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OPTIMAL STOPPING PROBLEMS WITH CONSTRAINTS FOR CONTINUOUS-TIME GENERAL MARKOV PROCESSES

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Abstract. In this work, we consider an optimal stopping time problem with a restriction that requires the stopping times to occur only when a given signal is observed. The dynamic of the stopping problem evolves as a continuous-time Markov process whose state space is a subset of a Polish space, whereas the signal is modeled by a piecewise deterministic Markov process, both processes are assumed to be independent of each other. To solve the problem, we employ the well-known discrete-time optimal control theory. In particular, we characterize the value function as the solution of a certain dynamic programming equation related to the discrete-time model and demonstrate the existence of an optimal stopping time through this equation. Our technique extends previous works that use more restrictive conditions on the state space of the process. As a byproduct, we intend to solve optimal stopping problems for a wider family of Markov processes.

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1. Introduction

Optimal stopping problems are perhaps one of the most interesting and studied problems in the theory of stochastic processes. Successful methods have been developed over decades to show the existence and several characterizations of optimal stopping times. The most studied methods to address these problems are definitely the theory of Snell envelopes and backward-reflected stochastic differential equations –see, for instance, El Karoui *et al.* [1–3], Shyryaev [4], Peskir and Shyryaev [5], Hamedène and Jeanblanc [6] –, but on the other hand, there is also another useful method that tackles stopping problems from a merely analytical viewpoint – see Bensoussan and Lions [7, 8], Menaldi [9, 10], Oksendal and Sulem [11], Robin [12, 13], among others.

In this work, we study an optimal stopping time problem constrained to apply the stopping only at certain random times so-called signals. While in classical optimal stopping problems (*e.g.* the models considered in Bensoussan [8] or in Shiryaev [4]), the range of stopping times lies in some fixed subset of the nonnegative real line (mostly unbounded), here we study a variation of this problem consisting that the dynamics can be only stopped at certain times specified by means of an exogenous random sequence. Pioneering studies in this direction were Dupuis and Wang [14] and Lempa [15], in which the authors studied a specific kind of dynamics, modeled by a geometric Brownian motion combined with an exogenous sequence of i.i.d. random variables

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representing the time between signals with common exponential distribution. Later, Menaldi and Robin [16, 17] extended this theory into a more general setting in which (a) the "signals" are put in terms of a piecewise deterministic Markov process representing the time elapsed since the last signal, (b) the (main) dynamic is of Markov-Feller type with compact state space and (c) the type of control also includes the way to impulse the process. More recently, Menaldi and Robin [18] extended their results in [16, 17] for the case of locally compact state space and also includes some other optimality criteria, such as the ergodic payoff. This paper is a continuation of [16] in the sense that the signals are still modeled by means of a piecewise deterministic Markov process representing the time elapsed since the last signal but the difference now relies on that the dynamic evolves as a Markov process with state space *not necessarily compact nor locally-compact.* This relaxation leads us to reformulate the method to provide optimality solutions. Namely, in lieu of applying martingale methods as in [16] and [18], we take advantage of recent advances given in the area of the discrete-time control processes a.k.a. Markov Decision Processes (MDPs). We will see that our original problem can be regarded as an MDP with an absorbent state and with an associated total cost criterion. Thus, optimal elements of this latter model will provide the solution to the original one.

The content of the paper can be synthesized as follows: after this introductory part, Section 2 gives a summary of the main results of the manuscript. Later, in Section 3, we introduce the type of dynamics for both: the Markov process over which we apply the stopping, and the signal process; the section concludes with the introduction of an embedded chain that will become the main dynamics within our analysis. Section 4 is devoted to introducing the set of admissible stopping times. It also shows the optimal stopping time problem (OSTP) viewed as a minimization problem. Later, in Section 5, we link a discrete-time control problem (DTCP) with the OSTP. Namely, a bijection on the set of admissible control policies for the DTCP and the set of admissible stopping times is given. We also render an equivalence between the objective functions of both problems. Finally, Section 6 analyzes the dynamic programming equations (DPE) associated with the DTCP, in particular, it provides both the existence and regularity of solutions to that DPE, whereas Section 7 is devoted to providing solution of OSTP by means of the optimality results of the DTCP. The last section of the manuscript is the Appendix A, containing key technical results that become fundamental to accomplishing our theory.

2. Main results

Given a strong Markov-Feller process $\{y(t,x)\}_{t\geq 0}$, and a piecewise deterministic Markov process $\{z(t,z_0)\}_{t\geq 0}$ with slope one and jumps to zero (see Fig. 1 below), both being continuous-time processes over the same filtrated probability space, with initial conditions x in some metric space and $z_0 \geq 0$ – which sometimes will be denoted as $y(\cdot,x)$ and $z(\cdot,z_0)$ for short notation –, we analyze the stopping time problem of the process $y(\cdot,x)$ constrained to stop it just when $z(\cdot,z_0)$ hits the real axis, which may occur infinite but denumerable times. In this sense, given the cost function f that is applied when the system maintains its course, and φ a cost for stopping the system, we aim to optimize the following total cost function:

$$J(x,z_0,\eta) = \mathrm{E}\bigg[\sum_{n=0}^{\eta-1} \mathrm{e}^{-\alpha\tau_n} f\left(y(\tau_n,x)\right)\mathbf{1}_{\{z(\tau_n,z_0)=0\}} + 1_{\{\tau_\eta<\infty\}}\mathrm{e}^{-\alpha\tau_\eta}\varphi\left(y(\tau_\eta,x)\right)\bigg] \tag{2.1}$$

over the set of "admissible" stopping times η to be defined later on. We will denote by $J^*(\cdot,\cdot) := \inf_\eta J(\cdot,\cdot,\eta)$ the optimal cost (a.k.a. value function). Our goal is to show the following three aspects:

(a) The value function satisfies the next dynamic programming equation

$$J^*(x,z_0) = \inf_{a\in\mathbb{A}(x,z_0,0)}\Big\{c(x,z_0,0,a) + \int_0^\infty \Phi_\alpha(t)J^*(x,0)\mu_{z_0}(\mathrm{d}t)\mathbf{1}_{\{0\}}(a)\Big\}, \tag{2.2}$$

where $\mathbb{A}(x,z_0,0)$ is a set consisting of just one of two elements: 0 (meaning the continuation of the process) and 1 (meaning to stop the process) that depends strongly on z_0. The measure μ_{z_0} is the distribution of

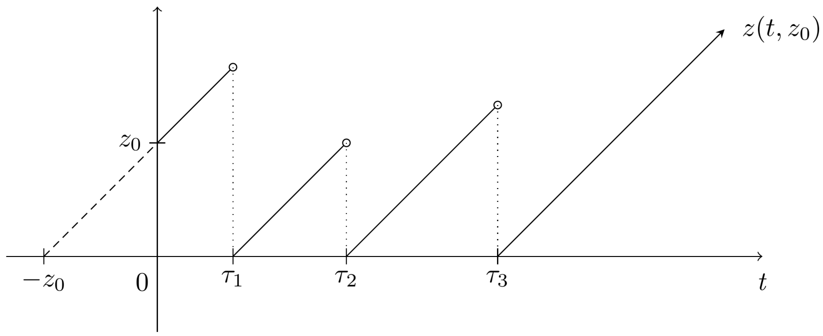

FIGURE 1. Process $z(\cdot, z_0)$ with initial elapsed time $z_0 > 0$ at $t = 0$.

the first hitting time of $z(\cdot, z_0)$ at the real axis, and $\Phi_\alpha(\cdot)$ is the α-discounted semigroup associated to $y(\cdot, x)$ with $\alpha > 0$. The function c encloses the information pertinent to the cost functions f and φ:

$$c(x, z, t, a) = \mathrm{e}^{-\alpha t}\big[f(x)\mathbf{1}_{\{0\}}(a) + \varphi(x)\mathbf{1}_{\{1\}}(a)\big]\mathbf{1}_{\{0\}}(z).$$

(b) If f and φ are continuous and nonnegative then J^* is also continuous and nonnegative.
(c) The existence of an optimal stopping time η^* that minimizes (2.1) in the sense of $J(x, z_0, \eta^*) = J^*(x, z_0)$ that will be linked with the minimizers of (2.2).

In the rest of the paper, we will establish with rigorous detail all the aspects necessary to give the answer to these three aforementioned goals.

3. Preliminaries

Let us consider an open subset $\mathbb{O}$ of a Polish space provided with its Borel σ-algebra $\mathcal{B}_{\mathbb{O}}$, and a fixed filtered probability space $\mathcal{E} \coloneqq (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$ satisfying the usual conditions (*i.e.*, the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous and $\mathcal{F}_0$ contains all subsets of the P-null sets). We consider a cádlág $\mathbb{O}$-valued homogeneous-time strong Markov process $\{y(t, x)\}_{t\geq 0}$ with initial condition $x \in \mathbb{O}$ (*i.e.* $\mathrm{P}\big(y(0, x) = x\big) = 1$), defined on the fixed filtered probability space $\mathcal{E}$ and whose transition probabilities satisfy

$$\mathrm{P}(y(s + t, x) \in B|\mathcal{F}_s) = \mathrm{P}(y(s + t, x) \in B|y(s, x)), \quad \forall B \in \mathcal{B}_{\mathbb{O}}. \tag{3.1}$$

With these elements, we can formulate a classical optimal stopping time problem, but in this work, we need an extra component: a sequence of nonnegative random variables τ_n representing the *signals* indicating the moments at which the process can be stopped. Our first target in this preliminary part is to provide a more detailed description of the underlying Markov process, which may have an infinite dimensional state space. Next, we will show that the process $\{(y(\tau_n, x), \tau_n)\}_{n\in\mathbb{N}_0}$ is a time-inhomogeneous Markov chain, provided that the elements of the sequence $\{\tau_n\}_{n\in\mathbb{N}_0}$ are regarded as stopping times. By the end of this section, we will include a so-called *signal process* with the purpose to fix the previously mentioned non-homogeneity.

Let us denote the homogeneous transition probabilities $\{\rho_t\}_{t\geq 0}$ of the aforementioned Markov process $\{y(t, x)\}_{t\geq 0}$ by:

$$\rho_t\big(B|y(s, x)\big) \coloneqq \mathrm{P}\big(y(s + t, x) \in B \mid \mathcal{F}_s\big), \quad \forall\, B \in \mathcal{B}_{\mathbb{O}}. \tag{3.2}$$

Also, consider the set $B(\mathbb{O})$ consisting of all measurable functions from $\mathbb{O}$ to $\mathbb{R}$. In the same manner, let $B_b(\mathbb{O})$ be the subspace of $B(\mathbb{O})$ given by all bounded measurable functions from $\mathbb{O}$ to $\mathbb{R}$ endowed with the uniform norm $\|\cdot\|$. Finally, let us denote by $C_b(\mathbb{O})$ the space of bounded and continuous functions over $\mathbb{O}$.

Assumption 3.1. We assume that for each $h \in B_b(\mathbb{O})$, the mapping

$$(t,x) \mapsto \int_{\mathbb{O}} h(w)\rho_t(\mathrm{d}w|x) = \mathrm{E}\big[h(y(t,x))\big]$$

is measurable.

The transition probabilities $\{\rho_t\}_{t\geq 0}$ satisfy the Chapman-Kolmogorov equation: for each $t,s \geq 0$ and $h \in B_b(\mathbb{O})$ we have

$$\int_{\mathbb{O}} h(w)\rho_{t+s}(\mathrm{d}w|x) = \int_{\mathbb{O}}\int_{\mathbb{O}} h(w)\rho_t(\mathrm{d}w|z)\rho_s(dz|x). \tag{3.3}$$

Note that (3.1) is equivalent to the following Markov property in the functional form:

$$\mathrm{E}\big[h(y(t+s,x))|\mathcal{F}_s\big] = \mathrm{E}\big[h(y(t+s,x))|y(s,x)\big] \quad \forall\ h \in B_b(\mathbb{O}). \tag{3.4}$$

Thus, we define a semigroup of operators $\{\Phi_\alpha(t)\}_{t\geq 0}$ over $B_b(\mathbb{O})$ with a given discount factor $\alpha > 0$ is fixed by

$$\Phi_\alpha(t)h(x) = \int_{\mathbb{O}} \mathrm{e}^{-\alpha t}h(w)\rho_t(\mathrm{d}w|x) = \mathrm{E}\big[\mathrm{e}^{-\alpha t}h(y(t,x))\big], \tag{3.5}$$

for each $t \geq 0$, $h \in B_b(\mathbb{O})$ and $x \in \mathbb{O}$. This semigroup satisfies the next properties:

(a) (semigroup property) $\Phi_\alpha(0) = \mathrm{I}$ and $\Phi_\alpha(t)\Phi_\alpha(s) = \Phi_\alpha(t+s)$, $\forall\ t,s \geq 0$.
(b) (contraction property) $\|\Phi_\alpha(t)h\| \leq \|h\|$, $\forall\ t \geq 0$ and $h \in B_b(\mathbb{O})$.
(c) (monotony) $\Phi_\alpha(t)h \geq 0$ whenever $h \geq 0$.

3.1. The signals

As we mentioned earlier, our system can only be stopped at some specific random times that we will call *signals*. In order to model these times, we first introduce a sequence of i.i.d. random variables $T_1, T_2, \ldots$ in $\mathbb{R}_+$ with common distribution μ_0 such that $\mu_0(\{0\}) = 0$ which in turn are independent of the process $\{y(t,x)\}_{t\geq 0}$. The sequence $T_1, T_2, \ldots$ represents the waiting times between two consecutive signals.

Let us consider a constant $z_0 \geq 0$, which will be interpreted as the *elapsed time* since the last signal happened up to time 0. Hence we define the *signals* after time 0 as follows: τ_0 is a reference initial time equal to zero; τ_1 is independent of the sequence T_k, $k = 2,3,\ldots$ with distribution $\mu_{z_0}(\cdot)$ on $\mathcal{B}_{\mathbb{R}_+}$ defined over any interval $[a,b) \subset \mathbb{R}$ by

$$\begin{aligned}\mu_{z_0}([a,b)) = \mathrm{P}\big(\tau_1 \in [a,b)\big) &:= \mathrm{P}\big(T_1 - z_0 \in [a,b) \mid T_1 \geq z_0\big)\\ &= \frac{\mathrm{P}\big(T_1 - z_0 \in [a,b), T_1 \geq z_0\big)}{\mathrm{P}(T_1 \geq z_0))} = \frac{\mu_0\big([a+z_0, b+z_0)\big)}{\mu_0([z_0,\infty))}.\end{aligned}$$

Finally, we define τ_n recursively by

$$\tau_n := \tau_{n-1} + T_n = \tau_1 + \sum_{k=2}^{n} T_k, \text{ for all } n \geq 2.$$

Throughout this work, we shall assume that $\tau_n \to \tau_\infty := \infty$ as $n \to \infty$.

Remark 3.2. (a) Due to $\mu_0(\{0\}) = 0$, the sequence $\{\tau_n\}_{n\geq 0}$ is strictly increasing a.s.

(b) The case when the elapsed time is $z_0 = 0$, reduces $\tau_n = \sum_{1\leq k\leq n} T_k$, for all $n \in \mathbb{N}$ and the distributions μ_{z_0} and μ_0 agree.

(c) On the contrary, when $z_0 > 0$ the signals turns out to be $-z_0, \tau_1, \tau_2, \ldots$ but by convention and in accordance with the starting time of the process $\{y(t,x)\}_{t\geq 0}$ we have defined $\tau_0 = 0$. In the case $z_0 = 0$, the signals $\tau_0, \tau_1, \ldots$ and the times $0, T_1, \cdots$ coincide.

(d) The assumption of $\tau_n \to \infty$ as $n \to \infty$ is a common property for specific instances of Markov processes such as the Poisson random measure of the well-known Levy's processes, see Remark 1.3 in [11], p. 2.

