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ON HOMOTOPY PROPERTIES OF SOLUTIONS OF SOME DIFFERENTIAL INCLUSIONS IN THE $W^{1,p}$-TOPOLOGY

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Abstract. We consider a differential inclusion on a manifold, defined by a field of open half-spaces whose boundary in each tangent space is the kernel of a one-form ω. We make the assumption that the corank one distribution associated to the kernel of ω is completely nonholonomic of step 2. We identify a subset of solutions of the differential inclusion, satisfying two endpoints and periodic boundary conditions, which are homotopy equivalent in the $W^{1,p}$-topology, for any $p \in [1, +\infty)$, to the based loop space and the free loop space respectively.

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1. Introduction

Let M be a connected, smooth manifold and let us consider the affine control system on M

$$\dot{\gamma} = X_0(\gamma) + \sum_{i=1}^{d} u_i X_i(\gamma), \tag{1.1}$$

where $X_0, X_i \in \Gamma(TM)$, $i = 1, \dots d$, are smooth vector fields on M such that the set $\{X_i\}_{i \in \{1,\dots,d\}}$ generates a distribution $\mathcal{D}$ of constant rank. The set of solutions $\gamma\colon I \subset \mathbb{R} \to M$ of (1.1) can be endowed with the Sobolev topology $W^{1,p}$ by taking controls u_i, for all $i \in \{1, \dots, d\}$, in the space $L^p(I, \mathbb{R})$.

Let us assume that $\{X_i\}_{i \in \{1,\dots,d\}}$ satisfy the Hörmander condition, *i.e.* for each $x \in M$, a finite number of their iterated brackets span the whole tangent space T_xM (we say also that $\mathcal{D}$ is completely nonholonomic). In a recent paper [1], the authors prove that the set of solutions of (1.1) connecting two given points in M is homotopy equivalent in the $W^{1,p}$-topology to the based loop space of M. If the drift X_0 is 0 then p can be any number in $[1, +\infty)$. In this case, the same result holds for periodic solutions and the free loop space as shown in [2]. If the drift does not vanish, the value of p in [1] is confined to an interval $[1, p_c)$ where p_c is at most $\sigma/(\sigma - 1)$, with σ equal to the step of the distribution, *i.e.* the minimal number plus 1 of Lie bracket iterations necessary to generate $\Gamma(TM)$ from the base fields $\{X_i\}_{i \in \{1,\dots,d\}}$.

It is then natural to ask if this last result can be improved by considering a variable drift, for example by multiplying X_0 with a positive control u_0. We are able to answer positively to this question at least when the

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distribution associated to $\{X_i\}_{i\in\{1,\dots,d\}}$ is generated by the kernel of a smooth, nowhere vanishing, one-form ω on M, provided that it is of step 2, and X_0 is transversal to it. The control problem we consider is equivalent to the differential inclusion

$$\dot\gamma \in \mathcal{H}, \tag{1.2}$$

where $\mathcal{H}$ is the field of open half-spaces $\{v \in TM : \omega(v) < 0\}$ with the union of the zero section of TM. Actually, the trajectories we consider are the ones associated with some specific controls $u : [0,1] \to \mathbb{R}^{d+1}$, $u(t) = \big(u_0(t), u_1(t), \dots, u_d(t)\big)$ that constitute a subset $\mathcal{A}$ of $L^\infty([0,1],\mathbb{R}^{d+1})$. More specifically, let D be the set of partition of the interval $[0,1]$, then

$$\mathcal{A} := \Big\{u \in L^\infty\big([0,1],\mathbb{R}^{d+1}\big)\,\big|\, u_0 \geq 0,\ \exists P \in D\colon \forall J \in P,\ u_0|_J = \text{const.} = \xi_J \text{ and, } \forall i \in \{1,\dots,d\},\ u_i|_J = \xi_j\alpha_{Ji}\Big\}.$$

The functions $\alpha_J : J \to \mathbb{R}^d$, $\alpha_J(t) = \big(\alpha_{J1}(t),\dots,\alpha_{Jd}(t)\big)$ have images in the closed ball of $\mathbb{R}^d$ centred at 0 and having radius K, where K is depends on $m = \dim M$, on the one-form ω, and on a sequence of local frame fields for $\Gamma(\mathcal{D})$ defined on neighbourhoods of M that cover M (see Rem. 2.1, Asms. (A1)–(A2) and (2.6)). The type of controls considered naturally align with the construction of a local *cross-section* σ for the endpoint map (see Prop. 2.13). Anyway, the main reason that let led us to consider the set $\mathcal{A}$ is that this class of controls enables us to obtain energy bounds on compact subsets (in the H^1-topology) of the trajectories space for certain singular Finsler metrics in [3] that generalize Kropina metrics (see, *e.g.*, [4]).

We emphasize that our main results, Theorems 3.2 and 3.3, remain valid for a compact manifold M, without assuming (A1)–(A2), as follows:

Theorem. *Let M be a connected, compact manifold and $\mathcal{D} \subset TM$ a corank one, completely nonholonomic distribution of step 2. Then the sets $\Omega^p_{x,y}$ and Λ^p are homotopy equivalent to $\Omega^{p,\mathrm{st}}_{x,y}$ and Λ^p_{st} respectively;* where, $\Omega^p_{x,y}$ and Λ^p are the subsets of controls in $\mathcal{A}$ corresponding, respectively, to the solutions of (1.2) connecting the points $x, y \in M$ and the ones having equal initial and final point, endowed with the L^p-topology, $1 \leq p < \infty$, and $\Omega^{p,\mathrm{st}}_{x,y}$ and Λ^p_{st} are respectively the manifold of the paths in M between x and y and the one of the free loops in M, both endowed with the $W^{1,p}$-topology.

2. Differential inclusions and Hurewicz fibrations

Let (M, g_0) be a connected, complete Riemannian manifold of dimension m. We consider a smooth, nowhere vanishing, one-form $\omega \in \Gamma(T^*M)$ and its kernel distribution

$$\mathcal{D} := \{v \in TM : \omega(v) = 0\}.$$

We assume that $\mathcal{D}$ is completely nonholonomic of step 2. It is not difficult to see that this property is equivalent to the form $\omega \wedge \mathrm{d}\omega$ being nowhere vanishing.

The pair $(\mathcal{D}, i)$ defines a *sub-Riemannian structure* in the sense of [5], Definition 3.2, where $i\colon \mathcal{D} \hookrightarrow TM$ denotes the inclusion. The Riemannian structure on $(\mathcal{D}, i)$ is $g_0|_{\mathcal{D}\times\mathcal{D}}$. According to [5], Corollary 3.27 every sub-Riemannian structure is equivalent to a *free* sub-Riemannian structure $(\mathbf{U}, f)$, *i.e.* a trivial vector bundle $\mathbf{U} \to M$ and a vector bundle morphism $f\colon \mathbf{U} \to TM$ such that there exists a surjective vector bundle morphism $\mathbf{p}\colon \mathbf{U} \to \mathcal{D}$ with $i \circ \mathbf{p} = f$ and the Riemannian metric on $\mathbf{U}$ satisfies

$$|v|_0 = \min\{|u| : u \in f^{-1}(v)\} \tag{2.1}$$

for all $v \in \mathcal{D}$ (here $|\cdot|_0$ and $|\cdot|$ are the norms associated, respectively, with the Riemannian metric g_0 and the bundle metric on $\mathbf{U}$).

