\widehat{W}(t) - W(t)| + |W(t) - \overline{\rho}| \\ &\leqslant |\widehat{W}(T_{2i+1}) - W(T_{2i+1})| + \frac{2R_0 + r_0}{3} \\ &\leqslant \frac{R_0 - r_0}{4} + \frac{2R_0 + r_0}{3} < R_0.\end{aligned}$$

Accordingly, $T^*_{2i+2} = T_{2i+2}$ when β is chosen as in (3.41) and $\mathcal{T}_1 = T_{2i+1} - T_{2i}$ is chosen as in (3.42).

• **Decay of the L^p-norms.**

By (3.13) and (3.37), we have, with $\mathcal{M}_\varepsilon$ as in (3.40), that for all $t \in [T_{2i+1}, T_{2i+2}]$,

$$\|\rho(t) - \overline{\rho}\|_{L^p(0,1)} = \|\widetilde{\rho}(t)\|_{L^p(0,1)} \leqslant \mathrm{e}^{\beta(1-\lambda_{\min}(t - T_{2i+1}))}\delta_p(2i) + \frac{\mathcal{M}_\varepsilon}{T_{2i+1} - T_{2i}}\delta_p(2i). \tag{3.43}$$

In particular, for $t = T_{2i+2}$ in (3.43), we obtain by (3.27) that

$$\delta_p(2i+2) \leqslant \mathrm{e}^{\beta(1-\lambda_{\min}(T_{2i+2} - T_{2i+1}))}\delta_p(2i) + \frac{\mathcal{M}_\varepsilon}{T_{2i+1} - T_{2i}}\delta_p(2i).$$

Therefore, if $\mathcal{T}_2 = T_{2i+2} - T_{2i+1}$ and $\mathcal{T}_1 = T_{2i+1} - T_{2i}$ satisfy, in addition to (3.42),

$$\mathrm{e}^{\beta(1-\lambda_{\min}\mathcal{T}_2)} \leqslant \frac{1}{4} \quad \text{and} \quad \frac{\mathcal{M}_\varepsilon}{\mathcal{T}_1} \leqslant \frac{1}{4},$$

i.e.

$$\mathcal{T}_2 \geqslant \frac{1}{\lambda_{\min}}\left(1 + \frac{\log 4}{\beta}\right) \quad \text{and} \quad \mathcal{T}_1 \geqslant 4\mathcal{M}_\varepsilon, \tag{3.44}$$

we get

$$\delta_p(2i+2) \leqslant \frac{1}{4}\delta_p(2i) + \frac{1}{4}\delta_p(2i) \leqslant \frac{1}{2}\delta_p(2i). \tag{3.45}$$

3.3.3. Complete Iteration: From T_{2i} to T_{2i+2}

For fixed $R_0 > r_0 > 0$ and $k \in (-1,1)$, we take $\beta > 0$ such that (3.41) holds, $\varepsilon > 0$ such that (3.11) holds, and, with $\mathcal{M}_\varepsilon$ given by (3.40), we set

$$\mathcal{T}_1 = \max\left\{ \frac{2r_0}{\gamma_g}, \frac{8r_0\|(g^{-1})'\|_\infty}{R_0 - r_0}, \frac{3\mathcal{M}_\varepsilon r_0}{R_0 - r_0}, 4\mathcal{M}_\varepsilon \right\}, \tag{3.46}$$

$$\mathcal{T}_2 = \frac{1}{\lambda_{\min}}\left(1 + \frac{\log 4}{\beta}\right). \tag{3.47}$$

By Sections 3.3.1 and 3.3.2, we obtain that for all $i \in \mathbb{N}$, the solution ρ of (1.1)–(1.2)–(1.3)–(1.4) with the feedback law given by (1.18) for $t \in (T_{2i}, T_{2i+1})$ and by (1.19) for $t \in (T_{2i+1}, T_{2i+2})$ satisfies, if the initial condition ρ_0 is such that $\|\rho_0 - \overline{\rho}\|_{L^p(0,1)} \leqslant r_0$, that for all $i \in \mathbb{N}$,

$$\delta_p(2i+1) = \delta_p(2i) \quad \text{and} \quad \delta_p(2i+2) \leqslant \frac{1}{2}\delta_p(2i).$$

In particular, we also get

$$\forall i \in \mathbb{N}, \quad \delta_p(2i) \leqslant 2^{-i}\delta_p(0). \tag{3.48}$$

Let us now explain how to obtain the estimate (1.20) from there. Recall that $T_0 = 0$ and

$$T_{2i+1} = (i+1)\mathcal{T}_1 + i\mathcal{T}_2 \quad \text{and} \quad T_{2i+2} = (i+1)\mathcal{T}_1 + (i+1)\mathcal{T}_2 \quad \text{for all} \quad i \in \mathbb{N}, \tag{3.49}$$

where $\mathcal{T}_1, \mathcal{T}_2$ be given by (3.46) and (3.47). Then for any $t \in [0,\infty)$, $j = \lfloor t/(\mathcal{T}_1 + \mathcal{T}_2) \rfloor$ is the integer such that $T_{2j} \leqslant t < T_{2j+2}$. If $t \in [T_{2j}, T_{2j+1}]$, we get from (3.24) and (3.48) that

$$\|\rho(t) - \overline{\rho}\|_{L^p(0,1)} = \delta_p(2j) \leqslant 2^{-j}\delta_p(0) = 2^{-j}\left\|\rho_0 - \overline{\rho}\right\|_{L^p(0,1)} \leqslant 2\mathrm{e}^{-\frac{\log 2}{\mathcal{T}_1+\mathcal{T}_2}t}\left\|\rho_0 - \overline{\rho}\right\|_{L^p(0,1)}.$$