Continuous time signal process. Alternatively, we can track continuously the evolution of the signal sequence $\{\tau_n\}_{n\geq 0}$ by means of the process $\{z(t,z_0)\}_{t\geq 0}$ that measures the *elapsed time since the last signal* with initial condition z_0 at time 0 (see Fig. 1). The definition of this process is given next:

$$z(t,z_0) := \begin{cases} t + z_0, & t \in [0,\tau_1); \\ t - \tau_n, & t \in [\tau_n, \tau_{n+1}), \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{3.6}$$

On the other hand, the elements of the signal times sequence $\{\tau_k\}_{k\in\mathbb{N}}$ can be regarded in terms of the process $z(\cdot, z_0)$ as follows:

$$\tau_k = \inf\{t > \tau_{k-1} \mid z(t,z_0) = 0\}, \quad k \in \mathbb{N} \tag{3.7}$$

which will be useful for Lemma 3.4 below.

Remark 3.3. Without loss of generality we assume that, for each $t \in \mathbb{R}_+$, $\mathcal{F}_t$ is large enough so that it contains the sigma-algebra $\sigma(z(s,z_0) : s \leq t)$ for all $t \geq 0$. Consequently, the process $z(t,z_0)$ is $\mathcal{F}_t$-adapted.

Lemma 3.4. *The nonnegative random variables $\{\tau_n\}_{n\geq 1}$ are stopping times with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.*

Proof. We will proceed by induction: For $n = 1$, due to (3.7) we have

$$\tau_1 := \inf\{t > 0 \mid z(t,z_0) = 0\}.$$

Using Theorem 2.36, p. 53, in [19] we get that τ_1 is a stopping time, because $\{z(t,z_0)\}_{t\geq 0}$ is right-continuous and $\{0\}$ is a closed set. Let us now consider $n > 1$ and suppose that τ_{n-1} is a stopping time. Since the signal times τ_k's are strictly increasing, we have that $z(t,z_0) = t - \tau_k$ if and only if $t \in [\tau_k, \tau_{k+1})$. Then

$$\begin{aligned} \tau_n \leq t &\iff t \in [\tau_{n+k}, \tau_{n+k+1}), \text{ for some } k \geq 0 \\ \iff z(t,z_0) = t - \tau_{n+k}, &\text{ for some } k \geq 0 \iff z(t,z_0) < t - \tau_{n-1}. \end{aligned}$$

Hence,

$$\{\tau_n \leq t\} = \{z(t,z_0) < t - \tau_{n-1}\} = \bigcup_{q\in\mathbb{Q}^+,\ q<t} \{z(t,z_0) < q < t - \tau_{n-1}\}. \tag{3.8}$$

The right-continuity of the filtration $\{\mathcal{F}_s\}_{s\geq 0}$ implies that τ_{n-1} is a stopping time relative to such filtration if and only if $\{\tau_{n-1} < s\}$ is $\mathcal{F}_s$-measurable for all $s \geq 0$. Hence, the right hand side of (3.8) is $\mathcal{F}_t$-measurable due to $\{z(t,z_0) < q\}$ is $\mathcal{F}_t$-measurable for all $q \in \mathbb{Q}^+,\ q < t$ and $\{\tau_{n-1} < t - q\}$ too, due to the induction hypothesis. So we conclude the proof. $\square$

Since the process $\{y(t,x)\}_{t\geq 0}$ is homogeneous-time strong Markovian, the process $\{(y(t,x),t)\}_{t\geq 0}$ is also homogeneous-time strong Markovian, in addition, its transition probabilities are given, for all $C\in\mathcal{B}_{\mathbb{O}\times\mathbb{R}_+}$, by

$$\mathrm{P}\big((y(t+s,x),t+s)\in C \mid (y(s,x),s)\big) = \int_{\mathbb{O}} \mathbf{1}_C(w,t+s)\rho_t\big(\mathrm{d}w \mid y(s,x)\big). \tag{3.9}$$

3.2. The embedded Markov chain

In the following paragraphs we analyze a discrete-time process $\{X_n\}_{n\in\mathbb{N}_0}$ that is defined through the observations of the continuous-time processes $\{(y(t,x),t)\}_{t\geq 0}$ and $\{z(t,z_0)\}_{t\geq 0}$ just at the signals τ_n; that is,

$$X_n := \big(y(\tau_n,x), z(\tau_n,z_0), \tau_n\big) \quad \forall\ n\in\mathbb{N}_0, \tag{3.10}$$

with $X_0=(x,z_0,0)$. Note that the state space of this process is $\mathbb{O}\times\mathbb{R}_+\times\mathbb{R}_+$ which we endow with the Borel σ-algebra $\mathcal{B}_{\mathbb{O}}\otimes\mathcal{B}_{\mathbb{R}_+}\otimes\mathcal{B}_{\mathbb{R}_+}$. In addition, we associate this process with its natural filtration $\mathcal{G}_n := \sigma(X_m : 0\leq m\leq n)$. This process is indeed a homogeneous Markov chain as is stated in Theorem 3.7 below. To prove this last assertion we first need the following lemmas.

Lemma 3.5. *For any fixed $n\in\mathbb{N}$ and $s\geq 0$, we have that the sigma algebras $\sigma(\mathcal{G}_n, y(\tau_n+s,x), \tau_n+s)$ and $\sigma(T_{n+1})$ are independent.*

Proof. Let $f\in C_b(\mathbb{O})$, and $g,h\in C_b(\mathbb{R}_+)$. By Lemma A.12 we can take a sequence of denumerable valued random times τ_n^k such that $\tau_n^k\downarrow\tau_n$ and τ_n^k are independent of T_{n+1}. So, we have that

$$\begin{aligned}
&\mathrm{E}\big[f(y(\tau_n^k,x))g(\tau_n)h(T_{n+1})\big] = \sum_s \mathrm{E}\big[\mathbf{1}_{\{\tau_n^k=s\}} f(y(s,x))g(\tau_n)h(T_{n+1})\big]\\
&= \sum_s \mathrm{E}\big[f(y(s,x))\big]\,\mathrm{E}\big[\mathbf{1}_{\{\tau_n^k=s\}} g(\tau_n)h(T_{n+1})\big]\\
&= \sum_s \mathrm{E}\big[f(y(s,x))\big]\,\mathrm{E}\big[\mathbf{1}_{\{\tau_n^k=s\}} g(\tau_n)\big]\,\mathrm{E}\big[h(T_{n+1})\big]\\
&= \sum_s \mathrm{E}\big[f(y(s,x))\mathbf{1}_{\{\tau_n^k=s\}} g(\tau_n)\big]\,\mathrm{E}\big[h(T_{n+1})\big] = \mathrm{E}\big[f(y(\tau_n^k,x))g(\tau_n)\big]\,\mathrm{E}\big[h(T_{n+1})\big].
\end{aligned}$$

Due to the right continuity of $y(s,x)$ together with the continuity of f, we get that $f(y(\tau_n^k,x))\to f(y(\tau_n,x))$ a.s. Hence, the boundedness of f leads us to apply dominated convergence to obtain

$$\mathrm{E}[f(y(\tau_n^k,x))g(\tau_n)h(T_{n+1})] \to \mathrm{E}[f(y(\tau_n,x))g(\tau_n)h(T_{n+1})]$$

on one hand, as well as

$$\mathrm{E}[f(y(\tau_n^k,x))g(\tau_n)]\,\mathrm{E}[h(T_{n+1})] \to \mathrm{E}[f(y(\tau_n,x))g(\tau_n)]\,\mathrm{E}[h(T_{n+1})]$$

on the other hand. Therefore by the uniqueness of the limits we obtain

$$\mathrm{E}[f(y(\tau_n,x))g(\tau_n)h(T_{n+1})] = \mathrm{E}[f(y(\tau_n,x))g(\tau_n)]\,\mathrm{E}[h(T_{n+1})]. \tag{3.11}$$

Let H be the set of all functions $f\in C_b(\mathbb{O})$ satisfying the relation (3.11), for fixed functions $g,h\in C_b(\mathbb{R}_+)$. This set will actually be the one given in part (1) of Corollary A.4. On the other hand, by taking a sequence of $f_n\in H\cap C_b^+(\mathbb{O})$ in such a way that $f_n\uparrow f$ for f finite, we can use a simple monotone convergence argument, to prove that $f \in H$, which guaranties part (2) of Corollary A.4. Thus, in virtue of this aforementioned corollary, we claim that the condition (3.11) can be extended to all measurable functions f. In the same manner, we can repeat the same last argument to extend the relation (3.11) for all measurable functions g and h.

The rest of the proof focuses on the above argument for a sequence of vectors $\{(\tau_0^k, \tau_1^k, \ldots, \tau_n^k, \tau_n^k + s)\}_{k\in\mathbb{N}}$ with denumerable-valued components that pointwise approximates the vector $(0, \tau_1, \ldots, \tau_n, \tau_n + s)$. Then, for suitable functions $f \in C_b(\mathbb{O}^{n+2})$, $g \in C_b(\mathbb{R}_+^{n+2})$, and $h \in C_b(\mathbb{R}_+)$, we can apply the arguments above to get

$$\begin{aligned}
&\mathrm{E}\big[f(y(\tau_0^k, x), \ldots, y(\tau_n^k + s, x))g(0, \ldots, \tau_n + s)h(T_{n+1})\big] \\
&= \mathrm{E}\big[f(y(\tau_0^k, x), \ldots, y(\tau_n^k + s, x))g\,(0, \ldots, \tau_n + s)\big]\,\mathrm{E}\big[h(T_{n+1})\big].
\end{aligned} \tag{3.12}$$

Letting $k \to \infty$ in (3.12) we deduce the independence

$$\begin{aligned}
&\mathrm{E}\left[f(y(\tau_0, x), \ldots, y(\tau_n + s, x))g(0, \ldots, \tau_n + s)h(T_{n+1})\right] \\
&= \mathrm{E}\left[f(y(\tau_0, x), \ldots, y(\tau_n + s, x))g\,(0, \ldots, \tau_n + s)\right]\mathrm{E}\left[h(T_{n+1})\right].
\end{aligned}$$

Using again Corollary A.4, we can also deduce the independence (3.12) for every measurable functions f, g, h in the same way as in the scalar case, and thus the proof is fulfilled. ☐

The next lemma concerns the Markov property of some elements of the process $\{X_n\}_{n\geq 0}$. Before establishing it we first introduce the following σ-algebra and the definition of the strong Markov property.

Given a stopping time τ in $\mathbb{R}_+$, we define $\mathcal{F}_\tau$ by

$$\mathcal{F}_\tau \coloneqq \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t,\ t \in \mathbb{R}_+\}.$$

The strong Markov property is expressed as

$$\mathrm{P}(y(\tau + t, x) \in B | \mathcal{F}_\tau) = \mathrm{P}(y(\tau + t, x) \in B | y(\tau, x)), \quad t \geq 0,\ B \in \mathcal{F},$$

for each stopping time τ in $\mathbb{R}_+$.

Lemma 3.6. *The discrete-time process $\{(y(\tau_n, x), \tau_n)\}_{n\in\mathbb{N}_0}$ is a time inhomogeneous Markov chain with respect to the filtration $\mathcal{G}_n$. Furthermore, for every $C \in \mathcal{B}_{\mathbb{O}} \otimes \mathcal{B}_{\mathbb{R}_+}$ and $n \geq 0$, we have*

$$\mathrm{P}\big((y(\tau_{n+1}, x), \tau_{n+1}) \ \in\ C \ \mid\ (y(\tau_n, x), \tau_n)\big) \ =\ \int_{\mathbb{R}_+}\int_{\mathbb{O}} \mathbf{1}_C(w, \tau_n + t)\rho_t\big(\mathrm{d}w \ \mid\ y(\tau_n, x)\big)\mu_{z(\tau_n, z_0)}(\mathrm{d}t). \tag{3.13}$$

Proof. We will start with providing the proof of (3.13). For this end, take $n \in \mathbb{N}$ as well as a countably valued sequence of random times T_{n+1}^k, such that $T_{n+1}^k \downarrow T_{n+1}$ (see Lem. A.12). Let $f \in C_b(\mathbb{O} \times \mathbb{R}_+)$ and let us write

$$g(t) \coloneqq \mathrm{E}\left[f(y(\tau_n + t, x), \tau_n + t) | (y(\tau_n, x), \tau_n)\right].$$

Using the T_{n+1}-measurability of T_{n+1}^k given in Lemma A.12 as well as the independence result of Lemma 3.5, we get

$$\begin{aligned}
&\mathrm{E}\big[f\big(y(\tau_n + T_{n+1}^k, x), \tau_n + T_{n+1}^k\big) \mid (y(\tau_n, x), \tau_n)\big] \\
&= \sum_s \mathrm{E}\Big[\mathbf{1}_{\{T_{n+1}^k = s\}} f(y(\tau_n + s, x), \tau_n + s) \ \Big|\ (y(\tau_n, x), \tau_n)\Big] \\
&= \sum_s \mathrm{E}\big[f\big(y(\tau_n + s, x), \tau_n + s\big) \mid (y(\tau_n, x), \tau_n)\big]\mathrm{P}\big(T_{n+1}^k = s\big) \\
&= \sum_s g(s)\mathrm{P}\big(T_{n+1}^k = s\big) = \mathrm{E}\big[g(T_{n+1}^k)\big].
\end{aligned}$$

Due to the continuity and boundedness of f as well as the right-continuity of $t \mapsto (y(\tau_n + t, x), \tau_n + t)$, the dominated convergence for conditional expectation (see [20], Thm. 5.5.5, p.223), yields to $g(T^k_{n+1}) \to g(T_{n+1})$, when $k \to \infty$. In addition, it is easy to verify that $\sup_k |g(T^k_{n+1})| \leq \|f\|$ and then, by dominated convergence, we can deduce $\mathrm{E}\big[g(T^k_{n+1})\big] \to \mathrm{E}\big[g(T_{n+1})\big]$; that is

$$\sum_s g(s)\mathrm{P}\big(T^k_{n+1} = s\big) \to \int_{\mathbb{R}_+} g(s)\mathrm{P}\big(T_{n+1} \in \mathrm{d}s\big). \tag{3.14}$$

On the other hand, by similar arguments, we get that

$$\mathrm{E}\big[f\big(y(\tau_n + T^k_{n+1}, x), \tau_n + T^k_{n+1}\big) \;\big|\; (y(\tau_n, x), \tau_n)\big] \;\to\; \mathrm{E}\big[f\big(y(\tau_n + T_{n+1}, x), \tau_n + T_{n+1}\big) \;\big|\; (y(\tau_n, x), \tau_n)\big],$$

when $k \to \infty$. Then, we deduce the next equality for all $f \in C_b(\mathbb{O} \times \mathbb{R}_+)$:

$$\begin{aligned} &\mathrm{E}\big[f\big(y(\tau_n + T_{n+1}, x), \tau_n + T_{n+1}\big) \;\big|\; (y(\tau_n, x), \tau_n)\big] \\ &= \int_{\mathbb{R}_+} \mathrm{E}\big[f\big(y(\tau_n + s, x), \tau_n + s\big) \;\big|\; (y(\tau_n, x), \tau_n)\big]\mathrm{P}\big(T_{n+1} \in \mathrm{d}s\big). \end{aligned} \tag{3.15}$$

Now let us proceed as in previous proofs. Namely, let H be the set of all functions $f \in C_b(\mathbb{O} \times \mathbb{R}_+)$ satisfying the relation (3.15). This set will be actually the one given in part (a) of Corollary A.4. On the other hand, by taking a sequence of $f_n \in H \cap C^+_b(\mathbb{O} \times \mathbb{R}_+)$ in such a way that $f_n \uparrow f$ for f finite, then we can use a simple monotone convergence argument, to prove that $f \in H$, which guaranties part (b) of Corollary A.4. Thus, in virtue of this aforementioned corollary, we claim that the condition (3.15) can be extended to all measurable functions $f : \mathbb{O} \times \mathbb{R}_+ \to \mathbb{R}$. Hence, by the strong Markov property and choosing $f = \mathbf{1}_C$ with $C \in \mathcal{B}_{\mathbb{O}} \otimes \mathcal{B}_{\mathbb{R}_+}$ in (3.15), we obtain (3.13) for the case $n \geq 1$. In the same manner, for $n = 0$ in (3.15), we get

$$\mathrm{E}[f(y(\tau_1, x), \tau_1) \mid x, 0] = \int_{\mathbb{R}_+} \mathrm{E}[f(y(s, x), s) \mid x, 0]\mathrm{P}(\tau_1 \in \mathrm{d}s),$$

from where we obtain again (3.13) by choosing $f = \mathbf{1}_C$.

