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ANALYSIS AND APPROXIMATION TO PARABOLIC OPTIMAL CONTROL PROBLEMS WITH MEASURE-VALUED CONTROLS IN TIME

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Abstract. In this paper, we investigate an optimal control problem governed by parabolic equations with measure-valued controls over time. We establish the well-posedness of the optimal control problem and derive the first-order optimality condition using Clarke's subgradients, revealing a sparsity structure in time for the optimal control. Consequently, these optimal control problems represent a generalization of impulse control for evolution equations. To discretize the optimal control problem, we employ the space-time finite element method. Here, the state equation is approximated using piecewise linear and continuous finite elements in space, alongside a Petrov–Galerkin method utilizing piecewise constant trial functions and piecewise linear and continuous test functions in time. The control variable is discretized using the variational discretization concept. For error estimation, we initially derive *a priori* error estimates and stabilities for the finite element discretizations of the state and adjoint equations. Subsequently, we establish weak-* convergence for the control under the norm $\mathcal{M}(\bar{I}_c; L^2(\omega))$, with a convergence order of $O(h^{\frac{1}{2}} + \tau^{\frac{1}{4}})$ for the state.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a convex polyhedron with boundary $\Gamma := \partial\Omega$, and $I := (0, T)$ with $T > 0$. In this paper we consider the following optimal control problem:

$$\min_{(u,q)\in X\times\mathcal{M}(\bar{I}_c;L^2(\omega))} J(u, q) = \frac{1}{2}\|u - u_d\|^2_{L^2(I;L^2(\Omega))} + \frac{\beta}{2}\|u(T) - u_T\|^2_{L^2(\Omega)} + \alpha\|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}, \tag{1.1}$$

where $\mathcal{M}(\bar{I}_c; L^2(\omega))$ is the control space of vector measures that will be defined in the subsequent section, $X := \{v \in L^2(I; L^2(\Omega)),\ v(T) \in L^2(\Omega)\}$ is the observation space, $u_d \in L^2(I; L^2(\Omega))$ and $u_T \in L^2(\Omega)$ are given observations or target states, $\alpha > 0$ is a regularization parameter, $\beta \geq 0$ is a weight parameter. The state u and the control $q \in \mathcal{M}(\bar{I}_c; L^2(\omega))$ in (1.1) are constrained by the following parabolic equation with initial data

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$u_0 \in L^2(\Omega)$ and source $f \in L^2(I; L^2(\Omega))$:

$$\begin{cases} \partial_t u - \Delta u = f + \chi_{I_c \times \omega} q & \text{in } \Omega \times I, \\ u = 0 & \text{on } \Gamma \times I, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\omega \subset \Omega$ and $I_c \subset\subset I$ (relatively compact) denote the spatial and temporal control domains, respectively. Here $\chi_{I_c \times \omega}$ is the characteristic function of $I_c \times \omega$ taking values 1 in $I_c \times \omega$ and 0 otherwise, which can be viewed as a zero extension operator.

In recent years, sparse controls of partial differential equations have garnered significant attention. Initially motivated by actuator placement, their application scope has since expanded to practical domains. Two main approaches have emerged to achieve sparsity: one involves L^1-norm regularization in the objective functional, while the other relies on measure-valued controls. The pioneering work in this direction is [1], which investigated $L^1(\Omega)$ control for linear elliptic equations. Subsequently, [2] addressed the spatio-temporally sparse optimal control problem of semilinear parabolic equations, introducing three different sparsity-promoting terms in the objective functional: $L^1(\Omega \times I)$, $L^2(I; L^1(\Omega))$, or $L^1(\Omega; L^2(I))$. For the latter case, [3] provided an error estimate for its fully discrete finite element approximation, with further refinement seen in [4]. Additionally, [5] explored directional sparse control for parabolic equations, where controls exhibit sparsity in space but not necessarily in time. Notably, the sparsity pattern remains constant over time.

For measure-valued control problems, [6] investigates elliptic equations with control space $\mathcal{M}(\Omega)$, while [7] addresses parabolic equations in $L^2(I; \mathcal{M}(\Omega))$, providing error estimates for finite element approximations. For parabolic control problems in space-time measure $\mathcal{M}(I \times \Omega)$, [8] is a relevant reference. Extending the directional sparsity concept [5] to measure spaces, [9] examines measure-valued directional sparsity for parabolic control problems with control space $\mathcal{M}(\Omega; L^2(I))$, deriving an *a priori* error estimate. Optimal control of the linear second-order wave equation with measure-valued controls in $\mathcal{M}(\Omega; L^2(I))$ is discussed in [10]. In [11], the authors explore measure-valued optimal control problems for 1D wave equations with control spaces of either measure-valued functions $L^2_{w^*}(I; \mathcal{M}(\Omega))$ or vector measures $\mathcal{M}(\Omega, L^2(I))$, deriving error estimates for the optimal state variable and the error measured in the cost functional. Additionally, [12] explores a variational discretization of a parabolic optimal control problem involving space-time measure controls, where the test functions in the weak form of the state equation must be continuous in both space and time. To achieve this, the authors propose a novel Petrov–Galerkin method using piecewise constant trial functions and piecewise linear continuous test functions, resulting in a conforming time discretization for the state equation. Consequently, this Petrov–Galerkin time discretization leads to a piecewise linear continuous time discretization for the adjoint state. This approach contrasts with the standard Petrov–Galerkin scheme for time discretization in parabolic optimal control problems, as discussed in [13], where either both the state and adjoint state are discontinuous in time, or the state is continuous while the adjoint state remains discontinuous. References [14–17] provide further insights into initial value identification of parabolic equations in measure spaces. For the theoretical and numerical analysis of optimal control problems with sparse structures we refer to [18–27], while [28–30] contain the related error estimates for parabolic optimal control problems.

For time-dependent systems, the control problem (1.1)-(1.2) posed in $\mathcal{M}(\bar{I}_c; L^2(\omega))$ yields controls with compact support in time. This characteristic allows for determining the optimal moments for control device actions, akin to a generalization of impulse control [31–41]. Recall that in impulse control problems, the control q in (1.1)-(1.2) is replaced by

$$q(x,t) = \sum_{i=1}^{m} q_i(x) \otimes \delta_{\tau_i}(t),$$

where δ_{τ_i} denotes the Dirac delta measure concentrated at $\tau_i \in I_c \subset\subset I$, and $q_i \otimes \delta_{\tau_i}$ are linear functionals on space $C(\bar{I}_c; L^2(\omega))$, with $q_i \in L^2(\omega), i = 1, 2, \cdots, m$. Here, the impulse strengths $q_i, i = 1, 2, \cdots, m$, are optimized at prescribed time nodes $\tau_i \in (0,T)$ [42]. However, in many cases, the interest lies in optimizing both the time nodes and the impulse strengths. This motivation leads to the formulation of the generalized impulse control problem as described by (1.1)–(1.2). We remark that impulse control belongs to a class of important control and has wide applications (see, for instance, [31, 39, 43]). In many cases impulse control is an interesting alternative to deal with systems that cannot be acted on by means of continuous control inputs.

The contributions of this article are threefold. Firstly, we investigate the well-posedness of both the state equation and the optimal control problem. Additionally, we derive the first-order optimality condition, revealing that the optimal control exhibits a sparsity structure independent of space. Secondly, for the state equation approximation, we utilize piecewise linear and continuous functions in space and a Petrov–Galerkin scheme from [44] in time. Specifically, we employ piecewise constant trial functions and piecewise linear and continuous test functions. We also employ a variational discretization concept for the control. Lastly, we provide an *a priori* error estimate for the finite element approximation of the control problem. Building upon the *a priori* error and stability estimates for finite element discretizations of the state and adjoint equations, we establish a convergence order of $O(h^{\frac{1}{2}} + \tau^{\frac{1}{4}})$ for the approximation of the state. Furthermore, we demonstrate weak-* convergence for the control under the $\mathcal{M}(\bar{I}_c; L^2(\omega))$ norm, with a convergence order of $O(h + \tau^{\frac{1}{2}})$ for the discrete cost functional.

The remainder of this paper is organized as follows. Section 2 presents some preliminary results, including the definition of very weak solutions to the state equation, as well as the global and local regularity and weak-* continuity of the state variable. In Section 3, we derive the first-order optimality system and investigate the sparse structure and regularity of the optimal control. Two discrete optimal control problems and their associated optimality systems are provided in Section 4. Section 5 primarily focuses on the error analysis for the optimal control problem.

2. Preliminaries

2.1. Notations for function spaces

Let $W^{k,p}(\Omega)$ ($k \in N_+ \cup \{0\}$, $1 \le p \le \infty$) be the usual Sobolev space defined in Ω with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$. Note that $W^{k,p}(\Omega) \cap W_0^{1,p}(\Omega)$ ($H^k(\Omega) \cap H_0^1(\Omega)$) is the closed subspace of $W^{k,p}(\Omega)$ ($H^k(\Omega)$) with null-traces on the boundary Γ. We abbreviate it by $H^k(\Omega) := W^{k,2}(\Omega)$ (resp. $H_0^k(\Omega) := W_0^{k,2}(\Omega)$) ($k \ge 1$) with norms $\|\cdot\|_{H^k(\Omega)}$, and $L^p(\Omega) := W^{0,p}(\Omega)$ that is the p-integrable function space in Ω with norms $\|\cdot\|_{L^p(\Omega)}$. Particularly, $L^2(D)$ ($D = \Omega, \omega$) is a Hilbert space with inner products $(\cdot,\cdot)$ and norms $\|\cdot\|$. Let $C(\bar{I}_c)$ be the Banach space consisting of continuous functions on $\bar{I}_c$ equipped with the supremum norm $\|\cdot\|_{C(\bar{I}_c)}$. Let $\mathcal{M}(\bar{I}_c)$ be the dual space of $C(\bar{I}_c)$ that is a Banach space under the norm

$$\|v\|_{\mathcal{M}(\bar{I}_c)} := \sup\left\{\int_{\bar{I}_c} w dv,\, \forall \omega \in C(\bar{I}_c)\,,\, \|w\|_{C(\bar{I}_c)} \le 1\right\} \quad \forall v \in \mathcal{M}(\bar{I}_c),$$

which can be identified with the space of regular Borel measures in I_c.

For a given positive measure $\mu \in \mathcal{M}(\bar{I}_b)$ the notation $L^p(I_b, \mu; L^2(\Omega))$ ($1 \le p$) denotes the set of all functions defined on a subset $I_b \subset I$ and valued in $L^2(\Omega)$, which is a Banach space endowed with the norm

$$\|v\|_{L^p(I_b,\mu;L^2(\Omega))} := \Big(\int_{I_b} \|v(t)\|^p \,\mathrm{d}\mu(t)\Big)^{\frac{1}{p}}$$

for $p < \infty$, and

$$\|v\|_{L^\infty(I_b,\mu;L^2(\Omega))} := \inf_{\substack{E \subseteq I_c \\ \mu(E)=0}} \sup_{t \in I_b \setminus E} \|v(t)\|_{L^2(\Omega)}$$

for $p = \infty$. If μ is the Lebesgue measure, we abbreviate $L^p(I_b, \mu; L^2(\Omega))$ as $L^p(I_b; L^2(\Omega))$.

The space $L^2(\omega;\mathcal{M}(\bar{I}_c))$, consisting of all weakly-* measurable functions $q:\omega\to\mathcal{M}(\bar{I}_c)$, is a Banach space endowed with the norm

$$\|q\|_{L^2(\omega;\mathcal{M}(\bar{I}_c))} := \Big(\int_\omega \|q(x)\|^2_{\mathcal{M}(\bar{I}_c)}\mathrm{d}x\Big)^{\frac{1}{2}} \quad \forall q\in L^2(\omega;\mathcal{M}(\bar{I}_c))$$

which can be identified with the dual of $L^2(\omega;C(\bar{I}_c))$, where $L^2(\omega;C(\bar{I}_c))$ denotes the Banach space consisting of all functions defined on ω and valued in $C(\bar{I}_c)$ with the norm

$$\|v\|_{L^2(\omega;C(\bar{I}_c))} := \Big(\int_\omega \|v(x)\|^2_{C(\bar{I}_c)}\mathrm{d}x\Big)^{\frac{1}{2}} \quad \forall v\in L^2(\omega;C(\bar{I}_c)).$$

For any given Banach space X, *e.g.*, $H^1_0(\Omega)$, $L^2(\Omega)$, *etc.*, the notation $C(\bar{I}_b;X)$ denotes the set of all continuous functions on $\bar{I}_b$ and valued in X, which is a Banach space under the supremum norm

$$\|v\|_{C(\bar{I}_b;X)} := \sup_{t\in\bar{I}_b}\|v(t)\|_X \quad \forall v\in C(\bar{I}_b;X).$$

Then, we define $\mathcal{M}(\bar{I}_c;L^2(\omega))$ as the space containing all countably additive measures with bounded total variations defined on the Borel sets $\mathcal{B}(\bar{I}_c)$ and valued in $L^2(\omega)$. For any $\mu\in\mathcal{M}(\bar{I}_c;L^2(\omega))$, the variation measure $|\mu|\in\mathcal{M}(\bar{I}_c)$ is defined as

$$|\mu|(B) := \sup\Big\{\sum_{n=1}^\infty \|\mu(B_n)\|_{L^2(\omega)} :\ \{B_n\}_{n=1}^\infty\subset\mathcal{B}(\bar{I}_c) \text{ is the disjoint partition of } B\Big\}$$

for any $B\in\mathcal{B}(\bar{I}_c)$, where $\mathcal{B}(\bar{I}_c)$ denotes the Borel set on $\bar{I}_c$. We denote by $|\mu|(\bar{I}_c)$ the total variation of μ. The space $\mathcal{M}(\bar{I}_c;L^2(\omega))$ endowed with the norm $\|\mu\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} = \||\mu|\|_{\mathcal{M}(\bar{I}_c)} = |\mu|(\bar{I}_c)$ is a Banach space (*cf.* [9] and [45], Chap. 12, Sect. 3), and that can be identified to the dual of $C(\bar{I}_c;L^2(\omega))$. In the following we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $\mathcal{M}(\bar{I}_c)$ and $C(\bar{I}_c)$, $\mathcal{M}(\bar{I}_c;L^2(\omega))$ and $C(\bar{I}_c;L^2(\omega))$, respectively.

For each $\mu\in\mathcal{M}(\bar{I}_c;L^2(\omega))$, the polar decomposition of μ consists of the variation measure $|\mu|\in\mathcal{M}(\bar{I}_c)$ and a space-time function $\mu'\in L^1(I_c,|\mu|;L^2(\omega))$, where the temporal support of μ' is included in the support of $|\mu|$ (*cf.* [9, 45]), such that

$$\mathrm{d}\mu = \mu' d|\mu|(t) \quad \text{and} \quad \langle\mu,w\rangle_{\bar{I}_c\times\omega} = \int_{I_c}(\mu'(t),w(t))d|\mu|(t)$$

for any $\mu\in\mathcal{M}(\bar{I}_c;L^2(\omega))$ and $w\in C(\bar{I}_c;L^2(\omega))$. Furthermore, similar to equation (2.3) in [9] we can show that $\mu'\in L^\infty(I_c,|\mu|;L^2(\omega))$ with $\|\mu'\|_{L^\infty(I_c,|\mu|;L^2(\omega))}\le 1$ and (*cf.* [9, 45])

$$\|\mu'(t)\|_{L^2(\omega)} = 1 \quad \text{for } |\mu| - \text{almost all } t\in\bar{I}_c. \tag{2.1}$$

Remark 2.1. Since $L^2(\omega;C(\bar{I}_c))\hookrightarrow C(\bar{I}_c;L^2(\omega))$ by Minkowski's inequality, we have $\mathcal{M}(\bar{I}_c;L^2(\omega))\hookrightarrow L^2(\omega;\mathcal{M}(\bar{I}_c))$. The difference is that the optimal control problem in $L^2(\omega;\mathcal{M}(\bar{I}_c))$ yields measure valued controls whose temporal supports are spatial dependent, while the ones for the former are spatial independent. We refer to [9] for a similar setting where parabolic optimal control problems in $\mathcal{M}(\Omega_c;L^2(I))$ were considered such that the spatial supports of measure valued controls are time independent, this is in contrast to the case in $L^2(I;\mathcal{M}(\bar{\Omega}_c))$ where the spatial supports of measure valued controls are time dependent. Based on the above embedding, we see that for each $\mu\in\mathcal{M}(\bar{I}_c;L^2(\omega))$, the representation $\mu(x)\in\mathcal{M}(\bar{I}_c)$ is well-defined for almost all

$x \in \omega$. Since $\mu' \in L^\infty(I_c, |\mu|; L^2(\omega))$, it thus also belongs to $L^2(I_c, |\mu|; L^2(\omega))$, or equivalently, $L^2(\omega; L^2(I_c, |\mu|))$. Therefore, $\mu'(x) \in L^2(I_c, |\mu|)$ for a.e. $x \in \omega$. We can now write (*cf.* [9], Eq. (2.5))

$$\mathrm{d}\mu(x) = \mu'(x) d|\mu| \quad \text{ a.e. } x \in \omega. \tag{2.2}$$

Lemma 2.2. *For given $g \in L^2(I; H^{-1}(\Omega))$ and $z_T \in L^2(\Omega)$, there exists a unique solution $z \in L^2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega))$ to the following problem*

$$\begin{cases} -\partial_t z - \Delta z = g & \text{in } \Omega \times (0, T), \\ z = 0 & \text{on } \Gamma \times (0, T), \\ z(T) = z_T & \text{in } \Omega. \end{cases} \tag{2.3}$$

Moreover, the following estimate holds

$$\|z\|_{C(\bar{I}; L^2(\Omega))} + \|\partial_t z\|_{L^2(I; H^{-1}(\Omega))} + \|z\|_{L^2(I; H_0^1(\Omega))} \leq C \left(\|g\|_{L^2(I; H^{-1}(\Omega))} + \|z_T\|_{L^2(\Omega)} \right). \tag{2.4}$$

If, in addition, $g \in L^2(I; L^2(\Omega))$ and *$z_T \in H_0^1(\Omega)$, then $z \in L^2(I; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I; L^2(\Omega)) \cap C(\bar{I}; H_0^1(\Omega))$ and there holds the stability estimate*

$$\|z\|_{C(\bar{I}; H_0^1(\Omega))} + \|\partial_t z\|_{L^2(I; L^2(\Omega))} + \|z\|_{L^2(I; H^2(\Omega))} \leq C \left(\|g\|_{L^2(I; L^2(\Omega))} + \|z_T\|_{H^1(\Omega)} \right). \tag{2.5}$$

The constants $C > 0$ is independent of g and z_T in the above two stability estimates.

Proof. The proof of the existence, uniqueness and stability estimate (2.4) of solutions can be found in [46], Theorem 5.1. The improved regularity and stability estimate are classical; see, *e.g.*, [47]. The regularity $z \in C(\bar{I}; H_0^1(\Omega))$ can be obtained from the fact that $L^2(I; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I; L^2(\Omega)) \hookrightarrow C(\bar{I}; H_0^1(\Omega))$. ☐

2.2. Well-posedness of the state equation

To begin with, we first investigate the well-posedness of the state equation (1.2). The very weak solution of equation (1.2) can be defined by transposition techniques (*cf.* [48]), which will be given in the following.