Let $\mathcal{F} := \{X_1, \dots X_d\} \subset \Gamma(\mathbf{U})$ be a global orthonormal frame field on $\mathbf{U}$. Then, from (2.1), for each $v \in \mathcal{D}$, it follows

$$|v|_0 = \min\left\{ \left(\sum_{i=1}^d u_i^2\right)^{1/2} \middle| \sum_{i=1}^d u_i X_i \in f^{-1}(v) \right\}. \tag{2.2}$$

Let $X_0 \in \Gamma(TM)$ be the smooth unit vector field on M which is orthogonal to $\mathcal{D}$, *i.e.* $g_0(X_0, X_0) = 1$ and $g_0(X_0, v) = 0$, for all $v \in \mathcal{D}$, and moreover $-\omega(X_0) = \|\omega\|$, where $\|\omega\|$ denotes the function on M given by $x \in M \mapsto \|\omega_x\|$, being $\|\omega_x\|$ the norm of ω_x w.r.t. g_0.

Remark 2.1. For each $z \in M$ we can consider a neighbourhood U_z of z and a frame field $\{Y_i\}_{i\in\{1,\dots,m-1\}}$ for $\Gamma(\mathcal{D})|_{U_z}$ such that $|(Y_i)_x|_0 = 1$ for all $x \in U_z$ and for all $i \in \{1, \dots m-1\}$. Being $\mathcal{D}$ completely nonholonomic of step 2, up to restricting U_z, there exist Y_j, Y_l in the basis, such that $[Y_j, Y_l]$ is transversal to $\mathcal{D}$ at each point in U_z and moreover

$$\lambda_z := \inf\big(-\omega([Y_j, Y_l])\big) = \inf \mathrm{d}\omega(Y_j, Y_l) > 0. \tag{2.3}$$

By rearranging the vector fields in the local frame, we can assume that the vector field Y_j, Y_l are the first two Y_1, Y_2.

We assume that

(A1)

$$\Omega := \sup \|\omega\| < +\infty; \tag{2.4}$$

(A2) there exists a countable covering U_{z_k}, $k \in \mathbb{N}$, of M made by neighbourhoods U_{z_h} as in Remark 2.1 and such that

$$\lambda := \inf_{k\in\mathbb{N}} \lambda_{z_k} > 0. \tag{2.5}$$

We notice that Assumptions (A1)–(A2) are satisfied if M is compact.

Let us multiply the vectors Y_i, $i \in \{1, \dots, m-1\}$, by the same factor

$$K := \big(5(m+3)\Omega/\lambda\big)^{1/2}. \tag{2.6}$$

Let us denote these rescaled vector fields still with Y_i. Hence, for the rescaled vector fields Y_i the number defined in (2.5), still denoted with λ, satisfies:

$$\lambda > 4(m+3)\Omega, \tag{2.7}$$

moreover

$$|(Y_i)_x|_0 = K, \quad x \in U_{z_k},\ k \in \mathbb{N}, \tag{2.8}$$

for all $i \in \{1, \dots m-1\}$. We consider

$$\mathcal{H} := ([X_0]^+ + \mathcal{D}) \cup 0,$$

where $[X_0]^+ := \{aX_0 : a \in (0, +\infty)\}$ and 0 denotes the zero section of TM.

Definition 2.2. Let $I \subset \mathbb{R}$ be an interval. An absolutely continuous curve $\gamma\colon I \to M$ is a *solution of the differential inclusion* (1.2) if $\dot\gamma(t) \in \mathcal{H}$, a.e. on I.

We are interested in solutions which also belong to the Sobolev spaces $W^{1,p}(I, M)$, for $p \in (1, +\infty)$ for a compact interval I. We recall that we can see M as isometrically embedded in a Euclidean space $\mathbb{R}^N$ by Nash's isometric embedding theorem. Consequently, $W^{1,p}(I, M)$ can be defined as the space $\{\gamma \in W^{1,p}(I, \mathbb{R}^N) : \gamma(I) \subset M\}$, since $W^{1,p}(I, \mathbb{R}^N)$ canonically embeds into $C^{0,1-1/p}(I, \mathbb{R}^N)$.

We notice that (1.2) is invariant by orientation preserving reparametrisations, thus we can assume that its solutions are defined on the interval $I = [0, 1]$.

Since for each $i \in \{1, \dots, d\}$, $|X_i| = 1$, by (2.2) we have $|f \circ X_i| \le 1$. We then deduce easily that an absolutely continuous curve $\gamma : I \to M$, with $\dot\gamma(s) \neq 0$ a.e., solves (1.2) and belongs to $W^{1,p}(I, M)$ if and only if there exist L^p-functions $(u_0, u_1, \dots, u_d) =: u\colon I \to \mathbb{R}^{d+1}$ (called *controls*) with $0 < u_0 = g_0(\dot\gamma, X_0)$ a.e. such that

$$\dot\gamma = u_0 X_0 \circ \gamma + \sum_{i=1}^{d} u_i f \circ X_i \circ \gamma. \tag{2.9}$$

In particular, since the isometric embedding can be taken closed (see [6]), if the controls are defined on I, so it is the curve γ. Vice versa, if $\gamma \in W^{1,p}(I, M)$ solves (1.2), a control $u := (u_0, u_1, \dots, u_d) \in L^p(I, \mathbb{R}^d)$ exists, and it is given by $u_0 = g_0(\dot\gamma, X_0)$ and $u_{\mathcal{D}} := \big(u_1, \dots, u_d\big)$ equal to the *minimal control* for $\dot\gamma - g_0(\dot\gamma, X_0 \circ \gamma) X_0 \circ \gamma$, *i.e.* $\big|\dot\gamma(t) - u_0(t) X_0(\gamma(t))\big|_0 = |u_{\mathcal{D}}(t)|$, for a.e. $t \in I$.

Let us introduce the subset $\mathcal{A}$ of controls that we consider.

Definition 2.3. Let D be the set of partitions of the interval I. We call a measurable curve $u\colon I \to \mathbb{R}^{d+1}$ *admissible* if there exists $P \in D$ such that for each $J \in P$, there exist a constant $\xi_J \ge 0$, and measurable functions $\alpha_{Ji}\colon J \to \mathbb{R}$, $i \in \{1, \dots, d\}$, with

$$\sup_{s \in J} \sum_{i=1}^{d} \alpha_{Ji}^2(s) \le K^2 \tag{2.10}$$

where K satisfies (2.8), such that

$$u_0|_J = \xi_J^2 \quad \text{and } u_i|_J = \xi_J \alpha_{Ji} \tag{2.11}$$

for all $i \in \{1, \dots, d\}$.

We define $\mathcal{A}$ to be the set of all admissible curves endowed with the L^p-topology for some $p \ge 1$.

Remark 2.4. Notice that $\mathcal{A}$ contains the zero function $u \equiv 0$, which corresponds to $\xi_J = 0$ for all $J \in P$; moreover we notice that u can assume the value zero only on a whole interval $\bar J \in P$, where $\xi_{\bar J} = 0$.

Definition 2.5. We define a *concatenation* operation $\star$ on $\mathcal{A}$: for all $u, v \in \mathcal{A}$, let $u \star v \in \mathcal{A}$ be

$$u \star v(s) := \begin{cases} u(2s), & s \in [0, 1/2] \\ v(2s-1), & s \in [1/2, 1] \end{cases}$$

We notice that $\star$ is continuous w.r.t. the L^p-topology, $1 \le p \le \infty$.