To prove the Markov property, we note that $\sigma(y(\tau_n, x), \tau_n) \subseteq \mathcal{G}_n \subseteq \mathcal{F}_{\tau_n}$. Namely, the first contention is straightforward whereas the second follows from Corollary A.15. Using this fact together with the strong Markov property of the process $(y(t, x), t)$, we get

$$\begin{aligned} &\mathrm{E}\big[f\big(y(\tau_n + s, x), \tau_n + s\big) \;\big|\; y(\tau_n, x), \tau_n\big] \\ &= \mathrm{E}\big[\mathrm{E}\big[f\big(y(\tau_n + s, x), \tau_n + s\big) \;\big|\; y(\tau_n, x), \tau_n\big] \;\big|\; \mathcal{G}_n\big] \\ &= \mathrm{E}\big[\mathrm{E}\big[f\big(y(\tau_n + s, x), \tau_n + s\big) \;\big|\; \mathcal{F}_{\tau_n}\big] \;\big|\; \mathcal{G}_n\big] = \mathrm{E}\big[f(y(\tau_n + s, x), \tau_n + s) \;\big|\; \mathcal{G}_n\big]. \end{aligned}$$

Hence $g(t) = \mathrm{E}[f(y(\tau_n + t, x), \tau_n + t) \mid \mathcal{G}_n]$ and thus by (3.14) and (3.15) we can prove as above that

$$\begin{aligned} \mathrm{E}\big[f\big(y(\tau_n + T_{n+1}, x), \tau_n + T_{n+1}\big)|\mathcal{G}_n\big] &= \lim_{k\to\infty} \mathrm{E}\big[f\big(y(\tau_n + T^k_{n+1}, x), \tau_n + T^k_{n+1}\big) \;\big|\; \mathcal{G}_n\big] \\ &= \lim_{k\to\infty} \sum_s g(s)\mathrm{P}(T^k_{n+1} = s) = \int_{\mathbb{R}_+} g(s)\mathrm{P}(T_{n+1} \in \mathrm{d}s) \\ &= \mathrm{E}\big[f\big(y(\tau_n + T_{n+1}, x), \tau_n + T_{n+1}\big) \;\big|\; y(\tau_n, x), \tau_n\big], \end{aligned}$$

from which we get the Markov property after taking $f = \mathbf{1}_C$ for $C \in \mathcal{B}_{\mathbb{O}} \otimes \mathcal{B}_{\mathbb{R}_+}$; *i.e.*,

$$\mathrm{P}\big((y(\tau_{n+1},x),\tau_{n+1}) \in C \mid y(\tau_n,x),\tau_n\big) = \mathrm{P}\big((y(\tau_{n+1},x),\tau_{n+1}) \in C \mid \mathcal{G}_n\big).$$

□

Equation (3.13) and Lemma 3.6, show that the process $\{(y(\tau_n,x),\tau_n)\}_{n\in\mathbb{N}_0}$, with initial condition $(y(\tau_0,x),\tau_0) = (x,0)$ is an *inhomogeneous* Markov chain, since its transition probabilities depend on the random variable $z(\tau_n,z_0)$ at stage n. However, by adding to this chain the extra variable $z(\tau_n,z_0)$ as part of its components (*i.e.* by considering now the process $X_n \coloneqq \big\{\big(y(\tau_n,x),z(\tau_n,z_0),\tau_n)\big)\big\}_{n\in\mathbb{N}_0}$ with $X_0 \coloneqq (x,z_0,0))$ it is sufficient to guarantee that this new one is homogeneous. This fact is established in the next theorem.

Theorem 3.7. *The process $\{X_n\}_{n\in\mathbb{N}_0}$ is a time-homogeneous Markov chain with transition probabilities given by*

$$\mathrm{P}(X_{n+1} \in D \mid X_n) = \int_{\mathbb{R}_+}\int_{\mathbb{O}} \mathbf{1}_D(w,0,\tau_n+t)\rho_t\big(\mathrm{d}w \mid y(\tau_n,x)\big)\mu_{z(\tau_n,z_0)}(\mathrm{d}t), \tag{3.16}$$

for all $D \in \mathcal{B}_{\mathbb{O}} \otimes \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{B}_{\mathbb{R}_+}$.

Proof. Take some $A \in \mathcal{B}_{\mathbb{O}}$, and $B, C \in \mathcal{B}_{\mathbb{R}_+}$. By Lemma 3.6 and recalling that $z(\tau_n,z_0) = 0$ for $n \geq 1$ (see (3.6)), we can deduce, for all $n \in \mathbb{N}_0$,

$$\begin{aligned}
&\mathrm{P}\big(X_{n+1} \in A\times B\times C \mid \mathcal{G}_n\big) = \mathbf{1}_{\{0\in B\}}\mathrm{P}\big(y(\tau_{n+1},x)\in A, \tau_{n+1}\in C \mid \mathcal{G}_n\big)\\
&= \mathbf{1}_{\{0\in B\}}\mathrm{P}\big(y(\tau_{n+1},x)\in A, \tau_{n+1}\in C \mid x, z_0, 0, y(\tau_1,x), 0, \tau_1, \ldots, y(\tau_n,x), 0, \tau_n\big)\\
&= \mathbf{1}_{\{0\in B\}}\mathrm{P}\big(y(\tau_{n+1},x)\in A, \tau_{n+1}\in C \mid y(\tau_n,x),\tau_n\big)\\
&= \mathbf{1}_{\{0\in B\}}\mathrm{P}\big(y(\tau_{n+1},x)\in A, \tau_{n+1}\in C \mid y(\tau_n,x), z(\tau_n,z_0),\tau_n\big)\\
&= \mathrm{P}\big(X_{n+1} \in A\times B\times C \mid X_n\big).
\end{aligned}$$

From the above and using again Lemma 3.6, we obtain

$$\begin{aligned}
&\mathrm{P}\big(X_{n+1} \in A\times B\times C \mid X_n\big)\\
&= \mathbf{1}_{\{0\in B\}}\int_{\mathbb{R}_+}\int_{\mathbb{O}} 1_{A\times C}(w,\tau_n+t)\rho_t\big(\mathrm{d}w \mid y(\tau_n,x)\big)\mu_{z(\tau_n,z_0)}(\mathrm{d}t)\\
&= \int_{\mathbb{R}_+}\int_{\mathbb{O}} 1_{A\times B\times C}(w,0,\tau_n+t)\rho_t\big(\mathrm{d}w \mid y(\tau_n,x)\big)\mu_{z(\tau_n,z_0)}(\mathrm{d}t).
\end{aligned}$$

Finally, we extend the above relation using monotone class arguments (see Lem. A.1). □

Note that (3.16) implies that

$$\mathrm{E}\big[h(X_1)\big] = \int_{\mathbb{R}_+}\int_{\mathbb{O}} h(w,0,t)\rho_t\big(\mathrm{d}w \mid x\big)\mu_{z_0}(\mathrm{d}t)$$

for any measurable function $h \in L_1(\Omega,\mathcal{F})$. In particular for $h(w,z,t) = \mathrm{e}^{-\alpha t}v(w,z)$ where v is a suitable measurable function, we get by (3.5) and the above relation

$$\mathrm{E}\big[h(X_1)\big] = \int_{\mathbb{R}_+} \mathrm{e}^{-\alpha t}\int_{\mathbb{O}} v(w,0)\rho_t\big(\mathrm{d}w \mid x\big)\mu_{z_0}(\mathrm{d}t) = \int_{\mathbb{R}_+} \Phi_\alpha(t)v(x,0)\mu_{z_0}(\mathrm{d}t). \tag{3.17}$$

4. The optimization problem

We consider now *the optimal stopping time problem with constraints* in which we can or never stop the process $y(s,x)$ at specific times given by the signal sequence $\{\tau_n\}_{n\in\mathbb{N}_0}$.

For the case in which we never stop the process, *i.e.* $\{\eta=\infty\}$, we shall assume $\tau_\eta=\infty$, which is consistent with the assumption mentioned in the preliminaries; namely, $\lim_{n\to\infty}\tau_n=\tau_\infty:=\infty$.

We begin this section with the following ancillary lemma that uses the following definitions:

$$\mathcal{G}_\infty := \sigma(\cup_{n\in\mathbb{N}_0}\mathcal{G}_n) \quad \text{and} \quad \mathcal{F}_\infty := \sigma(\cup_{t\in\mathbb{R}_+}\mathcal{F}_t).$$

and

$$\bar{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\} \quad \text{and} \quad \bar{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}.$$

Without loss of generality, we can take $\mathcal{F}=\mathcal{F}_\infty$ – this is because $\mathcal{F}$ is a σ-algebra containing all $\mathcal{F}_t$, $t\geq 0$.

Lemma 4.1. *Let η be a stopping time with respect to the filtration $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$ such that $z(\tau_\eta,z_0)=0$, a.s. on $\{\eta<\infty\}$. Then τ_η is a stopping time with respect to $\{\mathcal{F}_t\}_{t\in\bar{\mathbb{R}}_+}$.*

Proof. Let η be a stopping time with respect to $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$ such that $z(\tau_\eta,z_0)=0$ on $\{\eta<\infty\}$. By the proof of Lemma 3.6, we know that $\mathcal{G}_n\subseteq\mathcal{F}_{\tau_n}$ for $n\in\mathbb{N}_0$. But this is also true for $n=\infty$, in fact, we have that

$$\mathcal{F}_{\tau_\infty} = \{A\in\mathcal{F}_\infty : A\cap\{\tau_\infty\leq t\}\in\mathcal{F}_t,\ t\in\bar{\mathbb{R}}_+\}.$$

But,

$$\{\tau_\infty\leq t\} = \begin{cases} \emptyset, & t\in\mathbb{R}_+, \\ \Omega, & t=\infty. \end{cases}$$

Hence, given $A\in\mathcal{F}_\infty$ we get

$$A\cap\{\tau_\infty\leq t\} = \begin{cases} \emptyset, & t\in\mathbb{R}_+, \\ A, & t=\infty. \end{cases}$$

Therefore, $A\in\mathcal{F}_{\tau_\infty}$, which implies $\mathcal{F}_\infty=\mathcal{F}_{\tau_\infty}$. Then, since $\mathcal{G}_n\subseteq\mathcal{F}_{\tau_n}\subseteq\mathcal{F}_\infty$ for all $n\in\mathbb{N}_0$, we obtain $\mathcal{G}_\infty\subseteq\mathcal{F}_\infty=\mathcal{F}_{\tau_\infty}$ as desired.

On the other hand, for $t\in\bar{\mathbb{R}}_+$ fixed, we have

$$\{\tau_\eta\leq t\} = \bigcup_{n\in\bar{\mathbb{N}}_0}\{\eta=n,\tau_n\leq t\}. \tag{4.1}$$

Since $\{\eta=n\}\in\mathcal{F}_{\tau_n}$, for all $n\in\bar{\mathbb{N}}_0$, we get

$$\bigcup_{n\in\bar{\mathbb{N}}_0}\{\eta=n,\tau_n\leq t\}\in\mathcal{F}_t$$

which implies $\{\tau_\eta\leq t\}\in\mathcal{F}_t$, and thus τ_η is a stopping time w.r.t. $\{\mathcal{F}_t\}_{t\in\bar{\mathbb{R}}_+}$. □

Let us now introduce the next family of *admissible* stopping times defined over the signal sequence.

Definition 4.2. An *admissible stopping time with constraints* or simply *admissible stopping time* is any stopping time θ with respect to the filtration $\{\mathcal{F}_t\}_{t\in\bar{\mathbb{R}}}$ such that there is a stopping time η with respect to $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$ so that $\theta = \tau_\eta$ and $z(\tau_\eta, z_0) = 0$ on $\{\eta < \infty\}$.

Remark 4.3. (a) By Lemma 4.1, we can identify any admissible stopping time with constraints θ with a stopping time η with respect to $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$ such that $z(\tau_\eta, z_0) = 0$ on $\{\eta < \infty\}$. Whereas for the case $\theta = \infty$, we just take η equals ∞. Also note that we do not care about $z(\infty, z_0)$ because we never consider this evaluation on the cost functions (2.1) or (4.2) below. In this sense, we will use discrete-valued stopping times η instead of continuous-valued stopping times θ to represent the admissible stopping times.

(b) Whilst in [16–18] the authors distinguish between admissible stopping times and zero-admissible stopping times. However in our model the condition $z(\tau_\eta, z_0) = 0$ in the above lemma allows us to make no distinction by unifying the cases $z_0 = 0$ and $z_0 > 0$, for which η takes values in either $\bar{\mathbb{N}}_0$ or $\bar{\mathbb{N}}$, repectively.

Let us keep the notation $B(\mathbb{X})$ of all measurable functions on the (generic) set $\mathbb{X}$. The subspace of all nonnegative measurable functions will be similarly denoted by $B^+(\mathbb{X})$. With this in mind, let us now introduce the total cost function J that considers both a running cost function $f \in B^+(\mathbb{O})$ and a final cost function $\varphi \in B^+(\mathbb{O})$. It is worth mentioning that our problem assumes that only costs can be generated on the signals $\{\tau_n\}_{n\in\mathbb{N}_0}$ but not between them, then it is clear and net that our cost to be optimized becomes a (discrete) sum rather than an integral. The accurate definition is given next:

Definition 4.4. For all $x \in \mathbb{O}$, $z_0 \geq 0$, and admissible stopping time η, we define the total cost function

$$J(x, z_0, \eta) = \mathrm{E}\Bigg[\sum_{n=0}^{\eta-1} \mathrm{e}^{-\alpha\tau_n} f\left(y(\tau_n, x)\right) \mathbf{1}_{\{z(\tau_n, z_0)=0\}} + \mathbf{1}_{\{\eta<\infty\}} \mathrm{e}^{-\alpha\tau_\eta} \varphi\left(y(\tau_\eta, x)\right)\Bigg]. \tag{4.2}$$

Thus, if $z_0 = 0$, then $z(\tau_n, z_0) = 0$ for all $n \in \mathbb{N}_0$. Therefore, $\eta \geq 0$ and

$$J(x, 0, \eta) = \mathrm{E}\Bigg[\sum_{n=0}^{\eta-1} \mathrm{e}^{-\alpha\tau_n} f\left(y(\tau_n, x)\right) + \mathbf{1}_{\{\eta<\infty\}} \mathrm{e}^{-\alpha\tau_\eta} \varphi\left(y(\tau_\eta, x)\right)\Bigg].$$

On the other hand, if $z_0 > 0$, then $z(\tau_0, z_0) = z_0 > 0$ and $z(\tau_n, z_0) = 0$ for all $n \in \mathbb{N}$. Hence, $\eta \geq 1$ and

$$J(x, z_0, \eta) = \mathrm{E}\Bigg[\sum_{n=1}^{\eta-1} \mathrm{e}^{-\alpha\tau_n} f\left(y(\tau_n, x)\right) + \mathbf{1}_{\{\eta<\infty\}} \mathrm{e}^{-\alpha\tau_\eta} \varphi\left(y(\tau_\eta, x)\right)\Bigg].$$

The optimal cost is then

$$J^*(x, z_0) \coloneqq \inf\big\{J(x, z_0, \eta) \,\big|\, \eta \text{ is admissible}\big\}. \tag{4.3}$$

It is important to say that the above problem is well-defined as there exists more than one stopping time, say the constant (with respect to the variable ω) $\eta \equiv 1$, such that $J(x, z_0, 1) < \infty$ for every $(x, z_0) \in \mathbb{O} \times \mathbb{R}_+$ and provided that $\mathrm{E}[\mathrm{e}^{-\alpha\tau_1}\varphi(y(\tau_1, x))] < \infty$. Sufficient conditions to guarantee that the cost J is finite for every admissible stopping time will be given in later subsections, under mild assumptions on f and φ.