Definition 2.3. For any given $f \in L^2(I; L^2(\Omega))$, $q \in \mathcal{M}(\bar{I}_c; L^2(\omega))$ and $u_0 \in L^2(\Omega)$, a function $u \in L^2(I; L^2(\Omega))$ is called the very weak solution of equation (1.2) if it satisfies

$$(u, g)_{L^2(I; L^2(\Omega))} = \int_I (f, z_g)_{L^2(\Omega)} \mathrm{d}t + \langle q, z_g \rangle_{\bar{I}_c \times \omega} + (u_0, z_g(0))_{L^2(\Omega)} \quad \forall g \in L^2(I; L^2(\Omega)), \tag{2.6}$$

where $z_g \in C(\bar{I}_c; L^2(\omega))$ is the solution to equation (2.3) with the right-hand side g and $z_T = 0$, and $\langle \cdot, \cdot \rangle_{\bar{I}_c \times \omega}$ denotes the duality pairing between $\mathcal{M}(\bar{I}_c; L^2(\omega))$ and $C(\bar{I}_c; L^2(\omega))$.

Since $z_T = 0$ and Ω is convex, the solution of equation (2.3) satisfies $z \in L^2(I; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I; L^2(\Omega)) \hookrightarrow C(\bar{I}; H_0^1(\Omega))$ by Lemma 2.2. Therefore, the above definition is well-defined.

Theorem 2.4. *Assume that $f \in L^2(I; L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and $q \in \mathcal{M}(\bar{I}_c; L^2(\omega))$, then the parabolic equation (1.2) admits a unique very weak solution $u \in L^2(I; L^2(\Omega))$. Furthermore, $u \in L^2(I; H_0^1(\Omega)) \cap L^\infty(I; L^2(\Omega))$ and there holds the following estimate:*

$$\|u\|_{L^2(I; H_0^1(\Omega))} + \|u\|_{L^\infty(I; L^2(\Omega))} \leq C(\|f\|_{L^2(I; L^2(\Omega))} + \|q\|_{\mathcal{M}(\bar{I}_c; L^2(\omega))} + \|u_0\|_{L^2(\Omega)}), \tag{2.7}$$

where $C > 0$ is a constant independent of f, q and u_0.

In addition, assume that $I_c=(t_a,t_b)$ *with* $0<t_a<t_b<T$, *then there exist* t_c *and* t_d *satisfying* $t_b<t_c<t_d<T$, *i.e.,* $(t_d,T)\subseteq(t_c,T)\subseteq(t_b,T)$, *such that* $u|_{(t_d,T)}\in H^1((t_d,T);L^2(\Omega))\cap L^2((t_d,T);H^2(\Omega)\cap H_0^1(\Omega))\hookrightarrow C([t_d,T];H_0^1(\Omega))$ *and*

$$\begin{aligned}&\|u\|_{L^2((t_d,T);H^2(\Omega)\cap H_0^1(\Omega))}+\|u\|_{H^1((t_d,T);L^2(\Omega))}+\|u\|_{C([t_d,T];H_0^1(\Omega))}\\ \leq& C(\|f\|_{L^2((t_c,T);L^2(\Omega))}+\|u\|_{L^2((t_c,T);L^2(\Omega))})\\ \leq& C(\|f\|_{L^2(I;L^2(\Omega))}+\|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))}+\|u_0\|_{L^2(\Omega)}),\end{aligned}\tag{2.8}$$

where $C>0$ *is a constant independent of* f, q *and* u_0.

Proof. The proof of the existence of a unique very weak solution $u\in L^2(I;H_0^1(\Omega))\cap L^\infty(I;L^2(\Omega))$ can be found in, *e.g.*, [49], [42], Theorem 2.2, [50], Theorem 2.4 or [51], Theorem 2.4 for measure data in $\mathcal{M}([0,T])$. Here, we include a brief proof for completeness.

Since the state equation is linear, it suffices to consider the case either $u_0=0$, $f=0$ or $q=0$. If $q=0$, $u_0\in L^2(\Omega)$ and $f\in L^2(I;L^2(\Omega))$, it is obvious that problem (1.2) admits a unique weak solution $u\in L^2(I;H_0^1(\Omega))\cap L^\infty(I;L^2(\Omega))$ satisfying (*cf.* [52, 53])

$$\|u\|_{L^2(I;H_0^1(\Omega))}+\|u\|_{L^\infty(I;L^2(\Omega))}\leq C(\|f\|_{L^2(I;L^2(\Omega))}+\|u_0\|_{L^2(\Omega)}).$$

Now we consider the case $u_0=0$ and $f=0$. Let $\{q_n\}_n\subset C(\bar I_c\times\bar\omega)$ be the sequence converging weakly* to q in $\mathcal{M}(\bar I_c;L^2(\omega))$ and satisfy (we refer to, *e.g.* [54], for the existence of such sequence)

$$\|q_n\|_{L^1(I_c;L^2(\omega))}\leq\|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))}.$$

Let u_n be the solution of

$$\begin{cases}\partial_t u_n-\Delta u_n=\chi_{I_c\times\omega}q_n & \text{in }\ \Omega\times(0,T],\\ u_n=0 & \text{on }\partial\Omega\times(0,T],\\ u_n|_{t=0}=0 & \text{in }\Omega,\end{cases}\tag{2.9}$$

then one has $u_n\in L^2(I;H_0^1(\Omega))\cap H^1(I;H^{-1}(\Omega))$. Let z be the solution of problem (2.3) for given $g\in\mathcal{D}(I\times\Omega)$ and $z_T=0$, it follows from Lemma 2.2 that $z\in C(\bar I;H_0^1(\Omega))$. Then, using integration by parts we obtain

$$\begin{aligned}\int_I\int_\Omega gu_n\mathrm{d}x\mathrm{d}t&=\int_I\int_\Omega(-\partial_t z-\Delta z)u_n\mathrm{d}x\mathrm{d}t\\ &=\int_{I_c}(q_n,z)_{L^2(\omega)}\mathrm{d}t\\ &\leq\|q_n\|_{L^1(I_c;L^2(\omega))}\|z\|_{L^\infty(I_c;L^2(\omega))}\\ &\leq\|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))}\|z\|_{L^\infty(I_c;L^2(\omega))}.\end{aligned}\tag{2.10}$$

Combining the following standard estimates (*cf.* [42, 54]):

$$\|z\|_{L^\infty(I;L^2(\Omega))}\leq C\|g\|_{L^1(I;L^2(\Omega))},\quad \|z\|_{L^\infty(I;L^2(\Omega))}\leq C\|g\|_{L^2(I;H^{-1}(\Omega))},$$

we conclude that $\{u_n\}_n$ is bounded in the space $L^\infty(I;L^2(\Omega))$ by setting $g:=\psi_0\in\mathcal{D}(I\times\Omega)$ and using the density of $\mathcal{D}(I\times\Omega)$ in $L^1(I;L^2(\Omega))$, and also bounded in $L^2(I;H_0^1(\Omega))$ by setting $g:=\psi_0-\frac{\partial\psi_j}{\partial x_j}$, $\psi_j\in\mathcal{D}(I\times\Omega)$, $j=1,\ldots,d$ and using the density of $\mathcal{D}(I\times\Omega)$ in $L^2(I;H^{-1}(\Omega))$ (*cf.* [7, 8]), respectively. Thus, we can extract a subsequence, still denoted by $\{u_n\}_n$, such that $u_n \to u$ weakly in $L^2(I;H_0^1(\Omega))$ (using weak compactness of $L^2(I;H_0^1(\Omega))$) and $u_n \to u$ weakly* in $L^\infty(I;L^2(\Omega))$ (using weak* compactness of $L^\infty(I;L^2(\Omega))$).

For any $g \in L^2(I;L^2(\Omega))$, let $z_g \in H^1(I;L^2(\Omega)) \cap L^2(I;H^2(\Omega)\cap H_0^1(\Omega))$ be the solution of equation (2.3) with $z_T = 0$. Multiplying by z_g in both sides of equation (2.9) and integrating by parts give

$$(u_n, g)_{L^2(I;L^2(\Omega))} = \int_{I_c}\int_\omega z_g(x,t)\mathrm{d}x\mathrm{d}q_n(t),$$

which yields the identity (2.6) with $f = 0$, $u_0 = 0$ by passing to the limit in the above identity. Therefore, u is the very weak solution of equation (1.2). By the weak lower semicontinuity of $\|\cdot\|_{L^2(I;H_0^1(\Omega))}$ and the weak* lower semicontinuity of $\|\cdot\|_{L^\infty(I;L^2(\Omega))}$, we can obtain the estimate (2.7) of u.

Since $I_c = (t_a, t_b) \subseteq (0,T)$, there exist t_c and t_d satisfying $t_b < t_c < t_d < T$, such that $(t_d,T) \subseteq (t_c,T) \subseteq (t_b,T)$. Therefore, we consider a smooth cut-off function $\tilde\omega$ with the following properties:

$$\tilde\omega(t) \in [0,1] \quad \forall t \in [0,T]; \quad \tilde\omega(t) = 1 \quad \forall t \in (t_d,T); \quad \tilde\omega(t) = 0 \quad \forall t \in (0,t_c].$$

Let $\tilde u := \tilde\omega u$. Since $\bar I_c \cap \operatorname{supp}\tilde u = \emptyset$, $\tilde u$ satisfies the following equation:

$$\begin{cases} \partial_t \tilde u - \Delta \tilde u = F & \text{in } \Omega\times(t_c,T), \\ \tilde u = 0 & \text{on } \Gamma\times(t_c,T), \\ \tilde u(\hat t) = 0 & \text{in } \Omega, \end{cases} \tag{2.11}$$

where $F := \partial_t\tilde\omega u + \tilde\omega f$. Since $F \in L^2((t_c,T);L^2(\Omega))$, we can obtain $\tilde u \in L^2((t_c,T);H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1((t_c,T);L^2(\Omega))$ and there holds the following estimate:

$$\begin{aligned} \|\tilde u\|_{L^2((t_c,T);H^2(\Omega)\cap H_0^1(\Omega))} + \|\tilde u\|_{H^1((t_c,T);L^2(\Omega))} &\le C\|F\|_{L^2((t_c,T);L^2(\Omega))} \\ &\le C(\|f\|_{L^2((t_c,T);L^2(\Omega))} + \|u\|_{L^2((t_c,T);L^2(\Omega))}) \\ &\le C(\|f\|_{L^2(I;L^2(\Omega))} + \|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))} + \|u_0\|_{L^2(\Omega)}), \end{aligned}$$

where we have used the estimate (2.7). From the above inequality we obtain

$$\begin{aligned} \|u\|_{L^2((t_d,T);H^2(\Omega)\cap H_0^1(\Omega))} + \|u\|_{H^1((t_d,T);L^2(\Omega))} &= \|\tilde u\|_{L^2((t_d,T);H^2(\Omega)\cap H_0^1(\Omega))} + \|\tilde u\|_{H^1((t_d,T);L^2(\Omega))} \\ &\le \|\tilde u\|_{L^2((t_c,T);H^2(\Omega)\cap H_0^1(\Omega))} + \|\tilde u\|_{H^1((t_c,T);L^2(\Omega))} \\ &\le C(\|f\|_{L^2(I;L^2(\Omega))} + \|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))} + \|u_0\|_{L^2(\Omega)}). \end{aligned}$$

Therefore, we complete the proof of the estimate (2.8). □

With the help of Theorem 2.4, the identity (2.6) in Definition 2.3 is equivalent to the following one

$$(u,g)_* + (u(T),z_T)_{L^2(\Omega)} = \int_I (f,z)_{L^2(\Omega)}\mathrm{d}t + \langle q, z\rangle_{\bar I_c\times\omega} + (u_0, z(0))_{L^2(\Omega)} \quad \forall (g,z_T) \in S\times L^2(\Omega), \tag{2.12}$$

where $(u,g)_* := (u,g)_{L^2(I;L^2(\Omega))}$ for $S := L^2(I;L^2(\Omega))$ and $(u,g)_* := \langle g,u\rangle_{L^2(I;H^{-1}(\Omega)),L^2(I;H_0^1(\Omega))}$ defined by

$$\langle g,u\rangle_{L^2(I;H^{-1}(\Omega)),L^2(I;H_0^1(\Omega))} := \int_I \langle u, -\partial_t z\rangle_{H^1(\Omega),H^{-1}(\Omega)} + (\nabla u,\nabla z)_{L^2(\Omega),L^2(\Omega)}\mathrm{d}t \tag{2.13}$$

for $S := L^2(I;H^{-1}(\Omega))$, where $z \in C(\bar I_c;L^2(\omega))$ satisfies (2.3) with the right-hand side g and $z_T \in L^2(\Omega)$.

In fact, taking $\hat{z} = z - \tilde{z}$, where $\tilde{z}$ is the solution of equation (2.3) with $g = 0$ and initial data $\tilde{z}(T) = z(T) = z_T$, then $\hat{z}$ satisfies (2.3) with the right-hand side g and initial data $z_T = 0$. In other words, $\hat{z}$ can be chosen as a test function in Definition 2.3, *i.e*,

$$\begin{aligned}(u, g)_{L^2(I;L^2(\Omega))} &= \int_I (f, \hat{z})_{L^2(\Omega)} \mathrm{d}t + \langle q, \hat{z}\rangle_{\bar{I}_c\times\omega} + (u_0, \hat{z}(0))_{L^2(\Omega)} \\ &= \int_I (f, z)_{L^2(\Omega)} \mathrm{d}t + \langle q, z\rangle_{\bar{I}_c\times\omega} + (u_0, z(0))_{L^2(\Omega)} \\ &\quad - \left(\int_I (f, \tilde{z})_{L^2(\Omega)} \mathrm{d}t + \langle q, \tilde{z}\rangle_{\bar{I}_c\times\omega} + (u_0, \tilde{z}(0))_{L^2(\Omega)}\right) \\ &= \int_I (f, z)_{L^2(\Omega)} \mathrm{d}t + \langle q, z\rangle_{\bar{I}_c\times\omega} + (u_0, z(0))_{L^2(\Omega)} - \mathscr{L}(z_T),\end{aligned}$$

where $\mathscr{L}(z_T) := \int_I (f, \tilde{z})_{L^2(\Omega)} \mathrm{d}t + \langle q, \tilde{z}\rangle_{\bar{I}_c\times\omega} + (u_0, \tilde{z}(0))_{L^2(\Omega)}$. It is easy to check that $\mathscr{L}$ is a bounded linear functional of $z_T \in L^2(\Omega)$. Therefore, there exists a unique $\theta \in L^2(\Omega)$ such that $\mathscr{L}(z_T) = (\theta, z_T)_{L^2(\Omega)}$ by the Riesz representation theorem. Obviously, $\theta = u(T)$. Then, the identity (2.12) holds.

In order to show that the optimal control problem (1.1) has a unique solution, we have to provide a continuity property of the control-to-observation mapping under the weak-* topology.

Proposition 2.5. *Let $\{q_n\}_{n\in\mathbb{N}_+} \subset \mathcal{M}(\bar{I}_c; L^2(\omega))$ be a sequence of control variables such that $q_n \to q$ weakly* in $\mathcal{M}(\bar{I}_c; L^2(\omega))$. Assume that $u_n := u(q_n)$ and $u := u(q)$ are the corresponding solutions to the state equation (1.2) associated with q_n and q, respectively. Then we have*

$$u_n \to u \text{ in } L^2(I; L^2(\Omega)) \quad \text{and} \quad u_n(T) \to u(T) \text{ in } H^1(\Omega) \quad \text{for } n \to \infty.$$

Proof. The main idea of the proof is to apply the definition of very weak solutions to $u - u_n$. To this end, we note that $u - u_n$ satisfies the equation (1.2) with $f = 0$, $u_0 = 0$, and q replaced by $q - q_n$. Therefore, taking any $g \in L^2(I; L^2(\Omega))$ and $z_T = 0$, let $z \in C(\bar{I}; L^2(\Omega))$ be the solution of equation (2.3). Using (2.12) there holds

$$(g, u - u_n)_{L^2(I;L^2(\Omega))} = \langle q - q_n, z\rangle_{\bar{I}_c\times\omega} \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that $u_n \to u$ weakly in $L^2(I; L^2(\Omega))$.

Once again, we denote $z_n \in H^1(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega) \cap H_0^1(\Omega))$ as the solution of equation (2.3) with $g_n = u - u_n$ and $z_n(T) = 0$ for $n \in \mathbb{N}_+$. Since $u_n \to u$ weakly in $L^2(I; L^2(\Omega))$, we conclude that $z_n \to 0$ weakly in $H^1(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega) \cap H_0^1(\Omega))$ by using the continuity of the linear operator from $L^2(I; L^2(\Omega))$ to $H^1(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega) \cap H_0^1(\Omega))$ (*i.e.*, the solution operator of the backward parabolic equation). Using (2.12) there holds

$$\begin{aligned}\|u - u_n\|^2_{L^2(I;L^2(\Omega))} &= \langle q - q_n, z_n\rangle_{\bar{I}_c\times\omega} \\ &\le \|q - q_n\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} \|z_n\|_{C(\bar{I};L^2(\Omega))} \\ &\le C\|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} \|z_n\|_{C(\bar{I};L^2(\Omega))},\end{aligned}$$

where we have used the fact that $\|q_n\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}$ is bounded by $\|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}$. Since the embedding $H^1(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega) \cap H_0^1(\Omega)) \hookrightarrow C(\bar{I}; L^2(\Omega))$ is compact, we arrive at $\|z_n\|_{C(\bar{I};L^2(\Omega))} \to 0$ as $n \to \infty$. Therefore, we have obtained $u_n \to u$ in $L^2(I; L^2(\Omega))$.

Applying the estimate (2.8) in Theorem 2.4 to $(u - u_n)(T)$ yields the following estimate:

$$\|(u - u_n)(T)\|_{H^1(\Omega)} \le C\|u - u_n\|_{L^2((t_c,T);L^2(\Omega))} \le C\|u - u_n\|_{L^2(I;L^2(\Omega))},$$

which in turn implies $u_n(T) \to u(T)$ in $H_0^1(\Omega)$ by using the fact that $\|u_n - u\|_{L^2(I;L^2(\Omega))} \to 0$ for $n \to \infty$. This finishes the proof.

□

3. Optimal control problems

With the above preparations, we are in the position to study the existence and uniqueness of solutions to the optimal control problem (1.1)-(1.2), and derive the first order optimality system and regularity results of the solution.