If $t \in [T_{2j+1}, T_{2j+2}]$, we get from (3.39), (3.44) and (3.48) that

$$\begin{aligned}\|\rho(t) - \overline{\rho}\|_{L^p(0,1)} &\leqslant \mathrm{e}^{\beta}\left\|\widetilde{\rho}(t)\mathrm{e}^{-\beta x}\right\|_{L^p(0,1)} \leqslant \mathrm{e}^{\beta}\delta_p(2j) + \frac{1}{4}\delta_p(2j) \\ &\leqslant \left(\mathrm{e}^{\beta} + \frac{1}{4}\right)2^{-j}\delta_p(0) \leqslant 2\left(\mathrm{e}^{\beta} + \frac{1}{4}\right)\mathrm{e}^{-\frac{\log 2}{\mathcal{T}_1+\mathcal{T}_2}t}\left\|\rho_0 - \overline{\rho}\right\|_{L^p(0,1)}.\end{aligned}$$

These last two estimates conclude the proof of Theorem 1.1.

Remark 3.5. From the above proof, it is clear that Theorem 1.1 can be proven for any $\mathcal{T}_2 > 1/\lambda_{\min}$, by taking, with the notations in the above proof,

$$\mathcal{T}_1 = \max\left\{ \frac{2r_0}{\gamma_g}, \frac{8r_0\|(g^{-1})'\|_\infty}{R_0 - r_0}, \frac{3\mathcal{M}_\varepsilon r_0}{R_0 - r_0}, \frac{2}{1 - \mathrm{e}^{\beta(1-\lambda_{\min}\mathcal{T}_2)}}\mathcal{M}_\varepsilon \right\}.$$

4. Proof of Theorem 1.5

Similarly as in Section 3, we will distinguish the study of the acquisition steps from the stabilizing steps.

Since only the acquisition step differs from the one in Theorem 1.1, we will focus on it before proving Theorem 1.5.

4.1. The acquisition step in Theorem 1.5

In this section, we still consider the Cauchy problem (3.1).

Let us recall that the problem is that we only have an approximation $\widehat{W}_0$ of W_0, and our goal is to get a better approximation of W_0 based on the outflux y.

In particular, our goal is to explain why the value

$$\widetilde{W}_1 = \widehat{W}_0 - \frac{1}{1+\overline{\rho}\lambda'(\widehat{W}_0)/\lambda(\widehat{W}_0)}\left(\widehat{W}_0 - \int_0^{\widehat{T}_0} y(s)\,\mathrm{d}s\right), \tag{4.1}$$

or

$$\widehat{W}_1 = \widehat{W}_0 - \frac{1}{1+\overline{\rho}\lambda'(\overline{\rho})/\lambda(\overline{\rho})}\left(\widehat{W}_0 - \int_0^{\widehat{T}_0} y(s)\,\mathrm{d}s\right), \tag{4.2}$$

with

$$\widehat{T}_0 = \frac{1}{\lambda(\widehat{W}_0)}, \tag{4.3}$$

is a better approximation of W_0 than $\widehat{W}_0$, at least if the initial data are close to $\overline{\rho}$.

Theorem 4.1. *Assume that $\overline{\rho}\lambda'(\overline{\rho}) + \lambda(\overline{\rho}) \neq 0$ (equivalently $\overline{\rho}\lambda'(\overline{\rho})/\lambda(\overline{\rho}) \neq -1$). Let $\varepsilon_0 > 0$ be such that for all $s \in [\overline{\rho}-\varepsilon_0, \overline{\rho}+\varepsilon_0]$, $\overline{\rho}\lambda'(s) + \lambda(s) \neq 0$.*

Then, for $p \in (1,\infty)$ there exists $C > 0$ such that if $(\rho_0, \widehat{W}_0) \in L^p(0,1)\times\mathbb{R}$ satisfies

$$|\widehat{W}_0 - \overline{\rho}| + |W_0 - \overline{\rho}| \leqslant \varepsilon_0, \tag{4.4}$$

setting $(\widetilde{W}_1, \widehat{W}_1, \widehat{T}_0)$ as in (4.1)–(4.3), we have

$$|\widetilde{W}_1 - W_0| \leqslant C\left(|W_0 - \widehat{W}_0|^2 + \|\rho_0 - \overline{\rho}\|_{L^p(0,1)}|W_0 - \widehat{W}_0|^{1-1/p}\right), \tag{4.5}$$

and

$$|\widehat{W}_1 - W_0| \leqslant C\left(|W_0 - \widehat{W}_0|^2 + |\widehat{W}_0 - \overline{\rho}|^2 + \|\rho_0 - \overline{\rho}\|_{L^p(0,1)}|W_0 - \widehat{W}_0|^{1-1/p}\right). \tag{4.6}$$

Before going into the proof of Theorem 4.1, let us mention that although the estimate (4.5) is valid under the only condition of $|\widehat{W}_0 - W_0|$ being small enough, it expresses that $\widetilde{W}_1$ is closer to W_0 than $\widehat{W}_0$ only when $\|\rho_0 - \overline{\rho}\|_{L^p(0,1)}$ is at most of the order $|\widehat{W}_0 - W_0|^{1/p}$.

This means that the estimate (4.5) on $\widetilde{W}_1$ can be used mainly when ρ_0 is close to $\overline{\rho}$, in which case we can also use $\widehat{W}_1$ instead of $\widetilde{W}_1$, thus yielding to the estimate (4.6).

