ESAIM: COCV 31 (2025) 39
https://doi.org/10.1051/cocv/2025028

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

INVERTIBILITY OF SOBOLEV MAPS THROUGH APPROXIMATE INVERTIBILITY AT THE BOUNDARY AND TANGENTIAL POLYCONVEXITY

Carlos Mora-Corral[1,2] and David Mur-Callizo[1,*]

Abstract. We work in a class of Sobolev $W^{1,p}$ maps, with $p > d-1$, from a bounded open set $\Omega \subset \mathbb{R}^d$ to $\mathbb{R}^d$ that do not exhibit cavitation and whose trace on $\partial\Omega$ is also $W^{1,p}$. Under the assumptions that the Jacobian is positive and the deformation can be approximated on the boundary by injective maps, we show that the deformation is injective. We prove the existence of minimizers in this class for functionals accounting for a nonlinear elastic energy and a boundary energy. The energy density in Ω is assumed to be polyconvex, while the energy density in $\partial\Omega$ is assumed to be tangentially polyconvex, a new type of polyconvexity on $\partial\Omega$.

Mathematics Subject Classification. 49J40, 49J45, 74B20, 74G25, 74G65.

Received August 5, 2024. Accepted February 24, 2025.

1. Introduction

A classic problem in topology is to prove invertibility of a map from local invertibility and invertibility at the boundary. This question has also a long history in nonlinear elasticity theory, where a deformation map $u : \Omega \to \mathbb{R}^d$ is assumed to be Sobolev $W^{1,p}$ from a bounded open set $\Omega \subset \mathbb{R}^d$ representing the reference configuration. Local invertibility and preservation of orientation are modelled through the constraint $\det Du > 0$ a.e., where Du is the deformation gradient. Invertibility on the boundary is typically imposed with an adequate Dirichlet boundary condition. The goal is then to obtain invertibility for u, since this property is physically required in order to prevent interpenetration of matter.

The pioneering work of Ball [1] showed that when $p \geq d$, Sobolev maps are invertible a.e. or even homeomorphisms when they coincide in $\partial\Omega$ with an invertible map. Here, *invertibility a.e.* means that the restriction of u to a set of full measure is invertible. Further developments of invertibility in the context of nonlinear elasticity are [2–7].

In this article we focus on the approaches of Henao, Mora-Corral and Oliva [8] and Krömer [9], which we explain next. In [8] it was defined the class $\overline{\mathcal{A}}_p(\Omega)$ of Sobolev maps $W^{1,p}(\Omega, \mathbb{R}^d)$, with $p > d-1$, such that its trace belongs to $W^{1,p}(\partial\Omega, \mathbb{R}^d)$ and satisfy the *divergence identities* (see [5, 7, 10, 11])

$$\operatorname{Div}[\operatorname{adj} Du(x)\, g(u(x))] = \operatorname{div} g(u(x)) \det Du(x) \tag{1.1}$$

Keywords and phrases: Approximate invertibility, global invertibility, Sobolev maps, nonlinear elasticity, polyconvexity.

[1] Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain.
[2] Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, 28049 Madrid, Spain.

[*] Corresponding author: david.mur@estudiante.uam.es

up to the boundary, meaning that for all $\phi \in C^\infty(\bar\Omega)$ and $g \in C^1(\mathbb{R}^d, \mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ we have

$$\begin{aligned}\int_{\partial\Omega} \phi(x)\,(\operatorname{adj} Du(x)\, g(u(x))) \cdot n(x)\, \mathrm{d}\mathcal{H}^{d-1}(x) - \int_\Omega [\operatorname{adj} Du(x)\, g(u(x))] \cdot D\phi(x)\, \mathrm{d}x \\ = \int_\Omega \det Du(x)\, \phi(x) \operatorname{div} g(u(x))\, \mathrm{d}x,\end{aligned} \tag{1.2}$$

where n is the unit exterior normal of $\partial\Omega$. This class is the version *up to the boundary* of class $\mathcal{A}_p(\Omega)$ defined in [12] for which equality (1.2) is requested to hold only for $\phi \in C_c^\infty(\Omega)$ (and, hence, the integral on $\partial\Omega$ vanishes). It was shown there that maps in $\mathcal{A}_p(\Omega)$ enjoy extra regularity than a typical $W^{1,p}$ function, and, earlier in [4, 13], that they do not present cavitation. The main result in [8] is that deformations in $\overline{\mathcal{A}}_p(\Omega)$ that coincide with an invertible map on $\partial\Omega$ are themselves invertible a.e.