3.1. Well-posedness of the optimal control problem

Recall that $X := L^2(I; L^2(\Omega)) \times L^2(\Omega)$ is the observation space, then we introduce the control-to-observation operator $S : \mathcal{M}(\bar{I}_c; L^2(\omega)) \to X$ as

$$Sq := (S_1 q, S_2 q),$$

where $S_1 q := u_q$ and $S_2 q := u_q(T)$, and u_q solves equation (1.2) with the control variable q on the right-hand side. Theorem 2.4 and Proposition 2.5 imply that the operator S is well-defined, affine linear and bounded, and weak continuous under the weak-* topology in $\mathcal{M}(\bar{I}_c; L^2(\omega))$. Since $I_c \subseteq I$, by the unique continuation of heat equations the operator S is injective. With the help of the control-to-observation operator S the reduced cost functional of (1.1) can be defined as

$$j(q) := J_1(q) + J_2(q) \quad \forall q \in \mathcal{M}(\bar{I}_c; L^2(\omega)), \tag{3.1}$$

where

$$J_1(q) := \frac{1}{2}\|S_1 q - u_d\|^2_{L^2(I;L^2(\Omega))} + \frac{\beta}{2}\|S_2 q - u_T\|^2_{L^2(\Omega)}, \qquad J_2(q) := \alpha\|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}.$$

$J_1(q)$ is a quadratic functional of tracking type, which is continuous under the weak-* topology in $\mathcal{M}(\bar{I}_c; L^2(\omega))$ and strictly convex by the weak-* continuity and injection of the operator S. On the other hand, $J_2(q)$ is weakly-* lower semicontinuous in space $\mathcal{M}(\bar{I}_c; L^2(\omega))$. Therefore, we conclude that the reduced functional j is also weakly-* lower semicontinuous and strictly convex. With this observation we can provide the following result.

Theorem 3.1. *The optimal control problem (1.1)-(1.2) admits a unique solution $(\bar{u}, \bar{q}) \in X \times \mathcal{M}(\bar{I}_c; L^2(\omega))$, where $\bar{q}$ is an optimal control that minimizes the reduced cost functional (3.1) and $\bar{u}$ is the optimal state that solves the state equation (1.2) associated with $\bar{q}$.*

Proof. According to Theorem 2.4, the objective functional j is well defined on $\mathcal{M}(\bar{I}_c; L^2(\omega))$. For the existence of solutions, we follow the standard arguments. Since $j \geq 0$ is bounded from below on $\mathcal{M}(\bar{I}_c; L^2(\omega))$, we can find a minimizing sequence $\{q_n\}$ with

$$\lim_{n\to\infty} j(q_n) = \inf_{q \in \mathcal{M}(\bar{I}_c;L^2(\omega))} j(q) = j^* \quad \text{and} \quad \|q_n\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} \leq \frac{1}{\alpha} j(q_n) \leq C.$$

On the other hand, the predual space $C(\bar{I}_c; L^2(\omega))$ is separable, then the bounded set in $\mathcal{M}(\bar{I}_c; L^2(\omega))$ is weakly-* compact by the Banach–Alaoglu theorem. Hence, we can extract a weakly-* convergent subsequence, still denoted by $\{q_n\}$, such that $q_n \overset{*}{\rightharpoonup} \bar{q}$ in $\mathcal{M}(\bar{I}_c; L^2(\omega))$ for some $\bar{q} \in \mathcal{M}(\bar{I}_c; L^2(\omega))$. Let u_n and $\bar{u}$ be the state corresponding to q_n and $\bar{q}$, respectively. Then $u_n \to \bar{u}$ in X since the operator S is weakly-* continuous. It is easy to check that $(\bar{u}, \bar{q})$ is an optimal pair. In fact, j is weakly-* lower semicontinuous, then

$$j(\bar{q}) \le \liminf_{n\to\infty} \ j(q_n) = j^*,$$

which means that $\bar{q}$ is optimal, *i.e.*, $(\bar{u}, \bar{q})$ is an optimal pair.

Furthermore, the control-to-observation mapping S is injective, thus the objective functional j is strictly convex. Therefore, the optimal pair $(\bar{u}, \bar{q})$ is unique. □

Remark 3.2. As pointed out in [9], if the state observation is of the form $\chi_{I_o\times\Omega_o}(u - u_d) \in L^2(I_o; L^2(\Omega_o))$ with $\mathrm{dist}(I_o, I_c) > 0$ and $\beta = 0$ where $I_o \subset I$ is an observation time window. The objective functional is no longer strictly convex since the control-to-observation operator is not injective, and thus the optimal pair of the optimization problem (1.1)-(1.2) is not unique. However, in the current paper we do not consider this case and focus only on the situation of $I_o = I$, *i.e.*, $I_c \subset I_o$.

3.2. First order optimality system

Below, we are in the position to derive the first order optimality condition.

Theorem 3.3. *A control $\bar{q} \in \mathcal{M}(\bar{I}_c; L^2(\omega))$ and an associated state $\bar{u} \in L^2(I; H_0^1(\Omega)) \cap L^\infty(I; L^2(\Omega))$ are an optimal pair of the optimal control problem (1.1)-(1.2), if and only if there exists an adjoint state $\bar{\varphi} \in L^2(I; H_0^1(\Omega)) \cap H^1(I; L^2(\Omega)) \hookrightarrow C(\bar{I}; L^2(\Omega))$ satisfying*

$$\begin{cases} -\partial_t\bar{\varphi} - \Delta\bar{\varphi} = \bar{u} - u_d & \text{in } \Omega\times(0,T),\\ \bar{\varphi} = 0 & \text{on } \Gamma\times(0,T),\\ \bar{\varphi}(T) = \beta(\bar{u}(T) - u_T) & \text{in } \Omega,\end{cases} \tag{3.2}$$

where $u_T \in L^2(\Omega)$, $u_d \in L^2(I; L^2(\Omega))$, such that the following subgradient condition holds:

$$0 \in \bar{\varphi}|_{\bar{I}_c\times\omega} + \alpha\partial\|\cdot\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}(\bar{q}) \quad \text{in } (\mathcal{M}(\bar{I}_c; L^2(\omega)))^* \tag{3.3}$$

i.e.,

$$-\langle p - \bar{q}, \bar{\varphi}\rangle_{\bar{I}_c\times\omega} + \alpha\|\bar{q}\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} \le \alpha\|p\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} \quad \forall p \in \mathcal{M}(\bar{I}_c; L^2(\omega)), \tag{3.4}$$

where $\partial\|\cdot\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}(\bar{q})$ denotes the set of subgradients of $\|\cdot\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}$ at $\bar{q}$, which is nonempty since $\|\cdot\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}$ is a convex functional on $\mathcal{M}(\bar{I}_c; L^2(\omega))$.

Furthermore, from the condition (3.4) we can easily conclude the following relation between the optimal control $\bar{q}$ and the adjoint state $\bar{\varphi}$:

$$\alpha\|\bar{q}\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} + \langle\bar{q}, \bar{\varphi}\rangle_{\bar{I}_c\times\omega} = 0, \tag{3.5}$$

$$\|\bar{\varphi}\|_{C(\bar{I}_c;L^2(\omega))} \begin{cases} = \alpha & \text{if } \bar{q} \ne 0,\\ \le \alpha & \text{if } \bar{q} = 0.\end{cases} \tag{3.6}$$

Proof. We split the reduced cost functional j into the sum of two parts in (3.1), where J_1 is differentiable and J_2 is subdifferentiable. We use $J_1'(\bar{q})$ and $\partial J_2(\bar{q})$ to denote the Fréchet derivative of J_1 at $\bar{q}$ and subgradients of J_2 at $\bar{q}$, respectively. By the calculus rules of subdifferentials for convex functions, there holds (see, *e.g.*, [55], Sect. 5.3):

$$j(\bar{q}) = \min_{p\in\mathcal{M}(\bar{I}_c;L^2(\omega))} j(p) \quad \text{if and only if} \quad 0 \in J_1'(\bar{q}) + \partial J_2(\bar{q}), \tag{3.7}$$

where for any $p \in \mathcal{M}(\bar{I}_c; L^2(\omega))$, $J_1'(\bar{q})$ and $\partial J_2(\bar{q})$ satisfy

$$\begin{aligned} &J_1'(\bar{q})(p-\bar{q}) =(\bar{u}-u_d, u_p-\bar{u})_{L^2(I;L^2(\Omega))} + \beta(\bar{u}(T)-u_T, u_p(T)-\bar{u}(T)),\\ &\forall \xi \in \partial J_2(\bar{q}), \quad \langle \xi, p-\bar{q}\rangle_{(\mathcal{M}(\bar{I}_c;L^2(\omega)))^*,\mathcal{M}(\bar{I}_c;L^2(\omega))} \le J_2(p) - J_2(\bar{q}), \end{aligned} \tag{3.8}$$

respectively, where $(\mathcal{M}(\bar{I}_c;L^2(\omega)))^*$ denotes the topological dual of $\mathcal{M}(\bar{I}_c;L^2(\omega))$ and $\langle\cdot,\cdot\rangle_{(\mathcal{M}(\bar{I}_c;L^2(\omega)))^*,\mathcal{M}(\bar{I}_c;L^2(\omega))}$ denotes the duality pairing between $(\mathcal{M}(\bar{I}_c; L^2(\omega)))^*$ and $\mathcal{M}(\bar{I}_c; L^2(\omega))$, u_p is the solution of problem (1.2) with q replaced by p. In order to give an explicit representation of $J_1'(\bar{q})(p-\bar{q})$ with respect to $p-\bar{q}$, let $\bar{\varphi}$ be the solution of equation (2.3) with $g = \bar{u}-u_d,\ z_T = \beta(\bar{u}(T)-u_T)$, and then apply the identity (2.12) to the difference $u_p - \bar{u}$ to deduce

$$(\bar{u}-u_d, u_p-\bar{u})_{L^2(I;L^2(\Omega))} + \beta(\bar{u}(T)-u_T, u_p(T)-\bar{u}(T)) = \langle p-\bar{q}, \bar{\varphi}\rangle_{\bar{I}_c\times\omega} \tag{3.9}$$

for any $p \in \mathcal{M}(\bar{I}_c; L^2(\omega))$. Furthermore, we obtain

$$J_1'(\bar{q})(p) = \langle p, \bar{\varphi}\rangle_{\bar{I}_c\times\omega} \quad \forall p \in \mathcal{M}(\bar{I}_c; L^2(\omega)),$$

which means that $J_1'(\bar{q}) = \bar{\varphi}|_{\bar{I}_c\times\omega}$. Therefore, combining with (3.7) we deduce the optimality condition (3.3) which claims that $-\bar{\varphi}|_{\bar{I}_c\times\omega} \in \alpha\partial\|\cdot\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}$, *i.e.*, (3.4).

Testing (3.4) with $p = 2\bar{q}$ and $p = 0$ we arrive at (3.5). Furthermore, it follows from setting $p = \bar{q} - r$ in (3.4) for arbitrary $r \in \mathcal{M}(\bar{I}_c; L^2(\omega))$ that

$$\langle r, \varphi\rangle_{\bar{I}_c\times\omega} \le J_2(\bar{q}-r) - J_2(\bar{q}) \le J_2(r) = \alpha\|r\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} \quad \forall r \in \mathcal{M}(\bar{I}_c; L^2(\omega)).$$

Hence, we obtain

$$\|\varphi\|_{C(\bar{I}_c;L^2(\omega))} = \sup_{\|r\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}\le 1} \langle r, \varphi\rangle_{\bar{I}_c\times\omega} \le \alpha, \tag{3.10}$$

this verifies (3.6) in view of (3.5). □

In the following we will derive the sparsity structure in time of $\bar{q}$.

Theorem 3.4. *Let $\bar{q}$ be the optimal control of the optimization problem (1.1)-(1.2) and $\bar{\varphi}$ be the optimal adjoint state defined by equation (3.2), then there holds*

$$\operatorname{supp}|\bar{q}| \subset \{t \in \bar{I}_c : \|\bar{\varphi}(t)\|_{L^2(\omega)} = \alpha\}, \tag{3.11}$$

$$\bar{q}'(t,x) = -\frac{1}{\alpha}\bar{\varphi}(t,x) \quad \text{in } L^1(I_c, |\bar{q}|; L^2(\omega)), \tag{3.12}$$

where $\mathrm{d}\bar{q} = \bar{q}'d|\bar{q}|$ denotes the polar decomposition of $\bar{q}$.

Proof. The idea of proof follows from [9], Theorem 2.12, see also [7], Theorem 3.3. Here, we sketch it for completeness. Applying the polar decomposition of q in (3.5) we have

$$\int_{\bar{I}_c} (\alpha + (\bar{q}'(t), \bar{\varphi})_{L^2(\omega)}) d|\bar{q}|(t) = 0. \tag{3.13}$$

On the other hand, it follows from (2.1) and (3.6) that

$$(\bar{q}'(t), \bar{\varphi})_{L^2(\omega)} \geq -\|\bar{q}'(t)\|_{L^2(\omega)}\|\bar{\varphi}\|_{L^2(\omega)} \geq -\alpha \quad |\bar{q}| - \text{a.e. } t \in \bar{I}_c, \tag{3.14}$$

i.e., the integrand in (3.13) is nonnegative. Thus, it must be zero $|\bar{q}|$- almost everywhere, that is,

$$-(\bar{q}'(t), \bar{\varphi})_{L^2(\omega)} = \alpha \quad \text{for } |\bar{q}| - \text{almost all } t \in \bar{I}_c. \tag{3.15}$$

Therefore, in view of (3.14) we have the identity:

$$(\bar{q}'(t), \bar{\varphi})_{L^2(\omega)} = -\|\bar{q}'(t)\|_{L^2(\omega)}\|\bar{\varphi}\|_{L^2(\omega)} = -\alpha, \quad \text{for } |\bar{q}| - \text{almost all } t \in \bar{I}_c, \tag{3.16}$$

which is equivalent to

$$\|\bar{\varphi}(t)\|_{L^2(\omega)} = \alpha \quad \text{and} \quad \bar{\varphi}(t,x) = -\alpha\bar{q}'(t,x) \tag{3.17}$$

for $|\bar{q}|$-almost all $t \in \bar{I}_c$ and a.e. $x \in \omega$. Thus, we finish the proof of (3.12).

In view of (2.1), we have

$$\|\bar{\varphi}(t)\|_{L^2(\omega)} = \alpha\|\bar{q}'(t)\|_{L^2(\omega)} = \alpha \quad \text{for } |\bar{q}| \text{ -almost all } t \in \bar{I}_c.$$

Namely,

$$|\bar{q}|(\bar{I}_c) = |\bar{q}|(\text{supp } |\bar{q}|) = |\bar{q}|(\{t \in \bar{I}_c : \|\bar{\varphi}(t)\|_{L^2(\omega)} = \alpha\}),$$

which means that supp $|\bar{q}| \subseteq \{t \in \bar{I}_c : \|\bar{\varphi}(t)\|_{L^2(\omega)} = \alpha\}$. This finishes the proof. □

In view of (3.11) in Theorem 3.4, we find that the optimal control $\bar{q} \in \mathcal{M}(\bar{I}_c; L^2(\omega))$ has sparsity pattern in time that is independent of the spatial domain. Moreover, if $\|\bar{\varphi}(t)\|_{L^2(\omega)} = \alpha$ holds for a finite set of time instances, namely, $\{t \in \bar{I}_c : \|\varphi(t)\|_{L^2(\omega)} = \alpha\} = \{\tau_i\}_{i=1}^N$, then $\bar{q}$ has the representation $\bar{q}(t,x) = \sum_{i=1}^N \chi_\omega \bar{q}_i(x)\delta_{\tau_i}$ such that $\bar{q}_i \in L^2(\omega)$ (*cf.* [9]). This is exactly the impulse control problem, studied extensively in the literature, *e.g.*, [33–37], and the references therein.

Proposition 3.5. *There exists $\alpha_0 > 0$ such that the optimal control $\bar{q} = 0$ when $\alpha > \alpha_0$.*

Proof. The main idea follows from [7], Corollary 3.5, see also [6], Proposition 2.2, and we sketch it here. Note that

$$\frac{1}{2}\|\bar{u} - u_d\|^2_{L^2(I;L^2(\Omega))} + \frac{\beta}{2}\|\bar{u}(T) - u_T\|^2_{L^2(\Omega)} \leq J(\bar{q}) \leq J(0),$$

where $J(0)$ is independent of α. Then we have obtained a uniform upper bound of $\|\bar{u} - u_d\|_{L^2(I;L^2(\Omega))}$ and $\|\bar{u}(T) - u_T\|_{L^2(\Omega)}$ with respect to α. For any $u_d \in L^2(I; L^2(\Omega))$, $u_T \in L^2(\Omega)$, let $\bar{\varphi} \in H^1(I; H^{-1}(\Omega)) \cap L^2(I; H_0^1(\Omega)) \hookrightarrow C(\bar{I}; L^2(\Omega))$ be the optimal adjoint state defined by equation (3.2) with the following estimate:

$$\|\bar{\varphi}(t)\|_{L^2(\Omega)} \leq C(\|\bar{u} - u_d\|_{L^2(I;L^2(\Omega))} + \beta\|\bar{u}(T) - u_T\|_{L^2(\Omega)}) \leq 2CJ(0),$$

where we have used Lemma 2.2. Setting $\alpha_0 = 2CJ(0)$, it follows from (3.11) in Theorem 3.4 that $\bar{q} = 0$ for all $\alpha > \alpha_0$. This finishes the proof. □

3.3. The regularity of solutions

In this subsection we prepare to state the regularity of solutions to the optimality system, which will be used in the finite element approximation to the optimal state and adjoint state.

Theorem 3.6. *For any u_d, $f \in L^2(I; L^2(\Omega))$, u_0, $u_T \in L^2(\Omega)$, let $(\bar{q}, \bar{u}, \bar{\varphi})$ be the optimal solution of the optimal control problem (1.1)-(1.2), where $\bar{q}, \bar{u}$ and $\bar{\varphi}$ are the optimal control, optimal state and adjoint state, respectively. Then there hold*

$$\begin{aligned}
&\bar{u} \in L^2(I; H_0^1(\Omega)) \cap L^\infty(I; L^2(\Omega)), \quad \bar{\varphi} \in H^1(I; H^{-1}(\Omega)) \cap L^2(I; H_0^1(\Omega)), \quad \bar{q} \in \mathcal{M}(\bar{I}_c; H^1(\omega)),\\
&\bar{u}|_{(t_d,T)} \in L^2((t_d,T); H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1((t_d,T); L^2(\Omega)) \hookrightarrow C([t_d,T]; H_0^1(\Omega)),\\
&\bar{\varphi} \in L^2(I_c; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I_c; L^2(\Omega)) \hookrightarrow C(\bar{I}_c; H_0^1(\Omega)),
\end{aligned}$$

where $(t_d,T) \cap I_c = \emptyset$, and there hold the following stability estimates

$$\begin{aligned}
&\|\bar{u}\|_{L^2(I;H_0^1(\Omega))} + \|\bar{u}\|_{L^\infty(I;L^2(\Omega))}\\
&+\|\bar{\varphi}\|_{H^1(I;H^{-1}(\Omega))} + \|\bar{\varphi}\|_{L^2(I;H_0^1(\Omega))} + \|\bar{\varphi}\|_{C(\bar{I};L^2(\Omega))} + \|\bar{q}\|^{\frac{1}{2}}_{\mathcal{M}(\bar{I}_c;H^1(\omega))}\\
&+\|\bar{u}\|_{H^1((t_d,T);L^2(\Omega))} + \|\bar{u}\|_{L^2((t_d,T);H^2(\Omega)\cap H_0^1(\Omega))} + \|\bar{u}\|_{C([t_d,T];H_0^1(\Omega))}\\
&\leq C\left(\|f\|_{L^2(I;L^2(\Omega))} + \|u_d\|_{L^2(I;L^2(\Omega))} + \|u_T\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)}\right).
\end{aligned}$$

Moreover, if $u_T \in H_0^1(\Omega)$, then the adjoint state has the following improved regularity:

$$\bar{\varphi} \in H^1(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega) \cap H_0^1(\Omega))$$

with the estimate

$$\begin{aligned}
&\|\bar{\varphi}\|_{H^1(I;L^2(\Omega))} + \|\bar{\varphi}\|_{L^2(I;H^2(\Omega)\cap H_0^1(\Omega))} + \|\bar{\varphi}\|_{C(\bar{I};H_0^1(\Omega))}\\
&\leq C\left(\|f\|_{L^2(I;L^2(\Omega))} + \|u_d\|_{L^2(I;L^2(\Omega))} + \|u_T\|_{H^1(\Omega)} + \|u_0\|_{L^2(\Omega)}\right).
\end{aligned}$$

Proof. To begin with, we first consider the regularity of $\bar{u}$, $\bar{\varphi}$ and derive the associated estimates. By using Theorem 2.4 and Lemma 2.2, $\bar{u}$ and $\bar{\varphi}$ have the above mentioned regularity, but the local regularity of $\bar{\varphi}$ and the associated stability estimates depend on $\|\bar{q}\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}$. Therefore, we first provide the estimate for $\|\bar{q}\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}$.

