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MEAN FIELD TYPE CONTROL PROBLEMS, SOME HILBERT-SPACE-VALUED FBSDES, AND RELATED EQUATIONS

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Abstract. In this article, we provide an original systematic global-in-time analysis of mean field type control problems on $\mathbb{R}^n$ with generic cost functions allowing quadratic growth by a novel "lifting" approach which is not the same as the traditional lifting. As an alternative to the recent popular analytical method of tackling master equations, we resolve the control problem in a proper Hilbert subspace of the whole space of L^2 random variables, it can be regarded as a tangent space attached at the initial probability measure. The problem is linked to the global solvability of the Hilbert-space-valued forward–backward stochastic differential equation (FBSDE), which is solved by variational techniques here. We also rely on the Jacobian flow of the solution to this FBSDE to establish the regularity of the value function, including its linearly functional differentiability, which leads to the classical well-posedness of the Bellman equation. Together with the linear functional derivatives and the gradient of the linear functional derivatives of the solution to the FBSDE, we also obtain the classical well-posedness of the master equation. Our current approach imposes structural conditions directly on the cost functions. The contributions of adopting this framework in our study are twofold: (i) compared with imposing conditions on Hamiltonian, the structural conditions imposed in this work are easily verified, and less demanding on the cost functions while solving the master equation; and (ii) when the cost functions are not convex in the state variable or there is a lack of monotonicity of cost functions, an accurate lifespan can be provided for the local existence, which may not be that small in many cases.

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1. Introduction

1.1. Background and Existing literature

Modeling collective behavior in systems is challenging due to high computational costs. To address this, a mean field approach was introduced by Huang-Malhamé-Caines [48] and Lasry-Lions [49] independently. Every mean field type control problem, also known as the McKean–Vlasov control problem, involves a dynamical

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system whose state is affected by its own probability measure on $\mathbb{R}^n$. Numerous researchers have made significant contributions to the modeling and techniques in the form of books or journal papers. A fundamental model is the linear quadratic setting, which has been investigated by Bensoussan–Sung–Yam–Yung [1] (see also Bensoussan–Frehse–Yam [2]) and Carmona–Delarue–Lachapelle [3]. They solved both linear quadratic mean field control problems and mean field games, where the former handled the N-dimensional case and the latter was in the 1-dimensional case; further studies can be found in Pham [4], Yong [5], and other references. Most of these works resolved the problems by relating the control problems to Riccati-type differential equations, which were then solved under certain positive definiteness of the cost functions. See also [6] for the closed-loop solvability of the stochastic linear quadratic optimal control problem governed by stochastic evolution equations, and the explicit expression for the closed-loop optimal strategy by considering the Riccati equations.

One of the approaches to solving mean field control problems is to extend the stochastic maximum principle to the mean field setting, which allows us to specify the necessary conditions for optimality in terms of the Hamiltonian. However, due to the non-Markovian nature of the state dynamics involving the mean field term, this approach typically requires a certain convexity of the Hamiltonian. This direction was first studied by Andersson–Djehiche [7], Li [8] in the convex action space, and Buckdahn–Djehiche–Li [9] in general settings by using the spike variation. Many researchers have extended the stochastic maximum principle to various settings, including [10–13]. The sufficient condition for optimality involves the solvability of the forward dynamics together with the adjoint backward dynamics, subject to the first order condition. This system is called the mean field (or McKean–Vlasov) forward–backward stochastic differential equations (FBSDEs). The mean field stochastic differential equations (SDEs) were studied in Buckdahn–Li–Peng–Rainer [14]. The mean field backward stochastic differential equations (BSDEs) were first investigated in Buckdahn–Djehiche–Li–Peng [15] and Buckdahn–Li–Peng [16], while the well-posedness of the coupled mean-field FBSDEs was first studied in Carmona–Delarue [17] for rather general assumptions, where they did not require convexity of the coefficients but imposed certain boundedness on the coefficients with respect to the state variable. Later, Carmona–Delarue [12] removed this boundedness assumption and established the well-posedness of FBSDEs under the framework of linear forward dynamics and convex coefficients of the backward dynamics. Bensoussan–Yam–Zhang [18] also addressed the well-posedness of FBSDEs with another set of generic assumptions on the coefficients.

Another approach to addressing mean field control problems with general cost functions and state dynamics involves characterizing the problem using the infinite-dimensional Bellman equation. Bensoussan–Frehse–Yam [2, 19, 20] derived and discussed the Bellman equation, as well as the master equation for mean field problems, under the non-linear-quadratic framework, both with and without the presence of common noise. An independent approach in this direction was also provided by Laurière–Pironneau [21] in earlier work. However, due to the nonlinear dependence of the cost functions on the mean field term of the state, the state process may exhibit time inconsistency, rendering the usual Bellman optimality principle inapplicable. To tackle this issue, several researchers, including Cosso–Gozzi–Kharroubi–Pham–Rosestolato [22, 23], Djete–Possamaï-Tan [24], and Pham–Wei [25, 26], developed the dynamic programming principle in the general setting. For instance, Cosso–Gozzi–Kharroubi–Pham–Rosestolato [22] and Pham–Wei [25, 26] demonstrated the verification theorem and proved that the value function satisfies the Bellman equation in the sense of viscosity. Moreover, Pham–Wei [25, 26] established the uniqueness of the viscosity solution by using the standard comparison principle for viscosity solutions. Cosso–Gozzi–Kharroubi–Pham–Rosestolato [23] verified the uniqueness by employing the smooth variational principle in Wasserstein space. Wu–Zhang [27] introduced a new definition of viscosity solution, which differs from the Crandall–Lions definition by imposing the maximum/minimum condition on the compact set of semimartingale measures on the Wasserstein space. This modification aims to reduce the difficulty of establishing uniqueness. Readers can also find the weak solutions of master equations in Mou–Zhang [28] and Ciampa–Rossi [29]. The primary difference between the theory of classical solutions and that of viscosity solutions is that the latter may not be smooth enough for one to verify that it is the solution of the original control problem. Enhancing the regularity of viscosity solutions to meet classical regularity is usually very hard. To the best of our knowledge, before our work, results concerning the improvement of the regularity of viscosity solutions of master equations associated with mean field problems in the classical sense are not so immediate. Readers can also see some non-uniqueness results in Bardi-Fischer [30].

1.2. Our contributions and approach

One of the main contributions of this article is the novel "lifting" approach which provides a convenient method to establish the global well-posedness of classical solutions for the Bellman and master equations linked to the mean field-type control problems, see Theorems 5.1 and 5.6. This is done under the displacement monotonicity condition as outlined in Assumption $\mathbf{b(v)}^\dagger$'s (3.16), with generic cost functions exhibiting quadratic growth. Additionally, we offer a precise lifespan for well-posedness when dealing with non-convex cost functions. Let's electorate further into these contributions.

First, our lifting approach differs from the conventional one, it does not lift all the randomness to the L^2-space of random variables, but only the part corresponding to the Brownian motion evolving but not the initial. It gives a convenient way to write the Bellman and master equations, achieved by setting the initial data to be the identity. For instance, one can refer to the equivalent conditions in Assumption $\mathbf{b(v)}^\dagger$'s (3.16) and Assumption $\mathbf{B(v)}$'s (3.27) to observe that the latter (lifting version) takes a much simpler form than the former. The lifting approach allows us to use the variational technique to solve the mean field FBSDE by formulating it as a control problem, it streamlines a lot of technical issues of proving the well-posedness of the mean field FBSDE.

Second, comparing to imposing conditions on the Hamiltonian, the structural conditions (3.4)–(3.16), Assumptions (5.2)–(5.4) imposed on the cost functions in this work are easily verified, and less demanding on the cost functions while solving the master equation. When the cost functions are not convex in the state variable, the structural conditions also allow one to write an accurate lifespan for the local existence, which may not be that small in many cases, see (3.20). As this lifespan is determined from the convexity condition of the control problem rather than relying on a fixed-point theorem which typically necessitates a very small lifespan. Therefore, the lifespan provided in (3.20) should find applicability in various scenarios. This corresponding lifespan may be difficult to compute if we impose conditions on the Hamiltonian for the non-convex case.

Third, we employ the displacement monotonicity specified in Assumption $\mathbf{b(v)}^\dagger$'s (3.16). While the important work [31] also explores second-order master equations under displacement monotonicity, dealing with common noise and a non-separable Hamiltonian, but their framework excludes cost functions with quadratic growth and requires the Hamiltonian to be C^5. In contrast, our approach allows such cost functions and requires less regularity. Unlike [32], this paper allows generic cost functions involving the state and control variables with quadratic growth instead of explicit quadratic functions with only a control variable (without the state variable). More precisely, we adopt to $l(x, v) + F(m)$ in this paper instead of $v^2 + F(m)$ as in [32], where v is the control variable and F is the functional on $\mathcal{P}_2(\mathbb{R}^n)$. So, in [32], the feedback control can be explicitly expressed in terms of the costate, which is the solution to the BSDE, utilizing the first order condition of the control problem. However, in our current work, the feedback control can only be expressed by the state and costate using the implicit function theorem. This requires a more intensive effort in the estimate and analysis of this work, especially in establishing the regularity of the value function. Moreover, we also have to impose assumptions on the cost functions carefully, which allow quadratic growth and do not require the convexity with respect to the state variable, see (3.4), (3.6) (3.8), (3.9) and (3.30). With these assumptions, we can develop some new estimates, such as (3.41) and (A.49), to show the regularity of the value function. Consequently, the results of this paper are thus applicable to a much broader class of scenarios than [32], particularly for the models under and also beyond the standard linear quadratic setting. See, for example, applications in [1], such as the centralized socially optimal control problem in [33], the mean field congestion and aversion in pedestrian crowds in [34], and the particle filtering problem in [35]. Furthermore, the work [36] also permits cost functions with quadratic growth, but it adopts the Lasry–Lions monotonicity, whereas our approach is grounded on displacement monotonicity.

The current study involves two types of noises: the initial probability measures and the independent Wiener process that models the evolving local noise. The state of the system depends on both the initial data and the local noise. Unlike our former article [37], we cannot simply generalize the "deterministic case" dealt with in there, that is, the control problem cannot be naively handled in an arbitrary Hilbert space. To overcome this hurdle, we separate to study (1) the initial condition of a controlled diffusion process and (2) the evolutionary part of the process from two different perspectives; namely, we consider the initial condition of a probability measure as a parameter, while the evolving part generated by the local noise of Wiener process is treated as an element in a specially designated Hilbert space. The big advantage of this approach is that it remains a control problem in a Hilbert space, which is not the whole space of L^2 random variables, but a tangent space-like structure attached to the initial probability measure. However, a more common approach, such as that in [14] calculates the derivative of the value function with respect to m by using its lifted version on a Hilbert space, and this differs from our approach here.

The inspiring idea of "lifting" was first introduced by P.-L. Lions [38] in his lectures at Collège de France (see also Carmona–Delarue [39], Cardialaguet–Delarue–Lasry–Lions [40]), for the purpose of studying alternatively the Wasserstein derivatives on the space of measures; it reformulates the problem on the Wasserstein metric space of measures to another one but now on a linear space. Instead of finding a gradient flow of measures, one can look for an optimal flow in the Hilbert space of random variables, whose laws match the optimal flow of measures; we instead consider the controlled stochastic processes in a space of random vector fields, each of which is attached to the initial probability measure via a pushforward operation on the space of measures. These processes can be considered as tangents to the space of measures at the initial measure m. The optimal costate then gives the gradient of the value function in the same Hilbert space, which then projects down to a derivative on the Wasserstein space of measures by letting the initial random field be the identity map. This Hilbert space approach turns out to be quite powerful in obtaining the Bellman equation and the master equation of mean field type control problems. More precisely, we here aim to separate the derivation into two parts: the optimal control on a Hilbert space on the one hand, and the differentiation with respect to the initial distribution on the other, so that we can avoid certain stringent technical assumptions on the cost functions regarding the differentiability with respect to the measure argument. Besides, the regularity of the value function on the Hilbert space can be translated to that of the corresponding problem on the Wasserstein space, which guarantees the equivalence of solving the master equation in the Hilbert space with that lies in the Wasserstein space.

In mean field control problems under usual strict convexity settings, a unique optimal control exists, as demonstrated in Proposition 3.5, through the use of variational techniques. The maximum principle guarantees the existence of optimal control and is linked to the solvability of the associated FBSDEs for the dynamics of the state and costate (adjoint process). It is worth noting that in mean field games, quite differently, due to the lack of variational techniques warranted in mean field control setting, we have to solve the corresponding FBSDE by gluing consecutive local solutions to obtain a global one, through bounding the sensitivity of the adjoint process; see [41] for details. Our global results on an arbitrary time horizon rely on certain monotonicity assumptions on the cost functions with respect to the measure argument and the usual convexity in control of cost functions. Some of these assumptions exhibit similarity to the displacement monotonicity concept used in [42], particularly in relation to (3.16). Further details can be found in Remark 3.1.

Moreover, the assumption we make on the boundedness of the second-order Fréchet derivative of $F(X \otimes m)$ with respect to X in (3.23) is not that restrictive, where F is a functional on the space of probability measure and $X \otimes m$ is the pushforward of the probability measure m by X. Under this assumption, we see that $D_X^2 F(X \otimes m)$ is a bounded linear operator from $\mathcal{H}_m$ to $\mathcal{H}_m$. In the languages of linear functional derivative on $\mathbb{R}^n$, it is sufficient to assume that $\nabla_x^2 \frac{dF}{d\nu}(m)(x)$ and $\nabla_{\tilde{x}} \nabla_x \frac{d^2F}{d\nu^2}(m)(x, \widetilde{x})$ are bounded for any (m, x) and $(m, x, \widetilde{x})$, respectively. For instance, $F(m) := \int_{\mathbb{R}} \left(x^2 + e^{-x^2}\right) dm(x)$, clearly far from linear quadratic, satisfies our proposed assumptions. Another plausible but highly restrictive assumption is to assume that the third-order Fréchet derivative of $F(X \otimes m)$ with respect to X is a bounded operator from $\mathcal{H}_m \times \mathcal{H}_m$ to $\mathcal{H}_m$; if one adopts it, it slightly reduces the technicality, but it puts very restrictive limitations on F; indeed, under such assumption, the derivative $\nabla_x^3 \frac{dF}{d\nu}(m)(x)$ must be zero, which certainly reduces the family of cost functions to be that of linear quadratic only but nothing more. Therefore, we must not involve this here. Our present setting adopts $F(X \otimes m)$ with a bounded second-order Fréchet derivative only, which covers a wide range of practical applications, while also greatly simplifying the mathematical technicalities, and making the methodology more transparent. Most importantly, it motivates all the estimates involved when we formulate the problem on Wasserstein space, see also our author's latest work [43], in return to allow a more general setting, both generic in driving dynamics and cost functions, than the present one stated here, for instance, without assuming the boundedness of the second-order Fréchet derivative of $F(X \otimes m)$ but requiring the boundedness of the second-order Wasserstein gradient of $F(m)$, though more tedious analysis has to be involved as a sacrification. To this end, we also refer to another very interesting work [42] which handled master equations on Wasserstein space under the same dynamically controlled process as formulated here without Brownian noise.

Finally, we indicate some significant differences between this article and the very interesting work [36]. First, the current Hilbert space approach does not lift all the randomness to the L^2-space of random variables as in [36], but only the part corresponding to the Brownian motion evolving but not the initial. It gives a convenient way to write the Bellman and master equations as it can be done by simply putting the initial data to be the identity. Second, we allow a certain sense of non-convexity of cost functions in the state variable x, see Assumptions $\mathbf{A(v)}$'s (3.8), $\mathbf{A(vi)}$'s (3.9), $\mathbf{b(v)}^*$'s (3.15), $\mathbf{b(v)}^\dagger$'s (3.16) and also (3.30). Third, our master equation is different from that in (5.23) (see also Thm. 60 in [36]) while the former is more analytic (sufficient conditions) and the latter is more probabilistic (necessary conditions), see more discussion in Section 4.4 in [19] about the differences. Computations in this current work are tractable and self-contained. Fourth, we adopt the displacement monotonicity (3.16) instead of the Lasry–Lions monotonicity used in [36]. Fifth, when the cost functions are not convex in the state variable, we can calculate an accurate lifespan for the well-posedness, which may not be that small in many cases. Last but not least, we believe that the methodology of this work can be applied to more generic frameworks, we here choose a convenient setting so that most of the ideas can be illustrated clearly.

1.3. Organization of the Article

This article is organized as follows. In Section 2, we review basic calculus in the Wasserstein and Hilbert spaces. In Section 3, we introduce the mean field type control problem and establish its connection with the associated FBSDE. Section 4 verifies some regularities of the value function, which facilitates the formulation of the Bellman equation. We investigate the classical solvability of the Bellman and master equations in Section 5. Proofs of various assertions in Sections 2 to 5 are provided in the Appendix A. We would like to emphasize that the methodology presented in this article can be extended to study problems where common noise is present, as well as cases with a generic driving drift function and more general cost functions.

2. Calculus in hilbert space of L^2-random variables

2.1. 2-Wasserstein space of probability measures

Consider the space of probability measures m's on $\mathbb{R}^n$, each of which has a finite second moment, *i.e.* $\int_{\mathbb{R}^n} |x|^2 \mathrm{d}m(x) < \infty$. For any such m and m', the 2-Wasserstein metric of them is defined by

$$W_2(m, m') := \inf_{\pi \in \Pi_2(m,m')} \left[\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - x'|^2 \pi(\mathrm{d}x, \mathrm{d}x') \right]^{1/2},$$

where $\Pi_2(m, m')$ is the set of joint probability measures with respective marginals m and m'. This space equipped with the 2-Wasserstein metric is denoted by $\mathcal{P}_2(\mathbb{R}^n)$. The infimum is attainable such that there are random variables $\hat{X}_m$ and $\hat{X}_{m'}$ (they may be dependent) associated with m and m' respectively so that $W_2^2(m, m') = \mathbb{E}\left| \hat{X}_m - \hat{X}_{m'} \right|^2$. We recall the fact that a sequence of measures $\left\{ m_k \right\}_{k \in \mathbb{N}}$ converges to m in $\mathcal{P}_2(\mathbb{R}^n)$ if and only if it converges in the sense of weak convergence and simultaneously $\int_{\mathbb{R}^n} |x|^2 \mathrm{d}m_k(x) \longrightarrow \int_{\mathbb{R}^n} |x|^2 \mathrm{d}m(x)$.

2.2. Functionals and their derivatives on $\mathcal{P}_2(\mathbb{R}^n)$

Consider a continuous functional $F(m)$ on $\mathcal{P}_2(\mathbb{R}^n)$, we study the concept of several derivatives of $F(m)$. Firstly, the linear functional derivative of $F(m)$ at m is a function $(m, x) \in \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \longmapsto \frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$, being continuous under the product topology, satisfying $\int_{\mathbb{R}^n} \left| \frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x) \right|^2 \mathrm{d}m(x) \leq c(m)$, for some positive constant $c(m)$, depending locally on m, which is uniformly bounded on compacta; in the rest of this article, all other

$c(m)$'s represent different constants yet possessing the same boundedness nature. Besides, the Fréchet derivative L_F of $F(m)$ with respect to m (if exists) is given as usual: for any $m,\, m' \in \mathcal{P}_2(\mathbb{R}^n)$, we have

$$\left|\frac{F\big(m+\epsilon(m'-m)\big)-F(m)}{\epsilon}-L_F(m-m')\right|\longrightarrow 0 \qquad \text{as } \epsilon\to 0. \tag{2.1}$$

For if the densities of m and m' exist, denoted by f_m and $f_{m'}$ respectively. By Riesz representation theorem on $L^2(\mathbb{R}^n)$, there is a $L^2(\mathbb{R}^n)$ function $\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$, called the linear functional derivative, such that

$$L_F(m-m')=L_F(f_m-f_{m'})=\int_{\mathbb{R}^n}\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)\big[f_m(x)-f_{m'}(x)\big]\,\mathrm{d}x=\int_{\mathbb{R}^n}\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)\big[\mathrm{d}m(x)-\mathrm{d}m'(x)\big]. \tag{2.2}$$

The subset of $\mathcal{P}_2(\mathbb{R}^n)$ consisting of all measures $m\in\mathcal{P}_2(\mathbb{R}^n)$ with $L^2(\mathbb{R}^n)$ densities is clearly dense in $\mathcal{P}_2(\mathbb{R}^n)$, as long as L_F is closable, the representation in (2.2) can be extended to all measures. Regarding to (2.2), the linear functional derivative $\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$ is actually the linear functional derivative $\frac{\delta F}{\delta m}(m)(x)$ defined in Carmona–Delarue [39][1]. Further, (2.1) and (2.2) give

$$\frac{\mathrm{d}}{\mathrm{d}\theta}F\big(m+\theta(m'-m)\big)=\int_{\mathbb{R}^n}\frac{\mathrm{d}F}{\mathrm{d}\nu}\big(m+\theta(m'-m)\big)(x)\big[\mathrm{d}m'(x)-\mathrm{d}m(x)\big]. \tag{2.3}$$

We next turn to the notion of the second-order derivative, if $\frac{\mathrm{d}}{\mathrm{d}\theta}F\big(m+\theta(m'-m)\big)$ is continuously differentiable in θ, then we can write the formula

$$\frac{d^2F}{\mathrm{d}\theta^2}\big(m+\theta(m'-m)\big)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{d^2F}{\mathrm{d}\nu^2}\big(m+\theta(m'-m)\big)(x,\widetilde{x})\big[\mathrm{d}m'(x)-\mathrm{d}m(x)\big]\big[\mathrm{d}m'(\widetilde{x})-\mathrm{d}m(\widetilde{x})\big], \tag{2.4}$$

where the map $(m,x,\widetilde{x})\in\mathcal{P}_2(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n\longmapsto\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})$, called the second-order linear functional derivative, is continuous and satisfies

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right|^2\mathrm{d}m(x)\mathrm{d}m(\widetilde{x})\le c(m). \tag{2.5}$$

Moreover, we have the formula

$$\begin{aligned}F(m')-F(m)=&\int_{\mathbb{R}^n}\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)\Big[\mathrm{d}m'(x)-\mathrm{d}m(x)\Big]\\&+\int_0^1\int_0^1\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\theta\frac{d^2F}{\mathrm{d}\nu^2}\Big(m+\lambda\theta(m'-m)\Big)(x,\widetilde{x})\Big[\mathrm{d}m'(x)-\mathrm{d}m(x)\Big]\Big[\mathrm{d}m'(\widetilde{x})-\mathrm{d}m(\widetilde{x})\Big]\mathrm{d}\lambda\mathrm{d}\theta.\end{aligned}$$

Formulas (2.2) and (2.3) do not change if we add a constant to $\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$, and we assume the normalization that $\int_{\mathbb{R}^n}\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)\mathrm{d}m(x)=0$. Similarly, in Formulas (2.4) and (2.5), we can replace $\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})$ by $\frac{1}{2}\left(\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})+\frac{d^2F}{\mathrm{d}\nu^2}(m)(\widetilde{x},x)\right)$ without a change of value, and we simply assume that the function $(x,\widetilde{x})\longmapsto$

[1]Formula (2.8) yields, by replacing X by $\mathcal{I}_x$, $\frac{dF}{d\nu}(m)(x)-\frac{dF}{d\nu}(m)(0)=\int_0^1 D_XF(\mathcal{I}\otimes m)(\theta x)\cdot x\,\mathrm{d}\theta$, which can serve as a definition of the linear functional derivative when the Fréchet derivative exists.

$\frac{d^2F}{d\nu^2}(m)(x,\widetilde{x})$ is symmetric. Likewise, adding a function of the form $\varphi(x)+\varphi(\widetilde{x})$ to a symmetric function $\frac{d^2F}{d\nu^2}(m)(x,\widetilde{x})$ would result no change in value.

2.3. Hilbert space of L^2-random variables

Our working Hilbert space is $\mathcal{H}_m := L^2\big(\Omega,\mathcal{A},\mathbb{P};L^2_m(\mathbb{R}^n;\mathbb{R}^n)\big)$ of the atomless probability space $\big(\Omega,\mathcal{A},\mathbb{P}\big)$. An element of $\mathcal{H}_m$ is denoted by $X_x = X(x,\omega)$ which maps $\mathbb{R}^n\times\Omega$ to $\mathbb{R}^n$; for each x, X_x is a $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^n)$ random variable. The inner product over $\mathcal{H}_m$ is defined by $\langle X,Y\rangle_{\mathcal{H}_m} := \mathbb{E}\left[\int_{\mathbb{R}^n} X\cdot Y\mathrm{d}m(x)\right]$. The deterministic pushforward[2] probability measure of $m\otimes\mathbb{P}$ by X on $\mathbb{R}^n$ is denoted by $X\#(m\otimes\mathbb{P})\in\mathcal{P}_2(\mathbb{R}^n)$. As $\mathbb{P}$ is fixed, we simply suppress $\mathbb{P}$ and write $X\otimes m$ for short. For any test function φ, we have $\int_{\mathbb{R}^n}\varphi(\xi)dX\otimes m(\xi) = \mathbb{E}\left(\int_{\mathbb{R}^n}\varphi(X_x)\mathrm{d}m(x)\right)$. In addition, for any $X\in\mathcal{H}_m$, the norm of X is written as $\|X\|_{\mathcal{H}_m} = \sqrt{\langle X,X\rangle_{\mathcal{H}_m}}$. Given any functional F on $\mathcal{P}_2(\mathbb{R}^n)$, the map $X\longmapsto F(X\otimes m)$ is now a functional on $\mathcal{H}_m$. Since $X\longmapsto X\otimes m$ is continuous from $\mathcal{H}_m$ to $\mathcal{P}_2(\mathbb{R}^n)$, the functional will be continuous as long as $F(m)$ is continuous on $\mathcal{P}_2(\mathbb{R}^n)$. Its Fréchet derivative (if exists) $D_XF(X\otimes m)$, as an element of $\mathcal{H}_m$, satisfies

$$\frac{F\big((X+\epsilon Y)\otimes m\big)-F(X\otimes m)}{\epsilon}\to\Big\langle D_XF(X\otimes m),Y\Big\rangle_{\mathcal{H}_m},\quad\text{as }\epsilon\to 0,\text{ for any }Y\in\mathcal{H}_m.$$

Also, we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}F\big((X+\theta Y)\otimes m\big)=\Big\langle D_XF\big((X+\theta Y)\otimes m\big),Y\Big\rangle_{\mathcal{H}_m}. \tag{2.6}$$

If $F(m)$ has a linear functional derivative $\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$ which is differentiable in x for each $m\in\mathcal{P}_2(\mathbb{R}^n)$, then it is easy to convince oneself that when $(m,x)\longmapsto\nabla_x\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$ is continuous with the growth condition $\left|\nabla_x\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)\right|\le c(m)(1+|x|)$, where ∇_x is the usual differential operator with respect to x in $\mathbb{R}^n$ and $c(m)$ is bounded on compacta from $\mathcal{P}_2(\mathbb{R}^n)$, then

$$D_XF(X\otimes m)(\cdot,\omega)=\nabla_x\frac{\mathrm{d}F}{\mathrm{d}\nu}(X\otimes m)(x)\Bigg|_{x=X(\cdot,\omega)}, \tag{2.7}$$

which belongs $\mathcal{H}_m$; we refer readers to Section 2.1 in [20] for a rigorous proof of (2.7). Note that we see that $D_XF(X\otimes m)(u,\omega)$ is a $\sigma(X(\cdot,\cdot))$-measurable random variable, accounted in the augmented probability space $(\mathbb{R}^n\times\Omega,\mathcal{B}\times\mathcal{A},m\otimes\mathbb{P})$. On the other hand, it can also be viewed as a limiting function of the linear combinations of products of a function of $X(u,\omega)$ and a multivariate function of several integrals of different test functions against the measure $X\otimes m$. The linear functional derivative can be computed by the formula

$$\frac{\mathrm{d}F}{\mathrm{d}\nu}(X\otimes m)(X_u)-\frac{\mathrm{d}F}{\mathrm{d}\nu}(X\otimes m)(0)=\int_0^1 D_XF(X\otimes m)(\theta X_u)\cdot X_u\mathrm{d}\theta,\qquad\text{for any }u\in\mathbb{R}^n. \tag{2.8}$$

The identity function $\mathcal{I}_{\boldsymbol{\cdot}}$ is a particular choice of $X_{\boldsymbol{\cdot}}$ being a constant random variable, such that for any $x\in\mathbb{R}^n$, $\mathcal{I}_x=x$ and hence $\mathcal{I}_x\otimes m=m$. Therefore, (2.7) becomes $D_XF(m)(x,\omega)=D_XF(\mathcal{I}\otimes m)(x,\omega)=\nabla_x\dfrac{\mathrm{d}F}{\mathrm{d}\nu}(\mathcal{I}\otimes m)(x)=\nabla_x\dfrac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$, it is identical to the L-derivative $\partial_mF(m)(x)$ in Carmona–Delarue [39]; the L-derivative can be interpreted as the Fréchet derivative of the functional $F(X\otimes m)$ on $\mathcal{H}_m$ evaluated at $X=\mathcal{I}_x$.

[2] To follow the usual custom, we use the notation # to denote the pushforward operation in the measure space. Nevertheless, we use $\otimes$ instead of # to emphasize the bilinear structure of $X\otimes m$ in the space of random variables X and the space of signed measures m. These two notations have their own importance. In future work, we may use $\otimes$ and $\otimes$ interchangeably.

On the other hand, the L-derivative can also be realized by taking a X_x which is totally independent of x, by then $X \otimes m = \mathcal{L}_X$ (the law of X) for every m, so $F(X \otimes m) = F(\mathcal{L}_X)$, and then $D_X F(X \otimes m) = \partial_\nu F(\mathcal{L}_X)(X)$.

Now we consider all X_x such that $\frac{X_x}{(1+|x|^2)^{\frac{1}{2}}} \in L^\infty(\mathbb{R}^n; L^2\big(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)\big)$, which means that $\mathbb{E}|X_x|^2 \leq c(X)(1+|x|^2)$, where $c(X)$ is a positive constant depending on $X(\cdot,\cdot)$ but independent of x. Then $X \in \mathcal{H}_m$ for every $m \in \mathcal{P}_2(\mathbb{R}^n)$, and so X and m can be considered as two separate arguments in $F(X \otimes m)$. Further, we can consider the functional $m \longmapsto F(X \otimes m)$ and its linear functional derivative $\frac{\partial F}{\partial m}(X \otimes m)(x) = \frac{\delta F}{\delta \mu}(X \otimes \mu)(x)\Big|_{\mu=m}$, which is distinguished from $\frac{\mathrm{d}F}{\mathrm{d}\nu}(X \otimes m)(x) = \frac{\delta F}{\delta \nu}(\nu)(x)\Big|_{\nu = X\otimes m}$; in fact, by Proposition 2.12 in [32], when $m \longmapsto \frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$ is continuous and has at most quadratic growth $\left|\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)\right| \leq c(m)(1+|x|^2)$, we have $\frac{\partial F}{\partial m}(X \otimes m)(x) = \mathbb{E}\left[\frac{\mathrm{d}F}{\mathrm{d}\nu}(X \otimes m)(X_x)\right]$.

Remark 2.1. To facilitate further development, we allow the abuse of notations that $D_X F(Y \otimes m)$ always mean the Fréchet derivative of $F(\cdot \otimes m)$ with respect to $\cdot$; even for the extreme case, the symbol $D_X F(Y_X \otimes m)$ represents $D_X F(X \otimes m)\big|_{X=Y_X}$.

2.4. Second-order Gâteaux derivative in the Hilbert space

A functional $F(X \otimes m)$ is said to have a second-order Gâteaux derivative in $\mathcal{H}_m$, denoted by $D_X^2 F(X \otimes m) \in \mathcal{L}(\mathcal{H}_m; \mathcal{H}_m)$, if for any $Y \in \mathcal{H}_m$, it holds that

$$\frac{\Big\langle D_X F((X+\epsilon Y)\otimes m) - D_X F(X \otimes m), Y\Big\rangle_{\mathcal{H}_m}}{\epsilon} \longrightarrow \Big\langle D_X^2 F(X \otimes m)(Y), Y\Big\rangle_{\mathcal{H}_m}, \tag{2.9}$$

as $\epsilon \to 0$. For any Z and $W \in \mathcal{H}_m$, $D_X^2 F(X \otimes m)(Z)$ is generally defined by

$$\Big\langle D_X^2 F(X \otimes m)(Z), W\Big\rangle_{\mathcal{H}_m} = \frac{1}{4}\Big\langle D_X^2 F(X \otimes m)(Z+W), Z+W\Big\rangle_{\mathcal{H}_m} - \frac{1}{4}\Big\langle D_X^2 F(X \otimes m)(Z-W), Z-W\Big\rangle_{\mathcal{H}_m}.$$

Therefore, $D_X^2 F(X \otimes m)$ is clearly self-adjoint by definition. Also

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Big\langle D_X F((X+\theta Z)\otimes m), W\Big\rangle_{\mathcal{H}_m} = \Big\langle D_X^2 F((X+\theta Z)\otimes m)(Z), W\Big\rangle_{\mathcal{H}_m}. \tag{2.10}$$

Then, (2.10) and (2.6) imply $\frac{d^2 F}{\mathrm{d}\theta^2}\big((X+\theta Y)\otimes m\big) = \Big\langle D_X^2 F((X+\theta Y)\otimes m)(Y), Y\Big\rangle_{\mathcal{H}_m}$. Hence, we can rewrite the Taylor's expansion for F as follows:

$$F\Big((X+Y)\otimes m)\Big) = F(X \otimes m) + \Big\langle D_X F(X \otimes m), Y\Big\rangle_{\mathcal{H}_m} + \int_0^1 \int_0^1 \theta \Big\langle D_X^2 F((X+\theta\lambda Y)\otimes m)(Y), Y\Big\rangle_{\mathcal{H}_m} \mathrm{d}\theta \mathrm{d}\lambda. \tag{2.11}$$

From (2.7), it yields $\Big\langle D_X F(X \otimes m), W\Big\rangle_{\mathcal{H}_m} = \mathbb{E}\left[\int_{\mathbb{R}^n} \nabla_x \frac{\mathrm{d}F}{\mathrm{d}\nu}(X \otimes m)(X_x) \cdot W_x \mathrm{d}m(x)\right]$. It follows that

$$\begin{aligned}
&\frac{1}{\epsilon}\Big\langle D_X F((X+\epsilon Z)\otimes m) - D_X F(X \otimes m), W\Big\rangle_{\mathcal{H}_m} \\
&= \frac{1}{\epsilon}\mathbb{E}\left[\int_{\mathbb{R}^n}\left(\nabla_x \frac{\mathrm{d}F}{\mathrm{d}\nu}((X+\epsilon Z)\otimes m)(X_x+\epsilon Z_x) - \nabla_x \frac{\mathrm{d}F}{\mathrm{d}\nu}(X \otimes m)(X_x)\right)\cdot W_x \mathrm{d}m(x)\right],
\end{aligned}$$

further, by assuming the existences of $\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})$, its derivatives $\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})$ and $\nabla_x^2\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$, together with continuity properties, taking limits of both sides will give

$$\begin{aligned}\left\langle D_X^2F(X\otimes m)(Z),W\right\rangle_{\mathcal{H}_m} &= \mathbb{E}\left[\int_{\mathbb{R}^n}\nabla_x^2\frac{\mathrm{d}F}{\mathrm{d}\nu}(X\otimes m)(X_x)Z_x\cdot W_x\mathrm{d}m(x)\right]\\ &\quad+\mathbb{E}\left\{\widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(X\otimes m)(X_x,\widetilde{X}_{\tilde{x}})\widetilde{Z}_{\tilde{x}}\cdot W_x\mathrm{d}m(\widetilde{x})\mathrm{d}m(x)\right]\right\},\end{aligned}\tag{2.12}$$

in which $\left(\widetilde{X}_{\tilde{x}},\widetilde{Z}_{\tilde{x}}\right)$ is an independent copy of $\left(X_x,Z_x\right)$; also refer to the detailed discussions in Section 2.1 of [20]. We can write, consequently, that

$$D_X^2F(X\otimes m)(Z)=\nabla_x^2\frac{\mathrm{d}F}{\mathrm{d}\nu}(X\otimes m)(X_x)Z_x+\widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(X\otimes m)(X_x,\widetilde{X}_{\tilde{x}})\widetilde{Z}_{\tilde{x}}\mathrm{d}m(\widetilde{x})\right].\tag{2.13}$$

We notice the following measurability property of $D_X^2F(X\otimes m)(Z)$:

$$D_X^2F(X\otimes m)(Z)=M(X)Z+L(X,Z),$$

where $M(X)$ is a matrix-valued $\sigma(X(\cdot,\cdot))$-measurable random element, and $L(X,Z)$ is a vector-valued $\sigma(X(\cdot,\cdot))$-measurable random element, accounted in the augmented probability space $(\mathbb{R}^n\times\Omega,\mathcal{B}\times\mathcal{A},m\otimes\mathbb{P})$, which depends functionally on Z in a linear manner. Likewise, if we take $X_x=\mathcal{I}_x=x$, then we obtain

$$D_X^2F(m)(Z)=\nabla_x^2\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)Z_x+\widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\widetilde{Z}_{\tilde{x}}\mathrm{d}m(\widetilde{x})\right].\tag{2.14}$$

From Formula (2.13), it follows immediately that if Z is independent of X and $\mathbb{E}(Z)=0$, then the second term vanishes and so

$$D_X^2F(X\otimes m)(Z)=\nabla_x^2\frac{\mathrm{d}F}{\mathrm{d}\nu}(X\otimes m)(X_x)Z_x.\tag{2.15}$$

In order to get $D_X^2F(X\otimes m)(X)\in\mathcal{H}_m$, we need to assume $\left|\nabla_x\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)\right|\le c(m)$ and $\left|\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right|\le c(m)$, where $|\cdot|$ is the matrix norm defined by $|A|:=\sup_{x\in\mathbb{R}^n,|x|=1}|Ax|$ for any $A\in\mathbb{R}^{n\times n}$.