In [9] it is performed a topological study of maps that are *approximately invertible on the boundary*, meaning that their restriction to $\partial\Omega$ can be uniformly approximated by continuous invertible maps. The class of such maps is denoted by AIB. Then, it was shown that those deformations that, in addition, are Sobolev $W^{1,p}$ with $p \geq d$ and preserve the orientation are invertible a.e. An advantage of his approach is that the invertibility condition is only required on the boundary, which avoids the delicate issue of homeomorphic extension and allows for boundary conditions different from Dirichlet. Moreover, the notion of approximate invertibility on the boundary permits self-contact at the boundary, while forbiding self-interpenetration.

In this article we extend, in the class $\overline{\mathcal{A}}_p(\Omega)$ with $p > d-1$, the result of [9] on invertibility a.e. from approximate invertibility on the boundary. At the same time, it also generalizes the result of [8] inasmuch the boundary data is not required to be an invertible a.e. map on the whole Ω but only approximately invertible on the boundary.

After establishing the invertibility result, in view of its applications in nonlinear elasticity, we study energies of the form

$$I(u) = \int_\Omega W(x, u(x), Du(x))\mathrm{d}x + \int_{\partial\Omega} V(x, u(x), D^\tau u(x), n(x))\mathrm{d}\mathcal{H}^{d-1}(x)$$

in the class $\overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$. The integral in Ω is standard in nonlinear elasticity and accounts for the elastic energy plus volume forces. The usual assumption is that W is *polyconvex*, that is, convex in the minors of the derivative. The integral in $\partial\Omega$ has not received as much attention as the volume term. It accounts for the applied surface forces and, in general, the surface interaction potentials; see, *e.g.*, [14], Section 5.1 or [15], and, in the context of binary alloys, [16]. The function $D^\tau u$ is the tangential derivative of $u_{|\partial\Omega}$, and $n(x)$ is the normal to Ω.

A necessary and sufficient condition for the lower semicontinuity of the integral in $\partial\Omega$ is the tangential quasiconvexity of V; see [17]. In this article we introduce a natural sufficient condition: the *tangential polyconvexity*, which roughly consists of being convex in the minors of the tangential derivative of $u_{|\partial\Omega}$. With this, we prove the existence of minimizers of I in the class $\overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ under several boundary conditions, not necessarily Dirichlet. Moreover, we compare our notion of tangential polyconvexity with the related one of *interface polyconvexity* from [18]. Both concepts refer to maps that are convex in certain minors of the derivative of $u_{|\partial\Omega}$. The interface polyconvexity, originally defined as the supremum of a family of null Lagrangians, requires, in an *a posteriori* characterization, the positive 1-homogeneity. This is not needed in the definition of tangential polyconvexity, because it is based on the convexity on the minors of the tangential differential of $u_{|\partial\Omega}$, once a basis of the tangent space is chosen.

The outline of this article is as follows. In Section 2 we explain the general notation and preliminary concepts and results. Section 3 recalls the definition and main properties of class $\overline{\mathcal{A}}_p(\Omega)$. In Section 4 we prove the injectivity a.e. of deformations in $\overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$. In Section 5 we show, by means of a counterexample, that the injectivity a.e. does not hold in general if the class $\overline{\mathcal{A}}_p(\Omega)$ is replaced by the class $\mathcal{A}_p(\Omega)$. Section 6 shows the weak continuity in $W^{1,p}(\partial\Omega, \mathbb{R}^d)$ of the minors of the tangential derivative. In Section 7 we define tangential polyconvexity: it implies tangential quasiconvexity and is equivalent to polyconvexity of an extension.