Let $\tilde{u}_0$ be the solution of equation (1.2) with $q = 0$. Using the identity (3.5) there holds

$$\begin{aligned}
\alpha\|\bar{q}\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} &\leq J(\bar{u}, \bar{q}) \leq J(\tilde{u}_0, 0)\\
&\leq C(\|u_d\|^2_{L^2(I;L^2(\Omega))} + \|f\|^2_{L^2(I;L^2(\Omega))} + \|u_0\|^2_{L^2(\Omega)} + \beta\|u_T\|^2_{L^2(\Omega)}).
\end{aligned}$$

Below, we show that $\bar{\varphi} \in H^1(I_c; L^2(\Omega)) \cap L^2(I_c; H^2(\Omega) \cap H_0^1(\Omega)) \hookrightarrow C(\bar{I}_c; H_0^1(\Omega))$ and $\bar{q} \in \mathcal{M}(\bar{I}_c; H^1(\omega))$. Since $I_c = (t_a, t_b) \subseteq I$ is relatively compact, there exist t_e, t_f satisfying $t_e < t_a < t_b < t_f$, such that $I_c \subseteq (t_e, t_f)$. Let $\tilde{\omega}$ be a smooth cut-off function satisfying

$$\tilde{\omega}(t) \in [0,1] \quad \forall t \in [0,T]; \quad \tilde{\omega}(t) = 1 \quad \forall t \in I_c; \quad \tilde{\omega}(t) = 0 \quad \forall t \in \bar{I}\backslash(t_e, t_f),$$

and $\tilde{\varphi} := \bar{\varphi}\tilde{\omega}$, then $\tilde{\varphi}$ satisfies the following equation:

$$\begin{cases} -\partial_t\tilde{\varphi} - \Delta\tilde{\varphi} = -\partial_t\tilde{\omega}\bar{\varphi} + \tilde{\omega}(\bar{u} - u_d) & \text{in } \Omega \times (t_e, t_f), \\ \tilde{\varphi} = 0 & \text{on } \Gamma \times (t_e, t_f), \\ \tilde{\varphi}(\tilde{t}_2) = 0 & \text{in } \Omega. \end{cases} \tag{3.18}$$

Therefore, by using Lemma 2.2 we obtain $\tilde{\varphi} \in H^1((t_e, t_f); L^2(\Omega)) \cap L^2((t_e, t_f); H^2(\Omega) \cap H_0^1(\Omega))$, which implies that $\bar{\varphi} \in H^1(I_c; L^2(\Omega)) \cap L^2(I_c; H^2(\Omega) \cap H_0^1(\Omega)) \hookrightarrow C(\bar{I}_c; H_0^1(\Omega))$ and the following estimate holds:

$$\begin{aligned} &\|\bar{\varphi}\|_{H^1(I_c;L^2(\Omega))} + \|\bar{\varphi}\|_{L^2(I_c;H^2(\Omega)\cap H_0^1(\Omega))} + \|\bar{\varphi}\|_{C(\bar{I}_c;H_0^1(\Omega))} \\ \le\ &\|\tilde{\varphi}\|_{H^1((t_e,t_f);L^2(\Omega))} + \|\tilde{\varphi}\|_{L^2((t_e,t_f);H^2(\Omega))} + \|\tilde{\varphi}\|_{C([t_e,t_f];H_0^1(\Omega))} \\ \le\ &C(\|\bar{\varphi}\|_{L^2(I;L^2(\Omega))} + \|u_d\|_{L^2(I;L^2(\Omega))} + \|\bar{u}\|_{L^2(I;L^2(\Omega))}) \\ \le\ &C(\|f\|_{L^2(I;L^2(\Omega))} + \|u_d\|_{L^2(I;L^2(\Omega))} + \|u_T\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)}). \end{aligned} \tag{3.19}$$

Finally, we prove that $\bar{q} \in \mathcal{M}(\bar{I}_c; H^1(\omega))$. In view of Theorem 3.4, there holds the relation $\mathrm{d}\bar{q} = \bar{q}' d|\bar{q}|$, where $\bar{q}'(t,x) = -\frac{1}{\alpha}\bar{\varphi}(t,x)$ with $\bar{\varphi} \in C(\bar{I}_c; H_0^1(\Omega))$. Therefore, for any $\psi \in C(\bar{I}_c; (H^1(\omega))^*)$, there holds

$$\begin{aligned} \langle \bar{q}, \psi\rangle_{\bar{I}_c\times\omega} = &\int_{I_c} \langle \psi, \mathrm{d}\bar{q}\rangle = -\frac{1}{\alpha}\int_{I_c} \langle \psi, \bar{\varphi}\rangle d|\bar{q}| \\ \le &\frac{1}{\alpha}\|\psi\|_{C(\bar{I}_c;(H^1(\omega)^*))}\|\bar{\varphi}\|_{C(\bar{I}_c;H^1(\omega))}\|\bar{q}\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}. \end{aligned}$$

Thus, $\|\bar{q}\|_{\mathcal{M}(\bar{I}_c;H^1(\omega))} \le C\frac{1}{\alpha}\|\bar{\varphi}\|_{C(\bar{I}_c;H^1(\Omega))}\|\bar{q}\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}$ (*cf.* [56]). This combining with the estimate (3.19) yields the conclusion. We thus finish the proof. □

4. Finite element approximations

In this section we consider the space-time finite element approximation for optimal control problems.

4.1. Notations for finite element methods

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of quasi-uniform and shape regular triangulations of Ω in the sense of Ciarlet [57], such that $\bar{\Omega} = \cup_{K\in\mathcal{T}_h}\bar{K}$, where h is the mesh parameter. Define the piecewise linear and continuous finite element space

$$V_h := \left\{v_h \in H_0^1(\Omega) :\ v_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\right\},$$

where $P_1(K)$ denotes the space of linear functions in K.

For simplicity, we assume that $\omega \subseteq \Omega$ is polygonal and the restriction of $\mathcal{T}_h$ on ω gives a partition of ω. Thus, we define $U_h := V_h|_\omega$ consisting of piecewise linear and continuous functions in ω.

Next, we divide $[0,T]$ into a family of subintervals $I_m := (t_{m-1}, t_m]$, $m = 1, 2, \cdots, M-1$, $I_M := (t_{M-1}, t_M)$ with step size $\tau_m = t_m - t_{m-1}$ such that $[0,T] = \{0\} \cup \bigcup_{i=1}^{M} I_m \cup \{T\}$, where $0 = t_0 < t_1 < \cdots < t_M = T$. We assume that there exist $1 < k_a < k_b < M$ such that $t_{k_a} = t_a$, $t_{k_b} = t_b$ and $\bar{I}_c = [t_a, t_b] = \{t_{k_a}\} \cup \bigcup_{m=k_a}^{k_b} I_m$, this can be achieved by setting t_a and t_b as time nodes of the time partition. The maximal time step is defined by $\tau := \max_{1\le m\le M} \tau_m$.

Now, we are ready to define two time semi-discrete finite element spaces consisting of either piecewise constant or piecewise linear and continuous Ansatz. Define (*cf.* [44])

$$
\begin{aligned}
P_\tau &:= \{v \in C(\bar{I}; H_0^1(\Omega)) : \ v|_{I_m} \in \mathcal{P}_1(I_m; H_0^1(\Omega)), \ m = 1, 2, \ldots, M\},\\
Y_\tau &:= \{v \in L^2(I; H_0^1(\Omega)) : \ v|_{I_m} \in \mathcal{P}_0(I_m; H_0^1(\Omega)), \ m = 1, 2, \ldots, M, \ v(T) \in H_0^1(\Omega)\},
\end{aligned}
$$

where $\mathcal{P}_i(I_m; H_0^1(\Omega))$ $(i = 0, 1)$ denotes the set of polynomial functions of degree at most i on time interval I_m and valued in $H_0^1(\Omega)$, and let

$$P_\tau^0 := \{v_\tau \in P_\tau : \ v_\tau(T) = 0\}.$$

The notation $\sigma = (\tau, h)$ denotes the vector of two discretization parameters τ and h. In order to introduce the Petrov–Galerkin scheme for parabolic equations, we also need to define the following two time-space finite element spaces:

$$
\begin{aligned}
P_\sigma &:= \{v \in P_\tau : v|_{I_m} \in \mathcal{P}_1(I_m; V_h), \ m = 1, 2, \ldots, M\},\\
Y_\sigma &:= \{v \in Y_\tau : \ v|_{I_m} \in \mathcal{P}_0(I_m; V_h), \ m = 1, 2, \ldots, M, \ v(T) \in V_h\},
\end{aligned}
$$

where the definition of $\mathcal{P}_i(I_m; V_h)$ $(i = 0, 1)$ is similar to $\mathcal{P}_i(I_m; H_0^1(\Omega))$ $(i = 0, 1)$. We set

$$P_\sigma^0 := \{v_\sigma \in P_\sigma : \ v_\sigma(T) = 0\}.$$

It is clear that $P_\sigma \subset L^2(I; H_0^1(\Omega)) \cap H^1(I; L^2(\Omega))$ and $Y_\sigma \subseteq L^2(I; H_0^1(\Omega))$.

Let $\{x_j\}_{j=1}^{N_h}$ be the interior nodes of the mesh $\mathcal{T}_h$, and $\{\psi_{x_j}\}_{j=1}^{N_h}$, $\{e_{t_m}\}_{m=0}^{M}$ be the nodal basis functions of the piecewise linear and continuous finite element spaces approximating to $H^1(\Omega)$ and $H^1(I)$, respectively (*cf.* [57]). Obviously, P_σ and Y_σ can be rewritten as

$$
\begin{aligned}
P_\sigma &= \operatorname{span}\{\psi_{x_j} \otimes e_{t_m} : 1 \le j \le N_h, \ 0 \le m \le M\},\\
Y_\sigma &= \operatorname{span}\{\psi_{x_j} \otimes \chi_{I_m}, \ \psi_{x_j} \otimes \chi_{\{M\}} : 1 \le j \le N_h, \ 1 \le m \le M\},
\end{aligned}
$$

where χ_{I_m} denotes the characteristic function on $I_m, \ m = 1, 2, \cdots, m$ and $\chi_{\{M\}}$ denotes the characteristic function at M. We define the indices

$$\mathcal{I}_\sigma = \Big\{(j, m) : (x_j, t_m) \in \bar{\omega} \times \bar{I}_c, 1 \le j \le N_h, \ 0 \le m \le M\Big\}, \ \mathcal{I}_\tau = \big\{m : t_m \in \bar{I}_c, 0 \le m \le M\big\},$$

and the discrete spaces

$$
\begin{aligned}
U_\tau &= \operatorname{span}\{\delta_{t_m} : m \in \mathcal{I}_\tau\} \subset \mathcal{M}(\bar{I}_c), \quad V_\tau = \operatorname{span}\{e_{t_m}|_{\bar{I}_c} : m \in \mathcal{I}_\tau\} \subset C(\bar{I}_c),\\
U_\sigma &= \operatorname{span}\{\psi_{x_j}|_{\bar{\omega}} \otimes \delta_{t_m} : (j, m) \in \mathcal{I}_\sigma\}, \quad V_\sigma = \operatorname{span}\{\psi_{x_j} \otimes e_{t_m}|_{\bar{\omega} \times \bar{I}_c} : (j, m) \in \mathcal{I}_\sigma\}.
\end{aligned}
$$

For $q_\tau \in U_\tau$ and $v_\tau \in V_\tau$ we identify them with $\vec{q}_\tau = (q_1, \ldots, q_{M_c})$ and $\vec{v}_\tau = (v_1, \ldots, v_{M_c})$ where M_c is the cardinality of $\mathcal{I}_\tau$. The linear functions in V_τ attain their maximum and minimum at the nodes. Therefore, for $v_\tau \in V_\tau$ we define

$$\|v_\tau\|_{L^\infty(I_c)} = \max_{1 \le m \le M_c} |v_m|.$$

Similarly, we have for $q_\tau \in U_\tau$ that

$$\|q_\tau\|_{\mathcal{M}(\bar{I}_c)} = \sup_{v\in C(\bar{I}_c),\|v\|_{L^\infty(I_c)}\le 1} \sum_{m=1}^{M_c} q_m \langle \delta_{t_m}, v\rangle_{\bar{I}_c} = \sum_{m=1}^{M_c} |q_m|.$$

Given two functions $u \in L^2(I; H_0^1(\Omega))$ with $u(T) \in L^2(\Omega)$, $v \in L^2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega))$, we define a bilinear form $A(u, v) \to \mathbb{R}$ as follows:

$$A(u,v) := -\langle u, \partial_t v\rangle_{L^2(I;H_0^1,H^{-1})} + \int_I (\nabla u(t), \nabla v(t))\mathrm{d}t + (u(T), v(T)),$$

where $\langle\cdot,\cdot\rangle_{L^2(I;H_0^1,H^{-1})}$ denotes the duality pairing between $L^2(I; H_0^1(\Omega))$ and $L^2(I; H^{-1}(\Omega))$.

If, in addition, $u_\tau \in Y_\tau$ and $v_\tau \in P_\tau$, applying integration by parts to the bilinear form $A(u_\tau, v_\tau)$ we can obtain the following dual representation:

$$A(u_\tau, v_\tau) = \sum_{m=1}^{M} ([u_\tau]_m, v_\tau(t_m))_{L^2(\Omega)} + (u_{\tau,0}^+, v_\tau(0))_{L^2(\Omega)} + \int_I (\nabla u_\tau(t), \nabla v_\tau(t))\mathrm{d}t, \tag{4.1}$$

where

$$\begin{aligned}
&u_{\tau,m+1} = u_{\tau,m}^+ := \lim_{\epsilon\to 0^+} u_\tau(t_m+\epsilon), \quad m = 0,1,2,\cdots, M-1,\\
&u_{\tau,m} = u_{\tau,m}^- := \lim_{\epsilon\to 0^+} u_\tau(t_m-\epsilon), \quad m = 1,2,\cdots, M,\\
&[u]_m := u_{\tau,m}^+ - u_{\tau,m}^-,\ m = 1,2,\cdots, M-1,\ [u_\tau]_M = u_\tau(T) - u_{\tau,M}^-,
\end{aligned}$$

and $u_{\tau,m} := u_\tau|_{I_m}$, $m = 1,2,\cdots, M$.

In view of the identity (2.12), the very weak solution to the state equation (1.2) reads: Find $u \in L^2(I; H_0^1(\Omega))$ such that

$$A(u,v) = \int_I (f,v)\mathrm{d}t + \langle q, v\rangle_{\bar{I}_c\times\omega} + (u_0, v(x,0)) \quad \forall v \in L^2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega)).$$

Similarly, the weak formulation of the backward equation (2.3) can be rewritten as: Find $z \in L^2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega))$ such that

$$A(v,z) = \int_I (g,v)\mathrm{d}t + (v(T), z_T) \quad \forall v \in L^2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega)). \tag{4.2}$$

4.2. Interpolation and projection operators

In the following we introduce some interpolation operators defined in $\bar{I}_c$ and give their properties whose proofs are very similar to [6], Theorem 3.1, see also [8], Proposition 4.1.

Lemma 4.1. *Let the linear operators Λ_τ and Π_τ be defined as follows:*

$$\Lambda_\tau : \mathcal{M}(\bar{I}_c) \to U_\tau \subset \mathcal{M}(\bar{I}_c), \quad \Lambda_\tau q := \sum_{m\in\mathcal{I}_\tau} \delta_{t_m} \int_{\bar{I}_c} e_{t_m}\mathrm{d}q,$$

$$\Pi_\tau : C(\bar{I}_c) \to V_\tau \subset C(\bar{I}_c), \quad \Pi_\tau v := \sum_{m\in\mathcal{I}_\tau} v(t_m) e_{t_m}.$$

Then for every $q \in \mathcal{M}(\bar{I}_c)$, $v \in C(\bar{I}_c)$ and $v_\tau \in V_\tau$, there hold

$$\langle q, \Pi_\tau v\rangle_{\bar{I}_c} = \langle \Lambda_\tau q, v\rangle_{\bar{I}_c}, \tag{4.3}$$

$$\|\Lambda_\tau q\|_{\mathcal{M}(\bar{I}_c)} \le \|q\|_{\mathcal{M}(\bar{I}_c)}, \tag{4.4}$$

$$\Lambda_\tau q \xrightarrow{*} q \text{ in } \mathcal{M}(\bar{I}_c) \quad \textit{and } \|\Lambda_\tau q\|_{\mathcal{M}(\bar{I}_c)} \xrightarrow{\tau\to 0^+} \|q\|_{\mathcal{M}(\bar{I}_c)}, \tag{4.5}$$

$$\|q-\Lambda_\tau q\|_{(H^1(I_c))'} \le C\tau^{\frac{1}{2}}\|q\|_{\mathcal{M}(\bar{I}_c)}, \quad \|q-\Lambda_\tau q\|_{(W^{1,\infty}(I_c))'} \le C\tau\|q\|_{\mathcal{M}(\bar{I}_c)}. \tag{4.6}$$

Proof. The proofs of (4.3)–(4.5) are trivial, thus we only provide the proof for (4.6). For an arbitrary $v \in H^1(I_c)$, using (4.3) and the standard Lagrange interpolation error estimate we have for $s > 1$ that

$$\langle q-\Lambda_\tau q, v\rangle_{\bar{I}_c} = \langle q, v-\Pi_\tau v\rangle_{\bar{I}_c} \le \|q\|_{\mathcal{M}(\bar{I}_c)}\|v-\Pi_\tau v\|_{C(\bar{I}_c)} \le C\tau^{1-\frac{1}{s}}\|q\|_{\mathcal{M}(\bar{I}_c)}\|v\|_{W^{1,s}(I_c)}.$$

From the duality we can obtain two desired results by setting $s = 2$ and $s = \infty$. □

Let $\pi_h : L^2(\omega) \to V_h|_\omega$ be the usual L^2-projection defined by

$$(v-\pi_h v, \varphi_h)_{L^2(\omega)} = 0 \quad \forall \varphi_h \in V_h|_\omega,$$

then there holds

$$\|v-\pi_h v\|_{L^2(\omega)} + h\|v-\pi_h v\|_{H^1(\omega)} \le Ch^m\|v\|_{H^m(\omega)} \quad \forall v \in H^m(\omega),\ m = 1, 2,$$

where $C > 0$ is a constant independent of h and v. Note that the application of the operator π_h to time-dependent arguments has to be understood pointwisely in time. Below, we will extend π_h to a negative exponent Sobolev space that includes $L^2(\omega)$.