2.5. Stochastic calculus in the space of L^2-random variables

The probability space $(\Omega,\mathcal{A},\mathbb{P})$ under consideration is sufficiently large such that it contains a standard Wiener process in $\mathbb{R}^n$, denoted by $w(t)$, for $t\ge 0$, together with some additional random variables, for instance, serving as the initial data, can be independent of the filtration $\mathcal{W}_t^s:=\sigma\big(w(\tau)-w(t);\tau\in[t,s]\big)$ generated by the Wiener process. The latter random variables can be considered as $\mathcal{W}_0^t$- (or $\mathcal{W}^t$- for short) measurable for any $t\ge 0$. For each $t\in[0,T]$, we let $X_{tx}=X(x,\omega,t)$ be an arbitrary element in $\mathcal{H}_m$, independent of $\mathcal{W}_t^T$ ($\mathcal{W}_t$ for short). We shall refer $X_{t\boldsymbol{\cdot}}\otimes m$ or $X_t\otimes m$ for short the pushforward probability measure in $\mathcal{P}_2(\mathbb{R}^n)$. We also define $\mathcal{W}_{tX}^s:=\sigma(X_t)\bigvee\mathcal{W}_t^s$ and denote by $\mathcal{W}_{tX}$ the filtration generated by the $\sigma-$algebras $\mathcal{W}_{tX}^s$, for $s\ge t$. We denote by $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ the subspace of $L^2(t,T;\mathcal{H}_m)$ of all elements adapted to the filtration $\mathcal{W}_{tX}$. Based on Lemma 3.1 in [32], it is important to note that, because X_t is independent of $\mathcal{W}_t^T$, there exists an isometry between $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ and $L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X_t\otimes m})$; indeed, for $Y(\cdot)\in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$, there exists a random field $Y_{t\xi}(s)\in L^2_{\mathcal{W}_t}(t,T;\mathbb{R}^n)$ for $\xi\in\mathbb{R}^n$ and $s\ge t$ such that $Y(s)=Y_{t\xi}(s)\Big|_{\xi=X_{tx}}$ and so

$\mathbb{E}\big|Y(s)\big|^2 = \mathbb{E}\left(\mathbb{E}\big|Y_{t\xi}(s)\big|^2\Big|_{\xi=X_{tx}}\right)$. It follows that

$$\int_t^T \Big\|Y(s)\Big\|^2_{\mathcal{H}_m} ds = \int_t^T \mathbb{E}\left[\int_{\mathbb{R}^n} \Big|Y(s)\Big|^2 \mathrm{d}m(x)\right] ds = \int_t^T \mathbb{E}\left[\int_{\mathbb{R}^n} \mathbb{E}\Big|Y_{t\xi}(s)\Big|^2\Bigg|_{\xi=X_{tx}} \mathrm{d}m(x)\right] ds$$
$$= \int_t^T \mathbb{E}\left[\int_{\mathbb{R}^n} \Big|Y_{t\xi}(s)\Big|^2 \mathrm{d}\Big(X_t \otimes m\Big)(\xi)\right] ds.$$

Let us consider for $j = 1, 2, \ldots, n$, coefficient functions $\eta^j_{tX}(\cdot) \in L^2_{\mathcal{W}_{tX}}(t, T; \mathcal{H}_m)$, so that each of them corresponds to the random field $\eta^j_{t\xi}(\cdot) \in L^2_{\mathcal{W}_t}(t, T; \mathcal{H}_{X_t \otimes m})$, where $X_t \in \mathcal{H}_m$. We define the stochastic integral

$$\sum_{j=1}^n \int_t^s \eta^j_{tX}(\tau) dw_j(\tau) = \sum_{j=1}^n \int_t^s \eta^j_{t\xi}(\tau) dw_j(\tau)\Big|_{\xi=X},$$

which consequently defines a process in $L^2_{\mathcal{W}_{tX}}(t, T; \mathcal{H}_m)$ with

$$\mathbb{E}\left[\int_{\mathbb{R}^n} \left|\sum_{j=1}^n \int_t^s \eta^j_{tX}(\tau) dw_j(\tau)\right|^2 \mathrm{d}m(x)\right] = \int_{\mathbb{R}^n} \mathbb{E}\left[\left|\sum_{j=1}^n \int_t^s \eta^j_{t\xi}(\tau) dw_j(\tau)\right|^2 \mathrm{d}\Big(X_t \otimes m\Big)(\xi)\right]$$
$$= \int_{\mathbb{R}^n} \mathbb{E}\left[\sum_{j=1}^n \int_t^s \Big|\eta^j_{t\xi}(\tau)\Big|^2 \mathrm{d}\tau\, \mathrm{d}\Big(X_t \otimes m\Big)(\xi)\right] = \mathbb{E}\left[\int_{\mathbb{R}^n} \sum_{j=1}^n \int_t^s \Big|\eta^j_{tX}(\tau)\Big|^2 \mathrm{d}\tau \mathrm{d}m(x)\right].$$

Definition 2.2 (**Generalized Itô Process).** Given X measurable to $\mathcal{W}^t_0$ and $a_{tX}(s) \in L^2_{\mathcal{W}_{tX}}(t, T; \mathcal{H}_m)$, we can define the Itô process as, for a given $t \in [0, T)$,

$$\mathbb{X}_{tX}(s) = X + \int_t^s a_{tX}(\tau) \mathrm{d}\tau + \sum_{j=1}^n \int_t^s \eta^j_{tX}(\tau) dw_j(\tau), \qquad \text{for } s \in [t, T],$$

which belongs to $L^2_{\mathcal{W}_{tX}}(t, T; \mathcal{H}_m)$.

We can then verify the following differentiation rule

$$\frac{\mathrm{d}}{ds}\big\|\,\mathbb{X}_{tX}(s)\big\|^2_{\mathcal{H}_m} = 2\Big\langle \mathbb{X}_{tX}(s), a_{tX}(s)\Big\rangle_{\mathcal{H}_m} + \sum_{j=1}^n \Big\|\eta^j_{tX}(s)\Big\|^2_{\mathcal{H}_m}. \tag{2.16}$$

Fix $t \in [0, T)$, we now proceed to obtain an Itô's formula for the evolution of $F(\mathbb{X}_{tX}(s) \otimes m, s)$, where m is still the initial distribution at time 0 and $s \in [t, T]$. Furthermore, we assume the following:

(i) Let $L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m) \subset \mathcal{H}_m$ be the subspace of $\mathcal{H}_m$ of all elements independent of $\mathcal{W}_t$, consider the map $X \in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m) \longmapsto F(X \otimes m, s)$, and it is twice Gâteaux differentiable over $\mathcal{H}_m$; (2.17)

(ii) $\Big|F(X \otimes m, s + \epsilon) - F(X \otimes m, s)\Big| \leq C_T \epsilon \left(1 + \|X\|^2_{\mathcal{H}_m}\right)$; (2.18)

(iii) $D_X F(X \otimes m, s_k)$ converges to $D_X F(X \otimes m, s)$ in $\mathcal{H}_m$ as $s_k \downarrow s$; (2.19)

(iv) For any $X \in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m)$, $\big\|D_X F(X \otimes m, s)\big\|_{\mathcal{H}_m} \leq C_T\Big(1 + \|X\|_{\mathcal{H}_m}\Big)$ and $\big\|D^2_X F(X \otimes m, s)\big\|_{\mathcal{H}_m} \leq c.$ (2.20)

The second-order derivative is assumed to have the following continuity properties:

(v) For any sequence $\{X_k\}_{k\in\mathbb{N}} \subset L^2_{\mathcal{W}_t^{\perp\!\!\!\perp}}(\mathcal{H}_m)$ converging to $X \in L^2_{\mathcal{W}_t^{\perp\!\!\!\perp}}(\mathcal{H}_m)$ as $s_k \downarrow s$, then $D_X^2 F(X_k \otimes m, s_k)(Y) \to D_X^2 F(X \otimes m, s)(Y)$ uniformly in $\mathcal{H}_m$ for all Y in a bounded set. (2.21)

Furthermore, if the sequence $\{Y_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{H}_m$, then $D_X^2 F(X_k \otimes m, s_k)(Y_k) - D_X^2 F(X \otimes m, s_k)(Y_k) \to 0$ in $L^1\big(\Omega, \mathcal{A}, \mathbb{P}; L^1_m(\mathbb{R}^n;\mathbb{R}^n)\big)$; (2.22)

(vi) The processes $a_{tX}(s)$, $\eta^j_{tX}(s) \in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ satisfy the following additional properties:

$$\begin{aligned}
&\text{(a)}\quad \sup_{s\in[t,T]} \mathbb{E}\left[\int_{\mathbb{R}^n}\left(\sum_{j=1}^n \left|\eta^j_{tX}(s)\right|^2\right)^2 \mathrm{d}m(x)\right] < \infty;\\
&\text{(b)}\quad \frac{1}{\epsilon}\sum_{j=1}^n \mathbb{E}\left[\int_s^{s+\epsilon}\int_{\mathbb{R}^n}\left|\eta^j_{tX}(\tau) - \eta^j_{tX}(s)\right|^2 \mathrm{d}m(x)\mathrm{d}\tau\right] \to 0 \text{ as } \epsilon \to 0, \text{ a.e. } s \in [t,T];\\
&\text{(c)}\quad \frac{1}{\epsilon}\int_s^{s+\epsilon} a_{tX}(\tau)\mathrm{d}\tau \longrightarrow a_{tX}(s) \quad \text{in } \mathcal{H}_m, \text{ as } \epsilon \to 0, \text{ a.e. } s \in [t,T].
\end{aligned} \tag{2.23}$$

We then state the following Itô's lemma in the mean field setting with its proof in Appendix A.

Theorem 2.3 (**Mean Field Itô's Formula**). *Under the assumptions of (2.17) to (2.23), the function* $s \longmapsto F\big(\mathbb{X}_{tX}(s)\otimes m, s\big)$ *is a.e differentiable on* (t,T)*, and we also have the change-of-variable formula:*

$$\begin{aligned}
\frac{\mathrm{d}}{ds}F(\mathbb{X}_{tX}(s)\otimes m, s) = {}& \frac{\partial}{\partial s}F(\mathbb{X}_{tX}(s)\otimes m, s) + \big\langle D_X F(\mathbb{X}_{tX}(s)\otimes m, s), a_{tX}(s)\big\rangle_{\mathcal{H}_m}\\
&+ \frac{1}{2}\left\langle D_X^2 F(\mathbb{X}_{tX}(s)\otimes m, s)\left(\sum_{j=1}^n \eta^j_{tX}(s)\mathcal{N}^j_s\right), \sum_{j=1}^n \eta^j_{tX}(s)\mathcal{N}^j_s\right\rangle_{\mathcal{H}_m}, \quad \text{a.e. } s \in (t,T),
\end{aligned} \tag{2.24}$$

where $\mathcal{N}^j_s$*'s are (being independent of each other)*[3] *Gaussian random variables, everyone with a mean* 0 *and a unit variance, and each* $\mathcal{N}^j_s$ *is independent of the* σ*-algebra* $\mathcal{W}^s_{tX}$*. If one applies Formula (2.13) together with the discussion between (2.14) and (2.15), then for a.e.* $s \in (t,T)$*, we also have*

$$\begin{aligned}
\frac{\mathrm{d}}{ds}F\big(\mathbb{X}_{tX}(s)\otimes m, s\big) = {}& \frac{\partial}{\partial s}F\big(\mathbb{X}_{tX}(s)\otimes m, s\big) + \left\langle \nabla_x \frac{\mathrm{d}F}{\mathrm{d}\nu}\big(\mathbb{X}_{tX}(s)\otimes m, s\big)\big(\mathbb{X}_{tx}(s)\big), a_{tX}(s)\right\rangle_{\mathcal{H}_m}\\
&+ \frac{1}{2}\sum_{j=1}^n \left\langle \nabla_x^2 \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{X}_{tX}(s)\otimes m, s)(\mathbb{X}_{tX}(s))\eta^j_{tX}(s), \eta^j_{tX}(s)\right\rangle_{\mathcal{H}_m}.
\end{aligned} \tag{2.25}$$

3. Formulation and characterization of mean field type control problem

3.1. Problem formulation

The space of controls is set as the Hilbert space $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$. A control is denoted by $v_{tX}(s)$, where the initial data at time t is $X = \mathbb{X}_{tX}(t) \in \mathcal{H}_m$, for any X measurable to $\mathcal{W}^t_0$, being independent of $\mathcal{W}^s_t =$

[3] We can enlarge the initial σ-algebra that also contains all these $\mathcal{N}^j_s$'s and is independent of X_t.

$\sigma(w(\tau)-w(t): t\le\tau\le s)$ for any $s\in[t,T]$. So, we have a measurable random field $v_{t\xi}(s)$, $\xi\in\mathbb{R}^n$, and for each ξ, $v_{t\xi}(\cdot)\in L^2_{\mathcal{W}_t}(t,T;\mathbb{R}^n)$ such that $v_{t\xi}(s)\Big|_{\xi=X=\mathbb{X}_{tX}(t)}=v_{tX}(s)$; then we have the equivalence:

$$\int_t^T \|v_{tX}(s)\|^2_{\mathcal{H}_m}ds=\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^n}|v_{t\xi}(s)|^2 d(X\otimes m)(\xi)ds\right]=\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^n}|v_{tX}(s)|^2\mathrm{d}m(x)ds\right].$$

The state denoted by $\mathbb{X}_{tX}(s)=\mathbb{X}_{tX}(s;v_{tX}(\cdot))$ associated with a control $v_{tX}(\cdot)$ is defined as the Itô process in accordance with Section 2.5

$$\mathbb{X}_{tX}(s)=X+\int_t^s v_{tX}(\tau)\mathrm{d}\tau+\eta(w(s)-w(t)),\qquad \text{for all } s\in[t,T], \tag{3.1}$$

where η is a constant matrix in $\mathbb{R}^{n\times n}$. We see that $\mathbb{X}_{tX}(s)$ belongs to $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$. More precisely, $\mathbb{X}_{tX}(s)=\mathbb{X}_{t\xi}(s)\Big|_{\xi=X}=\mathbb{X}_{t\xi}(s;v_{t\xi})\Big|_{\xi=X}$, where

$$\mathbb{X}_{t\xi}(s)=\xi+\int_t^s v_{t\xi}(\tau)\mathrm{d}\tau+\eta(w(s)-w(t)),\quad \text{for } \xi\in\mathbb{R}^n. \tag{3.2}$$

The probability law on $\mathbb{R}^n$, $\mathbb{X}_{tX}(s)\otimes m$ is simply the probability law of $\mathbb{X}_{tX}(s)$. It is also the probability law of $\mathbb{X}_{t\xi}(s)$ when ξ is equipped with a probability $\mathbb{X}_t\otimes m$, more precisely $\mathbb{X}_{t\boldsymbol{\cdot}}\otimes m$. We aim to minimize:

$$\begin{aligned}J_{tX}(v_{tX})=&\int_t^T\int_{\mathbb{R}^n}\mathbb{E}\Big[l\Big(\mathbb{X}_{tX}(s),v_{tX}(s)\Big)\Big]\mathrm{d}m(x)+F\Big(\mathbb{X}_{tX}(s)\otimes m\Big)ds\\&+\int_{\mathbb{R}^n}\mathbb{E}\Big[h(\mathbb{X}_{tX}(T)\Big]dm(x)+F_T\Big(\mathbb{X}_{tX}(T)\otimes m\Big),\end{aligned} \tag{3.3}$$

over $v\in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$, where the cost functions $l(x,v):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and $h(x):\mathbb{R}^n\to\mathbb{R}$. The functional $J_{tX}(v_{tX})$ depends on X only through its probability distribution; in particular, if we are dealing with an Itô process $\mathbb{X}_{tX}(s)$ with $\mathbb{X}_{tX}(t)=X$, then $J_{tX}(v_{tX})$ depends on X only through $X\otimes m$. For a fixed X measurable to $\mathcal{W}_0^t$, the functional $v_{tX}\longmapsto J_{tX}(v_{tX})$ is defined on the Hilbert space $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)\subset L^2(t,T;\mathcal{H}_m)$.

3.2. Assumptions

All the following constants are positive except that c_l', c_h', c' and c_T' can be allowed to be non-positive. For any x, v, ξ and $\zeta\in\mathbb{R}^n$, we assume that $l(x,v)$ and $h(x)$ are twice differentiable and the following hold:

$$\textbf{A(i)}\quad |l(x,v)|\le c_l\Big(1+|x|^2+|v|^2\Big),\ |l_x(x,v)|,|l_v(x,v)|\le c_l\Big(1+|x|^2+|v|^2\Big)^{\frac{1}{2}}; \tag{3.4}$$

$$\textbf{A(ii)}\quad |l_{xx}(x,v)|,\quad |l_{vx}(x,v)|,\quad |l_{vv}(x,v)|\le c_l; \tag{3.5}$$

$$\textbf{A(iii)}\quad |h(x)|\le c_h\Big(1+|x|^2\Big),\quad |h_x(x)|\le c_h\Big(1+|x|^2\Big)^{\frac{1}{2}},\quad |h_{xx}(x)|\le c_h; \tag{3.6}$$

$$\textbf{A(iv)}\quad l_{xx}(x,v),l_{vx}(x,v),l_{vv}(x,v),h_{xx}(x)\ \text{are continuous}; \tag{3.7}$$

$$\textbf{A(v)}\quad l_{xx}(x,v)\xi\cdot\xi+2l_{vx}(x,v)\zeta\cdot\xi+l_{vv}(x,v)\zeta\cdot\zeta\ge\lambda|\zeta|^2-c_l'|\xi|^2; \tag{3.8}$$

$$\textbf{A(vi)}\quad h_{xx}\xi\cdot\xi\ge -c_h'|\xi|^2. \tag{3.9}$$

The notation l_{xv} represents the derivative $\partial_v\Big[\partial_x l(x,v)\Big]$. We describe next the assumptions on the functionals $m \longmapsto F(m)$ and $m \longmapsto F_T(m)$ on $\mathcal{P}_2(\mathbb{R}^n)$. We assume that, for any $m \in \mathcal{P}_2(\mathbb{R}^n)$,

$$|F(m)| \le c\left(1+\int_{\mathbb{R}^n}|x|^2\mathrm{d}m(x)\right), \quad |F_T(m)| \le c_T\left(1+\int_{\mathbb{R}^n}|x|^2\mathrm{d}m(x)\right), \tag{3.10}$$

where c and c_T are independent of m. For any $m \in \mathcal{P}_2(\mathbb{R}^n)$, $\widetilde{x} \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the functionals also satisfy

b(i) $$\left|\nabla_x \frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)\right| \le \frac{c}{\sqrt{2}}(1+|x|), \quad \left|\nabla_x \frac{dF_T}{\mathrm{d}\nu}(m)(x)\right| \le \frac{c_T}{\sqrt{2}}(1+|x|); \tag{3.11}$$

b(ii) $$\left|\nabla_x^2 \frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)\right| \le \frac{c}{2}, \quad \left|\nabla_x\nabla_{\tilde{x}} \frac{d^2F}{\mathrm{d}\nu^2}(m)(\widetilde{x},x)\right| \le \frac{c}{2}, \quad \left|\nabla_x^2 \frac{dF_T}{\mathrm{d}\nu}(m)(x)\right| \le \frac{c_T}{2}, \quad \left|\nabla_x\nabla_{\tilde{x}} \frac{d^2F_T}{\mathrm{d}\nu^2}(m)(\widetilde{x},x)\right| \le \frac{c_T}{2}; \tag{3.12}$$

b(iii) $\nabla_x^2 \frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$, $\nabla_x\nabla_{\tilde{x}} \frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})$ are continuous in (m,x) and $(m,x,\widetilde{x})$ respectively; (3.13)

b(iv) $\nabla_x^2 \frac{dF_T}{\mathrm{d}\nu}(m)(x)$, $\nabla_x\nabla_{\tilde{x}} \frac{d^2F_T}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})$ are continuous in (m,x) and $(m,x,\widetilde{x})$ respectively; (3.14)

b(v)* $$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n}\nabla_x^2\frac{\mathrm{d}F}{\mathrm{d}\nu}(X\otimes m)(X(x))Y(x)\cdot Y(x)dm(x) \ge -c'\mathbb{E}\int_{\mathbb{R}^n}|Y(x)|^2dm(x),\\
&\mathbb{E}\widetilde{\mathbb{E}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(X\otimes m)(\widetilde{X}(\widetilde{x}),X(x))\widetilde{Y}(\widetilde{x})\cdot Y(x)dm(\widetilde{x})dm(x) \ge 0,\\
&\mathbb{E}\int_{\mathbb{R}^n}\nabla_x^2\frac{dF_T}{\mathrm{d}\nu}(X\otimes m)(X(x))Y(x)\cdot Y(x)dm(x) \ge -c_T'\mathbb{E}\int_{\mathbb{R}^n}|Y(x)|^2dm(x),\\
&\mathbb{E}\widetilde{\mathbb{E}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F_T}{\mathrm{d}\nu^2}(X\otimes m)(\widetilde{X}(\widetilde{x}),X(x))\widetilde{Y}(\widetilde{x})\cdot Y(x)dm(\widetilde{x})dm(x) \ge 0,
\end{aligned} \tag{3.15}$$

or

b(v)† $$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n}\nabla_x^2\frac{\mathrm{d}F}{\mathrm{d}\nu}(X\otimes m)(X(x))Y(x)\cdot Y(x)dm(x)\\
&\quad+\mathbb{E}\widetilde{\mathbb{E}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(X\otimes m)(\widetilde{X}(\widetilde{x}),X(x))\widetilde{Y}(\widetilde{x})\cdot Y(x)dm(\widetilde{x})dm(x) \ge -c'\mathbb{E}\int_{\mathbb{R}^n}|Y(x)|^2dm(x),\\
&\mathbb{E}\int_{\mathbb{R}^n}\nabla_x^2\frac{dF_T}{\mathrm{d}\nu}(X\otimes m)(X(x))Y(x)\cdot Y(x)dm(x)\\
&\quad+\mathbb{E}\widetilde{\mathbb{E}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F_T}{\mathrm{d}\nu^2}(X\otimes m)(\widetilde{X}(\widetilde{x}),X(x))\widetilde{Y}(\widetilde{x})\cdot Y(x)dm(\widetilde{x})dm(x) \ge -c_T'\mathbb{E}\int_{\mathbb{R}^n}|Y(x)|^2dm(x),
\end{aligned} \tag{3.16}$$

for any $X, Y \in \mathcal{H}_m$, where $(\widetilde{X},\widetilde{Y})$ is an independent copy of (X,Y).

Remark 3.1. Although mean field control problems and mean field games are different in nature, they share some common structures in their master equations. We compare our assumptions with two commonly used monotonicity conditions in the literature dealing with master equations in mean field games, namely, the Lasry–Lions monotonicity ((2.5) in [40]) and the displacement monotonicity (Asm. 3.2 and 3.5 in [31]). The corresponding linear functional derivatives $\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$ and $\frac{dF_T}{\mathrm{d}\nu}(m)(x)$ of the running cost function F and the terminal cost function F_T in our formulation are equivalent to $F(x,m)$ and $G(x,m)$ introduced on page 20 in [40] respectively, in terms of the master equation (see the respective master equations in (5.9) in this work and (2.6) in [40]). The corresponding Lasry–Lions monotonicity if formulated in our setting is equivalent to the following condition (see Rem. 2.4 in [31]):

$$LLM_{\text{modified}} := \mathbb{E}\widetilde{\mathbb{E}}\Big[\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(X\otimes m)(X,\widetilde{X})\widetilde{Y}\cdot Y\Big] \geq 0, \tag{3.17}$$

for any $X,Y\in\mathcal{H}_m$, where $(\widetilde{X},\widetilde{Y})$ is an independent copy of (X,Y). Therefore, first and third conditions in Assumption $\mathbf{b(v)}^*$'s (3.15) are consistent with the modified Lasry–Lions monotonicity (3.17). For the displacement monotonicity, we have to first define the Hamiltonian by

$$\widetilde{H}(x,m,p) := l\big(x,u(x,p)\big) + u(x,p)\cdot p + \frac{dF}{\mathrm{d}\nu}(m)(x), \tag{3.18}$$

where $u(x,p)$ solves the first order condition $l_v\big(x,u(x,p)\big)+p=0$ (see Lem. 3.4 and Proposition 3.6). The corresponding displacement monotonicity if formulated in our setting is equivalent to the following condition (see Rem. 2.4 in [31]):

$$DM_{\text{modified}} := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big\langle \nabla_x\nabla_{\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}\nu}\widetilde{H}(x,m,\varphi(x))(\widetilde{x})v(\widetilde{x}) + \nabla_{xx}\widetilde{H}(x,m,\varphi(x))v(x),v(x)\Big\rangle_{\mathbb{R}^n}\mathrm{d}m(x)\mathrm{d}m(\widetilde{x}) \geq 0, \tag{3.19}$$

for any $m\in\mathcal{P}_2(\mathbb{R}^n)$, $\varphi\in C^1$, $v\in L^2_m$. Under our current setting, we substitute (3.18) into to (3.19) to see that

$$\begin{aligned}
DM_{\text{modified}} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Bigg\langle \nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})v(\widetilde{x}) \\
&\quad + \left[l_{xx}\Big(x,u(x,p)\Big) + l_{xv}\Big(x,u(x,p)\Big)\nabla_x u(x,p) + \nabla_x^2\frac{dF}{\mathrm{d}\nu}(m)(x)\right]v(x),v(x)\Bigg\rangle_{\mathbb{R}^n}\mathrm{d}m(x)\mathrm{d}m(\widetilde{x}) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Bigg\langle \nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})v(\widetilde{x}) \\
&\quad + \left[l_{xx}\Big(x,u(x,p)\Big) - l_{xv}\Big(x,u(x,p)\Big)\Big[l_{xv}\Big(x,u(x,p)\Big)\Big]^{-1}l_{vx}\Big(x,u(x,p)\Big) + \nabla_x^2\frac{dF}{\mathrm{d}\nu}(m)(x)\right]v(x),v(x)\Bigg\rangle_{\mathbb{R}^n} \\
&\qquad \mathrm{d}m(x)\mathrm{d}m(\widetilde{x}).
\end{aligned}$$

With the aid of Assumptions $\mathbf{A(v)}$'s (3.8) and $\mathbf{b(v)}^*$'s (3.15) (or $\mathbf{b(v)}^\dagger$'s (3.16)), we obtain that $DM_{\text{modified}}\geq -(c'+c'_l)\int_{\mathbb{R}^n}|v(x)|^2\mathrm{d}m(x)$. If $-(c'+c'_l)\geq 0$, then our assumptions are consistent with displacement monotonicity. If $-(c'+c'_l)<0$, the displacement monotonicity may not be satisfied and thus we may not have the well-posedness over an arbitrarily large time horizon. However, we can find a lifespan $T_0>0$ such that the master equation is still well-posed over the finite time horizon $[0,T_0]$ under our set of assumptions (3.4)–(3.16) and (3.30). This T_0 depends on λ, c'_T, c'_h, c'_l and c' such that $\lambda-(c'_T+c'_h)_+T-(c'_l+c')_+\frac{T^2}{2}>0$ (see (3.30)). Indeed, the lifespan T_0 can be chosen as

$$T_0 < (c'+c'_l)^{-1}\Big\{\big[2\lambda(c'+c'_l)+((c'_T+c'_h)_+)^2\big]^{1/2}-(c'_T+c'_h)_+\Big\}, \tag{3.20}$$

so it is not that small in many cases. The choice of T_0 in (3.20) is essentially the same as the assumption imposed in (3.30). In this situation, the displacement monotonicity is not required for the well-posedness of the master equation. Besides, the Hamiltonian in [31] is required to be C^5 and its third-order derivatives are uniformly bounded (translated to our setting this requirement is equivalent that $l(x,u)$, $\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x)$ are C^5 and their third-order derivatives are uniformly bounded), which enforces the cost functions to have linear growth at the most. In contrast, our assumptions allow models beyond linear quadratic, for example, $F(m) := \int_{\mathbb{R}} \left(x^2 + \mathrm{e}^{-x^2}\right) \mathrm{d}m(x)$ in Introduction. We also remark that the displacement monotonicity does not imply Lasry–Lions monotonicity, and vice versa, but they are overlapped in some situations, see more discussion in the work of [44]; see more discussion on the mean field monotonicity from the control perspective in the recent work [45].

Remark 3.2. The claims of this work remain valid if we assume F and F_T depend on x in addition to the measure argument m, that is $F = F(x,m) : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}$ and $F_T = F_T(x,m) : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}$, together with certain standard assumptions on the regularity in the spatial argument. It does not bring in any technical difficulty but here we assume $F = F(m)$ for the sake of convenience.

Remark 3.3. We provide the corresponding properties on the functionals $X \mapsto F(X \otimes m)$ and $X \mapsto F_T(X \otimes m)$ on $\mathcal{H}_m$ based on Assumptions **b(i)** to **b(v)**. The below assumptions are necessary but not sufficient for Assumptions **b(i)** to **b(v)**, in some sense, the following are weaker assumptions. For $m \in \mathcal{P}_2(\mathbb{R}^n)$, $X \in \mathcal{H}_m$,

$$|F(X \otimes m)| \leq c\Big(1 + \|X\|^2_{\mathcal{H}_m}\Big)\,, \quad |F_T(X \otimes m)| \leq c_T\Big(1 + \|X\|^2_{\mathcal{H}_m}\Big), \tag{3.21}$$

where c and c_T are the same as in (3.10). The functionals have Fréchet derivatives up to second order on $\mathcal{H}_m$ such that for any $X \in \mathcal{H}_m$,

$$\textbf{B(i)}\ \ \|D_X F(X \otimes m)\|_{\mathcal{H}_m} \leq c\Big(1 + \|X\|_{\mathcal{H}_m}\Big)\,,\ \ \|D_X F_T(X \otimes m)\|_{\mathcal{H}_m} \leq c_T\Big(1 + \|X\|_{\mathcal{H}_m}\Big); \tag{3.22}$$

$$\textbf{B(ii)}\ \ \Big\|D^2_X F(X \otimes m)\Big\|_{\mathcal{H}_m} \leq c, \quad \Big\|D^2_X F_T(X \otimes m)\Big\|_{\mathcal{H}_m} \leq c_T; \tag{3.23}$$

$$\textbf{B(iii)}\ \ X \longmapsto D^2_X F(X \otimes m)(Y) \text{ is continuous as a function from } \mathcal{H}_m \text{ to } \mathcal{H}_m \text{ for each fixed } Y; \tag{3.24}$$

$$\textbf{B(iv)}\ \ X \longmapsto D^2_X F_T(X \otimes m)(Y) \text{ is continuous as a function from } \mathcal{H}_m \text{ to } \mathcal{H}_m \text{ for each fixed } Y; \tag{3.25}$$

B(v) **(a)** if there is a sequence $\left\{X_k\right\}_{k\in\mathbb{N}}$ converging to X in $\mathcal{H}_m$, and the sequence $\{Y_k\}_{k\in\mathbb{N}}$ is bounded in norm of $\mathcal{H}_m$, then

$$\begin{aligned} &D^2_X F(X_k \otimes m)(Y_k) - D^2_X F(X \otimes m)(Y_k) \longrightarrow 0, \text{ in } L^1(\Omega, \mathcal{A}, \mathbb{P}; L^1_m(\mathbb{R}^n;\mathbb{R}^n)), \text{ and}^4 \\ &D^2_X F_T(X_k \otimes m)(Y_k) - D^2_X F_T(X \otimes m)(Y_k) \longrightarrow 0, \text{ in } L^1(\Omega, \mathcal{A}, \mathbb{P}; L^1_m(\mathbb{R}^n;\mathbb{R}^n)); \end{aligned} \tag{3.26}$$

(b) $$\Big\langle D^2_X F(X \otimes m)(Y), Y\Big\rangle_{\mathcal{H}_m} \geq -c'\|Y\|^2_{\mathcal{H}_m},\ \Big\langle D^2_X F_T(X \otimes m)(Y), Y\Big\rangle_{\mathcal{H}_m} \geq -c'_T\|Y\|^2_{\mathcal{H}_m}. \tag{3.27}$$

3.3. Differentiability and convexity of objective function

We now study the differentiability and convexity of the objective functional $v_{tX} \longmapsto J_{tX}(v_{tX})$ as one on the Hilbert space $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$, we have

Lemma 3.4. *Under the assumptions (3.4), (3.6) and (3.22), for any $v_{tX} \in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$, the functional $J_{tX}(v_{tX})$ has a Gâteaux derivative, given by:*

$$D_v J_{tX}(v_{tX})(s) = l_v(\mathbb{X}_{tX}(s), v_{tX}(s)) + \mathbb{Z}_{tX}(s), \quad \textit{for } s \in [t,T], \tag{3.28}$$

[4] It means that $\mathbb{E}\left[\int_{\mathbb{R}^n}\left|D^2_X F(X_k \otimes m)(Y_k) - D^2_X F(X \otimes m)(Y_k)\right| \mathrm{d}m(x)\right] \longrightarrow 0$ as $k \to 0$.

where $\mathbb{Z}_{tX}(s) \in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$*, being the solution to the backward stochastic differential equation (BSDE)*

$$\begin{cases} -d\mathbb{Z}_{tX}(s) = \Big[l_x\big(\mathbb{X}_{tX}(s), v_{tX}(s)\big) + D_X F(\mathbb{X}_{tX}(s)\otimes m)\Big] ds - \sum_{j=1}^n \mathbb{r}_{tX,j}(s) dw_j(s); \\ \mathbb{Z}_{tX}(T) = h_x\big(\mathbb{X}_{tX}(T)\big) + D_X F_T\big(\mathbb{X}_{tX}(T)\otimes m\big), \end{cases} \tag{3.29}$$

for the adapted process $\mathbb{r}_{tX,j}(s) \in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ *with* $j = 1, 2, \dots, n$*.*

Its proof can be found in Appendix A. We denote $\mathbb{r}_{tX}(s) := \big(\mathbb{r}_{tX,1}(s), \mathbb{r}_{tX,2}(s), \dots, \mathbb{r}_{tX,n}(s)\big)$ the matrix having $\mathbb{r}_{tX,j}(s)$ as its j-th column vector. We next state the existence and uniqueness results of the optimal control.

Proposition 3.5 (Existence and Uniqueness). *We assume (3.4), (3.5), (3.6), (3.8), (3.9), (3.21), (3.22), (3.23), (3.27) and*

$$c_0 := \lambda - (c'_T + c'_h)_+ T - (c'_l + c')_+ \frac{T^2}{2} > 0, \tag{3.30}$$

then the functional $J_{tX}(v_{tX})$ *is strictly convex in* v_{tX}*, and coercive in* v_{tX} *in the sense that* $J_{tX}(v_{tX}) \longrightarrow \infty$ *as* $\int_t^T \|v_{tX}(s)\|^2_{\mathcal{H}_m} ds \to \infty$*, here* $a_+ := \max\{a, 0\}$ *for any* $a \in \mathbb{R}$*. Consequently, there is a unique optimal control* $u_{tX}(s) \in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ *to the control problem that minimizes* $J_{tX}(v_{tX})$ *subject to* $v_{tX}(s) \in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ *and* $\mathbb{X}_{tX}(s; v_{tX})$ *satisfying (3.2).*

Its proof is in Appendix A. In the rest of the article, we assume that the condition (3.30) always holds.