In Section 8, through the language of multilinear algebra, we give insight into the interface polyconvexity and compare it to the tangential polyconvexity. Section 9 shows that typical examples of surfaces potentials used in the literature are tangentially polyconvex. Finally, in Section 10 we prove the existence of minimizers of I in $\overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ under the assumptions of polyconvexity of the energy density in Ω and tangential polyconvexity of the energy density on $\partial\Omega$.

2. Notation and preliminaries

We first specify the general notation used in the article.

We will use Ω to refer to a bounded open subset of $\mathbb{R}^d$, which most of the times will be assumed to have a Lipschitz boundary and that $\mathbb{R}^d \setminus \partial\Omega$ has exactly two connected components. Here $d \in \mathbb{N}$ is the dimension of the space, which will be assumed to be $d \geq 2$; otherwise, $\mathbb{R}^d \setminus \partial\Omega$ will have three connected components for an interval Ω. The issue of invertibility for $d = 1$ is essentially trivial, since for Sobolev functions is reduced to having positive derivative. The set Ω represents an elastic material in its reference configuration, and $u : \overline{\Omega} \to \mathbb{R}^d$ is the deformation of the body.

The notation for Sobolev $W^{1,p}$ and Lebesgue L^p spaces is standard. Most of the times, the exponent p will satisfy $p > d - 1$. For $u \in W^{1,p}(\Omega; \mathbb{R}^d)$, we denote by $u_{|\partial\Omega}$ the trace of u on $\partial\Omega$. It is known that $u_{|\partial\Omega} \in L^p(\partial\Omega; \mathbb{R}^d)$, and we will write $u \in W^{1,p}(\partial\Omega; \mathbb{R}^d)$ whenever its trace $u_{|\partial\Omega}$ belongs to $W^{1,p}(\partial\Omega; \mathbb{R}^d)$.

We will abbreviate *almost everywhere* as *a.e.*, which refers to the d-dimensional Lebesgue measure $\mathcal{L}^d$, unless otherwise specified. We say that two subsets of $\mathbb{R}^d$ are equal a.e. if its symmetric difference has $\mathcal{L}^d$-measure zero. We will also use $\mathcal{H}^{d-1}$ to refer to the $(d-1)$-dimensional Hausdorff measure. The set $S^{d-1} \subset \mathbb{R}^d$ is the d-dimensional unit sphere.

The set $\mathbb{R}^{d\times d}$ is the set of square matrices of order d, and its subset $\mathbb{R}^{d\times d}_+$ consists of those with positive determinant. The adjoint $\operatorname{adj} F$ of an $F \in \mathbb{R}^{d\times d}$ is the square matrix that satisfies $F \operatorname{adj} F = (\det F) I$, where $I \in \mathbb{R}^{d\times d}$ is the identity matrix. The cofactor $\operatorname{cof} F$ is the transpose of $\operatorname{adj} F$. We will use $\mathcal{L}(U; V)$ to denote the set of linear maps between two (finite-dimensional) vector spaces U and V.

A key concept studied in [9] is the *approximate invertibility on the boundary.*

Definition 2.1. Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $u \in C(\partial\Omega; \mathbb{R}^d)$. We say that u is approximate invertible on the boundary if there exists a sequence of injective maps $\{\varphi_k\}_{k\in\mathbb{N}} \subset C\left(\partial\Omega; \mathbb{R}^d\right)$ with $\varphi_k \to u$ uniformly on $\partial\Omega$. The class of all such maps is denoted by $\mathrm{AIB}(\Omega)$, or by AIB if Ω is clear from the context. We say that $u : \bar{\Omega} \to \mathbb{R}^d$ is approximate invertible on the boundary if so is $u_{|\partial\Omega}$.