5. Equivalent discrete-time Markov decision process

The aim of this section is to introduce an ancillary discrete-time control problem whose dynamic involves the discrete-time homogeneous Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$ defined before. We will explore our Markov decision process (MDP) under study and will see, among other facts, that such a process is quite robust in the sense that include both cases: when the signal is allowed at time zero ($z_0 = 0$) or when it does not ($z_0 > 0$). This paradigm somehow simplifies the modeling in previous literature as we do not have to distinguish between models in which it is mandatory to consider a signal at time zero and models whose stopping is not allowed at that point – see Menaldi and Robin [16–18].

Going back to the cost function (4.2), there are three elements in it that evolve from time to time. Namely, the process $y(\tau_n, x)$, the signal indicator $\mathbf{1}_{z(\tau_n,z_0)=0}$ and the discount factor $\mathrm{e}^{-\alpha\tau_n}$. So, our MDP model must be large enough in order to contain all these elements. With this in mind, we next introduce the elements conforming the aforementioned MDP.

(a) The state space is denoted by $\mathbb{E} = \mathbb{O}\times\mathbb{R}_+\times\mathbb{R}_+\cup\{\partial\}$, with Borel σ-algebra $\mathcal{B}_{\mathbb{E}}$, and where ∂ is an absorbent state not contained in $\mathbb{O}\times\mathbb{R}_+\times\mathbb{R}_+$.

(b) The action space is just $\mathbb{A} = \{0,1\}$, and given $\xi\in\mathbb{E}$ the admissible action space becomes

$$\mathbb{A}(\xi) = \begin{cases} \{0\}, & \xi = (x,z,t)\neq\partial, \text{ and } z>0,\\ \mathbb{A}, & \xi = (x,z,t)\neq\partial, \text{ and } z=0,\\ \{1\}, & \xi=\partial. \end{cases} \tag{5.1}$$

(c) A transition kernel Q defined as follows: for all $D\in\mathcal{B}_{\mathbb{E}},\ \xi\in\mathbb{E},\ a\in\mathbb{A}$

$$Q(D|\xi,a) = \begin{cases} \displaystyle\int_{\mathbb{O}\times\mathbb{R}_+} \mathbf{1}_D(y,0,t+r)\rho_r(\mathrm{d}y|x)\mu_z(dr)\mathbf{1}_{\{0\}}(a), & \xi=(x,z,t)\neq\partial,\\ \delta_{\{\partial\}}(D), & \xi=\partial \ \text{ or } \ a=1. \end{cases} \tag{5.2}$$

(d) A cost function $c:\mathbb{E}\times\mathbb{A}\to\mathbb{R}_+$ given by

$$c(\xi,a) = \begin{cases} \mathrm{e}^{-\alpha t}\big[f(x)\mathbf{1}_{\{0\}}(a) + \varphi(x)\mathbf{1}_{\{1\}}(a)\big]\mathbf{1}_{\{0\}}(z), & \xi=(x,z,t)\neq\partial,\\ 0, & \xi=\partial, \text{ or } z>0. \end{cases} \tag{5.3}$$

Let us define now the set of feasible state-action pairs by

$$\mathbb{K} \coloneqq \{(\xi,a)\mid \xi\in\mathbb{E}, a\in\mathbb{A}(\xi)\},$$

which is a measurable subset of $\mathbb{E}\times\mathbb{A}$. Observe that $\mathbb{K}$ contains at least the graph of a measurable function from E to A. Namely, consider the function $g(\xi)=0$ if $\xi\neq\partial$ and $g(\xi)=1$ if $\xi=\partial$. This function satisfies $\{(\xi,g(\xi)):\xi\in\mathbb{E}\}\subset\mathbb{K}$.

For each $n\in\mathbb{N}_0$ let us define the space $\mathbb{H}_n$ of admissible histories up to time n as $\mathbb{H}_0\coloneqq\mathbb{E}$ and

$$\mathbb{H}_n \coloneqq \mathbb{K}^n\times\mathbb{E} = \mathbb{K}\times\mathbb{H}_{n-1}, \quad \forall\, n\in\mathbb{N}. \tag{5.4}$$

Observe that for each n, $\mathbb{H}_n$ is a subspace of

$$\bar{\mathbb{H}}_n \coloneqq (\mathbb{E}\times\mathbb{A})^n\times\mathbb{E} = (\mathbb{E}\times\mathbb{A})\times\bar{\mathbb{H}}_{n-1}, \quad \forall\, n\in\mathbb{N}$$

and $\bar{\mathbb{H}}_0\coloneqq\mathbb{H}_0=\mathbb{E}$.

We also consider the set Π of *deterministic* polices $\pi=\{g_n\}_{n\in\mathbb{N}_0}$ given by measurable mappings $g_n:\mathbb{H}_n\to\mathbb{A}$ satisfying $g_n(h_n)\in\mathbb{A}(\xi_n)$ for all $h_n\in\mathbb{H}_n$ of the form $(\xi_0,a_0,\cdots,\xi_{n-1},a_{n-1},\xi_n)$. Let $\bar{\mathbb{H}}_\infty=(\mathbb{E}\times\mathbb{A})^\infty$ endowed with its corresponding product σ-algebra $\mathcal{H}_\infty$. A simple use of the Ionescu-Tulcea theorem (see for instance

Proposition C.10 in [21]) ensures that for any given measure ν over $\mathbb{E}$, and any policy $\pi \in \Pi$, there is a unique probability measure P_ν^π on $(\bar{\mathbb{H}}_\infty, \mathcal{H}_\infty)$ – indeed, it is supported on $\mathbb{H}_\infty := \mathbb{K}^\infty$ by definition of g_n, satisfying

$$\mathrm{P}_\nu^\pi(\xi_0 \in D) = \nu(D), \tag{5.5}$$
$$\mathrm{P}_\nu^\pi(a_n \in C \mid h_n) = \mathbf{1}_{\{g_n(h_n)\in C\}}, \tag{5.6}$$
$$\mathrm{P}_\nu^\pi(\xi_{n+1} \in D \mid h_n, a_n) = Q(D \mid \xi_n, a_n), \tag{5.7}$$

for all $D \in \mathcal{B}_\mathbb{E}$, $C \subseteq \mathbb{A}$, and where ξ_n, a_n stands for the projections of $\mathbb{H}_\infty$ onto its respective n-th coordinates, whilst h_n is the projection onto the history $\mathbb{H}_n$ given by:

$$h_n(\bar{\omega}) = (\xi_0(\bar{\omega}), a_0(\bar{\omega}), \ldots, a_{n-1}(\bar{\omega}), \xi_n(\bar{\omega})) = (\bar{\omega}_0, \bar{\omega}_1, \ldots \bar{\omega}_{2n-1}, \bar{\omega}_{2n}),\ \forall\ \bar{\omega} \in \mathbb{H}_\infty.$$

We denote by E_ν^π the expectation with respect to P_ν^π.

On the other hand, let us consider the following cost function

$$L(\nu, \pi) = \mathrm{E}_\nu^\pi\left[\sum_{n=0}^\infty c(\xi_n, a_n)\right], \quad \forall \pi \in \Pi. \tag{5.8}$$

Also, let the optimal cost (value function) be given by

$$L^*(\nu) = \inf\{L(\nu, \pi) \mid \pi \in \Pi\}. \tag{5.9}$$

Remark 5.1. The Markov decision process (MDP) above is formulated in a weak sense because the probability space involved in this model is just $(\mathbb{H}_\infty, \mathcal{H}_\infty, \mathrm{P}_\nu^\pi)$, and its structure was built in terms of "primitive" elements including the transition kernel Q, the initial distribution ν, and the control policy π. However, in what follows, we will see that, when the initial distribution in (5.5) is $\nu(\cdot) = \delta_{(x,z_0,0)}(\cdot)$, the Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$ together with an arbitrary stopping time η (with respect to the filtration $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$), both imply the existence of a deterministic policy $\pi \in \Pi$ and a new state-action process that fits with our MDP above, in particular, the relations (5.14) to (5.16) below are satisfied. In other words, each pair $(\{X_n\}_{n\in\mathbb{N}_0}, \eta)$ implies the existence of $\pi \in \Pi$, so that the paths of a state-action process (see (5.10)–(5.11) below) will belong to the set $\mathbb{H}_\infty$ and whose transition probabilities satisfy the aforementioned relations.

To begin with, let η be an admissible stopping time and $\mathcal{E} = (\Omega, \mathcal{F}, \mathrm{P})$. Think of a discrete-time model Y_n that tracks the evolution of X_n up to the stopping time η, and then enters into an absorbent state ∂. This is the MDP we will construct in the following lines, adding the action process $\{A_n\}_{n\in\mathbb{N}_0}$, which indicates whether the process moves forward or is stopped. Let us define the aforementioned processes by

$$A_n(\omega) := \mathbf{1}_{\{\eta\le n\}}(\omega) \tag{5.10}$$

for all $n \in \mathbb{N}_0,\ \omega \in \Omega$. Note that $A_n = 0$ for all $n \in \mathbb{N}_0$ if and only if $\eta = \infty$.

Also, we define

$$Y_n(\omega) := \begin{cases} X_0(\omega) = (x, z_0, 0), & n = 0, \\ X_n(\omega) = \big(y(\tau_n(\omega), x)(\omega), 0, \tau_n(\omega)\big), & A_{n-1}(\omega) = 0, \\ \partial, & A_{n-1}(\omega) = 1, \end{cases} \tag{5.11}$$

for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Note that A_n takes values on the measurable space $(\mathbb{A}, 2^{\mathbb{A}})$, whilst Y_n takes so in the space $\mathbb{E} = \mathbb{O} \times \mathbb{R}_+ \times \mathbb{R}_+ \cup \{\partial\}$ endowed with the σ-algebra $\mathcal{B}_{\mathbb{E}}$. Together with these processes, we also define

$$S_0 \coloneqq Y_0 \quad \text{and} \quad S_n(\omega) \coloneqq \big(Y_0(\omega), A_0(\omega), \dots, A_{n-1}(\omega), Y_n(\omega)\big) \quad \forall\, n \in \mathbb{N},\ \omega \in \Omega,$$

which is contained within the measurable state space $(\mathbb{H}_n, \mathcal{H}_n)$ defined in (5.4).

Next, we will prove that the process $\{S_n\}_{n\in\mathbb{N}_0}$ is $\{\mathcal{G}_n\}_{n\in\mathbb{N}_0}$-adapted. Essentially, the idea is that if we know the process has not been stopped up to time $n-1$, then Y_n is not at the absorbent state ∂, and vice versa, if we know Y_n is not at the absorbent state, then we can deduce that the stopping time η occurs strictly after $n-1$. For easy of notation, in what follows we will omit the variable ω. Indeed, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} \{\eta > n-1\} &= \{\eta > r, \quad \forall r = 0, \cdots, n-1\} \\ &= \{S_n = (X_0, 0, X_1, \dots, X_{n-1}, 0, X_n)\}, \end{aligned} \tag{5.12}$$

meanwhile

$$\{\eta \leq n-1\} \iff \{Y_n = \partial\}. \tag{5.13}$$

Hence, we have that for all $D \in \mathcal{B}_{\mathbb{E}}$ and $k \leq n$:

$$\{Y_k \in D\} \cap \{\eta > n-1\} = \{X_k \in D \setminus \{\partial\}\} \cap \{\eta > n-1\}$$

and

$$\{Y_k \in D\} \cap \{\eta \leq k-1\} = \{\partial \in D\} \cap \{\eta \leq k-1\};$$

which proves that $\{Y_k \in D\} \in \mathcal{G}_n$, since X_k is $\mathcal{G}_n$-measurable for all $k \leq n$ and η is a stopping time with respect to $\{\mathcal{G}_m\}_{m\in\bar{\mathbb{N}}_0}$. This latter fact also implies that $A_{k-1} = \mathbf{1}_{\{\eta \leq k-1\}}$ is $\mathcal{G}_n$-measurable. Hence, we have (A_{k-1}, Y_k) is $\mathcal{G}_n$-measurable for all $1 \leq k \leq n$, so in virtue of $S_0 = X_0$ is also $\mathcal{G}_n$ measurable we get S_n is $\mathcal{G}_n$-measurable for all $n \in \mathbb{N}_0$, which proves the above assertion.

The proof of the next proposition uses the following concept: We say that two σ-algebras $\mathcal{G}$ and $\mathcal{H}$ are equal on B if and only if $B \in \mathcal{G} \cap \mathcal{H}$ and $B \cap \mathcal{G} = B \cap \mathcal{H}$.

Proposition 5.2. *Consider the Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$ with initial condition $(x, z_0, 0)$, and take an arbitrary admissible stopping time η. Then, there exists a sequence of measurable functions $\{g_n\}_{n\in\mathbb{N}_0}$ from $\mathbb{H}_n$ to $\mathbb{A}$, associated with the state-action process $\{(Y_n, A_n)\}_{n\in\mathbb{N}_0}$ (5.10)–(5.11), so that the following relation holds in terms of the probability measure P in $\mathcal{E}$:*

$$\mathrm{P}(Y_0 \in D) = \delta_{(x,z_0,0)}(D), \tag{5.14}$$

$$\mathrm{P}(A_n \in B | S_n) = \mathbf{1}_{\{g_n(S_n) \in B\}}, \tag{5.15}$$

$$\mathrm{P}(Y_{n+1} \in D | S_n, A_n) = Q(D | Y_n, A_n), \tag{5.16}$$

and

$$g_n(S_n) \in \mathbb{A}(Y_n), \tag{5.17}$$

for all $n \in \mathbb{N}_0$, $D \in \mathcal{B}_{\mathbb{E}}, B \subseteq \mathbb{A}$.

Proof. The relation (5.14) is straightforward. To prove (5.15), we will invoke Lemma A.10, so in the following lines we will demonstrate that its corresponding hypotheses are satisfied. Indeed, by (5.12) and (5.10), we have that

$$\Delta := \{S_n = (X_0, 0, \ldots, X_{n-1}, 0, X_n)\} = \{\eta > n-1\} = \{A_{n-1} = 0\},$$

then $\Delta \in \sigma(S_n) \cap \mathcal{G}_n = \sigma(S_n)$. Now observe that $\Delta^{\mathrm{c}} \in \sigma(S_n)$ and

$$\Delta^{\mathrm{c}} = \{\eta \leq n-1\} = \{A_{n-1} = 1, A_n = 1\}. \tag{5.18}$$

Indeed, first note that $\eta \leq n-1$ if and only if $1 = \mathbf{1}_{\{\eta \leq n-1\}} = A_{n-1}$. Besides, we have

$$\{A_{n-1} = 1\} \subseteq \{A_n = 1\}.$$

Therefore, we get $\{\eta \leq n-1\} = \{A_{n-1} = 1\} = \{A_{n-1} = 1, A_n = 1\}$ as desired. Then, for all $B \subseteq \mathbb{A}$, we get

$$\begin{aligned} \mathbf{1}_{\Delta^{\mathrm{c}}} \mathrm{P}(A_n \in B \mid S_n) &= \mathrm{P}(\Delta^{\mathrm{c}}, A_n \in B \mid S_n) \\ &= \mathrm{P}(\Delta^{\mathrm{c}}, 1 \in B \mid S_n) = \mathbf{1}_{\{1 \in B\}} \mathbf{1}_{\Delta^{\mathrm{c}}} = \mathbf{1}_{\{A_n \in B\}} \mathbf{1}_{\Delta^{\mathrm{c}}}. \end{aligned} \tag{5.19}$$

