ESAIM: COCV 31 (2025) 91
https://doi.org/10.1051/cocv/2025077

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ACCESSIBILITY CONDITION FOR DISCRETE-TIME LINEAR SYSTEMS ON LIE GROUPS

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Abstract. In this paper, we characterize the accessibility condition for discrete-time linear systems on Lie groups and establish a sufficient condition for local controllability. Our approach extends the concept of the Fréchet derivative to the setting of Lie groups. We construct a subalgebra based on the infinitesimal automorphisms of the system and show that if the dimension of this subalgebra is maximal, then the system is accessible. Additionally, we examine an application concerning the existence of control sets and provide illustrative examples.

Mathematics Subject Classification. 93B03, 93B05, 93C05, 22E99.

Received July 2, 2024. Accepted September 17, 2025.

1. Introduction

In general, a control system can be regarded as a collection $(\mathbb{T}, M, U, \mathcal{U}, \varphi)$ consisting of a time set $\mathbb{T}$, a state space M, a control range U, a set of admissible controls $\mathcal{U}$, and a transition map φ, which describes the behavior of a dynamical system within the state space M under the influence of the controls (see *e.g.* Sontag [1]). One of the most fundamental concepts in control system theory is controllability, which means that the entire M can be reached by trajectories (in positive time) of the control system from any starting point. For the special case of linear control systems on $\mathbb{R}^n$ there exists a straightforward algebraic criterion for deciding controllability. In contrast, the currently available algebraic necessary and sufficient conditions for nonlinear systems remain incomplete and far more complex (see *e.g.* Elliot [2]). Moreover, even if such conditions are eventually established for the nonlinear case, they are expected to be quite difficult to verify (see Kawski [3] and Sontag [4]). Nevertheless, there are several important partial results (see Ayala and Da Silva [5], Colonius and Kliemann [6], Sontag [1] and references therein). Many of these results concern weaker notions of controllability. Two such weaker controllability concepts are local controllability, where the system can be steered from an initial state to any neighboring point and back, and the accessibility condition, which means that the system can be steered from an initial state to a full-dimensional final set, or equivalently, that the set of reachable points (from any initial state) has nonempty interior (see Dıaz-Seoane *et al.* [7] and Sontag [1]). Recall the fundamental relationship between accessibility and controllability: if a system is controllable, the trajectories (reachable sets)

Keywords and phrases: Discrete system, accessibility, reachable set, Lie groups.

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are all open, which implies accessibility. However, accessibility alone does not guarantee controllability; a system may be accessible without being controllable.

In the context of continuous-time linear systems on Lie groups (see (1.1) below), several studies from the 2000s began investigating conditions for accessibility and controllability (see, *e.g.*, Ayala and Tirao [8]; Ayala and Da Silva [5, 9]; Da Silva [10]; Do Rocio *et al.* [11]; Jouan [12]; and Ayala and San Martin [13]). A key concept related to controllability is the ad-rank condition (see *e.g.* [8]): consider the continuous-time linear systems on a Lie group G

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} X_j(g(t))u_j(t), \tag{1.1}$$

where the controls $u(t) = (u_1(t), u_2(t), ..., u_m(t))$ are piecewise constant maps with values in a compact neighborhood U of $0 \in \mathbb{R}^m$, $\mathcal{X}$ is a vector field on G contained in the normalizer of its Lie algebra $\mathfrak{g}$, and X_j are right-invariant vector fields. Denote the solution of the system starting at $g \in G$ by $\varphi(t, g, u)$. The reachable set from identity $e \in G$ at time $t > 0$ and the reachable set from e are given, respectively, by $\mathcal{R}_t(e) := \{\varphi(t, e, u) \mid u = (u_1, \dots, u_m)\}$ and $\mathcal{R} := \bigcup_{t>0} \mathcal{R}_t(e)$. Define the derivation of $\mathfrak{g}$ by $\mathcal{D}(X) = [\mathcal{X}, X]$ and let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\{X_j : j = 1, \dots, m\}$. The system (1.1) is said to satisfy the ad-rank condition if the vector subspace $V = \text{span}\{\mathcal{D}^j(X_i(e)) : i = 1, \dots, m, j \in \mathbb{N}_0\}$ coincides with $\mathfrak{g}$ (see [10]). In the setting of local controllability from $e \in G$ (*i.e.*, supposing $e \in \text{int}\mathcal{R}$), a well-known result (Thm. 3.5 in [8]) shows that if the system (1.1) satisfies the ad-rank condition, then it is locally controllable from e. If, instead of constraining the controls to lie within U, one allows them to take any value in $\mathbb{R}^m$—a setting referred to as the unbounded or unconstrained system—then a stronger criterion applies: if the subspace $\text{span}\{\mathcal{D}^j\mathfrak{h}; j \in \mathbb{N}_0\}$ coincides with $\mathfrak{g}$, then the system is locally controllable from e (see *e.g.* [12], Proposition 6. Some authors refer to this stronger condition as the ad-rank condition instead of the former.

More recently, Colonius *et al.* [14] introduced a discrete-time version of the above system and Cavalheiro *et al.* studied the controllability of these systems in [15] and [16]. The discrete-time linear system presented in [14] is a special case of the following well-known family of dynamical systems on a manifold M:

$$x_{k+1} = f(u_k, x_k), k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{1.2}$$

for which $f : U \times M \longrightarrow M$ is a map defined on a nonempty compact neighborhood U of 0 in $\mathbb{R}^m$. But if we consider the linear system on $\mathbb{R}^n$,

$$x_{k+1} = Ax_k + Bu_k, \tag{1.3}$$

where $A \in \text{Gl}(n, \mathbb{R}), B \in \mathbb{R}^{n\times m}$ and $U = \mathbb{R}^m$. Then the condition for controllability is well-known and the main result proved by Kalman, Ho, and Narendra [17] states that the controllability of the system (1.3) is equivalent to the requirement that the matrix $\mathcal{K} = (A^{n-1}B \ \dots \ AB \ B)$ has full rank. This condition, known as Kalman condition, also applies to the continuous-time version of the system (1.3).

In this paper, we focus on a special case of the system (1.2), the discrete-time linear systems on a connected Lie group G as introduced in [14], defined by:

$$g_{k+1} = f(u_k, g_k), k \in \mathbb{N}_0, \tag{1.4}$$

where $f : U \times G \longrightarrow G$ satisfies the following properties

1. $f(u, g) = f(u, e)f(0, g)$, for every $g \in G$ and $u \in U$,
2. The map $f_0 := f(0, \cdot) : G \longrightarrow G$ is an automorphism.

Note that this system is a discrete-time version of the continuous-time linear systems on a Lie group G (see *e.g.* [5] and references therein) and generalizes the system (1.3). In the continuous-time setting on a nilpotent Lie group, if the system is regular, that is, $\det(\mathcal{D}) \neq 0$, then accessibility is sufficient to ensure the existence of subsets with nonempty interior and approximate controllability, known as control sets (see [18]). For the system (1.4), however, stronger conditions are required to guarantee the existence of such sets (see [16]).

A result characterizing the accessibility for discrete control systems in a more general setting was proven by Jakubcyk and Sontag [19], in terms of the partial derivatives of the map f. Expanding on this foundation, we present the following two principal contributions in this work. The first characterizes the accessibility for a system using either the smallest subgroup containing the set $f(U, e)$ or the derivatives of the map $u \mapsto f(u, e)$ (see Thms. 3.9 and 3.18). The second result concerns local controllability from identity e providing a result analogous to the ad-rank condition from the constrained case, now adapted for discrete-time linear systems (see Thm. 3.20). To state and prove both results, we introduce a new notion of derivative for maps between Lie groups, using right-invariant vector fields as a reference. This derivative is compatible with common operations on Lie groups—such as left and right translations and group homomorphisms—making it particularly well-suited for computations in this setting. Our approach and results allow us to characterize accessibility through simpler conditions that align more closely with Lie theory than those employed in [19].

This paper is organized as follows. In Section 2, we introduce the foundational concepts of control systems, providing the main definitions and essential background. Section 3 presents the main results, beginning with the basic properties of linear systems on Lie groups. We then introduce the derivative $\hat{d}$, discuss its properties, and use it to prove the theorems on accessibility and the ad-rank condition. In Section 3, we illustrate the application of these results through examples involving the Heisenberg group, the special linear group $\mathrm{Sl}(2, \mathbb{R})$, and the affine two-dimensional Lie group $\mathrm{Aff}(2, \mathbb{R})$.

Notation. To avoid overcomplicating the notation with excessive indices, we frequently use the letter u (or v and w) to represent both individual elements of the control range U and control sequences $u = (u_i)_{i\in\mathbb{N}_0} \in \mathcal{U} := \prod_{i\in\mathbb{N}} U$. This slight abuse of notation will always be clear from context.

2. Preliminaries

In this section, we define key properties of discrete-time control systems and establish the terminology that will be used throughout the paper. We consider the discrete-time control system (1.2), where M is a smooth n-dimensional manifold and the control range $U \subset \mathbb{R}^m$ is bounded with connected interior satisfying $U \subset \overline{\mathrm{int}U}$. Additionally, we assume that U is either open or compact. Furthermore, for an open set $\hat{U}$ containing U, we require that $f : \hat{U} \times M \to M$ be of class $\mathcal{C}^\infty$. We also assume that for each $u \in U$, the map $f_u : M \to M$ is invertible, meaning that its inverse $f_u^{-1} : M \to M$ is well-defined.