Using the frame field $\mathcal{F}$ we can define

$$S\colon M \times \mathcal{A} \to W^{1,p}(I, M),$$

where $S(x,u)$ is the unique solution $\gamma\colon I \to M$ to the Cauchy problem

$$\begin{cases} \dot\gamma(t) = (u_0 X_0 \circ \gamma)(t) + \sum_{i=1}^{d} u_i (f \circ X_i \circ \gamma)(t) \\ \gamma(0) = x. \end{cases} \tag{2.12}$$

The following lemma and the subsequent proposition are quite standard; we include their proofs for the reader's convenience.

Lemma 2.6. *The solution operator $S\colon M \times \mathcal{A} \to W^{1,p}(I,M) \subset C^0(I,M)$ is a continuous map with respect to the L^p-topology, $1 \le p \le \infty$, in $\mathcal{A}$ and the C^0-topology in the target space.*

Proof. Consider a sequence (x_n,u_n) in $M \times \mathcal{A}$ such that $(x_n,u_n) \to (x,u) \in M \times \mathcal{A}$. By the Nash isometric embedding theorem, we can consider $\mathbb{R}^N$ with the Euclidean topology as the target space of the curves γ in (2.12). Since $x_n \to x$, we can take a relatively compact neighbourhood $U \subset \mathbb{R}^N$ of x such that $(x_n) \subset U$. The embedding being closed (see [6]), the vector fields X_0, $f \circ X_i$ can be extended outside $U \cap M$ to smooth bounded vector fields on $\mathbb{R}^N$ with compact support which are therefore globally Lipschitz on $\mathbb{R}^N$. Let us denote by L_U the maximum of the Lipschitz constants of these vector fields. Let us consider the system (2.12) with these modified vector fields and let us still denote by S the corresponding solution operator, $S : \mathbb{R}^N \times \mathcal{A} \to W^{1,p}(I,\mathbb{R}^N)$. Let then $C_U > 0$ the maximum of the C_0-norms of these modified vector fields. We have, for all t in the compact interval I:

$$\begin{aligned} |S(x_n,u_n)(t) - S(x,u)(t)| \le |x_n - x| + C_U \int_0^t |(u_0)_n - u_0| \mathrm{d}s + L_U \int_0^t |u_0||S(x_n,u_n) - S(x,u)| \mathrm{d}s \\ + C_U \sum_{i=1}^{d} \int_0^t |(u_i)_n - u_i| \mathrm{d}s + L_U \sum_{i=1}^{d} \int_0^t |u_i||S(x_n,u_n) - S(x,u)| \mathrm{d}s. \end{aligned}$$

By the Gronwall inequality we then get

$$|S(x_n,u_n)(t) - S(x,u)(t)| \le b_n(t) \mathrm{e}^{\int_0^t a(s) \mathrm{d}s},$$

where

$$\begin{aligned} b_n(t) &= |x_n - x| + C_U \Big(\int_0^t \big(|(u_0)_n - u_0| + \sum_{i=1}^{d} |(u_i)_n - u_i| \big) \mathrm{d}s \Big) \\ a(t) &= L_U \sum_{i=0}^{d} |u_i(t)|. \end{aligned}$$

Since

$$\int_I \big(|(u_0)_n - u_0| + \sum_{i=1}^{d} |(u_i)_n - u_i| \big) \mathrm{d}s \to 0$$

and $x_n \to x$, we conclude that $S(x_n,u_n) \to S(x,u)$ in the C^0-topology with the target space $\mathbb{R}^N$. Hence there exists $\delta > 0$ such that $S(x_n,u_n)$ and $S(x,u)$ restricted to $[0,\delta]$ have image contained in U and are then also solution of the initial system (2.12). We can then repeat the above argument with initial points $S(x_n,u_n)(\delta)$

and $S(x,u)(\delta)$ and a neighbourhood of $S(x,u)(\delta)$. Being $S(x,u)$ compact in M in a finite number of step we can conclude that $S(x_n,u_n)\to S(x,u)$ in the C^0-topology. □

Proposition 2.7. *$S\colon M\times\mathcal{A}\to W^{1,p}(I,M)$, $1\le p\le\infty$, is a continuous map.*

Proof. Let (x_n,u_n) in $M\times\mathcal{A}$ such that $(x_n,u_n)\to(x,u)\in M\times\mathcal{A}$. From Lemma 2.6 the curves $S(x_n,u_n)$ are contained in a compact subset K of M. This allows us to control the distance associated to g_0 both from below and above by the Euclidean distance in $\mathbb{R}^N$. Hence it is enough to prove the remaining convergence needed using the Euclidean metric in $\mathbb{R}^N$. The vector fields $X_0|_K$, $(f\circ X_i)|_K$, can be extended to smooth bounded vector fields on $\mathbb{R}^N$ with compact support inside an open neighbourhood of K in R^N. These extended vector fields (that will be denoted with the same symbols X_0, $f\circ X_i$) are therefore globally Lipschitz on $\mathbb{R}^N$. Let us denote by L and C the maximum of their Lipschitz constants and C^0 norms respectively.

Let us also denote the derivatives of the curves $S(x,u)$ and $S(x_n,u_n)$ by $\dot S(x,u)$ and $\dot S(x_n,u_n)$, respectively.

From the equation in (2.12) and the triangle inequality we obtain

$$\begin{aligned}|\dot S(x_n,u_n)-\dot S(x,u)| &\le |(u_0)_nX_0\circ S(x_n,u_n)-u_0X_0\circ S(x,u)|\\ &\qquad+\sum_{i=1}^d|(u_i)_nf\circ X_i\circ S(x_n,u_n)-u_if\circ X_i\circ S(x,u)|\\ &\le |(u_0)_n-u_0||X_0\circ S(x_n,u_n)|+|u_0||X_0\circ S(x_n,u_n)-X_0\circ S(x,u)|\\ &\qquad+\sum_{i=1}^d|(u_i)_n-u_i||f\circ X_i\circ S(x_n,u_n)|+\sum_{i=1}^d|u_i||f\circ X_i\circ S(x_n,u_n)-f\circ X_i\circ S(x,u)|.\end{aligned}$$

Using the Lipschitz constant and the bound on the vector fields it follows

$$|\dot S(x_n,u_n)-\dot S(x,u)|\le C\big((u_0)_n-u_0|+\sum_{i=1}^d|(u_i)_n-u_i|\big)+L\big(|u_0|+\sum_{i=1}^d|u_i|\big)|S(x_n,u_n)-S(x,u)|.$$

Finally with Jensen's inequality we get

$$\begin{aligned}|\dot S(x_n,u_n)-\dot S(x,u)|^p\le (2(d+1))^{p-1}\Big[&C^p\big(|(u_0)_n-u_0|^p+\sum_{i=1}^d|(u_i)_n-u_i|^p\big)\\ &+L^p\big(|u_0|^p+\sum_{i=1}^d|u_i|^p\big)|S(x_n,u_n)-S(x,u)|^p\Big],\end{aligned}$$

from which the convergence of $S(x_n,u_n)$ to $S(x,u)$ in $W^{1,p}(I,\mathbb{R}^N)$ and consequently in $W^{1,p}(I,M)$ easily follows. □

Let us define the *endpoint map*

$$F\colon M\times\mathcal{A}\to M\times M,\quad (x,u)\mapsto\big(x,S(x,u)(1)\big).$$

Remark 2.8. The uniform convergence in Proposition 2.7 especially implies that F is continuous.