3.4. Necessary and sufficient condition of optimality

According to Proposition 3.5, there is a unique optimal control $u_{tX}(s)$ which satisfies the first condition $D_v J_{tX}(u_{tX}) \equiv 0$ in Lemma 3.4. Particularly, the optimal control $u_{tX}(s)$ is a feedback one; to this end, we define the Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $L(x,v,p) := l(x,v) + v \cdot p$. Then, from Assumption **A(v)**'s (3.8), for each x, $p \in \mathbb{R}^n$, the function $v \longmapsto L(x,v,p)$ is strictly convex and coercive in v, therefore it attains its unique minimum at $v = u(x,p)$. We further define the Hamiltonian

$$H(x,p) := \inf_{v\in\mathbb{R}^n} L(x,v,p) = l\Big(x, u(x,p)\Big) + u(x,p)\cdot p \tag{3.31}$$

which is clearly $C^1(\mathbb{R}^n \times \mathbb{R}^n)$ with derivatives $H_x(x,p) = l_x(x, u(x,p))$ and $H_p(x,p) = u(x,p)$. Therefore, we can write $u_{tX}(s)$ in the following feedback form:

$$u_{tX}(s) = H_p\Big(\mathbb{Y}_{tX}(s), D_X V(\mathbb{Y}_{tX}(s)\otimes m, s)\Big) = H_p\Big(\mathbb{Y}_{tX}(s), \mathbb{Z}_{tX}(s)\Big),$$

where we shall verify the last equality in (4.1). Calling $\mathbb{Y}_{tX}(s)$ the optimal state and $\big(\mathbb{Z}_{tX}(s), \mathbb{r}_{tX}(s)\big)$ the solution to (3.29) under the influence of $\mathbb{Y}_{tX}$, we have the proposition:

Proposition 3.6. *For $t \in [0,T)$ and $X \in \mathcal{H}_m$ measurable to $\mathcal{W}_0^t$, the process u_{tX} is the optimal control to the control problem that minimizes $J_{tX}(v_{tX})$ subject to $v_{tX}(s) \in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ and (3.2) if and only if the process $\big(\mathbb{Y}_{tX}(s), \mathbb{Z}_{tX}(s), u_{tX}(s), \mathbb{r}_{tX,j}(s)\big)$ is the solution in $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ to the system,*

$$\begin{cases} \mathbb{Y}_{tX}(s) = X + \displaystyle\int_t^s u_{tX}(\tau)\mathrm{d}\tau + \eta(w(s)-w(t)), \quad \text{for } s \in [t,T]; & (3.32)\\ \mathbb{Z}_{tX}(s) = h_x(\mathbb{Y}_{tX}(T)) + D_X F_T(\mathbb{Y}_{tX}(T)\otimes m) + \displaystyle\int_s^T \Big[l_x\Big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\Big) + D_X F(\mathbb{Y}_{tX}(\tau)\otimes m)\Big]\mathrm{d}\tau \\ \qquad\qquad - \displaystyle\int_s^T \sum_{j=1}^n \mathbb{r}_{tX,j}(\tau) dw_j(\tau); & (3.33) \end{cases}$$

subject to

$$l_v\Big(\mathbb{Y}_{tX}(s), u_{tX}(s)\Big) + \mathbb{Z}_{tX}(s) = 0. \tag{3.34}$$

Proof. Suppose u_{tX} is the optimal control, Lemma 3.4 and Theorem 7.2.12 in [46] imply $D_v J_{tX}(v_{tX})(s) = l_v(\mathbb{X}_{tX}(s), v_{tX}(s)) + \mathbb{Z}_{tX}(s) = 0$, where the process $\big(\mathbb{Y}_{tX}(s), \mathbb{Z}_{tX}(s), u_{tX}(s), \mathbb{r}_{tX,j}(s)\big)$ satisfies (3.32)–(3.34). Suppose u_{tX} solves the first order condition (3.34) such that $l_v(\mathbb{X}_{tX}(s), v_{tX}(s)) + \mathbb{Z}_{tX}(s) = 0$, where $\big(\mathbb{Y}_{tX}(s), \mathbb{Z}_{tX}(s), \mathbb{r}_{tX,j}(s)\big)$ solves (3.32)–(3.33) correspondingly. If $J_{tX}(v)$ does not attain its minimum value at $v = u_{tX}$, then there is $u^\dagger$ such that $J_{tX}(u^\dagger) < J_{tX}(u_{tX})$. The convexity of $J_{tX}(v)$ in v implies that $J_{tX}(\theta u^\dagger + (1-\theta)u_{tX}) \le \theta J_{tX}(u^\dagger) + (1-\theta)J_{tX}(u_{tX})$ for any $\theta \in [0,1]$, that is,

$$\frac{J_{tX}(\theta u^\dagger + (1-\theta)u_{tX}) - J_{tX}(u^\dagger)}{\theta} \le J_{tX}(u^\dagger) - J_{tX}(u_{tX}) < 0.$$

It contradicts that $\frac{\mathrm{d}}{\mathrm{d}\theta} J_{tX}(u_{tX} + \theta(u^\dagger - u_{tX}))\big|_{\theta=0} = 0$. □

We note that there is a unique $u(x,z)$ such that $l_v\big(x,v\big)\Big|_{v=u(x,z)} + z = 0$ for x, $z \in \mathbb{R}^n$, based on Assumption $\mathbf{A(v)}$'s (3.8) and the inverse function theorem. Moreover, as discussed, $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ is isometric to $L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})$, there exist measurable random fields $\mathbb{Y}_{t\xi}(s)$, $\mathbb{Z}_{t\xi}(s)$, $u_{t\xi}(s)$, $\mathbb{r}_{t\xi,j}(s) \in L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X_t\otimes m})$ such that $\mathbb{Y}_{tX}(s) = \mathbb{Y}_{t\xi}(s)\big|_{\xi=X}$, $\mathbb{Z}_{tX}(s) = \mathbb{Z}_{t\xi}(s)\big|_{\xi=X}$, $u_{tX}(s) = u_{t\xi}(s)\big|_{\xi=X}$, $\mathbb{r}_{tX,j}(s) = \mathbb{r}_{t\xi,j}(s)\big|_{\xi=X}$. These random fields $\mathbb{Y}_{t\xi}(s), \mathbb{Z}_{t\xi}(s), u_{t\xi}(s), \mathbb{r}_{t\xi,j}(s)$ are solution to the FBSDE system,

$$\begin{cases} \mathbb{Y}_{t\xi}(s) = \xi + \displaystyle\int_t^s u_{t\xi}(\tau)\mathrm{d}\tau + \eta(w(s)-w(t)), \quad \text{for } s \in [t,T], & (3.35)\\ \mathbb{Z}_{t\xi}(s) = h_x(\mathbb{Y}_{t\xi}(T)) + D_X F_T(\mathbb{Y}_{t\cdot}(T)\otimes(X\otimes m)) \\ \qquad + \displaystyle\int_s^T \Big[l_x\Big(\mathbb{Y}_{t\xi}(\tau), u_{t\xi}(\tau)\Big) + D_X F(\mathbb{Y}_{t\cdot}(\tau)\otimes(X\otimes m))\Big]\mathrm{d}\tau - \int_s^T \sum_{j=1}^n \mathbb{r}_{t\xi,j}(\tau) dw_j(\tau), & (3.36) \end{cases}$$

subject to $l_v\Big(\mathbb{Y}_{t\xi}(s), u_{t\xi}(s)\Big) + \mathbb{Z}_{t\xi}(s) = 0$. We notice that $\mathbb{Y}_{t\xi}(s), \mathbb{Z}_{t\xi}(s), u_{t\xi}(s), \mathbb{r}_{t\xi,j}(s)$ depend on m, at time t, only through $X \otimes m = X_t \otimes m$.

Remark 3.7. Equations (3.32)–(3.34) are defined only up to a null set. If we consider infinitely many random variables X, the processes may not be defined. By considering the deterministic ξ in (3.35)–(3.36), we avoid this kind of measurability problem by putting $\xi = X$.

We can express the value function

$$V(X\otimes m,t)=\int_t^T\mathbb{E}\left[\int_{\mathbb{R}^n}l\big(\mathbb{Y}_{tX}(s),u_{tX}(s)\big)\mathrm{d}m(x)ds\right]+\int_t^T F(\mathbb{Y}_{tX}(s)\otimes m)ds$$
$$+\,\mathbb{E}\left[\int_{\mathbb{R}^n}h(\mathbb{Y}_{tX}(T))\mathrm{d}m(x)\right]+F_T(\mathbb{Y}_{tX}(T)\otimes m). \tag{3.37}$$

This quantity depends only on the probability measure $X\otimes m=X_t\otimes m$ and t. It can be written as follows

$$V(X\otimes m,t)=\int_t^T\mathbb{E}\left[\int_{\mathbb{R}^n}l\big(\mathbb{Y}_{t\xi}(s),u_{t\xi}(s)\big)\mathrm{d}\big(X\otimes m\big)(\xi)\right]+F(\mathbb{Y}_{t\cdot}(s)\otimes(X\otimes m))ds$$
$$+\,\mathbb{E}\left[\int_{\mathbb{R}^n}h(\mathbb{Y}_{t\xi}(T))\mathrm{d}\big(X\otimes m\big)(\xi)\right]+F_T(\mathbb{Y}_{t\cdot}(T)\otimes(X\otimes m)).$$

Remark 3.8. We may enlarge the subspace of $L^2(t,T;\mathcal{H}_m)$ of controls in which u_{tX} remains optimal; to this point, this will facilitate the relevant comparison principle and the study of the regularity of $X\longmapsto V(X\otimes m,t)$ in Sections 4 to 5, by introducing an additional initial element in $\mathcal{H}_m$, denoted by $\widetilde{X}$ which is also independent of $\mathcal{W}_t$ but not necessary with X. To see this, define the σ-algebras $\mathcal{W}^s_{tX\widetilde{X}}=\sigma(X,\widetilde{X})\bigvee\mathcal{W}^s_t$ and the filtration $\mathcal{W}_{tX\widetilde{X}}:=\mathcal{W}^T_{tX\widetilde{X}}$ generated by the σ-algebras $\mathcal{W}^s_{tX\widetilde{X}}$ for all $s\in[t,T]$. If we change the space of controls from $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$ to $L^2_{\mathcal{W}_{tX\widetilde{X}}}(t,T;\mathcal{H}_m)$, one checks easily that the system of Equations (3.32), (3.33) and (3.34) solve the necessary conditions of optimality of the control problem still with the enlarged σ-algebras, and by the uniqueness of the solution of the necessary conditions, the optimal control remains the same as before, except that we can now interpret the solution on the enlarged σ-algebras with all formulae unchanged. Of course, instead of adding one $\widetilde{X}$, one could add a finite number of elements of $\mathcal{H}_m$, as long as all of these are independent of $\mathcal{W}_t$.

3.5. Regularity of solutions to FBSDE (3.32)–(3.34) and its jacobian flow

We now give the upper bounds of $\mathbb{Y}_{tX}(s),\mathbb{Z}_{tX}(s),u_{tX}(s),\mathbb{r}_{tX,j}(s)$.

Proposition 3.9 (**Bounds of FBSDE's Solution).** *Under the assumptions of Proposition 3.5, for any $X\in\mathcal{H}_m$ measurable to $\mathcal{W}^t_0$ and $s\in(t,T)$, the solution quadruple $\big(\mathbb{Y}_{tX}(s),\mathbb{Z}_{tX}(s),u_{tX}(s),\mathbb{r}_{tX,j}(s)\big)$ to the FBSDE (3.32)–(3.34) on $[t,T]$ satisfies*

$$\|\mathbb{Y}_{tX}(s)\|_{\mathcal{H}_m},\ \|\mathbb{Z}_{tX}(s)\|_{\mathcal{H}_m},\ \|u_{tX}(s)\|_{\mathcal{H}_m},\ \left[\sum_{j=1}^n\int_t^T\left\|\mathbb{r}_{tX,j}(s)\right\|^2_{\mathcal{H}_m}ds\right]^{1/2}\le C_4\big(1+\|X\|_{\mathcal{H}_m}\big), \tag{3.38}$$

where C_4 is a positive constant depending only on n, λ, η, c, c_l, c_h, c_T, c', c'_l, c'_h, c'_T and T.

Proof. For simplicity, without any cause of ambiguity, we omit the subscripts tX in $\mathbb{Y}_{tX}(s)$, $\mathbb{Z}_{tX}(s)$, $u_{tX}(s)$ and $\mathbb{r}_{tX}(s)$. Recalling (3.33) that describes the dynamics of $\mathbb{Z}(s)$ and (3.34) that characterizes $\mathbb{Z}(s)$, an application of Itô's lemma to the inner product $\Big\langle \mathbb{Z}(s), \mathbb{Y}(s) - \eta\big(w(s) - w(t)\big)\Big\rangle_{\mathcal{H}_m}$ gives

$$\begin{aligned}\Big\langle \mathbb{Z}(t), X\Big\rangle_{\mathcal{H}_m} = &\int_t^T \Big\langle l_v(\mathbb{Y}(s), u(s)), u(s)\Big\rangle_{\mathcal{H}_m} ds + \int_t^T \Big\langle l_x(\mathbb{Y}(s), u(s)) + D_X F(\mathbb{Y}(s)\otimes m), \mathbb{Y}(s)\Big\rangle_{\mathcal{H}_m} ds \\ &- \int_t^T \Big\langle l_x(\mathbb{Y}(s), u(s)) + D_X F(\mathbb{Y}(s)\otimes m), \eta\big(w(s) - w(t)\big)\Big\rangle_{\mathcal{H}_m} ds \\ &+ \Big\langle h_x(\mathbb{Y}(T)) + D_X F_T(\mathbb{Y}(T)\otimes m), \mathbb{Y}(T) - \eta\big(w(T) - w(t)\big)\Big\rangle_{\mathcal{H}_m}.\end{aligned} \tag{3.39}$$

On the other hand, using the definition of dynamic of $\mathbb{Z}(s)$ in (3.33), we see

$$\Big\langle \mathbb{Z}(t), X\Big\rangle_{\mathcal{H}_m} = \Big\langle \int_t^T l_x(\mathbb{Y}(s), u(s)) + D_X F(\mathbb{Y}(s)\otimes m) ds, X\Big\rangle_{\mathcal{H}_m} + \Big\langle h_x(\mathbb{Y}(T)) + D_X F_T(\mathbb{Y}(T)\otimes m), X\Big\rangle_{\mathcal{H}_m}. \tag{3.40}$$

Therefore, subtracting (3.40) from (3.39), we have

$$\begin{aligned}0 = &\int_t^T \Big\langle l_v(\mathbb{Y}(s), u(s)), u(s)\Big\rangle_{\mathcal{H}_m} ds + \int_t^T \Big\langle l_x(\mathbb{Y}(s), u(s)) + D_X F(\mathbb{Y}(s)\otimes m), \mathbb{Y}(s) - X - \eta\big(w(s) - w(t)\big)\Big\rangle_{\mathcal{H}_m} ds \\ &+ \Big\langle h_x(\mathbb{Y}(T)) + D_X F_T(\mathbb{Y}(T)\otimes m), \mathbb{Y}(T) - X - \eta\big(w(T) - w(t)\big\rangle_{\mathcal{H}_m}.\end{aligned}$$

After telescoping, we obtain

$$\begin{aligned}0 = &\int_t^T \Big\langle l_v(\mathbb{Y}(s), u(s)) - l_v(X + \eta(w(s) - w(t)), 0), u(s)\Big\rangle_{\mathcal{H}_m} ds \\ &+ \int_t^T \Big\langle l_x(\mathbb{Y}(s), u(s)) - l_x(X + \eta(w(s) - w(t)), 0), \mathbb{Y}(s) - X - \eta\big(w(s) - w(t)\big)\Big\rangle_{\mathcal{H}_m} ds \\ &+ \int_t^T \Big\langle D_X F(\mathbb{Y}(s)\otimes m) - D_X F((X + \eta(w(s) - w(t)))\otimes m), \mathbb{Y}(s) - X - \eta\big(w(s) - w(t)\big)\Big\rangle_{\mathcal{H}_m} ds \\ &+ \Big\langle h_x(\mathbb{Y}(T)) - h_x\Big(X + \eta\big(w(T) - w(t)\big)\Big), \mathbb{Y}(T) - X - \eta(w(T) - w(t))\Big\rangle_{\mathcal{H}_m} \\ &+ \Big\langle D_X F_T(\mathbb{Y}(T)\otimes m) - D_X F_T\Big((X + \eta(w(T) - w(t)))\otimes m\Big), \mathbb{Y}(T) - X - \eta(w(T) - w(t))\Big\rangle_{\mathcal{H}_m} \\ &+ \int_t^T \Big\langle l_v(X + \eta(w(s) - w(t)), 0), u(s)\Big\rangle_{\mathcal{H}_m} ds \\ &+ \int_t^T \Big\langle l_x(X + \eta(w(s) - w(t)), 0) + D_X F((X + \eta(w(s) - w(t)))\otimes m), \mathbb{Y}(s) - X - \eta\big(w(s) - w(t)\big)\Big\rangle_{\mathcal{H}_m} ds \\ &+ \Big\langle h_x\Big(X + \eta\big(w(T) - w(t)\big)\Big) + D_X F_T\Big((X + \eta(w(T) - w(t)))\otimes m\Big), \int_t^T u(s) ds\Big\rangle_{\mathcal{H}_m}.\end{aligned}$$

From the mean value theorem, Assumptions **A(iii)**'s (3.6), **A(iv)**'s (3.8), **A(vi)**'s (3.9) and **B(i)**'s (3.22), we have

$$\begin{aligned}\lambda\int_t^T\|u(s)\|^2_{\mathcal{H}_m}ds\le{}&(c_l'+c')\int_t^T\|\mathbb{Y}(s)-X-\eta(w(s)-w(t))\|^2_{\mathcal{H}_m}ds+(c_h'+c_T')\left\|\int_t^T u(s)ds\right\|^2_{\mathcal{H}_m}\\&+c_l\big(1+\|X+\eta(w(T)-w(t))\|_{\mathcal{H}_m}\big)\int_t^T\|u(s)\|_{\mathcal{H}_m}ds\\&+(c_l+c)\big(1+\|X+\eta(w(T)-w(t))\|_{\mathcal{H}_m}\big)\int_t^T\|\mathbb{Y}(s)-X-\eta(w(s)-w(t))\|_{\mathcal{H}_m}ds\\&+(c_h+c_T)\big(1+\|X+\eta(w(T)-w(t))\big\|_{\mathcal{H}_m}\big)\int_t^T\|u(s)\|_{\mathcal{H}_m}ds.\end{aligned}$$

Applications of Cauchy–Schwarz and Young's inequalities further give

$$\begin{aligned}\lambda\int_t^T\|u(s)\|^2_{\mathcal{H}_m}ds\le{}&(c_l'+c')_+\frac{T^2}{2}\int_t^T\|u(s)\|^2_{\mathcal{H}_m}ds+\left[(c_h'+c_T')_++\frac{1}{4\kappa_4}\right]T\int_t^T\|u(s)\|^2_{\mathcal{H}_m}ds\\&+\kappa_4c_l^2\Big(1+\|X\|_{\mathcal{H}_m}+\big\|\eta(w(T)-w(t))\big\|_{\mathcal{H}_m}\Big)^2\\&+\kappa_5(c_l+c)^2\Big(1+\|X\|_{\mathcal{H}_m}+\big\|\eta(w(T)-w(t))\big\|_{\mathcal{H}_m}\Big)^2+\frac{T^2}{4\kappa_5}\int_t^T\|u(s)\|^2_{\mathcal{H}_m}ds\\&+\kappa_6(c_h+c_T)^2\Big(1+\|X\|_{\mathcal{H}_m}+\big\|\eta(w(T)-w(t))\big\|_{\mathcal{H}_m}\Big)^2+\frac{T}{4\kappa_6}\int_t^T\|u(s)\|^2_{\mathcal{H}_m}ds,\end{aligned}$$

for some positive constants κ_i determined later, with $i=3,4,5,6$. It is equivalent to

$$\begin{aligned}&\left\{\lambda-\left[(c_h'+c_T')_++\frac{1}{4\kappa_4}+\frac{T}{4\kappa_5}+\frac{1}{4\kappa_6}\right]T-(c_l'+c')_+\frac{T^2}{2}\right\}\int_t^T\|u(s)\|^2_{\mathcal{H}_m}ds\\&\le\kappa_4c_l^2\Big(1+\|X\|_{\mathcal{H}_m}+|\eta|\sqrt{nT}\Big)^2+\kappa_5(c_l+c)^2\Big(1+\|X\|_{\mathcal{H}_m}+|\eta|\sqrt{nT}\Big)^2+\kappa_6(c_h+c_T)^2\Big(1+\|X\|_{\mathcal{H}_m}+|\eta|\sqrt{nT}\Big)^2\\&\le2\Big[\kappa_4c_l^2+\kappa_5(c_l+c)^2+\kappa_6(c_h+c_T)^2\Big]\|X\|^2_{\mathcal{H}_m}+4\Big[\kappa_4c_l^2+\kappa_5(c_l+c)^2+\kappa_6(c_h+c_T)^2\Big](1+|\eta|^2nT).\end{aligned}\tag{3.41}$$

Taking suitable κ_i with $i=3,4,5,6$ and using (3.30) to ensure that the coefficient in the first line of (3.41) is strictly positive, we thus conclude that

$$\int_t^T\|u(s)\|^2_{\mathcal{H}_m}ds\le A_1\Big(1+\|X\|^2_{\mathcal{H}_m}\Big),\tag{3.42}$$

for some $A_1>0$ depending on n, λ, η, c, c_l, c_h, c_T, c', c_l', c_h', c_T' and T. Next, from the dynamics of $\mathbb{Y}(s)$ in (3.32), it yields that

$$\|\mathbb{Y}(s)-X\|^2_{\mathcal{H}_m}\le(s-t)(1+\kappa_7)\int_t^T\|u(\tau)\|^2_{\mathcal{H}_m}\mathrm{d}\tau+n|\eta|^2(s-t)\left(1+\frac{1}{\kappa_7}\right),\tag{3.43}$$

for some $\kappa_7 > 0$. From (3.42) and (3.43), we deduce that

$$\sup_{s\in(t,T)} \big\| \mathbb{Y}(s)\big\|^2_{\mathcal{H}_m} \leq 2 \sup_{s\in(t,T)} \big\| \mathbb{Y}(s) - X\big\|^2_{\mathcal{H}_m} + 2\big\| X\big\|^2_{\mathcal{H}_m} \leq A_2\Big(1 + \|X\|^2_{\mathcal{H}_m}\Big). \tag{3.44}$$

From (3.29) and Itô's formula in (2.16), we obtain

$$\begin{aligned}\big\| \mathbb{Z}(s)\big\|^2_{\mathcal{H}_m} + \sum_{j=1}^n \int_s^T \big\| \mathbb{r}_j(\tau)\big\|^2_{\mathcal{H}_m} \mathrm{d}\tau =& \Big\| h_x(\mathbb{Y}(T)) + D_X F_T(\mathbb{Y}(T)\otimes m)\Big\|^2_{\mathcal{H}_m} \\ &+ 2\int_s^T \Big\langle \mathbb{Z}(\tau), l_x(\mathbb{Y}(\tau), u(\tau)) + D_X F(\mathbb{Y}(\tau)\otimes m)\Big\rangle_{\mathcal{H}_m} \mathrm{d}\tau.\end{aligned}$$

Using Assumptions **A(i)**'s (3.4), **A(iii)**'s (3.6), **B(i)**'s (3.22) and (3.34), it follows that

$$\begin{aligned}&\sup_{s\in(t,T)} \big\| \mathbb{Z}(s)\big\|^2_{\mathcal{H}_m} + \sum_{j=1}^n \int_t^T \big\| \mathbb{r}_j(s)\big\|^2_{\mathcal{H}_m} ds \\ &\leq 2(c_h^2 + 2c_T^2)\Big(1 + \big\| Y(T)\big\|^2_{\mathcal{H}_m}\Big) + 3c_l^2 \int_t^T 1 + \big\| \mathbb{Y}(s)\big\|^2_{\mathcal{H}_m} + \big\| u(s)\big\|^2_{\mathcal{H}_m} ds + 4c^2 \int_t^T 1 + \big\| \mathbb{Y}(s)\big\|^2_{\mathcal{H}_m} ds.\end{aligned} \tag{3.45}$$

Using (3.44) and (3.42), we obtain the required upper bounds of $\sup_{s\in(t,T)} \big\| \mathbb{Z}(s)\big\|^2_{\mathcal{H}_m}$ and $\sum_{j=1}^n \int_t^T \big\| \mathbb{r}_j(s)\big\|^2_{\mathcal{H}_m} ds$. Finally, from (3.34), the assumptions (3.4), (3.8), as well as Young's inequality, we have

$$\lambda\big\| u(s)\big\|^2_{\mathcal{H}_m} \leq \frac{\lambda}{4}\big\| u(s)\big\|^2_{\mathcal{H}_m} + \frac{c_l^2}{\lambda}\Big(1 + \|\mathbb{Y}(s)\|^2_{\mathcal{H}_m}\Big) + \frac{1}{\lambda}\big\| \mathbb{Z}(s)\big\|^2_{\mathcal{H}_m} + \frac{\lambda}{4}\big\| u(s)\big\|^2_{\mathcal{H}_m}.$$

Results in (3.44) and (3.45) imply $\sup_{s\in(t,T)} \|u(s)\|_{\mathcal{H}_m} \leq A_3\Big(1 + \|X\|_{\mathcal{H}_m}\Big)$. This concludes Proposition 3.9. □

Given X, $\Psi \in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m) \subset \mathcal{H}_m$, we aim to check the existence of the Gâteaux derivative of the solution, with respect to the initial data X in the direction of Ψ, over $[t,T]$. Define $\mathcal{W}^s_{tX\Psi} := \sigma(X,\Psi)\bigvee \mathcal{W}^s_t$ and denote $\mathcal{W}_{tX\Psi}$ the filtration generated by the σ-algebras $\mathcal{W}^s_{tX\Psi}$. For $\epsilon > 0$, we define the difference processes

$$\begin{aligned}\Delta^\epsilon_\Psi \mathbb{Y}_{tX}(s) &:= \frac{\mathbb{Y}_{t,X+\epsilon\Psi}(s) - \mathbb{Y}_{tX}(s)}{\epsilon},\ \ \Delta^\epsilon_\Psi \mathbb{Z}_{tX}(s) := \frac{\mathbb{Z}_{t,X+\epsilon\Psi}(s) - \mathbb{Z}_{tX}(s)}{\epsilon}, \\ \Delta^\epsilon_\Psi u_{tX}(s) &:= \frac{u_{t,X+\epsilon\Psi}(s) - u_{tX}(s)}{\epsilon},\ \ \Delta^\epsilon_\Psi \mathbb{r}_{tX,j}(s) := \frac{\mathbb{r}_{t,X+\epsilon\Psi,j}(s) - \mathbb{r}_{tX,j}(s)}{\epsilon}.\end{aligned} \tag{3.46}$$

We next aim to bound them.

Lemma 3.10. *Let X, $\Psi \in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m) \subset \mathcal{H}_m$, $\epsilon > 0$ and $s \in [t,T]$. Under the assumptions of Proposition 3.5, the difference processes defined in (3.46) satisfy*

$$\big\|\Delta^\epsilon_\Psi \mathbb{Y}_{tX}(s)\big\|_{\mathcal{H}_m}, \big\|\Delta^\epsilon_\Psi \mathbb{Z}_{tX}(s)\big\|_{\mathcal{H}_m}, \big\|\Delta^\epsilon_\Psi u_{tX}(s)\big\|_{\mathcal{H}_m}, \left[\sum_{j=1}^n \int_t^T \big\|\Delta^\epsilon_\Psi \mathbb{r}_{tX,j}(s)\big\|^2_{\mathcal{H}_m} ds\right]^{1/2} \leq C_4'\|\Psi\|_{\mathcal{H}_m}, \tag{3.47}$$

where C_4' is a positive constant depending only on λ, η, c, c_l, c_h, c_T, c', c_l', c_h', c_T' and T.

Proof. From (3.32)–(3.34), the quadruple $\left(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s), \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s), \Delta_\Psi^\epsilon u_{tX}(s), \Delta_\Psi^\epsilon \mathbb{r}_{tX}(s)\right)$ solves the system

$$\begin{cases} \Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s) = \Psi + \int_t^s \Delta_\Psi^\epsilon u_{tX}(\tau)\mathrm{d}\tau; \\ \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s) = \int_0^1 h_{xx}\Big(\mathbb{Y}_{tX}(T) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T)\Big)\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T) + D_X^2 F_T\Big((\mathbb{Y}_{tX}(T) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T))\otimes m\Big)(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T))\mathrm{d}\theta \\ \qquad + \int_s^T \int_0^1 l_{xx}\Big(\mathbb{Y}_{tX}(\tau) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\Big)\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(\tau) \\ \qquad + l_{xv}\Big(\mathbb{Y}_{t,X+\epsilon\Psi}(\tau), u_{tX}(\tau) + \theta\epsilon\Delta_\Psi^\epsilon u_{tX}(\tau)\Big)\Delta_\Psi^\epsilon u_{tX}(\tau)\mathrm{d}\theta\mathrm{d}\tau \\ \qquad + \int_s^T \int_0^1 D_X^2 F\Big((\mathbb{Y}_{tX}(\tau) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(\tau))\otimes m\Big)(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(\tau))\mathrm{d}\theta\mathrm{d}\tau - \int_s^T \sum_{j=1}^n \Delta_\Psi^\epsilon \mathbb{r}_{tX,j}(\tau)dw_j(\tau), \end{cases}$$

meanwhile,

$$\int_0^1 l_{vx}\Big(\mathbb{Y}_{tX}(s) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s), u_{tX}(s)\Big)\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s) + l_{vv}\Big(\mathbb{Y}_{t,X+\epsilon\Psi}(s), u_{tX}(s) + \theta\epsilon\Delta_\Psi^\epsilon u_{tX}(s)\Big)\Delta_\Psi^\epsilon u_{tX}(s)\mathrm{d}\theta + \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s) = 0.$$

The proof actually follows the same steps as in the proof for Proposition 3.9 by applying Itô lemma to the inner product $\left\langle \Delta_\Psi^{\epsilon_j} \mathbb{Z}_{tX}(s), \Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(s)\right\rangle_{\mathcal{H}_m}$ and using the above equations, we omit it here. □

Lemma 3.11. *Let X, $\Psi \in L^2_{\mathcal{W}_t^{\perp\!\!\!\perp}}(\mathcal{H}_m)$. Under the assumptions of Proposition 3.5, the processes $\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s), \Delta_\Psi^\epsilon u_{tX}(s)$ and $\Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s)$ converge in $L^\infty_{\mathcal{W}_{tX\Psi}}(t,T;\mathcal{H}_m)$, and $\Delta_\Psi^\epsilon \mathbb{r}_{tX,j}(s)$ converge in $L^2_{\mathcal{W}_{tX\Psi}}(t,T;\mathcal{H}_m)$, as $\epsilon \to 0$. They are called the Jacobian flow (Gâteaux derivative) of the solution to the FBSDE (3.32)–(3.34). The limit of $\left(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s),\ \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s),\ \Delta_\Psi^\epsilon \mathbb{r}_{tX,j}(s)\right)$ is the unique solution in $L^2_{\mathcal{W}_{tX\Psi}}(t,T;\mathcal{H}_m)$ of the alternative FBSDE*

$$\begin{aligned} D\mathbb{Y}_{tX}^\Psi(s) &= \Psi + \int_t^s \Big[\nabla_y u(\mathbb{Y}_{tX},\mathbb{Z}_{tX})(\tau)\Big] D\mathbb{Y}_{tX}^\Psi(\tau)\mathrm{d}\tau + \int_t^s \Big[\nabla_z u(\mathbb{Y}_{tX},\mathbb{Z}_{tX})(\tau)\Big] D\mathbb{Z}_{tX}^\Psi(\tau)\mathrm{d}\tau; \\ D\mathbb{Z}_{tX}^\Psi(s) &= h_{xx}(\mathbb{Y}_{tX}(T))D\mathbb{Y}_{tX}^\Psi(T) + D_X^2 F_T(\mathbb{Y}_{tX}(T)\otimes m)(D\mathbb{Y}_{tX}^\Psi(T)) \\ &\quad + \int_s^T \Bigg\{ l_{xx}(\mathbb{Y}_{tX}(\tau),u(\tau))D\mathbb{Y}_{tX}^\Psi(\tau) + l_{xv}(\mathbb{Y}_{tX}(\tau),u(\tau))\Big[\nabla_y u(\mathbb{Y}_{tX},\mathbb{Z}_{tX})(\tau)\Big] D\mathbb{Y}_{tX}^\Psi(\tau) \\ &\qquad + l_{xv}(\mathbb{Y}_{tX}(\tau),u(\tau))\Big[\nabla_z u(\mathbb{Y}_{tX},\mathbb{Z}_{tX})(\tau)\Big] D\mathbb{Z}_{tX}^\Psi(\tau) \\ &\qquad + D_X^2 F(\mathbb{Y}_{tX}(\tau)\otimes m)\Big(D\mathbb{Y}_{tX}^\Psi(\tau)\Big)\Bigg\}\mathrm{d}\tau - \int_s^T \sum_{j=1}^n D\mathbb{r}_{tX,j}^\Psi(\tau)dw_j(\tau). \end{aligned} \tag{3.48}$$

Remark 3.12. We call that the control process $u_{tX}(s)$ satisfying the first order condition (3.34) is a map $u(y,z)$ evaluated at $(y,z) = (\mathbb{Y}_{tX}(s),\mathbb{Z}_{tX}(s))$ such that it is solved implicitly in the equation $l_v(y,u(y,z)) + z = 0$. The Jacobian matrices $\nabla_y u$ and $\nabla_z u$, with respect to the first and second arguments respectively, are bounded by $\Big|\nabla_y u\Big| \leq \frac{c_l}{\lambda}$ and $\Big|\nabla_z u\Big| \leq \frac{1}{\lambda}$, which can be verified by differentiating the first order condition (3.34).

The proof of Lemma 3.11 can be found in Appendix A.

Lemma 3.13. *Let X, $\Psi \in L^2_{\mathcal{W}_t^{\perp\!\!\!\perp}}(\mathcal{H}_m) \subset \mathcal{H}_m$ and $s \in [t,T]$. Under the assumptions of Proposition 3.5, the Jacobian flow $\left(D\mathbb{Y}_{tX}^\Psi(s), D\mathbb{Z}_{tX}^\Psi(s), Du_{tX}^\Psi(s),\ D\mathbb{r}_{tX,j}^\Psi(s)\right)$ is linear in Ψ, and each of them is partially continuous* *in X for a given Ψ. Thus, their Fréchet derivatives exist, and they are denoted by $D\mathbb{Y}_{tX}(s), D\mathbb{Z}_{tX}(s), Du_{tX}(s)$, $D\mathbb{r}_{tX,j}(s)$, respectively.*

The proof can be found in Appendix A.

4. Regularities of value function and its functional derivatives

In this section, the quadruple $\big(\mathbb{Y}_{tX}(s), \mathbb{Z}_{tX}(s), u_{tX}(s), \mathbb{r}_{tX}(s)\big)$ is the solution to the FBSDE (3.32)–(3.34) and $V(X\otimes m,t)$ is the corresponding value function defined in (3.37).

4.1. Growth and regularities of value function and its derivatives

Due to the quadratic growths in x of the running and terminal cost functions, we have the same growth for the value function in X.

Proposition 4.1 (Quadratic Growth of V in X). *Under the assumptions of Proposition 3.5, for any $X\in\mathcal{H}_m$ measurable to $\mathcal{W}_0^t$, we have $|V(X\otimes m,t)|\le C_5\big(1+\|X\|^2_{\mathcal{H}_m}\big)$, where C_5 is a positive constants depending only on n, λ, η, c, c_l, c_h, c_T, c', c_l', c_h', c_T' and T.*

Proof. Recalling the value function in Formula (3.37) and applying Assumptions **A(i)**'s (3.4), **A(iii)**'s (3.6), (3.21) and the bounds of $\mathbb{Y}_{tX}$, u_{tX} in Proposition 3.9, we can then deduce the estimate in Propositions 4.1. ☐

Next, we have the Lipschitz nature of the derivative $D_XV(X\otimes m,t)$ with respect to $X\in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m)$ which is defined in (2.17).

Proposition 4.2 (Lipschitz Continuity of D_XV in X). *Under the assumptions in Proposition 3.5, the functional $X\in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m)\longmapsto V(X\otimes m,t)$ is Fréchet differentiable and $D_XV(X\otimes m,t)=\mathbb{Z}_{tX}(t)$. Moreover, for any $X^1,X^2\in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m)$, the derivative is Lipschitz continuous such that $\big\|D_XV(X^1\otimes m,t)-D_XV(X^2\otimes m,t)\big\|_{\mathcal{H}_m}\le C_6\big\|X^1-X^2\big\|_{\mathcal{H}_m}$, where C_6 depends only on λ, c, c_l, c_h, c_T, c', c_h', c_T', c_l' and T.*

The proof can be found in Appendix A. As a consequence of the optimality principle mentioned in Section 5.1, we have the time consistency, namely, the control $u_{tX}(\tau)$, for $\tau\in[s,T)$ with $s\in[t,T)$, remains optimal for the problem with initial conditions $(s,\mathbb{Y}_{tX}(s))$ and thus we also have

$$D_XV(\mathbb{Y}_{tX}(s)\otimes m,s)=\mathbb{Z}_{s\mathbb{Y}_{tX}(s)}(s)=\mathbb{Z}_{tX}(s), \tag{4.1}$$

due to the flow property (5.2). We then proceed on the regularity in time of the value function.

Proposition 4.3 (Hölder Continuity in Time of V and D_XV). *With the assumptions in Proposition 3.5, it holds that for any $X\in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m)$ and t_1, $t_2\in[0,T]$,*

$$\Big|V(X\otimes m,t_1)-V(X\otimes m,t_2)\Big|\le C_7\Big(1+\|X\|^2_{\mathcal{H}_m}\Big)\Big|t_1-t_2\Big| \quad \textit{and} \tag{4.2}$$

$$\Big\|D_XV(X\otimes m,t_1)-D_XV(X\otimes m,t_2)\Big\|_{\mathcal{H}_m}\le C_8\left(\Big|t_1-t_2\Big|^{\frac{1}{2}}+\|X\|_{\mathcal{H}_m}\Big|t_1-t_2\Big|\right), \tag{4.3}$$

where C_7 and C_8 depend only on n, λ, η, c, c_l, c_h, c_T, c', c_l', c_h', c_T' and T.