Proof of Theorem 4.1. In order to prove Theorem 4.1, it is convenient to rewrite the system (3.1) on $\mathbb{R}$ with a initial datum ρ_0 extended by 1-periodicity to $\mathbb{R}$. Indeed, since $u = y$ in (3.1), W is independent of time (recall Lem. 3.1), and the velocity $\lambda(W(t))$ is thus constant equal to $\lambda(W_0)$, denoted by λ_0 up to the end. It is then clear that, given $\rho_0 \in L^1(0,1)$, the solution ρ of

$$\begin{cases} \partial_t\rho + \lambda_0\partial_x\rho = 0, & (x,t) \in (0,1)\times(0,\infty), \\ \rho(0,t) = \rho(1,t), & t \in (0,\infty), \\ \rho(x,0) = \rho_0(x), & x \in (0,1), \end{cases}$$

coincides, on $(0,1)\times(0,\infty)$, with the solution ρ of

$$\begin{cases} \partial_t\rho + \lambda_0\partial_x\rho = 0, & (x,t) \in \mathbb{R}\times(0,\infty), \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{4.7}$$

where, with a slight abuse of notation, we denote the same for ρ_0 and its 1-periodic extension on $\mathbb{R}$.

We then work on the solutions ρ of (4.7). It is obvious that

$$\rho(x,t) = \rho_0(x - t\lambda_0), \qquad (x,t) \in \mathbb{R}\times(0,\infty).$$

Accordingly,

$$\begin{aligned} \int_0^{\widehat{T}_0} y(s)\,\mathrm{d}s &= \int_0^{\widehat{T}_0} \lambda_0\rho_0(1 - t\lambda_0)\,\mathrm{d}s = \int_{1-\lambda_0\widehat{T}_0}^1 \rho_0(x)\,\mathrm{d}x \\ &= \int_0^1 \rho_0(x)\,\mathrm{d}x - \int_0^{1-\lambda_0\widehat{T}_0} \rho_0(x)\,\mathrm{d}x \\ &= W_0 - \overline{\rho}(1 - \lambda_0\widehat{T}_0) - \int_0^{1-\lambda_0\widehat{T}_0} (\rho_0 - \overline{\rho})\,\mathrm{d}x \\ &= W_0 - \overline{\rho}\left(1 - \frac{\lambda(W_0)}{\lambda(\widehat{W}_0)}\right) - \int_0^{1-\lambda(W_0)/\lambda(\widehat{W}_0)} (\rho_0 - \overline{\rho})\,\mathrm{d}x. \end{aligned} \tag{4.8}$$

We then estimate the two last terms. On one hand, we get

$$\left|\left(1 - \frac{\lambda(W_0)}{\lambda(\widehat{W}_0)}\right) + \frac{\lambda'(\widehat{W}_0)}{\lambda(\widehat{W}_0)}(W_0 - \widehat{W}_0)\right| \leqslant \frac{\sup_{[\overline{\rho}-\varepsilon_0,\overline{\rho}+\varepsilon_0]}\lambda''}{\inf_{[\overline{\rho}-\varepsilon_0,\overline{\rho}+\varepsilon_0]}\lambda}|W_0 - \widehat{W}_0|^2.$$

On the other hand, we have

$$\begin{aligned} \left|\int_0^{1-\lambda(W_0)/\lambda(\widehat{W}_0)} (\rho_0 - \overline{\rho})\,\mathrm{d}x\right| &\leqslant \|\rho_0 - \overline{\rho}\|_{L^p(0,1)} \left|\left(1 - \frac{\lambda(W_0)}{\lambda(\widehat{W}_0)}\right)\right|^{1-1/p} \\ &\leqslant \|\rho_0 - \overline{\rho}\|_{L^p(0,1)}|W_0 - \widehat{W}_0|^{1-1/p}\left(\frac{\sup_{[\overline{\rho}-\varepsilon_0,\overline{\rho}+\varepsilon_0]}\lambda'}{\inf_{[\overline{\rho}-\varepsilon_0,\overline{\rho}+\varepsilon_0]}\lambda}\right)^{1-1/p}. \end{aligned}$$

By plugging these two estimates into the identity (4.8), we get

$$\left|\int_0^{\widehat{T}_0} y(s)\,\mathrm{d}s + \frac{\overline{\rho}\lambda'(\widehat{W}_0)}{\lambda(\widehat{W}_0)}\widehat{W}_0 - W_0\left(1+\frac{\overline{\rho}\lambda'(\widehat{W}_0)}{\lambda(\widehat{W}_0)}\right)\right| \leqslant C\left(|W_0-\widehat{W}_0|^2 + \|\rho_0-\overline{\rho}\|_{L^p(0,1)}|W_0-\widehat{W}_0|^{1-1/p}\right). \quad (4.9)$$

We then immediately deduce the estimate (4.5).