Given an initial condition $x_0 \in M$ and considering the set $\mathcal{U} = \prod_{i\in\mathbb{N}} U$, the solution $\varphi : \mathbb{N}_0 \times M \times \mathcal{U} \longrightarrow M$ of the system (1.2) is given by

$$\varphi(k, x_0, u) = \begin{cases} f_{u_{k-1}} \circ \cdots \circ f_{u_0}(x_0), & k > 0 \\ x_0, & k = 0 \end{cases}.$$

Take the map $\Theta : \mathbb{N}_0 \times \mathcal{U} \longrightarrow \mathcal{U}$, defined by $\Theta_k((u_i)_{i\in\mathbb{N}}) = (u_{i+k})_{i\in\mathbb{N}}$, then the solution φ satisfies the cocycle property, which means that $\varphi(k+t, x, u) = \varphi(k, \varphi(t, x, u), \Theta_t(u)) = \varphi(t, \varphi(k, x, u), \Theta_k(u)), \forall k, t \in \mathbb{N}_0$. The solution φ also satisfies the following property: given $t, s \in \mathbb{N}$ $u, v \in \mathcal{U}$, there is a $w \in \mathcal{U}$ such that $\varphi(t, \varphi(s, x, u), v) = \varphi(t + s, x, w), \forall x \in M$.

Definition 2.1. For $x \in M$, the set of points reachable and controllable from x up to time $k \in \mathbb{N}_0$ are

$$\mathcal{R}_k(x) = \{y \in M : \text{ there is } u \in \mathcal{U} \text{ with } \varphi(k, x, u) = y\},$$
$$\mathcal{C}_k(x) = \{y \in M : \text{ there is } u \in \mathcal{U} \text{ with } \varphi(k, y, u) = x\},$$

respectively. The sets $\mathcal{R}(x) = \bigcup_{k\in\mathbb{N}_0} \mathcal{R}_k(x)$ and $\mathcal{C}(x) = \bigcup_{k\in\mathbb{N}_0} \mathcal{C}_k(x)$ denote, respectively, the reachable and the controllable set from x.

Hence, we have the following definition.

Definition 2.2. The system (1.2) is said to be forward accessible (respectively, backward accessible) if $\text{int}\mathcal{R}(x) \neq \emptyset$ (resp. $\text{int}\mathcal{C}(x) \neq \emptyset$), for all $x \in M$. It is called accessible if both conditions are satisfied.

Another important definition in this work is the ad-rank condition for the system (1.2). However, since its introduction requires some preliminary constructions, we present this definition later (see the following Def. 3.7).

One of the most important concepts in control system theory is controllability.

Definition 2.3. We say that the system (1.2) is controllable if for any $x, y \in M$, there are $k \in \mathbb{N}$ and $u \in \mathcal{U}$ such that $\varphi(k, x, u) = y$.

When global controllability is not achievable, one may instead consider local controllability or identify maximal regions where approximate controllability holds. These notions also play a significant role in control theory (see *e.g.* [6] and [20]).

Definition 2.4. The system (1.2) is locally controllable from $x \in M$ if the set $\mathcal{R}(x)$ is a neighborhood of x.

Definition 2.5. Consider the control system (1.2). A control set D is a set satisfying the following properties

1. For any $x \in D$, there is an $u \in \mathcal{U}$ such that $\varphi(k, x, u) \in D$, for any $k \in \mathbb{N}$;
2. $D \subset \overline{\mathcal{R}(x)}$, for any $x \in D$;
3. D is maximal with such properties.

In the remainder of this section, we present two results regarding the systems considered in this study. Consider the discrete-time linear system (1.4). In this case the map f can be defined using the translations of G. Recall that given $g \in G$, the left and right translations $L_g : G \to G$ and $R_g : G \to G$ are defined by $L_g(h) = gh$ and $R_g(h) = hg$, respectively. Then for $u \in U$, as $f_u(e) \in G$, we can write $f_u(g)$ as

$$f_u(g) = f_u(e)f_0(g) = L_{f_u(e)}(f_0(g)), \tag{2.1}$$

where $L_{f_u(e)}$ is the left translation by the element $f_u(e)$. Considering the expression above, the inverse of f_u is given by

$$(f_u)^{-1}(g) = f_0^{-1} \circ L_{(f_u(e))^{-1}}(g) = f_0^{-1}((f_u(e))^{-1} \cdot g). \tag{2.2}$$

Thus we conclude that f_u is a diffeomorphism of G, for any $u \in U$. The solutions at $g \in G$ can also be defined in terms of translations of the solution at the neutral element by the automorphism f_0, see the next proposition proved in (Colonius, *et al.* [14]).

Proposition 2.6. *Consider the above discrete-time linear system defined on a Lie group G. Then for all $g \in G$ and $u = (u_i)_{i\in\mathbb{N}_0} \in \mathcal{U}$*

$$\varphi(k, g, u) = \varphi(k, e, u)f_0^k(g).$$

For the sake of completeness, we present the following relationship between the system and its time-reversed counterpart

$$g_{k+1} = \hat{f}_{u_k}(g_k), k \in \mathbb{N}_0, \tag{2.3}$$

given by the map $\hat{f}_u(g) = f_u^{-1}(g) = f_0^{-1}(f_u^{-1}(e)g)$. Denote by $\mathcal{R}_k^*$ and $\mathcal{C}_k^*$ its reachable and controllable sets from e up to time k of the system (2.3), then Lemma 14 in [15] establishes the following result.

Lemma 2.7. *It holds that* $\mathcal{R}_k^* = \mathcal{C}_k$ *and* $\mathcal{R}_k = \mathcal{C}_k^*$, *for all* $k \in \mathbb{N}$.

3. Conditions for accessibility

In this section, we introduce key notations for the remainder of the paper and present our main results, *i.e.*, we prove necessary and sufficient conditions for accessibility of the discrete-time linear system (1.4). Denote by $\mathfrak{g}$ the Lie algebra of G and take U a neighborhood of $0 \in \mathbb{R}^m$. We assume U to be either open or compact. As said before, the map $f_u : G \longrightarrow G$ is a diffeomorphism for any $u \in U$ and f_0 is an automorphism of G. Since this system is a special case of the system (1.2), the construction of the solution is similar and it is also denoted by φ. Note that the system is defined for any $k \in \mathbb{Z}$. Recall the Lie bracket of $\mathfrak{g}$

$$[X, Y] = \frac{\partial^2}{\partial t \partial s} \left(X_{-t} \circ Y_s \circ X_t \right) \Big|_{t=s=0} \tag{3.1}$$

where X_t and Y_t are the solutions of X and Y at the time $t \in \mathbb{R}$, respectively. It is well-known (see [21]) that $[X, Y] = 0$ if, and only if, $\exp(tX)\exp(sY) = \exp(sY)\exp(tX)$ for all $t, s \in \mathbb{R}$.

Our first result regarding accessibility involves the concept of reachability, then we discuss some properties of this concept. Considering the reachable sets $\mathcal{R}_k(e)$, $\mathcal{R}_{\leq k}(e) = \{\varphi(t, e, u) : t \in [0, k] \cap \mathbb{N}, u \in \mathcal{U}\}$ and $\mathcal{R}(e) = \bigcup_{k \in \mathbb{N}} \mathcal{R}_k(e)$, it is easy to see that $e \in \mathcal{R}_k(e)$ for any $k \in \mathbb{N}$. Besides, using the notation $\mathcal{R}(e) = \mathcal{R}$, $\mathcal{R}_k(e) = \mathcal{R}_k$ and $\mathcal{R}_{\leq k} = \mathcal{R}_{\leq k}(e)$, we obtain the following property (see [15] for the proof).

Proposition 3.1. *The reachable set* $\mathcal{R}$ *satisfies the following properties:*

1. *For any* $\tau \in \mathbb{N}$ *with* $\tau \geq 1$, *we have* $\mathcal{R}_\tau = \mathcal{R}_{\leq \tau}$.
2. *If* $0 < \tau_1 \leq \tau_2$ *with* $\tau_1, \tau_2 \in \mathbb{N}$, *then* $\mathcal{R}_{\tau_1} \subset \mathcal{R}_{\tau_2}$.
3. *For any* $g \in G$, *it holds that* $\mathcal{R}_\tau(g) = \mathcal{R}_\tau f_0^\tau(g)$.
4. *For any* $\tau_1, \tau_2 \in \mathbb{N}$, *we have* $\mathcal{R}_{\tau_1+\tau_2} = \mathcal{R}_{\tau_1} f_0^{\tau_1}(\mathcal{R}_{\tau_2}) = \mathcal{R}_{\tau_2} f_0^{\tau_2}(\mathcal{R}_{\tau_1})$.
5. *For all* $u \in \mathcal{U}$, $g \in G$ *and* $k \in \mathbb{N}$, *we have* $\varphi(k, \mathcal{R}(g), u) \subset \mathcal{R}(g)$
6. *The identity element* e *belongs to the interior of* $\mathcal{R}$ *if and only if* $\mathcal{R}$ *is open.*

Denoting $\mathcal{C}_k(e) = \mathcal{C}_k$ and $\mathcal{C}(e) = \mathcal{C}$, we have the following propositions related to accessibility.

Proposition 3.2. *Consider the system (1.4). Then* $\text{int}\mathcal{R}_k \neq \emptyset$ *if, and only if,* $\text{int}\mathcal{C}_k \neq \emptyset$.