Let $V := H^1(\omega)$ (resp. $H_0^1(\Omega)$), $H := L^2(\omega)$ (resp. $L^2(\Omega)$), then the inclusion $V \subseteq H$ is dense and continuous. Note that $V \hookrightarrow H = H^* \hookrightarrow V^*$ is a Gelfand triple, where $H \hookrightarrow V^*$ is given by $y \in H \to (y, \cdot)_{L^2(\Omega)} \in H^* \subseteq V^*$. Therefore, we extend in the following definition the usual projection π_h from H to V^*.

Definition 4.2. Define the action of the L^2-projection π_h on V^* as

$$\pi_h : V^* \to V_h|_\omega \quad v \to \pi_h v,$$

where $\pi_h v \in V_h|_\omega$ satisfies

$$(\pi_h v, \varphi_h)_H = \langle v, \varphi_h\rangle_{V^*,V} \quad \forall \varphi_h \in V_h|_\omega \subset V. \tag{4.7}$$

Furthermore, we also define the following two interpolation operators:

$$\Lambda_\sigma : \mathcal{M}(\bar{I}_c; H) \to U_\sigma \subset \mathcal{M}(\bar{I}_c; H), \quad \Lambda_\sigma q := \pi_h(\Lambda_\tau q) = \sum_{(j,m)\in\mathcal{I}_\sigma} q_{j,m}\delta_{t_m} \otimes \psi_{x_j}|_{\bar{\omega}},$$

$$\Pi_\sigma : C(\bar{I}_c; V^*) \to V_\sigma \subset C(\bar{I}_c; V^*), \quad \Pi_\sigma v := \pi_h(\Pi_\tau v) = \sum_{(j,m)\in\mathcal{I}_\sigma} v_{j,m} e_{t_m} \otimes \psi_{x_j}|_{\bar{I}_c\times\bar{\omega}},$$

where $q_{j,m} := \pi_h(\int_{\bar{I}_c} e_{t_m}\mathrm{d}q)(x_j)$ and $v_{j,m} := \pi_h(v(t_m))(x_j)$.

It is easy to check that there exists a unique $\pi_h v \in V_h|_\omega$ such that the identity (4.7) holds for any $v \in V^*$. Moreover, $\pi_h|_H : H \to V_h|_\omega$ is consistent with the usual L^2-projection and π_h is stable, *i.e.*,

$$\|\pi_h v\|_{V^*} \le \|v\|_{V^*}, \quad \|\pi_h v\|_H \le \|v\|_H$$

for $v \in V^*$ and $v \in H$, respectively. The definition of above two interpolation operators Λ_σ and Π_σ are very similar to [7], Theorem 4.2, see also [8], Proposition 4.2.

Lemma 4.3. *For every $q_\sigma \in U_\sigma$ and $v_\sigma \in V_\sigma$ one have*

$$\Lambda_\sigma q_\sigma = q_\sigma \quad \textit{and} \quad \Pi_\sigma v_\sigma = v_\sigma. \tag{4.8}$$

Moreover, there hold

$$\langle q, \Pi_\sigma v\rangle_{\bar I_c\times\omega} = \langle \Lambda_\sigma q, v\rangle_{\bar I_c\times\omega} \quad \forall\,(q,v) \in \mathcal{M}(\bar I_c; S)\times C(\bar I_c; S^*), \tag{4.9}$$

$$\|\Lambda_\sigma q\|_{\mathcal{M}(\bar I_c;S)} \le \|q\|_{\mathcal{M}(\bar I_c;S)}, \quad \|\Pi_\sigma v\|_{C(\bar I_c;S^*)} \le \|v\|_{C(\bar I_c;S^*)}, \tag{4.10}$$

$$\Lambda_\sigma q \stackrel{*}{\rightharpoonup} q \in \mathcal{M}(\bar I_c; H), \qquad \|\Lambda_\sigma q\|_{\mathcal{M}(\bar I_c;H)} \stackrel{|\sigma|\to 0}{\longrightarrow} \|q\|_{\mathcal{M}(\bar I_c;H)}, \tag{4.11}$$

where $S = H$ or V.

Proof. A simple calculation gives

$$\begin{aligned}
\langle q, \Pi_\sigma v\rangle_{\bar I_c\times\omega} &= \langle q, \pi_h(\Pi_\tau v)\rangle_{\bar I_c\times\omega} = \Big\langle q, \pi_h\big(\sum_m v(t_m)e_{t_m}\big)\Big\rangle_{\bar I_c\times\omega}\\
&= \Big\langle \sum_m \pi_h(v(t_m)), \int_{I_m} e_{t_m}\,dq\Big\rangle_{S^*,S}\\
&= \Big\langle \sum_m v(t_m), \pi_h(\int_{I_m} e_{t_m}\,dq)\Big\rangle_{S^*,S}\\
&= \Big\langle v, \sum_m \pi_h(\int_{I_m} e_{t_m}\,dq)\otimes\delta_{t_m}\Big\rangle\\
&= \langle \Lambda_\sigma q, v\rangle_{\bar I_c\times\omega}
\end{aligned} \tag{4.12}$$

for any $(q, v) \in \mathcal{M}(\bar I_c; S)\times C(\bar I_c; S^*)$. Moreover, by using (4.8) and (4.9) we have $\langle q, v_\sigma\rangle_{\bar I_c\times\omega} = \langle q, \Pi_\sigma v_\sigma\rangle_{\bar I_c\times\omega} = \langle \Lambda_\sigma q, v_\sigma\rangle_{\bar I_c\times\omega}$.

For the second inequality in (4.10), we use the stability of the projection π_h to conclude

$$\begin{aligned}
\|\Pi_\sigma v\|_{C(\bar I_c;S^*)} &= \sup_{t\in\bar I_c}\|\pi_h(\Pi_\tau v)(t)\|_{S^*} \le \max_{1\le m\le M_c}\|\pi_h v(t_m)\|_{S^*}\\
&\le \max_{1\le m\le M_c}\|v(t_m)\|_{S^*} \le \|v\|_{C(\bar I_c;S^*)}.
\end{aligned}$$

Next, we prove the first inequality in (4.10)

$$\begin{aligned}
\|\Lambda_\sigma q\|_{\mathcal{M}(\bar I_c;S)} &= \sup_{\|v\|_{C(\bar I_c;S^*)}\le 1}\langle \Lambda_\sigma q, v\rangle_{\bar I_c\times\omega} = \sup_{\|v\|_{C(\bar I_c;S^*)}\le 1}\langle q, \Pi_\sigma v\rangle_{\bar I_c\times\omega}\\
&\le \sup_{\|v\|_{C(\bar I_c;S^*)}\le 1}\|q\|_{\mathcal{M}(\bar I_c;S)}\|\Pi_\sigma v\|_{C(\bar I_c;S^*)}\\
&\le \|q\|_{\mathcal{M}(\bar I_c;S)},
\end{aligned}$$

where we have used the second inequality in (4.10). This proves the first inequality in (4.10), while (4.11) is obvious. This finishes the proof. □

Next, we will introduce the following interpolation and projection operators in time (*cf.* [44]). Define the L^2-projection $\mathcal{P}_{Y_\tau} : L^2(I; H_0^1(\Omega)) \to Y_\tau$ such that $\mathcal{P}_{Y_\tau} z \in Y_\tau$ satisfies

$$\mathcal{P}_{Y_\tau} z|_{I_m} := \frac{1}{\tau_m} \int_{I_m} z \mathrm{d}t \quad \forall z \in L^2(I; H_0^1(\Omega)),\ m = 1, \cdots, M,\ \text{and } \mathcal{P}_{Y_\tau} v(T) := 0.$$

In addition, we also need the following interpolation operators. Define the Lagrange interpolation operator $\Pi_{P_\tau} : C(\bar{I}; H_0^1(\Omega)) \to P_\tau$ and the piecewise constant in time interpolation operator $\Pi_{Y_\tau} : C(\bar{I}; H_0^1(\Omega)) \to Y_\tau$ such that $\Pi_{P_\tau} v \in P_\tau$, $\Pi_{Y_\tau} v \in Y_\tau$ satisfy

$$\Pi_{P_\tau} v := \sum_{m=0}^{M} v(t_m) e_{t_m}, \quad \Pi_{Y_\tau} v := \sum_{m=1}^{M} v(t_m) \chi_{I_m}, \quad \Pi_{Y_\tau} v(T) := 0,$$

where $\{e_{t_m}\}$ and $\{\chi_{I_m}\}$ are the families of node basis functions of $\mathcal{P}_1$ and $\mathcal{P}_0$ elements on the time interval I, respectively.

Lemma 4.4. *For arbitrary $v \in L^2(I; H_0^1(\Omega)) \cap H^1(I; L^2(\Omega))$, there hold the following standard interpolation error estimates (cf. [13, 44]):*

$$\|v - \mathcal{P}_{Y_\tau} v\|_{L^2(I;L^2(\Omega))} \le C\tau \|\partial_t v\|_{L^2(I;L^2(\Omega))}, \tag{4.13}$$

$$\|v - \Pi_{P_\tau} v\|_{L^2(I;L^2(\Omega))} \le C\tau \|\partial_t v\|_{L^2(I;L^2(\Omega))}, \tag{4.14}$$

$$\|v - \Pi_{P_\tau} v\|_{L^\infty(I;L^2(\Omega))} \le C\tau^{\frac{1}{2}} \|\partial_t v\|_{L^2(I;L^2(\Omega))}. \tag{4.15}$$

We also have the following half an order interpolation error estimates for any $v \in H^1(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega))$ (cf. [9], Lem. 3.13)

$$\|v - \Pi_{Y_\tau} v\|_{L^2(I;H_0^1(\Omega))} \le C\tau^{\frac{1}{2}} (\|\partial_t v\|_{L^2(I;L^2(\Omega))} + \|\Delta v\|_{L^2(I;L^2(\Omega))}), \tag{4.16}$$

$$\|v - \Pi_{P_\tau} v\|_{L^2(I;H_0^1(\Omega))} \le C\tau^{\frac{1}{2}} (\|\partial_t v\|_{L^2(I;L^2(\Omega))} + \|\Delta v\|_{L^2(I;L^2(\Omega))}), \tag{4.17}$$

where $C > 0$ is a constant independent of τ and v.

4.3. Discretization of the optimal control problem

With the above preparation the discrete optimal control problem reads:

$$\min_{\substack{q \in \mathcal{M}(\bar{I}_c; L^2(\omega)) \\ u_\sigma \in Y_\sigma}} J_\sigma(u_\sigma, q) := \frac{1}{2} \|u_\sigma - u_d\|^2_{L^2(I;L^2(\Omega))} + \frac{\beta}{2} \|u_\sigma(T) - u_T\|^2_{L^2(\Omega)} + \alpha \|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}, \tag{4.18}$$

where $u_\sigma \in Y_\sigma$ is the discrete state variable satisfying the following discrete state equation:

$$A(u_\sigma, v_\sigma) = \int_I (f, v_\sigma) \mathrm{d}t + \langle q, v_\sigma \rangle_{\bar{I}_c \times \omega} + (u_0, v_\sigma(0)) \quad \forall v_\sigma \in P_\sigma. \tag{4.19}$$

Note that in the above discrete optimal control problem (4.18) the control variable is not explicitly discretized, but in the following we will see that the discretization of the adjoint state indeed automatically yields the discretization of the control variable, which is the so-called variational discretization for optimal control problems proposed in [58].

Similar to Subsection 3.1, we can easily check that the discrete optimal control problem (4.18) exists at least one optimal pair. However, the optimal control is not unique in general, since the discrete control-to-state operator is not injective. In fact, we have $\langle \Lambda_\sigma q, v_\sigma\rangle_{\bar I_c\times\omega} = \langle q, v_\sigma\rangle_{\bar I_c\times\omega}$ for any $v_\sigma\in P_\sigma$, which implies that the cost functional J_σ is not strictly convex on $\mathcal{M}(\bar I_c; L^2(\omega))$. Fortunately, we find that the discrete optimal control problem (4.18) does indeed have a unique optimal control in the subspace $U_\sigma\subseteq \mathcal{M}(\bar I_c; L^2(\omega))$.

To derive the discrete first order optimality condition, we denote the solution of (4.19) by $u_\sigma(q)\in Y_\sigma$, then we can obtain the following reduced optimization problem:

$$\min_{q\in\mathcal{M}(\bar I_c;L^2(\omega))} j_\sigma(q) := j_{\sigma,1}(q) + j_{\sigma,2}(q), \tag{4.20}$$

where

$$j_{\sigma,1}(q) := \frac{1}{2}\|u_\sigma(q)-u_d\|^2_{L^2(I;L^2(\Omega))} + \frac{\beta}{2}\|u_\sigma(q)(T)-u_T\|^2_{L^2(\Omega)}, \quad j_{\sigma,2}(q) := \alpha\|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))}.$$

Obviously, $j_{\sigma,1}:\mathcal{M}(\bar I_c; L^2(\omega))\to\mathbb{R}$ is a quadratic functional of tracking type that is differentiable, and $j_{\sigma,2}: \mathcal{M}(\bar I_c; L^2(\omega))\to\mathbb{R}$ is convex and subdifferentiable. We remark that $j_{\sigma,2}\equiv J_2$.

By straightforward calculations, we obtain for any $q,\, p\in\mathcal{M}(\bar I_c; L^2(\omega))$ that

$$j'_{\sigma,1}(q)p = \left(u_\sigma(q)-u_d, \tilde u_\sigma(p)\right)_{L^2(I;L^2(\Omega))} + \beta\left(u_\sigma(q)(T)-u_T, \tilde u_\sigma(p)(T)\right),$$

where $\tilde u_\sigma(p)\in Y_\sigma$ is the finite element approximation of the state equation (1.2) with $f=0$, $u_0=0$, $q=p$, *i.e.*, satisfying equation (4.19) with the assumed data. On the other hand, we define a discrete adjoint variable as follows: Find $\varphi_\sigma\in P_\sigma$ such that

$$A(w_\sigma,\varphi_\sigma) = \int_I (u_\sigma(q)-u_d, w_\sigma)\mathrm{d}t + \beta(u_\sigma(q)(\cdot,T)-u_T, w_\sigma(T)) \quad \forall w_\sigma\in Y_\sigma. \tag{4.21}$$

Then, we obtain for any $q,\, p\in\mathcal{M}(\bar I_c; L^2(\omega))$ that

$$\begin{aligned} j'_{\sigma,1}(q)p &= \left(u_\sigma(q)-u_d, \tilde u_\sigma(p)\right)_{L^2(I;L^2(\Omega))} + \beta\left(u_\sigma(q)(T)-u_T, \tilde u_\sigma(p)(T)\right) \\ &= A(\tilde u_\sigma(p),\varphi_\sigma) = \langle p,\varphi_\sigma\rangle_{\bar I_c\times\omega}, \end{aligned} \tag{4.22}$$

which implies that $j'_{\sigma,1}(q) = \varphi_\sigma|_{\bar I_c\times\omega}$ with φ_σ given by (4.21).

Now we are in the position to derive the first order optimality condition for the discrete optimization problem (4.20).

Theorem 4.5. *Let $\hat q_\sigma\in\mathcal{M}(\bar I_c; L^2(\omega))$ be an optimal control and $\bar u_\sigma\in Y_\sigma$ be the corresponding optimal state of the optimal control problem (4.18). Then there exists an adjoint state $\bar\varphi_\sigma\in P_\sigma$ solving (4.21) with $u_\sigma(q)$ replaced by $\bar u_\sigma$ on the right-hand side, such that the following subgradient condition holds*

$$0\in\bar\varphi_\sigma|_{\bar I_c\times\omega} + \alpha\partial\|\cdot\|_{\mathcal{M}(\bar I_c;L^2(\omega))}(\hat q_\sigma) \ \text{ in } \ (\mathcal{M}(\bar I_c; L^2(\omega)))^*, \tag{4.23}$$

i.e.,

$$-\langle p-\hat q_\sigma, \bar\varphi_\sigma\rangle_{\bar I_c\times\omega} + \alpha\|\hat q_\sigma\|_{\mathcal{M}(\bar I_c;L^2(\omega))} \le \alpha\|p\|_{\mathcal{M}(\bar I_c;L^2(\omega))} \quad \forall p\in\mathcal{M}(\bar I_c; L^2(\omega)), \tag{4.24}$$

where $\partial\|\cdot\|_{\mathcal{M}(\bar I_c;L^2(\omega))}(\hat q_\sigma)$ *denotes the set of subgradients for* $\|\cdot\|_{\mathcal{M}(\bar I_c;L^2(\omega))}$ *at* $\hat q_\sigma$ *which is nonempty since* $\|\cdot\|_{\mathcal{M}(\bar I_c;L^2(\omega))}$ *is convex on* $\mathcal{M}(\bar I_c;L^2(\omega))$*, and* $(\mathcal{M}(\bar I_c;L^2(\omega)))^*$ *denotes the topological dual of* $\mathcal{M}(\bar I_c;L^2(\omega))$*.*

Furthermore, from the above condition (4.24) *we can easily conclude the following relation between* $\hat q_\sigma$ *and* $\bar\varphi_\sigma$*:*

$$\alpha\|\hat q_\sigma\|_{\mathcal{M}(\bar I_c;L^2(\omega))} + \langle \hat q_\sigma, \bar\varphi_\sigma\rangle_{\bar I_c\times\omega} = 0, \tag{4.25}$$

$$\|\bar\varphi_\sigma\|_{C(\bar I_c;L^2(\omega))}\begin{cases} = \alpha & \text{if } \hat q_\sigma \neq 0,\\ \leq \alpha & \text{if } \hat q_\sigma = 0.\end{cases} \tag{4.26}$$

In addition, since the discrete control-to-state mapping has an infinite-dimensional kernel, the discrete optimal control problem (4.18) admits more than one optimal control $\hat q_\sigma \in \mathcal{M}(\bar I_c;L^2(\omega))$ *corresponding to the identical optimal state* $\bar u_\sigma \in Y_\sigma$*. Among these optimal controls there exists a unique one* $\bar q_\sigma \in U_\sigma$*, such that for any other optimal control* $\hat q_\sigma \in \mathcal{M}(\bar I_c;L^2(\omega))$ *there holds* $\bar q_\sigma = \Lambda_\sigma \hat q_\sigma$*, i.e.,*

$$j_\sigma(\bar q_\sigma) = j_\sigma(\hat q_\sigma) = \min_{q\in\mathcal{M}(\bar I_c;L^2(\omega))} j_\sigma(q).$$

In other words, $(\bar q_\sigma, \bar u_\sigma) \in U_\sigma \times Y_\sigma$ *is the unique optimal pair of the following fully discrete optimal control problem:*

$$\min_{\substack{q_\sigma\in U_\sigma\\ u_\sigma\in Y_\sigma}} J_\sigma(u_\sigma,q_\sigma) := \frac12\|u_\sigma-u_d\|^2_{L^2(I;L^2(\Omega))} + \frac{\beta}{2}\|u_\sigma(T)-u_T\|^2_{L^2(\Omega)} + \alpha\|q_\sigma\|_{\mathcal{M}(\bar I_c;L^2(\omega))}, \tag{4.27}$$

where $u_\sigma \in Y_\sigma$ *is the discrete state variable satisfying* (4.19) *with the discrete control* $q_\sigma \in U_\sigma$*, which can be computed in practice.*

Proof. Similar to Theorem 3.3, we can easily obtain (4.23)–(4.26) by recalling that $j'_{\sigma,1}(\bar q_\sigma) = \bar\varphi_\sigma|_{\bar I_c\times\omega}$ in (4.22).