The proof can be found in Appendix A.

Remark 4.4. With $t_1 < t_2$, since $\mathbb{Z}_{t_1X}(t_2) = \mathbb{Z}_{t_2,\mathbb{Y}_{t_1X}(t_2)}(t_2)$ (see (5.2)), together with Proposition 4.2, (4.3) and (3.43), we have $\left\|\mathbb{Z}_{t_1X}(t_2) - \mathbb{Z}_{t_2X}(t_2)\right\|_{\mathcal{H}_m} \leq C_6\left\|\mathbb{Y}_{t_1X}(t_2) - X\right\|_{\mathcal{H}_m}$ and $\left\|\mathbb{Z}_{t_1X}(t_2) - \mathbb{Z}_{t_1X}(t_1)\right\|_{\mathcal{H}_m} \leq C_8'\Big(\left|t_1 - t_2\right|^{\frac{1}{2}} + \left|t_1 - t_2\right| \|X\|_{\mathcal{H}_m}\Big)$.

4.2. Second-order Gâteaux derivative

In the following proposition, we characterize the second-order Gâteaux derivative of the value function by the Jacobian flow as the solution to the backward equation in (3.48).

Proposition 4.5. *Let X, $\Psi \in \mathcal{H}_m \in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m)$. Under the assumptions (3.4)–(3.9), (3.21)–(3.27) and (3.30), the value function $V(X \otimes m, t)$ has a second-order Gâteaux derivative with respect to X satisfying*

$$D_X^2 V(X \otimes m, t)(\Psi) = D\mathbb{Z}_{tX}^{\Psi}(t) \quad \textit{and} \quad \left\|D_X^2 V(X \otimes m, t)(\Psi)\right\|_{\mathcal{H}_m} \leq C_9 \|\Psi\|_{\mathcal{H}_m}, \tag{4.4}$$

where C_9 is a positive constant depending only on n, λ, η, c, c_l, c_h, c_T, c', c_l', c_h', c_T' and T.

Proof. Since $\mathbb{Z}_{tX}(t) = D_X V(X \otimes m, t)$ due to Proposition 4.2, by differentiating both sides and using the definition of Gâteaux second derivative in (2.9) and the strong convergence in Lemma 3.11, we obtain

$$\begin{aligned}\left\langle D_X^2 V(X \otimes m, t)(\Psi), \Psi\right\rangle_{\mathcal{H}_m} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\langle D_X V((X+\epsilon\Psi)\otimes m, t) - D_X V(X\otimes m, t), \Psi\right\rangle_{\mathcal{H}_m} &= \lim_{\epsilon\to 0}\left\langle \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(t), \Psi\right\rangle_{\mathcal{H}_m} \\ &= \left\langle D\mathbb{Z}_{tX}^{\Psi}(t), \Psi\right\rangle_{\mathcal{H}_m},\end{aligned}$$

together with the bound in (3.47), as a limit, we also obtain the upper bound in (4.4). □

We then state additional pointwise continuity properties for the second-order derivative.

Proposition 4.6. *Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence decreasing to $t \in [0, T)$. Suppose that $\{X_k\}_{k\in\mathbb{N}}$ and $\{\Psi_k\}_{k\in\mathbb{N}}$ are sequences converging to X and Ψ in $\mathcal{H}_m$, respectively, such that each X_k, $\Psi_k \in L^2_{\mathcal{W}_{t_k}^{\perp\!\!\perp}}(\mathcal{H}_m)$. Under the assumptions of Proposition 4.5, we have $D_X^2 V(X_k \otimes m, t_k)(\Psi_k) \longrightarrow D_X^2 V(X \otimes m, t)(\Psi)$ in $\mathcal{H}_m$.*

Remark 4.7. As a consequence, using Cauchy–Schwarz inequality, we can deduce the L^1-convergence

$$D_X^2 V(X_k \otimes m, t_k)(\Psi_k) - D_X^2 V(X \otimes m, t_k)(\Psi_k) \longrightarrow 0 \quad \text{in } L^1(\Omega, \mathcal{A}, \mathbb{P}; L^1_m(\mathbb{R}^n));$$

indeed, the quantity

$$\begin{aligned}&\int_{\mathbb{R}^n} \mathbb{E}\left[\left| D_X^2 V(X_k \otimes m, t_k)(\Psi_k) - D_X^2 V(X \otimes m, t_k)(\Psi_k)\right|\right] \mathrm{d}m(x) \\ &\leq \left\|D_X^2 V(X_k \otimes m, t_k)(\Psi_k) - D_X^2 V(X \otimes m, t)(\Psi)\right\|_{\mathcal{H}_m} + \left\|D_X^2 V(X \otimes m, t)(\Psi) - D_X^2 V(X \otimes m, t_k)(\Psi_k)\right\|_{\mathcal{H}_m}\end{aligned}$$

tends to 0 as $k \to \infty$. It ensures a valid application of Itô lemma (Thm. 2.3) in Section 5.

The proof of Proposition 4.6 can be found in Appendix A.

4.3. Linear functional derivatives

In the discussion before, we use the notations of the processes $\mathbb{Y}_{tX}(s)$, $\mathbb{Z}_{tX}(s)$, $u_{tX}(s)$, $\mathbb{r}_{tX,j}(s)$ to emphasize the dependence on the initial random variable X. In this section, we take $X = \mathcal{I}_x = x$, and then we denote the processes $\mathbb{Y}_{t\mathcal{I}}(s)$, $\mathbb{Z}_{t\mathcal{I}}(s)$, $u_{t\mathcal{I}}(s)$, $\mathbb{r}_{t\mathcal{I},j}(s)$ by $\mathbb{Y}_{txm}(s)$, $\mathbb{Z}_{txm}(s)$, $u_{txm}(s)$, $\mathbb{r}_{txm,j}(s)$ so as to emphasize the dependence on the initial distribution m and the initial point x. With the new notations, the system (3.32)–(3.34) can be rewritten as

$$\begin{cases} \mathbb{Y}_{txm}(s) = x + \int_t^s u_{txm}(\tau)\mathrm{d}\tau + \eta(w(s) - w(t)); & (4.5) \\ \mathbb{Z}_{txm}(s) = h_x(\mathbb{Y}_{txm}(T)) + \nabla_x \frac{dF_T}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(T) \otimes m)\Big(\mathbb{Y}_{txm}(T)\Big) + \int_s^T l_x\Big(\mathbb{Y}_{txm}(\tau), u_{txm}(\tau)\Big)\mathrm{d}\tau \\ \qquad + \int_s^T \nabla_x \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(\tau) \otimes m)\Big(\mathbb{Y}_{txm}(\tau)\Big)\mathrm{d}\tau - \int_s^T \sum_{j=1}^n \mathbb{r}_{txm,j}(\tau) dw_j(\tau); & (4.6) \end{cases}$$

subject to
$$l_v\Big(\mathbb{Y}_{txm}(s), u_{txm}(s)\Big) + \mathbb{Z}_{txm}(s) = 0. \tag{4.7}$$

In order to study the master equation in Section 5, it is necessary to consider the linear functional derivative of the value function with respect to m. Clearly $\mathcal{I}_x \otimes m = m$, we write $V(m,t) = V(\mathcal{I} \otimes m, t)$, which is given by

$$V(m,t) = \int_t^T \mathbb{E}\left[\int_{\mathbb{R}^n} l\Big(\mathbb{Y}_{txm}(s), u_{txm}(s)\Big)\mathrm{d}m(x)\right] + F(\mathbb{Y}_{t\cdot m}(s) \otimes m)ds + \mathbb{E}\left[\int_{\mathbb{R}^n} h(\mathbb{Y}_{txm}(T))\mathrm{d}m(x)\right] + F_T(\mathbb{Y}_{t\cdot m}(T) \otimes m).$$

Recalling from Proposition 4.2, the Gâteaux derivative of V equals $D_X V(m,t) = D_X V(X \otimes m)\big|_{X=\mathcal{I}} = \mathbb{Z}_{t\mathcal{I}}(t) = \mathbb{Z}_{txm}(t)$. Next, we want to discuss the linear functional derivative $\frac{dV}{\mathrm{d}\nu}(m,t)$.

Proposition 4.8. *Under the assumptions of Proposition 4.5, the value function $V(m,t)$ has the linear functional derivative $\frac{dV}{\mathrm{d}\nu}(m,t)(x)$ satisfying*

(a)
$$\begin{aligned} \frac{dV}{\mathrm{d}\nu}(m,t)(x) = & \int_t^T \left\{\mathbb{E}\Big[l(\mathbb{Y}_{txm}(s), u_{txm}(s))\Big] + \mathbb{E}\left[\frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(s) \otimes m)(\mathbb{Y}_{txm}(s))\right]\right\} ds \\ & + \mathbb{E}\Big[h(\mathbb{Y}_{txm}(T))\Big] + \mathbb{E}\left[\frac{dF_T}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(T) \otimes m)(\mathbb{Y}_{txm}(T))\right], \end{aligned} \tag{4.8}$$

with all the expectations $\mathbb{E}$'s being taken with respect to the Brownian motion only;

(b)
$$\nabla_x \frac{dV}{\mathrm{d}\nu}(m,t)(x) = \mathbb{Z}_{txm}(t); \tag{4.9}$$

(c)
$$\frac{dV}{\mathrm{d}\nu}(m,t)(x) = \int_0^1 \mathbb{Z}_{t,\theta x,m}(t) \cdot x\mathrm{d}\theta + C(t), \tag{4.10}$$

where $C(t)$ depends only on t but not x, and which can be chosen to be 0 if the normalization condition $\int_{\mathbb{R}^n} \frac{dV}{\mathrm{d}\nu}(m,t)(x)\mathrm{d}m(x) = 0$ is taken.

Remark 4.9. Formulae (4.1) and (4.9) tell us that $\mathbb{Z}_{txm}(s) = \nabla_x \frac{dV}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(s) \otimes m, s)(\mathbb{Y}_{txm}(s))$ for $s \in (t,T]$.

Proof. Since the Gâteaux derivative $D_X V(X \otimes m, t)$ exists due to Proposition 4.2 and satisfies the regularity in (3.38), the linear functional derivative also exists and is given by (2.8), because of Proposition 4.1 in [37]. Utilizing (2.7) and the fact that $\mathbb{Z}_{tX}(t) = D_X V(X \otimes m, t)$, we know that (4.9) holds, and a simple integration and the mean value theorem give (4.10). We now turn to prove (4.8).

Denote the function on the right hand side of (4.8) by $\mathcal{S}(x,m,t)$, it suffices to show that

$$\mathbb{Z}_{txm}(t) = \nabla_x \mathcal{S}(x,m,t) \tag{4.11}$$

since the linear functional derivative is defined up to a constant function independent of x, but possibly depending on t (see also (2.2)). To this end, we consider the control problem described as follows: for any control $\alpha_{tx} \in L^2_{\mathcal{W}_t}(t,T;\mathbb{R}^n)$, consider the state process by

$$\mathbb{A}_{tx}(s) = x + \int_t^s \alpha_{tx}(\tau)\mathrm{d}\tau + \eta(w(s) - w(t)). \tag{4.12}$$

We aim to minimize the following objective function:

$$\begin{aligned} \mathcal{S}^*_{tx}(\alpha_{tx}) := \mathbb{E}\left[\int_t^T l(\mathbb{A}_{tx}(s), \alpha_{tx}(s))ds\right] + \int_t^T \mathbb{E}\left[\frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(s)\otimes m)(\mathbb{A}_{tx}(s))ds\right] \\ + \mathbb{E}\Big[h(\mathbb{A}_{tx}(T))\Big] + \mathbb{E}\left[\frac{dF_T}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(T)\otimes m)(\mathbb{A}_{tx}(T))\right], \end{aligned} \tag{4.13}$$

where $\mathbb{Y}_{txm}(s)$ is given by the system (4.5)–(4.7). The necessary conditions of optimality, working similarly to Lemma 3.4, is given by

$$l_v(\mathbb{A}_{tx}(s), \alpha_{tx}(s)) + \mathbb{K}_{tx}(s) = 0, \quad \text{and} \tag{4.14}$$

$$\begin{cases} -\mathrm{d}\mathbb{K}_{tx}(s) = \left[l_x\Big(\mathbb{A}_{tx}(s), \alpha_{tx}(s)\Big) + \nabla_x \dfrac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(s)\otimes m)\Big(\mathbb{A}_{tx}(s)\Big)\right]ds - \displaystyle\sum_{j=1}^n \kappa_{tx,j}(s)dw_j(s); \\ \mathbb{K}_{tx}(T) = h_x\Big(\mathbb{A}_{tx}(T)\Big) + \nabla_x \dfrac{dF_T}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{A}_{tx}(T)\Big), \end{cases} \tag{4.15}$$

for some adapted processes $\kappa_{tx,j}(s) \in L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_m)$ with $j = 1,2,\ldots,n$. It is clear that $\mathbb{A}_{tx} = \mathbb{Y}_{txm}$, $\mathbb{K}_{tx} = \mathbb{Z}_{txm}$, $\alpha_{tx} = u_{txm}$, $\kappa_{tx,j} = \mathbb{r}_{txm,j}$ satisfy (4.12), (4.15) and (4.14) respectively. Since (3.30) holds by the assumption of this proposition, it implies the strict convexity of the objective function $\mathcal{S}^*_{tx}(\alpha_{tx})$, and thus the optimal controlled process is unique. Therefore, the optimal control is $u_{txm}(s)$, and the optimal state is $\mathbb{Y}_{txm}(s)$. The Bellman equation for (4.12)–(4.13) reads

$$\begin{cases} \dfrac{\partial \Phi}{\partial s} + H(x, \nabla_x\Phi) + \dfrac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(s)\otimes m)(x) + \dfrac{1}{2}\displaystyle\sum_{j=1}^n D_x^2\Phi\eta^j\cdot\eta^j = 0, \\ \Phi(x,T) = h(x) + \dfrac{dF_T}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(T)\otimes m)(x), \end{cases}$$

where η^j is the j-th column vector of the matrix η. By the uniqueness of the Bellman equation, we see that $\Phi(x,t) = \mathcal{S}(x,m,t)$ and $\mathbb{Z}_{txm}(t) = \nabla_x\mathcal{S}(x,m,t)$ which concludes (4.11). □

We next study the second-order derivative of the linear functional derivative of the value function V and its connection with the second-order Gâteaux derivative of V. With $X = \mathcal{I}_x = x$, just like above, we adopt the notation of Jacobian flow $D\mathbb{Y}^\Psi_{tX}(s) = D\mathbb{Y}^\Psi_{txm}(s)$, $D\mathbb{Z}^\Psi_{tX}(s) = D\mathbb{Z}^\Psi_{txm}(s)$, $Du^\Psi_{tX}(s) = Du^\Psi_{txm}(s)$ and $D\mathbb{r}^\Psi_{tX}(s) =$

$D\mathbb{r}^{\Psi}_{txm}(s)$ such that the governing system (3.48) can be rewritten as

$$\begin{cases} D\mathbb{Y}^{\Psi}_{txm}(s) =\Psi + \displaystyle\int_t^s \Big[\nabla_y u(\mathbb{Y}_{txm},\mathbb{Z}_{txm})(\tau)\Big] D\mathbb{Y}^{\Psi}_{txm}(\tau)\mathrm{d}\tau + \int_t^s \Big[\nabla_z u(\mathbb{Y}_{txm},\mathbb{Z}_{txm})(\tau)\Big] D\mathbb{Z}^{\Psi}_{txm}(\tau)\mathrm{d}\tau; \\ D\mathbb{Z}^{\Psi}_{txm}(s) =h_{xx}(\mathbb{Y}_{txm}(T))D\mathbb{Y}^{\Psi}_{txm}(T) + D_X^2 F_T(\mathbb{Y}_{txm}(T)\otimes m)(D\mathbb{Y}^{\Psi}_{txm}(T)) \\ \qquad + \displaystyle\int_s^T \Big\{ l_{xx}(\mathbb{Y}_{txm}(\tau),u(\tau))D\mathbb{Y}^{\Psi}_{txm}(\tau) + l_{xv}(\mathbb{Y}_{txm}(\tau),u(\tau))\Big[\nabla_y u(\mathbb{Y}_{txm},\mathbb{Z}_{txm})(\tau)\Big] D\mathbb{Y}^{\Psi}_{txm}(\tau) \\ \qquad + l_{xv}(\mathbb{Y}_{txm}(\tau),u(\tau))\Big[\nabla_z u(\mathbb{Y}_{txm},\mathbb{Z}_{txm})(\tau)\Big] D\mathbb{Z}^{\Psi}_{txm}(\tau) \\ \qquad + D_X^2 F(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(D\mathbb{Y}^{\Psi}_{txm}(\tau)\Big)\Big\}\mathrm{d}\tau - \displaystyle\int_s^T \sum_{j=1}^n D\mathbb{r}^{\Psi}_{txm,j}(\tau)dw_j(\tau). \end{cases} \tag{4.16}$$

The existence of solution to (4.16) is established in Lemma 3.11. Result (4.4) can also be rewritten as $D\mathbb{Z}^{\Psi}_{txm}(t) = D_X^2 V(m,t)(\Psi)$. On the other hand, we consider the derivative of $\Big(\mathbb{Y}_{txm}(s),\mathbb{Z}_{txm}(s),u_{txm}(s),\mathbb{r}_{txm,j}(s)\Big)$ with respect to x, for instance, the notation $\nabla_x\mathbb{Y}_{txm}(s)$ denotes the matrix-valued random process with (i,j) entry $\Big[D\mathbb{Y}^{e_j}_{txm}(s)\Big]_i$, here the l-th coordinate of e_j is δ_{jl}. Without involving the test random vector Ψ, (4.16) can be rewritten as

$$\begin{cases} \nabla_x\mathbb{Y}_{txm}(s) =Id + \displaystyle\int_t^s \Big[\nabla_y u(\mathbb{Y}_{txm},\mathbb{Z}_{txm})(\tau)\Big] \nabla_x\mathbb{Y}_{txm}(\tau)\mathrm{d}\tau + \int_t^s \Big[\nabla_z u(\mathbb{Y}_{txm},\mathbb{Z}_{txm})(\tau)\Big] \nabla_x\mathbb{Z}_{txm}(\tau)\mathrm{d}\tau; \\ \nabla_x\mathbb{Z}_{txm}(s) =h_{xx}(\mathbb{Y}_{txm}(T))\nabla_x\mathbb{Y}_{txm}(T) + D_X^2 F_T(\mathbb{Y}_{txm}(T)\otimes m)(\nabla_x\mathbb{Y}_{txm}(T)) \\ \qquad + \displaystyle\int_s^T \Big\{ l_{xx}(\mathbb{Y}_{txm}(\tau),u(\tau))\nabla_x\mathbb{Y}_{txm}(\tau) + l_{xv}(\mathbb{Y}_{txm}(\tau),u(\tau))\Big[\nabla_y u(\mathbb{Y}_{txm},\mathbb{Z}_{txm})(\tau)\Big] \nabla_x\mathbb{Y}_{txm}(\tau) \\ \qquad + l_{xv}(\mathbb{Y}_{txm}(\tau),u(\tau))\Big[\nabla_z u(\mathbb{Y}_{txm},\mathbb{Z}_{txm})(\tau)\Big] \nabla_x\mathbb{Z}_{txm}(\tau) \\ \qquad + D_X^2 F(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\nabla_x\mathbb{Y}_{txm}(\tau)\Big)\Big\}\mathrm{d}\tau - \displaystyle\int_s^T \sum_{k=1}^n \nabla_x\mathbb{r}_{txm,k}(\tau)dw_j(\tau). \end{cases} \tag{4.17}$$

Due to the uniqueness of the system (3.48) and Lemma 3.13, we compare (4.16) and (4.17) to obtain that

$$D\mathbb{Y}^{\Psi}_{txm}(s) = \nabla_x\mathbb{Y}_{txm}(s)\Psi\ , D\mathbb{Z}^{\Psi}_{txm}(s) = \nabla_x\mathbb{Z}_{txm}(s)\Psi\ , Du^{\Psi}_{txm}(s) = \nabla_x u_{txm}(s)\Psi\ , D\mathbb{r}^{\Psi}_{txm,k}(s) = \nabla_x\mathbb{r}_{txm,k}(s)\Psi, \tag{4.18}$$

where each term on the right hand side is the matrix product of the gradient matrix and the random vector Ψ. Recalling from (2.13), we have

$$\begin{aligned} D_X^2 F(\mathbb{Y}_{t\cdot m}(s)\otimes m)(D\mathbb{Y}^{\Psi}_{txm}(s)) = {} & \nabla_x^2\frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}\otimes m)(\mathbb{Y}_{txm})D\mathbb{Y}^{\Psi}_{txm}(s) \\ & + \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(s)\otimes m)(\mathbb{Y}_{txm}(s),\widetilde{\mathbb{Y}}_{t\widetilde{x}m}(s))D\widetilde{\mathbb{Y}}^{\widetilde{\Psi}}_{t\widetilde{x}m}(s)\mathrm{d}m(\widetilde{x})\right]. \end{aligned}$$

Here $(\widetilde{\mathbb{Y}}_{t\widetilde{x}m}(s), D\widetilde{\mathbb{Y}}^{\widetilde{\Psi}}_{t\widetilde{x}m}(s))$ is an independent copy of $(\mathbb{Y}_{txm}(s), D\mathbb{Y}^{\Psi}_{txm}(s))$. A similar expression holds for F_T. If Ψ is independent of $\mathcal{W}^s_0$ with $\mathbb{E}(\Psi) = 0$, then
(a) we use (4.18) to obtain

$$\begin{aligned}
&D^2_X F(\mathbb{Y}_{t\cdot m}(s)\otimes m)(D\mathbb{Y}^{\Psi}_{txm}(s))\\
&= \nabla^2_x \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}\otimes m)(\mathbb{Y}_{txm})D\mathbb{Y}^{\Psi}_{txm}(s) + \int_{\mathbb{R}^n} \widetilde{\mathbb{E}}\left[\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(s)\otimes m)(\mathbb{Y}_{txm}(s), \widetilde{\mathbb{Y}}_{t\widetilde{x}m}(s))\nabla_x\widetilde{\mathbb{Y}}_{t\widetilde{x}m}(s)\right]\widetilde{\mathbb{E}}\Big(\widetilde{\Psi}\Big)\mathrm{d}m(\widetilde{x})\\
&= \nabla^2_x \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}\otimes m)(\mathbb{Y}_{txm})D\mathbb{Y}^{\Psi}_{txm}(s);
\end{aligned} \tag{4.19}$$

(b) moreover, under the assumptions of Proposition 4.5, similar to (4.19), Proposition 4.5 and (4.18) imply

$$\nabla^2_x \frac{dV}{\mathrm{d}\nu}(m,t)(x)\Psi = D^2_X V(m,t)(\Psi) = D\mathbb{Z}^{\Psi}_{txm}(t) = \nabla_x \mathbb{Z}_{txm}(t)\Psi. \tag{4.20}$$

5. Bellman and master equations

5.1. Optimality principle and Bellman equation

The dynamic programming is the key to establishing the Bellman equation, it is read as: for $\epsilon \in (0, T-t)$,

$$V(X\otimes m, t) = \int_t^{t+\epsilon}\int_{\mathbb{R}^n} \mathbb{E}\Big[l\Big(\mathbb{Y}_{tX}(s), u_{tX}(s)\Big)\Big]\,\mathrm{d}m(x) + F(\mathbb{Y}_{tX}(s)\otimes m)\,ds + V(\mathbb{Y}_{tX}(t+\epsilon)\otimes m, t+\epsilon). \tag{5.1}$$

The above follows by the usual arguments consisting of showing that for $X^\epsilon = \mathbb{Y}_{tX}(t+\epsilon)$ and $s \in [t+\epsilon, T]$, the flow property of

$$\mathbb{Y}_{t+\epsilon, X^\epsilon}(s) = \mathbb{Y}_{tX}(s),\ \mathbb{Z}_{t+\epsilon, X^\epsilon}(s) = \mathbb{Z}_{tX}(s),\ u_{t+\epsilon, X^\epsilon}(s) = u_{tX}(s),\ \mathbb{r}_{t+\epsilon, X^\epsilon, j}(s) = \mathbb{r}_{tX, j}(s), \tag{5.2}$$

which is clearly true due to the uniqueness of solution to the FBSDE (3.32)–(3.34). It implies that

$$\begin{aligned}
V(\mathbb{Y}_{tX}(t+\epsilon)\otimes m, t+\epsilon) = &\int_{t+\epsilon}^T\int_{\mathbb{R}^n} \mathbb{E}\Big[l\Big(\mathbb{Y}_{tX}(s), u_{tX}(s)\Big)\Big]\,\mathrm{d}m(x) + F(\mathbb{Y}_{tX}(s)\otimes m)\,ds\\
&+ \int_{\mathbb{R}^n} \mathbb{E}\Big[h(\mathbb{Y}_{tX}(T)\Big]\,dm(x) + F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big).
\end{aligned}$$

We can now state the following.

Theorem 5.1. *Let $X \in L^2_{\mathcal{W}^{\perp\!\!\perp}_t}(\mathcal{H}_m)$. Under the assumptions (3.4)–(3.9), (3.21)–(3.27), (3.30), we have:*

(a) The value function $V(X\otimes m, t)$ defined in (3.37) is a solution to the Bellman equation

$$\begin{cases}
\dfrac{\partial\Phi}{\partial t}(X\otimes m, t) + \mathbb{E}\left[\displaystyle\int_{\mathbb{R}^n} H\Big(X, D_X\Phi(X\otimes m, t)\Big)\,\mathrm{d}m(x)\right] + F(X\otimes m)\\
+ \dfrac{1}{2}\left\langle D^2_X\Phi(X\otimes m, t)\left(\displaystyle\sum_{j=1}^n \eta^j\mathcal{N}^j_t\right), \displaystyle\sum_{j=1}^n \eta^j\mathcal{N}^j_t\right\rangle_{\mathcal{H}_m} = 0;\\
\Phi(X\otimes m, T) = \mathbb{E}\left[\displaystyle\int_{\mathbb{R}^n} h(X)\mathrm{d}m(x)\right] + F_T(X\otimes m),
\end{cases} \tag{5.3}$$

where η^j is the j-th column vector of the matrix η, H is the Hamiltonian defined in (3.31); and $\mathcal{N}_t^j$'s are Gaussian random variables each of them having a mean 0 and a unit variance, and they are independent of each other and also of X;

(b) Among all functions satisfying the regularity properties in Propositions 4.1, Proposition 4.2, (4.3), (4.2), (4.4) and Proposition 4.6, $V(X\otimes m,t)$ is the unique solution to the Bellman equation (5.3).

Its proof is placed in Appendix A. Particularly, when $X=\mathcal{I}_x$, thanks to (4.20), we have

$$\begin{aligned}\left\langle D_X^2 V(m,t)\left(\sum_{j=1}^n \eta^j\mathcal{N}_t^j\right),\sum_{j=1}^n \eta^j\mathcal{N}_t^j\right\rangle_{\mathcal{H}_m} &= \left\langle \nabla_x^2\frac{dV}{\mathrm{d}\nu}(m,t)(x)\sum_{j=1}^n \eta^j\mathcal{N}_t^j,\sum_{j=1}^n \eta^j\mathcal{N}_t^j\right\rangle_{\mathcal{H}_m}\\ &= \int_{\mathbb{R}^n}\sum_{j=1}^n \nabla_x^2\frac{dV}{\mathrm{d}\nu}(m,t)(x)\eta^j\cdot\eta^j\mathrm{d}m(x),\end{aligned}$$

by then the Bellman equation in this case reads

$$\begin{cases}\dfrac{\partial V}{\partial t}(m,t)+\dfrac{1}{2}\displaystyle\sum_{j=1}^n\int_{\mathbb{R}^n}\nabla_x^2\frac{dV}{\mathrm{d}\nu}(m,t)(x)\eta^j\cdot\eta^j\mathrm{d}m(x)+\int_{\mathbb{R}^n}H\left(x,\nabla_x\frac{dV}{\mathrm{d}\nu}(m,t)(x)\right)\mathrm{d}m(x)+F(m)=0,\\ V(m,T)=\displaystyle\int_{\mathbb{R}^n}h(x)\mathrm{d}m(x)+F_T(m).\end{cases}\tag{5.4}$$

5.2. Master equation

Up to the last section, the mean field type control problem (3.2)–(3.3) has already been completely resolved. In this section, we introduce the master equation of the mean field type control problem, and show how we apply the assertions we have so far to solve the master equation. By using (4.20), the Bellman equation in (5.4) is written as follows,

$$-\frac{\partial V}{\partial t}(m,t)-\frac{1}{2}\sum_{j=1}^n\int_{\mathbb{R}^n}\nabla_\xi\mathbb{Z}_{t\xi m}(t)\eta^j\cdot\eta^j\mathrm{d}m(\xi)=\int_{\mathbb{R}^n}H(\xi,\mathbb{Z}_{t\xi m}(t))\mathrm{d}m(\xi)+F(m).\tag{5.5}$$

For if the linear functional derivatives of $\mathbb{Z}_{t\xi m}(s)$ and $\nabla_\xi\mathbb{Z}_{t\xi m}(s)$ with respect to m exist (their existence will be established in Prop. 5.3 and 5.5 respectively), which are denoted by $\frac{\mathrm{d}\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s)$ and $\frac{\mathrm{d}\nabla_\xi\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s)$ respectively, further, if we set $U(x,m,t):=\frac{dV}{\mathrm{d}\nu}(m,t)(x)$, by differentiating (5.5) both sides with respect to m, we can write

$$\begin{aligned}&-\frac{\partial U}{\partial t}(x,m,t)-\frac{1}{2}\sum_{j=1}^n\nabla_x\mathbb{Z}_{txm}(t)\eta^j\cdot\eta^j-\frac{1}{2}\sum_{j=1}^n\int_{\mathbb{R}^n}\frac{\mathrm{d}\nabla_\xi\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,t)\eta^j\cdot\eta^j\mathrm{d}m(\xi)\\&=H(x,\mathbb{Z}_{txm}(t))+\int_{\mathbb{R}^n}H_p(\xi,\mathbb{Z}_{t\xi m}(t))\frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,t)\mathrm{d}m(\xi)+\frac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x).\end{aligned}\tag{5.6}$$

The relation of (4.9) implies that

$$\nabla_x U(x,m,t)=\mathbb{Z}_{txm}(t)\quad\text{and}\quad\nabla_x^2 U(x,m,t)=\nabla_x\mathbb{Z}_{txm}(t).\tag{5.7}$$

In particular, we have used Lemma 3.13 to justify the Fréchet differentiability of the second term in (5.7), which is obtained by putting $X=\mathcal{I}_x=x$ in $D\mathbb{Z}_{tX}(t)$ mentioned in Lemma 3.13 as shown in (4.18), and the equation of $D\mathbb{Z}_{tX}(t)$ reduces to that of $\nabla_x\mathbb{Z}_{txm}(s)$ in (4.17). It further implies, by differentiating with respect to m that,

$$\nabla_\xi \frac{dU}{\mathrm{d}\nu}(\xi,m,t)(x) = \frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,t),\ \ \nabla_\xi^2 \frac{dU}{\mathrm{d}\nu}(\xi,m,t)(x) = \frac{\mathrm{d}\nabla_\xi \mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,t). \tag{5.8}$$

Therefore, plugging (5.8) into (5.6), we finally obtain the Cauchy problem of the master equation

$$\begin{cases} \dfrac{\partial U}{\partial t}(x,m,t) + \dfrac{1}{2}\displaystyle\sum_{j=1}^n \nabla_x^2 U(x,m,t)\eta^j\cdot\eta^j + \dfrac{1}{2}\sum_{j=1}^n \int_{\mathbb{R}^n} \nabla_\xi^2 \dfrac{dU}{\mathrm{d}\nu}(\xi,m,t)(x)\eta^j\cdot\eta^j \mathrm{d}m(\xi) \\ + H(x,\nabla_x U(x,m,t)) + \displaystyle\int_{\mathbb{R}^n} H_p(\xi,\nabla_\xi U(\xi,m,t))\nabla_\xi \dfrac{dU}{\mathrm{d}\nu}(\xi,m,t)(x)\mathrm{d}m(\xi) + \dfrac{\mathrm{d}F}{\mathrm{d}\nu}(m)(x) = 0; \\ U(x,m,T) = h(x) + \dfrac{dF_T}{\mathrm{d}\nu}(m)(x), \end{cases} \tag{5.9}$$

see also the discussions in [19, 20]. We next aim to justify the existences of the derivatives $\frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s)$ and $\frac{\mathrm{d}\nabla_\xi\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s)$. To this end, we consider the linear system for the linear functional derivatives of $\mathbb{Y}_{t\xi m}(s)$, $\mathbb{Z}_{t\xi m}(s)$, $u_{t\xi m}(s)$, $\mathbb{r}_{t\xi m,j}(s)$, which are denoted by $\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,s)$, $\frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s)$, $\frac{du_{t\xi m}}{\mathrm{d}\nu}(x,s)$, $\frac{\mathrm{d}\mathbb{r}_{t\xi m,j}}{\mathrm{d}\nu}(x,s)$, respectively. From the system (4.5)–(4.6), using the rules of differentiation with respect to m, we obtain

$$\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,s) = \int_t^s \frac{du_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau; \tag{5.10}$$

$$\begin{aligned} \frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s) =\ & h_{xx}(\mathbb{Y}_{t\xi m}(T))\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,T) + \nabla_x^2\frac{dF_T}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{t\xi m}(T)\Big)\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,T) \\ & + \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F_T}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{t\xi m}(T),\widetilde{\mathbb{Y}}_{t\tilde{\xi}m}(T)\Big)\frac{d\widetilde{\mathbb{Y}}_{t\tilde{\xi}m}}{\mathrm{d}\nu}(x,T)\mathrm{d}m(\tilde{\xi})\right] \\ & + \widetilde{\mathbb{E}}\left[\nabla_x\frac{d^2F_T}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{t\xi m}(T),\widetilde{\mathbb{Y}}_{txm}(T)\Big)\right] \\ & + \int_s^T\left[l_{xx}\Big(\mathbb{Y}_{t\xi m}(\tau),u_{t\xi m}(\tau)\Big) + \nabla_x^2\frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau)\Big)\right]\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau \\ & + \int_s^T l_{xv}\Big(\mathbb{Y}_{t\xi m}(\tau),u_{t\xi m}(\tau)\Big)\frac{du_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau \\ & + \int_s^T\widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n}\nabla_x\nabla_{\tilde{x}}\frac{d^2F}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau),\widetilde{\mathbb{Y}}_{t\tilde{\xi}m}(\tau)\Big)\frac{d\widetilde{\mathbb{Y}}_{t\tilde{\xi}m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}m(\tilde{\xi})\right]\mathrm{d}\tau \\ & + \int_s^T\widetilde{\mathbb{E}}\left[\nabla_x\frac{d^2F}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau),\widetilde{\mathbb{Y}}_{txm}(\tau)\Big)\right]\mathrm{d}\tau - \int_s^T\sum_{j=1}^n\frac{\mathrm{d}\mathbb{r}_{t\xi m,j}}{\mathrm{d}\nu}(x,\tau)dw_j(\tau). \end{aligned} \tag{5.11}$$

To ensure the validity of the system (5.10)–(5.11), we assume further regularity conditions on F and F_T.

Assumption 5.2. For any x, $\widetilde{x}\in\mathbb{R}^n$, we assume

(i) the regularities of the second-order linear functional derivative:

$$\left|\nabla_x\frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right| \le c(1+|\widetilde{x}|),\qquad \left|\nabla_x\frac{d^2F_T}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right| \le c_T(1+|\widetilde{x}|); \tag{5.12}$$

$$\left|\nabla_x \nabla_{\tilde{x}} \frac{d^2 F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right| \le c, \quad \left|\nabla_x \nabla_{\tilde{x}} \frac{d^2 F_T}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right| \le c_T; \tag{5.13}$$

(ii) all the derivatives in (i) and (ii) are continuous in (m,x) and $(m,x,\widetilde{x})$ respectively.

Proposition 5.3. *Under the assumptions (3.4)–(3.9), (3.21)–(3.27), (3.30) and Assumption 5.2, the system (5.10)–(5.11) has the unique solution* $\left(\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,s), \frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s), \frac{du_{t\xi m}}{\mathrm{d}\nu}(x,s), \frac{\mathrm{d}\mathbb{r}_{t\xi m,j}}{\mathrm{d}\nu}(x,s)\right)$*, and they are the linear functional derivatives of* $\mathbb{Y}_{t\xi m}(s), \mathbb{Z}_{t\xi m}(s), u_{t\xi m}(s), \mathbb{r}_{t\xi m,j}(s)$*. Moreover, for any* $s \in [t,T]$*, they satisfies the following* L^2*-boundedness*

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^n}\left|\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,s)\right|^2 \mathrm{d}m(\xi)\right], \quad \mathbb{E}\left[\int_{\mathbb{R}^n}\left|\frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s)\right|^2 \mathrm{d}m(\xi)\right], \\
&\mathbb{E}\left[\int_{\mathbb{R}^n}\left|\frac{du_{t\xi m}}{\mathrm{d}\nu}(x,s)\right|^2 \mathrm{d}m(\xi)\right], \quad \sum_{j=1}^n \int_t^T \mathbb{E}\left[\int_{\mathbb{R}^n}\left|\frac{d\mathbb{r}_{t\xi m,j}}{\mathrm{d}\nu}(x,s)\right|^2 \mathrm{d}m(\xi)\right] ds \le C_{10}(1+|x|^2),
\end{aligned} \tag{5.14}$$

where C_{10} *is a positive constant depending only on* δ_1, n, λ, η, c, c_l, c_h, c_T, c', c'_l, c'_h, c'_T *and* T.