To prove the estimate (4.6), we proceed similarly, the only difference being in the estimate of the second term in (4.8), which is now as follows:

$$\left|\left(1-\frac{\lambda(W_0)}{\lambda(\widehat{W}_0)}\right) + \frac{\lambda'(\overline{\rho})}{\lambda(\overline{\rho})}(W_0-\widehat{W}_0)\right| \leqslant \sup_{[\overline{\rho}-\varepsilon_0,\overline{\rho}+\varepsilon_0]}\left\{\left(\frac{\lambda'}{\lambda}\right)'\right\}|\widehat{W}_0-\overline{\rho}||\widehat{W}_0-W_0| + \frac{\sup_{[\overline{\rho}-\varepsilon_0,\overline{\rho}+\varepsilon_0]}\lambda''}{\inf_{[\overline{\rho}-\varepsilon_0,\overline{\rho}+\varepsilon_0]}\lambda}|W_0-\widehat{W}_0|^2.$$

The end of the proof of estimate (4.6) then follows line to line the one of (4.5). □

For later use, let us also remark that due to the estimate

$$\forall a,b\geqslant 0,\quad ab\leqslant \frac{(p+1)}{2p}a^{2p/(p+1)} + \frac{(p-1)}{2p}b^{2p/(p-1)},$$

and $|W_0-\overline{\rho}|\leqslant\|\rho_0-\overline{\rho}\|_{L^p(0,1)}$, under the setting of Theorem 4.1, we get the existence of a constant C depending on p such that

$$|\widehat{W}_1-W_0|\leqslant C\left(|\widehat{W}_0-W_0|^2+\|\rho_0-\overline{\rho}\|_{L^p(0,1)}^{2p/(p+1)}\right). \quad (4.10)$$

4.2. Proof of Theorem 4.1

We assume the setting of Theorem 1.5. We choose ε_0 as in Theorem 4.1, and we introduce

$$\lambda_{\min} = \inf_{s\in[\overline{\rho}-\varepsilon_0,\overline{\rho}+\varepsilon_0]}\{\lambda(s)\}, \quad (4.11)$$

$$\|\lambda'\|_\infty = \sup_{s\in[\overline{\rho}-\varepsilon_0,\overline{\rho}+\varepsilon_0]}\{\lambda'(s)\}. \quad (4.12)$$

We assume that $\rho_0\in L^p(0,1)$ $(p>1)$ and $\widehat{W}_0\in\mathbb{R}$ are such that

$$\|\rho(0)-\overline{\rho}\|_{L^p(0,1)}+|\widehat{W}_0-W_0|\leqslant r_0,$$

for a bound $r_0>0$ to be determined. Let $(\rho,\widehat{W})$ be the solution of (1.1)–(1.2)–(1.3)–(1.4) with the feedback law given by (1.23) for $t\in(T_{2i},T_{2i+1})$ and by (1.19) for $t\in(T_{2i+1},T_{2i+2})$.

We then set for all $i\in\mathbb{N}$,

$$\delta_p(i)=\|\rho(T_i)-\overline{\rho}\|_{L^p(0,1)},\quad \text{and}\quad e(i)=|\widehat{W}(T_i)-W(T_i)|. \quad (4.13)$$

We assume that $r_0\leqslant\varepsilon_0$, with ε_0 as in Theorem 4.1, so that we get

$$\delta_p(0)+e(0)\leqslant r_0\leqslant\varepsilon_0. \quad (4.14)$$

Let us take $i \in \mathbb{N}$ such that

$$\delta_p(2i) + e(2i) \leqslant r_0 \tag{4.15}$$

and let us show that, under suitable smallness assumption on r_0, we can guarantee that

$$\delta_p(2i+2) + e(2i+2) \leqslant (\delta_p(2i) + e(2i))\gamma, \tag{4.16}$$

for some $\gamma \in (0,1)$.

During the acquisition step, since W is constant on (T_{2i}, T_{2i+1}) from Lemma 3.1, we get from (4.10) that

$$e(2i+1) \leqslant C\left(e(2i)^2 + \delta_p(2i)^{2p/(p+1)}\right), \quad \text{and} \quad \delta_p(2i+1) = \delta_p(2i). \tag{4.17}$$

During the stabilizing step, as argued in the proof of Theorem 1.1, let us defined T^*_{2i+2} as the largest time larger than T_{2i+1} and smaller than T_{2i+2} such that

$$\forall t \in (T_{2i+1}, T^*_{2i+2}), \quad \|\rho(t) - \overline{\rho}\|_{L^p(0,1)} + |\widehat{W}(t) - \overline{\rho}| \leqslant \varepsilon_0.$$

Then we have

$$\forall t \in (T_{2i+1}, T^*_{2i+2}), \quad |\widehat{W}(t) - W(t)| = e(2i+1). \tag{4.18}$$

Besides, taking $\beta > 0$ and $\varepsilon > 0$ such that (3.11) holds, applying Proposition 3.4, we get that for all $t \in (T_{2i+1}, T^*_{2i+2})$,

$$\begin{aligned} \|\rho(t) - \overline{\rho}\|_{L^p(0,1)} &\leqslant \mathrm{e}^{\beta}\delta_p(2i+1) + (C_p(\varepsilon))^{1/p}\mathrm{e}^{\beta}\left(1 + \frac{1}{\beta}\right)\|h\|_{L^\infty(T_{2i+1}, T^*_{2i+2})} \\ &\leqslant \mathrm{e}^{\beta}\delta_p(2i) + (C_p(\varepsilon))^{1/p}\mathrm{e}^{\beta}\left(1 + \frac{1}{\beta}\right)\frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}} e(2i+1), \end{aligned} \tag{4.19}$$

where the second estimate comes from (3.37), and we used the notations (4.11)–(4.12).

This guarantees that for all $t \in (T_{2i+1}, T^*_{2i+2})$,

$$\begin{aligned} |W(t) - \overline{\rho}| &\leqslant \mathrm{e}^{\beta}\delta_p(2i) + C(C_p(\varepsilon))^{1/p}\mathrm{e}^{\beta}\left(1 + \frac{1}{\beta}\right)\frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}}\left(e(2i)^2 + \delta_p(2i)^{2p/(p+1)}\right), \\ |\widehat{W}(t) - \overline{\rho}| &\leqslant \mathrm{e}^{\beta}\delta_p(2i) + C\left(1 + (C_p(\varepsilon))^{1/p}\mathrm{e}^{\beta}\left(1 + \frac{1}{\beta}\right)\frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}}\right)\left(e(2i)^2 + \delta_p(2i)^{2p/(p+1)}\right), \end{aligned}$$

where the second estimate is deduced from the first one, together with (4.17) and (4.18).