Next, we state the non-uniqueness of optimal controls and the unique solvability of problem (4.27). Let $\hat q_\sigma \in \mathcal{M}(\bar I_c;L^2(\omega))$ be any optimal control of problem (4.18). Setting $\bar q_\sigma := \Lambda_\sigma \hat q_\sigma$, it follows from (4.9) and (4.10) that $\langle \bar q_\sigma, v_\sigma\rangle_{\bar I_c\times\omega} = \langle \hat q_\sigma, v_\sigma\rangle_{\bar I_c\times\omega}$ for any $v_\sigma \in P_\sigma$ and

$$J_\sigma(\bar q_\sigma) \leq J_\sigma(\hat q_\sigma).$$

This means $\bar q_\sigma \in U_\sigma$ is also optimal, and $\bar q_\sigma \neq \hat q_\sigma$ unless $\hat q_\sigma \in U_\sigma$. On the other hand, the functional j_σ is not strictly convex on $\mathcal{M}(\bar I_c;L^2(\omega))$. Therefore, the discrete optimization problem (4.18) admits more than one solution in $\mathcal{M}(\bar I_c;L^2(\omega))$.

Obviously, for any given $q_\sigma \in U_\sigma$ satisfying $\langle q_\sigma, v_\sigma\rangle_{\bar I_c\times\omega} = 0,\ \forall v_\sigma \in P_\sigma$, we have $q_\sigma = 0$. Therefore, the control-to-state mapping is injective on U_σ, which implies that the discrete cost functional J_σ is strictly convex on U_σ. Therefore, the optimization problem (4.27) admits a unique optimal pair $(\bar q_\sigma, \bar u_\sigma) \in U_\sigma \times Y_\sigma$. The uniqueness of optimal controls for the cost functional J_σ in U_σ ensures that any other optimal control $\hat q_\sigma \in \mathcal{M}(\bar I_c;L^2(\omega))$ satisfies $\bar q_\sigma = \Lambda_\sigma \hat q_\sigma$. This finishes the proof. □

5. Error estimates for the state and adjoint equations

In this section we intend to derive *a priori* error estimates for finite element solutions of the discrete state equation (4.19) and the discrete adjoint equation (4.21). Here we assume that the data of the optimal control problem (1.1) satisfy the following assumptions.

Assumption 5.1. Assume that $f \in L^2(I;H^1(\Omega)),\ u_d \in L^2(I;L^2(\Omega)),\ u_0 \in H^1_0(\Omega)$ and $u_T \in H^1_0(\Omega)$.

In the following we also need the stability of the semi-discrete in time approximation to the state equation (1.2). Therefore, we introduce here the semi-discrete state equation for given f, u_0 and q: Find $u_\tau \in Y_\tau$ such that

$$A(u_\tau, v_\tau) = \int_I (f, v_\tau)\mathrm{d}t + \langle q, v_\tau\rangle_{\bar{I}_c\times\omega} + (u_0, v_\tau(0)) \quad \forall v_\tau \in P_\tau. \tag{5.1}$$

Similarly, for any given g and z_T, the semi-discrete approximation to problem (2.3) reads: Find $z_\tau \in P_\tau$ such that

$$A(v_\tau, z_\tau) = \int_I (g, v_\tau)\mathrm{d}t + (v_\tau(T), z_T) \quad \forall v_\tau \in Y_\tau. \tag{5.2}$$

Below in Subsections 5.1 and 5.2 we will present some stability and error estimates for the state and adjoint state equations. The similar results can be found in [13], Sections 4 and 5 where the state and adjoint state have higher regularity compared to our case, say $H^1(I; H^2(\Omega)\cap H_0^1(\Omega)) \cap H^2(I; L^2(\Omega))$. On the other hand, the stability estimates for the spatial and time derivatives of the state and adjoint state, as well as the error estimates for the discrete state and adjoint state, are measured in the norm $\|\cdot\|_{L^2(I;L^2(\Omega))}$ in [13]. However, in our case the error estimates for the discrete state or adjoint state are measured in norms $\|\cdot\|_{C(\bar{I};L^2(\Omega))}$ and $\|\cdot\|_{L^2(I;L^2(\Omega))}$, whereas the stability estimates for the spatial and time derivatives of the state and adjoint state are measured in norms $\|\cdot\|_{C(\bar{I};L^2(\Omega))}$, $\|\cdot\|_{L^2(I;H^{-1}(\Omega))}$ and $\|\cdot\|_{L^2(I;L^2(\Omega))}$.

5.1. Stability estimates for the discrete state and adjoint state

We first give a stability result for the semi-discrete approximation of the backward parabolic equation (2.3).

Lemma 5.2. *Let $z_\tau \in P_\tau$ solve (5.2) for given $g \in L^2(I; H^{-1}(\Omega))$ and $z_T \in L^2(\Omega)$, then there exists a constant $C > 0$, independent of τ, such that*

$$\begin{aligned}
&\|z_\tau\|_{L^2(I;L^2(\Omega))} \le C\sqrt{T}\|z_\tau\|_{C(\bar{I};L^2(\Omega))},\\
&\|z_\tau\|_{C(\bar{I};L^2(\Omega))} + \|\partial_t z_\tau\|_{L^2(I;H^{-1}(\Omega))} \le C\|g\|_{L^2(I;H^{-1}(\Omega))} + \|z_T\|_{L^2(\Omega)}.
\end{aligned} \tag{5.3}$$

In addition, if $g \in L^2(I; L^2(\Omega))$ and $z_T \in H_0^1(\Omega)$, then there hold

$$\begin{aligned}
&\|\nabla z_\tau\|_{L^2(I;L^2(\Omega))} \le C\sqrt{T}\|\nabla z_\tau\|_{C(\bar{I};L^2(\Omega))},\\
&\|\nabla z_\tau\|_{C(\bar{I};L^2(\Omega))} + \|\partial_t z_\tau\|_{L^2(I;L^2(\Omega))} \le C\|g\|_{L^2(I;L^2(\Omega))} + \|\nabla z_T\|_{L^2(\Omega)}.
\end{aligned} \tag{5.4}$$

Proof. For any fixed $m_0 \in \{1, 2, \dots, M\}$, setting $v_\tau|_{I_m} = 0$ for $m = 1, \cdots, m_0 - 1$, $v_\tau|_{I_m} = -\partial_t(-\Delta)^{-1} z_\tau|_{I_m}$ for $m = m_0, \cdots, M$ and $v_\tau(\cdot, T) = z_\tau(T) = z_T$ in (5.2), it is clear that such $v_\tau \in Y_\tau$ and we have

$$\begin{aligned}
A(v_\tau, z_\tau) &= \|\nabla\partial_t(-\Delta)^{-1} z_\tau\|^2_{L^2(I';L^2(\Omega))} + \frac{1}{2}(\|z_\tau(t_{m_0-1})\|^2_{L^2(\Omega)} + \|z_\tau(T)\|^2_{L^2(\Omega)})\\
&= -\int_{t_{m_0-1}}^T \langle g, \partial_t(-\Delta)^{-1} z_\tau\rangle \mathrm{d}t + \|z_T\|^2_{L^2(\Omega)}\\
&\le C\|g\|^2_{L^2(I';H^{-1}(\Omega))} + \frac{1}{2}\|\nabla\partial_t(-\Delta)^{-1} z_\tau\|^2_{L^2(I';L^2(\Omega))} + \|z_T\|^2_{L^2(\Omega)},
\end{aligned}$$

where $I' := \cup_{m=m_0}^M I_m$, this implies that

$$\|\nabla\partial_t(-\Delta)^{-1} z_\tau\|^2_{L^2(I';L^2(\Omega))} + \|z_\tau(t_{m_0-1})\|^2_{L^2(\Omega)} \le C\|g\|^2_{L^2(I;H^{-1}(\Omega))} + \|z_T\|^2_{L^2(\Omega)}.$$

That is, for any t_m $(m = 0, \cdots, M-1)$ we have the stability

$$\|z_\tau(t_m)\|_{L^2(\Omega)} \leq C\|g\|_{L^2(I;H^{-1}(\Omega))} + \|z_T\|_{L^2(\Omega)},$$

i.e., $\|z_\tau\|_{L^\infty(I;L^2(\Omega))} \leq C\|g\|_{L^2(I;H^{-1}(\Omega))} + \|z_T\|_{L^2(\Omega)}$. This finishes the proof of (5.3).

Next, we prove the estimate (5.4). Similar to the above procedure, for any fixed $m_0 \in \{1, 2, \cdots, M\}$, setting $v_\tau|_{I_m} = 0$ for $m = 1, \cdots, m_0 - 1$, $v_\tau|_{I_m} = -\partial_t z_\tau|_{I_m}$ for $m = m_0, \cdots, M$ and $v_\tau(\cdot, T) = z_\tau(T) = z_T$ in (5.2), it is clear that $v_\tau \in Y_\tau$ and we have

$$\begin{aligned} A(v_\tau, z_\tau) &= \|\partial_t z_\tau\|^2_{L^2(I';L^2(\Omega))} + \frac{1}{2}(\|\nabla z_\tau(t_{m_0-1})\|^2_{L^2(\Omega)} - \|\nabla z_\tau(T)\|^2_{L^2(\Omega)}) + \|z_\tau(T)\|^2_{L^2(\Omega)} \\ &= -\int_{t_{m_0-1}}^T (g, \partial_t z_\tau)\mathrm{d}t + \|z_\tau(T)\|^2_{L^2(\Omega)} \\ &\leq C\|g\|^2_{L^2(I';L^2(\Omega))} + \frac{1}{2}\|\partial_t z_\tau\|^2_{L^2(I';L^2(\Omega))} + \|z_\tau(T)\|^2_{L^2(\Omega)}. \end{aligned}$$

This implies that

$$\|\partial_t z_\tau\|^2_{L^2(I';L^2(\Omega))} + \|\nabla z_\tau(t_{m_0-1})\|^2_{L^2(\Omega)} \leq C\|g\|^2_{L^2(I';L^2(\Omega))} + \|\nabla z_T\|^2_{L^2(\Omega)},$$

then the estimate (5.4) follows. This finishes the proof. □

Then we will prove the following stability result for the semi-discrete state approximation.

Lemma 5.3. *Let $u_\tau \in Y_\tau$ solve (5.1) for given $f \in L^2(I; L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and $q \in \mathcal{M}(\bar{I}_c; L^2(\omega))$. Then there exists a constant $C > 0$, independent of τ, such that*

$$\|\nabla u_\tau\|_{L^2(I;L^2(\Omega))} + \|u_\tau(T)\|_{L^2(\Omega)} \leq C(\|f\|_{L^2(I;L^2(\Omega))} + \|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} + \|u_0\|_{L^2(\Omega)}). \tag{5.5}$$

Moreover, if $f \in L^2(I; H^1(\Omega))$, $u_0 \in H^1_0(\Omega)$ and $q \in \mathcal{M}(\bar{I}_c; H^1(\omega))$, then there holds

$$\|\Delta u_\tau\|_{L^2(I;L^2(\Omega))} + \|\nabla u_\tau(T)\|_{L^2(\Omega)} \leq C(\|f\|_{L^2(I;H^1(\Omega))} + \|q\|_{\mathcal{M}(\bar{I}_c;H^1(\omega))} + \|u_0\|_{H^1(\Omega)}), \tag{5.6}$$

where $C > 0$ is a constant independent of τ.

Proof. We first prove the estimate (5.5). Recall that $\|g\|^2_{L^2(I;H^{-1}(\Omega))} = \|\nabla u_\tau\|^2_{L^2(I;L^2(\Omega))} = \langle -\Delta u_\tau, u_\tau\rangle_{L^2(I;H^{-1},H^1)}$ with $g = -\Delta u_\tau$, and $\|h\|_{L^2(\Omega)} = \|u_\tau(T)\|_{L^2(\Omega)}$ with $h = u_\tau(T)$. We denote by $z_\tau \in P_\tau$ the semi-discrete approximation defined in (5.2) with $g = -\Delta u_\tau$ and $z_T = u_\tau(T)$, and test (5.2) with u_τ, then

$$\begin{aligned} &\int_I \langle g, u_\tau\rangle \mathrm{d}t + (z_T, u_\tau(T)) = A(u_\tau, z_\tau) \\ = &\int_I (f, z_\tau)\mathrm{d}t + \langle q, z_\tau\rangle_{\bar{I}_c\times\omega} + (u_0, z_\tau(0)) \\ \leq &\|f\|_{L^2(I;L^2(\Omega))}\|z_\tau\|_{L^2(I;L^2(\Omega))} + \|u_0\|_{L^2(\Omega)}\|z_\tau(0)\|_{L^2(\Omega)} + \|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}\|z_\tau\|_{C(\bar{I}_c;L^2(\omega))} \\ \leq &C(\|f\|_{L^2(I;L^2(\Omega))} + \|u_0\|_{L^2(\Omega)} + \|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))})\|z_\tau\|_{C(\bar{I};L^2(\Omega))} \\ \leq &C(\|f\|_{L^2(I;L^2(\Omega))} + \|u_0\|_{L^2(\Omega)} + \|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))})(\|g\|_{L^2(I;H^{-1}(\Omega))} + \|z_T\|_{L^2(\Omega)}), \end{aligned}$$

where we used the scheme (5.1) and the estimate (5.3) in Lemma 5.2. Therefore, we obtain the estimate (5.5).

Now, we prepare to show (5.6). First, we denote by $u_{\tau,m} := u_\tau|_{I_m}$, $m = 1, 2, \cdots, M$, and $u_{\tau,M+1} := u_\tau(T)$. Thus there holds (*cf.* [44])

$$
\begin{aligned}
&(u_{\tau,1} - u_0, v) + \frac{1}{2}(\tau_1 \nabla u_1, \nabla v) = (\tilde{f}_0, v)_{L^2(I_1;L^2(\Omega))},\\
&(u_{\tau,m+1} - u_{\tau,m}, v) + \frac{1}{2}(\tau_{m+1}\nabla u_{\tau,m+1} + \tau_m \nabla u_{\tau,m}, \nabla v) = (\tilde{f}_m, v),\\
&(u_{\tau,M+1} - u_{\tau,M}, v) + \frac{1}{2}(\tau_M \nabla u_M, \nabla v) = (\tilde{f}_M, v)
\end{aligned}
\tag{5.7}
$$

for arbitrary $v \in H_0^1(\Omega)$, where

$$
\tilde{f}_0 = (f, e_{t_1})_{L^2(I_1)}, \quad \tilde{f}_M = (f, e_{t_M})_{L^2(I_M)},
$$
$$
\tilde{f}_m = (f, e_{t_m})_{L^2(I_m \cup I_{m+1})} + \int_{(I_m \cup I_{m+1}) \cap \bar{I}_c} e_{t_m} \mathrm{d}q(t) \in L^2(\Omega) \quad m = 1, 2, \cdots, M-1.
$$

Since $f \in L^2(I; H^1(\Omega))$ and $q \in \mathcal{M}(\bar{I}_c; H^1(\omega))$, we have $\tilde{f}_m \in H^1(\Omega)$ for $m = 0, 1, \cdots, M$. Then from the first two expressions in (5.7) it follows that $u_{\tau,m} \in H^2(\Omega) \cap H_0^1(\Omega)$ $(m = 1, 2, \cdots, M)$ by the regularity of elliptic equations. By the last expression, there has $u_{\tau,M+1} = \tilde{f}_M + u_{\tau,M} + \frac{1}{2}\tau_M \Delta u_{\tau,M}$, which implies that $u_{\tau,M+1} \in L^2(\Omega)$. Therefore, we have $-\Delta u_{\tau,m} \in L^2(\Omega)$, $1 \le m \le M$ and $-\Delta u_{\tau,M+1} \in H^{-2}(\Omega)$ in the sense of distributions. Using the similar idea we can show that for any $g \in L^2(I; L^2(\Omega))$ and $z_T \in H_0^1(\Omega)$, the semi-discrete solution of (5.2) satisfies $z_\tau \in L^2(I; H^2(\Omega)) \cap C(\bar{I}; H^1(\Omega))$ and $z_\tau(0) \in H^1(\Omega)$.