Proof. As in the proof of Lemma 3.11, by considering the finite differences of the processes, we can show that the unique solution to (5.10)–(5.11) equals the linear functional derivatives of $\left(\mathbb{Y}_{t\xi m}(s), \mathbb{Z}_{t\xi m}(s), u_{t\xi m}(s), \mathbb{r}_{t\xi m,j}(s)\right)$. We omit the proof here. To establish the estimates in (5.14), we differentiate the first order condition (3.34) with respect to m such that

$$l_{vx}\Big(\mathbb{Y}_{t\xi m}(s), u_{t\xi m}(s)\Big)\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,s) + l_{vv}\Big(\mathbb{Y}_{t\xi m}(s), u_{t\xi m}(s)\Big)\frac{du_{t\xi m}}{\mathrm{d}\nu}(x,s) + \frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s) = 0. \tag{5.15}$$

We then apply Itô lemma to the inner product $\left\langle \frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s), \frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,s)\right\rangle_{\mathcal{H}_m}$ and then integrate from t to T, together with (5.15), to obtain

$$\begin{aligned}
&\Bigg\langle h_{xx}(\mathbb{Y}_{txm}(T))\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,T) + \nabla_x^2 \frac{dF_T}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{txm}(T)\Big)\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,T) \\
&\quad + \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n} \nabla_x \nabla_{\tilde{x}} \frac{d^2 F_T}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{t\xi m}(T), \widetilde{\mathbb{Y}}_{t\tilde{\xi} m}(T)\Big)\frac{d\widetilde{\mathbb{Y}}_{t\tilde{\xi} m}}{\mathrm{d}\nu}(x,T)\mathrm{d}m(\widetilde{\xi})\right] \\
&\qquad + \widetilde{\mathbb{E}}\left[\nabla_x \frac{d^2 F_T}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{t\xi m}(T), \widetilde{\mathbb{Y}}_{txm}(T)\Big)\right], \frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,T)\Bigg\rangle_{\mathcal{H}_m} \\
&= -\int_t^T \Bigg\langle \left[l_{xx}\Big(\mathbb{Y}_{t\xi m}(\tau), u_{t\xi m}(\tau)\Big) + \nabla_x^2 \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau)\Big)\right]\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,\tau) + l_{xv}\Big(\mathbb{Y}_{t\xi m}(\tau), u_{t\xi m}(\tau)\Big)\frac{du_{t\xi m}}{\mathrm{d}\nu}(x,\tau) \\
&\qquad + \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n} \nabla_x \nabla_{\tilde{x}} \frac{d^2 F}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau), \widetilde{\mathbb{Y}}_{t\tilde{\xi} m}(\tau)\Big)\frac{d\widetilde{\mathbb{Y}}_{t\tilde{\xi} m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}m(\widetilde{\xi})\right]
\end{aligned}$$

$$+\widetilde{\mathbb{E}}\left[\nabla_x \frac{d^2F}{d\nu^2}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau),\widetilde{\mathbb{Y}}_{txm}(\tau)\Big)\right],\frac{d\mathbb{Y}_{t\xi m}}{d\nu}(x,\tau)\Bigg\rangle_{\mathcal{H}_m} d\tau$$

$$-\int_t^T\Bigg\langle l_{vx}\Big(\mathbb{Y}_{t\xi m}(\tau),u_{t\xi m}(\tau)\Big)\frac{d\mathbb{Y}_{t\xi m}}{d\nu}(x,\tau)+l_{vv}\Big(\mathbb{Y}_{t\xi m}(\tau),u_{t\xi m}(\tau)\Big)\frac{du_{t\xi m}}{d\nu}(x,\tau),\frac{du_{t\xi m}}{d\nu}(x,\tau)\Bigg\rangle_{\mathcal{H}_m} d\tau.$$

Note the integration on $\mathbb{R}^n$ involved in the $\mathcal{H}_m$-inner product corresponds to the variable ξ. Using the assumptions in (5.12)–(5.13), Assumptions $\mathbf{A(v)}$'s (3.8), $\mathbf{A(vi)}$'s (3.9), $\mathbf{b(i)}$'s (3.11), $\mathbf{b(ii)}$'s (3.12), $\mathbf{b(v)}$'s (3.15), $\mathbf{B(v)(b)}$'s (3.27), we obtain

$$\lambda\int_t^T\left\|\frac{du_{t\xi m}}{d\nu}(x,\tau)\right\|^2_{\mathcal{H}_m}d\tau\le\Big(c'+c'_l\Big)\int_t^T\left\|\frac{d\mathbb{Y}_{t\xi m}}{d\nu}(x,\tau)\right\|^2_{\mathcal{H}_m}d\tau+\Big(c'_h+c'_T\Big)\left\|\frac{d\mathbb{Y}_{t\xi m}}{d\nu}(x,T)\right\|^2_{\mathcal{H}_m}$$
$$+c\int_t^T\left\|\frac{d\mathbb{Y}_{t\xi m}}{d\nu}(x,\tau)\right\|_{\mathcal{H}_m}\widetilde{\mathbb{E}}\left[1+\left|\widetilde{\mathbb{Y}}_{txm}(\tau)\right|\right]d\tau+c_T\left\|\frac{d\mathbb{Y}_{t\xi m}}{d\nu}(x,T)\right\|_{\mathcal{H}_m}\widetilde{\mathbb{E}}\left[1+\left|\widetilde{\mathbb{Y}}_{txm}(T)\right|\right].\tag{5.16}$$

The equation in (5.10) implies that

$$\left\|\frac{d\mathbb{Y}_{t\xi m}}{d\nu}(x,T)\right\|^2_{\mathcal{H}_m}\le T\int_t^T\left\|\frac{du_{t\xi m}}{d\nu}(x,\tau)\right\|^2_{\mathcal{H}_m}d\tau\ \text{ and }\ \int_t^T\left\|\frac{d\mathbb{Y}_{t\xi m}}{d\nu}(x,\tau)\right\|^2_{\mathcal{H}_m}d\tau\le\frac{T^2}{2}\int_t^T\left\|\frac{du_{t\xi m}}{d\nu}(x,\tau)\right\|^2_{\mathcal{H}_m}d\tau.\tag{5.17}$$

By repeating the proof of Proposition 3.9, we also have the fact that $\sup_{\tau\in(t,T)}\mathbb{E}\Big(|\mathbb{Y}_{txm}(\tau)|^2\Big)\le C_4(1+|x|^2)$ which is put into the third and fourth lines of (5.16), together with (5.17), the inequality in (5.16) can be rewritten as

$$\left[\lambda-\Big(c'_h+c'_T\Big)_+T-\Big(c'+c'_l\Big)_+\frac{T^2}{2}-\delta_1c_TT-\delta_1cT^2/2\right]\int_t^T\left\|\frac{du_{t\xi m}}{d\nu}(x,\tau)\right\|^2_{\mathcal{H}_m}d\tau\le\frac{(cT+c_T)}{2\delta_1}(1+C_4+C_4|x|^2),\tag{5.18}$$

for some small $\delta_1\in(0,1)$. The required estimate for $\frac{du_{t\xi m}}{d\nu}(x,\tau)$ thus holds by the assumption in (3.30), and similarly, the estimates for $\frac{d\mathbb{Y}_{t\xi m}}{d\nu}(x,s)$, $\frac{d\mathbb{Z}_{t\xi m}}{d\nu}(x,s)$, $\frac{d\mathbb{r}_{t\xi m,j}}{d\nu}(x,s)$ also hold by following the step as in the proof of Proposition 3.9 and using (5.18). This completes the proof. □

To further proceed to the existence of the gradient of the linear functional derivatives, we assume:

Assumption 5.4. For any x, $\widetilde{x}$, $v\in\mathbb{R}^n$,

(i) the boundedness of the derivatives:

$$|l_{xxx}(x,v)|,\ \ |l_{xxv}(x,v)|,\ \ |l_{xvv}(x,v)|,\ \ |l_{vvv}(x,v)|\le c,\ \ |h_{xxx}(x)|\le c_T;\tag{5.19}$$

(ii) the regularities of the first order linear functional derivatives:

$$\left|\nabla_x^3\frac{dF}{d\nu}(m)(x)\right|\le c,\ \ \left|\nabla_x^3\frac{dF_T}{d\nu}(m)(x)\right|\le c_T;\tag{5.20}$$

(iii) the regularities of the second-order linear functional derivatives:

$$\left|\nabla_x^2 \frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right| \le c(1+|\widetilde{x}|)\,, \quad \left|\nabla_x^2 \frac{d^2F_T}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right| \le c_T(1+|\widetilde{x}|); \tag{5.21}$$

$$\left|\nabla_x^2 \nabla_{\tilde{x}} \frac{d^2F}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right| \le c, \quad \left|\nabla_x^2 \nabla_{\tilde{x}} \frac{d^2F_T}{\mathrm{d}\nu^2}(m)(x,\widetilde{x})\right| \le c_T; \tag{5.22}$$

(iv) all the derivatives in (i), (ii) and (iii) are continuous in (x,v), (m,x) and $(m,x,\widetilde{x})$, respectively.

From (5.10)–(5.11), by taking the gradient in ξ, we obtain the system

$$\nabla_\xi \frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,s) = \int_t^s \nabla_\xi \frac{du_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau; \tag{5.23}$$

$$\begin{aligned}
&\nabla_\xi \frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s)\\
&= \left[h_{xx}(\mathbb{Y}_{t\xi m}(T)) + \nabla_x^2 \frac{dF_T}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{t\xi m}(T)\Big)\right] \nabla_\xi \frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,T)\\
&+ \left[h_{xxx}(\mathbb{Y}_{t\xi m}(T)) + \nabla_x^3 \frac{dF_T}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{t\xi m}(T)\Big)\right] \nabla_\xi \mathbb{Y}_{t\xi m}(T) \frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,T)\\
&+ \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n} \nabla_x^2\nabla_{\tilde{x}} \frac{d^2F_T}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{t\xi m}(T), \widetilde{\mathbb{Y}}_{t\widetilde{\xi} m}(T)\Big)\nabla_\xi \mathbb{Y}_{t\xi m}(T)\frac{d\widetilde{\mathbb{Y}}_{t\widetilde{\xi} m}}{\mathrm{d}\nu}(x,T)\mathrm{d}m(\widetilde{\xi})\right]\\
&+ \widetilde{\mathbb{E}}\left[\nabla_x^2 \frac{d^2F_T}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(T)\otimes m)\Big(\mathbb{Y}_{t\xi m}(T), \widetilde{\mathbb{Y}}_{txm}(T)\Big)\right]\nabla_\xi \mathbb{Y}_{t\xi m}(T)\\
&+ \int_s^T \left[l_{xx}\Big(\mathbb{Y}_{t\xi m}(\tau), u_{t\xi m}(\tau)\Big) + \nabla_x^2 \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau)\Big)\right] \nabla_\xi \frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau\\
&+ \int_s^T l_{xxx}\Big(\mathbb{Y}_{t\xi m}(\tau), u_{t\xi m}(\tau)\Big)\nabla_\xi \mathbb{Y}_{t\xi m}(\tau)\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau + \int_s^T \nabla_x^3 \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau)\Big)\nabla_\xi \mathbb{Y}_{t\xi m}(\tau)\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau\\
&+ \int_s^T \left[l_{xxv}\Big(\mathbb{Y}_{t\xi m}(\tau), u_{t\xi m}(\tau)\Big)\right]\nabla_\xi u_{t\xi m}(\tau)\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau\\
&+ \int_s^T l_{xv}\Big(\mathbb{Y}_{t\xi m}(\tau), u_{t\xi m}(\tau)\Big)\nabla_\xi \frac{du_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau + \int_s^T l_{xvx}\Big(\mathbb{Y}_{t\xi m}(\tau), u_{t\xi m}(\tau)\Big)\nabla_\xi \mathbb{Y}_{t\xi m}(\tau)\frac{du_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau\\
&+ \int_s^T l_{xvv}\Big(\mathbb{Y}_{t\xi m}(\tau), u_{t\xi m}(\tau)\Big)\nabla_\xi u_{t\xi m}(\tau)\frac{du_{t\xi m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}\tau\\
&+ \int_s^T \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n} \nabla_x^2\nabla_{\tilde{x}} \frac{d^2F}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau), \widetilde{\mathbb{Y}}_{t\widetilde{\xi} m}(\tau)\Big)\nabla_\xi \mathbb{Y}_{t\xi m}(\tau)\frac{d\widetilde{\mathbb{Y}}_{t\widetilde{\xi} m}}{\mathrm{d}\nu}(x,\tau)\mathrm{d}m(\widetilde{\xi})\right]\mathrm{d}\tau\\
&+ \int_s^T \widetilde{\mathbb{E}}\left[\nabla_x^2 \frac{d^2F}{\mathrm{d}\nu^2}(\mathbb{Y}_{t\cdot m}(\tau)\otimes m)\Big(\mathbb{Y}_{t\xi m}(\tau), \widetilde{\mathbb{Y}}_{txm}(\tau)\Big)\nabla_\xi \mathbb{Y}_{t\xi m}(\tau)\right]\mathrm{d}\tau - \int_s^T \sum_{j=1}^n \nabla_\xi \frac{\mathrm{d}\mathbb{r}_{t\xi m,j}}{\mathrm{d}\nu}(x,\tau)dw_j(\tau).
\end{aligned} \tag{5.24}$$

We can then state:

Proposition 5.5. *Under the assumptions (3.4)–(3.9), (3.21)–(3.27), (3.30), Assumptions 5.2 and 5.4, the system (5.23)–(5.24) has the unique solution, and they are the gradient of* $\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,s)$, $\frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s)$, $\frac{du_{t\xi m}}{\mathrm{d}\nu}(x,s)$,

$\frac{\mathrm{d}\mathbb{r}_{t\xi m,j}}{\mathrm{d}\nu}(x,s)$ *with respect to* ξ *such that*

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^n}\left|\nabla_\xi\frac{d\mathbb{Y}_{t\xi m}}{\mathrm{d}\nu}(x,s)\right|^2\mathrm{d}m(\xi)\right],\quad \mathbb{E}\left[\int_{\mathbb{R}^n}\left|\nabla_\xi\frac{d\mathbb{Z}_{t\xi m}}{\mathrm{d}\nu}(x,s)\right|^2\mathrm{d}m(\xi)\right],\\
&\mathbb{E}\left[\int_{\mathbb{R}^n}\left|\nabla_\xi\frac{du_{t\xi m}}{\mathrm{d}\nu}(x,s)\right|^2\mathrm{d}m(\xi)\right],\quad \sum_{j=1}^n\int_t^T\mathbb{E}\left[\int_{\mathbb{R}^n}\left|\nabla_\xi\frac{d\mathbb{r}_{t\xi m,j}}{\mathrm{d}\nu}(x,s)\right|^2\mathrm{d}m(\xi)\right]\le C_{11}(1+|x|^2),
\end{aligned}$$

where C_{11} *is a positive constant depending only on* δ_1, n, λ, η, c, c_l, c_h, c_T, c', c'_l, c'_h, c'_T *and* T.

Proof. The proof is similar to that of Proposition 5.3. Note that we also need the L^4-estimate of the linear functional derivatives and the Jacobian flow of the process $\big(\mathbb{Y}_{t\xi m}(s),\mathbb{Z}_{t\xi m}(s),u_{t\xi m}(s)\big)$, it can be established by the arguments of Lemma 5.5 of [41]. ☐

This justifies the differentiation with respect to m of the Bellman equation, and hence $U(x,m,t)=\frac{dV}{\mathrm{d}\nu}(m,t)(x)$ is a solution to the Master equation in (5.9). If $U^*(x,m,t)$ is another solution to (5.9), then we define $\mathbb{Z}^*_{txm}(t):=\nabla_xU^*(x,m,t)$, $\mathbb{Y}^*_{txm}(s):=\int_t^s u\Big(\mathbb{Y}^*_{txm}(\tau),\mathbb{Z}^*_{\tau\mathbb{Y}^*_{txm}(\tau)m}(\tau)\Big)\mathrm{d}\tau+\eta\Big(w(s)-w(t)\Big)$ and $\mathbb{Z}^*_{txm}(\tau):=\mathbb{Z}^*_{\tau\mathbb{Y}^*_{txm}(\tau)m}(\tau)=\nabla_xU^*(\mathbb{Y}^*_{txm}(\tau),m,\tau)$. We then differentiate (5.9) with respect to x and evaluate the resulting equation at $x=\mathbb{Y}^*_{txm}(s)$. By using the mean-field Itô lemma in Theorem 2.3, we work backward and realize that $\mathbb{Z}^*_{txm}(s)$ satisfies the backward dynamics (3.29). Since the FBSDE (3.32)–(3.33) has the unique solution, therefore we know $\mathbb{Y}_{txm}(s)=\mathbb{Y}^*_{txm}(s)$, $\mathbb{Z}_{txm}(s)=\mathbb{Z}^*_{txm}(s)$ and hence $U=U^*$ due to the terminal condition in (5.9). The well-posedness of the master equation in (5.9) is concluded by the proposition.

Theorem 5.6. *Under the assumptions (3.4)–(3.10),* (3.30), *(3.21)–(3.27) and (3.30), the value function* $V(m,t)$ *satisfies the Bellman equation in (5.1) classically. Furthermore, if Assumptions 5.2 and 5.4 are also fulfilled, the linear functional derivative* $U(x,m,t)=\frac{dV}{\mathrm{d}\nu}(m,t)(x)$ *of* $V(m,t)$ *is the unique classical solution to the master equation in (5.9) in the pointwise sense, with all the derivatives* $\frac{\partial U}{\partial t}(x,m,t)$, $\nabla_xU(x,m,t)$, $\nabla_x^2U(x,m,t)$, $\nabla_\xi\frac{dU}{d\nu}(\xi,m,t)(x)$ *and* $\nabla_\xi^2\frac{dU}{d\nu}(\xi,m,t)(x)$ *being existed.*

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Data availability statement

The research data associated with this article are included in the article.

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Appendix A.

A.1 Proof of Statements in Section 2

A.1.1 Proof of Theorem 2.3

We note that the constant C in the following proof may be different from line to line, but we still denote by the same symbol C without ambiguity. We begin with some preliminaries. If the sequence $\{Y_k\}_{k\in\mathbb{N}}$ in assumption (2.22) further satisfies

$$\sup_k \mathbb{E}\left[\int_{\mathbb{R}^n} |Y_k|^4 \mathrm{d}m(x)\right] < \infty, \tag{A.1}$$

then

$$\left\langle D_X^2 F(X_k \otimes m, s_k)(Y_k) - D_X^2 F(X \otimes m, s_k)(Y_k), Y_k \right\rangle_{\mathcal{H}_m} \longrightarrow 0 \quad \text{as } k \to \infty; \tag{A.2}$$

indeed, denote $\mathscr{I}_k$ for the expression (A.2), we write $\mathscr{I}_k =: \mathscr{I}^1_{\epsilon k} + \mathscr{I}^2_{\epsilon k}$, for any $\epsilon > 0$, where

$$\mathscr{I}^1_{\epsilon k} := \left\langle D_X^2 F(X_k \otimes m, s_k)(Y_k) - D_X^2 F(X \otimes m, s_k)(Y_k), \frac{Y_k}{1+\epsilon|Y_k|} \right\rangle_{\mathcal{H}_m},$$

$$\mathscr{I}^2_{\epsilon k} := \epsilon \left\langle D_X^2 F(X_k \otimes m, s_k)(Y_k) - D_X^2 F(X \otimes m, s_k)(Y_k), \frac{Y_k|Y_k|}{1+\epsilon|Y_k|} \right\rangle_{\mathcal{H}_m}.$$

Thanks to Assumptions (2.20) and (A.1), we have $\left|\mathscr{I}^2_{\epsilon k}\right| \leq C\epsilon$ for some $C > 0$ independent of ϵ; and thanks to (2.22), $\mathscr{I}^1_{\epsilon k} \longrightarrow 0$, as $k \to \infty$, for each $\epsilon > 0$. Combining the two convergences, we get $\mathscr{I}_k \longrightarrow 0$ as $k \to \infty$. Next, using (2.11), we write

$$\frac{1}{\epsilon}\left[F\Big(\mathbb{X}_{tX}(s+\epsilon) \otimes m, s+\epsilon\Big) - F\Big(\mathbb{X}_{tX}(s) \otimes m, s+\epsilon\Big)\right] =: \mathrm{I}_\epsilon + \mathrm{II}_\epsilon + \mathrm{III}_\epsilon + \mathrm{IV}_\epsilon,$$

where

$$\mathrm{I}_\epsilon := \frac{1}{\epsilon}\left\langle D_X F\Big(\mathbb{X}_{tX}(s) \otimes m, s+\epsilon\Big), \int_s^{s+\epsilon} a_{tX}(\tau)\mathrm{d}\tau + \sum_{j=1}^n \int_s^{s+\epsilon} \eta^j_{tX}(\tau) dw_j(\tau) \right\rangle_{\mathcal{H}_m};$$

$$\begin{aligned}\mathrm{II}_\epsilon := \frac{1}{\epsilon}\int_0^1\int_0^1 \theta \Bigg\langle & D_X^2 F\Big(\Big\{\mathbb{X}_{tX}(s) + \theta\lambda\Big[\mathbb{X}_{tX}(s+\epsilon) - \mathbb{X}_{tX}(s)\Big]\Big\} \otimes m, s+\epsilon\Big)\left(\int_s^{s+\epsilon} a_{tX}(\tau)\mathrm{d}\tau\right), \\ & \int_s^{s+\epsilon} a_{tX}(\tau)\mathrm{d}\tau + 2\sum_{j=1}^n \int_s^{s+\epsilon} \eta^j_{tX}(\tau) dw_j(\tau) \Bigg\rangle_{\mathcal{H}_m} d\theta d\lambda;\end{aligned}$$

$$\begin{aligned}\mathrm{III}_\epsilon := \frac{1}{\epsilon}\int_0^1\int_0^1 \theta \Bigg\langle & D_X^2 F\Big(\Big\{\mathbb{X}_{tX}(s) + \theta\lambda\Big[\mathbb{X}_{tX}(s+\epsilon) - \mathbb{X}_{tX}(s)\Big]\Big\} \otimes m, s+\epsilon\Big)\left(\sum_{j=1}^n \int_s^{s+\epsilon} \eta^j_{tX}(\tau) dw_j(\tau)\right) \\ & - D_X^2 F(\mathbb{X}_{tX}(s) \otimes m, s+\epsilon)\left(\sum_{j=1}^n \int_s^{s+\epsilon} \eta^j_{tX}(\tau) dw_j(\tau)\right), \sum_{j=1}^n \int_s^{s+\epsilon} \eta^j_{tX}(\tau) dw_j(\tau) \Bigg\rangle_{\mathcal{H}_m} \mathrm{d}\theta\mathrm{d}\lambda;\end{aligned}$$

$$\mathrm{IV}_\epsilon := \frac{1}{2\epsilon}\left\langle D_X^2 F(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon)\left(\sum_{j=1}^n \int_s^{s+\epsilon} \eta_{tX}^j(\tau)dw_j(\tau)\right), \sum_{j=1}^n \int_s^{s+\epsilon} \eta_{tX}^j(\tau)dw_j(\tau)\right\rangle_{\mathcal{H}_m}.$$

Firstly, we expand the first term I_ϵ by writing

$$\begin{aligned}\mathrm{I}_\epsilon = &\Big\langle D_X F(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon), a_{tX}(s)\Big\rangle_{\mathcal{H}_m} + \left\langle D_X F(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon), \frac{1}{\epsilon}\int_s^{s+\epsilon} a_{tX}(\tau)\mathrm{d}\tau - a_{tX}(s)\right\rangle_{\mathcal{H}_m}\\ &+ \left\langle D_X F\Big(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon\Big), \frac{1}{\epsilon}\sum_{j=1}^n \int_s^{s+\epsilon} \eta_{tX}^j(\tau)dw_j(\tau)\right\rangle_{\mathcal{H}_m}.\end{aligned}$$

From Assumption (c) of (2.23), together with (2.20), the second line tends to zero. While for the third line, due to the future Brownian increment of the stochastic integral over $[s, s+\epsilon]$, it vanishes as well after an application of tower property. Finally, for the remaining term, in light of Assumption (2.19), we get $\mathrm{I}_\epsilon \longrightarrow \Big\langle D_X F(\mathbb{X}_{tX}(s)\otimes m, s), a_{tX}(s)\Big\rangle_{\mathcal{H}_m}$ as $\epsilon \to 0$. For the term II_ϵ, we apply the second assumption in (2.20), assumptions (b) and (c) of (2.23), we have $\mathrm{II}_\epsilon \longrightarrow 0$ as $\epsilon \to 0$. For the term III_ϵ, the use of Burkholder–Davis–Gundy and Cauchy–Schwarz inequalities gives $\frac{1}{\epsilon^2}\mathbb{E}\left[\left|\sum_{j=1}^n \int_s^{s+\epsilon} \eta_{t\xi}^j(\tau)dw_j(\tau)\right|^4\right] \le \frac{4}{\epsilon}\mathbb{E}\left[\int_s^{s+\epsilon}\left(\sum_{j=1}^n \Big|\eta_{t\xi}(\tau)\Big|^2\right)^2 \mathrm{d}\tau\right]$, for any $\xi \in \mathbb{R}^n$. Therefore,

$$\begin{aligned}\frac{1}{\epsilon^2}\mathbb{E}_{\mathcal{W}_t}\left[\int_{\mathbb{R}^n}\left|\sum_{j=1}^n \int_s^{s+\epsilon} \eta_{tX}^j(\tau)dw_j(\tau)\right|^4 \mathrm{d}m(x)\right] &\le \frac{C}{\epsilon}\mathbb{E}_{\mathcal{W}_t}\left[\int_{\mathbb{R}^n}\int_s^{s+\epsilon}\left(\sum_{j=1}^n |\eta_{tX}(\tau)|^2\right)^2 \mathrm{d}\tau \mathrm{d}m(x)\right]\\ &\le C\sup_{s\in[t,T]}\mathbb{E}_{\mathcal{W}_t}\left[\int_{\mathbb{R}^n}\left(\sum_{j=1}^n |\eta_{tX}(s)|^2\right)^2 \mathrm{d}m(x)\right]\\ &\le C\end{aligned}$$

by the first assumption (a) of (2.23). Based on this bound, from Assumption (2.22), one can use the claim (A.2) to conclude the convergence of the integrand of III_ϵ to zero for each (λ, θ). Next, under the second assumption of (2.20), a standard application of the bounded convergence theorem asserts that $\mathrm{III}_\epsilon \longrightarrow 0$ as $\epsilon \to 0$. Moreover, from the second assumption (2.23), we have $\frac{1}{\sqrt{\epsilon}}\sum_{j=1}^n \int_s^{s+\epsilon}\Big[\eta_{tX}^j(\tau) - \eta_{tX}^j(s)\Big]dw_j(\tau) \longrightarrow 0$ as $\epsilon \to 0$ in $\mathcal{H}_m$ and thus by using the second assumption of (2.20)

$$\mathrm{IV}_\epsilon - \frac{1}{2}\left\langle D_X^2 F(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon)\left(\sum_{j=1}^n \eta_{tX}^j(s)\frac{w_j(s+\epsilon)-w_j(s)}{\sqrt{\epsilon}}\right), \sum_{j=1}^n \eta_{tX}^j(s)\frac{w_j(s+\epsilon)-w_j(s)}{\sqrt{\epsilon}}\right\rangle_{\mathcal{H}_m} \longrightarrow 0.$$

On the other hand, by formula (2.12), it holds that

$$\begin{aligned}&\left\langle D_X^2 F(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon)\left(\sum_{j=1}^n \eta_{tX}^j(s)\frac{w_j(s+\epsilon)-w_j(s)}{\sqrt{\epsilon}}\right), \sum_{j=1}^n \eta_{tX}^j(s)\frac{w_j(s+\epsilon)-w_j(s)}{\sqrt{\epsilon}}\right\rangle_{\mathcal{H}_m}\\ &= \left\langle D_X^2 F(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon)\left(\sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right), \sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right\rangle_{\mathcal{H}_m},\end{aligned}$$

for any standard normal distributed $\mathcal{N}_s^j$ being independent of $\mathcal{W}_{tX}^s$ since the law of $\frac{w_j(s+\epsilon)-w_j(s)}{\sqrt{\epsilon}}$ is normally distributed with a mean 0 and variance 1, and hence is independent of the choice of ϵ.

We have proven, from assumption (2.21), we conclude

$$\left\langle D_X^2 F(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon)\left(\sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right), \sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right\rangle_{\mathcal{H}_m} \longrightarrow \left\langle D_X^2 F(\mathbb{X}_{tX}(s)\otimes m, s)\left(\sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right), \sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right\rangle_{\mathcal{H}_m},$$

as $\epsilon \to 0$. Collecting all these results, we have proven that

$$\begin{aligned}&\frac{1}{\epsilon}\Big[F(\mathbb{X}_{tX}(s+\epsilon)\otimes m, s+\epsilon) - F(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon)\Big] \longrightarrow \\ &\Big\langle D_X F(\mathbb{X}_{tX}(s)\otimes m, s), a_{tX}(s)\Big\rangle_{\mathcal{H}_m} + \frac{1}{2}\left\langle D_X^2 F(\mathbb{X}_{tX}(s)\otimes m, s)\left(\sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right), \sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right\rangle_{\mathcal{H}_m}.\end{aligned}$$

Finally, from assumption (2.18), in accordance with Rademacher's theorem, $\frac{1}{\epsilon}\big[F(\mathbb{X}_{tX}(s)\otimes m, s+\epsilon) - F(\mathbb{X}_{tX}(s)\otimes m, s)\big] \longrightarrow \frac{\partial}{\partial s}F(\mathbb{X}_{tX}(s)\otimes m, s)$ a.e. $s \in [t,T]$. Therefore, the result in (2.24) is obtained. If the formula (2.13) applies, by combining the discussion between (2.14) and (2.15),

$$\begin{aligned}D_X^2 F(\mathbb{X}_{tX}(s)\otimes m, s)\left(\sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right) &= \nabla_x^2 \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{X}_{tX}(s)\otimes m, s)(\mathbb{X}_{tX}(s))\left(\sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right) \\ &\quad + \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}^n} \nabla_x \nabla_{\tilde{x}} \frac{d^2F}{\mathrm{d}\nu^2}(\mathbb{X}_{tX}(s)\otimes m, s)\Big(\mathbb{X}_{tX}(s), \widetilde{\mathbb{X}}_{t\widetilde{X}}(s)\Big)\left(\sum_{j=1}^n \eta_{t\widetilde{X}}^j(s)\widetilde{\mathcal{N}}_s^j\right)\mathrm{d}m(\widetilde{x})\right] \\ &= \nabla_x^2 \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{X}_{tX}(s)\otimes m, s)\Big(\mathbb{X}_{tX}(s)\Big)\left(\sum_{j=1}^n \eta_{tX}^j(s)\mathcal{N}_s^j\right),\end{aligned}$$

where $\widetilde{\mathcal{N}}_s^j$ is independent of $\mathcal{W}_{t\widetilde{X}}^s$. This concludes the proof. ■

A.1.2 Complements to Theorem 2.3

If we choose $X_t = \mathcal{I}_x$ in the formula (2.25), then the process $\mathbb{X}_{t\mathcal{I}}(s)$ is simply a classical Itô process indexed by a parameter x, which is the initial data at the time t, denoted by $\mathbb{X}_{tx}(s)$. It reads $\mathbb{X}_{tx}(s) = x + \int_t^s a_{tx}(\tau)\mathrm{d}\tau + \sum_{j=1}^n \int_t^s \eta_{tx}^j(\tau)dw_j(\tau)$ for $s \geq t$. Both the processes $a_{tx}(s)$ and $\eta_{tx}^j(s)$ are adapted to the filtration $\mathcal{W}_t^s$, and such that $\mathbb{E}\left[\int_t^T \int_{\mathbb{R}^n} |a_{tx}(s)|^2 ds\, \mathrm{d}m(x)\right]$ and $\mathbb{E}\left[\int_t^T \int_{\mathbb{R}^n} |\eta_{tx}^j(s)|^2 ds\, \mathrm{d}m(x)\right] < \infty$. We can write (2.25) as

$$\begin{aligned}\frac{\mathrm{d}}{ds}F(\mathbb{X}_{tx}(s)\otimes m, s) =& \frac{\partial}{\partial s}F(\mathbb{X}_{tx}(s)\otimes m, s) + \mathbb{E}\left\{\int_{\mathbb{R}^n} \nabla_x \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{X}_{tx}(s)\otimes m, s)(\mathbb{X}_{tx}(s))\cdot a_{tx}(s)\mathrm{d}m(x)\right\} \\ &+ \mathbb{E}\left\{\int_{\mathbb{R}^n} \frac{1}{2}\sum_{j=1}^n \nabla_x^2 \frac{\mathrm{d}F}{\mathrm{d}\nu}(\mathbb{X}_{tx}(s)\otimes m, s)(\mathbb{X}_{tx}(s))\eta_{tx}^j(s)\cdot \eta_{tx}^j(s)\mathrm{d}m(x)\right\}. \end{aligned} \tag{A.3}$$

For instance, taking $F(m,s) = \int_{\mathbb{R}^n} \Psi(x,s)\mathrm{d}m(x)$, then $F(\mathbb{X}_{tx}(s) \otimes m, s) = \mathbb{E}\Big[\int_{\mathbb{R}^n} \Psi(\mathbb{X}_{tx}(s),s)\mathrm{d}m(x)\Big]$ and $\frac{\mathrm{d}F}{\mathrm{d}\nu}(m,s)(x) = \Psi(x,s)$. Therefore, (A.3) reduces to

$$\frac{\mathrm{d}}{ds}\mathbb{E}\left[\int_{\mathbb{R}^n} \Psi(\mathbb{X}_{tx}(s),s)\mathrm{d}m(x)\right] = \mathbb{E}\Bigg\{ \int_{\mathbb{R}^n} \Bigg[\frac{\partial}{\partial s}\Psi(\mathbb{X}_{tx}(s),s) + \nabla_x \Psi(\mathbb{X}_{tx}(s),s)\cdot a_{tx}(s) + \frac{1}{2}\sum_{j=1}^{n} \nabla_x^2 \Psi(\mathbb{X}_{tx}(s),s)\eta_{tx}^j(s)\cdot \eta_{tx}^j(s)\Bigg]\mathrm{d}m(x)\Bigg\},$$

which is the same as the classical Itô's formula when applies to an arbitrary test function Ψ.