On the other hand, given sets $D_0, \ldots, D_n \in \mathcal{B}_{\mathbb{E}}$ and $B_0, \ldots, B_{n-1} \in \mathbb{A}$ we have

$$\begin{aligned} &\Delta \cap \{S_n \in D_0 \times B_0 \times \ldots \times D_{n-1} \times B_{n-1} \times D_n\} \\ &= \Delta \cap \{(X_0, 0, \ldots, X_{n-1}, A_n, X_n) \in D_0 \times B_0 \times \ldots \times D_{n-1} \times B_{n-1} \times D_n\} \\ &= \Delta \cap \{X_0 \in D_0, \ldots, X_n \in D_n\} \cap \{0 \in B_0, \ldots, 0 \in B_{n-1}\}, \end{aligned}$$

which is in $\Delta \cap \mathcal{G}_n$, and so, $\Delta \cap \sigma(S_n) \subseteq \Delta \cap \mathcal{G}_n$. Also, we get

$$\Delta \cap \{X_0 \in D_0, \ldots, X_n \in D_n\} = \Delta \cap \{Y_0 \in D_0, \ldots, Y_n \in D_n\},$$

which is in $\Delta \cap \sigma(S_n)$. Thus $\Delta \cap \mathcal{G}_n \subseteq \Delta \cap \sigma(S_n)$, and so, $\sigma(S_n) = \mathcal{G}_n$ on Δ. Hence, by Lemma A.10, we obtain the first equality in (5.20). Additionally, since A_n is $\mathcal{G}_n$-measurable and by properties of conditional expectation, we obtain the second equality in (5.20)

$$\mathbf{1}_{\Delta} \mathrm{P}(A_n \in B \mid S_n) = \mathbf{1}_{\Delta} \mathrm{P}(A_n \in B \mid \mathcal{G}_n) = \mathbf{1}_{\{A_n \in B\}} \mathbf{1}_{\Delta}. \tag{5.20}$$

Using the above and (5.19) we get

$$\mathrm{P}(A_n \in B \mid S_n) = \mathbf{1}_{\{A_n \in B\}}, \quad \forall\, n \in \mathbb{N}_0, \tag{5.21}$$

which proves that A_n is S_n-measurable. As a consequence, there exists a measurable function $g_n : \mathbb{H}_n \to \mathbb{A}$ such that $g_n(S_n) = A_n$ for each $n \in \mathbb{N}_0$. Now, we will show that $g_n(S_n) \in \mathbb{A}(Y_n)$ for all $n \in \mathbb{N}_0$ (see (5.1) for the definition of $\mathbb{A}(Y_n)$). Due to (5.13), we get $Y_n = \partial$ if and only if $A_{n-1} = 1$ which implies $A_n = 1 = g_n(S_n) \in \mathbb{A}(Y_n)$. We know that when $Y_n \neq \partial$ the second component of Y_n is not zero only when $n = 0$. So, we analyze this case: let $z_0 > 0$, then $\eta \geq 1$ and so $A_0 = 0 = g_0(S_0) \in \mathbb{A}(S_0)$. Therefore,

$$g_n(S_n) \in \mathbb{A}(Y_n), \ \forall\, n \in \mathbb{N}_0.$$

Thus, we obtain relation (5.17).

It remains to prove the relation (5.16). First take $D \in \mathcal{B}_{\mathbb{E}}$ and note that $\{A_n = 1\} = \{A_n = 1, Y_{n+1} = \partial\}$, then

$$\mathbf{1}_{\{A_n=1\}}\mathrm{P}(Y_{n+1} \in D \mid S_n, A_n) = \mathrm{P}(A_n = 1,\ \partial \in D \mid S_n, A_n) = \mathbf{1}_{\{\partial\in D\}}\mathbf{1}_{\{A_n=1\}}. \tag{5.22}$$

In the same manner, we obtain $\{A_n = 0\} = \{\eta > n\} = \{\eta \le n\}^{\mathrm{c}} = \{A_n = 1\}^{\mathrm{c}} \in \sigma(S_n, A_n) \cap \mathcal{G}_n$, because $\{\eta \le n\}^{\mathrm{c}} \in \mathcal{G}_n$ and $\{A_n = 1\}^{\mathrm{c}} \in \sigma(S_n, A_n)$; and as in the first part of the proof, we can show that $\sigma(S_n, A_n) = \mathcal{G}_n$ on $\{A_n = 0\}$ that is the same to say that when $A_n = 0$ the information provided by $\sigma(S_n, A_n)$ is equal to $\mathcal{G}_n$. Furthermore, we have that $\mathbf{1}_{\{Y_{n+1}\in D\}} = \mathbf{1}_{\{X_{n+1}\in D\}}$ on $\{A_n = 0\}$ by (5.11) then, again by Lemma A.10 and the Markov property of $\{X_n\}_{n\in\mathbb{N}_0}$, we get

$$\begin{aligned}\mathbf{1}_{\{A_n=0\}}\mathrm{P}(Y_{n+1} \in D \mid S_n, A_n) &= \mathbf{1}_{\{A_n=0\}}\mathrm{P}(X_{n+1} \in D \mid \mathcal{G}_n)\\ &= \mathbf{1}_{\{A_n=0\}}\mathrm{P}(X_{n+1} \in D \mid X_n).\end{aligned} \tag{5.23}$$

Comparing (5.22) and (5.23) with (5.2) and using the homogeneity of the chain $\{X_n\}_{n\in\mathbb{N}_0}$ we get (5.16). □

The next theorem is a key point for giving solution to our optimal stopping problem. It provides a kind of bijection between the set of optimal stopping times with respect to the filtration $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$ and the set of deterministic control policies Π in terms of the cost functions (4.2) and (5.8).

Theorem 5.3. *Consider the cost functions J and L defined in (4.2) and (5.8), respectively. Then, for any initial condition of type $\xi = (x, z_0, 0) \in \mathbb{E} \setminus \{\partial\}$, the following statements holds true:*

(a) *For any admissible stopping time η with respect to $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$, there exists a deterministic policy $\pi \in \Pi$ such that*

$$J(x, z_0, \eta) = L(\xi, \pi). \tag{5.24}$$

(b) *For any deterministic policy Π there exists an admissible stopping time η in $\bar{\mathbb{N}}_0$ with respect to $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$ such that the equation (5.24) remains unchanged.*

As a consequence, we obtain

$$J^*(x, z_0) = L^*(x, z_0, 0). \tag{5.25}$$

Proof. (a) Let us take an admissible stopping time η. Then, the discrete-time process $Y_0, A_0, Y_1, \ldots$ is well defined, which depends on η as in (5.10)–(5.11) and we take $\{g_n\}_{n\in\mathbb{N}_0}$ as in Proposition 5.2.

The mapping $\psi_\eta : \Omega \to \mathbb{H}_\infty$ given by $\psi_\eta(\omega) = (Y_0(\omega), A_0(\omega), \ldots)$ is measurable due to Lemma A.9. Let P_η be the induced measure over $(\mathbb{H}_\infty, \mathcal{H}_\infty)$ given by

$$\mathrm{P}_\eta(C) := \mathrm{P}(\psi_\eta \in C), \quad C \in \mathcal{H}_\infty. \tag{5.26}$$

Denote also by E_η its corresponding expectation operator. We note that for any random variable ζ on $(\mathbb{H}_\infty, \mathcal{H}_\infty, \mathrm{P}_\eta)$ we have

$$\mathrm{E}_\eta\,\zeta = \int \zeta\ d\mathrm{P}_\eta = \int \zeta\ d\big(\mathrm{P} \circ \psi_\eta^{-1}\big) = \int \zeta \circ \psi_\eta\ d\mathrm{P} = \mathrm{E}[\zeta \circ \psi_\eta]. \tag{5.27}$$

We will prove that P_η satisfies (5.5)–(5.7). First, note that

$$\mathrm{P}_\eta(\xi_0 \in B) = \mathrm{E}_\eta[\mathbf{1}_{\xi_0 \in B}] = \mathrm{E}[\mathbf{1}_{\xi_0 \in B} \circ \psi_\eta] = \mathrm{E}[\mathbf{1}_{Y_0 \in B}] = \mathrm{P}(Y_0 \in B) = \nu(B). \tag{5.28}$$

Also, we get

$$\begin{aligned}\mathrm{E}_\eta\big[\mathbf{1}_{\{a_n \in C\}}\mathbf{1}_{\{h_n \in D\}}\big] &= \mathrm{E}\big[\mathbf{1}_{\{A_n \in C\}}\mathbf{1}_{\{S_n \in D\}}\big] \\ &= \mathrm{E}\big[\mathbf{1}_{\{g_n(S_n) \in C\}}\mathbf{1}_{\{S_n \in D\}}\big] = \mathrm{E}_\eta\big[\mathbf{1}_{\{g_n(h_n) \in C\}}\mathbf{1}_{\{h_n \in D\}}\big],\end{aligned}$$

yielding

$$\mathrm{P}_\eta(a_n \in C \mid h_n) = \mathbf{1}_{\{g_n(h_n) \in C\}}. \tag{5.29}$$

Moreover, using (5.26), (5.14)–(5.16) in Proposition 5.2, and the concept of conditional expectation, we get

$$\begin{aligned}&\mathrm{E}_\eta\big[\mathbf{1}_{\{\xi_{n+1} \in D\}}\mathbf{1}_{\{h_n \in B\}}\mathbf{1}_{\{a_n \in C\}}\big] = \mathrm{E}\big[\mathbf{1}_{\{Y_{n+1} \in D\}}\mathbf{1}_{\{S_n \in B\}}\mathbf{1}_{\{A_n \in C\}}\big] \\ &= \mathrm{E}\big[Q(D \mid Y_n, A_n)\mathbf{1}_{\{S_n \in B\}}\mathbf{1}_{\{A_n \in C\}}\big] = \mathrm{E}_\eta\big[Q(D \mid \xi_n, a_n)\mathbf{1}_{\{h_n \in B\}}\mathbf{1}_{\{a_n \in C\}}\big],\end{aligned}$$

which implies

$$\mathrm{P}_\eta(\xi_{n+1} \in D \mid h_n, a_n) = Q(D \mid \xi_n, a_n). \tag{5.30}$$

Besides, by (5.17) we obtain

$$\mathrm{P}_\eta(g_n(h_n) \notin \mathbb{A}(\xi_n)) = \mathrm{P}(g_n(S_n) \notin \mathbb{A}(Y_n)) = 0.$$

Hence, essentially $\pi := \{g_n\}_{n \in \mathbb{N}_0}$ is an admissible policy. Now, considering the distribution $\nu(D) := \delta_{(x,z_0,0)}(D)$, the transition kernel Q and the policy π, we can construct the probability measure P^π_ν over $(\bar{\mathbb{H}}_\infty, \mathcal{H}_\infty)$ satisfying (5.5)–(5.7). But due to (5.28)–(5.30), P_η also satisfies the same relations, therefore, by the uniqueness of P^π_ν due to Iunescu-Tulcea theorem, we assert that

$$\mathrm{P}^\pi_\nu = \mathrm{P}_\eta, \tag{5.31}$$

and so using (5.27) along with the definition of the cost function c in (5.3) we obtain

$$\begin{aligned}\mathrm{E}^\pi_\nu\Big[\sum_{n=0}^\infty c(\xi_n, a_n)\Big] &= \mathrm{E}\Big[\sum_{n=0}^\infty c(\xi_n, a_n) \circ \psi_\eta\Big] = \mathrm{E}\Big[\sum_{n=0}^\infty c(Y_n, A_n)\Big] \\ &= \mathrm{E}\Big[\sum_{n=0}^{\eta-1} \mathrm{e}^{-\alpha\tau_n} f(y(\tau_n, x))\mathbf{1}_{\{z(\tau_n, z_0)=0\}} + \mathbf{1}_{\{\eta<\infty\}}\mathrm{e}^{-\alpha\tau_\eta}\varphi(y(\tau_\eta, x))\Big],\end{aligned}$$

which proves (a).

(b) Consider a deterministic policy $\pi = \{g_n(h_n)\}_{n \in \mathbb{N}_0} \in \Pi$. Now define the process $(Y_0, A_0, Y_1, \dots)$ as $Y_0 = X_0$, $A_0 = g_0(Y_0) = g_0(X_0)$, and for $n \in \mathbb{N}$ as

$$Y_n = \begin{cases} X_n, & A_{n-1} = 0, \\ \partial, & A_{n-1} = 1, \end{cases}$$

and

$$A_n = g_n(Y_0, A_0, \dots, Y_{n-1}, A_{n-1}, Y_n). \tag{5.32}$$

Let us define the random time

$$\eta \coloneqq \inf\{n \in \mathbb{N}_0 \mid A_n = 1\}, \tag{5.33}$$

with the convention that $\inf \emptyset = \infty$, which is the case when $A_n = 0$ for all $n \in \mathbb{N}_0$. Now we will prove that η is indeed a stopping time with respect to $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$. Namely, first consider $n \in \mathbb{N}_0$ and note that

$$\{A_{n-1} = 0\} = \{Y_n = X_n\} = \{Y_n = X_n, \dots, Y_1 = X_1\} = \{A_{n-1} = 0, \dots, A_0 = 0\}. \tag{5.34}$$

The first equality follows from the definition of Y and A. The second one follows from the fact that if $Y_j = \partial$ for some $j < n$, then $A_j = g_j(Y_0, \dots, A_j, \partial) \in \mathbb{A}(\partial) = \{1\}$, which would imply $Y_{j+1} = \partial$ and inductively $Y_n = \partial$. The third equality comes again from the definitions of Y and A.

Our aim is to prove that $\{\eta = n\} = \{A_{n-1} = 0, A_n = 1\}$ is $\mathcal{G}_n$-measurable and the way to proceed is by using mathematical induction on n. Namely, by definition, we know that $A_0 = g_0(X_0)$ is $\mathcal{G}_0$-measurable. For $n = 1$ we have that

$$\begin{aligned}\{A_0 = 0, A_1 \in C\} &= \{Y_1 = X_1\} \cap \{g_1(Y_0, A_0, Y_1) \in C\}\\ &= \{g_0(X_0) = 0\} \cap \{g_1(X_0, 0, X_1) \in C\}\end{aligned}$$

is $\mathcal{G}_1$-measurable for any $C \subseteq \mathbb{A}$. Let us assume that $\{A_{n-1} = 0, A_n \in C\}$ is $\mathcal{G}_n$-measurable. We have that

$$\begin{aligned}\{A_n = 0, A_{n+1} \in C\} &= \{Y_{n+1} = X_{n+1}, \dots, Y_1 = X_1\} \cap \{g_{n+1}(X_0, 0, \dots, X_{n+1}) \in C\}\\ &= \{A_0 = 0, \dots, A_n = 0\} \cap \{g_{n+1}(X_0, 0, \dots, 0, X_{n+1}) \in C\},\end{aligned}$$

is $\mathcal{G}_{n+1}$-measurable because by induction hypothesis $\{A_0 = 0, \dots, A_n = 0\}$ is $\mathcal{G}_n$-measurable. In particular, we get

$$\{\eta = n\} = \{A_{n-1} = 0, A_n = 1\} \in \mathcal{G}_n,$$

and so η is a stopping time with respect to $\{\mathcal{G}_n\}_{n\in\mathbb{N}_0}$.

We will now analyze the case $n = \infty$. For this end, note that $\{\eta = \infty\} = \bigcap_{n\in\mathbb{N}_0}\{A_n = 0\}$. Based on previous arguments, we have that $\{A_n = 0\} \in \mathcal{G}_n$ for each $n \in \mathbb{N}_0$. Hence, $\{\eta = \infty\} \in \mathcal{G}_\infty$, which entails that η is a stopping time w.r.t. $\{\mathcal{G}_n\}_{n\in\bar{\mathbb{N}}_0}$.

Now, we will prove that in fact $A_n = \mathbf{1}_{\{\eta\le n\}}$ for all $n \in \mathbb{N}_0$. For this purpose, note that $\mathbf{1}_{\{\eta\le n\}} = 0$ if and only if $\eta > n$ if and only if $A_0 = \ldots = A_n = 0$, and also $\mathbf{1}_{\{\eta\le n\}} = 1$ if and only if $Y_n = \partial$ if and only if $A_n = 1$; that is $\mathbf{1}_{\{\eta\le n\}} = A_n$ for all $n \in \mathbb{N}_0$. So, it is clear that $z(\tau_\eta, z_0) = 0$ if $z_0 = 0$, however, if $z_0 > 0$ then $A_0 = 0$ and necessarily $\eta \ge 1$, which imlplies $z(\tau_\eta, z_0) = 0$. Hence, by Remark 4.3, η is admissible.