Therefore, if $r_0 > 0$ is small enough (recall that r_0 quantifies the smallness condition in (4.15)), we can guarantee that for all $t \in (T_{2i+1}, T^*_{2i+2})$,

$$|W(t) - \overline{\rho}| + |\widehat{W}(t) - \overline{\rho}| \leqslant \frac{\varepsilon_0}{2}.$$

Combined with (4.19) and Theorem 2.5, we deduce that $T^*_{2i+2} = T_{2i+2}$. We also have

$$e(2i+2) = e(2i+1),$$

and, with $\beta > 0$ and $\varepsilon > 0$ such that (3.11) holds, applying Proposition 3.4, we get that for all $t \in (T_{2i+1}, T_{2i+2})$,

$$\|\rho(t) - \overline{\rho}\|_{L^p(0,1)} \leqslant \mathrm{e}^{\beta(1-\lambda_{\min}(t-T_{2i+1}))}\delta_p(2i) + (C_p(\varepsilon))^{1/p}\mathrm{e}^{\beta}\left(1+\frac{1}{\beta}\right)\frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}}e(2i+1). \tag{4.20}$$

Taking $t = T_{2i+2}$, we get

$$\begin{aligned} e(2i+2) + \delta_p(2i+2) \leqslant \mathrm{e}^{\beta(1-\lambda_{\min}\mathcal{T}_2)}\delta_p(2i) \\ + C\left(1 + (C_p(\varepsilon))^{1/p}\mathrm{e}^{\beta}\left(1+\frac{1}{\beta}\right)\frac{(1-k)\overline{\rho}\,\|\lambda'\|_\infty}{\lambda_{\min}}\right)\left(e(2i)^2 + \delta_p(2i)^{2p/(p+1)}\right). \end{aligned}$$

It is then clear that, for $\mathcal{T}_2$ such that

$$\lambda_{\min}\mathcal{T}_2 > 1,$$

there exists r_0 small enough and $\gamma \in (0,1)$ such that, if the condition (4.15) holds at time T_{2i}, we have (4.16), and in particular the condition (4.15) holds at time $T_{2(i+1)} = T_{2i+2}$.

Accordingly, by induction, we get that, if

$$e(0) + \delta_p(0) \leqslant r_0,$$

then, for all $i \in \mathbb{N}$,

$$e(2i) + \delta_p(2i) \leqslant (e(0) + \delta_p(0))\gamma^{2i}.$$

Since the times $T_{2i+1} - T_{2i}$ are uniformly bounded from above by $1/\lambda_{\min}$, we can argue as in the proof of Theorem 1.1 to show the exponential stability (1.20) for solutions of (1.1)–(1.2)–(1.3)–(1.4) with the feedback law given by (1.23) for $t \in (T_{2i}, T_{2i+1})$ and by (1.19) for $t \in (T_{2i+1}, T_{2i+2})$. Details are left to the reader.

5. Numerical simulations

In this section, we give some numerical simulations which illustrate our theoretical results established before.

5.1. Numerical implementations

The numerical scheme is implemented as follows.

For $N \in \mathbb{N}^*$, the space interval $(0,1)$ is divided into N points, corresponding to a mesh size $\Delta x = 1/N$. The corresponding time discretization parameter is $\Delta t = \Delta x/\lambda_M$, where $\lambda_M = \max\{\lambda(s), s \in \mathbb{R}\}$, so that the CFL condition is satisfied.

The function ρ is then approximated by a sequence of vectors $(\vec{\rho}^{\,n})_{n\in\mathbb{N}} = (\rho_0^n, \cdots, \rho_N^n)_{n\in\mathbb{N}}$, in the sense that ρ_m^n denotes the discrete approximation of the value of ρ at (t_n, x_m) where $t_n = n\Delta t$ and $x_m = m\Delta x$. Similarly, $(W^n)_{n\in\mathbb{N}}$ is a sequence of real numbers approximating W at time t_n and given by

$$W^n = \Delta x \sum_{k=1}^{N} \rho_k^n, \qquad (n \in \mathbb{N}).$$

The sequence $(\widehat{W}^n)_{n\in\mathbb{N}}$ corresponds to the approximation of $\widehat{W}$ at times $(t_n = n\Delta t)_{n\in\mathbb{N}}$.

Corresponding to the algorithm presented in Theorem 1.1, we use the following explicit solver, given for $n \in \mathbb{N}$ by