Proof. Suppose $\text{int}\mathcal{R}_k \neq \emptyset$ and consider the automorphism f_0 of the system (1.4). Take $g \in \text{int}\mathcal{R}_k$. Hence, there is an $u \in \mathcal{U}$ such that $g = \varphi(k, e, u)$, that is $\varphi(k, e, u)g^{-1} = e$. Using the properties of φ, $\varphi(k, e, u)g^{-1} = \varphi(k, e, u)f_0^k(f_0^{-k}(g^{-1})) = \varphi(k, f_0^{-k}(g^{-1}), u) = e$, that is, $f_0^{-k}(g^{-1}) \in \mathcal{C}_k$. Now, consider $\mathcal{N}$ a neighborhood of g such that $g \in \mathcal{N} \subset \mathcal{R}_k$. By the arguments above, $f_0^{-k}(\mathcal{N}^{-1}) \subset \mathcal{C}_k$. The map f_0^{-k} is an automorphism of G and $\mathcal{N}^{-1}$ is a neighborhood of g^{-1}. Then $f_0^{-k}(g^{-1}) \in \text{int}\mathcal{C}_k$.

Now, if $\text{int}\mathcal{C}_k \neq \emptyset$ take $g \in \text{int}\mathcal{C}_k$. Then, there are $k \in \mathbb{N}$ and $u \in \mathcal{U}$ such that $\varphi(k, g, u) = e$. Hence $\varphi(k, g, u) = \varphi(k, e, u)f_0^k(g) = e$, and consequently $\varphi(k, e, u) = f_0^k(g^{-1})$. If $g \in \mathcal{N} \subset \mathcal{C}_k$, for some neighborhood $\mathcal{N}$ of g, by the previous argument, $f_0^k(\mathcal{N}^{-1}) \subset \mathcal{R}_k$. Therefore $\text{int}\mathcal{R}_k \neq \emptyset$. □

Remark 3.3. In [22], it is stated that if $\text{int}\mathcal{R} \neq \emptyset$ then there exists some $k_0 \geq 1$ such that $\text{int}\mathcal{R}_k \neq \emptyset$ for every $k \geq k_0$. This follows from two key observations. First, note that if $\mathcal{R}_k$ has nonempty interior, then so does $\mathcal{R}_{k+1}$. Therefore, it suffices to show that there exists some $k_0 \in \mathbb{N}$ for which $\mathcal{R}_{k_0}$ has nonempty interior. Now, if U is compact, then each $\mathcal{R}_k$ is also compact. Since G is a Baire space, the statement follows. On the other hand, if U is open, it can be expressed as a countable union of compact sets, and consequently, the same holds for each $\mathcal{R}_k$. The result follows by an analogous argument. As a consequence, the proposition above ensures that $\text{int}\mathcal{R} \neq \emptyset$ if and only if $\text{int}\mathcal{C} \neq \emptyset$. Moreover, if $e \in \text{int}\mathcal{R}$, a similar argument shows that $e \in \text{int}\mathcal{C}$. Therefore, $\mathcal{R}$ is open if and only if $\mathcal{C}$ is open.

Proposition 3.4. *The system (1.4) is accessible if and only if* $\text{int}\mathcal{R} \neq \emptyset$.

Proof. Suppose $\operatorname{int}\mathcal{R} \neq \emptyset$, then there is a $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, $\operatorname{int}\mathcal{R}_k \neq \emptyset$ (see [22], pp. 3). Consider $\mathcal{O}$ an open set such that $\mathcal{O} \subset \mathcal{R}_k$. For any $g \in G$, the set $\mathcal{O}(f_0^k(g))$ is an open set and

$$\mathcal{O}(f_0^k(g)) \subset \mathcal{R}_k(f_0^k(g)) = \mathcal{R}_k(g). \tag{3.2}$$

Then $\operatorname{int}\mathcal{R}_k(g) \neq \emptyset$, for every $k \geq k_0$. This proves that the system (1.4) is forward accessible if $\operatorname{int}\mathcal{R} \neq \emptyset$. For backward accessibility, since $\mathcal{C}_k(g) = \mathcal{C}_k(f_0^{-k}(g))$, Proposition (3.2) ensures that (1.4) is accessible (forward and backward) if $\operatorname{int}\mathcal{R} \neq \emptyset$. The converse is true by definition. □

Now we introduce some concepts and algebraic generalities necessary to prove the main results. In particular, we present a generalization of the concept of Fréchet derivative to Lie groups.

Whenever we say differentiable we mean continuously differentiable. If M is a manifold and $x \in M$, recall that any vector in the tangent space T_xM of M can be written as $c'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} c(t)$ for some differentiable curve $c : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = x$.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Given $g \in G$, consider that the left and right translations $L_g : G \to G$ and $R_g : G \to G$ where $L_g(h) = gh$ and $R_g(h) = hg$. It is well known that for $X \in \mathfrak{g}$ and $g \in G$ we have $X_R(g) = d(R_g)_e(X) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{e}^{tX} g$, where X_R denotes the only right invariant vector field on G satisfying $X_R(e) = X$. In the following, R_g^G and R_h^H denote the right translation on G and H, respectively.

Definition 3.5. Take H and G Lie groups, and $\mathfrak{h}$ and $\mathfrak{g}$ their Lie algebras. Consider $\eta : O \subset H \to G$ a differentiable map with $O \subset H$ an open subset. For any $h \in O$, we define the derivative $\hat{d}\eta_h : \mathfrak{h} \to \mathfrak{g}$ at h by

$$\hat{d}\eta_h(X) = d(R^G_{\eta(h)^{-1}})_{\eta(h)} \circ d\eta_h \circ d(R^H_h)_{e_H}(X),$$

where $d\eta_h : T_hH \to T_{\eta(h)}G$ denotes the usual derivative.

Equivalently, $\hat{d}\eta_h(X)$ is defined as only element $Y \in \mathfrak{g}$ which satisfies

$$Y_R(\eta(h)) = d\eta_h(X_R(h)).$$

Since the derivative $\hat{d}$ is defined directly from the usual derivative, it shares most of its properties, such as linearity, the chain rule, and the product rule.

Remark 3.6. Take $H = \mathbb{R}^n$. Recall that $\mathfrak{h}$ can be naturally identified with $\mathbb{R}^n$ itself. Under this association, we have $X_R(g) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(g + tX) \in T_g\mathbb{R}^n$ for all $g, X \in \mathbb{R}^n$. Hence, if $c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is a differentiable curve with $c(0) = g$, and $c'(0) \in T_g\mathbb{R}^n$ denotes the tangent vector in the sense of manifolds, then $d(R^H_{c(0)^{-1}})_{c(0)}(c'(0)) = \lim_{t\to 0} \frac{c(t)-c(0)}{t} \in \mathbb{R}^n$. Now considering $G = \mathbb{R}^m$, the derivative in the classical sense is recovered as follows: If $\eta : O \subset \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable map from an open subset O then, for each $g \in O, X \in \mathbb{R}^n$,

$$\hat{d}\eta_g(X) = d(R^G_{\eta(g)^{-1}})_{\eta(g)} \left(d\eta_g \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g + tX \right) \right) = \lim_{t\to 0} \frac{\eta(g + tX) - \eta(g)}{t},$$

and $\hat{d}\eta_g$ coincides with the Fréchet derivative of η.

We now discuss some properties of this derivative. Take $\rho : G_1 \to G_2$ an arbitrary isomorphism between Lie groups and $\rho_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ its infinitesimal isomorphism. For any $X \in \mathfrak{g}_1$ and $g \in G_1$ we have that

$$d\rho_g(X_R(g)) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \rho(\mathrm{e}^{tX} g) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{e}^{t\rho_*(X)} \rho(g) = (\rho_*(X))_R(\rho(g)),$$

thus $\hat{d}\rho_g = \rho_*$ for all $g \in G_1$.

Take $g \in G$ then for any $X \in \mathfrak{g}$ and $h \in G$, $d(R_g)_e(X_R(h)) = X_R(hg)$. Thus, $\hat{d}(R_g)_h$ coincides with the identity $I : \mathfrak{g} \to \mathfrak{g}$ for all $h \in G$. On the other hand, consider the adjoint representation of the Lie group G on its Lie algebra $\mathfrak{g}$, $\mathrm{Ad} : G \to \mathrm{Gl}(\mathfrak{g})$, given by $\mathrm{Ad}(g) = d(L_g \circ R_{g^{-1}})_e = d(R_{g^{-1}} \circ L_g)_e = (dL_g)_{g^{-1}} \circ (dR_{g^{-1}})_e = (dR_{g^{-1}})_g \circ (dL_g)_e$. Then, for any $h \in G$ and $X \in \mathfrak{g}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} L_g(\mathrm{e}^{tX}h) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g\mathrm{e}^{tX}g^{-1}gh = (\mathrm{Ad}(g)(X))_R(gh).$$

Hence, $\hat{d}(L_g)_h = \mathrm{Ad}(g)$ for all $h \in G$. Now we can define the ad-rank condition. Consider the discrete-time linear system (1.4) and its solution $\varphi : \mathbb{Z} \times G \times \mathcal{U} \longrightarrow G$, where $\mathcal{U} = \prod_{i\in\mathbb{N}} U$. Denote by $\psi : G \to G$ the automorphism $f_0 = \varphi(1, \cdot, 0)$. The flow satisfies $\varphi(1, g, u) = \varphi(1, e, u)\psi(g)$, and, more generally, $\varphi(n, g, u) = \varphi(n, e, u)\psi^n(g)$.

Therefore, taking $\psi_* : \mathfrak{g} \to \mathfrak{g}$ the infinitesimal automorphism given by ψ and defining the map $F : U \longrightarrow G$ as $F(u) := f_u(e) = \varphi(1, e, u)$, we have the following definition.