With the above spatial regularity of semi-discrete solutions to (5.2) and (5.1) we can rewrite the semi-discrete schemes (5.2) and (5.1) as: Find $u_\tau \in Y_\tau$ such that for any $v_\tau \in P_\tau$

$$
\int_I -(u_\tau, \partial_t v_\tau) - (\Delta u_\tau, v_\tau)\mathrm{d}t + (u_\tau(T), v_\tau(T)) = \int_I (f, v_\tau)\mathrm{d}t + \langle q, v_\tau \rangle_{\bar{I}_c \times \omega} + (u_0, v_\tau(0)), \tag{5.8}
$$

and find $z_\tau \in P_\tau$ such that for any $w_\tau \in Y_\tau$

$$
\int_I -(w_\tau, \partial_t z_\tau) + (w_\tau, -\Delta z_\tau)\mathrm{d}t + (w_\tau(T), z_\tau(T)) = \int_I (g, w_\tau)\mathrm{d}t + (z_T, w_\tau(T)). \tag{5.9}
$$

Since there are no spatial derivatives for the test functions in schemes (5.8) and (5.9), the formulations (5.8) and (5.9) hold not only for all $v_\tau \in P_\tau$ and $w_\tau \in Y_\tau$, but also hold, by the dense of $H_0^1(\Omega)$ in $L^2(\Omega)$, for all $v_\tau \in \tilde{P}_\tau$ and $w_\tau \in \tilde{Y}_\tau$, respectively, where

$$
\tilde{P}_\tau := \{v_\tau \in C(\bar{I}; L^2(\Omega)) : v_\tau|_{I_m} \in \mathcal{P}_1(I_m; L^2(\Omega)),\ m = 1, 2, \cdots, M\},
$$
$$
\tilde{Y}_\tau := \{v_\tau \in L^2(I; L^2(\Omega)) : v_\tau|_{I_m} \in \mathcal{P}_0(I_m; L^2(\Omega)), m = 1, 2, \cdots, M,\ v_\tau(T) \in H^{-1}(\Omega)\}.
$$

We denote by z_τ the semi-discrete approximation to the backward equation (2.3) defined by (5.2), or equivalently, (5.9), for arbitrary $g \in L^2(I; L^2(\Omega))$ and $z_T \in C_0^\infty(\Omega)$. Similarly, we test (5.9) with $-\Delta u_\tau$, then

$$
\begin{aligned}
&\int_I (g, -\Delta u_\tau)\mathrm{d}t + \langle z_T, -\Delta u_\tau(T)\rangle = A(-\Delta u_\tau, z_\tau)\\
&= A(u_\tau, -\Delta z_\tau)\\
&= \int_I \langle f, -\Delta z_\tau \rangle \mathrm{d}t + \langle q, -\Delta z_\tau \rangle_{\bar{I}_c \times \omega} + \langle u_0, -\Delta z_\tau(0)\rangle\\
&\le \|f\|_{L^2(I;H^1(\Omega))} \|\nabla z_\tau\|_{L^2(I;L^2(\Omega))} + \|\nabla u_0\|_{L^2(\Omega)} \|\nabla z_\tau(0)\|_{L^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&+C\|q\|_{\mathcal{M}(\bar{I}_c;H^1(\omega))}\|\nabla z_\tau\|_{C(\bar{I}_c;L^2(\Omega))}\\
&\le C(\|f\|_{L^2(I;H^1(\Omega))}+\|u_0\|_{H^1(\Omega)}+\|q\|_{\mathcal{M}(\bar{I}_c;H^1(\omega))})\|\nabla z_\tau\|_{C(\bar{I};L^2(\Omega))}\\
&\le C(\|f\|_{L^2(I;H^1(\Omega))}+\|u_0\|_{H^1(\Omega)}+\|q\|_{\mathcal{M}(\bar{I}_c;H^1(\omega))})(\|g\|_{L^2(I;L^2(\Omega))}+\|\nabla z_T\|_{L^2(\Omega)}),
\end{aligned}
$$

where we have used the scheme (5.8) and the estimate (5.4) in Lemma 5.2. By the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ we obtain

$$
\|\Delta u_\tau\|_{L^2(I;L^2(\Omega))}+\|\Delta u_\tau(T)\|_{H^{-1}(\Omega)}\le C(\|f\|_{L^2(I;H^1(\Omega))}+\|u_0\|_{H^1(\Omega)}+\|q\|_{\mathcal{M}(\bar{I}_c;H^1(\omega))}),
$$

which confirms the estimate (5.6) by using the fact $\|\Delta u_\tau(T)\|_{H^{-1}(\Omega)}\approx\|\nabla u_\tau(T)\|_{L^2(\Omega)}$. This finishes the proof. □

Furthermore, the fully discrete finite element approximation of the backward parabolic equation (2.3) can be defined as: Find $z_\sigma\in P_\sigma$ such that

$$
A(v_\sigma,z_\sigma)=\int_I(g,v_\sigma)\mathrm{d}t+(v_\sigma(T),z_T)\qquad\forall v_\sigma\in Y_\sigma. \tag{5.10}
$$

Similar to Lemma 5.2 we have the following stability result on the fully discrete approximation of backward parabolic equations.

Lemma 5.4. *Let $z_\sigma\in P_\sigma$ solve (5.10) for given $g\in L^2(I;H^{-1}(\Omega))$ and $z_T\in L^2(\Omega)$. Then there exists a constant $C>0$, independent of σ, such that*

$$
\left\|\nabla\partial_t(-\Delta_h)^{-1}z_\sigma\right\|_{L^2(I;L^2(\Omega))}+\|z_\sigma\|_{C(\bar{I};L^2(\Omega))}\le C\left(\|g\|_{L^2(I;H^{-1}(\Omega))}+\|z_T\|_{L^2(\Omega)}\right), \tag{5.11}
$$

where $-\Delta_h:V_h\to V_h$ is defined as $(-\Delta_h v_h,w_h):=(\nabla v_h,\nabla w_h)$ for any $w_h\in V_h$.

Proof. Setting $v_\sigma\in Y_\sigma$ such that $v_\sigma|_{I_m}=-\partial_t(-\Delta_h)^{-1}z_\sigma|_{I_m}\in Y_\sigma$ $(m=m_0,\cdots,M)$, $v_\sigma|_{I_m}=0$ $(m=1,2,\cdots,m_0-1)$ and $v_\sigma(T)=z_\sigma(T)$ in (5.10) for arbitrary $1\le m_0\le M$, there holds

$$
\begin{aligned}
A(v_\sigma,z_\sigma)&=\|\nabla\partial_t(-\Delta_h)^{-1}z_\sigma\|^2_{L^2(I';L^2(\Omega))}+\frac12(\|z_\sigma(t_{m_0-1})\|^2_{L^2(\Omega)}+\|z_\sigma(T)\|^2_{L^2(\Omega)})\\
&=\int_{t_{m_0-1}}^T\langle g,-\partial_t(-\Delta_h)^{-1}z_\sigma\rangle\mathrm{d}t+(z_T,z_\sigma(T))\\
&\le C\|g\|^2_{L^2(I';H^{-1}(\Omega))}+\frac12\|\nabla\partial_t(-\Delta_h)^{-1}z_\sigma\|^2_{L^2(I';L^2(\Omega))}+C\|z_T\|^2_{L^2(\Omega)}+\frac12\|z_\sigma(T)\|^2_{L^2(\Omega)},
\end{aligned}
$$

which implies that

$$
\|\nabla\partial_t(-\Delta_h)^{-1}z_\sigma\|_{L^2(I';L^2(\Omega))}+\|z_\sigma(t_{m_0-1})\|_{L^2(\Omega)}\le C(\|g\|_{L^2(I';H^{-1}(\Omega))}+\|z_T\|_{L^2(\Omega)})
$$

for any $1\le m_0\le M$. Therefore, we can derive the conclusion. This finishes the proof. □

Now we prove the following stability for the fully discrete approximation to the state equation.

Lemma 5.5. *For given $f\in L^2(I;L^2(\Omega))$, $u_0\in L^2(\Omega)$ and $q\in\mathcal{M}(\bar{I}_c;L^2(\omega))$, let $u_\sigma(q)\in Y_\sigma$ solve (4.19). Then there holds the following estimate:*

$$
\|\nabla u_\sigma(q)\|_{L^2(I;L^2(\Omega))}+\|u_\sigma(q)(T)\|_{L^2(\Omega)}\le C(\|f\|_{L^2(I;L^2(\Omega))}+\|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}+\|u_0\|_{L^2(\Omega)}), \tag{5.12}
$$

where $C>0$ is a constant independent of σ.

Proof. Similar to Lemma 5.3, we denote by z_σ the discrete approximation defined in (5.10) with $g = u_\sigma$ and $z_T = u_\sigma(T)$, then

$$\begin{aligned}
&\int_I (g, u_\sigma)\mathrm{d}t + (z_T, u_\sigma(T)) = A(u_\sigma, z_\sigma)\\
&= \int_I (f, z_\sigma)\mathrm{d}t + \langle q, z_\sigma\rangle_{\bar I_c\times\omega} + (u_0, z_\sigma(0))\\
&\le \|f\|_{L^2(I;L^2(\Omega))}\|z_\sigma\|_{L^2(I;L^2(\Omega))} + \|u_0\|_{L^2(\Omega)}\|z_\sigma(0)\|_{L^2(\Omega)} + \|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))}\|z_\sigma\|_{C(\bar I_c;L^2(\omega))}\\
&\le C(\|f\|_{L^2(I;L^2(\Omega))} + \|u_0\|_{L^2(\Omega)} + \|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))})\|z_\sigma\|_{C(\bar I;L^2(\Omega))}\\
&\le C(\|f\|_{L^2(I;L^2(\Omega))} + \|u_0\|_{L^2(\Omega)} + \|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))})(\|g\|_{L^2(I;H^{-1}(\Omega))} + \|z_T\|_{L^2(\Omega)}),
\end{aligned}$$

where we have used Lemma 5.4. Then we can obtain the result by canceling the common term. This finishes the proof. □

5.2. A priori error estimates for the state and adjoint equations

In this subsection we are now able to give *a priori* error estimates for the finite element solutions to the state and adjoint equations.

Theorem 5.6. *For arbitrary $f \in L^2(I;H^1(\Omega))$, $q \in \mathcal{M}(\bar I_c;H^1(\omega))$ and $u_0 \in H_0^1(\Omega)$, let $u \in L^2(I;L^2(\Omega))$ be the solution to problem (1.2) and $u_\sigma \in Y_\sigma$ be its discretization defined in (4.19). Then there exists a positive constant C, independent of σ, such that*

$$\begin{aligned}
&\|u - u_\sigma(q)\|_{L^2(I;L^2(\Omega))} + \|(u - u_\sigma(q))(T)\|_{L^2(\Omega)}\\
&\le C(h + \tau^{\frac12})(\|f\|_{L^2(I;H^1(\Omega))} + \|q\|_{\mathcal{M}(\bar I_c;H^1(\omega))} + \|u_0\|_{H^1(\Omega)}).
\end{aligned} \tag{5.13}$$

Proof. We split the fully discrete error estimate into the temporal and spatial parts. To begin with, let $u_\tau \in Y_\tau$ be the semi-discrete solution to problem (5.1). Then we estimate respectively $\|u - u_\tau\|_{L^2(I;L^2(\Omega))} + \|u(T) - u_\tau(T)\|_{L^2(\Omega)}$ and $\|u_\tau - u_\sigma\|_{L^2(I;L^2(\Omega))} + \|u_\tau(T) - u_\sigma(T)\|_{L^2(\Omega)}$. To prove the two estimates we use the duality argument (*cf.* [7, 42]).

We first prove the estimate for $\|u - u_\tau\|_{L^2(I;L^2(\Omega))} + \|u(T) - u_\tau(T)\|_{L^2(\Omega)}$. Let $z \in H^1(I;L^2(\Omega)) \cap L^2(I;H^2(\Omega)\cap H_0^1(\Omega))$ be the solution to the backward parabolic equation (2.3) with $g := u - u_\tau$, $z_T := u(T) - u_\tau(T)$. Setting $\tilde z_\tau := \Pi_{P_\tau} z \in P_\tau$, then there holds $(z - \tilde z_\tau)(T) = 0$ and

$$(v_\tau, \partial_t(z - \tilde z_\tau))_{L^2(I;L^2(\Omega))} = 0 \quad \forall v_\tau \in Y_\tau.$$

From (4.2) and (5.1) we have

$$\begin{aligned}
&\int_I (g, u - u_\tau)\mathrm{d}t + (z_T, u(T) - u_\tau(T)) = A(u, z) - A(u_\tau, z_\tau)\\
&= A(u, z) - A(u_\tau, z) - \int_I (f, \tilde z_\tau)\mathrm{d}t - \langle q, \tilde z_\tau\rangle_{\bar I_c\times\omega} - (u_0, \tilde z_\tau(0)) + A(u_\tau, \tilde z_\tau)\\
&= \int_I (f, z - \tilde z_\tau)\mathrm{d}t + \langle q, z - \tilde z_\tau\rangle_{\bar I_c\times\omega} + (u_0, z(0) - \tilde z_\tau(0)) - A(u_\tau, z - \tilde z_\tau)\\
&= \int_I (f, z - \tilde z_\tau)\mathrm{d}t + \langle q, z - \tilde z_\tau\rangle_{\bar I_c\times\omega} - \int_I (\nabla u_\tau, \nabla(z - \tilde z_\tau))\mathrm{d}t\\
&\le \left(\|f\|_{L^2(I;L^2(\Omega))} + \|\Delta u_\tau\|_{L^2(I;L^2(\Omega))}\right)\|z - \tilde z_\tau\|_{L^2(I;L^2(\Omega))} + \|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))}\|z - \tilde z_\tau\|_{C(\bar I_c;L^2(\omega))}\\
&\le C\tau\left(\|f\|_{L^2(I;L^2(\Omega))} + \|\Delta u_\tau\|_{L^2(I;L^2(\Omega))}\right)\|z\|_{H^1(I;L^2(\Omega))} + C\tau^{\frac12}\|q\|_{\mathcal{M}(\bar I_c;L^2(\omega))}\|z\|_{H^1(I_c;L^2(\Omega))}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\tau \left(\|f\|_{L^2(I;L^2(\Omega))} + \|\Delta u_\tau\|_{L^2(I;L^2(\Omega))}\right)\left(\|g\|_{L^2(I;L^2(\Omega))} + \|z_T\|_{H^1(\Omega)}\right)\\
&\quad + C\tau^{\frac{1}{2}}\|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}\left(\|g\|_{L^2(I;L^2(\Omega))} + \|z_T\|_{L^2(\Omega)}\right)\\
&\leq C\tau H\left(\|g\|_{L^2(I;L^2(\Omega))} + H\right) + C\tau^{\frac{1}{2}}\|q\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}\left(\|g\|_{L^2(I;L^2(\Omega))} + \|z_T\|_{L^2(\Omega)}\right)\\
&\leq C\tau(1+\tau)H^2 + C\tau\|q\|^2_{\mathcal{M}(\bar{I}_c;L^2(\omega))} + \frac{1}{2}(\|g\|^2_{L^2(I;L^2(\Omega))} + \|z_T\|^2_{L^2(\Omega)}),
\end{aligned}
$$

where we have used Lemma 5.3, Theorem 2.4 and $H := \|f\|_{L^2(I;H^1(\Omega))} + \|q\|_{\mathcal{M}(\bar{I}_c;H^1(\omega))} + \|u_0\|_{H^1(\Omega)}$. Therefore, we have obtained

$$
\begin{aligned}
&\|u - u_\tau\|_{L^2(I;L^2(\Omega))} + \|u(T) - u_\tau(T)\|_{L^2(I;L^2(\Omega))}\\
&\leq C\tau^{\frac{1}{2}}(\|f\|_{L^2(I;H^1(\Omega))} + \|q\|_{\mathcal{M}(\bar{I}_c;H^1(\omega))} + \|u_0\|_{H^1(\Omega)}).
\end{aligned} \tag{5.14}
$$

Next, we estimate $\|u_\tau - u_\sigma\|_{L^2(I;L^2(\Omega))} + \|u_\tau(T) - u_\sigma(T)\|_{L^2(\Omega)}$. Note that there exists the following splitting:

$$
u_\tau - u_\sigma = u_\tau - \mathcal{R}_h u_\tau + \mathcal{R}_h u_\tau - u_\sigma := \eta_\sigma + \xi_\sigma,
$$

where $\mathcal{R}_h : H^1_0(\Omega) \to V_h$ is the standard spatial Ritz projection (*cf.* [57]).

Let $z_\sigma \in P_\sigma$ be the fully discrete solution to problem (5.10) with $g := \xi_\sigma$ and $z_T := \xi_\sigma(T)$. Taking $v_\sigma = \xi_\sigma \in Y_\sigma$ in the scheme (5.10) and applying the Galerkin orthogonality, one obtains

$$
\begin{aligned}
&\int_I (g, \xi_\sigma)\mathrm{d}t + (z_T, \xi_\sigma(T)) = A(\xi_\sigma, z_\sigma) = -A(\eta_\sigma, z_\sigma)\\
&= \int_I (\eta_\sigma, \partial_t z_\sigma) - (\nabla\eta_\sigma, \nabla z_\sigma)\mathrm{d}t - (\eta_\sigma(T), z_\sigma(T))\\
&= \int_I (\eta_\sigma, \partial_t z_\sigma)\mathrm{d}t - (\eta_\sigma(T), z_\sigma(T))\\
&\leq \|\nabla\eta_\sigma\|_{L^2(I;L^2(\Omega))}\|\nabla\partial_t(-\Delta_h)^{-1}z_\sigma\|_{L^2(I;L^2(\Omega))} + \|\eta_\sigma(T)\|_{L^2(\Omega)}\|z_\sigma(T)\|_{L^2(\Omega)}\\
&\leq C(\|\nabla\eta_\sigma\|_{L^2(I;L^2(\Omega))} + \|\eta_\sigma(T)\|_{L^2(\Omega)})(\|g\|_{L^2(I;L^2(\Omega))} + \|z_T\|_{L^2(\Omega)}),
\end{aligned}
$$

where we have used Lemma 5.4. Therefore, we have

$$
\begin{aligned}
\|u_\tau - u_\sigma\|_{L^2(I;L^2(\Omega))} + \|u_\tau(T) - u_\sigma(T)\|_{L^2(\Omega)} &\leq C(\|\nabla\eta_\sigma\|_{L^2(I;L^2(\Omega))} + \|\eta_\sigma(T)\|_{L^2(\Omega)})\\
&\leq Ch(\|\Delta u_\tau\|_{L^2(I;L^2(\Omega))} + \|\nabla u_\tau(T)\|_{L^2(\Omega)})\\
&\leq Ch(\|f\|_{L^2(I;H^1(\Omega))} + \|q\|_{\mathcal{M}(\bar{I}_c;H^1(\omega))} + \|u_0\|_{H^1(\Omega)}),
\end{aligned} \tag{5.15}
$$

where we have used Lemma 5.3. Combining the above two estimates we finish the proof. □

Theorem 5.7. *For any $z_T \in H^1_0(\Omega)$ and $g \in L^2(I;L^2(\Omega))$, let $z_\sigma \in P_\sigma$ be the solution of the discrete scheme (5.10), and $z \in H^1(I;L^2(\Omega)) \cap L^2(I;H^2(\Omega) \cap H^1_0(\Omega))$ be the solution of equation (2.3). Then there exists a positive constant $C > 0$, independent of σ, such that*

$$
\|z - z_\sigma\|_{C(\bar{I};L^2(\Omega))} \leq C(h + \tau^{\frac{1}{2}})(\|g\|_{L^2(I;L^2(\Omega))} + \|z_T\|_{H^1(\Omega)}). \tag{5.16}
$$

Proof. Let $e_\sigma := z - z_\sigma = (z - \pi_h\Pi_{P_\tau}z) + (\pi_h\Pi_{P_\tau}z - z_\sigma) =: \eta_\sigma + \zeta_\sigma$, then by the Galerkin orthogonality there holds for any $v_\sigma \in Y_\sigma$ that

$$\begin{aligned} A(v_\sigma, \zeta_\sigma) = -A(v_\sigma, \eta_\sigma) &= \int_I (v_\sigma, \partial_t\eta_\sigma) - (\nabla v_\sigma, \nabla\eta_\sigma)\mathrm{d}t - (v_\sigma(T), \eta_\sigma(T)) \\ &= \sum_{m=1}^{M} \Big(v_\sigma|_{I_m}, (z - \pi_h\Pi_{P_\tau}z)(t_m) - (z - \pi_h\Pi_{P_\tau}z)(t_{m-1})\Big) \\ &\quad -(v_\sigma(T), \eta_\sigma(T)) - \int_I (\nabla v_\sigma, \nabla\eta_\sigma)\mathrm{d}t \\ &= -\int_I (\nabla v_\sigma, \nabla\eta_\sigma)\mathrm{d}t, \end{aligned} \tag{5.17}$$

i.e., ζ_σ satisfies the following variational problem: Find $\zeta_\sigma \in P_\sigma^0$ such that

$$\int_I -(v_\sigma, \partial_t\zeta_\sigma) + (\nabla v_\sigma, \nabla\zeta_\sigma)\mathrm{d}t = -\int_I (\nabla v_\sigma, \nabla\eta_\sigma)\mathrm{d}t \quad \forall v_\sigma \in Y_\sigma. \tag{5.18}$$