Finally, we can apply the arguments in part (a) of this proof in order to define first P_η and hence use Proposition 5.2 as long with Ionescu-Tulcea Theorem to get $\mathrm{P}^\pi_\nu = \mathrm{P}_\eta$. Later use (5.27) to finally obtain (5.24).

Once we have established the equivalence in (5.24), we can easily conclude the veracity of expression (5.25). □

6. Dynamic programming equations

In this section, we will analyze the dynamic programming equation (DPE) (6.5) below associated with the Markov decision process (MDP) studied in the previous section. This analysis involves showing the existence of solutions to this equation and its regularity in the sense to get a solution in the space $C_b(\mathbb{O})$.

Recall that τ_1, appearing as an element of the sequence of signals $\{\tau_n\}_{n\in\mathbb{N}_0}$, has distribution

$$\mu_{z_0}([a,b)) = \mathrm{P}\big(\tau_1 \in [a,b)\big) = \frac{\mu_0\big([a+z_0, b+z_0)\big)}{\mu_0([z_0,\infty))}, \tag{6.1}$$

and so, the expectation of τ_1 depends on z_0. This dependence will be reflected in the next lemma. Also remember that $\mu_0(\{0\}) = 0$.

Lemma 6.1. *For each $z_0 \in \mathbb{R}_+$ and $\alpha > 0$ the next inequality holds true*

$$\varrho_\alpha^{z_0} := \mathrm{E}[\mathrm{e}^{-\alpha\tau_1}] < 1. \tag{6.2}$$

Proof. We have

$$\varrho_\alpha^{z_0} = \int_0^\infty \mathrm{e}^{-\alpha t}\mu_{z_0}(\mathrm{d}t) = \frac{1}{\mu_0([z_0,\infty))}\int_{z_0}^\infty \mathrm{e}^{-\alpha(t-z_0)}\mu_0(\mathrm{d}t) \tag{6.3}$$

$$= \frac{\mathrm{e}^{\alpha z_0}}{\mu_0([z_0,\infty))}\int_{z_0}^\infty \mathrm{e}^{-\alpha t}\mu_0(\mathrm{d}t) \tag{6.4}$$

for all $z_0 \geq 0$, since by (6.1)

$$\int_0^\infty \mathbf{1}_{[a,b)}(t)\mu_{z_0}(\mathrm{d}t) = \frac{1}{\mu_0([z_0,\infty))}\int_{z_0}^\infty \mathbf{1}_{[a,b)}(t-z_0)\mu_0(\mathrm{d}t).$$

If $z_0 = 0$, then given that $\mu_0(\{0\}) = 0$ we get $t \mapsto \mathrm{e}^{-\alpha t} < 1$ μ_0-a.s, and hence

$$\varrho_\alpha^0 = \int_0^\infty \mathrm{e}^{-\alpha t}\mu_0(\mathrm{d}t) < \int_0^\infty \mu_0(\mathrm{d}t) = 1.$$

On the other side, for $z_0 > 0$ we have that

$$\int_{z_0}^\infty \mathrm{e}^{-\alpha t}\mu_0(\mathrm{d}t) \leq \mathrm{e}^{-\alpha z_0}\mu_{z_0}([z_0,\infty)) < 1.$$

Finally, using (6.3) and the above inequalities, we get

$$\varrho_\alpha^{z_0} = \frac{\mathrm{e}^{\alpha z_0}}{\mu_0([z_0,\infty))}\int_{z_0}^\infty \mathrm{e}^{-\alpha t}\mu_0(\mathrm{d}t) < 1.$$

□

6.1. Regularity under continuous costs

In this subsection, we shall restrict our study to the case when both the running cost f and terminal cost φ introduced in (5.3) satisfy a certain grade of regularity (continuity and boundedness). It turns out that, assuming the Feller property of process $\{y(t,x)\}_{t\geq 0}$, we can prove the same regularity properties for the solution of the DPE (6.5) (or (6.9)) below. We begin with the setup of the aforementioned assumptions.

Assumption 6.2. **(a)** The process $\{y(t,x)\}_{t\geq 0}$ satisfies the Feller property. That is, the mapping $x \mapsto \int_{\mathbb{O}} \mathrm{e}^{-\alpha t}\psi(y)\rho_t(\mathrm{d}y \mid x) = \Phi_\alpha(t)\psi(x)$ belongs to $C_b(\mathbb{O})$ for every continuous and bounded function $\psi \in C_b(\mathbb{O})$ and all $t \geq 0$.

(b) The running and terminal costs $f, \varphi : \mathbb{O} \mapsto \mathbb{R}_+$ belong to the set of nonnegative, bounded, and continuous functions $C_b^+(\mathbb{O})$.

We shall be interested in the next DPE:

$$v(x,z_0) = \inf_{a\in\mathbb{A}(x,z_0,0)} \Big\{ c(x,z_0,0,a) + \int\limits_{\mathbb{E}\setminus\{\partial\}} \mathrm{e}^{-\alpha t} v(y,z) Q(\mathrm{d}y \times dz \times \mathrm{d}t \mid x, z_0, 0, a)\mathbf{1}_{\{0\}}(a)\Big\}, \tag{6.5}$$

for each $x \in \mathbb{O}$ and $z_0 \in \mathbb{R}_+$.

Remark 6.3. Equation (6.5) can be viewed as a specific case of the more commonly used equation in the theory of Markov decision processes (MDPs) with undiscounted costs – see, for instance [22], Section 9.

$$\hat{v}(\xi_0) = \begin{cases} \inf\limits_{a\in\mathbb{A}(\xi_0)} \Big\{ c(\xi_0,a) + \int_{\mathbb{E}} \hat{v}(\xi) Q(d\xi \mid \xi_0, a)\Big\}, & \xi_0 = (x,z_0,0) \in \mathbb{E}\setminus\{\partial\}; \\ 0, & \xi_0 = \partial. \end{cases} \tag{6.6}$$

for $x \in \mathbb{O}$ and $z_0 \in \mathbb{R}_+$. In the equation above, note that we restrict the initial condition to take only values of the form $\xi = (x, z_0, 0)$. Namely, the relation between the solutions of (6.5) and (6.6) trivially obeys to the following:

$$\hat{v}(\xi) = \begin{cases} \mathrm{e}^{-\alpha t} v(y,z), & \xi = (y,z,t) \in \mathbb{E}\setminus\{\partial\}, \\ 0, & \xi = \partial, \end{cases} \tag{6.7}$$

where v is the solution of the DPE (6.5). In fact, departing from equation (6.5), we can then pose this equation in terms of $\hat{v}$ as in (6.7) to obtain

$$\begin{aligned} \hat{v}(\xi_0) &= \inf_{a\in\mathbb{A}(\xi_0)} \Big\{ c(\xi_0,a) + \int_{\mathbb{E}\setminus\{\partial\}} \hat{v}(\xi) Q(d\xi \mid \xi_0, a)\}\mathbf{1}_{\{0\}}(a)\Big\} \\ &= \inf_{a\in\mathbb{A}(\xi_0)} \Big\{ c(\xi_0,a) + \int_{\mathbb{E}} \hat{v}(\xi) Q(d\xi \mid \xi_0, a)\Big\}. \end{aligned} \tag{6.8}$$

The second equality in (6.8) is due to the fact $Q(d\xi \mid \xi_0, 1) = \delta_{\{\partial\}}(d\xi)$ and $\hat{v}(\partial) = 0$.

We will use equation (6.5) to seek existence, uniqueness, and regularity of the value functions J^*, meanwhile, we shall utilize the equation (6.6) to achieve the link between the DPE and our optimal stopping problem with constraints and also to guarantee the existence of an optimal stopping time.

The definition of Q suggests another way to write the integral above; namely, by (5.2) and (3.17)

$$\begin{aligned} \int_{\mathbb{E}\setminus\{\partial\}} \mathrm{e}^{-\alpha t} v(y,z) Q(\mathrm{d}y \times dz \times \mathrm{d}t \mid x, z_0, 0, 0) &= \int_{\mathbb{O}\times\mathbb{R}_+} \mathrm{e}^{-\alpha t} v(y,0)\rho_t(\mathrm{d}y \mid x)\mu_{z_0}(\mathrm{d}t) \\ &= \int_0^\infty \Phi_\alpha(t) v(x,0)\mu_{z_0}(\mathrm{d}t), \end{aligned}$$

then we obtain the next form of the DPE:

$$v(x,z_0)=\inf_{a\in\mathbb{A}(x,z_0,0)}\Big\{c(x,z_0,0,a)+\int_0^\infty \Phi_\alpha(t)v(x,0)\mu_{z_0}(\mathrm{d}t)\mathbf{1}_{\{0\}}(a)\Big\}. \tag{6.9}$$

For each fixed $z_0\in\mathbb{R}_+$, we define the operator $T_{z_0}: B_b^+(\mathbb{O})\mapsto B_b^+(\mathbb{O})$ as the right-hand side of the DPE; *i.e.*,

$$T_{z_0}v(x)\coloneqq\inf_{a\in\mathbb{A}(x,z_0,0)}\Big\{c(x,z_0,0,a)+\int_0^\infty \Phi_\alpha(t)v(x)\mu_{z_0}(\mathrm{d}t)\mathbf{1}_{\{0\}}(a)\Big\}\quad \forall\ x\in\mathbb{O}. \tag{6.10}$$

The next theorem provides some properties of the operator T_{z_0}:

Theorem 6.4. *Under Assumption 6.2, we have for each $z_0\in\mathbb{R}_+$:*

- **(a)** *The operator T_{z_0} leaves invariant the space $C_b^+(\mathbb{O})$; i.e., for every $v\in C_b^+(\mathbb{O})$, we have $T_{z_0}v\in C_b^+(\mathbb{O})$.*
- **(b)** *T_{z_0} is a contraction operator in the space $C_b^+(\mathbb{O})$. Furthermore, it is monotone non-decreasing on $B_b^+(\mathbb{O})$; i.e, $T_{z_0}u\geq T_{z_0}v$ for all $u\geq v$.*

Proof. (a) Take $v\in C_b^+(\mathbb{O})$. Note that $c(\cdot,z_0,0,a)\in C_b^+(\mathbb{O})$ since $f,\varphi\in C_b^+(\mathbb{O})$. Then, by Assumption 6.2, boundedness of v and dominated convergence we see that the mapping

$$x\mapsto c(x,z_0,0,a)+\int_0^\infty \Phi_\alpha(t)v(x)\mu_{z_0}(\mathrm{d}t)\mathbf{1}_{\{0\}}(a) \tag{6.11}$$

is nonnegative, bounded, and continuous for each $z_0\in\mathbb{R}_+$ and $a\in\mathbb{A}(x,z_0,0)$. Noting also that the set $\mathbb{A}$ in (5.1) is finite, so it is $\mathbb{A}(x,z_0,0)$ for every x,z_0. Then, the continuity and nonnegativeness of the right-hand side of (6.10) on $\mathbb{O}$ remains valid, in other words, the mapping $T_{z_0}v$ lies in $C_b^+(\mathbb{O})$, which proves (a).

(b) Let $v,v'\in C_b^+(\mathbb{O})$. Firstly, note that when $z_0=0$,

$$T_{z_0}v(x)=c(x,0,0,1)\wedge\Big\{c(x,0,0,0)+\int_0^\infty \Phi_\alpha(t)v(x)\mu_0(\mathrm{d}t)\Big\};$$

whilst $z_0>0$, we obtain

$$T_{z_0}v(x)=c(x,z_0,0,0)+\int_0^\infty \Phi_\alpha(t)v(x)\mu_{z_0}(\mathrm{d}t).$$

Therefore, using Lemmas A.16 and 6.1, we deduce, for every $z_0\geq 0$ that

$$|T_{z_0}v(x)-T_{z_0}v'(x)|\leq\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{E}\big|v(y(t,x))-v'(y(t,x))\big|\mu_{z_0}(\mathrm{d}t).$$

Hence,

$$|T_{z_0}v(x)-T_{z_0}v'(x)|\leq\int_0^\infty \mathrm{e}^{-\alpha t}\mu_{z_0}(\mathrm{d}t)\,\|v-v'\|\leq\varrho_\alpha^{z_0}\,\|v-v'\|\,,\quad\forall\ x\in\mathbb{O} \tag{6.12}$$

where the last inequality in the above procedure is due to the bound in (6.2). This implies that T_{z_0} is a contraction operator over $C_b^+(\mathbb{O})$. The monotony of T_{z_0} follows easily by the definition of this operator. ☐

In virtue of the contraction property of the operator T_{z_0} in Theorem 6.4, and the fact of the completeness of the space $C_b^+(\mathbb{O})$, we can apply Banach's fixed point theorem to ensure the existence of a function, say v_{z_0} in $C_b^+(\mathbb{O})$ so that

$$v_{z_0} = T_{z_0} v_{z_0}, \tag{6.13}$$

which in turn guarantees the solution of the DPE (6.9). With this in mind, we claim the following corollary.

Corollary 6.5. *The solution $v(\cdot, z_0) := v_{z_0}(\cdot)$ of the DPE (6.9), belongs to the space $C_b^+(\mathbb{O})$, for each $z_0 \in \mathbb{R}_+$.*

7. Solution of the stopping time problem

We have reserved this section as a connection between our optimal control problem (5.8)–(5.9), the optimal stopping problem (4.2)–(4.3), and the solution of the DPE (6.5).

As a first remark, we can verify that Lemma 6.1 and Assumption 6.2, as long with the definition of c in (5.3), all together imply that there is a control policy $\pi \in \Pi$ such that the function L in (5.8) is finite under this policy when the initial distribution is $\nu(\cdot) = \delta_{\{\xi\}}(\cdot)$, $\xi \in \mathbb{E}$, and so the value function (5.9) is finite too. Namely, assume first Assumption 6.2 and take $\pi := \{g_n(h_n)\}_{n \in \mathbb{N}_0}$, with $g_0 \equiv 0$, and $g_n \equiv 1$ for all $n \geq 1$. Plugging this policy into (5.8), we obtain for all $\xi = (x, z_0, 0) \in \mathbb{E}$:

$$L(\xi, \pi) = \mathrm{E}_\xi^\pi \Big[\sum_{n=0}^{\infty} c(\xi_n, a_n) \Big] = f(x) + \mathrm{E}[\mathrm{e}^{-\alpha\tau_1} \varphi(X_1)] \leq f(x) + \|\varphi\| \, \mathrm{E}[\mathrm{e}^{-\alpha\tau_1}] < \infty, \tag{7.1}$$

where the last inequality is due to (6.2) as long with the transformation (5.27) and (5.31). We note that $L^* \leq L(\xi, \pi) < \infty$, which demonstrates the finiteness of the value function (5.9) when the initial distribution is deterministic.

Next, we establish a technical result useful to show our main result later. Consider the function $\hat{v}$ defined in (6.7). This function allows to simplify the notation for the DPE and related expressions.