Condition AIB models possible self-contact at the boundary without interpenetration in the interior, hence it is a realistic class to pose problems in nonlinear elasticity. Note that if $u : \partial\Omega \to \mathbb{R}^d$ is continuous and injective then $u \in \mathrm{AIB}$; and if $u \in \mathrm{AIB}$, then u is continuous on $\partial\Omega$. The following lemma is a variant of [9], Lemma 2.3 and its proof is elementary relying on the fact that $W^{1,p}(\partial\Omega; \mathbb{R}^d)$ is compactly embedded in $C(\partial\Omega; \mathbb{R}^d)$ for $p > d - 1$.

Lemma 2.2. *Let Ω be a Lipschitz domain. Let $p > d - 1$, let $u \in W^{1,p}\left(\partial\Omega; \mathbb{R}^d\right)$ and let $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\partial\Omega; \mathbb{R}^d) \cap \mathrm{AIB}$ be such that $u_k \rightharpoonup u$ in $W^{1,p}(\partial\Omega; \mathbb{R}^d)$. Then $u \in \mathrm{AIB}$.*

We say that a function $u : \Omega \to \mathbb{R}^d$ is *injective a.e.* if there exists some $\widetilde{\Omega} \subset \Omega$ such that $\mathcal{L}^d(\Omega \setminus \widetilde{\Omega}) = 0$ and u is injective in $\widetilde{\Omega}$.

The following proposition is a version of Federer's area formula [19]; this specific formulation can be found in [6], Proposition 2.6.

Proposition 2.3. *Let $u \in W^{1,1}(\Omega; \mathbb{R}^d)$. Then there exists a measurable set $\Omega_0 \subset \Omega$, with $\mathcal{L}^d\left(\Omega \setminus \Omega_0\right) = 0$, such that the following property holds. For any measurable $A \subset \Omega$, define $\mathcal{N}_{u,A} : \mathbb{R}^d \to \mathbb{N} \cup \{\infty\}$ as follows: $\mathcal{N}_{u,A}(y)$ equals the number of $x \in \Omega_0 \cap A$ such that $u(x) = y$. Then $\mathcal{N}_{u,A}$ is measurable and for any measurable*

$\varphi : \mathbb{R}^d \to \mathbb{R}$,

$$\int_A \varphi(u(x))|\det Du(x)|\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(y)\mathcal{N}_{u,A}(y)\mathrm{d}y,$$

whenever either integral exists.

Here, Du stands for the distributional derivative of the Sobolev function u. We will mainly use $\mathcal{N}_{u,\Omega}$, which will be denoted by $\mathcal{N}_u$.

We define the concept of the geometric image of a set $\Omega \subset \mathbb{R}^d$ under a function u (see [6]).

Definition 2.4. Let $u \in W^{1,p}\left(\Omega; \mathbb{R}^d\right)$ and let Ω_0 be the set of Proposition 2.3. We define the geometric image of Ω under u as $\operatorname{im}_{\mathrm{G}}(u;\Omega) := u\left(\Omega_0\right)$.

Note that the set Ω_0 described in Proposition 2.3 is not uniquely defined. In particular, if Ω_1 is another set with the same properties, then for any measurable $A \subset \Omega$, we have that $u(A \cap \Omega_0) = u(A \cap \Omega_1)$ a.e. and the two definitions of $\mathcal{N}_{u,A}$ that come from these sets coincide a.e. For example, in [12], Ω_0 is chosen as the set of approximate differentiability points of u.

A fundamental tool in this article, as well as in the context of nonlinear elasticity, is Brouwer's degree; see, *e.g.*, [20], Chapter 1. The degree on Ω of (the continuous representative of) a map in $W^{1,p}(\partial\Omega;\mathbb{R}^d)$ is defined as the degree of any continuous extension to $\overline{\Omega}$. Another important concept is the *topological image* (see [6, 7]).

Definition 2.5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $u \in C\left(\partial\Omega; \mathbb{R}^d\right)$. We define the topological image of Ω with respect to u as

$$\operatorname{im}_{\mathrm{T}}(u;\Omega) := \left\{ y \in \mathbb{R}^d \setminus u\left(\partial\Omega\right) : \deg(u;\Omega;y) \neq 0 \right\}.$$

Note that $\deg\left(u;\Omega;y\right) = 0$ for all y in the unbounded component of $\mathbb{R}^d \setminus u\left(\partial\Omega\right)$. Therefore $\operatorname{im}_{\mathrm{T}}(u;\Omega)$ is a bounded set, and is also open because of the continuity of the degree.