The space under consideration here is

$$\Lambda^p:=F^{-1}(\triangle_M),$$

where $\triangle_M:=\{(x,x)\in M\times M\}$ denotes the diagonal in the product $M\times M$ together with the induced topology.

Definition 2.9. Let E, B be topological spaces. A continuous map $\pi\colon E \to B$ is a *Hurewicz fibration* if it satisfies the *homotopy lifting property*, *i.e.* for every topological spaces Z and every homotopy $H\colon Z \times [0,1] \to B$ with a continuous lift $\tilde{H}_0\colon Z \to E$ of $H(\cdot, 0)\colon Z \to B$, relative to π, *i.e.* $\pi \circ \tilde{H}_0 = H(\cdot, 0)$, there exists a continuous lift $\tilde{H}\colon Z \times [0,1] \to E$ of $H\colon Z \times [0,1] \to E$ relative to π.

The main result of this section is the following:

Theorem 2.10. *Let Assumptions (A1)–(A2) hold. Then the endpoint map*

$$F|_{\Lambda^p}\colon \Lambda^p \to \triangle_M$$

is a Hurewicz fibration for every $1 \leq p < \infty$.

Theorem 2.10 was proved in [2] for $p = 2$ and $X_0 = 0$; the space of paths between two points satisfying an affine control system of the type (2.9), with u_0 fixed equal to 1, has been studied in [7] for $p = 1$ and in [1] for $p > 1$ (p below a "critical" exponent related to the step of the distribution $\mathcal{D}$); the case of horizontal paths defined by a completely nonholonomic distribution and endowed with the uniform topology was previously considered in [8], where credit was given to [9] for certain ideas employed.

For the proof, we need the following proposition, which can be seen as the counterpart for the differential inclusion (1.2) of [7], Lemma 1 and [1], Proposition 2 valid for an affine control system. We emphasize that in our case the upper bound on p, existing for affine control systems, is not present. We use a non-smooth implicit function theorem developed by F. H. Clarke. Let us introduce some related notions.

Let $A \subset \mathbb{R}^n$ be an open subset and $G : A \to \mathbb{R}^m$ be a Lipschitz map. Let $\Omega_G \subset A$ be the set of points where G is not differentiable. Let $J_G(y)$ be the Jacobian matrix of G at a point $y \in A \setminus \Omega_G$. Following [10], Definition 2.6.1, we say that the *generalized Jacobian* at $x \in A$, denoted with $\partial G(x)$ is the convex hull of all the matrices obtained as limits of $J_G(x_i)$ as $x_i \to x$, $x_i \notin \Omega_G$. The generalized Jacobian at x is said to be of *maximal rank* if all the matrices in $\partial G(x)$ have maximal rank. The following theorem is [10], Theorem 7.1.1.

Theorem 2.11 (Non-smooth inverse function theorem by Clarke)**.**
Let $\partial G(x_0)$ be of maximal rank, then there exist neighborhoods V and U of x_0 and $G(x_0)$, respectively, and a Lipschitz function $H : U \to \mathbb{R}^n$ such that

(i) $G\big(H(u)\big) = u$ *for every* $u \in U$*;*
(ii) $H\big(G(v)\big) = v$ *for every* $v \in V$*.*

We use Theorem 2.11 in the proof of Lemma 2.12 below, where we show local injectivity at $0 \in \mathbb{R}^m$ of a map $G(x, \cdot) : \mathbb{R}^m \to M$ given as the composition of endpoint maps associated to the flows of vector fields that are sections of $\mathcal{H}$ in (1.1). Let us introduce the map G.

Recalling Remark 2.1, let $h \in \mathbb{N}$ such that $x_0 \in U_{z_h}$ and let us take a local basis $(Y_j)_{j \in \{1,\dots,m-1\}}$ for $\Gamma(\mathcal{D})$ defined in U_{z_h}, and $Y_m := [Y_1, Y_2]$. For each $j \in \{1, \dots, m-1\}$, let $(a_{ji})_{i \in \{1,\dots,d\}} : U_{z_h} \to \mathbb{R}$ such that on U_{z_h} we have both

$$Y_j = \sum_{i=1}^{d} a_{ji} f \circ X_i, \quad \text{and} \quad \big|(Y_j)\big|_0^2 = \sum_{i=1}^{d} a_{ji}^2 \tag{2.13}$$

(recall (2.2)). We notice that the vector field $a_j\colon U_{z_h} \to \mathbb{R}^d$, having components $\big(a_{j1}(x), \dots, a_{jd}(x)\big)$ and which realizes (2.13), is smooth. In fact, for each $x \in U_{z_h}$, $a_j(x)$ is the vector in the affine subspace $f^{-1}(Y_j(x))$ of $\mathbb{R}^d$ which is orthogonal to the kernel of f on the fibre $\mathbf{U}_x$. Since f and Y_j are both smooth, it follows that this vector smoothly varies with $x \in U_{z_h}$.

We extend the vector fields Y_j (after restricting them to a smaller open set U_{z_h}) without changing the supremum of their g_0-norms, to obtain global sections Y_j of $\mathcal{D}$.

For each $j \in \{1, \dots, m\}$, any $\xi_j, \xi_{m_1}, \xi_{m_2} \in \mathbb{R}$ and any $x \in M$ let us consider

$$\begin{aligned} Q_j(\xi_j)(x) &:= \mathrm{e}^{\xi_j^2 \tilde{X}_0 + \xi_j \tilde{Y}_j}(x), \qquad \text{for } j = 1, \dots, m-1, \\ Q_m(\xi_{m_1}, \xi_{m_2})(x) &:= \mathrm{e}^{\xi_{m_2}^2 \tilde{X}_0 + \xi_{m_2} \tilde{Y}_2} \circ \mathrm{e}^{\xi_{m_1}^2 \tilde{X}_0 + \xi_{m_1} \tilde{Y}_1} \circ \mathrm{e}^{\xi_{m_2}^2 \tilde{X}_0 - \xi_{m_2} \tilde{Y}_2} \circ \mathrm{e}^{\xi_{m_1}^2 \tilde{X}_0 - \xi_{m_1} \tilde{Y}_1}(x), \end{aligned} \tag{2.14}$$

being, for $X \in \Gamma(TM)$, $\mathrm{e}^X(x)$ the value at $t = 1$ of the flow of X that passes through x at $t = 0$; the vector fields $\tilde{X}_0$ and $\tilde{Y}_j$ are respectively equal to $X_0/(m+3)$, and $Y_j/(m+3)$, for $j \in \{1, \dots, m-1\}$. We define

$$G : M \times \mathbb{R}^m \to M, \qquad G(x, \xi) = Q_m(\mathrm{sgn}(\xi_m)\sqrt{|\xi_m|}, \sqrt{|\xi_m|}) \circ Q_{m-1}(\xi_{m-1}) \circ \dots \circ Q_1(\xi_1)(x) \tag{2.15}$$

Lemma 2.12. *Let Assumptions (A1)–(A2) hold. Then for each $x_0 \in M$, there exists a neighbourhood V of x_0 and a continuous map $\psi\colon V \times V \to \mathbb{R}^m$,such that $G(x, \psi(x, y)) = y$ and $\psi(x, x) = 0$ for all $x, y \in V$.*