Definition 3.7. The discrete-time linear system (1.4) is said to satisfy the ad-rank condition if the smallest ψ_* invariant subspace containing the image of $\hat{d}F_0$ coincides with $\mathfrak{g}$, where $\hat{d}F_0 : \mathbb{R}^m \to \mathfrak{g}$ is defined above.

The main results, regarding the conditions for accessibility of the system (1.4), are divided according to the type of the control range. Hence, we first consider the control range U as a connected open neighborhood of the origin. After, we take convex and compact control ranges.

3.1. Open control range

Suppose U is a connected open set containing 0. Let $\mathfrak{W} \subset \mathfrak{g}$ be the smallest subspace containing the images $\hat{d}F_u(\mathbb{R}^m)$, for $u \in U$. Equivalently, $\mathfrak{W}$ is the subspace spanned by the set

$$\{\hat{d}F_u(y); u \in U, y \in \mathbb{R}^m\}.$$

Consider $\mathfrak{V} \subset \mathfrak{g}$ the smallest ψ_* invariant subspace containing $\mathfrak{W}$. If B is a matrix whose columns form a basis of $\mathfrak{W}$, we have that this space coincides with the image of the Kalman matrix

$$\begin{pmatrix} \psi_*^{n-1}B & \psi_*^{n-2}B & \cdots & B \end{pmatrix},$$

where $n = \dim\mathfrak{g}$. Let $\mathfrak{h}$ the Lie subalgebra generated by $\mathfrak{V}$ and note that $\mathfrak{h}$ is ψ_* invariant. In fact, $\psi_*(\mathfrak{h})$ is the subalgebra generated by $\psi_*(\mathfrak{V})$, since ψ_* is an automorphism. However, $\psi_*(\mathfrak{V}) = \mathfrak{V}$. Equivalently, $\mathfrak{h}$ is the smallest ψ_* invariant subalgebra of $\mathfrak{g}$ containing the image of all $\hat{d}F_u$, for $u \in U$. The next proposition gives us another way to define the subalgebra $\mathfrak{h}$, in terms of the image of F instead of its derivatives.

Proposition 3.8. *Let $H \subset G$ be the Lie subgroup generated by $\mathfrak{h}$. The image of F is contained in H.*

Proof. Let $u \in U$ be arbitrary. Since U is connected and contains the origin, there is a differentiable path $\xi : [0, 1] \to U$ such that $\xi(0) = 0$ and $\xi(1) = u$. Define $c : [0, 1] \to G$ by $c(t) = F(\xi(t))$. Denoting $\xi'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\xi(t)$ we have that c satisfies the initial valued problem

$$\begin{cases} c(0) = e \\ \dot{c}(t) = (\hat{d}F_{\xi(t)}(\xi'(t)))_R(c(t)) \end{cases}. \tag{3.3}$$

Note that this is a right-invariant initial value problem, where the right-invariant vector fields vary differentiably and lie in a compact subset of $\mathfrak{g}$. Thus, it admits a unique solution.

Since $\hat{d}F_{\xi(t)}(\xi'(t)) \in \mathfrak{h}$ for all $t \in [0, 1]$, the vector fields of the initial valued problem are tangent to H, and, thus, (3.3) can be restricted to an initial valued problem on H. This initial valued problem admits a solution

$\tilde{c}(t) : [0,1] \to H$, which is also a solution on G. Since (3.3) admits a unique solution, we have $c = \tilde{c}$. In particular, $F(u) = \tilde{c}(1) \in H$. □

Now let H^* be the smallest subgroup containing the image of F. This subgroup is path-connected, and, therefore, a Lie subgroup. Let $\mathfrak{h}^* \subset \mathfrak{g}$ the Lie subalgebra associated to it, and $\mathfrak{h}_2$ the smallest ψ_* invariant subalgebra containing $\mathfrak{h}^*$. By the previous proposition, $H^* \subset H$ and, then $\mathfrak{h}^* \subset \mathfrak{h}$. Thus, $\mathfrak{h}_2 \subset \mathfrak{h}$. On the other hand, since the image of F is contained in H^*, the image of $\hat{d}F_u$ is contained in $\mathfrak{h}^*$ for all $u \in U$. Consequently, $\mathfrak{h} \subset \mathfrak{h}_2$, and these two subalgebras coincide.

Therefore, $\mathfrak{h}$ can equivalently be defined as the smallest ψ_* invariant Lie algebra containing $\mathfrak{h}^*$, where $\mathfrak{h}^*$ is the Lie subalgebra associated with the smallest subgroup containing the image of F.

The next theorem states our accessibility criterion in case of a connected open control range containing 0.

Theorem 3.9. *Consider the discrete-time linear system (1.4), and let $\psi_* : \mathfrak{g} \to \mathfrak{g}$ the infinitesimal automorphism associated to F_0. Assume U to be a connected open subset of $\mathbb{R}^m$ containing the origin, and let $\mathfrak{h} \subset \mathfrak{g}$ be the smallest ψ_* invariant subalgebra containing the set*

$$\{\hat{d}F_u(y); u \in U, y \in \mathbb{R}^m\}.$$

Then the system is accessible if, and only if, $\mathfrak{h} = \mathfrak{g}$.

The next results are the steps to prove Theorem 3.9. As before, we denote by $H \subset G$ the subgroup generated by $\mathfrak{h}$.

Proposition 3.10. *For all $k \in \mathbb{N}$, $\mathcal{R}_k$ is contained in H.*

Proof. We begin by showing that $\mathcal{R}_1 \subset H$. In fact, $\mathcal{R}_1$ is the set of all $\varphi(1, e, u) = F(u(0))$, thus $\mathcal{R}_1$ is the image of F. By the previous proposition, this image is contained in H.

We now prove the proposition by induction. Since $\mathcal{R}_0 = \{e\} \subset H$, it is trivial for $\mathcal{R}_0$. Assume that $\mathcal{R}_k \subset H$. Then, $\mathcal{R}_{k+1} = \mathcal{R}_1\psi(\mathcal{R}_k)$.

However, H is invariant by ψ, as it is generated by the ψ_* invariant algebra $\mathfrak{h}$. Thus, $\psi(\mathcal{R}_k) \subset H$. By the previous argument, $\mathcal{R}_1$ is also contained in H. Therefore their product is contained in H. □

The proposition above shows that the equality $\mathfrak{h} = \mathfrak{g}$ is a necessary condition for accessibility of the system. In fact, if $\mathfrak{h} \neq \mathfrak{g}$ then H is a subgroup of lower dimension and has empty interior in G, therefore $\mathcal{R} \subset H$ also has empty interior in G.

Now we prove the sufficient condition. For this, we will use the following notations. For each $k \in \mathbb{N}$ let $\mathcal{P}(k, d)$ the set of differentiable maps $p : \mathbb{R}^d \to G$ such that $p(\mathbb{R}^d) \subset \mathcal{R}_k$. Let

$$\mathcal{P}_k = \bigcup_{d \in \mathbb{N}} \mathcal{P}(k, d) \quad \text{and} \quad \mathcal{P} = \bigcup_{k \in \mathbb{N}} \mathcal{P}_k.$$

Let $W \subset \mathfrak{g}$ be a subspace that arises as the image of $\hat{d}p_0$ for some $p \in \mathcal{P}$, and assume that $\dim(W)$ is maximal with this property. By the previous proposition, $\mathcal{R}_k \subset H$ for all $k \in \mathbb{N}$, thus $W \subset \mathfrak{h}$. The results that follow will establish that this inclusion is, in fact, an equality.

Fix an element $p \in \mathcal{P}(k, d)$ such that $\hat{d}p_0 = W$.

Lemma 3.11. *W is invariant by ψ_* and by $\mathrm{Ad}(p(0))$.*

Proof. Recall that $\varphi(1, g, 0) = \psi(g)$ for all $g \in G$. Thus, $\psi(\mathcal{R}_k) \subset \mathcal{R}_{k+1}$ for all $k \in \mathbb{N}$. Considering the maps

$$\tilde{p} : \mathbb{R}^d \times \mathbb{R}^d \to G, \tilde{p}(x, y) = p(x)\psi^k p(y), \text{ and } \tilde{p}_1 : \mathbb{R}^d \times \mathbb{R}^d \to G \text{ with } \tilde{p}_1(x, y) = p(x)\psi^{k+1}(p(y)),$$

then $\tilde{p}(x, y) \in \mathcal{R}_k\psi^k(\mathcal{R}_k) = \mathcal{R}_{2k}$ and $\tilde{p}_1(x, y) \in \mathcal{R}_k\psi^k\psi\mathcal{R}_k \subset \mathcal{R}_{2k+1}$ for all (x, y), thus $\tilde{p}, \tilde{p}_1 \in \mathcal{P}$.

Note that for $x, y \in \mathbb{R}^d$, $\tilde{p}(x,0) = p(x)g = R_g(p(x))$ where $g = \psi^k p(0)$, and $\tilde{p}(0,y) = L_{p(0)} \circ \psi^k(p(y))$. Thus, for all $x, y \in \mathbb{R}^n$ $\hat{d}(\tilde{p})_0(x,0) = \hat{d}p_0(x)$ and $\hat{d}(\tilde{p})_0(0,y) = \mathrm{Ad}(p(0))(\psi_*^k \hat{d}p_0(y))$. Similarly, $\hat{d}(\tilde{p}_1)_0(x,0) = \hat{d}p_0(x)$ and $\hat{d}(\tilde{p}_1)_0(0,y) = \mathrm{Ad}(p(0))(\psi_*^{k+1}\hat{d}p_0(y))$. Therefore, the image of $\hat{d}(\tilde{p})_0$ coincides with the sum $W + V$ where $V = \mathrm{Ad}(p(0))(\psi_*^k(W))$. Since $\dim(W)$ is maximal, then $\dim(W+V) \leq \dim(W)$, which implies $W + V = W$. Furthermore, $\dim(V) = \dim(W)$ since the linear maps $\mathrm{Ad}(p(0))$ and ψ_*^k are automorphisms. Then $V = W$. Hence, $\mathrm{Ad}(p(0))(\psi_*^k(W)) = W$. Applying a similar argument to $\tilde{p}_1$ one also concludes that $\mathrm{Ad}(p(0))(\psi_*^{k+1}(W)) = W$.