A.2 Proof of statements in Section 3

A.2.1 Proof of Lemma 3.4

For the perturbed control $v_{tX}(s) + \theta\widetilde{v}_{tX}(s)$ with $\theta\in\mathbb{R}$ and $\widetilde{v}_{tX}(s) \in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$, the corresponding state is $\mathbb{X}_{tX}(s) + \theta\int_t^s \widetilde{v}_{tX}(\tau)\mathrm{d}\tau$. Using (3.4), (3.6), (3.22) and dominated convergence theorem, we first check that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}J_{tX}\big(v_{tX}+\theta\widetilde{v}_{tX}\big)\bigg|_{\theta=0} &= \int_t^T \big\langle l_v\big(\mathbb{X}_{tX}(s),v_{tX}(s)\big), \widetilde{v}_{tX}(s)\big\rangle_{\mathcal{H}_m} ds \\ &\quad+ \int_t^T \left\langle l_x\big(\mathbb{X}_{tX}(s),v_{tX}(s)\big) + D_X F\big(\mathbb{X}_{tX}(s)\otimes m\big), \int_t^s \widetilde{v}_{tX}(\tau)\mathrm{d}\tau\right\rangle_{\mathcal{H}_m} ds \\ &\quad+ \left\langle h_x\big(\mathbb{X}_{tX}(T)\big) + D_X F_T\big(\mathbb{X}_{tX}(T)\otimes m\big), \int_t^T \widetilde{v}_{tX}(\tau)\mathrm{d}\tau\right\rangle_{\mathcal{H}_m}.\end{aligned} \tag{A.4}$$

To deal with the third and fourth lines in (A.4), we define Γ_{tX} by the following:

$$\Gamma_{tX} := \int_t^T l_x\big(\mathbb{X}_{tX}(s),v_{tX}(s)\big) + D_X F\big(\mathbb{X}_{tX}(s)\otimes m\big)ds + h_x\big(\mathbb{X}_{tX}(T)\big) + D_X F_T\big(\mathbb{X}_{tX}(T)\otimes m\big),$$

which is clearly a random variable taking values in $\mathbb{R}^n$, and is $\mathcal{W}_{tX}^T$-measurable, so it can be written as a $\mathcal{W}_t^T$-measurable random field $\Gamma_{t\xi}$, for almost every $\xi\in\mathbb{R}^n$, and then we substitute ξ by X. Now, $\mathbb{E}\big(\Gamma_{tX}\big|\mathcal{W}_{tX}^s\big) = \mathbb{E}\big(\Gamma_{t\xi}\big|\mathcal{W}_t^s\big)\big|_{\xi=X}$. By the standard martingale representation theorem adapted to the Wiener filtration, we can write $\mathbb{E}(\Gamma_{t\xi}|\mathcal{W}_t^s) = \mathbb{E}\big(\Gamma_{t\xi}\big) + \sum_{j=1}^n \int_t^s \mathbb{r}_{t\xi,j}(\tau)dw_j(\tau)$. Therefore, we can write $\mathbb{E}\big(\Gamma_{tX}\big|\mathcal{W}_{tX}^s\big) = \mathbb{E}\big(\Gamma_{tX}\big|X\big) + \sum_{j=1}^n \int_t^s \mathbb{r}_{tX,j}(\tau)dw_j(\tau)$, with $\mathbb{r}_{tX,j}(s)\in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$. Next, we define the Itô process:

$$\mathbb{Z}_{tX}(s) := \mathbb{E}(\Gamma_{tX}|X) - \int_t^s l_x\big(\mathbb{X}_{tX}(\tau),v_{tX}(\tau)\big) + D_X F\big(\mathbb{X}_{tX}(\tau)\otimes m\big)\mathrm{d}\tau + \sum_{j=1}^n \int_t^s \mathbb{r}_{tX,j}(\tau)dw_j(\tau).$$

We note that $\mathbb{Z}_{tX}(T) = h_x\big(\mathbb{X}_{tX}(T)\big) + D_X F_T\big(\mathbb{X}_{tX}(T)\otimes m\big)$ since Γ_{tX} is independent of $\mathcal{W}_{tX}^T$. This fact together with integration by parts yields

$$\int_t^T \big\langle \mathbb{Z}_{tX}(s), \widetilde{v}_{tX}(s)\big\rangle_{\mathcal{H}_m} ds = \int_t^T \left\langle l_x\big(\mathbb{X}_{tX}(s),v_{tX}(s)\big) + D_X F\big(\mathbb{X}_{tX}(s)\otimes m\big), \int_t^s \widetilde{v}_{tX}(\tau)\mathrm{d}\tau\right\rangle_{\mathcal{H}_m} ds$$

$$+\left\langle h_x(\mathbb{X}_{tX}(T))+D_XF_T(\mathbb{X}_{tX}(T)\otimes m),\int_t^T \widetilde{v}_{tX}(\tau)\mathrm{d}\tau\right\rangle_{\mathcal{H}_m}.$$

Hence $\frac{\mathrm{d}}{\mathrm{d}\theta}J_{tX}\big(v_{tX}+\theta\widetilde{v}_{tX}\big)\big|_{\theta=0}=\int_t^T\big\langle l_v\big(\mathbb{X}_{tX}(s),v_{tX}(s)\big)+\mathbb{Z}_{tX}(s),\widetilde{v}_{tX}(s)\big\rangle_{\mathcal{H}_m}ds$, which gives the result (3.28) when we take the first variation with respect to v_{tX}. ■

A.2.2 Proof of Proposition 3.5

Part 1A. Convexity:
Consider two controls $v^1_{tX}(s),v^2_{tX}(s)\in L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$. We are going to verify the strong convexity, in the sense that

$$\int_t^T\Big\langle D_vJ_{tX}\big(v^1_{tX}\big)(s)-D_vJ_{tX}\big(v^2_{tX}\big)(s),v^1_{tX}(s)-v^2_{tX}(s)\Big\rangle_{\mathcal{H}_m}ds\geq c_0\int_t^T\left\|v^1_{tX}(s)-v^2_{tX}(s)\right\|^2_{\mathcal{H}_m}ds. \tag{A.5}$$

for some constant $c_0>0$. Then, the claim in the proposition will follow immediately. To simplify the notations, we write $v^1(s)=v^1_{tX}(s)$, $v^2(s)=v^2_{tX}(s)$, $\mathbb{X}^1(s)=\mathbb{X}_{tX}\big(s;v^1_{tX}\big)$ and $\mathbb{X}^2(s)=\mathbb{X}_{tX}\big(s;v^2_{tX}\big)$. Let $\mathbb{Z}^1(s)$ and $\mathbb{Z}^2(s)$ be the corresponding solutions of (3.29) with respect to $\mathbb{X}^1(s)$ and $\mathbb{X}^2(s)$ respectively. From the formula (3.28), we use the fact that $v^1(s)-v^2(s)=\dfrac{\mathrm{d}}{ds}\big[\mathbb{X}^1(s)-\mathbb{X}^2(s)\big]$ and $\mathbb{X}^1(t)-\mathbb{X}^2(t)=0$ to obtain

$$\begin{aligned}&\int_t^T\Big\langle D_vJ_{tX}(v^1)(s)-D_vJ_{tX}(v^2)(s),v^1(s)-v^2(s)\Big\rangle_{\mathcal{H}_m}ds\\
=&\int_t^T\Big\langle l_v\Big(\mathbb{X}^1(s),v^1(s)\Big)-l_v\Big(\mathbb{X}^2(s),v^2(s)\Big),v^1(s)-v^2(s)\Big\rangle_{\mathcal{H}_m}+\Big\langle l_x(\mathbb{X}^1(s),v^1(s))-l_x(\mathbb{X}^2(s),v^2(s)),\mathbb{X}^1(s)-\mathbb{X}^2(s)\Big\rangle_{\mathcal{H}_m}ds\\
&+\int_t^T\Big\langle D_XF(\mathbb{X}^1(s)\otimes m)-D_XF(\mathbb{X}^2(s)\otimes m),\mathbb{X}^1(s)-\mathbb{X}^2(s)\Big\rangle_{\mathcal{H}_m}ds+\Big\langle h_x(\mathbb{X}^1(T))-h_x(\mathbb{X}^2(T)),\mathbb{X}^1(T)-\mathbb{X}^2(T)\Big\rangle_{\mathcal{H}_m}\\
&+\Big\langle D_XF_T(\mathbb{X}^1(T)\otimes m)-D_XF_T(\mathbb{X}^2(T)\otimes m),\mathbb{X}^1(T)-\mathbb{X}^2(T)\Big\rangle_{\mathcal{H}_m}.\end{aligned}$$

The mean value theorem, Assumptions **A(ii)**, **A(v)**, **A(vi)** and **B(v)(b)** tell us that

$$\begin{aligned}&\int_t^T\Big\langle D_vJ_{tX}(v^1)(s)-D_vJ_{tX}(v^2)(s),v^1(s)-v^2(s)\Big\rangle_{\mathcal{H}_m}ds\\
\geq&\int_t^T\inf_{x,v\in\mathbb{R}^n}\Bigg[\Big\langle l_{vv}(x,v)\big[v^1(s)-v^2(s)\big]+l_{vx}(x,v)\big[\mathbb{X}^1(s)-\mathbb{X}^2(s)\big],v^1(s)-v^2(s)\Big\rangle_{\mathcal{H}_m}\\
&\qquad+\Big\langle l_{xv}(x,v)\big[v^1(s)-v^2(s)\big]+l_{xx}(x,v)\big[\mathbb{X}^1(s)-\mathbb{X}^2(s)\big],\mathbb{X}^1(s)-\mathbb{X}^2(s)\Big\rangle_{\mathcal{H}_m}\Bigg]ds\\
&-c'\int_t^T\big\|\mathbb{X}^1(s)-\mathbb{X}^2(s)\big\|^2_{\mathcal{H}_m}ds-(c'_T+c'_h)\big\|\mathbb{X}^1(T)-\mathbb{X}^2(T)\big\|^2_{\mathcal{H}_m}\\
\geq&\lambda\int_t^T\big\|v^1(s)-v^2(s)\big\|^2_{\mathcal{H}_m}ds-(c'+c'_l)\int_t^T\big\|\mathbb{X}^1(s)-\mathbb{X}^2(s)\big\|^2_{\mathcal{H}_m}ds-(c'_T+c'_h)\big\|\mathbb{X}^1(T)-\mathbb{X}^2(T)\big\|^2_{\mathcal{H}_m}.\end{aligned}\tag{A.6}$$

The fact that $\mathbb{X}^1(s) - \mathbb{X}^2(s) = \int_t^s v^1(\tau) - v^2(\tau)\mathrm{d}\tau$ for any $s \in (t,T]$, together with a simple application of Cauchy–Schwarz inequality implies

$$\left\|\mathbb{X}^1(s) - \mathbb{X}^2(s)\right\|_{\mathcal{H}_m}^2 \leq (s-t)\int_t^s \left\|v^1(\tau) - v^2(\tau)\right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau. \tag{A.7}$$

Bringing (A.7) into (A.6), the strict convexity is proven for $c_0 := \lambda - (c_T' + c_h')_+ T - (c_l' + c')_+ \dfrac{T^2}{2} > 0$.

Part 1B. Coercivity:

For the coercivity, from the formula (A.5), we have

$$\int_t^T \left\langle D_v J_{tX}\big(v_{tX}\big)(s) - D_v J_{tX}(0)(s), v_{tX}(s)\right\rangle_{\mathcal{H}_m} ds \geq c_0 \int_t^T \left\| v_{tX}(s)\right\|_{\mathcal{H}_m}^2 ds. \tag{A.8}$$

Combined with (A.8), we obtain

$$\begin{aligned} J_{tX}\big(v_{tX}\big) - J_{tX}(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} J_{tX}\big(\theta v_{tX}\big) d\theta &= \int_0^1 \int_t^T \left\langle D_v J_{tX}(\theta v_{tX})(s), v_{tX}(s)\right\rangle_{\mathcal{H}_m} ds\mathrm{d}\theta \\ &\geq \int_t^T \left\langle D_v J_{tX}(0)(s), v_{tX}(s)\right\rangle_{\mathcal{H}_m} ds + \frac{c_0}{2}\int_t^T \left\| v_{tX}(s)\right\|_{\mathcal{H}_m}^2 ds. \end{aligned}$$

Part 2. Existence and Uniqueness of the Optimal Control:

Since l, h, F and F_T are continuous according to Assumptions **A(i)**'s (3.4), **A(iii)**'s (3.6) and **B(i)**'s (3.22), together with the fact that $\mathbb{X}_{tX}$ is continuous in v_{tX} under the strong $L^2_{\mathcal{W}_{tX}}(t,T;\mathcal{H}_m)$-norm by (3.1), the functional $J_{tX}\big(v_{tX}\big)$ is clearly continuous in v_{tX}. Combining the convexity and coercivity of the functional, the existence and uniqueness of the optimal control is then guaranteed by Theorem 7.2.12. in [46]. ■

A.2.3 Proof of Lemma 3.11

Step 1. Weak Convergence:

From (3.32)–(3.34), the quadruple $\big(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s), \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s), \Delta_\Psi^\epsilon u_{tX}(s), \Delta_\Psi^\epsilon \mathbb{r}_{tX}(s)\big)$ solves the system

$$\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s) = \Psi + \int_t^s \Delta_\Psi^\epsilon u_{tX}(\tau)\mathrm{d}\tau; \tag{A.9}$$

$$\begin{aligned} \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s) = &\int_0^1 h_{xx}\Big(\mathbb{Y}_{tX}(T) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T)\Big)\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T)d\theta \\ &+ \int_0^1 D_X^2 F_T\Big((\mathbb{Y}_{tX}(T) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T))\otimes m\Big)(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T))\mathrm{d}\theta \\ &+ \int_s^T \int_0^1 l_{xx}\Big(\mathbb{Y}_{tX}(s) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s), u_{tX}(s) + \theta\epsilon\Delta_\Psi^\epsilon u_{tX}(s)\Big)\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(\tau)\mathrm{d}\theta\mathrm{d}\tau \\ &+ \int_s^T \int_0^1 l_{xv}\Big(\mathbb{Y}_{tX}(s) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s), u_{tX}(s) + \theta\epsilon\Delta_\Psi^\epsilon u_{tX}(s)\Big)\Delta_\Psi^\epsilon u_{tX}(\tau)\mathrm{d}\theta\mathrm{d}\tau \\ &+ \int_s^T \int_0^1 D_X^2 F\Big((\mathbb{Y}_{tX}(\tau) + \theta\epsilon\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(\tau))\otimes m\Big)(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(\tau))\mathrm{d}\theta\mathrm{d}\tau - \int_s^T \sum_{j=1}^n \Delta_\Psi^\epsilon \mathbb{r}_{tX,j}(\tau)dw_j(\tau), \end{aligned} \tag{A.10}$$

meanwhile,

$$\int_0^1 l_{vx}\Big(\mathbb{Y}_{tX}(s)+\theta\epsilon\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(s),u_{tX}(s)+\theta\epsilon\Delta^\epsilon_\Psi u_{tX}(s)\Big)\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(s) \\ +l_{vv}\Big(\mathbb{Y}_{tX}(s)+\theta\epsilon\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(s),u_{tX}(s)+\theta\epsilon\Delta^\epsilon_\Psi u_{tX}(s)\Big)\Delta^\epsilon_\Psi u_{tX}(s)\mathrm{d}\theta+\Delta^\epsilon_\Psi\mathbb{Z}_{tX}(s)=0. \tag{A.11}$$

Since $\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(s)$, $\Delta^\epsilon_\Psi\mathbb{Z}_{tX}(s)$ and $\Delta^\epsilon_\Psi u_{tX}(s)$ are uniformly bounded in $L^\infty_{\mathcal{W}_{tX\Psi}}(t,T;\mathcal{H}_m)$ for all choices of ϵ by (3.47), then by Banach–Alaoglu theorem, these finite difference processes converge to the weak limits $\mathscr{D}\mathbb{Y}^\Psi_{tX}(s)$, $\mathscr{D}\mathbb{Z}^\Psi_{tX}(s)$ and $\mathscr{D}u^\Psi_{tX}(s)$ in $L^2_{\mathcal{W}_{tX\Psi}}(t,T;\mathcal{H}_m)$, as $\epsilon\to 0$ along a subsequence. Equation (A.9) becomes, as $\epsilon\to 0$ along the subsequence,

$$\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(s)\longrightarrow\mathscr{D}\mathbb{Y}^\Psi_{tX}(s)=\ \Psi+\int_t^s\mathscr{D}u^\Psi_{tX}(\tau)\mathrm{d}\tau,\quad\text{weakly in } L^2_{\mathcal{W}_{tX\Psi}}(t,T;\mathcal{H}_m), \tag{A.12}$$

where $\mathscr{D}u^\Psi_{tX}(\tau)$ can be expressed as

$$\mathscr{D}u^\Psi_{tX}(\tau)=\Big[\nabla_y u(\mathbb{Y}_{tX},\mathbb{Z}_{tX})(\tau)\Big]\Big[\mathscr{D}\mathbb{Y}^\Psi_{tX}(\tau)\Big]+\Big[\nabla_z u(\mathbb{Y}_{tX},\mathbb{Z}_{tX})(\tau)\Big]\Big[\mathscr{D}\mathbb{Z}^\Psi_{tX}(\tau)\Big] \tag{A.13}$$

due to the first order condition (3.34) and Remark 3.12. We also have

$$\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(T)\longrightarrow\mathscr{D}\mathbb{Y}^\Psi_{tX}(T)=\ \Psi+\int_t^T\mathscr{D}u^\Psi_{tX}(\tau)\mathrm{d}\tau, \tag{A.14}$$

weakly in $L^2_{\mathcal{W}_{tX\Psi}}(\mathcal{H}_m)$ up to a subsequence. Moreover, $\Delta^\epsilon_\Psi\mathbb{r}_{tX,j}(s)$ is uniformly bounded for all choices of ϵ in $L^2_{\mathcal{W}_{tX\Psi}}(t,T;\mathcal{H}_m)$ by (3.47), then it converges to the weak limits $\mathscr{D}\mathbb{r}^\Psi_{tX,j}(s)$, up to a subsequence. Define $\mathbb{Y}_{\theta\epsilon}(s):=\mathbb{Y}_{tX}(s)+\epsilon\theta\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(s)$ and $u_{\theta\epsilon}(s):=u_{tX}(s)+\epsilon\theta\Delta^\epsilon_\Psi u_{tX}(s)$, we rewrite the right hand side of (A.10) by augmenting the pointwisely bounded test random variable $\varphi\in L^\infty_{\mathcal{W}_{tX\Psi}}(t,T;\mathcal{H}_m)$ under the inner product, after telescoping,

$$\begin{aligned}
&\left\langle\int_0^1\left[h_{xx}(\mathbb{Y}_{\theta\epsilon}(T))+D_X^2F_T\Big(\mathbb{Y}_{\theta\epsilon}(T)\otimes m\Big)\right](\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(T))\mathrm{d}\theta,\varphi\right\rangle_{\mathcal{H}_m}\\
&\quad+\left\langle\int_s^T\int_0^1 l_{xv}(\mathbb{Y}_{\theta\epsilon}(\tau),u_{\theta\epsilon}(\tau))\Delta^\epsilon_\Psi u_{tX}(\tau)+\left[l_{xx}(\mathbb{Y}_{\theta\epsilon}(\tau),u_{\theta\epsilon}(\tau))+D_X^2F(\mathbb{Y}_{\theta\epsilon}(\tau)\otimes m)\right](\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(\tau))\mathrm{d}\theta\mathrm{d}\tau,\varphi\right\rangle_{\mathcal{H}_m}\\
&=\left\langle\int_0^1\left[h_{xx}(\mathbb{Y}_{\theta\epsilon}(T))-h_{xx}(\mathbb{Y}_{tX}(T))\right](\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(T))\mathrm{d}\theta,\varphi\right\rangle_{\mathcal{H}_m}\\
&\quad+\left\langle\int_0^1\left[D_X^2F_T\Big(\mathbb{Y}_{\theta\epsilon}(T)\otimes m\Big)-D_X^2F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)\right](\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(T))\mathrm{d}\theta,\varphi\right\rangle_{\mathcal{H}_m}\\
&\quad+\left\langle\int_0^1\left[h_{xx}(\mathbb{Y}_{tX}(T))+D_X^2F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)\right](\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(T))\mathrm{d}\theta,\varphi\right\rangle_{\mathcal{H}_m}\\
&\quad+\left\langle\int_s^T\int_0^1\left[l_{xv}(\mathbb{Y}_{\theta\epsilon}(\tau),u_{\theta\epsilon}(\tau))-l_{xv}(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau))\right]\Delta^\epsilon_\Psi u_{tX}(\tau)\mathrm{d}\theta\mathrm{d}\tau,\varphi\right\rangle_{\mathcal{H}_m}\\
&\quad+\left\langle\int_s^T\int_0^1 l_{xv}(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau))\Delta^\epsilon_\Psi u_{tX}(\tau)\mathrm{d}\theta\mathrm{d}\tau,\varphi\right\rangle_{\mathcal{H}_m}\\
&\quad+\left\langle\int_s^T\int_0^1\left[l_{xx}(\mathbb{Y}_{\theta\epsilon}(\tau),u_{\theta\epsilon}(\tau))-l_{xx}(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau))+D_X^2F(\mathbb{Y}_{\theta\epsilon}(\tau)\otimes m)-D_X^2F(\mathbb{Y}_{tX}(\tau)\otimes m)\right](\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(\tau))\mathrm{d}\theta\mathrm{d}\tau,\varphi\right\rangle_{\mathcal{H}_m}\\
&\quad+\left\langle\int_s^T\int_0^1\left[l_{xx}(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau))+D_X^2F(\mathbb{Y}_{tX}(\tau)\otimes m)\right](\Delta^\epsilon_\Psi\mathbb{Y}_{tX}(\tau))\mathrm{d}\theta\mathrm{d}\tau,\varphi\right\rangle_{\mathcal{H}_m}.
\end{aligned} \tag{A.15}$$

We claim that the following terms

$$\left\langle \int_0^1 \left[h_{xx}(\mathbb{Y}_{\theta\epsilon}(T)) + D_X^2 F_T\Big(\mathbb{Y}_{\theta\epsilon}(T)\otimes m\Big) - h_{xx}(\mathbb{Y}_{tX}(T)) - D_X^2 F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)\right](\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T))\mathrm{d}\theta, \varphi \right\rangle_{\mathcal{H}_m}; \tag{A.16}$$

$$\left\langle \int_s^T \int_0^1 \left[l_{xv}(\mathbb{Y}_{\theta\epsilon}(\tau), u_{\theta\epsilon}(\tau)) - l_{xv}(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau))\right] \Delta_\Psi^\epsilon u_{tX}(\tau)\mathrm{d}\theta\mathrm{d}\tau, \varphi \right\rangle_{\mathcal{H}_m}; \tag{A.17}$$

$$\left\langle \int_s^T \int_0^1 \left[l_{xx}(\mathbb{Y}_{\theta\epsilon}(\tau), u_{\theta\epsilon}(\tau)) + D_X^2 F(\mathbb{Y}_{\theta\epsilon}(\tau)\otimes m) - l_{xx}(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)) - D_X^2 F(\mathbb{Y}_{tX}(\tau)\otimes m)\right](\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(\tau))\mathrm{d}\theta\mathrm{d}\tau, \varphi \right\rangle_{\mathcal{H}_m}, \tag{A.18}$$

converge to zero, as $\epsilon \to 0$ up to a subsequence. For instance, for (A.16),

$$\left| \left\langle \int_0^1 \left[h_{xx}(\mathbb{Y}_{\theta\epsilon}(T)) - h_{xx}(\mathbb{Y}_{tX}(T))\right](\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T))\mathrm{d}\theta, \varphi \right\rangle_{\mathcal{H}_m} \right| \leq \|\varphi\|_{L^\infty} \cdot \|\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T)\|_{\mathcal{H}_m} \cdot \int_0^1 \Big\| h_{xx}(\mathbb{Y}_{\theta\epsilon}(T)) - h_{xx}(\mathbb{Y}_{tX}(T))\Big\|_{\mathcal{H}_m} \mathrm{d}\theta.$$

The definition of $\mathbb{Y}_{\theta\epsilon}(\tau)$ and the bounds in (3.47) tell us that

$$\int_0^1 \Big\| \mathbb{Y}_{\theta\epsilon}(T) - \mathbb{Y}_{tX}(T)\Big\|_{\mathcal{H}_m} \mathrm{d}\theta = \int_0^1 \epsilon\theta \Big\| \Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T)\Big\|_{\mathcal{H}_m} \mathrm{d}\theta \leq \frac{\epsilon C_4'}{2}\|\Psi\|_{\mathcal{H}_m} \longrightarrow 0 \quad \text{as } \epsilon \to 0. \tag{A.19}$$

The strong convergence in (A.19) and Borel–Cantelli lemma show that there is a subsequence of ϵ such that

$$\mathbb{Y}_{\theta\epsilon}(T) - \mathbb{Y}_{tX}(T) \longrightarrow 0, \quad m\otimes\mathbb{P}\text{-a.s., a.e. } \theta\in[0,1], \text{ as } \epsilon\to 0, \text{ and} \tag{A.20}$$

$$\Big\| \mathbb{Y}_{\theta\epsilon}(T) - \mathbb{Y}_{tX}(T)\Big\|_{\mathcal{H}_m} \longrightarrow 0, \quad \text{for a.e. } \theta\in[0,1] \text{ as } \epsilon\to 0. \tag{A.21}$$

Along this subsequence, the convergence in (A.20) and the continuity of h_{xx} in Assumptions **A(iv)**'s (3.7) tell us that

$$h_{xx}(\mathbb{Y}_{\theta\epsilon}(T)) - h_{xx}(\mathbb{Y}_{tX}(T)) \longrightarrow 0 \quad m\otimes\mathbb{P}\text{-a.s., a.e. } \theta\in[0,1], \text{ as } \epsilon\to 0. \tag{A.22}$$

Since h_{xx} is pointwisely bounded due to **A(iii)**'s (3.6), then $|h_{xx}(\mathbb{Y}_{\theta\epsilon}(T)) - h_{xx}(\mathbb{Y}_{tX}(T))| \leq 2\sup_{x\in\mathbb{R}^n} |h_{xx}(x)| \leq 2c_h$, $m\otimes\mathbb{P}$-a.s., a.e. $\theta\in[0,1]$. Using the dominated convergence theorem and (A.22), we see that as $\epsilon\to 0$, it holds that

$$\int_0^1 \|h_{xx}(\mathbb{Y}_{\theta\epsilon}(T)) - h_{xx}(\mathbb{Y}_{tX}(T))\|_{\mathcal{H}_m} \mathrm{d}\theta \longrightarrow 0. \tag{A.23}$$

By the $\mathcal{H}_m$-boundedness of $\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(\tau)$ in (3.47) and the strong convergence in (A.21) for a.e. $\theta\in[0,1]$, Assumption **B(v)(a)**'s (3.26) implies

$$\mathbb{E}\left[\int_{\mathbb{R}^n} \Big| D_X^2 F_T\Big(\mathbb{Y}_{\theta\epsilon}(T)\otimes m\Big)\big(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T)\big) - D_X^2 F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)\big(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(T)\big)\Big| \mathrm{d}m(x)\right] \longrightarrow 0,$$

for a.e. $\theta\in[0,1]$ as $\epsilon\to 0$. By Assumption **B(ii)**'s (3.23), Cauchy–Schwarz inequality and (3.47), we see also that

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^n}\left|D_X^2F_T\Big(\mathbb{Y}_{\theta\epsilon}(T)\otimes m\Big)\left(\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(T)\right)-D_X^2F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)\left(\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(T)\right)\right|\mathrm{d}m(x)\right]\\
&\le\left\|D_X^2F_T\Big(\mathbb{Y}_{\theta\epsilon}(T)\otimes m\Big)\left(\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(T)\right)\right\|_{\mathcal{H}_m}+\left\|D_X^2F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)\left(\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(T)\right)\right\|_{\mathcal{H}_m}\\
&\le 2c_TC_4'\|\Psi\|_{\mathcal{H}_m}
\end{aligned}\tag{A.24}$$

is uniformly bounded for a.e. $\theta\in[0,1]$. We apply the dominated convergence theorem to obtain that as $\epsilon\to 0$

$$\begin{aligned}
&\left|\left\langle\int_0^1\Big[D_X^2F_T\Big(\mathbb{Y}_{\theta\epsilon}(T)\otimes m\Big)-D_X^2F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)\Big](\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(T))\mathrm{d}\theta,\varphi\right\rangle_{\mathcal{H}_m}\right|\\
&\le\|\varphi\|_{L^\infty}\cdot\int_0^1\mathbb{E}\left[\int_{\mathbb{R}^n}\left|D_X^2F_T\Big(\mathbb{Y}_{\theta\epsilon}(T)\otimes m\Big)\left(\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(T)\right)-D_X^2F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)\left(\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(T)\right)\right|\mathrm{d}m(x)\right]\mathrm{d}\theta\\
&\longrightarrow 0.
\end{aligned}\tag{A.25}$$

Combining (A.23) and (A.25), we see that (A.16) converges to zero as $\epsilon\to 0$. Therefore, in a similar manner, (A.17)–(A.18) also converge to zero as $\epsilon\to 0$. Hence, by the weak convergences of $\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(\tau)$, $\Delta_\Psi^\epsilon u_{tX}(\tau)$, the weak convergence in (A.14) and the convergences of (A.16)–(A.18), (A.15) converges, as $\epsilon\to 0$ along a subsequence,

$$\begin{aligned}
&\int_t^T\left\langle\int_0^1\Big[h_{xx}(\mathbb{Y}_{\theta\epsilon}(T))+D_X^2F_T\Big(\mathbb{Y}_{\theta\epsilon}(T)\otimes m\Big)\Big](\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(T))\mathrm{d}\theta,\varphi\right\rangle_{\mathcal{H}_m}ds\\
&\quad+\int_t^T\int_s^T\left\langle\int_0^1 l_{xv}(\mathbb{Y}_{\theta\epsilon}(\tau),u_{\theta\epsilon}(\tau))\Delta_\Psi^\epsilon u_{tX}(\tau)+\Big[l_{xx}(\mathbb{Y}_{\theta\epsilon}(\tau),u_{\theta\epsilon}(\tau))+D_X^2F(\mathbb{Y}_{\theta\epsilon}(\tau)\otimes m)\Big](\Delta_\Psi^\epsilon\mathbb{Y}_{tX}(\tau))\mathrm{d}\theta,\varphi\right\rangle_{\mathcal{H}_m}\mathrm{d}\tau ds\\
&\longrightarrow\int_t^T\left\langle\Big[h_{xx}(\mathbb{Y}_{tX}(T))+D_X^2F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)\Big](\mathscr{D}\mathbb{Y}_{tX}^\Psi(T)),\varphi\right\rangle_{\mathcal{H}_m}ds\\
&\quad+\int_t^T\int_s^T\left\langle l_{xv}(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau))\mathscr{D}u_{tX}^\Psi(\tau)+\Big[l_{xx}(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau))+D_X^2F(\mathbb{Y}_{tX}(\tau)\otimes m)\Big](\mathscr{D}\mathbb{Y}_{tX}^\Psi(\tau)),\varphi\right\rangle_{\mathcal{H}_m}\mathrm{d}\tau ds,
\end{aligned}\tag{A.26}$$

since the remaining terms in (A.15) only involve linear operators acting on $\Delta_\Psi^\epsilon\mathbb{Y}_{tX}$ and $\Delta_\Psi^\epsilon u_{tX}$. The backward equation of (A.10) and the convergence of (A.26) show that

$$\begin{aligned}
\mathscr{D}\mathbb{Z}_{tX}^\Psi(s)=\;&h_{xx}\Big(\mathbb{Y}_{tX}(T)\Big)\mathscr{D}\mathbb{Y}_{tX}^\Psi(T)+D_X^2F_T\Big((\mathbb{Y}_{tX}(T))\otimes m\Big)(\mathscr{D}\mathbb{Y}_{tX}^\Psi(T))\\
&+\int_s^T l_{xx}\Big(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau)\Big)\mathscr{D}\mathbb{Y}_{tX}^\Psi(\tau)+l_{xv}\Big(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau)\Big)\mathscr{D}u_{tX}^\Psi(\tau)\mathrm{d}\tau\\
&+\int_s^T D_X^2F\Big((\mathbb{Y}_{tX}(\tau)\otimes m\Big)(\mathscr{D}\mathbb{Y}_{tX}^\Psi(\tau))\mathrm{d}\tau-\int_s^T\sum_{j=1}^n\mathscr{D}\mathbb{r}_{tX,j}^\Psi(\tau)dw_j(\tau).
\end{aligned}\tag{A.27}$$

Similarly, we also note that as $\epsilon\to 0$ along the subsequence,

$$
\begin{aligned}
\Delta_\Psi^\epsilon \mathbb{Z}_{tX}(t) \longrightarrow & \, h_{xx}\Big(\mathbb{Y}_{tX}(T)\Big)\mathscr{D}\mathbb{Y}_{tX}^\Psi(T) + D_X^2 F_T\Big((\mathbb{Y}_{tX}(T))\otimes m\Big)(\mathscr{D}\mathbb{Y}_{tX}^\Psi(T)) \\
& + \int_t^T l_{xx}\Big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\Big)\mathscr{D}\mathbb{Y}_{tX}^\Psi(\tau) + l_{xv}\Big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\Big)\mathscr{D}u_{tX}^\Psi(\tau)\mathrm{d}\tau \\
& + \int_t^T D_X^2 F\Big((\mathbb{Y}_{tX}(\tau)\otimes m\Big)(\mathscr{D}\mathbb{Y}_{tX}^\Psi(\tau))\mathrm{d}\tau - \int_t^T \sum_{j=1}^n \mathscr{D}\mathbb{r}_{tX,j}^\Psi(\tau)dw_j(\tau) \\
= & \, \mathscr{D}\mathbb{Z}_{tX}^\Psi(t), \qquad \text{weakly in } L^2_{\mathcal{W}_{tX\Psi}}(\mathcal{H}_m).
\end{aligned} \tag{A.28}
$$

Subject to the first order condition (3.34), we see that (3.48) is a linear system and thus its solution can be shown to be unique, where this claim will be proven below. Taking this for granted, we see that the solution of (A.12) and (A.27) is unique and equal to the weak limit of $\Big(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s), \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s), \Delta_\Psi^\epsilon \mathbb{r}_{tX}(s)\Big)$, with $u_{tX}(s)$ satisfying the first order condition (3.34). As a consequence, it implies that the weak limit of $\Big(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s), \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s), \Delta_\Psi^\epsilon \mathbb{r}_{tX}(s)\Big)$ along any subsequence as $\epsilon \to 0$ is the same, if it exists. Therefore, we can now identify $\Big(\mathscr{D}\mathbb{Y}_{tX}^\Psi(s), \mathscr{D}\mathbb{Z}_{tX}^\Psi(s), \mathscr{D}u_{tX}^\Psi(s), \mathscr{D}\mathbb{r}_{tX}^\Psi(s)\Big)$ with $\Big(D\mathbb{Y}_{tX}^\Psi(s), D\mathbb{Z}_{tX}^\Psi(s), Du_{tX}^\Psi(s), D\mathbb{r}_{tX}^\Psi(s)\Big)$ which is the Jacobian flow indeed.

Uniqueness of (3.48)[5]:

If we have two sets of solution with the same initial data Ψ to the system (3.48), then it is enough to show that the differences between them are zero; also note that these differences would satisfy the following first order condition (A.30). Since the system (3.48) is linear, it is sufficient to show that any solution $\Big(D\mathbb{Y}^*(s), D\mathbb{Z}^*(s), Du^*(s), D\mathbb{r}^*(s)\Big)$ to the system with a zero initial data, that is,

$$
\begin{cases}
D\mathbb{Y}^*(s) = \displaystyle\int_t^s Du^*(\tau)\mathrm{d}\tau; \\
D\mathbb{Z}^*(s) = h_{xx}(\mathbb{Y}_{tX}(T))D\mathbb{Y}^*(T) + D_X^2 F_T(\mathbb{Y}_{tX}(T)\otimes m)(D\mathbb{Y}^*(T)) \\
\qquad\qquad + \displaystyle\int_s^T \Big\{ l_{xx}(\mathbb{Y}_{tX}(\tau), u(\tau))D\mathbb{Y}^*(\tau) + l_{xv}(\mathbb{Y}_{tX}(\tau), u(\tau))Du^*(\tau)\Big\}\mathrm{d}\tau \\
\qquad\qquad + \displaystyle\int_t^T D_X^2 F(\mathbb{Y}_{tX}(\tau)\otimes m)\Big(D\mathbb{Y}^*(\tau)\Big)\mathrm{d}\tau - \int_s^T \sum_{j=1}^n D\mathbb{r}_j^*(\tau)dw_j(\tau),
\end{cases} \tag{A.29}
$$

must be vanished. From the first order condition in (3.34), we obtain that

$$
l_{vx}\Big(\mathbb{Y}_{tX}(s), u_{tX}(s)\Big)D\mathbb{Y}^*(s) + l_{vv}\Big(\mathbb{Y}_{tX}(s), u_{tX}(s)\Big)Du^*(s) + D\mathbb{Z}^*(s) = 0. \tag{A.30}
$$

We consider the inner product $\Big\langle D\mathbb{Y}^*(s), D\mathbb{Z}^*(s)\Big\rangle_{\mathcal{H}_m}$, together with (A.30), we obtain the following equation

$$
\begin{aligned}
& \Big\langle h_{xx}(\mathbb{Y}_{tX}(T))D\mathbb{Y}^*(T) + D_X^2 F_T\Big(\mathbb{Y}_{tX}(T)\otimes m\Big)(D\mathbb{Y}^*(T)), D\mathbb{Y}^*(T)\Big\rangle_{\mathcal{H}_m} \\
= & -\int_t^T \Big\langle l_{vx}\Big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\Big)D\mathbb{Y}^*(\tau) + l_{vv}\Big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\Big)Du^*(\tau), Du^*(\tau)\Big\rangle_{\mathcal{H}_m}\mathrm{d}\tau \\
& - \int_t^T \Big\langle l_{xv}(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau))Du^*(\tau) + l_{xx}(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau))D\mathbb{Y}^*(\tau), D\mathbb{Y}^*(\tau)\Big\rangle_{\mathcal{H}_m}\mathrm{d}\tau
\end{aligned}
$$

[5] It is well-known that the uniqueness of general linear FBSDE is not true, for instance, we consider the system $dX_t = Y_t$ and $dY_t = -X_t$ with $X_0 = Y_{\pi/2} = 0$. The uniqueness claim of (3.48) relies heavily on certain convexity structures of the FBSDE (3.30).

$$- \int_t^T \left\langle D_X^2 F(\mathbb{Y}_{tX}(\tau) \otimes m)(D\mathbb{Y}^*(\tau)), D\mathbb{Y}^*(\tau) \right\rangle_{\mathcal{H}_m} \mathrm{d}\tau.$$

Assumptions **A(v)**'s (3.8), **A(vi)**'s (3.9), **B(v)(b)**'s (3.27) imply

$$\int_t^T \lambda \left\| Du^*(s) \right\|_{\mathcal{H}_m}^2 - (c_l' + c') \left\| D\mathbb{Y}^*(s) \right\|_{\mathcal{H}_m}^2 ds - (c_h' + c_T') \left\| D\mathbb{Y}^*(T) \right\|_{\mathcal{H}_m}^2 \leq 0. \tag{A.31}$$

The equation of $D\mathbb{Y}^*(s)$ in (A.29) with a simple application of Cauchy–Schwarz inequality gives

$$\left\| D\mathbb{Y}^*(s) \right\|_{\mathcal{H}_m}^2 \leq s \int_t^T \left\| Du^*(\tau) \right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau, \int_t^T \left\| D\mathbb{Y}^*(\tau) \right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau \leq \frac{T^2}{2} \int_t^T \left\| Du^*(\tau) \right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau. \tag{A.32}$$

Putting (A.32) into (A.31), we have $\left[\lambda - (c_h' + c_T')_+ T - (c_l' + c')_+ \frac{T^2}{2}\right] \int_t^T \left\| Du^*(s) \right\|_{\mathcal{H}_m}^2 ds \leq 0$. The condition in (3.30) implies $\int_t^T \left\| Du^*(s) \right\|_{\mathcal{H}_m}^2 ds = 0$ which further deduces $Du^*(s) = 0$, $m \otimes \mathbb{P}$-a.s. for a.e. $s \in [t,T]$. Therefore, since the processes $D\mathbb{Y}^*(s)$ and $D\mathbb{Z}^*(s)$ are continuous in time s, we easily see that $D\mathbb{Y}^*(s) = D\mathbb{Z}^*(s) = 0$, $m \otimes \mathbb{P}$-a.s. for all $s \in [t,T]$, and $D\mathbb{r}^*(s) = 0$, $m \otimes \mathbb{P}$-a.s. for a.e. $s \in [t,T]$, from (A.29) and (A.30).