In this article we will assume that $\mathbb{R}^d \setminus \partial\Omega$ has exactly two connected components, which excludes the case $d = 1$. By the Jordan separation theorem, $\mathbb{R}^d \setminus \partial\Omega$ has exactly two connected components if $\partial\Omega$ is homeomorphic to the sphere, but the converse is not true, as shown by the classic example of the Warsaw circle. In addition, Ω is assumed to have a Lipschitz boundary. The following proposition clarifies an implication of these assumptions. This is probably a known result, but we have not found a specific reference.

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded, with a Lipschitz boundary and such that $\mathbb{R}^d \setminus \partial\Omega$ has exactly two connected components. Then Ω and $\partial\Omega$ are connected.*

Proof. Let us prove that $U := \mathbb{R}^d \setminus \overline{\Omega}$ and Ω are the two connected components of $\mathbb{R}^d \setminus \partial\Omega$. Clearly, U and Ω are open, disjoint and their union is $\mathbb{R}^d \setminus \partial\Omega$. This implies that any connected set of $\mathbb{R}^d \setminus \partial\Omega$ is contained either in Ω or in U.

Let Ω_1 be a connected component of Ω and let Ω_2 be the connected component of $\mathbb{R}^d \setminus \partial\Omega$ containing Ω_1. Then $\Omega_1 \subset \Omega_2 \subset \Omega$ and since Ω_1 is a connected component of Ω and Ω_2 is connected, then $\Omega_1 = \Omega_2$. This means that any connected component of Ω is also a connected component of $\mathbb{R}^d \setminus \partial\Omega$. Analogously, any connected component of U is a connected component of $\mathbb{R}^d \setminus \partial\Omega$. As $\mathbb{R}^d \setminus \partial\Omega$ has exactly two connected components, they have to be Ω and U.

Now we show that $\mathbb{R}^d \setminus \Omega = \overline{U}$. Since $U \subset \mathbb{R}^d \setminus \Omega$ and $\mathbb{R}^d \setminus \Omega$ is closed, then $\overline{U} \subset \mathbb{R}^d \setminus \Omega$. In addition, as Ω is a Lipschitz domain, every neighborhood of every point of $\partial\Omega$ has points in U. Therefore, $\partial\Omega \subset \overline{U}$ and, hence, $\mathbb{R}^d \setminus \Omega = U \cup \partial\Omega \subset \overline{U}$. Thus, $\mathbb{R}^d \setminus \Omega = \overline{U}$, which is connected as the closure of a connected set. The result in [21] shows that $\partial\Omega$ is connected. □

In [9], Theorem 4.2 it is proved that, when $\mathbb{R}^d \setminus \partial\Omega$ has exactly two connected components, for any $u \in C(\overline{\Omega};\mathbb{R}^d) \cap \mathrm{AIB}$, the function $\deg(u;\Omega;\cdot)$ is constantly 1 or -1 in $\operatorname{im}_{\mathrm{T}}(u;\Omega)$. By Tietze's extension theorem and the fact that Brouwer's degree only depends on the boundary values, we can give a version of that result for $u \in C(\partial\Omega; \mathbb{R}^d)$.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set such that $\mathbb{R}^d \setminus \partial\Omega$ has exactly two connected components and let $u \in C(\partial\Omega; \mathbb{R}^d) \cap \mathrm{AIB}$. Then there exists $\gamma \in \{\pm 1\}$ such that $\deg(u; \Omega; y) = \gamma$ for every $y \in \mathrm{im_T}(u; \Omega)$.*

3. Class $\overline{\mathcal{A}}_p\,(\Omega)$

In this section we recall the functional class $\overline{\mathcal{A}}_p\,(\Omega)$, which was introduced in [8].