Then $\mathrm{Ad}(p(0))(\psi_*^{k+1}(W)) = \mathrm{Ad}(p(0))(\psi_*^k(W))$ and hence $\psi_*(W) = W$, since $\mathrm{Ad}(p(0))$ and ψ_* are invertible, which shows that W is invariant by ψ_*. Therefore,

$$W = \mathrm{Ad}(p(0))(\psi_*^k(W)) = \mathrm{Ad}(p(0))(W).$$

□

Lemma 3.12. *The subspace W is invariant under $\mathrm{Ad}(g)$ for all $g \in \mathcal{R}$.*

Proof. Consider $g \in \mathcal{R}$, then $g \in \mathcal{R}_l$ for some $l \in \mathbb{N}$. Define $p_1 : \mathbb{R}^d \times \mathbb{R}^d \to G$ by $p_1(x,y) = p(x)\psi^k(g\psi^l(p(y)))$. Hence $p_1 \in \mathcal{R}_{2k+l}$ and, for $x, y \in \mathbb{R}^d$, $\hat{d}(p_1)_0(x,0) = \hat{d}p_0(x)$ and $\hat{d}(p_1)_0(0,y) = \mathrm{Ad}(p(0)) \circ \psi_*^k \circ \mathrm{Ad}(g) \circ \psi_*^l \circ \hat{d}p_0(y)$. We conclude that the image of $d(p_1)_0$ is the vector subspace $W + V$, where $V = \mathrm{Ad}(p(0)) \circ \psi_*^k \circ \mathrm{Ad}(g) \circ \psi_*^l(W)$. Therefore, $V = W$ and $W = \mathrm{Ad}(p(0)) \circ \psi_*^k \circ \mathrm{Ad}(g) \circ \psi_*^l(W)$. Since W is invariant by $\mathrm{Ad}(p(0))$ and ψ_*, it is also invariant by the inverse of these automorphisms, we obtain $W = \psi_*^{-k} \circ \mathrm{Ad}(p(0))^{-1}(W) = \mathrm{Ad}(g) \circ \psi_*^l(W) = \mathrm{Ad}(g)(W)$. □

Lemma 3.13. *W is a subalgebra.*

Proof. For each $X \in \mathfrak{g}$ consider the map $\mathfrak{C}_X : G \to \mathfrak{g}$, $g \mapsto \mathrm{Ad}(g)(X)$. Taking $\mathfrak{g} \simeq \mathbb{R}^n$ as its own Lie algebra, for all $Y \in \mathfrak{g}$ and $g \in G$ we have

$$\hat{d}(\mathfrak{C}_X)_g(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}(\mathrm{e}^{tY}g)(X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{\mathrm{ad}(tY)}\mathrm{Ad}(g)(X) = \mathrm{ad}(Y)(\mathrm{Ad}(g)(X)).$$

Now let $X, Y \in W$, $y \in \mathbb{R}^d$ such that $\hat{d}p_0(y) = Y$, and $Z = \mathrm{Ad}(p(0))^{-1}(X)$. Note that $Z \in W$ because W is invariant by $\mathrm{Ad}(p(0))$. Consider the curve $c : \mathbb{R} \to \mathfrak{g}$ with $c(t) = \mathfrak{C}_Z(p(ty)) = \mathrm{Ad}(p(ty))(Z)$. The image of c is contained in W since $p(ty) \in \mathcal{R}$ for all $t \in \mathbb{R}$. Thus, the image of c' is contained in W. Furthermore, $c'(0) = \hat{d}(\mathfrak{C}_Z)_{p(0)}\hat{d}p_0(y) = \mathrm{ad}(Y)(\mathrm{Ad}(p(0))(Z)) = \mathrm{ad}(Y)(X)$. Then $\mathrm{ad}(Y)(X) \in W$. Since $X, Y \in W$ are arbitrary, we conclude that W is a subalgebra. □

Lemma 3.14. *For all $\tilde{p} \in \mathcal{P}$, W contains the image of $\hat{d}(\tilde{p})_0$.*

Proof. Let $\tilde{p} \in \mathcal{P}(l, r)$. Define $p_2 : \mathbb{R}^r \times \mathbb{R}^d$ by $p_2(x,y) = \tilde{p}(x)\psi^l(p(y))$. Then, for $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^d$ we obtain that $\hat{d}(p_2)_0(x,0) = \hat{d}(\tilde{p})_0(x)$ and $\hat{d}(p_2)_0(0,y) = \mathrm{Ad}(\tilde{p}(0))\psi_*^l\hat{d}p_0(y)$. If V denotes the image of $\hat{d}(\tilde{p})_0$, then the image of $\hat{d}(p_2)_0$ coincides with the sum $V + \mathrm{Ad}(\tilde{p}(0))\psi_*^d(W) = V + W$. Since the dimension of W is maximal, V is contained in W. □

Lemma 3.15. *For all $u \in U$ and $y \in \mathbb{R}^m$, W contains $\hat{d}F_u(y)$.*

Proof. Consider the curve $c : \mathbb{R} \to G$, given by $c(t) = F(u + \alpha(t)y)$ where α is a diffeomorfism from $\mathbb{R}$ to $(-\varepsilon, \varepsilon)$ and ε is sufficiently small so that $u + (-\varepsilon, \varepsilon)y \in U$. Then the image of c is contained in $\mathcal{R}_1$ and $\hat{d}c_0 = \alpha'(0)\hat{d}F_u(y)$. Note that $\alpha'(0)$ is nonzero since α is a diffeomorphism, therefore the image of $\hat{d}c_0$ is the subspace spanned by $\hat{d}F_u(y)$. By the previous lemma, W contains this subspace and, therefore, contains $\hat{d}F_u(y)$. □

The previous lemmas show that W is invariant by ψ_*, is a subalgebra, and contains the set $\{\hat{d}F_u(y); u \in U, y \in \mathbb{R}^m\}$. Therefore, $\mathfrak{h} \subset W$. Since the reverse inclusion also holds, we conclude that $W = \mathfrak{h}$. Consequently, there exists $p \in \mathcal{P}$ such that $\mathfrak{h}$ is the image of $\hat{d}p_0$. Hence, we can now prove Theorem 3.9.

Proof ***(Thm. 3.9)****:* We see previously that equality $\mathfrak{h} = \mathfrak{g}$ is necessary for accessibility. For the other implication, assume that $\mathfrak{h} = \mathfrak{g}$. Then, from the previous lemmas, there is $p \in \mathcal{P}_{k,d}$ such that $\hat{dp}_0$ is surjective. Consequently, by the Submersion Theorem, $p(0)$ is in the interior of the image of p, and, then, $p(0) \in \text{int}(p(\mathbb{R}^d)) \subset \text{int}(\mathcal{R}_k)$. Therefore, $\text{int}(\mathcal{R}_k)$ is nonempty. □

3.2. Convex and compact control range

We now establish the accessibility result under the assumption that U is a convex and compact neighborhood of the origin. In addition, we prove the ad-rank theorem and, as a consequence, derive conditions for the existence of control sets. We begin by noting that the following lemma allows us to apply the previous results in this setting.

Lemma 3.16. *Let $c : [0, 1] \to G$ a continuous curve on a Lie group G, $H \subset G$ a connected Lie subgroup and assume that $c(t) \in H$ for all $t < 1$. Then $c(1) \in H$.*

Proof. Let $r = \dim(H)$ and $n = r + d = \dim(G)$. Then there is a local chart $\phi : V_1 \subset G \to V_2 \subset \mathbb{R}^r \times \mathbb{R}^d$ centered in $c(1)$ ($\phi(c(1)) = 0$) such that, for each $y \in \mathbb{R}^d$, the inverse image $\phi^{-1}(\mathbb{R}^r \times \{y\})$ is an open set of some cosset Hg. Thus, if $(x, y) \in V_2$ is such that $\phi^{-1}(x, y) \in H$ then $\phi^{-1}(\mathbb{R}^r \times y)$ is open and closed in $H \cap V_1$.