Step 2. Strong Convergence:

Now we next prove that the finite difference process $\left(\Delta_\Psi^\epsilon \mathbb{Y}_{tX}(s), \Delta_\Psi^\epsilon \mathbb{Z}_{tX}(s), \Delta_\Psi^\epsilon u_{tX}(s), \Delta_\Psi^\epsilon \mathbb{r}_{tX}(s)\right)$ converges to $\left(D\mathbb{Y}_{tX}^\Psi(s), D\mathbb{Z}_{tX}^\Psi(s), Du_{tX}^\Psi(s), D\mathbb{r}_{tX}^\Psi(s)\right)$ strongly in $L^2_{\mathcal{W}_{tX\Psi}}(t,T;\mathcal{H}_m)$ as $\epsilon \to 0$. For if not the case, there is a sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ such that $\epsilon_k \longrightarrow 0$ and, without loss of generality, we assume

$$\lim_{k\to\infty} \sup_{s\in[t,T]} \left\| \Delta_\Psi^{\epsilon_k} \mathbb{Y}_{tX}(s) - D\mathbb{Y}_{tX}^\Psi(s) \right\|_{\mathcal{H}_m} > 0. \tag{A.33}$$

According to Step 1 in this proof, we can extract a subsequence $\{\epsilon_{k_j}\}_{j\in\mathbb{N}}$ from $\{\epsilon_k\}_{k\in\mathbb{N}}$ such that $\left(\Delta_\Psi^{\epsilon_{k_j}} \mathbb{Y}_{tX}(s), \Delta_\Psi^{\epsilon_{k_j}} \mathbb{Z}_{tX}(s), \Delta_\Psi^{\epsilon_{k_j}} u_{tX}(s), \Delta_\Psi^{\epsilon_{k_j}} \mathbb{r}_{tX}(s)\right)$ converges weakly to $\left(D\mathbb{Y}_{tX}^\Psi(s), D\mathbb{Z}_{tX}^\Psi(s), Du_{tX}^\Psi(s), D\mathbb{r}_{tX}^\Psi(s)\right)$ as $j \to 0$, which is the unique solution of (3.48) with $Du_{tX}^\Psi(s)$ satisfying (A.13). For simplicity, we write ϵ_j in place of ϵ_{k_j} to avoid cumbersome notations without affecting the main arguments in the rest of our proof.

By noting that $\Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(s)$ is a finite variation process, we apply the traditional Itô lemma to $\left\langle \Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(s), \Delta_\Psi^{\epsilon_j} \mathbb{Z}_{tX}(s) \right\rangle_{\mathcal{H}_m}$ and use the first order condition in (A.11) to give the equation

$$\begin{aligned}
&\left\langle \Delta_\Psi^{\epsilon_j} \mathbb{Z}_{tX}(t), \Psi \right\rangle_{\mathcal{H}_m} \\
&= \left\langle \int_0^1 h_{xx}(\mathbb{Y}_{\theta\epsilon_j}(T)) \Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(T) + D_X^2 F_T\left(\mathbb{Y}_{\theta\epsilon_j}(T) \otimes m\right) (\Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(T)) \mathrm{d}\theta, \Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(T) \right\rangle_{\mathcal{H}_m} \\
&\quad + \int_t^T \left\langle \int_0^1 l_{vx}\left(\mathbb{Y}_{\theta\epsilon_j}(\tau), u_{\theta\epsilon_j}(\tau)\right) \Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(\tau) + l_{vv}\left(\mathbb{Y}_{\theta\epsilon_j}(\tau), u_{\theta\epsilon_j}(\tau)\right) \Delta_\Psi^{\epsilon_j} u_{tX}(\tau) \mathrm{d}\theta, \Delta_\Psi^{\epsilon_j} u_{tX}(\tau) \right\rangle_{\mathcal{H}_m} \mathrm{d}\tau \\
&\quad + \int_t^T \left\langle \int_0^1 l_{xv}\left(\mathbb{Y}_{\theta\epsilon_j}(\tau), u_{\theta\epsilon_j}(\tau)\right) \Delta_\Psi^{\epsilon_j} u_{tX}(\tau) + l_{xx}\left(\mathbb{Y}_{\theta\epsilon_j}(\tau), u_{\theta\epsilon_j}(\tau)\right) \Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(\tau) \right. \\
&\qquad \left. + D_X^2 F(\mathbb{Y}_{\theta\epsilon_j}(\tau) \otimes m)(\Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(\tau)) \mathrm{d}\theta, \Delta_\Psi^{\epsilon_j} \mathbb{Y}_{tX}(\tau) \right\rangle_{\mathcal{H}_m} \mathrm{d}\tau.
\end{aligned} \tag{A.34}$$

By defining the linear operators $Q_{1\epsilon_j} := h_{xx}(\mathbb{Y}_{\theta\epsilon_j}(T)) + D_X^2 F_T\Big(\mathbb{Y}_{\theta\epsilon_j}(T) \otimes m\Big)$, $Q_{2\epsilon_j} := l_{vx}\Big(\mathbb{Y}_{\theta\epsilon_j}(\tau), u_{\theta\epsilon_j}(\tau)\Big)$, $Q_{3\epsilon_j} := l_{vv}\Big(\mathbb{Y}_{\theta\epsilon_j}(\tau), u_{\theta\epsilon_j}(\tau)\Big)$, $Q_{4\epsilon_j} := l_{xv}\Big(\mathbb{Y}_{\theta\epsilon_j}(\tau), u_{\theta\epsilon_j}(\tau)\Big)$, $Q_{5\epsilon_j} := l_{xx}\Big(\mathbb{Y}_{\theta\epsilon_j}(\tau), u_{\theta\epsilon_j}(\tau)\Big) + D_X^2 F(\mathbb{Y}_{\theta\epsilon_j}(\tau) \otimes m)$, we can also write, by using the backward equation of Jacobian flow of (A.27) and Itô's lemma directly,

$$\begin{aligned}
\Big\langle D\mathbb{Z}_{tX}^\Psi(t), \Psi\Big\rangle_{\mathcal{H}_m} &= \Big\langle Q_1^*(D\mathbb{Y}_{tX}^\Psi(T)), D\mathbb{Y}_{tX}^\Psi(T)\Big\rangle_{\mathcal{H}_m} + \int_t^T \Big\langle Q_2^*(D\mathbb{Y}_{tX}^\Psi(\tau)), Du_{tX}^\Psi(\tau)\Big\rangle_{\mathcal{H}_m} d\tau \\
&\quad + \int_t^T \Big\langle Q_3^*(Du_{tX}^\Psi(\tau)), Du_{tX}^\Psi(\tau)\Big\rangle_{\mathcal{H}_m} d\tau + \int_t^T \Big\langle Q_4^*(Du_{tX}^\Psi(\tau)), D\mathbb{Y}_{tX}^\Psi(\tau)\Big\rangle_{\mathcal{H}_m} \mathrm{d}\tau \\
&\quad + \int_t^T \Big\langle Q_5^*(D\mathbb{Y}_{tX}^\Psi(\tau)), D\mathbb{Y}_{tX}^\Psi(\tau)\Big\rangle_{\mathcal{H}_m} \mathrm{d}\tau,
\end{aligned} \tag{A.35}$$

where Q_i^* is defined by replacing $\mathbb{Y}_{\theta\epsilon_j}$, $u_{\theta\epsilon_j}$ in $Q_{i\epsilon_j}$ by $\mathbb{Y}_{tX}$, u_{tX} respectively, for example, $Q_1^* = h_{xx}(\mathbb{Y}_{tX}(T)) + D_X^2 F_T(\mathbb{Y}_{tX}(T) \otimes m)$.

Step 2A. Estimate of $\Big\langle \Delta_\Psi^{\epsilon_j}\mathbb{Z}_{tX}(t), \Psi\Big\rangle_{\mathcal{H}_m} - \Big\langle D\mathbb{Z}_{tX}^\Psi(t), \Psi\Big\rangle_{\mathcal{H}_m}$:

Referring to (A.34) and (A.35), we want to study the limit of $\Big\langle \Delta_\Psi^{\epsilon_j}\mathbb{Z}_{tX}(t), \Psi\Big\rangle_{\mathcal{H}_m} - \Big\langle D\mathbb{Z}_{tX}^\Psi(t), \Psi\Big\rangle_{\mathcal{H}_m}$ by first checking the term

$$\begin{aligned}
&\int_0^1 \Big\langle Q_{1\epsilon_j}(\Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T)), \Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T)\Big\rangle_{\mathcal{H}_m} - \Big\langle Q_1^*(D\mathbb{Y}_{tX}^\Psi(T)), D\mathbb{Y}_{tX}^\Psi(T)\Big\rangle_{\mathcal{H}_m} \mathrm{d}\theta \\
&= \int_0^1 \Big\langle Q_{1\epsilon_j}\Big(\Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T) - D\mathbb{Y}_{tX}^\Psi(T)\Big), \Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T) - D\mathbb{Y}_{tX}^\Psi(T)\Big\rangle_{\mathcal{H}_m} \\
&\quad + \Big\langle Q_{1\epsilon_j}(D\mathbb{Y}_{tX}^\Psi(T)), \Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T) - D\mathbb{Y}_{tX}^\Psi(T)\Big\rangle_{\mathcal{H}_m} + \Big\langle Q_{1\epsilon_j}(\Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T)) - Q_1^*(D\mathbb{Y}_{tX}^\Psi(T)), D\mathbb{Y}_{tX}^\Psi(T)\Big\rangle_{\mathcal{H}_m} \mathrm{d}\theta.
\end{aligned} \tag{A.36}$$

The first term of third line of (A.36) reads

$$\begin{aligned}
\int_0^1 \Big\langle Q_{1\epsilon_j}(D\mathbb{Y}_{tX}^\Psi(T)), \Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T) - D\mathbb{Y}_{tX}^\Psi(T)\Big\rangle_{\mathcal{H}_m} \mathrm{d}\theta &= \int_0^1 \Big\langle (Q_{1\epsilon_j} - Q_1^*)(D\mathbb{Y}_{tX}^\Psi(T)), \Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T) - D\mathbb{Y}_{tX}^\Psi(T)\Big\rangle_{\mathcal{H}_m} \\
&\quad + \Big\langle Q_1^*(D\mathbb{Y}_{tX}^\Psi(T)), \Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T) - D\mathbb{Y}_{tX}^\Psi(T)\Big\rangle_{\mathcal{H}_m} \mathrm{d}\theta.
\end{aligned} \tag{A.37}$$

The second line of (A.37) converges to zero since $\Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T)$ weakly converges to $D\mathbb{Y}_{tX}^\Psi(T)$ by (A.14). The first term on the right hand side of (A.37) can be estimated by

$$\begin{aligned}
&\int_0^1 \left|\Big\langle (Q_{1\epsilon_j} - Q_1^*)(D\mathbb{Y}_{tX}^\Psi(T)), \Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T) - D\mathbb{Y}_{tX}^\Psi(T)\Big\rangle_{\mathcal{H}_m}\right| \mathrm{d}\theta \\
&\leq \int_0^1 \Big\|\Big[h_{xx}(\mathbb{Y}_{\theta\epsilon_j}(T)) - h_{xx}(\mathbb{Y}_{tX}(T))\Big](D\mathbb{Y}_{tX}^\Psi(T))\Big\|_{\mathcal{H}_m} \big\|\Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T) - D\mathbb{Y}_{tX}^\Psi(T)\big\|_{\mathcal{H}_m} \\
&\quad + \Big\|\Big[D_X^2 F_T\Big(\mathbb{Y}_{\theta\epsilon_j}(T) \otimes m\Big) - D_X^2 F_T\Big(\mathbb{Y}_{tX}(T) \otimes m\Big)\Big](D\mathbb{Y}_{tX}^\Psi(T))\Big\|_{\mathcal{H}_m} \cdot \big\|\Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T) - D\mathbb{Y}_{tX}^\Psi(T)\big\|_{\mathcal{H}_m} \mathrm{d}\theta.
\end{aligned}$$

By (3.47), in light of the nature of linear operator norm, Fatou's lemma and the weak convergence of $\Delta_\Psi^{\epsilon_j}\mathbb{Y}_{tX}(T)$ by (A.14), we see that

$$\big\|D\mathbb{Y}_{tX}^\Psi(T)\big\|_{\mathcal{H}_m} \leq C_4'\|\Psi\|_{\mathcal{H}_m}. \tag{A.38}$$

Thus

$$\begin{aligned}
&\int_0^1 \left|\Big\langle (Q_{1\epsilon_j} - Q_1^*)(D\mathbb{Y}^\Psi_{tX}(T)), \Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(T) - D\mathbb{Y}^\Psi_{tX}(T)\Big\rangle_{\mathcal{H}_m}\right| \mathrm{d}\theta \\
&\leq 2C_4'\|\Psi\|_{\mathcal{H}_m} \int_0^1 \left\|\big[h_{xx}(\mathbb{Y}_{\theta\epsilon_j}(T)) - h_{xx}(\mathbb{Y}_{tX}(T))\big](D\mathbb{Y}^\Psi_{tX}(T))\right\|_{\mathcal{H}_m} \mathrm{d}\theta \qquad \text{(A.39)}\\
&\quad + 2C_4'\|\Psi\|_{\mathcal{H}_m} \int_0^1 \left\|\big[D^2_X F_T\big(\mathbb{Y}_{\theta\epsilon_j}(T)\otimes m\big) - D^2_X F_T\big(\mathbb{Y}_{tX}(T)\otimes m\big)\big](D\mathbb{Y}^\Psi_{tX}(T))\right\|_{\mathcal{H}_m} \mathrm{d}\theta. \qquad \text{(A.40)}
\end{aligned}$$

Since $|h_{xx}|$ is also bounded due to Assumptions **A(iii)**'s (3.6), then by (A.22), (A.38) and dominated convergence theorem, we see that (A.39) converges to zero as $\epsilon_j \to 0$ up to a subsequence. Due to the strong convergence of $\mathbb{Y}_{\theta\epsilon_j}(T)$ in (A.21), the boundedness in (A.38) and (A.24), an application of Assumptions **B(v)(a)**'s (3.26) shows that

$$\int_0^1 \mathbb{E}\left[\int_{\mathbb{R}^n} \Big| \big[D^2_X F_T\big(\mathbb{Y}_{\theta\epsilon_j}(T)\otimes m\big) - D^2_X F_T\big(\mathbb{Y}_{tX}(T)\otimes m\big)\big](D\mathbb{Y}^\Psi_{tX}(T))\Big| \mathrm{d}m(x)\right] \mathrm{d}\theta \to 0,$$

which further implies that there is a subsequence of ϵ_j such that $\big[D^2_X F_T\big(\mathbb{Y}_{\theta\epsilon_j}(T)\otimes m\big) - D^2_X F_T\big(\mathbb{Y}_{tX}(T)\otimes m\big)\big](D\mathbb{Y}^\Psi_{tX}(T)) \longrightarrow 0$, $m\otimes\mathbb{P}$-a.s. for a.e. $\theta\in[0,1]$ as $\epsilon_j \to 0$. Therefore, by the boundedness of $D^2_X F_T$ in Assumption **B(ii)**'s (3.23), (A.38) and the dominated convergence theorem, we see that (A.40) also converges to 0 up to a subsequence. It concludes that (A.37) converges to 0 as $\epsilon_j \to 0$ up to a subsequence. The second term of the third line of (A.36) can be handled similarly to obtain its convergence to 0 up to a subsequence of ϵ_j. Hence, it yields that

$$\begin{aligned}
\int_0^1 &\Big\langle Q_{1\epsilon_j}(\Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(T)), \Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(T)\Big\rangle_{\mathcal{H}_m} - \Big\langle Q_1^*(D\mathbb{Y}^\Psi_{tX}(T)), D\mathbb{Y}^\Psi_{tX}(T)\Big\rangle_{\mathcal{H}_m} \\
&- \Big\langle Q_{1\epsilon_j}\big(\Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(T) - D\mathbb{Y}^\Psi_{tX}(T)\big), \Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(T) - D\mathbb{Y}^\Psi_{tX}(T)\Big\rangle_{\mathcal{H}_m} \mathrm{d}\theta
\end{aligned}$$

tends to 0 as $\epsilon_j \to 0$ up to a subsequence. By estimating the remaining terms in $\big\langle \Delta^{\epsilon_j}_\Psi \mathbb{Z}_{tX}(t), \Psi\big\rangle_{\mathcal{H}_m} - \big\langle D\mathbb{Z}^\Psi_{tX}(t), \Psi\big\rangle_{\mathcal{H}_m}$ involving $Q_{2\epsilon_j}, \dots, Q_{5\epsilon_j}$ similarly, we can deduce that as $\epsilon_j \to 0$ up to a subsequence,

$$\Big\langle \Delta^{\epsilon_j}_\Psi \mathbb{Z}_{tX}(t), \Psi\Big\rangle_{\mathcal{H}_m} - \Big\langle D\mathbb{Z}^\Psi_{tX}(t), \Psi\Big\rangle_{\mathcal{H}_m} - \mathscr{J}^\dagger_{\epsilon_j} \longrightarrow 0, \qquad \text{(A.41)}$$

where

$$\begin{aligned}
\mathscr{J}^\dagger_{\epsilon_j} := \int_0^1 &\Big\langle Q_{1\epsilon_j}\big(\Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(T) - D\mathbb{Y}^\Psi_{tX}(T)\big), \Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(T) - D\mathbb{Y}^\Psi_{tX}(T)\Big\rangle_{\mathcal{H}_m} \\
&+ \int_t^T \Big\langle Q_{2\epsilon_j}\big(\Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(s) - D\mathbb{Y}^\Psi_{tX}(s)\big) + Q_{3\epsilon_j}\big(\Delta^{\epsilon_j}_\Psi u_{tX}(s) - Du^\Psi_{tX}(s)\big), \Delta^{\epsilon_j}_\Psi u_{tX}(s) - Du^\Psi_{tX}(s)\Big\rangle_{\mathcal{H}_m} \\
&\quad + \Big\langle Q_{4\epsilon_j}\big(\Delta^{\epsilon_j}_\Psi u_{tX}(s) - Du^\Psi_{tX}(s)\big) + Q_{5\epsilon_j}\big(\Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(s) - D\mathbb{Y}^\Psi_{tX}(s)\big), \Delta^{\epsilon_j}_\Psi \mathbb{Y}_{tX}(s) - D\mathbb{Y}^\Psi_{tX}(s)\Big\rangle_{\mathcal{H}_m} ds\mathrm{d}\theta.
\end{aligned}$$

Step 2B. Estimate of $\mathscr{J}^\dagger_{\epsilon_j}$ and Conclusion:
Together with the convergences in (A.41) and (A.28), we have $\mathscr{J}^\dagger_{\epsilon_j} \longrightarrow 0$ as $\epsilon_j \to 0$ up to a subsequence. The convexity conditions of h_{xx}, $D^2_X F_T$, $D^2_X F$ and the second-order derivatives of l in Assumptions **A(v)**'s (3.8),

A(vi)'s (3.9) and **B(v)(b)**'s (3.27) imply that

$$\begin{aligned}\mathscr{J}^{\dagger}_{\epsilon_j} \geq \int_t^T \lambda \big\| \Delta^{\epsilon_j}_{\Psi} u_{tX}(s) - Du^{\Psi}_{tX}(s) \big\|^2_{\mathcal{H}_m} - (c_l' + c') \big\| \Delta^{\epsilon_j}_{\Psi} \mathbb{Y}_{tX}(s) - D\mathbb{Y}^{\Psi}_{tX}(s) \big\|^2_{\mathcal{H}_m} ds \\ - (c_h' + c_T') \big\| \Delta^{\epsilon_j}_{\Psi} \mathbb{Y}_{tX}(T) - D\mathbb{Y}^{\Psi}_{tX}(T) \big\|^2_{\mathcal{H}_m}. \end{aligned} \tag{A.42}$$

With an application of Cauchy–Schwarz inequality, the equation of $\Delta^{\epsilon_j}_{\Psi} \mathbb{Y}_{tX}(s) - D\mathbb{Y}^{\Psi}_{tX}(s)$ implies

$$\big\| \Delta^{\epsilon_j}_{\Psi} \mathbb{Y}_{tX}(s) - D\mathbb{Y}^{\Psi}_{tX}(s) \big\|^2_{\mathcal{H}_m} \leq (s-t) \int_t^s \big\| \Delta^{\epsilon_j}_{\Psi} u_{tX}(\tau) - Du^{\Psi}_{tX}(\tau) \big\|^2_{\mathcal{H}_m} \mathrm{d}\tau. \tag{A.43}$$

Putting (A.43) into (A.42), together with the assumption in (3.30), we see that as $\epsilon_j \to 0$ up to the subsequence,

$$\int_t^T \big\| \Delta^{\epsilon_j}_{\Psi} u_{tX}(s) - Du^{\Psi}_{tX}(s) \big\|^2_{\mathcal{H}_m} ds \longrightarrow 0. \tag{A.44}$$

We also bring (A.44) into (A.43) to yield that as $\epsilon_j \to 0$ up to a subsequence,

$$\big\| \Delta^{\epsilon_j}_{\Psi} \mathbb{Y}_{tX}(s) - D\mathbb{Y}^{\Psi}_{tX}(s) \big\|^2_{\mathcal{H}_m} \longrightarrow 0 \quad \text{uniformly for all } s \in [t,T]. \tag{A.45}$$

It contradicts (A.33), therefore the strong convergence of $\Delta^{\epsilon_j}_{\Psi} \mathbb{Y}_{tX}(s)$ should follow.

The strong convergences of $\Delta^{\epsilon_j}_{\Psi} \mathbb{Z}_{tX}(s)$ and $\Delta^{\epsilon_j}_{\Psi} \mathbb{r}_{tX}(s)$ are concluded by subtracting the equation of $\Delta^{\epsilon_j}_{\Psi} \mathbb{Z}_{tX}(s)$ in (A.10) from the equation of $D\mathbb{Z}^{\Psi}_{tX}(s)$ in (3.48) and then using Itô's lemma, together with the convergences in (A.44) and (A.45). Finally, the strong convergence of $\Delta^{\epsilon_j}_{\Psi} u_{tX}(s)$ is deduced by the first order condition in (3.34) and the strong convergences of $\Delta^{\epsilon_j}_{\Psi} \mathbb{Y}_{tX}(s)$, $\Delta^{\epsilon_j}_{\Psi} \mathbb{Z}_{tX}(s)$ and $\Delta^{\epsilon_j}_{\Psi} \mathbb{r}_{tX}(s)$ just obtained.

■

A.2.4 Proof of Lemma 3.13

Part 1. Linearity in Ψ:
The linearity follows easily from the linearity structure of the FBSDE in (3.48) and also its uniqueness proved in Lemma 3.11.

Part 2. Partial continuity in X:
For $X,\ \Psi \in L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m) \subset \mathcal{H}_m$, we consider a sequence $\{X_k\}_{k\in\mathbb{N}} \subset L^2_{\mathcal{W}_t^{\perp\!\!\perp}}(\mathcal{H}_m)$ such that $X_k \longrightarrow X$ in $\mathcal{H}_m$. Applying Itô's lemma to the inner product $\big\langle D\mathbb{Z}^{\Psi}_{tX_k}(s) - D\mathbb{Z}^{\Psi}_{tX}(s), D\mathbb{Y}^{\Psi}_{tX_k}(s) - D\mathbb{Y}^{\Psi}_{tX}(s) \big\rangle_{\mathcal{H}_m}$, together with the first order condition in (3.34), we have

$$\begin{aligned}
&\Big\langle h_{xx}(\mathbb{Y}_{tX_k}(T)) D\mathbb{Y}^{\Psi}_{tX_k}(T) - h_{xx}(\mathbb{Y}_{tX}(T)) D\mathbb{Y}^{\Psi}_{tX}(T), D\mathbb{Y}^{\Psi}_{tX_k}(T) - D\mathbb{Y}^{\Psi}_{tX}(T) \Big\rangle_{\mathcal{H}_m} \\
&\quad + \Big\langle D^2_X F_T(\mathbb{Y}_{tX_k}(T)\otimes m)\Big(D\mathbb{Y}^{\Psi}_{tX_k}(T)\Big) - D^2_X F_T(\mathbb{Y}_{tX}(T)\otimes m)\Big(D\mathbb{Y}^{\Psi}_{tX}(T)\Big), D\mathbb{Y}^{\Psi}_{tX_k}(T) - D\mathbb{Y}^{\Psi}_{tX}(T) \Big\rangle_{\mathcal{H}_m} \\
&= -2\int_t^T \Big\langle l_{vx}\Big(\mathbb{Y}_{tX_k}(\tau), u_{tX_k}(\tau)\Big) D\mathbb{Y}^{\Psi}_{tX_k}(\tau) - l_{vx}\Big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\Big) D\mathbb{Y}^{\Psi}_{tX}(\tau), Du^{\Psi}_{tX_k}(\tau) - Du^{\Psi}_{tX}(\tau) \Big\rangle_{\mathcal{H}_m} \mathrm{d}\tau \\
&\quad - \int_t^T \Big\langle l_{vv}\Big(\mathbb{Y}_{tX_k}(\tau), u_{tX_k}(\tau)\Big) Du^{\Psi}_{tX_k}(\tau) - l_{vv}\Big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\Big) Du^{\Psi}_{tX}(\tau), Du^{\Psi}_{tX_k}(\tau) - Du^{\Psi}_{tX}(\tau) \Big\rangle_{\mathcal{H}_m} \mathrm{d}\tau \\
&\quad - \int_t^T \Big\langle l_{xx}(\mathbb{Y}_{tX_k}(\tau), u_{tX_k}(\tau)) D\mathbb{Y}^{\Psi}_{tX_k}(\tau) - l_{xx}(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)) D\mathbb{Y}^{\Psi}_{tX}(\tau), D\mathbb{Y}^{\Psi}_{tX_k}(\tau) - D\mathbb{Y}^{\Psi}_{tX}(\tau) \Big\rangle_{\mathcal{H}_m} \mathrm{d}\tau
\end{aligned}$$

$$
-\int_t^T \left\langle D_X^2 F(\mathbb{Y}_{tX_k}(\tau)\otimes m)(D\mathbb{Y}_{tX_k}^\Psi(\tau)) - D_X^2 F(\mathbb{Y}_{tX}(\tau)\otimes m)(D\mathbb{Y}_{tX}^\Psi(\tau)), D\mathbb{Y}_{tX_k}^\Psi(\tau) - D\mathbb{Y}_{tX}^\Psi(\tau)\right\rangle_{\mathcal{H}_m} \mathrm{d}\tau. \tag{A.46}
$$

The first line in (A.46) can be estimated by using Young's inequality and Assumption **A(vi)**'s (3.9) such that we have

$$
\begin{aligned}
&\left\langle h_{xx}(\mathbb{Y}_{tX_k}(T))D\mathbb{Y}_{tX_k}^\Psi(T) - h_{xx}(\mathbb{Y}_{tX}(T))D\mathbb{Y}_{tX}^\Psi(T), D\mathbb{Y}_{tX_k}^\Psi(T) - D\mathbb{Y}_{tX}^\Psi(T)\right\rangle_{\mathcal{H}_m}\\
&= \left\langle h_{xx}(\mathbb{Y}_{tX_k}(T))\left[D\mathbb{Y}_{tX_k}^\Psi(T) - D\mathbb{Y}_{tX}^\Psi(T)\right] + \left[h_{xx}(\mathbb{Y}_{tX_k}(T)) - h_{xx}(\mathbb{Y}_{tX}(T))\right]D\mathbb{Y}_{tX}^\Psi(T), D\mathbb{Y}_{tX_k}^\Psi(T) - D\mathbb{Y}_{tX}^\Psi(T)\right\rangle_{\mathcal{H}_m}\\
&\geq -c_h'\left\|D\mathbb{Y}_{tX_k}^\Psi(T) - D\mathbb{Y}_{tX}^\Psi(T)\right\|_{\mathcal{H}_m}^2 - \kappa_8\left\|\left[h_{xx}(\mathbb{Y}_{tX_k}(T)) - h_{xx}(\mathbb{Y}_{tX}(T))\right]D\mathbb{Y}_{tX}^\Psi(T)\right\|_{\mathcal{H}_m}^2\\
&\quad - \frac{1}{4\kappa_8}\left\|D\mathbb{Y}_{tX_k}^\Psi(T) - D\mathbb{Y}_{tX}^\Psi(T)\right\|_{\mathcal{H}_m}^2,
\end{aligned}
$$

for some $\kappa_8 > 0$ to be determined later. All the other terms in (A.46) can be decomposed and estimated in a similar manner as above, then together with Assumptions **A(v)**'s (3.8), **B(v)(b)**'s (3.27), we have

$$
\begin{aligned}
&\int_t^T \lambda\left\|Du_{tX_k}^\Psi(\tau) - Du_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 - (c_l' + c')\left\|D\mathbb{Y}_{tX_k}^\Psi(\tau) - D\mathbb{Y}_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau - (c_h' + c_T')\left\|D\mathbb{Y}_{tX_k}^\Psi(T) - D\mathbb{Y}_{tX}^\Psi(T)\right\|_{\mathcal{H}_m}^2\\
&\leq \frac{1}{4\kappa_8}\left\|D\mathbb{Y}_{tX_k}^\Psi(T) - D\mathbb{Y}_{tX}^\Psi(T)\right\|_{\mathcal{H}_m}^2 + \kappa_8\left\|\left[h_{xx}(\mathbb{Y}_{tX_k}(T)) - h_{xx}(\mathbb{Y}_{tX}(T))\right]D\mathbb{Y}_{tX}^\Psi(T)\right\|_{\mathcal{H}_m}^2\\
&\quad + \frac{1}{4\kappa_9}\left\|D\mathbb{Y}_{tX_k}^\Psi(T) - D\mathbb{Y}_{tX}^\Psi(T)\right\|_{\mathcal{H}_m}^2 + \kappa_9\left\|\left[D_X^2 F(\mathbb{Y}_{tX_k}(T)\otimes m) - D_X^2 F(\mathbb{Y}_{tX}(T)\otimes m)\right](D\mathbb{Y}_{tX}^\Psi(T))\right\|_{\mathcal{H}_m}^2\\
&\quad + \int_t^T \frac{1}{2\kappa_{10}}\left\|Du_{tX_k}^\Psi(\tau) - Du_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 + 2\kappa_{10}\left\|\left[l_{vx}\big(\mathbb{Y}_{tX_k}(\tau), u_{tX_k}(\tau)\big) - l_{vx}\big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\big)\right]D\mathbb{Y}_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau\\
&\quad + \int_t^T \frac{1}{4\kappa_{11}}\left\|Du_{tX_k}^\Psi(\tau) - Du_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 + \kappa_{11}\left\|\left[l_{vv}\big(\mathbb{Y}_{tX_k}(\tau), u_{tX_k}(\tau)\big) - l_{vv}\big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\big)\right]Du_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau\\
&\quad + \int_t^T \frac{1}{4\kappa_{12}}\left\|D\mathbb{Y}_{tX_k}^\Psi(\tau) - D\mathbb{Y}_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 + \kappa_{12}\left\|\left[l_{xx}\big(\mathbb{Y}_{tX_k}(\tau), u_{tX_k}(\tau)\big) - l_{xx}\big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\big)\right]D\mathbb{Y}_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau\\
&\quad + \int_t^T \frac{1}{4\kappa_{13}}\left\|D\mathbb{Y}_{tX_k}^\Psi(\tau) - D\mathbb{Y}_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 + \kappa_{13}\left\|\left[D_X^2 F(\mathbb{Y}_{tX_k}(\tau)\otimes m) - D_X^2 F(\mathbb{Y}_{tX}(\tau)\otimes m)\right]D\mathbb{Y}_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau,
\end{aligned} \tag{A.47}
$$

for some positive constants $\kappa_9, \dots, \kappa_{13}$ to be determined. With an application of Cauchy–Schwarz inequality, the equation of $D\mathbb{Y}_{tX_k}^\Psi(\tau) - D\mathbb{Y}_{tX}^\Psi(\tau)$ implies that

$$
\left\|D\mathbb{Y}_{tX_k}^\Psi(s) - D\mathbb{Y}_{tX}^\Psi(s)\right\|_{\mathcal{H}_m}^2 \leq s\int_t^s \left\|Du_{tX_k}^\Psi(\tau) - Du_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau, \tag{A.48}
$$

Substituting (A.48) into (A.47), we have

$$
\begin{aligned}
&\int_t^T \Bigg[\lambda - (c_h' + c_T')_+ T - (c_l' + c')_+\frac{T^2}{2} - \frac{1}{2\kappa_{10}}\\
&\quad - \frac{1}{4\kappa_{11}} - \left(\frac{1}{4\kappa_8} + \frac{1}{4\kappa_9}\right)T - \left(\frac{1}{4\kappa_{12}} + \frac{1}{4\kappa_{13}}\right)\frac{T^2}{2}\Bigg]\left\|Du_{tX_k}^\Psi(\tau) - Du_{tX}^\Psi(\tau)\right\|_{\mathcal{H}_m}^2 \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \kappa_8\Big\|\big[h_{xx}(\mathbb{Y}_{tX_k}(T))-h_{xx}(\mathbb{Y}_{tX}(T))\big]D\mathbb{Y}^{\Psi}_{tX}(T)\Big\|^2_{\mathcal{H}_m}+\kappa_9\Big\|\big[D^2_XF(\mathbb{Y}_{tX_k}(T)\otimes m)-D^2_XF(\mathbb{Y}_{tX}(T)\otimes m)\big](D\mathbb{Y}^{\Psi}_{tX}(T))\Big\|^2_{\mathcal{H}_m}\\
&\quad+\int_t^T 2\kappa_{10}\Big\|\Big[l_{vx}\Big(\mathbb{Y}_{tX_k}(\tau),u_{tX_k}(\tau)\Big)-l_{vx}\Big(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau)\Big)\Big]D\mathbb{Y}^{\Psi}_{tX}(T)\Big\|^2_{\mathcal{H}_m}\mathrm{d}\tau\\
&\quad+\int_t^T \kappa_{11}\Big\|\Big[l_{vv}\Big(\mathbb{Y}_{tX_k}(\tau),u_{tX_k}(\tau)\Big)-l_{vv}\Big(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau)\Big)\Big]Du^{\Psi}_{tX}(T)\Big\|^2_{\mathcal{H}_m}\mathrm{d}\tau\\
&\quad+\int_t^T \kappa_{12}\Big\|\Big[l_{xx}\Big(\mathbb{Y}_{tX_k}(\tau),u_{tX_k}(\tau)\Big)-l_{xx}\Big(\mathbb{Y}_{tX}(\tau),u_{tX}(\tau)\Big)\Big]D\mathbb{Y}^{\Psi}_{tX}(T)\Big\|^2_{\mathcal{H}_m}\mathrm{d}\tau\\
&\quad+\int_t^T \kappa_{13}\Big\|\big[D^2_XF(\mathbb{Y}_{tX_k}(\tau)\otimes m)-D^2_XF(\mathbb{Y}_{tX}(\tau)\otimes m)\big](D\mathbb{Y}^{\Psi}_{tX}(\tau))\Big\|^2_{\mathcal{H}_m}\mathrm{d}\tau.
\end{aligned}
\tag{A.49}
$$

We next prove that the sequence of processes $\Big(D\mathbb{Y}^{\Psi}_{tX_k}(s),D\mathbb{Z}^{\Psi}_{tX_k}(s),Du^{\Psi}_{tX_k}(s),D\mathbb{r}^{\Psi}_{tX_k}(s)\Big)$ converges strongly in norm to $\Big(D\mathbb{Y}^{\Psi}_{tX}(s),D\mathbb{Z}^{\Psi}_{tX}(s),Du^{\Psi}_{tX}(s),D\mathbb{r}^{\Psi}_{tX}(s)\Big)$ as $k\to\infty$. For if not the case, there is a subsequence, without relabelling for simplicity, such that, for instance,

$$
\lim_{k\to\infty}\sup_{s\in[t,T]}\Big\|D\mathbb{Y}^{\Psi}_{tX_k}(s)-D\mathbb{Y}^{\Psi}_{tX}(s)\Big\|_{\mathcal{H}_m}>0. \tag{A.50}
$$

By setting $\epsilon=1$ and taking $\Psi=X_k-X$ in the bound in (3.47), we have

$$
\Big\|\mathbb{Y}_{tX_k}(T)-\mathbb{Y}_{tX}(T)\Big\|_{\mathcal{H}_m}\leq C_4'\|X_k-X\|_{\mathcal{H}_m}\qquad\text{and,} \tag{A.51}
$$

$$
\int_t^T\Big\|\mathbb{Y}_{tX_k}(\tau)-\mathbb{Y}_{tX}(\tau)\Big\|_{\mathcal{H}_m}\mathrm{d}\tau,\ \int_t^T\Big\|u_{tX_k}(\tau)-u_{tX}(\tau)\Big\|_{\mathcal{H}_m}\mathrm{d}\tau\leq C_4'T\|X_k-X\|_{\mathcal{H}_m}, \tag{A.52}
$$

where C_4' mentioned in (3.47) is independent of the sequence $\{X_k\}_{k\in\mathbb{N}}$ and X. The strong convergences in (A.51) and (A.52) imply that there is a subsequence such that $\mathbb{Y}_{tX_k}(T)$ converges to $\mathbb{Y}_{tX}(T)$, $m\otimes\mathbb{P}$-a.s., $\mathbb{Y}_{tX_k}(\tau)$ converges to $\mathbb{Y}_{tX}(\tau)$, $m\otimes\mathbb{P}$-a.s. for a.e. $\tau\in[t,T]$, and $u_{tX_k}(\tau)$ converges to $u_{tX}(\tau)$, $m\otimes\mathbb{P}$-a.s. for a.e. $\tau\in[t,T]$. Similar to Step 2A in the proof of Lemma 3.11, by the continuities and boundedness of D^2_XF, $D^2_XF_T$, h_{xx}, the second-order derivatives of l in Assumptions **B(iii)**'s (3.24), **B(iv)**'s (3.25), **A(iv)**'s (3.7), **B(ii)**'s (3.23), **A(ii)**'s (3.5), **A(iii)**'s (3.6), by applying the dominated convergence theorem to (A.49) deduces the subsequential convergence of the right hand side of (A.49) to zero and

$$
\begin{aligned}
&\int_t^T\left[\lambda-(c_h'+c_T')_+T-(c_l'+c')_+\frac{T^2}{2}-\frac{1}{2\kappa_{10}}-\frac{1}{4\kappa_{11}}-\left(\frac{1}{4\kappa_8}+\frac{1}{4\kappa_9}\right)T-\left(\frac{1}{4\kappa_{12}}+\frac{1}{4\kappa_{13}}\right)\frac{T^2}{2}\right]\\
&\quad\times\Big\|Du^{\Psi}_{tX_k}(\tau)-Du^{\Psi}_{tX}(\tau)\Big\|^2_{\mathcal{H}_m}\mathrm{d}\tau
\end{aligned}
$$

converges to 0 as $k\to\infty$ up to a subsequence. Choosing small enough κ_i's, the condition in (3.30) yields that

$$
\int_t^T\Big\|Du^{\Psi}_{tX_k}(\tau)-Du^{\Psi}_{tX}(\tau)\Big\|^2_{\mathcal{H}_m}\mathrm{d}\tau\longrightarrow 0,\qquad\text{as } k\to\infty \text{ up to a subsequence.} \tag{A.53}
$$

Together with (A.48), we have $\sup_{s\in[t,T]}\Big\|D\mathbb{Y}^{\Psi}_{tX_k}(s)-D\mathbb{Y}^{\Psi}_{tX}(s)\Big\|_{\mathcal{H}_m}$ converges to zero as $k\to\infty$ up to a subsequence. This contradicts (A.50), therefore the strong convergence of $D\mathbb{Y}^{\Psi}_{tX_k}(s)$ and the continuity of $D\mathbb{Y}^{\Psi}_{tX}(s)$ with respect to X in norm should follow.