Proof. From the Baker-Campbell-Hausdorff formula (see, *e.g.*, [11], p. 28) for ξ_{m_1}, ξ_{m_2} small enough, we have

$$Q_m(\xi_{m_1}, \xi_{m_2}) = \mathrm{e}^{2(\xi_{m_1}^2 + \xi_{m_2}^2)\tilde{X}_0 - \xi_{m_1}\xi_{m_2}[\tilde{Y}_1, \tilde{Y}_2] + \text{terms of degree } \geq 3 \text{ in } \xi_{m_1},\ \xi_{m_2}}.$$

Thus, G is Lipschitz on a neighbourhood of $(x_0, 0)$ and $G(x, 0) = x$. We notice that actually G is differentiable w.r.t. ξ, for each $\xi \neq 0$. Let us then consider the map

$$\tilde{G}\colon M \times \mathbb{R}^m \to M \times M, \quad \tilde{G}(x, \xi) = \big(x, G(x, \xi)\big).$$

It admits generalized Jacobian $\partial \tilde{G}(x, 0)$ given by

$$\begin{bmatrix} I & 0 \\ \partial_x G(x, 0) & A(\lambda, x, 0) \end{bmatrix}$$

where $A(\lambda, x, 0)$ is the linear matrix pencil

$$\lambda \in [0, 1] \mapsto \begin{bmatrix} \tilde{Y}_1(x) \\ \dots \\ \tilde{Y}_{m-1}(x) \\ -4\lambda \tilde{X}_0(x) + 4(1-\lambda)\tilde{X}_0(x) - \tilde{Y}_m(x) \end{bmatrix}$$

We notice that $\partial \tilde{G}(x_0, 0)$ is of maximal rank. In fact, as both $\omega_{x_0}(\tilde{X}_0)$ and $\omega_{x_0}(\tilde{Y}_m)$ are negative (recall (2.3)), then $-4\tilde{X}_0(x_0) - \tilde{Y}_m(x_0)$ is transversal to $\mathcal{D}_{x_0}$. Since $\mathcal{D}_{x_0}$ is generated by the vectors in the first $m-1$ rows of $A(\lambda, x, 0)$, the maximality of the rank of $\partial \tilde{G}(x_0, 0)$ is ensured if $\omega_{x_0}(4\tilde{X}_0 - \tilde{Y}_m) > 0$. Recalling (2.3)–(2.5), this follows from (2.7) as

$$\omega_{x_0}(4\tilde{X}_0 - \tilde{Y}_m) = -\frac{4}{m+3}\|\omega_{x_0}\| - \frac{1}{(m+3)^2}\omega_{x_0}([Y_1, Y_2]) > -\frac{4\Omega}{m+3} + \frac{\lambda}{(m+3)^2}.$$

By Theorem 2.11 applied to $\tilde{G}$, there exists then a neighbourhood V of x_0 and a Lipschitz function $\psi\colon V \times V \to \mathbb{R}^m$ such that

$$G(x, \psi(x, y)) = y, \quad \text{for all } x, y \in V.$$

□

From Lemma 2.12 we get the existence of a local cross-section for the endpoint map F.

Proposition 2.13. *Let Assumptions (A1)–(A2) hold. Then for each $x_0 \in M$, there exists a neighbourhood V of x_0 and a continuous maps $\sigma\colon V \times V \to \mathcal{A}$, such that $F(x, \sigma(x, y)) = (x, y)$ and $\sigma(x, x) = 0$ for all $x, y \in V$.*

Proof. Let $\psi = (\psi_1, \ldots, \psi_m)$ the map in Lemma 2.12. It defines, for each $x, y \in V$, a vector $\xi \in \mathbb{R}^m$ which appears in the definition of G in (2.15). We can then associate to this vector the continuous piecewise smooth curve $\tilde{\gamma}_{x,y} : [0, m+3] \to M$ connecting x to y, obtained as the concatenation of the flow lines, defined on intervals of length 1 of the $m + 3$ vector fields appearing in (2.14). Since ψ is Lipschitz, these flow lines, depending on x and y, uniformly converge on their unit interval of definition (see the proof of [12], Thm. 2.1, p. 94), *i.e.* the map

$$(x, y) \in V \times V \mapsto \tilde{\gamma}_{x,y} \in C^0([0, m + 3], M),$$

where $C^0([0, m + 3], M)$ is endowed with uniform convergence topology, is continuous. We can reparametrise $\tilde{\gamma}$ on the interval $[0, 1]$, by taking $\gamma_{x,y}(t) := \tilde{\gamma}_{x,y}((m + 3)t)$. Notice that $\gamma_{x,y}$ is the concatenation of the flow lines of the vector fields appearing in (2.14) with X_0 and Y_j that replace, respectively, $\tilde{X}_0$, and $\tilde{Y}_j$; moreover each flow line is parametrized on an interval of length $1/(m + 3)$. Let us then define (recall (2.13)–(2.15))

$$(x, y) \in V \times V \mapsto \sigma(x, y) : [0, 1] \to \mathbb{R}^{d+1}$$

$$\sigma(x,y)(t) = \begin{cases} \big(\psi_j^2(x,y), \psi_j(x,y)a_{j1}(\gamma_{x,y}(t)), \ldots, \psi_j(x,y)a_{jd}(\gamma_{x,y}(t))\big), \\ \qquad\qquad t \in \big[(j-1)/(m+3), j/(m+3)\big),\ j \in \{1, \ldots, m-1\} \\ \big(|\psi_m(x,y)|, -\epsilon_m\sqrt{|\psi_m(x,y)|}a_{11}(\gamma_{x,y}(t)), \ldots, -\epsilon_m\sqrt{|\psi_m(x,y)|}a_{1d}(\gamma_{x,y}(t))\big), \\ \qquad\qquad t \in \big[(m-1)/(m+3), m/(m+3)\big) \\ \big(|\psi_m(x,y)|, -\sqrt{|\psi_m(x,y)|}a_{21}(\gamma_{x,y}(t)), \ldots, -\sqrt{|\psi_m(x,y)|}a_{2d}(\gamma_{x,y}(t))\big), \\ \qquad\qquad t \in \big[m/(m+3), (m+1)/(m+3)\big) \\ \big(|\psi_m(x,y)|, \epsilon_m\sqrt{|\psi_m(x,y)|}a_{11}(\gamma_{x,y}(t)), \ldots, \epsilon_m\sqrt{|\psi_m(x,y)|}a_{1d}(\gamma_{x,y}(t))\big), \\ \qquad\qquad t \in \big[(m+1)/(m+3), (m+2)/(m+3)\big) \\ \big(|\psi_m(x,y)|, \sqrt{|\psi_m(x,y)|}a_{21}(\gamma_{x,y}(t)), \ldots, \sqrt{|\psi_m(x,y)|}a_{2d}(\gamma_{x,y}(t))\big), \\ \qquad\qquad t \in \big[(m+2)/(m+3), 1\big] \end{cases}$$

where $\epsilon_m := \text{sign}(\psi_m(x, y))$. Let J_j, $j \in \{1, \ldots, m + 3\}$ be one of the intervals of the variable t in the definition above. We notice that by construction (recall (2.8) and (2.13))

$$\sup_{t \in J_j} \sum_{i=1}^{d} \big(a_{ji}(\gamma_{x,y}(t))\big)^2 \leq K^2,$$

hence $\sigma(x, y) \in \mathcal{A}$. Moreover, since the map $(x, y) \in V \times V \mapsto \gamma_{x,y}|_J \in C^0(J, M)$, is continuous and the functions a_{ji} are smooth, each map

$$(x, y) \in V \times V \mapsto a_{ij} \circ (\gamma_{x,y})|_J \in C^0(J, \mathbb{R}),$$

is continuous (the target space is endowed with the topology of uniform convergence). Then the map

$$(x, y) \in V \times V \mapsto \sigma(x, y) \in \mathcal{A}$$

is continuous when $\mathcal{A}$ is endowed with the L^p-topology, $1 \le p \le \infty$. □

We are now ready to prove Theorem 2.10.