$$\begin{cases} \rho_m^{n+1} = \rho_m^n - \dfrac{\Delta t \lambda(W^n)}{\Delta x}(\rho_m^n - \rho_{m-1}^n), \qquad m = 1, \cdots, N, \\ \rho_0^{n+1} = \begin{cases} \rho_N^{n+1}, & \text{if } t_{n+1} \in [T_{2i}, T_{2i+1}), \\ k\rho_N^{n+1} + (1-k)\overline{\rho}\dfrac{\lambda(\widehat{W}^{n+1})}{\lambda(W^{n+1})}, & \text{if } t_{n+1} \in [T_{2i+1}, T_{2i+2}), \end{cases} \\ \widehat{W}^{n+1} = \begin{cases} \widehat{W}^n, & \text{if } t_{n+1} \in [T_{2i}, T_{2i+1}) \text{ for some } i \in \mathbb{N}, \\ g^{-1}\left(\dfrac{\sum_{j=v_i}^n \lambda(W^j)\rho_N^j}{n - v_i}\right), & \text{if } t_n < T_{2i+1} \leqslant t_{n+1} \text{ for some } i \in \mathbb{N}, \\ & \quad \text{with } v_i \in \mathbb{N} \text{ such that } T_{2i} < t_{v_i} \leqslant T_{2i} + \Delta t, \\ \widehat{W}^n + \Delta t\, \lambda(W^n)(\rho_0^n - \rho_N^n), & \text{if } T_{2i+1} \leqslant t_n < t_{n+1} < T_{2i+2}, \end{cases} \\ \rho_m^0 = \rho_0(m\Delta x), \qquad \text{for } m = 0, \cdots, N, \\ \widehat{W}^0 = \overline{\rho}, \\ \text{with } W^n = \Delta x \displaystyle\sum_{k=1}^N \rho_k^n. \end{cases} \tag{5.1}$$

Corresponding to the algorithm presented in Theorem 1.5, we do as in (5.1) when t_n belongs to intervals of the form $[T_{2i}, T_{2i+1})$ and when t_n belongs to intervals of the form $[T_{2i+1}, T_{2i+2}]$, except that:

- when $t_n < T_{2i+1} \leqslant t_{n+1}$ for some i, we set

$$\widehat{W}^{n+1} = \widehat{W}^n - \frac{1}{1+d}\left(\widehat{W}^n - \Delta t \sum_{j=v_i}^n \lambda(W^j)\rho_N^j\right),$$

  where $v_i \in \mathbb{N}$ is such that $T_{2i} < t_{v_i} \leqslant T_{2i} + \Delta t$.
- T_{2i+1} is computed when t_n becomes larger than T_{2i} with the formula

$$T_{2i+1} = T_{2i} + \frac{1}{\lambda(\widehat{W}^n)}.$$

We also fix a stopping time T_{stop} given by the stopping criterion $\|\vec{\rho}^{\,n} - \overline{\rho}\vec{e}\|_{\ell^2_N} \leq 10^{-2}$, where $\vec{e} = (1, \cdots, 1)$ and the ℓ^2 norm of $\vec{a} = (a_0, \cdots, a_N)$ is defined as

$$\|\vec{a}\|_{\ell^2_N} = \sqrt{\Delta x \sum_{i=1}^N a_j^2}.$$

5.2. Example 1

We choose the velocity function λ, the steady state $\overline{\rho}$ and the initial datum ρ_0 as follows:

$$\lambda(s) = \frac{1}{1+s^2}, \quad s \in \mathbb{R}, \qquad \overline{\rho} = 0.3, \qquad \rho_0(x) = \sin(\pi x), \quad x \in (0,1).$$

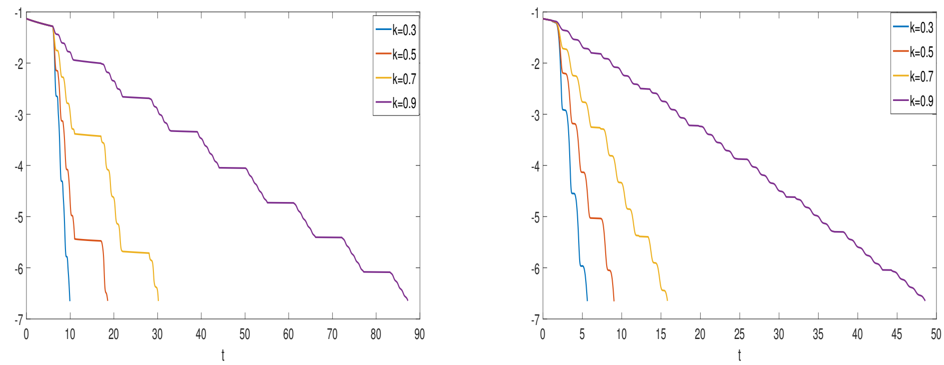

FIGURE 1. Convergence as $t \to \infty$ of $\log \|\vec{\rho} - 0.3\vec{e}\|_{\ell^2_N}$ for different values of k in Example 1 using, in the left, the algorithm of Theorem 1.1, and in the right, the algorithm of Theorem 1.5.

TABLE 1. Stopping times T_{stop} for $k = 0.3, 0.5, 0.7, 0.9$ in Example 1.

	k = 0.3	k=0.5	k=0.7	k=0.9
T_{stop} with the algorithm of Theorem 1.1	9.94	18.59	30.18	87.26
T_{stop} with the algorithm of Theorem 1.5	5.64	9.03	15.82	48.58
T_{stop} with the proportional control (1.6)	4.4	7.61	14.5	48.23

With this example, we show that the feedback control (1.18)–(1.19) of Theorem 1.1 can stabilize the system under consideration (Note that in this case, the proportional control (1.6) in [27] also applies locally, since $d = \overline{\rho}\lambda'(\overline{\rho})/\lambda(\overline{\rho}) = -18/109 > -1$ in this case).

Since $\|\rho_0 - \overline{\rho}\|_{L^1(0,1)} \leqslant 2/\pi < 0.65$, Theorem 1.1 applies, with $r_0 = 0.65$ and $R_0 = 0.7$, and we choose $\mathcal{T}_1 = 6$ and $\mathcal{T}_2 = 5$. Theorem 1.5 also applies, at least locally, with the same choice of $\mathcal{T}_2$ since $\mathcal{T}_2 > 1/\lambda(0.3) = 1.09$. Figure 1 left shows the convergence of $\|\vec{\rho} - 0.3\vec{e}\|_{\ell^2_N}$ to 0 for various choices of the parameter k when using the algorithm of Theorem 1.1, and Figure 1 right plots the same numerical experiments but when using the algorithm of Theorem 1.5.