Lemma 7.1. *Under Assumption 6.2, for each $\xi_0 = (x, z_0, 0)$ we have that:*

(a) *For all $n \in \mathbb{N}_0$ and $\pi \in \Pi$, the next holds true*

$$\hat{v}(\xi_n) \leq c(\xi_n, a_n) + \mathrm{E}_{\xi_0}^\pi \big[\hat{v}(\xi_{n+1}) \bigm| h_n, a_n \big]. \tag{7.2}$$

(b) *There exists a minimizer $g^*(\cdot, z_0) : \mathbb{O} \to \mathbb{A}$ of (6.8) such that $g^*_{z_0}(x) \in \mathbb{A}(x, z_0, 0)$ for all $x \in \mathbb{O}$.*

(c) *Let $\pi^* = \{g_n^*\}$ given by $g_n^*(h_n) := g^*(x_n, z_n)$ if $\xi_n = (x_n, z_n, t_n)$, and $g_n^*(h_n) := 1$ if $\xi_n = \partial$, for all $h_n = (\xi_0, a_0, \ldots, \xi_{n-1}, a_{n-1}, \xi_n) \in \mathbb{H}_n$ and $n \in \mathbb{N}_0$. Then, we have*

$$\hat{v}(\xi_n) = c(\xi_n, a_n) + \mathrm{E}_{\xi_0}^{\pi^*} \big[\hat{v}(\xi_{n+1}) \bigm| h_n, a_n \big] \quad \mathrm{P}_{\xi_0}^{\pi^*}\text{-a.s.} \quad \forall\, n \in \mathbb{N}_0. \tag{7.3}$$

(d) *For every $\pi \in \Pi$, we have*

$$\mathrm{E}_{\xi_0}^\pi \big[\hat{v}(\xi_n) \big] \to 0, \quad \text{as} \quad n \to \infty. \tag{7.4}$$

Proof. (a) From (6.8), if $\xi_n = (x_n, z_n, t_n)$, then we deduce

$$\mathrm{e}^{-\alpha t_n} \hat{v}(x_n, z_n, 0) \leq \mathrm{e}^{-\alpha t_n} c(x_n, z_n, 0, a_n) + \mathrm{e}^{-\alpha t_n} \int_{\mathbb{E}} \hat{v}(\xi) Q(d\xi \mid x_n, z_n, 0, a_n).$$

But in virtue of (5.2), we obtain

$$\begin{aligned}
&\mathrm{e}^{-\alpha t_n}\int_{\mathbb{E}} \hat{v}(\xi)Q(d\xi|x_n, z_n, 0, a_n)\mathbf{1}_{\{0\}}(a_n)\\
&= \int_{\mathbb{O}\times\mathbb{R}_+} \mathrm{e}^{-\alpha(t_n+t)}\hat{v}(y,0,0)\rho_t(\mathrm{d}y|x_n)\mu_{z_n}(\mathrm{d}t)\\
&= \int_{\mathbb{O}\times\mathbb{R}_+} \hat{v}(y,0,t_n+t)\rho_t(\mathrm{d}y \mid x_n)\mu_{z_n}(\mathrm{d}t)\\
&= \int_{\mathbb{E}} \hat{v}(\xi)Q(d\xi \mid x_n, z_n, t_n, a_n)\mathbf{1}_{\{0\}}(a_n) = \mathrm{E}^{\pi}_{\xi_0}\big[\hat{v}(\xi_{n+1}) \mid h_n, a_n\big]\mathbf{1}_{\{0\}}(a_n).
\end{aligned}$$

In the last equality above we have used (5.7). The cases $a_n = 1$ and $\xi_n = \partial$ are trivial. Thus we deduce (7.2)

(b) It is clear that the action set $\mathbb{A}$ in (5.1) is finite, so is $\mathbb{A}(\xi_0)$ for every $\xi_0 = (x, z_0, 0)$. Furthermore, the definition of the cost c (5.3), leads to ensuring that the mapping $a \mapsto c(\xi_0, a) + \int_{\mathbb{E}} \hat{v}(\xi)Q(d\xi \mid \xi_0, a)$ is continuous on $\mathbb{A}$ endowed with the discrete topology, then the result follows by the selection measure theorem in Proposition D.5 in [21].

(c) From (6.8) and part (b) we obtain

$$\hat{v}(\xi_n) = c(\xi_n, g^*_n(h_n)) + \int_{\mathbb{E}} \hat{v}(\xi)Q(d\xi \mid \xi_n, g^*_n(h_n))], \quad \forall\, n \in \mathbb{N}_0.$$

Then, following the steps in part (a) we get (7.3).

(d) The idea behind this is to use the operator $\mathcal{R}^{z_0}_\alpha : B_w(\mathbb{O}) \to B_w(\mathbb{O})$, defined by

$$\mathcal{R}^{z_0}_\alpha\, h(x) := \int_0^\infty \Phi_\alpha(t)h(x)\mu_{z_0}(\mathrm{d}t), \tag{7.5}$$

which maps $C^+_b(\mathbb{O})$ into itself by Theorem 6.4 and (6.11). First, it is easy to show that

$$\|\mathcal{R}^{z_0}_\alpha\, h\| \le \varrho^{z_0}_\alpha\, \|h\|\,, \text{ if } h \in B_b(\mathbb{O}). \tag{7.6}$$

Using (5.7), (5.2), (6.9) and (7.5) we get

$$\begin{aligned}
\mathrm{E}^{\pi}_{\xi_0}\big[\hat{v}(\xi_n)\big] &= \mathrm{E}^{\pi}_{\xi_0}\big[\mathrm{E}^{\pi}_{\xi_0}\big[\hat{v}(\xi_n) \bigm| h_{n-1}, a_{n-1}\big]\big]\\
&= \mathrm{E}^{\pi}_{\xi_0}\Big[\int_{\mathbb{E}} \hat{v}(\xi)Q(d\xi \mid \xi_{n-1}, 0)\mathbf{1}_{\{0\}}(a_{n-1})\Big]\\
&= \mathrm{E}^{\pi}_{\xi_0}\Big[\int_0^\infty \Phi_\alpha(t)\hat{v}(x_{n-1}, 0, t_{n-1})\mu_{z_{n-1}}(\mathrm{d}t)\mathbf{1}_{\{0\}}(a_{n-1})\Big]\\
&= \mathrm{E}^{\pi}_{\xi_0}\Big[\mathrm{e}^{-\alpha t_{n-1}}\int_0^\infty \Phi_\alpha(t)\hat{v}(x_{n-1}, 0, 0)\mu_{z_{n-1}}(\mathrm{d}t)\mathbf{1}_{\{0\}}(a_{n-1})\Big]\\
&\le \mathrm{E}^{\pi}_{\xi_0}\big[\mathrm{e}^{-\alpha t_{n-1}}\,\mathcal{R}^{z_{n-1}}_\alpha\, \hat{v}(x_{n-1}, 0, 0)\big].
\end{aligned}$$

As above, we can apply the same argument to the mapping

$$\xi_{n-1} = (x_{n-1}, z_{n-1}, t_{n-1}) \mapsto \mathrm{e}^{-\alpha t_{n-1}}\,\mathcal{R}^{z_{n-1}}_\alpha\, \hat{v}(x_{n-1}, 0, 0)$$

and get a similar inequality as above. Hence, iterating the latter inequality, we get

$$\mathrm{E}^{\pi}_{\xi_0}\big[\hat{v}(\xi_n)\big] \le \mathcal{R}^{z_0}_{\alpha}(\mathcal{R}^{0}_{\alpha})^{n-1}\hat{v}(x,0,0).$$

Since $\hat{v}(\cdot,0,0) = v(\cdot,0) \in C^{+}_{b}(\mathbb{O})$, we use (7.6) and get

$$\mathcal{R}^{z_0}_{\alpha}(\mathcal{R}^{0}_{\alpha})^{n-1}\hat{v}(x,0,0) \le (\varrho^{0}_{\alpha})^{n-1}\,\|\hat{v}(\cdot,0,0)\| \to 0, \quad \text{when} \quad n \to \infty,$$

due to $\varrho^{0}_{\alpha} < 1$.

□

The next result ensures that the value function L^* in (5.9) is the unique solution of the DPE (6.9) in the following sense.

Theorem 7.2. *Let Assumption 6.2 hold true. Then, for each $z_0 \in \mathbb{R}_+$, we have:*

(a) *The value function $L^*(\cdot, z_0, 0)$ in (5.9) is the unique nonnegative solution that satisfies the DPE (6.9); i.e.,*

$$L^*(x,z_0,0) = \inf_{a\in\mathbb{A}(x,z_0,0)}\Big\{c(x,z_0,0,a) + \int_0^{\infty}\Phi_{\alpha}(t)L^*(x,0,0)\mu_{z_0}(\mathrm{d}t)\mathbf{1}_{\{0\}}(a)\Big\}.$$

for all $x \in \mathbb{O}$.

(b) *The mapping $x \mapsto L^*(x,z_0,0)$ belongs to the space $C^{+}_{b}(\mathbb{O})$.*

(c) *There exists an optimal control policy $\pi^* \in \Pi$ for the optimal control problem (5.8)–(5.9), when the initial distribution is $\nu(\cdot) = \delta_{(x,z_0,0)}(\cdot)$.*

Proof. (a) Fixed $\pi \in \Pi$, we proceed to take expectation in (7.2) and then arrange terms to get

$$\mathrm{E}^{\pi}_{\xi_0}\big[\hat{v}(\xi_n)\big] \le \mathrm{E}^{\pi}_{\xi_0}\big[c(\xi_n,a_n) + \hat{v}(\xi_{n+1})\big].$$

Iterating the last expression, we obtain

$$\hat{v}(\xi_0) \le \mathrm{E}^{\pi}_{\xi_0}\Big[\sum_{k=0}^{n} c(\xi_k,a_k) + \hat{v}(\xi_{n+1})\Big]$$

$$\iff \hat{v}(\xi_0) - \mathrm{E}^{\pi}_{\xi_0}\big[\hat{v}(\xi_{n+1})\big] \le \mathrm{E}^{\pi}_{\xi_0}\Big[\sum_{k=0}^{n} c(\xi_k,a_k)\Big].$$

Taking the limit when $n \to \infty$ in the right-hand side inequality above, and then using (7.4), the nonnegativeness of v and c, and thus the monotone convergence, we obtain

$$\hat{v}(\xi_0) \le \mathrm{E}^{\pi}_{\xi_0}\Big[\sum_{k=0}^{\infty} c(\xi_k,a_k)\Big] = L(\xi_0,\pi).$$

Since $\pi \in \Pi$ was taken arbitrarily, we can deduce

$$v(\xi_0) \le L^*(\xi_0).$$

Now, consider the policy π^* as in Lemma 7.1 (c). Applying expectation to both sides but now to expression (7.3), we have

$$\mathrm{E}_{\xi_0}^{\pi^*}\big[\hat{v}(\xi_n)\big] = \mathrm{E}_{\xi_0}^{\pi^*}\big[c(\xi_n, a_n) + \hat{v}(\xi_{n+1})\big].$$

Again, iterating we obtain

$$\hat{v}(\xi_0) = \mathrm{E}_{\xi_0}^{\pi^*}\Big[\sum_{k=0}^{n} c(\xi_k, a_k) + \hat{v}(\xi_{n+1})\Big] = \mathrm{E}_{\xi_0}^{\pi^*}\Big[\sum_{k=0}^{n} c(\xi_k, a_k)\Big] + \mathrm{E}_{\xi_0}^{\pi^*}\big[\hat{v}(\xi_{n+1})\big].$$

Due to both the monotone convergence theorem and the fact that when $n \to \infty$ then $\mathrm{E}_{\xi_0}^{\pi^*}\big[\hat{v}(\xi_{n+1})\big] \to 0$, we obtain

$$\hat{v}(\xi_0) = \mathrm{E}_{\xi_0}^{\pi^*}\Big[\sum_{k=0}^{\infty} c(\xi_k, a_k)\Big] = L(\xi_0, \pi^*),$$

so we conclude

$$\hat{v}(\xi_0) = L^*(\xi_0).$$

(b) This part follows from Corollary 6.5.
(c) Use the policy $\pi^* \in \Pi$ defined in part (c) in Lemma 7.1. Then, by part (b) of this theorem, we know that $L(\xi_0, \pi^*) = L^*(\xi_0)$, which proves that π^* is optimal for the control problem (5.8)–(5.9). □

We have arrived at our main result concerning the regularity of the cost function u in (4.3) as well as the existence of an optimal stopping time η^* that minimizes J introduced in (4.2). The result is established in the sequel.

Theorem 7.3. *Under Assumption 6.2, the following assertions hold true for each $z_0 \in \mathbb{R}_+$*

(a) *J^* satisfies (6.9) and $J^*(\cdot, z_0)$ belongs to $C_b(\mathbb{O})$.*
(b) *There exists an admissible stopping time $\eta^* \in \mathbb{N}_0$ that minimizes (4.3); i.e., $J(x, z_0, \eta^*) = \inf_{\eta \in \mathbb{N}_0} J(x, z_0, \eta)$, for all $x \in \mathbb{O}$.*

Proof. Part (a) follows from Theorems 7.2(b) and 5.3, respectively; while part (b) is a direct consequence of Theorems 7.2(c) and 5.3, indeed the optimal stopping time is given by

$$\eta^* = \inf\{n \geq 0 \mid A_n^* = 1\} \tag{7.7}$$

where A_n^* comes from the equation (5.32) taking the optimal policy π^* in Theorem 7.2(c). □

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Appendix A.

This appendix shows some technical but important results for the theory developed in the manuscript. Most of this material is well-known; see for instance Kallenberg [23], Blumenthal *et al.* [24], and Tudor [25].

A.1 Monotone class theorems

Consider a nonempty set $\mathbb{X}$.

A π-system on $\mathbb{X}$ is a collection of subsets of $\mathbb{X}$ that is closed under finite intersections. A λ-system (on $\mathbb{X}$) is a collection of subsets of $\mathbb{X}$ that contains the total set $\mathbb{X}$, and is closed under proper differences and increasing sequences; that is, if A, B are in a λ-system and $A \subset B$ then $B \setminus A$ is in the same λ-system, and if $\{A_n\}_{n\in\mathbb{N}}$ is in a λ-system and $\mathbf{1}_{A_n} \uparrow \mathbf{1}_A$, where $A = \cup_n A_n$ pointwise, then A is in the same λ-system.

Lemma A.1. *Given a set $\mathbb{X}$, let $\mathcal{C}$ be a π-system and $\mathcal{D}$ be a λ-system both subsets of $2^{\mathbb{X}}$. Suppose that $\mathcal{C} \subset \mathcal{D}$, then $\sigma(\mathcal{C}) \subset \mathcal{D}$.*

Proof. See Theorem 1.1 in [23], p. 10. □

Theorem A.2. *Let $\mathcal{M} \subset 2^{\mathbb{X}}$ and H be a vector space of real-valued functions over $\mathbb{X}$. Let us assume that*

(a) *$\mathcal{M}$ is a non-empty π-system.*
(b) *$\mathbf{1} \in H$ and $\mathbf{1}_B \in H$, for all $B \in \mathcal{M}$.*
(c) *If f_n is an increasing sequence of nonnegative functions in H such that $f = \lim_n f_n$ is finite (resp. bounded), then $f \in H$.*

Under these assumptions H contains all real-valued $\sigma(\mathcal{M})$-measurable (resp. bounded) functions on $\mathbb{X}$.

Proof. See Theorem 2.3 in [24], p. 6. □

Lemma A.3. *Let $\mathbb{X}$ be a metric space. Then for any open set B in $\mathbb{X}$, there exists a sequence of continuous functions over $\mathbb{X}$ such that $f_n \to \mathbf{1}_B$.*

Proof. For any non-empty open set $B \subseteq \mathbb{X}$ we define

$$d(x, B) := \inf_{b\in B} d(x, b).$$

Then $x \mapsto d(x, B)$ is continuous since $|d(x, B) - d(y, B)| \leq d(x, y)$. Let us assume that $B \neq \mathbb{X}$. Let $D_n = \{x \in \mathbb{X} : d(x, \mathbb{X} \setminus B) \geq \frac{1}{n}\}$, then we have $D_n \subseteq D_{n+1}$. It is also true that $B = \bigcup_{n=1}^{\infty} D_n$ since B is open and D_n is closed for all $n \in \mathbb{N}$. The continuous functions

$$f_n(x) = \frac{d(x, X \setminus B)}{d(x, X \setminus B) + d(x, D_n)}$$

satisfy $0 \leq f_n \leq 1$ and converge pointwise and monotonically to $\mathbf{1}_B$. □

Corollary A.4. *Let $\mathbb{X}$ be a metric space and H a vector space of measurable (resp. bounded measurable) maps $f : \mathbb{X} \to \mathbb{R}$ such that:*

(1) *H contains the set $C_b(\mathbb{X})$.*
(2) *If $0 \le f_n \uparrow f$, $f_n \in H$, f finite (resp. bounded), then $f \in H$.*

Hence, H contains all measurable (resp. bounded measurable) maps $f : (\mathbb{X}, \mathcal{B}_{\mathbb{X}}) \to \mathbb{R}$.