Proof of Theorem 2.10. Since $\triangle_M \cong M$ is a metric space, by the Hurewicz uniformization theorem [13], it is enough to show that $F|_{\Lambda^p}$ is locally a Hurewicz fibration. Thus, it suffices to prove that the covering homotopy condition (recall Def. 2.9) holds locally, *i.e.* for any $x \in M$ there exists a neighbourhood $W \subset M$, such that given a topological space Z, a homotopy $H\colon Z \times [0,1] \to \triangle_W$ and a lift $\tilde{H}_0\colon Z \to F^{-1}(\triangle_W)$ of $H(\cdot,0)$, there exists a continuous map $\tilde{H}\colon Z \times [0,1] \to F^{-1}(\triangle_W)$ with $F \circ \tilde{H} = H$ and $\tilde{H}(\cdot,0) = \tilde{H}_0$.

Let then $x_0 \in M$ be given and choose W according to Proposition 2.13.

Let $h := \mathrm{pr}_1 \circ H$, where $\mathrm{pr}_1 : M \times M \to M$ is the projection on the first factor, and let $\tilde{h}_0 : Z \to \mathcal{A}$ be the second component of $\tilde{H}_0$. Consider, for each $\zeta \in Z$ and $s \in [0,1]$,

$$\sigma\big(h(\zeta,s), h(\zeta,0)\big) \star \tilde{h}_0(\zeta) \in \mathcal{A},$$

where $\star$ is the concatenation operation in Definition 2.5. We notice that $S\Big(h(\zeta,s), \sigma\big(h(\zeta,s), h(\zeta,0)\big) \star \tilde{h}_0(\zeta)\Big)$ is a path, satisfying (1.2), from $h(\zeta,s)$ to $\mathrm{pr}_1\big(H(\zeta,0)\big)$, *i.e.*

$$F\Big(h(\zeta,s), \sigma\big(h(\zeta,s), h(\zeta,0)\big) \star \tilde{h}_0(\zeta)\Big) = \big(h(\zeta,s), h(\zeta,0)\big).$$

Thus, we can complete the control $\sigma\big(h(\zeta,s), h(\zeta,0)\big) \star \tilde{h}_0(\zeta)$ to a loop control $c(\zeta,s)$ pointed at $h(\zeta,s)$ by taking

$$c(\zeta,s) := \Big(\sigma\big(h(\zeta,s), h(\zeta,0)\big) \star \tilde{h}_0(\zeta)\Big) \star \sigma\big(h(\zeta,0), h(\zeta,s)\big). \tag{2.16}$$

Let $\tilde{H}_1 : Z \times [0,1] \to \mathcal{A}$ be

$$\tilde{H}_1(\zeta,s)(t) := \begin{cases} c(\zeta,s)(t/s) & t \in [0, s/4) \\ c(\zeta,s)\big((t+1-s)/(4-3s)\big) & t \in [s/4, 1-s/2] \\ c(\zeta,s)(t/s - 1/s + 1) & t \in (1-s/2, 1] \end{cases} \tag{2.17}$$

Let us observe that $\tilde{H}_1(\zeta,1) = c(\zeta,1)$ and $\tilde{H}_1(\zeta,0)(t) = c(\zeta,0)\big((t+1)/4\big)$. As a piecewise reparametrization of concatenation maps by affine functions, $\tilde{H}_1$ is continuous on $Z \times (0,1]$ (this can be proved by applying the same argument used to estimate integral (2.21) below). In order to conclude the proof we need to show the continuity of $\tilde{H}_1$ at $(\bar{\zeta},0)$ for all $\bar{\zeta} \in Z$. Indeed, defining

$$\tilde{H} : Z \times [0,1] \to F^{-1}(\triangle_W), \quad \tilde{H}(\zeta,s) := \big(h(\zeta,s), \tilde{H}_1(\zeta,s)\big),$$

we notice that $F(\tilde{H}(\zeta,s)) = (h(\zeta,s), h(\zeta,s)) = H(\zeta,s)$; moreover the second component of $\tilde{H}(\zeta,0)$ is equal to $\tilde{H}_1(\zeta,0) = c(\zeta,0)\big((t+1)/4\big)$ which is equal to $\tilde{h}_0(\zeta)$ by (2.16), hence $\tilde{H}(\zeta,0) = \tilde{H}_0(\zeta)$.

Claim. *The map $\tilde{H}_1 : Z \times [0,1] \to \mathcal{A}$ in (2.17) is continuous, with $\mathcal{A}$ endowed with the L^p-topology, $1 \le p < \infty$.*

Let $(\zeta,s) \in Z \times [0,1]$. If $s = 0$ then

$$\int_0^1 |\tilde{H}_1(\zeta,0) - \tilde{H}_1(\bar{\zeta},0)|^p \mathrm{d}t = \int_0^1 |\tilde{h}_0(\zeta) - \tilde{h}_0(\bar{\zeta})|^p \mathrm{d}t, \tag{2.18}$$

and since $\tilde{h}_0$ is continuous in the L^p -topology of the target space, (2.18) is less than any $\epsilon > 0$ in a neighbourhood U_1 of $\bar{\zeta}$. Let now $s > 0$ and let us consider

$$\int_0^1 |\tilde{H}_1(\zeta, s) - \tilde{H}_1(\bar{\zeta}, 0)|^p \mathrm{d}t \tag{2.19}$$

Decomposing (2.19) in the sum of three integrals on the intervals $[0, s/4]$, $[s/4, 1-s/2]$ and $[1-s/2, 1]$ and changing the variable in the first and the last one, we get

$$\begin{aligned}
&\int_0^1 |\tilde{H}_1(\zeta, s) - \tilde{H}_1(\bar{\zeta}, 0)|^p \mathrm{d}t \\
&= \int_0^{1/4} s\big|c(\zeta, s)(\tau) - c(\bar{\zeta}, 0)\big((s\tau + 1)/4\big)\big|^p \mathrm{d}\tau \qquad (2.20)\\
&\quad + \int_{s/4}^{1-s/2} \big|c(\zeta, s)\big((t + 1 - s)/(4 - 3s)\big) - c(\bar{\zeta}, 0)\big((t + 1)/4\big)\big|^p \mathrm{d}t \qquad (2.21)\\
&\quad + \int_{1/2}^{1} s\big|c(\zeta, s)(\tau) - c(\bar{\zeta}, 0)\big((s(\tau - 1) + 2)/4\big)\big|^p \mathrm{d}\tau \qquad (2.22)
\end{aligned}$$