In Table 1 below, we also compare the different stopping time for the feedback proposed in Theorem 1.1, Theorem 1.5 and the simpler proportional feedback (1.6) in [27] with the same value k, and we do it for several choices of k.

As expected since $d = \overline{\rho}\lambda'(\overline{\rho})/\lambda(\overline{\rho}) = -18/109 > -1$ in this case, the proportional feedback controller (1.6) stabilizes the system and provides a faster decay of the solutions of the system compared to the algorithms of Theorem 1.1 and 1.5, since these both algorithms use acquisition steps during which the stabilization process is momentarily stopped.

However, as underlined in Figures 1, the stopping times for the algorithm of Theorem 1.5 are significantly reduced compared to those of Theorem 1.1, in which the acquisition step is larger. In fact, the performance of Theorem 1.5 is rather close to that of the proportional feedback controller (1.6) in most cases.

It is notable that, as expected, for the three algorithms presented in Table 1, the convergence of the solution is faster when taking k closer to 0.

Let us finally mention that the algorithm of Theorem 1.5, which theoretically should provide stabilization only locally, is working well in all the above mentioned numerical experiments, although the initial datum is not in a small neighborhood of $\overline{\rho}$ (recall $\|\rho_0 - \overline{\rho}\|_{L^1(0,1)} \leqslant 2/\pi < 0.65$).

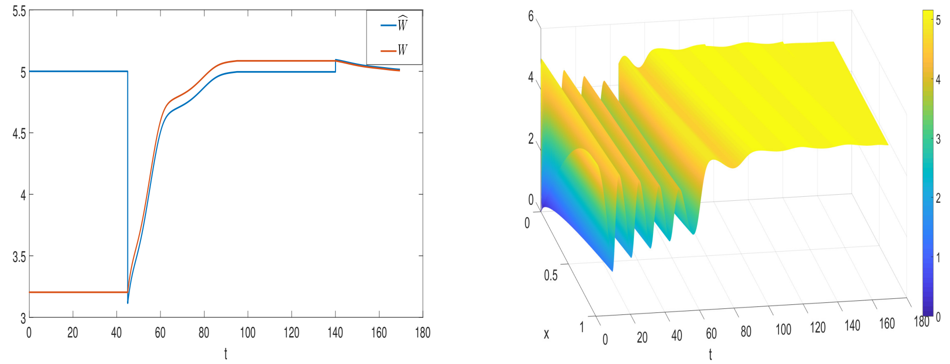

FIGURE 2. Example 2 with the algorithm of Theorem 1.1 and $k = 0.25$. Left, plot of W and its approximation $\widehat{W}$ versus time. Right, the 3-d plot of the solution ρ.

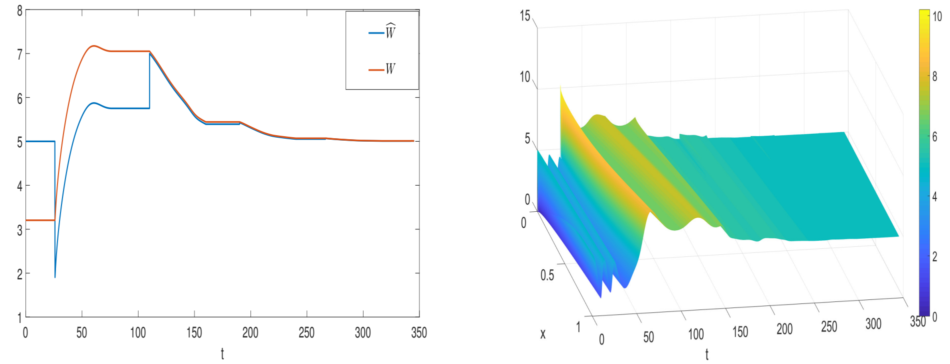

FIGURE 3. Example 2 with the algorithm of Theorem 1.5 and $k = 0.25$. Left, plot of W and its approximation $\widehat{W}$ versus time. Right, the 3-d plot of the solution ρ.

5.3. Example 2

We choose the velocity function λ, the steady state $\overline{\rho}$ and the initial datum ρ_0 as follows:

$$\lambda(s) = \frac{1}{1+s^2}, \quad s \in \mathbb{R}, \qquad \overline{\rho} = 5, \qquad \rho_0(x) = 5\sin\left(\frac{\pi x}{2}\right), \quad x \in (0,1). \tag{5.2}$$

In this case, explicit computations show that

$$d = \frac{\overline{\rho}\lambda'(\overline{\rho})}{\lambda(\overline{\rho})} = -\frac{25}{13} < -1,$$

so that the proportional feedback controller (1.6) does not stabilize the system (1.1)–(1.2)–(1.3)–(1.4) (recall [27]).

Despite this, the feedback control proposed in Theorem 1.1 and Theorem 1.5 still work. Note that $\|\rho_0 - \overline{\rho}\|_{L^1(0,1)} = 5 - 10/\pi < 3$, so we can choose $r_0 = 3, R_0 = 4$, and we choose $\mathcal{T}_1 = 45$ and $\mathcal{T}_2 = 50$ ($\mathcal{T}_2 > 1/\lambda(\overline{\rho})$ with this choice) in the algorithm of Theorem 1.1. Similarly, we choose $\mathcal{T}_2 = 50$ in the algorithm of Theorem 1.5. Regarding the algorithm of Theorem 1.1, Figure 2 left shows how $\widehat{W}$ approximates W and Figure 2 right shows the 3D plot of ρ; the stopping time is 169.51. Figure 3 shows the same plots for the algorithm of Theorem 1.5; the stopping time is then 345.3.