Since $c(1) \in V_1$ and c is continuous, there is $\varepsilon > 0$ such that $c(1 - \varepsilon, 1) \subset V_1$. Since $(1 - \varepsilon, 1)$ is connected, the previous argument implies that $c(1 - \varepsilon, 1)$ is entirely contained in a set $\phi^{-1}(\mathbb{R}^r \times y)$ for some $y \in \mathbb{R}^d$, and, since $\phi(c(1)) = 0$, then $y = 0$, which implies that $c(1) \in H$. □

Corollary 3.17. *The smallest subgroup containing $F(U)$ coincides with the smallest subgroup containing $F(\text{int}(U))$.*

Proof. Since U is convex and has nonempty interior, for each u in the boundary $\partial(U)$ there is a continuous curve $c : [0, 1] \to U$ such that $c(1) = u$ and $c(t) \in \text{int}(U)$ for all $t < 1$. According to the previous proposition, $F(u)$ is contained in the smallest subgroup containing $F(\text{int}(U))$. □

Then, in the case where U is convex and compact, with the origin in its interior, we define $\mathfrak{h} \subset \mathfrak{g}$ as the smallest ψ_* invariant subalgebra containing the images $\hat{dF}_u(\mathbb{R}^m)$ for $u \in \text{int}(U)$. This approach avoids the need to evaluate derivatives at boundary points, which may depend on the choice of the differentiable extension of f. By the proposition above, together with a previous argument, $\mathfrak{h}$ is equivalently defined as the smallest ψ_* invariant subalgebra containing $\mathfrak{h}^*$, where $\mathfrak{h}^*$ is the subalgebra associated to the smallest subgroup containing the image of F. In this case, we obtain a similar result:

Theorem 3.18. *Consider the discrete-time linear system (1.4), and let $\psi_* : \mathfrak{g} \to \mathfrak{g}$ be the infinitesimal automorphism associated to F_0. Assume that U is a convex and compact neighborhood of the origin, and let $\mathfrak{h} \subset \mathfrak{g}$ be the smallest ψ_* invariant subalgebra containing the set*

$$\{\hat{dF}_u(y); u \in \text{int}(U), y \in \mathbb{R}^m\}.$$

Then the system is accessible if, and only if, $\mathfrak{h} = \mathfrak{g}$.

Proof. A similar argument to Proposition 3.10 shows that $\mathcal{R}$ is contained in the subgroup H generated by $\mathfrak{h}$, therefore, if $\mathfrak{h} \neq \mathfrak{g}$ then the system is not accessible. If $\mathfrak{h} = \mathfrak{g}$ then we can consider the system with controls restricted to $\text{int}(U)$. This system is accessible by the previous arguments, therefore, the original system is also accessible. □

Remark 3.19. The hypothesis of convexity is only used in Corollary 3.17 to ensure that for any u in the boundary of U there is a curve $c : [0, 1] \to U$ such that $c(t) \in \text{int}(U)$ for all $t \in [0, 1)$ and $c(1) = u$. One could, instead, assume the existence of such curves and replace convexity by the assumption that U is a neighborhood of 0 with connected interior.

The following result is a discrete-time version of the ad-rank theorem (see Thm. 3.5 in [8]).

Theorem 3.20. *(Ad-rank) Consider the discrete-time linear system (1.4), and let ψ_* be the infinitesimal automorphism associated to $\psi = f_0$. Let V be the smallest ψ_* invariant subspace containing the image of $\hat{d}F_0$. If $V = \mathfrak{g}$, then $e \in \operatorname{int}(\mathcal{R}_n)$, where $n = \dim(\mathfrak{g})$. In particular, the system is locally controllable from e.*

Proof. Let V be the smallest ψ_* invariant subspace containing the image of $\hat{d}F_0$. Then V is spanned by the set $\{\psi_*^i \circ \hat{d}F_0(y); y \in \mathbb{R}^m, i \in \{0, 1, \dots, n-1\}\}$. Define $L : U^n \to G$ as $L(u_0, u_1, \dots, u_{n-1}) = \varphi(n, e, u_{\mathbf{x}}) = F(u_{n-1})\psi F(u_{n-2})\psi^2 \cdots F(u_1)\psi^{n-1}F(u_0)$, where $u_{\mathbf{x}}$ is defined by $u_{\mathbf{x}}(k) = u_k$ if $k \in \{0, 1, 2, \dots, n-1\}$ or $u_{\mathbf{x}}(k) = 0$ otherwise. Then $L(0, 0, \dots, 0) = e$, and the image of L is contained in $\mathcal{R}_n$. Furthermore, if $(0, 0, \dots, u_i, \dots, 0)$ is a vector in $(\mathbb{R}^m)^n$ (each coordinate is a vector of $\mathbb{R}^m$) where the nonzero entry u_i is the i-th entry, then $\hat{d}L_{(0,0,\dots,0)}(0, 0, \dots, u_i, \dots, 0) = \psi_*^i \circ \hat{d}F_0(u_i)$. Thus, the image of $\hat{d}L_{(0,0,\dots,0)}$ coincides with V. If hypothesis $V = \mathfrak{g}$ is satisfied, then by the submersion theorem $e = L(0, 0, \dots, 0) \in \operatorname{int}(\operatorname{im}(L)) \subset \operatorname{int}(\mathcal{R}_n)$. □

Remark 3.21. By applying the previous result to the time-reversed system, we also obtain that $e \in \operatorname{int}(\mathcal{C}_n)$. This implies local controllability from e, and, as a consequence, that e is contained in the interior of a control set.

Let

$$X_i = \frac{\partial}{\partial u_i} F(0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(1, e, te_i)$$

where $e_i \in \mathbb{R}^m$ denotes the canonical basis vector. Then, the space described in the theorem above coincides with

$$\operatorname{span}\{\psi_*^k(X_i); i \in \{1, 2, \dots, m\}, k \in \{0, 1, \dots, n-1\}\}.$$

Remark 3.22. Since $F(0) = e$ and $\mathfrak{g} = T_eG$ then the image of $\hat{d}F_0$ coincides with the image of dF_0. Thus, the theorem above can equivalently be stated as: If the smallest ψ_* invariant subspace containing the image of dF_0 coincides with $\mathfrak{g}$ then $e \in \operatorname{int}(\mathcal{R}_n)$.

Note that the smallest ψ_* invariant subspace containing the image of $\hat{d}F_0$ coincides with the image of the matrix

$$\mathcal{V} = \begin{pmatrix} \psi_*^{n-1}\hat{d}F_0 & \psi_*^{n-2}\hat{d}F_0 & \cdots & \hat{d}F_0 \end{pmatrix}.$$

Thus, if the matrix above has full rank, the system is locally controllable from e.

As an application of the previous results, we have the following criterion for the existence of control sets.

Proposition 3.23. *Consider a system (1.4) defined over the Lie group G satisfying the ad-rank condition. Then, there exists a control set D, with nonempty interior such that $D = \overline{\mathcal{R}} \cap \mathcal{C}$.*

Proof. If $\mathcal{R}$ is open, according to Remark (3.3), $\mathcal{C}$ is open. Now, as $e \in \mathcal{R} \cap \mathcal{C}$, there is a neighborhood V of e such that $e \in V \subset \mathcal{R} \cap \mathcal{C}$. Then, the neighborhood V is completely controllable. This proves that V satisfies the properties 1 and 2 of the definition of control sets. Hence there is a control set D such that $V \subset D$. As V is open, $e \in \operatorname{int} D$. This also proves that $e \in \operatorname{int}\mathcal{C}$. As $\mathcal{R}$ is open, the system (1.4) is accessible. From [22], Proposition 6, D has the form $D = \mathcal{C} \cap \overline{\mathcal{R}}$. □

4. Examples

In this section, we analyze the accessibility and ad-rank conditions for systems on certain low-dimensional Lie groups. Consequently, we examine controllability in these contexts. We begin by recalling a useful definition from [22].

Definition 4.1. For each $k \in \mathbb{N}$, consider the map $q_k(x,u) = \varphi(k,x,u)$. We denote by

$$\hat{\mathcal{R}}_k(x) = \{\varphi(k,x,u) : (x,u) \in M \times \text{int}U^k \text{ such that rank } \left[\frac{\partial}{\partial u} q_k(x,u)\right] = \dim M\}.$$

the regular reachable set from $x \in G$ up to time $k \in \mathbb{N}$ and $\hat{\mathcal{R}}(x) = \bigcup_{k\in\mathbb{N}} \hat{\mathcal{R}}_k(x)$ the regular reachable set from $x \in M$.

In particular, the set $\hat{\mathcal{R}}(x)$ is open for every $x \in M$.

Example 4.2. Let $G = \text{Sl}(2,\mathbb{R})$ be the semisimple, connected Lie group of 2×2 real matrices with determinant 1, and denote its Lie algebra by $\mathfrak{sl}(2,\mathbb{R})$. It is known that $\text{Aut}(\mathfrak{sl}(2,\mathbb{R})) = \text{Inn}(\mathfrak{sl}(2,\mathbb{R}))$, that is, every automorphism of $\mathfrak{sl}(2,\mathbb{R})$ is inner in the following sense: if $T \in \text{Aut}(\mathfrak{sl}(2,\mathbb{R}))$, there are $Y_1,\dots,Y_n \in \mathfrak{gl}(2,\mathbb{R})$ such that $T(X) = \text{e}^{\text{ad}\,Y_1}\cdots\text{e}^{\text{ad}\,Y_n}(X)$. By [21], Proposition 5.15, we have that the differential at the identity of the conjugation $C_h(g) = hgh^{-1}$ is T, where $h = \text{e}^{Y_1}\cdots\text{e}^{Y_n} \in \text{Gl}(2,\mathbb{R})$. Then we can define the discrete-time linear systems on $\text{Sl}(2,\mathbb{R})$. Given $h \in \text{Gl}(2,\mathbb{R})$, take a map $f : U \times \text{Sl}(2,\mathbb{R}) \longrightarrow \text{Sl}(2,\mathbb{R})$, $f_u(g) = A(u,e)\cdot hgh^{-1}$, where $A(0,e) = \text{Id}$ and $\det A(u,e) = 1$ for all $u \in U$. Considering the discrete-time system

$$g_{k+1} = f_{u_k}(g_k), k \in \mathbb{N}_0, \tag{4.1}$$

it is not difficult to prove that this is a linear system on $\text{Sl}(2,\mathbb{R})$. Moreover, every discrete-time linear system on $\text{Sl}(2,\mathbb{R})$ has the form (4.1).