Let us analyse the integral in (2.20). Since c is continuous in $(\bar{\zeta}, 0)$ there exists a neighbourhood $U_2 \times [0, \delta)$ of $(\bar{\zeta}, 0)$ such that c is bounded in the L^p-norm and therefore, up to decreasing δ, $\int_0^{1/4} s|c(\zeta, s)(\tau)|^p \mathrm{d}\tau$ is less than any fixed $\epsilon > 0$ on such a neighbourhood. On the other hand,

$$\int_0^{1/4} s\big|c(\bar{\zeta}, 0)\big((s\tau + 1)/4\big)\big|^p \mathrm{d}\tau = 4\int_{1/4}^{(s+4)/16} |c(\bar{\zeta}, 0)(w)|^p \mathrm{d}w$$

which is also arbitrarily small for s small enough. The integral in (2.22) can be estimated analogously. For the integral in (2.21), we add and subtract $c(\bar{\zeta}, s)\big((t + 1 - s)/(4 - 3s)\big)$ inside the absolute value and we notice that

$$\int_{s/4}^{1-s/2} \big|[c(\zeta, s) - c(\bar{\zeta}, s)]\big((t + 1 - s)/(4 - 3s)\big)\big|^p \mathrm{d}t = (4 - 3s)\int_{1/4}^{1/2} \big|\big(c(\zeta, s) - c(\bar{\zeta}, s)\big)(\tau)\big|^p \mathrm{d}\tau$$

can be made arbitrarily small in a neighbourhood of $(\bar{\zeta}, 0)$ by the continuity of c in the L^p-topology of the target space. It remains to show that

$$\int_{s/4}^{1-s/2} \big|c(\bar{\zeta}, s)\big((t + 1 - s)/(4 - 3s)\big) - c(\bar{\zeta}, 0)\big((t + 1)/4\big)\big|^p \mathrm{d}t$$

is arbitrarily small for s in a sufficiently small neighbourhood of 0. It is enough to show that for any sequence (s_n) converging to 0, it holds

$$\int_{s_n/4}^{1-s_n/2} \big|c(\bar{\zeta}, s_n)\big((t + 1 - s_n)/(4 - 3s_n)\big) - c(\bar{\zeta}, 0)\big((t + 1)/4\big)\big|^p \mathrm{d}t \to 0.$$

Let $\chi_{[1/4,1/2]}$ be the characteristic function of the interval $[1/4, 1/2]$, and, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\phi_n : [0, 1] \to \mathbb{R}, \quad \phi_n(t) := (t + 1 - s_n)/(4 - 3s_n),\\
&f_n : [0, 1] \to [0, +\infty), \quad f_n := \chi_{[1/4,1/2]} \circ \phi_n\big|c(\bar{\zeta}, s_n) \circ \phi_n\big|^p.
\end{aligned}$$

Let $A \subset [0,1]$ be a measurable set. Then we have

$$\begin{aligned}\int_A f_n(t)\mathrm{d}t &= (4-3s_n)\int_{\phi_n^{-1}(A)} \chi_{[1/4,1/2]}|c(\bar\zeta,s_n)|^p \mathrm{d}\tau \\ &\leq 2^p(4-3s_n)\int_{\phi_n^{-1}(A)} \chi_{[1/4,1/2]}|c(\bar\zeta,s_n)-c(\bar\zeta,0)|^p\mathrm{d}\tau + 2^p(4-3s_n)\int_{\phi_n^{-1}(A)}\chi_{[1/4,1/2]}|c(\bar\zeta,0)|^p\mathrm{d}\tau\end{aligned}$$

Since the measures of the sets $\phi_n^{-1}(A)$ are controlled from above by the measure of A, we can use the absolute continuity of the last integral and the fact that $c(\bar\zeta,s_n)\to c(\bar\zeta,0)$ in the L^p-norm, to conclude that the sequence f_n is uniformly integrable on $[0,1]$. This implies that the sequence

$$\{\chi_{[1/4,1/2]}\circ\phi_n(t)\big|c(\bar\zeta,s_n)\circ\phi_n(t)-c(\bar\zeta,0)\big((t+1)/4\big)\big|^p\}_n$$

is also uniformly integrable on $[0,1]$. As $c(\bar\zeta,s_n)\to c(\bar\zeta,0)$ in L^p, there exists a subsequence, still denoted by s_n, such that $c(\bar\zeta,s_n)\to c(\bar\zeta,0)$ a.e. on $[0,1]$. Hence, for any subsequence of (s_n), we have that there exists another subsequence such that

$$\chi_{[1/4,1/2]}\circ\phi_n(t)\big|c(\bar\zeta,s_n)\circ\phi_n(t)-c(\bar\zeta,0)\big((t+1)/4\big)\big|^p\to 0 \ \text{ a.e. on } [0,1].$$

By Vitali convergence theorem (see [14], p. 133) we conclude that

$$\begin{aligned}&\int_{s_n/4}^{1-s_n/2}\big|c(\bar\zeta,s_n)\big((t+1-s_n)/(4-3s_n)\big)-c(\bar\zeta,0)\big((t+1)/4\big)\big|^p\mathrm{d}t \\ &\qquad = \int_0^1 \chi_{[1/4,1/2]}\circ\phi_n(t)\big|c(\bar\zeta,s_n)\circ\phi_n(t)-c(\bar\zeta,0)\big((t+1)/4\big)\big|^p\mathrm{d}t\to 0.\end{aligned}$$

□

We notice that Proposition 2.13 gives also the following

Proposition 2.14. *Let Assumptions (A1)–(A2) hold. Then for each $\bar x\in M$ the endpoint map*

$$F_{\bar x}:\mathcal{A}\to M,\qquad F_{\bar x}(u):=S(\bar x,u)(1)$$

is a Hurewicz fibration.

Proof. As in the proof of Theorem 2.10, it is enough to show that the covering homotopy condition holds locally. Using the same notation as in the above proof, replacing $\triangle_W$ with W and $F^{-1}(\triangle_W)$ with $F_{\bar x}^{-1}(W)$, we consider now

$$c(\zeta,s):=\tilde H_0(\zeta)\star\sigma\big(H(\zeta,0),H(\zeta,s)\big).$$

and

$$\tilde H(\zeta,s)(t):=\begin{cases} c(\zeta,s)\big(t/(2-s)\big) & t\in[0,1-s/2]\\ c(\zeta,s)\big((t-1+s/2)/s+1/2\big) & t\in(1-s/2,1]\end{cases}$$

□

3. Homotopy equivalences of the based and the free loop space

Let us denote by $\Lambda^p_{\rm st}$ the free loop space of M endowed with the $W^{1,p}$ topology, *i.e.*

$$\Lambda^p_{\rm st} := \{\gamma \in W^{1,p}(I,M) : \gamma(0) = \gamma(1)\}.$$

Let $x, y \in M$. We consider also the set

$$\Omega^{p,\rm st}_{x,y} := \{\gamma \in W^{1,p}(I,M) : \gamma(0) = x,\ \gamma(1) = y\}$$

endowed with the $W^{1,p}$-topology and

$$\Omega^p_{x,y} := F_x^{-1}(y) \subset \mathcal{A}.$$

Let ΩM be the based loop space of M with base point $\bar{x} \in M$ endowed with the compact-open topology and let $i\colon W^{1,p}(I,M) \to C^0(I,M)$ be the canonical embedding.

Finally, let

$$\Omega_{F_{\bar{x}}} := \{(u,\omega) \in \mathcal{A} \times M^I : F_{\bar{x}}(u) = \omega(0)\},$$

where M^I is the space of parametrized paths in M endowed with the compact-open topology. Let us denote by Λ a lifting function belonging to $F_{\bar{x}}$, *i.e.*

$$\Lambda : \Omega_{F_{\bar{x}}} \to \mathcal{A}^I \quad \text{such that} \quad \Lambda(u,\omega)(0) = u \quad \text{and} \quad F_{\bar{x}}\big(\Lambda(u,\omega)(t)\big) = \omega(t),$$

for each $t \in I$ (see [13]), where $\mathcal{A}^I$ is the path space in $\mathcal{A}$ endowed with the compact-open topology. We have the following lemma whose proof is an application of [7], Lemma 2.