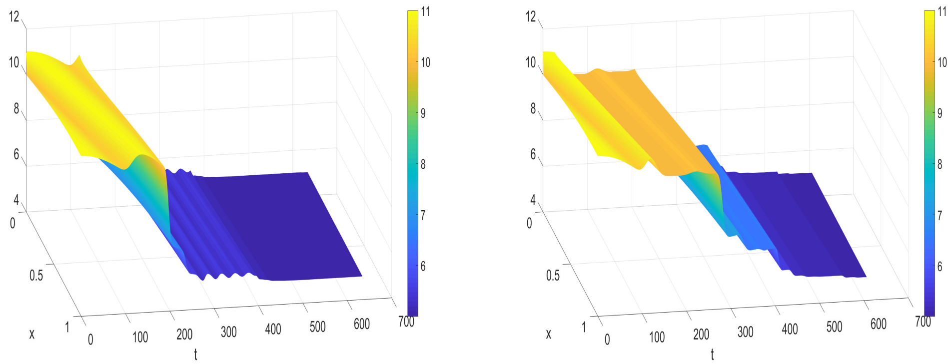

FIGURE 4. Example 3. Left, the 3-d plot of the solution ρ provided by the algorithm of Theorem 1.1 with $k = 0.3$, $\mathcal{T}_1 = 120$, and $\mathcal{T}_2 = 130$. Right, the 3-d plot of the solution ρ provided by the algorithm of Theorem 1.5 with $k = 0.3$, and $\mathcal{T}_2 = 130$.

It might seem surprising that the algorithm of Theorem 1.5 performs worse than the algorithm of Theorem 1.1 on that example. In fact, to understand this, one should look at the way both algorithms approximate W, that is Figures 2 and 3 left. Since the initial datum is rather far from the equilibrium, the algorithm of Theorem 1.5 has difficulty to suitably estimate W at the beginning of the process, while $\widehat{W}$ computed with the algorithm of Theorem 1.1 catches W rather fast.

Note that this example is the one corresponding to Example 2 in [28], Section 5.2, in which the proportional control (1.6) drives the solution ρ to the equilibrium $\widetilde{\rho} = 0.2$ instead of the desired equilibrium $\overline{\rho} = 5$. The reason lies in the fact that the function $g : s \mapsto s\lambda(s)$ coincides at $\widetilde{\rho}$ and $\overline{\rho}$, but the value of $d = \overline{\rho}\lambda'(\overline{\rho})/\lambda(\overline{\rho})$ at $\overline{\rho}$ is strictly less than -1, while its value at $\widetilde{\rho}$ is strictly larger than -1, which implies in particular that the proportional feedback controller given by (1.6) is not locally stabilizing around $\overline{\rho}$.

5.4. Example 3

We choose the velocity function λ, the steady state $\overline{\rho}$ and the initial datum ρ_0 as follows:

$$\lambda(s) = \frac{1}{1+s^2}, \quad s \in \mathbb{R}, \qquad \overline{\rho} = 5, \qquad \rho_0(x) = 10 + \sin\left(\frac{\pi x}{2}\right). \tag{5.3}$$

Note that example (5.3) differs from example (5.2) from the choice of the initial datum. This example is the one given in [28], Section 5.2, where it is shown that the proportional control (1.6) does not work and the solution diverges.

However, the feedback controls provided by Theorem 1.1 and Theorem 1.5 still work provided that $\mathcal{T}_2$ is large enough, as it is shown in Figure 4 plotting ρ with $k = 0.3$. In both numerical tests, we choose $\mathcal{T}_2 = 130$, so that it is larger than $1/\lambda(\overline{\rho})$ and $1/\lambda(W_0)$ (Since $\rho_0(x) \leqslant 11$ for all $x \in (0,1)$, $1/\lambda(W_0) \leqslant 1 + 11^2$). The stopping times are then 634.01 for the algorithm of Theorem 1.1 and 606.04 for the one of Theorem 1.5.

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DATA AVAILABILITY STATEMENT

The research data associated with this article are included in the article.

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Appendix A. A useful estimate

Lemma A.1. *Let $p \in [1, \infty)$. For all $\varepsilon \in (0,1)$, there exists $C_p(\varepsilon) > 0$ such that*

$$\forall (a,b) \in (\mathbb{R}_+)^2, \quad (a+b)^p \leqslant (1+\varepsilon)^p a^p + C_p(\varepsilon) b^p. \tag{A.1}$$

Proof. For a or b equal to 0 or $p = 1$, (A.1) is obvious. Let us then choose $p \in (1, \infty)$ and remark that $C_p(\varepsilon)$ could be defined as

$$C_p(\varepsilon) = \sup_{a,b>0} \left\{ \frac{(a+b)^p - (1+\varepsilon)^p a^p}{b^p} \right\} = \sup_{\tau > 0} \left\{ (1+\tau)^p - (1+\varepsilon)^p \tau^p \right\}.$$

Since the function $\alpha_p : \tau \mapsto (1+\tau)^p - (1+\varepsilon)^p \tau^p$ defined and continuous on $\mathbb{R}_+$ goes to $-\infty$ as $\tau \to \infty$ and equals 1 as $\tau = 0$, $C_p(\varepsilon)$ is finite. $\square$