Now, take $h, h^{-1} \in \text{Gl}(2,\mathbb{R})$ with

$$h = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \text{ and } h^{-1} = \frac{1}{h_{11}h_{22} - h_{21}h_{12}} \begin{bmatrix} h_{22} & -h_{21} \\ -h_{12} & h_{11} \end{bmatrix},$$

and an element $g = [g_{ij}] \in \text{Sl}(2,\mathbb{R})$. We have $f_u(g) = A(u,e)\cdot f_0(g)$, where $f_0(g) = hgh^{-1}$ is given by

$$f_0(g) = \begin{bmatrix} \frac{g_{12}h_{11}h_{21}+g_{22}h_{12}h_{21}-g_{11}h_{11}h_{22}-g_{21}h_{12}h_{22}}{h_{12}h_{21}-h_{11}h_{22}} & \frac{(g_{12}h_{11}^2-h_{12}(g_{11}h_{11}-g_{22}h_{11}+g_{21}h_{12}))}{(-h_{12}h_{21}+h_{11}h_{22})} \\ \frac{(-g_{12}h_{21}^2+h_{22}(g_{11}h_{21}-g_{22}h_{21}+g_{21}h_{22}))}{(-h_{12}h_{21}+h_{11}h_{22})} & \frac{(g_{12}h_{11}h_{21}+g_{22}h_{11}h_{22}-h_{12}(g_{11}h_{21}+g_{21}h_{22}))}{(-h_{12}h_{21}+h_{11}h_{22})} \end{bmatrix}.$$

The derivative df_0 (with f_0 viewed as a map $f_0 : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$) is

$$df_0 = \begin{bmatrix} -\frac{h_{11}h_{22}}{h_{12}h_{21}-h_{11}h_{22}} & \frac{h_{11}h_{21}}{h_{12}h_{21}-h_{11}h_{22}} & -\frac{h_{12}h_{22}}{h_{12}h_{21}-h_{11}h_{22}} & \frac{h_{12}h_{21}}{h_{12}h_{21}-h_{11}h_{22}} \\ -\frac{h_{11}h_{12}}{h_{11}h_{22}-h_{12}h_{21}} & \frac{h_{11}^2}{h_{11}h_{22}-h_{12}h_{21}} & -\frac{h_{12}^2}{h_{11}h_{22}-h_{12}h_{21}} & \frac{h_{11}h_{12}}{h_{11}h_{22}-h_{12}h_{21}} \\ \frac{h_{21}h_{22}}{h_{11}h_{22}-h_{12}h_{21}} & -\frac{h_{21}^2}{h_{11}h_{22}-h_{12}h_{21}} & \frac{h_{22}^2}{h_{11}h_{22}-h_{12}h_{21}} & -\frac{h_{21}h_{22}}{h_{11}h_{22}-h_{12}h_{21}} \\ -\frac{h_{12}h_{21}}{h_{11}h_{22}-h_{12}h_{21}} & \frac{h_{11}h_{21}}{h_{11}h_{22}-h_{12}h_{21}} & -\frac{h_{12}h_{22}}{h_{11}h_{22}-h_{12}h_{21}} & \frac{h_{11}h_{22}}{h_{11}h_{22}-h_{12}h_{21}} \end{bmatrix}.$$

In particular, let $U \subset \mathbb{R}$ be a compact convex neighborhood of 0 and

$$h = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{with} \quad f_u(e) = \begin{bmatrix} 1+u & -u \\ u & 1-u \end{bmatrix}.$$

Take the system defined by

$$g_{k+1} = f_{u_k}(g_k), k \in \mathbb{N}_0. \tag{4.2}$$

Considering the nilpotent matrix

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

it follows that $\mathrm{e}^M = \sum_{n\in\mathbb{N}} \frac{M^n}{n!} = I + M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = h$. The map f_0 is defined by $f_0(g_{11}, g_{12}, g_{21}, g_{22}) = (g_{11} + g_{21}, -g_{11} + g_{12} - g_{21} + g_{22}, g_{21}, g_{22} - g_{21})$ with derivative given by

$$df_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{4.3}$$

We now verify the accessibility of the system (4.2). Clearly, $f_u(e) \in \mathrm{Sl}(2,\mathbb{R})$, for every $u \in U$. Following the definition (4.1), we claim that $e \in \hat{\mathcal{R}}$. In fact, for $(u, v, w) \in \mathrm{int}U^3$, $k = 3$, and using the notation $f_{u,v,w} = f_u \circ f_v \circ f_w$, we have $f_{u,v,w}(e) = f_u(e)C_h(f_v(e))C_{h^2}(f_w(e))$. The resulting matrix is

$$f_{u,v,w}(e) = \begin{bmatrix} (w+1)((u+1)(2v+1) - uv) & (-u(1-2v) - 4(1+u)v) + (-uv + (1+u)(1+2v)) \\ ((1-u)v + u(1+2v))(1+w) & ((1-u)(1-2v) - 4uv) + ((1-u)v + u(1+2v)) \end{bmatrix},$$

where $f_{u,v,w} = f_u \circ f_v \circ f_w$, and its derivative is given by

$$\frac{\partial}{\partial(u,v,w)} f_u \circ f_v \circ f_w(e) = \begin{bmatrix} \begin{bmatrix} (1+v)(1+w) \\ (2+u)(1+w) \\ 1+2v+u(1+v) \end{bmatrix} & \begin{bmatrix} -v \\ -2-u \\ 0 \end{bmatrix} \\ \begin{bmatrix} (1+v)(1+w) \\ (1+u)(1+w) \\ v+u(1+v) \end{bmatrix} & \begin{bmatrix} -v \\ -1-u \\ 0 \end{bmatrix} \end{bmatrix}. \tag{4.4}$$

The vectors of the matrix above generate a $3-$dimensional subspace. Hence, this matrix has rank 3, for every $u \in \mathrm{int}U$. Then $e \in \hat{\mathcal{R}}_3 \subset \hat{\mathcal{R}}$. As the set $\hat{\mathcal{R}}$ is open and $\hat{\mathcal{R}} \subset \mathcal{R}$, we have $e \in \mathrm{int}\mathcal{R}$. By the Proposition (3.1), item 6, the set $\mathcal{R}$ is open, which also implies that the system is accessible.

Next, we show that the subalgebra $\mathfrak{h}$ defined in the Theorem (3.9) coincides with $\mathfrak{sl}(2,\mathbb{R})$. Consider the map $F : U \longrightarrow \mathrm{Sl}(2,\mathbb{R})$ defined by $F(u) = f_u(e)$, we compute the set

$$W = \mathrm{span}\{\hat{dF}_u(X) : u \in U, X \in \mathbb{R}\},$$

where $\hat{dF}_u(X) = d(R^s_{F(u)^{-1}})_{F(u)} \circ d(F)_u \circ d(R^r_u)_e(X)$, and R^s and R^r denote the right-translation maps on the Lie groups $\mathrm{Sl}(2,\mathbb{R})$ and $\mathbb{R}$, respectively. We have

$$d(R^s_X)_Y(Z) = ZX.$$

As the right-invariant vector fields on $\mathbb{R}$ are constant fields, we obtain that

$$\hat{dF}_u(X) = X \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1-u & u \\ -u & 1+u \end{bmatrix} = X \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{4.5}$$

Therefore

$$W = \left\{ k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \in \mathfrak{sl}(2,\mathbb{R}) : k \in \mathbb{R} \right\},$$

which is one-dimensional. Now consider the Kalman matrix

$$\mathcal{K} = (df_0^2 X_0 \ df_0 X_0 \ X_0),$$

with $X_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and df_0 defined in (4.3). Then

$$\mathcal{K} = \begin{bmatrix} 3 & 2 & 1 \\ -9 & -4 & -1 \\ 1 & 1 & 1 \\ -3 & -2 & -1 \end{bmatrix} \tag{4.6}$$

and it is easy to show that $rank(\mathcal{K}) = 3$. Since $\mathfrak{h}$ is the smallest subalgebra containing the image of $\mathcal{K}$, it follows that $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$. Hence, the system is accessible. By the ad-rank condition (see Thm. (3.20)), the vector

$$X_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

is the matrix of $\hat{dF}_0$ (see expression (4.5)). Therefore, the matrix $\mathcal{V} = (\psi_*^2 \hat{dF}_0 \ \psi_* \hat{dF}_0 \ \hat{dF}_0)$ coincides with the matrix $\mathcal{K}$ in (4.6), whose rank is 3 — the dimension of $\mathfrak{sl}(2,\mathbb{R})$ - thereby implying local controllability.