Lemma 3.1. *The map* $\eta\colon \Omega M \to F_{\bar{x}}^{-1}(\bar{x})$, $\eta(\gamma) := \Lambda\big(0,\gamma\big)(1)$, *is a homotopy equivalence with a homotopy inverse given by the map* $i \circ \big(S(\bar{x},\cdot)|_{F_{\bar{x}}^{-1}(\bar{x})}\big)$.

Proof. By Proposition 2.14 and [7], Lemma 2, it is enough to show that $\mathcal{A}$ is contractible. This follows immediately by considering the map

$$H\colon \mathcal{A} \times [0,1] \to \mathcal{A}, \qquad H(u,s)(t) := \begin{cases} u(t) & t \in [0, 1-s] \\ 0 & t \in (1-s, 1] \end{cases}$$

It is easy to see that H is continuous and it is a homotopy between the identity of $\mathcal{A}$ and the constant control $0 \in \mathcal{A}$. □

Using Lemma 3.1, we get:

Theorem 3.2. *Let Assumptions (A1)–(A2) hold. Then for all* $1 \le p < \infty$ *and any* $x,\ y \in M$, *the space* $\Omega^p_{x,y}$ *has the homotopy type of a CW-complex. In particular, the map* $S(x,\cdot)|_{\Omega^p_{x,y}}$ *is a homotopy equivalence between* $\Omega^p_{x,y}$ *and* $\Omega^{p,\rm st}_{x,y}$.

Proof. It is well-known that the inclusion $j\colon \Omega^{p,\rm st}_{\bar{x},\bar{x}} \to \Omega M$ is a weak homotopy equivalence and both the spaces $\Omega^{p,\rm st}_{\bar{x},\bar{x}}$ and ΩM have the homotopy type of a CW-complex, thus j is also a homotopy equivalence. From Lemma 3.1 and by the arbitrariness of the base point $\bar{x}$, we have that $\Omega^p_{x,x}$ has then the homotopy type of a CW-complex for every $x \in M$. Since all the fibers in a Hurewicz fibration with path connected base have the same homotopy type (see, *e.g.*, [15], Prop. 4.61) we have that $\Omega^p_{x,y}$ has the homotopy type of a CW-complex. Let us consider the following commutative diagram between Hurewicz fibrations

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F_x} & M \\ \big\downarrow{\scriptstyle S(x,\cdot)} & & \big\downarrow{\scriptstyle i_M} \\ W^{1,p}_x(I,M) & \xrightarrow{\tilde F_x} & M \end{array}$$

where $W^{1,p}_x(I,M)$ is the space of paths in $W^{1,p}(I,M)$ starting at x and $\tilde F_x$ is the endpoint map on $W^{1,p}_x(I,M)$. By the naturality of the long exact sequences of Hurewicz fibrations (see, *e.g.* [16], p. 6) we then get, for each $x, y \in M$ and $k \in \mathbb{N}$, the following commutative diagram:

$$\begin{array}{ccccccc} \longrightarrow \pi_{k+1}(M) & \longrightarrow & \pi_k(\Omega^p_{x,y}) & \longrightarrow & \pi_k(\mathcal{A}) & \xrightarrow{(F_x)_*} & \pi_k(M) \\ \big\downarrow & & \big\downarrow{\scriptstyle (S(x,\cdot)|_{\Omega^p_{x,y}})_*} & & \big\downarrow{\scriptstyle (S(x,\cdot))_*} & & \big\downarrow \\ \longrightarrow \pi_{k+1}(M) & \longrightarrow & \pi_k(\Omega^{p,\mathrm{st}}_{x,y}) & \longrightarrow & \pi_k(W^{1,p}_x(I,M)) & \xrightarrow{(\tilde F_x)_*} & \pi_k(M) \end{array}$$

Since both $\mathcal{A}$ and $W^{1,p}_x(I,M)$ are contractible the above diagram reduces to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{k+1}(M) & \longrightarrow & \pi_k(\Omega^p_{x,y}) & \longrightarrow & 0 & \longrightarrow & \pi_k(M) \\ & & \big\downarrow & & \big\downarrow{\scriptstyle (S(x,\cdot)|_{\Omega^p_{x,y}})_*} & & & & \big\downarrow \\ 0 & \longrightarrow & \pi_{k+1}(M)) & \longrightarrow & \pi_k(\Omega^{p,\mathrm{st}}_{x,y}) & \longrightarrow & 0 & \longrightarrow & \pi_k(M) \end{array} \tag{3.1}$$

which from the five lemma implies that $S(x,\cdot)|_{\Omega^p_{x,y}}$ is a weak homotopy equivalence and then by Whitehead theorem a homotopy equivalence. □

From Theorems 2.10 and 3.2, we get the following analogous for the space Λ of [2], Theorem 10, which concerns the horizontal free loop space.

Theorem 3.3. *Let Assumptions (A1)–(A2) hold. Then for all $1 \le p < \infty$, Λ^p and Λ^p_{st} are homotopy equivalent via the map $S|_{\Lambda^p}$.*

Proof. Since $F|_{\Lambda^p}$ is a Hurewicz fibration, after identifying $\triangle_M$ with M, and $F^{-1}(x,x)$ with $\Omega^p_{x,x}$ we have the following long exact sequence

$$\ldots \to \pi_k(\Omega^p_{x,x}) \to \pi_k(\Lambda^p) \xrightarrow{(F|_{\Lambda^p})_*} \pi_k(M) \to \pi_{k-1}(\Omega^p_{x,x}) \to \ldots$$

From (3.1), we see that $\pi_k(\Omega^p_{x,x})$ is isomorphic to $\pi_{k+1}(M)$, for each $k \ge 0$. Moreover, the map $x \in M \mapsto (x,0) \in \Lambda^p$ is continuous and then $(F|_{\Lambda^p})_*$ is surjective. Thus the long exact sequence above splits as

$$0 \to \pi_{k+1}(M) \to \pi_k(\Lambda^p) \to \pi_k(M) \to 0.$$

Now, Λ^p has the homotopy type of a CW-complex since both M and any fiber $\Omega^p_{x,x}$ do have it as well, thus the fact that the map $S|_{\Lambda^p}$ is a homotopy equivalence follows, by Whitehead's theorem, if the induced map $(S|_{\Lambda^p})_* \colon \pi_k(\Lambda^p) \to \pi_k(\Lambda^p_{\mathrm{st}})$ is an isomorphism for each $k \ge 0$. This is a consequence of the following diagram, which is commutative by the naturality of the long exact sequences of Hurewicz fibrations:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{k+1}(M) & \longrightarrow & \pi_k(\Lambda^p) & \longrightarrow & \pi_k(M) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle (S|_{\Lambda^p})_*} & & \big\downarrow & & \\ 0 & \longrightarrow & \pi_{k+1}(M) & \longrightarrow & \pi_k(\Lambda^p_{\mathrm{st}}) & \longrightarrow & \pi_k(M) & \longrightarrow & 0 \end{array}$$

□

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Data availability statement

The research data associated with this article are included in the article.

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