For arbitrary $1 \le m_0 \le M$, taking the test function v_σ satisfying $v_\sigma|_{I_m} = -\partial_t(-\Delta_h)^{-1}\zeta_\sigma|_{I_m}$, $m = m_0, m_0 + 1, \cdots, M$ and $v_\sigma|_{I_m} = 0$, $m = 1, 2, \cdots, m_0 - 1$, $v_\sigma(T) = 0$ in the above identity (5.18), then we have

$$\begin{aligned} \big\|\nabla\partial_t(-\Delta_h)^{-1}\zeta_\sigma\big\|^2_{L^2(I';L^2(\Omega))} + \frac{1}{2}\|\zeta_\sigma(t_{m_0-1})\|^2_{L^2(\Omega)} &= \int_{t_{m_0-1}}^{T} (\nabla\partial_t(-\Delta_h)^{-1}\zeta_\sigma, \nabla\eta_\sigma)\mathrm{d}t \\ &\le \frac{1}{2}\|\nabla\partial_t(-\Delta_h)^{-1}\zeta_\sigma\|^2_{L^2(I';L^2(\Omega))} + \frac{1}{2}\|\nabla\eta_\sigma\|^2_{L^2(I';L^2(\Omega))}. \end{aligned}$$

Therefore, we obtain

$$\|\zeta_\sigma(t_{m_0-1})\|^2_{L^2(\Omega)} \le \|\nabla\eta_\sigma\|^2_{L^2(I';L^2(\Omega))}$$

for arbitrary $1 \le m_0 \le M$, *i.e.*,

$$\|\zeta_\sigma\|_{C(\bar{I};L^2(\Omega))} = \max_{1\le m\le M-1} \|\zeta_\sigma(t_m)\|_{L^2(\Omega)} \le \|\nabla\eta_\sigma\|_{L^2(I;L^2(\Omega))}.$$

Combining with the expression $e_\sigma = \eta_\sigma + \zeta_\sigma$ one deduces

$$\|e_\sigma\|_{C(\bar{I};L^2(\Omega))} \le \|\eta_\sigma\|_{C(\bar{I};L^2(\Omega))} + \|\nabla\eta_\sigma\|_{L^2(I;L^2(\Omega))}. \tag{5.19}$$

Then it suffices to bound the two terms on the right-hand side.

The first term on the right-hand side of (5.19) can be bounded by

$$\begin{aligned} \|\eta_\sigma\|_{C(\bar{I};L^2(\Omega))} &= \|z - \pi_h\Pi_{P_\tau}z\|_{C(\bar{I};L^2(\Omega))} \\ &\le \|z - \pi_h z\|_{C(\bar{I};L^2(\Omega))} + \|\pi_h(z - \Pi_{P_\tau}z)\|_{C(\bar{I};L^2(\Omega))} \\ &\le \|z - \pi_h z\|_{C(\bar{I};L^2(\Omega))} + C\|z - \Pi_{P_\tau}z\|_{C(\bar{I};L^2(\Omega))} \\ &\le Ch\|z\|_{C(\bar{I};H^1(\Omega))} + C\tau^{\frac{1}{2}}\|z\|_{H^1(I;L^2(\Omega))} \\ &\le C(h + \tau^{\frac{1}{2}})(\|z\|_{L^2(I;H^2(\Omega))} + \|z\|_{H^1(I;L^2(\Omega))}) \\ &\le C(h + \tau^{\frac{1}{2}})(\|g\|_{L^2(I;L^2(\Omega))} + \|z_T\|_{H^1(\Omega)}), \end{aligned} \tag{5.20}$$

where we have used the stability of the L^2-projection π_h. On the other hand, $\|\nabla\eta_\sigma\|_{L^2(I;L^2(\Omega))}$ can be estimated by

$$\begin{aligned}\|\nabla\eta_\sigma\|_{L^2(I;L^2(\Omega))} &= \|\nabla(z-\pi_h\Pi_{P_\tau}z)\|_{L^2(I;L^2(\Omega))}\\ &\le \|\nabla(z-\pi_h z)\|_{L^2(I;L^2(\Omega))} + \|\nabla\pi_h(z-\Pi_{P_\tau}z)\|_{L^2(I;L^2(\Omega))}\\ &\le \|\nabla(z-\pi_h z)\|_{L^2(I;L^2(\Omega))} + \|\nabla(z-\Pi_{P_\tau}z)\|_{L^2(I;L^2(\Omega))}\\ &\le Ch\|z\|_{L^2(I;H^2(\Omega))} + C\tau^{\frac{1}{2}}(\|z\|_{L^2(I;H^2(\Omega))} + \|z\|_{H^1(I;L^2(\Omega))})\\ &\le C(h+\tau^{\frac{1}{2}})(\|g\|_{L^2(I;L^2(\Omega))} + \|z_T\|_{H^1(\Omega)}),\end{aligned} \tag{5.21}$$

where we have used the H^1-stability of the L^2-projection π_h and the estimate (4.17) in Lemma 4.4. Combining the two estimates (5.20) and (5.21) we finish the proof. □

5.3. Error estimates for the optimal control problem

At first we prove a plain convergence for the solution of problem (4.18) to that of problem (1.1) as $|\sigma| := \tau + h \to 0^+$.

Theorem 5.8. *Let $\{\hat{q}_\sigma\} \subseteq \mathcal{M}(\bar{I}_c; L^2(\omega))$ be the set of optimal controls for the discrete optimal control problem (4.18), and $\bar{u}_\sigma \in Y_\sigma$ be the unique discrete optimal state associated to $\{\hat{q}_\sigma\}$. Let $(\bar{q}, \bar{u}) \in \mathcal{M}(\bar{I}_c; L^2(\omega)) \times X$ be the unique optimal pair of the continuous problem (1.1), where $\bar{q}$ is the optimal control and $\bar{u}$ is the optimal state. Then we obtain*

$$q_\sigma \overset{*}{\rightharpoonup} \bar{q} \in \mathcal{M}(\bar{I}_c; L^2(\omega)) \qquad \forall q_\sigma \in \{\hat{q}_\sigma\}, \tag{5.22}$$
$$\|q_\sigma\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} \to \|\bar{q}\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))} \qquad \forall q_\sigma \in \{\hat{q}_\sigma\}, \tag{5.23}$$
$$\|\bar{u}_\sigma - \bar{u}\|_{L^2(I;L^2(\Omega))} + \|(\bar{u}_\sigma - \bar{u})(T)\|_{L^2(\Omega)} \to 0, \tag{5.24}$$
$$J_\sigma(q_\sigma) \to J(\bar{q}) \qquad \forall q_\sigma \in \{\hat{q}_\sigma\}, \tag{5.25}$$

when $|\sigma| \to 0^+$.

Proof. The main ideas follow from [7], Theorem 4.9 and [12], Theorem 1.2, see also [6], Theorem 3.5. Similar to Theorem 3.6, since q_σ is optimal, we can easily show that the sequence $\{q_\sigma\}$ is uniformly bounded in $\mathcal{M}(\bar{I}_c; L^2(\omega))$ with respect to σ. Then there exists a subsequence, still denoted by $\{q_\sigma\}$, such that $q_\sigma \overset{*}{\rightharpoonup} \tilde{q}$ in $\mathcal{M}(\bar{I}_c; L^2(\omega))$ for some $\tilde{q}$ when $\sigma \to 0^+$. Below, we show that $\tilde{q} = \bar{q}$.

Let $u_{\tilde{q}}$ be the solution of equation (1.2) with q replaced by $\tilde{q}$. We first show that

$$\|u_{\tilde{q}} - \bar{u}_\sigma\|_{L^2(I;L^2(\Omega))} + \|(u_{\tilde{q}} - \bar{u}_\sigma)(T)\|_{L^2(\Omega)} \to 0 \quad \text{as } |\sigma| \to 0. \tag{5.26}$$

In fact, from the triangle inequality we have

$$\begin{aligned}\|u_{\tilde{q}} - \bar{u}_\sigma\|_{L^2(I;L^2(\Omega))} + \|(u_{\tilde{q}} - \bar{u}_\sigma)(T)\|_{L^2(\Omega)} &\le \|u_{\tilde{q}} - u_{q_\sigma}\|_{L^2(I;L^2(\Omega))} + \|(u_{\tilde{q}} - u_{q_\sigma})(T)\|_{L^2(\Omega)}\\ &\quad + \|u_{q_\sigma} - \bar{u}_\sigma\|_{L^2(I;L^2(\Omega))} + \|(u_{q_\sigma} - \bar{u}_\sigma)(T)\|_{L^2(\Omega)}.\end{aligned} \tag{5.27}$$

Applying Proposition 2.5 to the first two terms and Theorem 5.6 to the latter two terms on the right-hand side of the above estimate yields the result.

Next, there holds

$$j(\tilde{q}) \le \liminf_{|\sigma|\to 0} j_\sigma(q_\sigma) \le \limsup_{|\sigma|\to 0} j_\sigma(q_\sigma) \le \limsup_{|\sigma|\to 0} j_\sigma(\Lambda_\sigma \bar{q}) = j(\bar{q}), \tag{5.28}$$

where in the first inequality we have used the weakly-* lower semicontinuity of the cost functional j, in the third inequality we used the optimality of q_σ, while in the last equality we have used (4.11) and (5.26). Therefore, $\tilde{q}$ is also optimal, so $\tilde{q} = \bar{q}$ since $\bar{q}$ is unique, *i.e.*, $q_\sigma \overset{*}{\rightharpoonup} \bar{q} \in \mathcal{M}(\bar{I}_c; L^2(\omega))$, then $\bar{u} = u_{\tilde{q}}$ by the unique solvability of the state equation, which proves (5.22) and (5.24). Obviously, (5.25) can be concluded from (5.28). Lastly, (5.23) follows from (5.24) and (5.25). □

A second convergence result concerns the convergence order of the objective functional.

Theorem 5.9. *Let $q_\sigma \in \{\hat{q}_\sigma\} \subseteq \mathcal{M}(\bar{I}_c; L^2(\omega))$ be any optimal control to the discrete problem (4.18) and $\bar{q} \in \mathcal{M}(\bar{I}_c; L^2(\omega))$ be the unique optimal control for the continuous problem (1.1). Then there exists a constant $C > 0$, independent of σ, such that*

$$|j(\bar{q}) - j_\sigma(q_\sigma)| \le C(h + \tau^{\frac{1}{2}}). \tag{5.29}$$

Proof. The proof uses the approach of [7], Theorem 5.1, see also [6], Theorem 4.1. It follows from the optimality of $\bar{q}$ and q_σ that

$$j(\bar{q}) - j_\sigma(\bar{q}) \le j(\bar{q}) - j_\sigma(q_\sigma) \le j(q_\sigma) - j_\sigma(q_\sigma),$$

which means that

$$|j(\bar{q}) - j_\sigma(q_\sigma)| \le \max\{|j(\bar{q}) - j_\sigma(\bar{q})|, |j(q_\sigma) - j_\sigma(q_\sigma)|\}. \tag{5.30}$$

Now it remains to estimate the two terms on the right-hand side.

For arbitrary $\tilde{q} \in \mathcal{M}(\bar{I}_c; L^2(\omega))$, we denote by $u_{\tilde{q}}$ and $u_\sigma(\tilde{q})$ the unique solutions to problems (1.2) and (4.19), respectively. Then we obtain from Theorem 5.6 that

$$\|u_{\tilde{q}} - u_\sigma(\tilde{q})\|_{L^2(I;L^2(\Omega))} + \|(u_{\tilde{q}} - u_\sigma(\tilde{q}))(T)\|_{L^2(\Omega)} \le C(h + \tau^{\frac{1}{2}}).$$

It is straightforward to show that

$$\begin{aligned}
|j(\tilde{q}) - j_\sigma(\tilde{q})| \le\ & \frac{1}{2}\Big| \|u_{\tilde{q}} - u_d\|^2_{L^2(I;L^2(\Omega))} - \|u_\sigma(\tilde{q}) - u_d\|^2_{L^2(I;L^2(\Omega))} \Big| \\
& + \frac{\beta}{2}\Big| \|u_{\tilde{q}}(T) - u_T\|^2_{L^2(\Omega)} - \|u_\sigma(\tilde{q})(T) - u_T\|^2_{L^2(\Omega)} \Big| \\
\le\ & C(\|u_{\tilde{q}}\|_{L^2(I;L^2(\Omega))} + \|u_\sigma(\tilde{q})\|_{L^2(I;L^2(\Omega))} + \|u_d\|_{L^2(I;L^2(\Omega))})\|u_{\tilde{q}} - u_\sigma(\tilde{q})\|_{L^2(I;L^2(\Omega))} \\
& + C(\|u_{\tilde{q}}(T)\|_{L^2(\Omega)} + \|u_\sigma(\tilde{q})(T)\|_{L^2(\Omega)} + \|u_T\|_{L^2(\Omega)})\|(u_{\tilde{q}} - u_\sigma(\tilde{q}))(T)\|_{L^2(\Omega)} \\
\le\ & C(h + \tau^{\frac{1}{2}}),
\end{aligned}$$

where we have used Lemma 5.5 and Theorem 2.4. By setting $\tilde{q} = q$ and $\tilde{q} = q_\sigma$ in the above error estimate we finish the proof by considering (5.30). □

The last convergence result is about the approximation of the state equation.

Theorem 5.10. *Let $\bar{u} \in L^2(I; L^2(\Omega))$ be the optimal state of the continuous optimal control problem (1.1) and $\bar{u}_\sigma \in Y_\sigma$ be the discrete optimal state of the discrete optimization problem (4.18). Then there exists a constant C, independent of σ, such that*

$$\|\bar{u} - \bar{u}_\sigma\|^2_{L^2(I;L^2(\Omega))} + \beta\|(\bar{u} - \bar{u}_\sigma)(T)\|^2_{L^2(\Omega)} \le C(h + \tau^{\frac{1}{2}}). \tag{5.31}$$

Proof. In order to obtain the above estimate (5.31), we first introduce two auxiliary variables. The first one is the finite element approximation to the state equation (1.2) with the optimal control $\bar{q}$: Find $\hat{u}_\sigma \in Y_\sigma$ such that

$$A(\hat{u}_\sigma, v_\sigma) = \int_I (f, v_\sigma)\mathrm{d}t + \langle \bar{q}, v_\sigma \rangle_{\bar{I}_c\times\omega} + (u_0, v_\sigma(0)) \quad \forall v_\sigma \in P_\sigma,$$

while the second is the finite element approximation to the adjoint equation (3.2) with the optimal state $\bar{u}$: Find $\hat{\varphi} \in P_\sigma$ such that

$$A(\omega_\sigma, \hat{\varphi}_\sigma) = \int_I (\bar{u} - u_d, \omega_\sigma)\mathrm{d}t + \beta(\bar{u}(T) - u_T, \omega_\sigma(T)) \quad \forall \omega_\sigma \in Y_\sigma.$$

Taking $p = \hat{q}_\sigma$ in the continuous optimality condition (3.4) and $p = \bar{q}$ in the discrete optimality condition (4.24), where $\hat{q}_\sigma$ is any optimal control for the discrete optimization problem (4.20), then adding them up we obtain

$$\begin{aligned}
0 &\le \langle \hat{q}_\sigma - \bar{q}, \bar{\varphi} - \bar{\varphi}_\sigma \rangle_{\bar{I}_c\times\omega} \\
&= \langle \hat{q}_\sigma - \bar{q}, \bar{\varphi} - \hat{\varphi}_\sigma \rangle_{\bar{I}_c\times\omega} + \langle \hat{q}_\sigma - \bar{q}, \hat{\varphi}_\sigma - \bar{\varphi}_\sigma \rangle_{\bar{I}_c\times\omega} \\
&= \langle \hat{q}_\sigma - \bar{q}, \bar{\varphi} - \hat{\varphi}_\sigma \rangle_{\bar{I}_c\times\omega} + A(\bar{u}_\sigma - \hat{u}_\sigma, \hat{\varphi}_\sigma - \bar{\varphi}_\sigma) \\
&= \langle \hat{q}_\sigma - \bar{q}, \bar{\varphi} - \hat{\varphi}_\sigma \rangle_{\bar{I}_c\times\omega} + (\bar{u} - \bar{u}_\sigma, \bar{u}_\sigma - \hat{u}_\sigma) + \beta((\bar{u} - \bar{u}_\sigma)(T), (\bar{u}_\sigma - \hat{u}_\sigma)(T)) \\
&= \langle \hat{q}_\sigma - \bar{q}, \bar{\varphi} - \hat{\varphi}_\sigma \rangle_{\bar{I}_c\times\omega} + (\bar{u} - \bar{u}_\sigma, \bar{u}_\sigma - \bar{u}) + \beta((\bar{u} - \bar{u}_\sigma)(T), (\bar{u}_\sigma - \bar{u})(T)) \\
&\quad + (\bar{u} - \bar{u}_\sigma, \bar{u} - \hat{u}_\sigma) + \beta((\bar{u} - \bar{u}_\sigma)(T), (\bar{u} - \hat{u}_\sigma)(T)).
\end{aligned}$$

Therefore, there holds

$$\begin{aligned}
&\|\bar{u} - \bar{u}_\sigma\|^2_{L^2(I;L^2(\Omega))} + \beta\|(\bar{u} - \bar{u}_\sigma)(T)\|^2_{L^2(\Omega)} \\
&\le 2\langle \hat{q}_\sigma - \bar{q}, \bar{\varphi} - \hat{\varphi}_\sigma \rangle_{\bar{I}_c\times\omega} + \|\bar{u} - \hat{u}_\sigma\|^2_{L^2(I;L^2(\Omega))} + \beta\|(\bar{u} - \hat{u}_\sigma)(T)\|^2_{L^2(\Omega)} \\
&\le 2\|\hat{q}_\sigma - \bar{q}\|_{\mathcal{M}(\bar{I}_c;L^2(\omega))}\|\bar{\varphi} - \hat{\varphi}_\sigma\|_{C(\bar{I}_c;L^2(\Omega))} + \|\bar{u} - \hat{u}_\sigma\|^2_{L^2(I;L^2(\Omega))} + \beta\|(\bar{u} - \hat{u}_\sigma)(T)\|^2_{L^2(\Omega)} \\
&\le C(h + \tau^{\frac{1}{2}}),
\end{aligned}$$

where we have used Theorems 5.6 and 5.7. This finishes the proof. □

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