Proof. It follows by Theorem A.2 and Lemma A.3. □

The following Proposition A.5 and Theorem A.6 can be found in [24], p. 6. The preamble is the following: let $\mathbb{X}$ be a set and $(\mathbb{S}_i, \mathcal{S}_i)_{i\in I}$ be a family of measurable spaces indexed by an arbitrary set I. For each $i \in I$, let $\mathcal{M}_i$ be a π-system generating $\mathcal{S}_i$ and let f_i be a map from $\mathbb{X}$ to $\mathbb{S}_i$. Using this setup we state the following two results.

Proposition A.5. *Let $\mathcal{M}$ be a collection consisting of all sets of the following form $\bigcap_{i\in J} f_i^{-1}(A_i)$ where $A_i \in \mathcal{M}_i$ for $i \in J$ and J ranges over all finite subsets of I. Then $\mathcal{M}$ is a π-system on $\mathbb{X}$ and $\sigma(\mathcal{M}) = \sigma(f_i : i \in I)$.*

Theorem A.6. *Let $\mathbb{X}$ be a nonempty set, $(\mathbb{X}_\alpha, \mathcal{F}_\alpha)$ a family of measurable spaces, $f_\alpha : \mathbb{X} \to \mathbb{X}_\alpha$, for all $\alpha \in I$ and H a vector space of functions $f : \mathbb{X} \to \mathbb{R}$ satisfying the following conditions:*

(a) $\mathbf{1} \in H$
(b) *If $0 \le f_n \uparrow f$, $f_n \in H$, f finite (resp. bounded), then $f \in H$.*
(c) *For each $\alpha \in I$ there is a π-system $\mathcal{M}_\alpha \subset \mathcal{F}_\alpha$ with $\sigma(\mathcal{M}_\alpha) = \mathcal{F}_\alpha$ and H contains the functions of the form*

$$\prod_{\alpha\in J} \mathbf{1}_{A_\alpha} \circ f_\alpha, \quad J \subset I, J \text{ finite and } A_\alpha \in \mathcal{M}_\alpha. \tag{A.1}$$

Then H contains the measurable (resp. bounded) functions $f : (\mathbb{X}, \sigma(f_\alpha : \alpha \in I)) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

A.2 Canonical Markov processes

Given a measurable space $(\mathbb{S}, \mathcal{S})$, and T a subset of $\mathbb{R}_+$, let $\mathbb{S}^T$ be the product set $\prod_{t\in t} \mathbb{S}$ and denote by $\mathcal{B}_{\mathbb{S}}^T$ the σ-algebra generated by finite cylinders of type: $\prod_{t\in T} B_t$, $B_t \in \mathcal{S}$, and $\{t \in T \mid B_t \neq \mathbb{S}\}$ is finite. In addition, let $\pi_t : \mathbb{S}^T \to \mathbb{S}$ be the mapping, a.k.a. projection mapping, given by $\pi_t(\omega) = \omega(t)$ for all $\omega \in \mathbb{S}^T$ and $t \in T$.

The following results are mainly part of Tudor [25]. For self-contained purposes, however, we have included the sketch of their respective proofs.

We denote by $\sigma(\pi_t : t \in T)$ the minimal σ-algebra that makes measurable all the projections π_t for $t \in T$. We can see that this σ-algebra is generated by the family $\{\pi_t^{-1}(B) \mid B \in \mathcal{S}, t \in T\}$.

Lemma A.7. *The σ-algebra $\mathcal{B}_{\mathbb{S}}^T$ coincides with the σ-algebra generated by the projection mappings $\{\pi_t \colon t \in T\}$.*

Proof. The inclusion $\sigma(\pi_t : t \in T) \subseteq \mathcal{B}_{\mathbb{S}}^T$ holds true because $\pi_t^{-1}(B) = \{\omega \in \mathbb{S}^T : \omega(t) \in B\} = \prod_{s\in T} B_s$, where $B_s = \mathbb{S}$ for all $s \neq t$ and $B_t = B$, is a finite cylinder for any $t \in T$ and $B \in \mathcal{S}$. On the other hand, for any finite cylinder $\prod_{s\in T} B_t$ in $\mathcal{B}_{\mathbb{S}}^T$ with $B_t = \mathbb{S}$ for all $t \neq t_1, \ldots, t_n$ we have that $\prod_{s\in T} B_t = \pi_{t_1}^{-1}(B_{t_1}) \cap \ldots \cap \pi_{t_n}^{-1}(B_{t_n})$ which is in $\sigma(\pi_t : t \in T)$. Therefore, $\sigma(\pi_t : t \in T) = \mathcal{B}_{\mathbb{S}}^T$. □

Lemma A.8. *Fix some set $\mathbb{X}$ and a σ-algebra $\mathcal{G}$ over $\mathbb{X}$. Let $B \subset \mathbb{X}$ and $\mathcal{G}_B$ be the σ-algebra over B given by $\mathcal{G}_B = \{B \cap A : A \in \mathcal{G}\}$, which is commonly called the trace of $\mathcal{G}$ on B. If $\mathcal{G}$ is generated by a family $\mathcal{C}$ of subsets of $\mathbb{X}$ that contains $\mathbb{X}$, then $\mathcal{G}_B$ is generated by $\mathcal{C} \cap B = \{B \cap C \mid C \in \mathcal{C}\}$.*

Proof. We clearly have that $\mathcal{C} \cap B \subseteq \mathcal{G}_B$. On the other hand, let $\mathcal{E} = \{A \subseteq X \mid A \cap B \in \sigma(\mathcal{C} \cap B)\}$, then $\mathcal{E}$ is a σ-algebra, in fact, $X \cap B \in \mathcal{C} \cap B \subseteq \sigma(\mathcal{C} \cap B)$ since $X \in \mathcal{C}$. On the other hand, given $A \in \mathcal{E}$ we have $A^c \cap B = B \setminus (A \cap B)$ that is in $\sigma(\mathcal{C} \cap B)$. Finally, the property $\cup_{n\in\mathbb{N}} A_n \in \mathcal{E}$ for any sequence $A_n \in \mathcal{E}$, follows from the distributive property between unions and intersections and the fact $B \cap A_n \in \sigma(\mathcal{C} \cap B)$. Now, we note that $\mathcal{G} = \sigma(\mathcal{C}) \subseteq \mathcal{E}$ since $\mathcal{C} \subseteq \mathcal{E}$ and $\mathcal{E}$ is a σ-algebra, and so, $\mathcal{G}_B = \mathcal{G} \cap B \subseteq \sigma(\mathcal{C} \cap B)$ and we conclude the result. □

Let us consider the case $T = \mathbb{N}$. Then by Lemma A.7 we get $\mathcal{B}_{\mathbb{S}}^{\mathbb{N}} = \sigma(\pi_t : t \in \mathbb{N})$. Hence, $\mathcal{B}_{\mathbb{S}}^{\mathbb{N}}$ is generated by $\{\pi_t^{-1}(B) \mid t \in \mathbb{N}, B \in \mathcal{B}_{\mathbb{S}}\}$ but this family is not a π-system, in aim to show the next lemma, we consider the following π-system

$$\mathcal{C}_0^{\pi} := \Big\{ \bigcap_{n=1}^{k} \pi_{t_n}^{-1}(B_n) \;\Big|\; k \in \mathbb{N},\, t_n \in \mathbb{N},\, B_n \in \mathcal{S},\, \forall\, n = 1, \dots, k \Big\}.$$

Given a discrete-time stochastic process $\{X_n\}_{n\in\mathbb{N}}$ with state space $(\mathbb{S}, \mathcal{S})$ we define the map $\psi_X : (\Omega, \mathcal{F}) \to (\mathbb{S}^{\mathbb{N}}, \mathcal{B}_{\mathbb{S}}^{\mathbb{N}})$ given by $\psi_X(\omega) = (X_1(\omega), X_2(\omega), \dots) \in \mathbb{S}^{\mathbb{N}}$.

Lemma A.9. *The mapping ψ_X is measurable.*

Proof. Let $\mathcal{D} := \big\{ B \in \mathcal{B}_{\mathbb{S}}^{\mathbb{N}} \;\big|\; \psi_X^{-1}(B) \in \mathcal{F} \big\}$. This set is a λ-system, indeed it is a σ-algebra for basic properties of inverse image. We will show that $\mathcal{C}_0^{\pi} \subseteq \mathcal{D}$. Let $k \in \mathbb{N}$, $t_1, \dots, t_k \in \mathbb{N}$ and $B_1, \dots, B_k \in \mathcal{S}$. We have that

$$\psi_X^{-1}\Big(\bigcap_{n=1}^{k} \pi_{t_n}^{-1}(B_n)\Big) = \bigcap_{n=1}^{k} \psi_X^{-1}\big(\pi_{t_n}^{-1}(B_n)\big) = \bigcap_{n=1}^{k} \big(\pi_{t_n} \circ \psi_X\big)^{-1}(B_n) = \bigcap_{n=1}^{k} X_{t_n}^{-1}(B_n)$$

which is in $\mathcal{F}$ by the measurability of X_{t_n}. Therefore $\mathcal{C}_0^{\pi} \subseteq \mathcal{D}$. The required contention $\mathcal{B}_{\mathbb{S}}^{\mathbb{N}} \subseteq \mathcal{D}$ follows by the equality $\mathcal{B}_{\mathbb{S}}^{\mathbb{N}} = \sigma(\mathcal{C}_0^{\pi})$ and by Lemma A.1. □

A.3 Conditional expectation and stopping times

The following lemma, taken from [23], Lemma 8.3, p. 166, is known as the local property for the conditional expectation and it will be useful to simplify some proofs. Given two σ-algebras $\mathcal{G}$ and $\mathcal{H}$, we say that $\mathcal{G} = \mathcal{H}$ on B if $B \in \mathcal{G} \cap \mathcal{H}$ and $B \cap \mathcal{G} = B \cap \mathcal{H}$.

Lemma A.10. *Consider two sub-σ-algebras $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ and random variables ξ, η such that $\mathrm{E}\,|\xi| \vee \mathrm{E}\,|\eta| < \infty$, and let $F \in \mathcal{G} \cap \mathcal{H}$. Then*

$$\mathcal{G} = \mathcal{H},\ \xi = \eta \quad a.s.\ on \quad F \quad \Longrightarrow \quad \mathrm{E}[\xi \mid \mathcal{G}] = \mathrm{E}[\eta \mid \mathcal{H}] \quad a.s.\ on \quad F.$$

We define a new filtration $\{\mathcal{F}_t^+\}_{t\geq 0}$ by $\mathcal{F}_t^+ := \bigcap_{s>t} \mathcal{F}_t$, and say that $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous if $\mathcal{F}_t = \mathcal{F}_t^+$ for all $t \geq 0$. We say that a nonnegative random variable τ is weakly $\mathcal{F}$-stopping time if $\{\tau < t\} \in \mathcal{F}_t$ for every $t > 0$. The notions of stopping time and weakly stopping time agree when $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous:

Lemma A.11. *A nonnegative random variable τ is weakly stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if and only if it is stopping time with respect to $\{\mathcal{F}_t^+\}_{t\geq 0}$.*

The following lemma is based on Lemma 9.4 in [23], p.188.

Lemma A.12. *For any nonnegative random variable τ with values in $\bar{\mathbb{R}}_+$, there exists a sequence of countably valued nonnegative random variables τ_n such that $\tau_n \downarrow \tau$ pointwise. Moreover, τ_n is measurable with respect to $\sigma(\tau)$ for all n. Besides, if τ is a stopping time with respect to $\{\mathcal{F}_t\}_{t\in\bar{\mathbb{R}}_+}$ which we assume to be right-continuous and $\mathcal{F}_\infty := \{\sigma(\mathcal{F}_t) : t \geq 0\}$, then every τ_n is also a stopping time w.r.t. $\{\mathcal{F}_t\}_{t\in\bar{\mathbb{R}}_+}$.*

Proof. Let

$$\tau_k := 2^{-k}\lfloor 2^k\tau + 1\rfloor \in 2^{-k}\mathbb{N}_0 \cup \{\infty\}, \quad k \in \mathbb{N}_0.$$

For $m \in \mathbb{N}_0$ we have that

$$\tau_k = 2^{-k}\lfloor 2^k\tau + 1\rfloor \leq 2^{-k}m \iff \lfloor 2^k\tau + 1\rfloor \leq m \iff 2^k\tau < m \iff \tau < 2^{-k}m.$$

Hence, $\{\tau_k \leq 2^{-k}m\} = \{\tau < 2^{-k}m\}$. Now, choose $m = \infty$ then $\{\tau_k = \infty\} = \{\tau = \infty\}$ which implies that τ_k is $\sigma(\tau)$-measurable. But if τ is stopping time with respect to $\{\mathcal{F}_t\}_{t\in\bar{\mathbb{R}}_+}$ then by Lemma A.11 we get $\{\tau_k \leq 2^{-k}m\} = \{\tau <$

$2^{-k}m\} \in \mathcal{F}_{2^{-k}m}$, and $\{\tau_k \leq \infty\} = \cup_{m\in\mathbb{N}_0}\{\tau_k \leq 2^{-k}m\} \cup \{\tau_k = \infty\} = \cup_{m\in\mathbb{N}_0}\{\tau < 2^{-k}m\} \cup \{\tau = \infty\}$ and so τ_k is also stopping time for all $k \in \mathbb{N}_0$. On the other hand, if $x \in \mathbb{R}_+$ then we have that $\lfloor x+1 \rfloor = x+1 > x = \lceil x \rceil$ and if $x \in (m, m+1)$ for some $m \in \mathbb{N}$ then $\lfloor x+1 \rfloor = m+1 = \lceil x \rceil$, both cases imply that $\lfloor x+1 \rfloor \geq \lceil x \rceil$. Using this we get that

$$\tau_k = 2^{-k}\lfloor 2^k\tau + 1 \rfloor \geq 2^{-k}\lceil 2^{-k}\tau \rceil \geq \tau.$$

Therefore, on $\{\tau < \infty\}$ we obtain

$$0 \leq \tau_k - \tau = \frac{1}{2^k}\left(\lfloor 2^k\tau + 1 \rfloor - 2^k\tau\right) \leq \frac{1}{2^k} \to 0, \quad \text{when} \quad k \to \infty.$$

Besides, if $\tau = \infty$ then we just get $\tau_k = \infty$. □

A.4 Progressively measurable processes

We say that a process X is progressively measurable or simply progressive if its restriction to $\Omega \times [0,t]$ is $\mathcal{F}_t \otimes \mathcal{B}_{[0,t]}$-measurable for every $t \geq 0$. Note that any progressive process is adapted. Conversely, we may approximate from the left or right to show that any adapted and left or right-continuous process is progressive as is stated in the following proposition taken from [26] (Prop. 1.13, p. 5), whose proof assumes a state space given by $\mathbb{R}^d$ but that can be adapted for a general metric state space.

Proposition A.13. *If the stochastic process X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and every sample path is right-continuous, then X is also progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.*

Lemma A.14. *Consider a filtration $\{\mathcal{F}_t\}_{t\in T}$ and a process X with index set $T \subseteq \mathbb{R}_+ \cup \{\infty\}$ and values in a measurable space $(\mathbb{S}, \mathcal{S})$, also, let τ a stopping time in T. Then X_τ is $\mathcal{F}_\tau$-measurable under each of these conditions:*

(1) T is countable and X is adapted,
(2) $T = \mathbb{R}_+$ and X is progressive.

Proof. See Lemma 9.5 in [23], p. 189. □

Corollary A.15. *Let X be an adapted and right-continuous process and τ a stopping time in $\mathbb{R}_+$ both with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then X_τ is $\mathcal{F}_\tau$-measurable.*

Proof. It follows by Proposition A.13 and Lemma A.14. □

A.5 Miscelaneous results

Lemma A.16. *The following inequality holds for every measurable functions f, g and h in $B(\mathbb{O})$:*

$$|f \wedge h - g \wedge h| \leq |f - g|.$$

Proof. We have both $-|f-g| + g \wedge h \leq f - g + g = f$ and $-|f-g| + g \wedge h \leq h$ that imply $-|f-g| + g \wedge h \leq f \wedge h$. Analogously, we have $-|f-g| + f \wedge h \leq g \wedge h$, and joining the two obtained inequalities we get $|f \wedge h - g \wedge h| \leq |f-g|$. □