The strong convergences of $D\mathbb{Z}^{\Psi}_{tX_k}(s)$ and $D\mathfrak{r}^{\Psi}_{tX_k}(s)$ are concluded by subtracting their equations and then using Itô's lemma, together with the convergences in (A.53), that of $D\mathbb{Y}^{\Psi}_{tX_k}(s)$, continuities and boundedness of D_X^2F, $D_X^2F_T$, h_{xx}, the second-order derivatives of l in Assumptions **B(iii)**'s (3.24), **B(iv)**'s (3.25), **A(iv)**'s (3.7), **B(ii)**'s (3.23), **A(ii)**'s (3.5), **A(iii)**'s (3.6). Finally, the strong convergence of $Du^{\Psi}_{tX_k}(s)$ is deduced by differentiating the first order condition in (3.34), continuities of the second-order derivatives of l in Assumption **A(iv)**'s (3.7), and the strong convergences of $D\mathbb{Y}^{\Psi}_{tX}(s)$, $D\mathbb{Z}^{\Psi}_{tX_k}(s)$ just obtained.

Part 3. Existence of Gâteaux Derivative:
To conclude, since $(D\mathbb{Y}^{\Psi}_{tX}(s), D\mathbb{Z}^{\Psi}_{tX}(s), Du^{\Psi}_{tX}(s), D\mathfrak{r}^{\Psi}_{tX,j}(s))$ is linear in Ψ and continuous in X for a given Ψ, therefore, by Proposition 3.2.15 in [46] together with the separability of the Hilbert space $\mathcal{H}_m$, we obtain the existence of the Fréchet derivatives. ■

A.3 Proof of statements in Section 4

A.3.1 Proof of Proposition 4.2

Let the initial random variables $X^1, X^2 \in L^2_{\mathcal{W}_t^{\perp\!\!\!\perp}}(\mathcal{H}_m)$, for $\nu = 1, 2$, and consider the corresponding solution to the FBSDE (3.32)–(3.34). For simplicity, we write $\mathbb{Y}^\nu(s) = \mathbb{Y}_{tX^\nu}(s)$, $\mathbb{Z}^\nu(s) = \mathbb{Z}_{tX^\nu}(s)$, $u^\nu(s) = u_{tX^\nu}(s)$ and $\mathfrak{r}^\nu(s) = \mathfrak{r}_{tX^\nu}(s)$, for $\nu = 1, 2$. We therefore consider the corresponding systems

$$\begin{cases} \mathbb{Y}^\nu(s) = X^\nu + \int_t^s u^\nu(\tau)\mathrm{d}\tau + \eta(w(s) - w(t)); & \text{(A.54)} \\ \mathbb{Z}^\nu(s) = h_x(\mathbb{Y}^\nu(T)) + D_X F_T(\mathbb{Y}^\nu(T) \otimes m) + \int_s^T \Big[l_x(\mathbb{Y}^\nu(\tau), u^\nu(\tau)) + D_X F(\mathbb{Y}^\nu(\tau) \otimes m) \Big] \mathrm{d}\tau - \int_s^T \sum_{j=1}^n \mathfrak{r}^\nu_j(\tau) dw_j(\tau), & \text{(A.55)} \end{cases}$$

subject to
$$l_v(\mathbb{Y}^\nu(s), u^\nu(s)) + \mathbb{Z}^\nu(s) = 0. \tag{A.56}$$

Part 1. Differentiability of $V(X \otimes m, t)$ in X:
According to Remark 3.8, we can use a common space of controls $L^2_{\mathcal{W}_{tX^1X^2}}(t, T; \mathcal{H}_m)$ for two problems (A.54)–(A.56) with $\nu = 1, 2$. Consider the dynamics (3.32)–(3.33) using the control $u^2(s)$ with the initial data X^1, the corresponding trajectory is $\mathbb{Y}^2(s) + X^1 - X^2$, and then by definition

$$\begin{aligned} & V(X^1 \otimes m, t) - V(X^2 \otimes m, t) \\ & \leq \int_t^T \left\{ \mathbb{E}\left[\int_{\mathbb{R}^n} l\Big(\mathbb{Y}^2(s) + X^1 - X^2, u^2(s)\Big) - l\Big(\mathbb{Y}^2(s), u^2(s)\Big) \mathrm{d}m(x) \right] + F((\mathbb{Y}^2(s) + X^1 - X^2) \otimes m) - F(\mathbb{Y}^2(s) \otimes m)) \right\} ds \\ & \quad + \mathbb{E}\left[\int_{\mathbb{R}^n} h(\mathbb{Y}^2(T) + X^1 - X^2) - h(\mathbb{Y}^2(T)) \mathrm{d}m(x) \right] + F_T((\mathbb{Y}^2(T) + X^1 - X^2) \otimes m) - F(\mathbb{Y}^2(T) \otimes m) \\ & = \int_0^1 \int_t^T \Big\langle l_x(\mathbb{Y}^2(s) + \theta(X^1 - X^2), u^2(s)) + D_X F((\mathbb{Y}^2(s) + \theta(X^1 - X^2)) \otimes m), X^1 - X^2 \Big\rangle_{\mathcal{H}_m} ds \mathrm{d}\theta \\ & \quad + \int_0^1 \Big\langle h_x(\mathbb{Y}^2(T) + \theta(X^1 - X^2)) + D_X F_T((\mathbb{Y}^2(T) + \theta(X^1 - X^2)) \otimes m), X^1 - X^2 \Big\rangle_{\mathcal{H}_m} \mathrm{d}\theta. \end{aligned}$$

From Assumptions **A(ii)**'s (3.5), **A(iii)**'s (3.6) and **B(ii)**'s (3.23), we further obtain

$$\begin{aligned} & V(X^1 \otimes m, t) - V(X^2 \otimes m, t) \\ & \leq \left\langle \int_t^T l_x(\mathbb{Y}^2(s), u^2(s)) + D_X F(\mathbb{Y}^2(s) \otimes m) ds + h_x(\mathbb{Y}^2(T)) + D_X F_T(\mathbb{Y}^2(T) \otimes m), X^1 - X^2 \right\rangle_{\mathcal{H}_m} \\ & \quad + \Big[c_h + c_T + (c_l + c)T \Big] \|X^1 - X^2\|^2_{\mathcal{H}_m}. \end{aligned} \tag{A.57}$$

Substituting (A.55) into (A.57), we see that

$$V(X^1\otimes m,t)-V(X^2\otimes m,t)\le\left\langle \mathbb{Z}^2(t),X^1-X^2\right\rangle_{\mathcal{H}_m}+\left[c_h+c_T+(c_l+c)T\right]\|X^1-X^2\|^2_{\mathcal{H}_m}. \tag{A.58}$$

By interchanging the role of X^1 and X^2, we obtain the reverse inequality

$$\begin{aligned}V(X^1\otimes m,t)-V(X^2\otimes m,t)\ge&\left\langle \mathbb{Z}^2(t),X^1-X^2\right\rangle_{\mathcal{H}_m}-\left[c_h+c_T+(c_l+c)T\right]\left\|X^1-X^2\right\|^2_{\mathcal{H}_m}\\&+\left\langle \mathbb{Z}^1(t)-\mathbb{Z}^2(t),X^1-X^2\right\rangle_{\mathcal{H}_m}.\end{aligned} \tag{A.59}$$

Next, using the first order conditions (A.56), the fact that $\mathbb{Y}^1(s)-\mathbb{Y}^2(s)$ is of finite variation and Itô's formula, we have

$$\begin{aligned}&\left\langle \mathbb{Z}^1(t)-\mathbb{Z}^2(t),X^1-X^2\right\rangle_{\mathcal{H}_m}\\&=\left\langle h_x(\mathbb{Y}^1(T))-h_x(\mathbb{Y}^2(T))+D_XF_T(\mathbb{Y}^1(T)\otimes m)-D_XF_T(\mathbb{Y}^2(T)\otimes m),\mathbb{Y}^1(T)-\mathbb{Y}^2(T)\right\rangle_{\mathcal{H}_m}\\&\quad+\int_t^T\left\langle l_v(\mathbb{Y}^1(s),u^1(s))-l_v(\mathbb{Y}^2(s),u^2(s)),u^1(s)-u^2(s)\right\rangle_{\mathcal{H}_m}ds\\&\quad+\int_t^T\left\langle l_x(\mathbb{Y}^1(s),u^1(s))-l_x(\mathbb{Y}^2(s),u^2(s))+D_XF(\mathbb{Y}^1(s)\otimes m)-D_XF(\mathbb{Y}^2(s)\otimes m),\mathbb{Y}^1(s)-\mathbb{Y}^2(s)\right\rangle_{\mathcal{H}_m}ds\\&\ge-(c_h'+c_T')\left\|\mathbb{Y}^1(T)-\mathbb{Y}^2(T)\right\|^2_{\mathcal{H}_m}+(\lambda-2c_l\kappa_{14})\int_t^T\left\|u^1(s)-u^2(s)\right\|^2_{\mathcal{H}_m}ds\\&\quad-\left(c_l'+\frac{c_l}{2\kappa_{14}}+c'\right)\int_t^T\left\|\mathbb{Y}^1(s)-\mathbb{Y}^2(s)\right\|^2_{\mathcal{H}_m}ds,\end{aligned}$$

for some positive constant κ_{14}. From the fact that $\mathbb{Y}^1(s)-\mathbb{Y}^2(s)=X^1-X^2+\int_t^s u^1(\tau)-u^2(\tau)\mathrm{d}\tau$, we get

$$\sup_{s\in[t,T]}\left\|\mathbb{Y}^1(s)-\mathbb{Y}^2(s)\right\|^2_{\mathcal{H}_m}\le(1+\kappa_{15})T\int_t^T\left\|u^1(s)-u^2(s)\right\|^2_{\mathcal{H}_m}ds+\left(1+\frac{1}{\kappa_{15}}\right)\left\|X^1-X^2\right\|^2_{\mathcal{H}_m} \tag{A.60}$$

for some $\kappa_{15}>0$, therefore we obtain

$$\begin{aligned}&\left\langle \mathbb{Z}^1(t)-\mathbb{Z}^2(t),X^1-X^2\right\rangle_{\mathcal{H}_m}\\&\ge\left[\lambda-2c_l\kappa_{14}-(c_h'+c_T')_+(1+\kappa_{15})T-\left(c_l'+\frac{c_l}{2\kappa_{14}}+c'\right)_+(1+\kappa_{15})\frac{T^2}{2}\right]\int_t^T\left\|u^1(s)-u^2(s)\right\|^2_{\mathcal{H}_m}ds\\&\quad-\left[\left(c_l'+\frac{c_l}{2\kappa_{14}}+c'\right)_+T\left(1+\frac{1}{\kappa_{15}}\right)+(c_h'+c_T')_+\left(1+\frac{1}{\kappa_{15}}\right)\right]\left\|X^1-X^2\right\|^2_{\mathcal{H}_m}.\end{aligned} \tag{A.61}$$

Employing the assumption (3.30) and choosing suitable κ_{14} and κ_{15} so that the coefficient of the first term on the right hand side of (A.61) is positive. Combining the inequalities in (A.59) and (A.58), together with (A.61), it follows that

$$\left|V(X^1\otimes m,t)-V(X^2\otimes m,t)-\left\langle \mathbb{Z}^2(t),X^1-X^2\right\rangle_{\mathcal{H}_m}\right|\le A_4\left\|X^1-X^2\right\|^2_{\mathcal{H}_m},$$

where A_4 is positive and clearly independent of X^1 and X^2. This proves immediately the first part of Proposition 4.2.
Part 2. Lipschitz continuity of $D_X V(X \otimes m, t)$ in X:
Note from (A.61) that we have also proven the following estimate

$$\int_t^T \left\| u^1(s) - u^2(s) \right\|_{\mathcal{H}_m}^2 ds \leq A_5 \left\langle \mathbb{Z}^1(t) - \mathbb{Z}^2(t), X^1 - X^2 \right\rangle_{\mathcal{H}_m} + A_6 \left\| X^1 - X^2 \right\|_{\mathcal{H}_m}^2, \tag{A.62}$$

for constants A_5 and A_6 depending only on λ, c_l, c', c'_h, c'_T, c'_l and T. Next, Young's inequality also gives

$$\begin{aligned}
&\left\langle \mathbb{Z}^1(t) - \mathbb{Z}^2(t), X^1 - X^2 \right\rangle_{\mathcal{H}_m} \\
&= \left\langle \int_t^T l_x(\mathbb{Y}^1(s), u^1(s)) - l_x(\mathbb{Y}^2(s), u^2(s)) + D_X F(\mathbb{Y}^1(s) \otimes m) - D_X F(\mathbb{Y}^2(s) \otimes m) ds, X^1 - X^2 \right\rangle_{\mathcal{H}_m} \\
&\quad + \left\langle h_x(\mathbb{Y}^1(T)) - h_x(\mathbb{Y}^2(T)) + D_X F_T(\mathbb{Y}^1(T) \otimes m) - D_X F(\mathbb{Y}^2(T) \otimes m), X^1 - X^2 \right\rangle_{\mathcal{H}_m} \\
&\leq \kappa_{16} c_l^2 \int_t^T \left\| u^1(s) - u^2(s) \right\|_{\mathcal{H}_m}^2 ds + \frac{1}{4} \left(\frac{1}{\kappa_{16}} + \frac{1}{\kappa_{17}} + \frac{1}{\kappa_{18}} + \frac{1}{\kappa_{19}} + \frac{1}{\kappa_{20}} \right) \left\| X^1 - X^2 \right\|_{\mathcal{H}_m}^2 \\
&\quad + (\kappa_{17} c_l^2 + \kappa_{18} c^2) \int_t^T \left\| \mathbb{Y}^1(s) - \mathbb{Y}^2(s) \right\|_{\mathcal{H}_m}^2 ds + (\kappa_{19} c_h^2 + \kappa_{20} c_T^2) \left\| \mathbb{Y}^1(T) - \mathbb{Y}^2(T) \right\|_{\mathcal{H}_m}^2.
\end{aligned} \tag{A.63}$$

Substituting the inequalities in (A.60), (A.63) into (A.62), we see

$$\begin{aligned}
&\left(1 - \kappa_{16} c_l^2 A_5 - (\kappa_{17} c_l^2 + \kappa_{18} c^2) A_5 (1 + \kappa'_{15}) \frac{T^2}{2} - (\kappa_{19} c_h^2 + \kappa_{20} c_T^2) A_5 (1 + \kappa'_{15}) T \right) \int_t^T \left\| u^1(s) - u^2(s) \right\|_{\mathcal{H}_m}^2 ds \\
&\leq \left[A_6 + \frac{A_5}{4} \sum_{i=16}^{20} \frac{1}{\kappa_i} + A_5 (\kappa_{17} c_l^2 + \kappa_{18} c^2) \left(1 + \frac{1}{\kappa'_{15}} \right) T + A_5 (\kappa_{19} c_h^2 + \kappa_{20} c_T^2) \left(1 + \frac{1}{\kappa'_{15}} \right) \right] \left\| X^1 - X^2 \right\|_{\mathcal{H}_m}^2.
\end{aligned}$$

Choosing κ_i's small enough, we have

$$\int_t^T \left\| u^1(s) - u^2(s) \right\|_{\mathcal{H}_m}^2 ds \leq A_7 \left\| X^1 - X^2 \right\|_{\mathcal{H}_m}^2, \tag{A.64}$$

for some $A_7 > 0$ depending only on λ, c, c_l, c_h, c_T, c', c'_h, c'_T, c'_l and T. Plugging back (A.64) into (A.60), it implies

$$\sup_{s \in [t,T]} \left\| \mathbb{Y}^1(s) - \mathbb{Y}^2(s) \right\|_{\mathcal{H}_m}^2 \leq A_9 \left\| X^1 - X^2 \right\|_{\mathcal{H}_m}^2 \text{ and } \int_t^T \left\| \mathbb{Y}^1(s) - \mathbb{Y}^2(s) \right\|_{\mathcal{H}_m}^2 ds \leq A_{10} \left\| X^1 - X^2 \right\|_{\mathcal{H}_m}^2. \tag{A.65}$$

Hence, by using Proposition 4.2, we have $\left\| D_X V(X^1 \otimes m, t) - D_X V(X^2 \otimes m, t) \right\|_{\mathcal{H}_m} = \left\| \mathbb{Z}^1(t) - \mathbb{Z}^2(t) \right\|_{\mathcal{H}_m}$, together with (3.45), (A.65), the backward equations of (A.55), Assumption **A(ii)**'s (3.5), **A(iii)**'s (3.6), **B(ii)**'s (3.23), the result follows. ■

A.3.2 Proof of Proposition 4.3

We begin with (4.2). For the ease of notation, we take $t_1 = t + \epsilon$ and $t_2 = t$ with $t \in [0, T)$ and $\epsilon \in [0, T - t]$. From the optimality principle and (5.1), we have

$$V(X \otimes m, t) - V(X \otimes m, t+\epsilon) = \int_t^{t+\epsilon} \mathbb{E}\left[\int_{\mathbb{R}^n} l(\mathbb{Y}_{tX}(s), u_{tX}(s))\mathrm{d}m(x)\right] + F(\mathbb{Y}_{tX}(s) \otimes m)\, ds \\ + V(\mathbb{Y}_{tX}(t+\epsilon) \otimes m, t+\epsilon) - V(X \otimes m, t+\epsilon). \tag{A.66}$$

The second line of (A.66) can be estimated by using Assumption **A(ii)**'s (3.4), (3.21) and (3.38) to obtain

$$\int_t^{t+\epsilon} \left|\mathbb{E}\left[\int_{\mathbb{R}^n} l(\mathbb{Y}_{tX}(s), u_{tX}(s)\mathrm{d}m(x)\right]\right| + \left|F(\mathbb{Y}_{tX}(s) \otimes m)\right|\, ds \leq \epsilon(c_l + c) \sup_{s\in[t,T]} \left(1 + \left\|\mathbb{Y}_{tX}(s)\right\|^2_{\mathcal{H}_m} + \left\|u_{tX}(s)\right\|^2_{\mathcal{H}_m}\right) \\ \leq \epsilon A_{11}\left(1 + \left\|X\right\|^2_{\mathcal{H}_m}\right), \tag{A.67}$$

for some $A_{11} > 0$ depending on n, λ, η, c, c_l, c_h, c_T, c', c'_h, c'_T, c'_l and T. The Lipschitz continuity of $D_X V$ in Proposition 4.2 and the relation in (2.6) imply that

$$\begin{aligned} &\left|V(\mathbb{Y}_{tX}(t+\epsilon) \otimes m, t+\epsilon) - V(X \otimes m, t+\epsilon) - \left\langle D_X V(X \otimes m, t+\epsilon), \mathbb{Y}_{tX}(t+\epsilon) - X\right\rangle_{\mathcal{H}_m}\right| \\ &= \left|\int_0^1 \left\langle D_X V\big((X + \theta(\mathbb{Y}_{tX}(t+\epsilon) - X)) \otimes m, t+\epsilon\big) - D_X V(X \otimes m, t+\epsilon), \mathbb{Y}_{tX}(t+\epsilon) - X\right\rangle_{\mathcal{H}_m} d\theta\right| \\ &\leq C_6 \left\|\mathbb{Y}_{tX}(t+\epsilon) - X\right\|^2_{\mathcal{H}_m}. \end{aligned} \tag{A.68}$$

Since $D_X V(X \otimes m, t+\epsilon)$ is $\sigma(X)$-measurable and $X \in L^2_{\mathcal{W}_t^{\perp\!\!\!\perp}}(\mathcal{H}_m)$, we use Proposition 4.2, (3.38) and remove the stochastic integral by taking the expectation so as to obtain

$$\left|\left\langle D_X V(X \otimes m, t+\epsilon), \mathbb{Y}_{tX}(t+\epsilon) - X\right\rangle_{\mathcal{H}_m}\right| = \left|\left\langle D_X V(X \otimes m, t+\epsilon), \int_t^{t+\epsilon} u_{tX}(s) ds\right\rangle_{\mathcal{H}_m}\right| \leq \epsilon A_{12}\left(1 + \left\|X\right\|^2_{\mathcal{H}_m}\right). \tag{A.69}$$

Substituting (A.67), (A.68), (A.69) into (A.66), we obtain

$$\left|V(X \otimes m, t) - V(X \otimes m, t+\epsilon)\right| \leq \epsilon A_{11}\left(1 + \left\|X\right\|^2_{\mathcal{H}_m}\right) + C_6\left\|\mathbb{Y}_{tX}(t+\epsilon) - X\right\|^2_{\mathcal{H}_m} + \epsilon A_{12}\left(1 + \left\|X\right\|^2_{\mathcal{H}_m}\right),$$

which leads to the result (4.2) by using (3.43) to deal with the middle term on the right hand side.

We now turn to the proof of (4.3) by applying (4.1), which is equivalent to establish the inequality $\left\|\mathbb{Z}_{t+\epsilon,X}(t+\epsilon) - \mathbb{Z}_{tX}(t)\right\|_{\mathcal{H}_m} \leq C_8\left(\epsilon^{\frac{1}{2}} + \epsilon\|X\|_{\mathcal{H}_m}\right)$. From the BSDE in (3.33), we can write

$$\mathbb{Z}_{t+\epsilon,X}(t+\epsilon) - \mathbb{Z}_{tX}(t) = \mathbb{Z}_{t+\epsilon,X}(t+\epsilon) - \mathbb{E}\big(\mathbb{Z}_{tX}(t+\epsilon)\big| X\big) - \int_t^{t+\epsilon} \mathbb{E}\left[l_x\big(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau)\big) + D_X F\big(\mathbb{Y}_{tX}(\tau) \otimes m\big)\Big| X\right] \mathrm{d}\tau.$$

Applying (3.38), Assumption **A(i)**'s (3.4) and **B(i)**'s (3.22), it yields

$$\left\|\mathbb{Z}_{t+\epsilon,X}(t+\epsilon) - \mathbb{Z}_{tX}(t)\right\|_{\mathcal{H}_m} \leq \left\|\mathbb{Z}_{t+\epsilon,X}(t+\epsilon) - \mathbb{Z}_{tX}(t+\epsilon)\right\|_{\mathcal{H}_m} + A_{13}\epsilon\big(1 + \|X\|_{\mathcal{H}_m}\big).$$

Note that $\mathbb{Z}_{tX}(t+\epsilon) = \mathbb{Z}_{t+\epsilon,\mathbb{Y}_{tX}(t+\epsilon)}(t+\epsilon)$, using Proposition 4.2, we further rewrite

$$\left\|\mathbb{Z}_{t+\epsilon,X}(t+\epsilon) - \mathbb{Z}_{tX}(t)\right\|_{\mathcal{H}_m} \leq C_6\left\|\mathbb{Y}_{tX}(t+\epsilon) - X\right\|_{\mathcal{H}_m} + A_{13}\epsilon\Big(1 + \|X\|_{\mathcal{H}_m}\Big)$$

and thus the result (4.3) is obtained immediately due to (3.43). ■

A.3.3 Proof of Proposition 4.6

The proof of this proposition is a modification of the proof of Lemma 3.11. By using (4.4), we need to prove that as $k \to \infty$, $D\mathbb{Z}^{\Psi_k}_{t_k X_k}(t_k) \longrightarrow D\mathbb{Z}^{\Psi}_{tX}(t)$ in $\mathcal{H}_m$. Fixing $s \in (t,T]$, we can assume that $t_k < s$ for large enough k. From (3.47) and Lemma 3.13, the linearity of the system (3.48) deduces that $\left\|D\mathbb{Z}^{\Psi_k}_{t_k X_k}(t_k) - D\mathbb{Z}^{\Psi}_{t_k X_k}(t_k)\right\|_{\mathcal{H}_m} \leq C_4'\|\Psi_k - \Psi\|_{\mathcal{H}_m}$, where C_4' is independent of k. Note that $D\mathbb{Z}^{\Psi_k}_{t_k X_k}(t_k) - D\mathbb{Z}^{\Psi}_{tX}(t) = \left[D\mathbb{Z}^{\Psi_k}_{t_k X_k}(t_k) - D\mathbb{Z}^{\Psi}_{t_k X_k}(t_k)\right] + \left[D\mathbb{Z}^{\Psi}_{t_k X_k}(t_k) - D\mathbb{Z}^{\Psi}_{tX}(t)\right]$, in order to prove the convergence, it is sufficient to prove that the second term of the right hand side converges to 0, that is, in the $\mathcal{H}_m$-sense,

$$D\mathbb{Z}^{\Psi}_{t_k X_k}(t_k) \longrightarrow D\mathbb{Z}^{\Psi}_{tX}(t) \quad \text{ in } \mathcal{H}_m. \tag{A.70}$$

From (3.47), (3.43), and the flow property (5.2), we obtain that, for $s \in [t_k, T]$,

$$\begin{aligned}\left\|\mathbb{Y}_{t_k X_k}(s) - \mathbb{Y}_{tX}(s)\right\|_{\mathcal{H}_m} &\leq \left\|\mathbb{Y}_{t_k X_k}(s) - \mathbb{Y}_{t_k X}(s)\right\|_{\mathcal{H}_m} + \left\|\mathbb{Y}_{t_k X}(s) - \mathbb{Y}_{t_k \mathbb{Y}_{tX}(t_k)}(s)\right\|_{\mathcal{H}_m}\\ &\leq A_{14}\Big[\left\|X_k - X\right\|_{\mathcal{H}_m} + (t_k - t)^{1/2}\Big]\end{aligned} \tag{A.71}$$

and similar estimates hold for $\left\|\mathbb{Z}_{t_k X_k}(s) - \mathbb{Z}_{tX}(s)\right\|_{\mathcal{H}_m}$ and $\left\|u_{t_k X_k}(s) - u_{tX}(s)\right\|_{\mathcal{H}_m}$ by arguing in the same way as in (A.71), with A_{14} independent of k. In order to facilitate the comparison against X_k's and Ψ_k's, we introduce the σ-algebras, for $s \in (t,T]$, $\widetilde{\mathcal{W}}^s_{tX\Psi} := \bigvee_{\{j\in\mathbb{N}|t_j<s\}}\left(\mathcal{W}^s_{t_j X_j \Psi_j}\bigvee \mathcal{W}^{t_j}_t\right)$. Note that X is $\mathcal{W}^s_{tX\Psi}$-measurable, so X is also $\widetilde{\mathcal{W}}^s_{tX\Psi}$-measurable as $\mathcal{W}^s_{tX\Psi} \subset \widetilde{\mathcal{W}}^s_{tX\Psi}$. For simplicity, we denote the respective processes $D\mathbb{Y}^{\Psi}_{t_k X_k}(s)$, $D\mathbb{Z}^{\Psi}_{t_k X_k}(s)$, $Du^{\Psi}_{t_k X_k}(s)$, $D\mathbb{r}^{\Psi}_{t_k X_k}(s)$ by $D\mathbb{Y}_k(s)$, $D\mathbb{Z}_k(s)$, $Du_k(s)$, $D\mathbb{r}_k(s)$. Fix a $\tau^* \in (t,T]$, no matter how close to t, we can find $N^* \in \mathbb{N}$ large enough such that $t_k < \tau^*$ for any $k > N^*$. The processes $D\mathbb{Y}_k(s)$, $Du_k(s)$ and $D\mathbb{Z}_k(s)$ remain bounded in $L^{\infty}_{\widetilde{\mathcal{W}}_{tX\Psi}}(t_k,T;\mathcal{H}_m)$, and $D\mathbb{r}_{k,j}(s)$ remains bounded in $L^2_{\widetilde{\mathcal{W}}_{tX\Psi}}(t_k,T;\mathcal{H}_m)$, due to (3.47) (the bounds hold no matter what the filtration is chosen).

To prove the convergence in (A.70), we suppose by contradiction that there is a subsequence of k, namely, k_l, such that $D\mathbb{Z}_{k_l}$ diverges from $D\mathbb{Z}^{\Psi}_{tX}$ in $\mathcal{H}_m$ as $l \to \infty$; if there were a subsequence of the subsequence k_l, namely, k_j', such that $D\mathbb{Z}_{k_j'}$ converges to $D\mathbb{Z}_{tX}$ in $\mathcal{H}_m$ as $j \to \infty$, then the assumption is violated. Therefore, without loss of generality, in light of Banach–Alaoglu theorem, we can pick subsequences of the processes, without relabeling, such that they converge weakly to $\mathscr{D}\mathbb{Y}_{\infty}(s)$, $\mathscr{D}u_{\infty}(s)$, $\mathscr{D}\mathbb{Z}_{\infty}(s)$, $\mathscr{D}\mathbb{r}_{\infty,j}(s)$ in $L^2_{\widetilde{\mathcal{W}}_{tX\Psi}}(\tau^*,T;\mathcal{H}_m)$, respectively, and then it is sufficient to prove the strong convergence in $\mathcal{H}_m$ and show that $\mathscr{D}\mathbb{Y}_{\infty}(s) = D\mathbb{Y}^{\Psi}_{tX}(s)$, $\mathscr{D}\mathbb{Z}_{\infty}(s) = D\mathbb{Z}^{\Psi}_{tX}(s)$, $\mathscr{D}u_{\infty}(s) = Du^{\Psi}_{tX}(s)$, $\mathscr{D}\mathbb{r}_{\infty,j}(s) = D\mathbb{r}^{\Psi}_{tX,j}(s)$. The remaining arguments are the same as in the proof of Lemma 3.11, we omit this here. We refer to Section 6.3.3 in [47] for more detail. ■

A.4 Proof of statements in Section 5

A.4.1 Proof of Theorem 5.1

Part 1. Proof of (a):

From the dynamic programming principle or optimality principle, we can write
$0 = \frac{1}{\epsilon}\int_s^{s+\epsilon}\int_{\mathbb{R}^n}\mathbb{E}\Big[l(\mathbb{Y}_{tX}(\tau), u_{tX}(\tau))\Big]\mathrm{d}m(x) + F(\mathbb{Y}_{tX}(\tau)\otimes m)\mathrm{d}\tau + \frac{1}{\epsilon}\Big[V(\mathbb{Y}_{tX}(s+\epsilon)\otimes m, s+\epsilon) - V(\mathbb{Y}_{tX}(s)\otimes$

$m, s)\Big]$. From the continuity of functions $s \longmapsto \mathbb{Y}_{tX}(s), u_{tX}(s)$, we have

$$\mathbb{E}\left[\int_{\mathbb{R}^n} l\Big(\mathbb{Y}_{tX}(s), u_{tX}(s)\Big)\,\mathrm{d}m(x)\right] + F(\mathbb{Y}_{tX}(s)\otimes m) + \frac{\mathrm{d}}{ds}V(\mathbb{Y}_{tX}(s)\otimes m, s) = 0. \tag{A.72}$$

The assumptions in (2.17) to (2.23) are fulfilled by the regularity properties of V mentioned in Propositions 4.1, Proposition 4.2, (4.3), (4.2), (4.4) and Proposition 4.6 by Propositions 4.1–4.6. Thus, it is legitimate to use Theorem 2.3 to replace the total derivative in time by the partial one, that is, for a.e. $s \in (t, T)$,

$$\begin{aligned} 0 = {} & \mathbb{E}\left[\int_{\mathbb{R}^n} l\Big(\mathbb{Y}_{tX}(s), u_{tX}(s)\Big)\,\mathrm{d}m(x)\right] + F(\mathbb{Y}_{tX}(s)\otimes m) + \frac{\partial}{\partial s}V(\mathbb{Y}_{tX}(s)\otimes m, s) \\ & + \Big\langle D_X V(\mathbb{Y}_{tX}(s)\otimes m, s), u_{tX}(s)\Big\rangle_{\mathcal{H}_m} + \frac{1}{2}\sum_{j=1}^{n}\Big\langle D_X^2 V(\mathbb{Y}_{tX}(s)\otimes m, s)\Big(\eta^j \mathcal{N}_s^j\Big), \eta^j \mathcal{N}_s^j\Big\rangle_{\mathcal{H}_m}. \end{aligned} \tag{A.73}$$

Recall from (4.1) and the definition of Hamiltonian in (3.31), the first term in the first line of (A.73) and the term in the second line of (A.73) are combined to give the Hamiltonian, we thus obtain the equation in (5.3).

Part 2. Proof of (b):

If a functional $V^*(X\otimes m, t)$ solves (5.3) and satisfies the regularity properties in Propositions 4.1-4.6, then $V^*(\mathbb{Y}_{tX}(s)\otimes m, s)$ also satisfies (A.72) by working backwards through the use of mean-field Itô lemma in Theorem 2.3 and then the definition of Hamiltonian, together with (5.3). By integrating (A.72) from t to T, we obtain $V^*(X\otimes m, t) = J_{tX}(u_{tX})$. Since the right hand side is the value function which is uniquely defined, the solution to (5.3) is necessarily unique. Note that X must be independent of $\mathcal{W}_t$ by its definition; indeed, we can always construct the Gaussian random variables $\mathcal{N}_t^j$ (for example, the construction in the proof of Theorem 2.3; see also Remark 5.4 in [32]), so that this condition is satisfied. This concludes the proof. ∎