Example 4.3. The real abelian solvable Lie groups of dimension 2 are $\mathbb{R}^2$, $\mathbb{T} \times \mathbb{R}$ and $\mathbb{T}^2$ (see *e.g.* [23]). In the non-abelian case, the unique (up to isomorphism) real solvable Lie group is the open half plane $G = \mathbb{R}^+ \ltimes \mathbb{R}$, endowed with the product $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_2 + x_2 y_1)$. The Lie group $(G, \cdot)$ is called affine group and is denoted by $\mathrm{Aff}(2,\mathbb{R})$. The Lie algebra of $\mathrm{Aff}(2,\mathbb{R})$ is given by the set $\mathbb{R}^2$, equipped with the Lie bracket $[(x_1, y_1), (x_2, y_2)] = (0, x_1 y_2 - y_1 x_2)$ which is known as affine Lie algebra, denoted by $\mathfrak{aff}(2,\mathbb{R})$. In particular, the automorphisms of $\mathrm{Aff}(2,\mathbb{R})$ are given by

$$\varphi(x, y) = (x, a(x-1) + \mathrm{d}y), \tag{4.7}$$

with $d \in \mathbb{R} \setminus \{0\}$ and $a \in \mathbb{R}$. Then, the discrete-time linear systems on $\mathrm{Aff}(2,\mathbb{R})$ can be defined by the maps

$$f((x, y), u) := f_u(x, y) = (\gamma(u)x, a(x-1) + \mathrm{d}y + \delta(u)x), \tag{4.8}$$

where $\gamma : \mathbb{R}^m \to \mathbb{R}^+$ and $\delta : \mathbb{R}^m \to \mathbb{R}$ are $\mathcal{C}^\infty$ maps satisfying $\gamma(0) = 1$ and $\delta(0) = 0$. Consider the system

$$x_{k+1} = f(x_k, u_k), \ k \in \mathbb{N}, \ u \in U, \tag{4.9}$$

where U is a compact, convex neighborhood of $0 \in \mathbb{R}^m$ and f is given by the expression in (4.8). Define the map $F : U \subset \mathbb{R}^m \longrightarrow \mathrm{Aff}(2,\mathbb{R})$ by $F(u) = f_u(1, 0)$. Then $F(u) = (\gamma(u), \delta(u))$. If $R^a_{(x,y)}$ denotes the right-translation

map on $\mathrm{Aff}(2,\mathbb{R})$, we obtain

$$R^a_{(x,y)}(x_1,y_1) = (x_1,y_1)\cdot(x,y) = (x_1x, y+xy_1),$$

which implies that

$$d(R^a_{(x,y)})_{(x_1,y_1)} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}.$$

Knowing that $(\gamma(u),\delta(u))^{-1} = (\frac{1}{\gamma(u)}, -\frac{\delta(u)}{\gamma(u)})$, for any $u \in \mathrm{int}U$ we have

$$\begin{aligned} \hat{d}F_u(X) &= d(R^a_{F(u)^{-1}})_{F(u)} \circ d(F)_u \circ d(R^r_u)_e(X) \\ &= X \begin{bmatrix} \frac{\gamma'(u)}{\gamma(u)} \\ \frac{\delta'(u)}{\gamma(u)} \end{bmatrix}, \end{aligned}$$

where R^r represents the right-translation on $\mathbb{R}^m$.

Consider the case $\gamma(u) = \mathrm{e}^u$, $\delta(u) = u^2$ and $U = [-1,1]$. Then

$$\hat{d}F_u(X) = \begin{bmatrix} X \\ 2Xu\mathrm{e}^{-u} \end{bmatrix}.$$

Therefore, the vector subspace W can be spanned by the vectors $(1,0)$ and $(1,2)$, which implies that $\dim W = 2$. Hence, the Kalman matrix $\mathcal{K} = (df_0 B \ B)$, with $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, has rank 2 and thus $\mathfrak{h} = \mathfrak{aff}(2,\mathbb{R})$. Note that in this case, there is no need to construct the subalgebra $\mathfrak{h}$, since $\dim W = \dim \mathfrak{aff}(2,\mathbb{R})$.

Now we verify the ad-rank condition. Considering that $\hat{d}F_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have that

$$\mathcal{V} = (\psi_* \hat{d}F_0 \ \hat{d}F_0) = \begin{bmatrix} 1 & 1 \\ a & 0 \end{bmatrix}.$$

Thus, the ad-rank condition (see Thm. (3.20)) is satisfied if, and only if, $a \neq 0$. Since all automorphisms of $\mathrm{Aff}(2,\mathbb{R})$ are of the form given in (4.7), if we take f_0 to be an inner automorphism and set $d = 1$, it follows from [15], Theorem 28 that the system (4.9) is controllable.

Example 4.4. Consider the Heisenberg group

$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} ; \ x_1,x_2,x_3 \in \mathbb{R} \right\},$$

which is diffeomorphic to the Euclidean space $\mathbb{R}^3$ endowed with the product $(x_1,x_2,x_3)\cdot(y_1,y_2,y_3) = (x_1 + y_1 + x_2y_3, x_2+y_2, x_3+y_3)$. The Lie algebra of $\mathbb{H}$ is

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}(3,\mathbb{R}) : X = \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}, (x,y,z) \in \mathbb{R}^3 \right\}$$

with the matrix Lie bracket.

Let U be a compact and connected neighborhood of $0 \in \mathbb{R}$ and $f : \mathbb{H} \times U \to \mathbb{H}$ given by $f_u(x_1, x_2, x_3) = \left(x_1 + x_2 + \frac{x_2^2}{2} + ux_2 + ux_3 - \frac{u}{2} - \frac{u^2}{3}, x_2 + u, x_2 + x_3 + \frac{u}{2}\right)$. Then $f_0 : \mathbb{H} \longrightarrow \mathbb{H}$ is defined by

$$f_0(x_1, x_2, x_3) = \left(x_1 + x_2 + \frac{x_2^2}{2}, x_2, x_2 + x_3\right) \tag{4.10}$$

and $F : U \longrightarrow \mathbb{H}$ by $F(u) = \left(-\frac{u}{2} - \frac{u^2}{3}, u, \frac{u}{2}\right)$. It follows that

$$g_{k+1} = f_{u_k}(g_k), \ u_k \in U \tag{4.11}$$

is a discrete-time linear system on $\mathbb{H}$. We will prove the accessibility property (see Thm. 3.18), that is, $\mathfrak{h} = \mathfrak{g}$. The right-translation on $\mathbb{H}$ is given by $R^h_{(y_1,y_2,y_3)}(x_1, x_2, x_3) = (x_1 + y_1 + x_2y_3, x_2 + y_3, x_3 + y_3)$ whose derivative is

$$d(R^h_{(y_1,y_2,y_3)})_{(x_1,x_2,x_3)} = \begin{bmatrix} 1 & y_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $F(u)^{-1} = \left(\frac{u}{2} + \frac{5u^2}{6}, -u, -\frac{u}{2}\right)$ for every $u \in U$, we have that

$$\hat{d}F_u(X) = \begin{bmatrix} 1 & -\frac{u}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} - \frac{2}{3}u \\ 1 \\ \frac{1}{2} \end{bmatrix} X = X \begin{bmatrix} -\frac{1}{2} - \frac{7u}{6} \\ 1 \\ \frac{1}{2} \end{bmatrix}. \tag{4.12}$$

Then, the spanned subspace W is given by

$$W = \text{span}\left\{ X \begin{bmatrix} -\frac{1}{2} - \frac{7u}{6} \\ 1 \\ \frac{1}{2} \end{bmatrix} \in \mathbb{R}^3 : u \in U, X \in \mathbb{R} \right\}$$

and therefore $\dim W = 2$. Now, considering the expression for f_0 given in (4.10), we obtain

$$d(f_0)^n_{(0,0,0)} = \begin{bmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & n & 1 \end{bmatrix}, \forall n \in \mathbb{N}. \tag{4.13}$$

Taking $U = [-1, 1]$ and $B = \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, the Kalman matrix $\mathcal{K} = (df_0^2 B \ df_0 B \ B)$ is given by

$$\mathcal{K} = \begin{bmatrix} \frac{3}{2} & 2 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{5}{2} & \frac{5}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then $rank(\mathcal{K}) = 3$, which implies that $\mathfrak{h} = \mathfrak{g}$. Therefore, the system is accessible. We now proceed to verify the conditions of Theorem (3.20). In fact, from the expression (4.12) we have $\hat{d}F_0 = [-\frac{1}{2} \ 1 \ \frac{1}{2}]^T$. Moreover, using

the expression (4.13), we obtain

$$\mathcal{V} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 1 \\ \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

which implies that $rank(\mathcal{V}) = 3$ as well. Therefore, the system (4.11) is locally controllable.

In particular, since $\mathcal{R}$ is open and all eigenvalues of $d(f_0)_{(0,0,0)}$ are equal 1, it follows by [15], Lemma 19 that $G \subset \mathcal{R} \cap \mathcal{C}$, which implies that the system (4.11) is controllable. This reinforces the importance of the assumption that $e \in \text{int}\mathcal{R}$ in our setting.

5. Conclusion

In conclusion, this paper presents an algebraic condition for the accessibility of discrete-time linear systems on Lie groups, using an adapted notion of the derivative for the map $u \mapsto f(u, e)$. We also establish the *ad-rank condition* for these systems, inspired by its counterpart in the continuous setting. Under this condition, we demonstrate the existence of a control set D with a nonempty interior such that $D = \mathcal{C} \cap \overline{\mathcal{R}}$. Our findings are illustrated with examples of systems on the affine two-dimensional Lie group $\text{Aff}(2, \mathbb{R})$, the special linear group $\text{Sl}(2, \mathbb{R})$, and the three-dimensional Heisenberg group, all of which satisfy the stated accessibility and ad-rank conditions.

This work builds on previous studies of continuous-time systems and extends key concepts to the discrete-time domain, thereby bridging an essential gap in the literature on Lie group systems. Future research could extend the analysis to more general structures such as smooth manifolds, establish conditions for controllability in specific Lie groups, or determine conditions for the uniqueness of control sets with nonempty interiors. By establishing these accessibility and ad-rank properties, our work provides a foundational understanding that can be applied to various engineering and computational problems where such systems are prevalent.

Acknowledgments

The authors are grateful to the anonymous referees for their valuable suggestions and thoughtful corrections, which have significantly improved the clarity, quality, and presentation of the paper.

Funding

Second author partially supported by CNPq grant no. 309409/2023-3, third author supported by CAPES grant no. 88887.669835/2022-00.

Data availability statement

The research data associated with this article are included in the article.

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