Consider a map $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and denote by $u_{|\partial\Omega}$ its trace on $\partial\Omega$. If $u_{|\partial\Omega}$ belongs to $W^{1,p}(\partial\Omega; \mathbb{R}^d)$, with a small abuse of notation we write $u \in W^{1,p}(\partial\Omega; \mathbb{R}^d)$ and $u \in W^{1,p}(\Omega; \mathbb{R}^d) \cap W^{1,p}(\partial\Omega; \mathbb{R}^d)$. The derivative of $u_{|\partial\Omega}$ will be denoted by $D^\tau u$. That the divergence identities (1.1) hold in Ω means that for all $\phi \in C^1_c(\Omega)$ and $g \in C^1_c(\mathbb{R}^d, \mathbb{R}^d)$ we have

$$\int_\Omega [\operatorname{adj} Du(x)\, g(u(x))] \cdot D\phi(x)\, \mathrm{d}x + \int_\Omega \det Du(x)\, \phi(x) \operatorname{div} g(u(x))\, \mathrm{d}x = 0, \tag{3.1}$$

while that they hold in $\bar\Omega$ means that for all $\phi \in C^1(\bar\Omega)$ and $g \in C^1_c(\mathbb{R}^d, \mathbb{R}^d)$ we have

$$\begin{aligned} &\int_\Omega \left([\operatorname{adj} Du(x)\, g(u(x))] \cdot D\phi(x) + \det Du(x)\, \phi(x) \operatorname{div} g(u(x)) \right) \mathrm{d}x \\ &\qquad = \int_{\partial\Omega} \phi(x) \left(\operatorname{adj} D^\tau u(x)\, g(u(x)) \right) \cdot n(x)\, \mathrm{d}\mathcal{H}^{d-1}(x), \end{aligned} \tag{3.2}$$

where n is the unit exterior normal of $\partial\Omega$. Clearly, if (3.2) holds for every $\phi \in C^1(\bar\Omega)$ then (3.1) holds for every $\phi \in C^1_c(\Omega)$. The geometric meaning of maps satisfying (3.1) was shown in [4, 13] to be that they do not present cavitation or create new surface. Moreover, we know from [12] that they enjoy a great part of the regularity properties that maps in $W^{1,p}$ with $p > n$ do. Furthermore, the examples of [8] suggest that property (3.2) implies that cavitation of u is also excluded at the boundary.

Definition 3.1. Let $p \geq d - 1$. The class $\overline{\mathcal{A}}_p\,(\Omega)$ consists of those maps $u \in W^{1,p}(\Omega; \mathbb{R}^d) \cap W^{1,p}(\partial\Omega; \mathbb{R}^d)$ such that $\det Du \in L^1\,(\Omega)$ and (3.2) holds for all $\phi \in C^1(\overline{\Omega})$ and $g \in C^1_c(\mathbb{R}^d; \mathbb{R}^d)$.

The following result is [8], Proposition 8.4 and constitutes an important step to prove injectivity of maps.

Proposition 3.2. *Let $p > d - 1$. If $u \in \overline{\mathcal{A}}_p(\Omega)$ with $\det Du \geq 0$ a.e., then $\deg(u; \Omega; \cdot) = \mathcal{N}_u$ a.e., $\mathrm{im_G}(u; \Omega) = \mathrm{im_T}(u; \Omega)$ a.e. and $u \in L^\infty(\Omega; \mathbb{R}^d)$.*

4. Injectivity of maps in $\overline{\mathcal{A}}_p \cap \mathrm{AIB}$

Our aim is to prove a refined version of the following theorem, which can be found in [8], Theorem 9.1.

Theorem 4.1. *Let $p > d - 1$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz open set. Let $u, u_0 \in \overline{\mathcal{A}}_p(\Omega)$ satisfy $u_{|\partial\Omega} = u_{0|\partial\Omega}$, $\det Du > 0$ a.e., $\det Du_0 \geq 0$ a.e. and u_0 is injective a.e. Then u is injective a.e. and $\mathrm{im_G}(u; \Omega) = \mathrm{im_G}(u_0; \Omega)$ a.e.*

To be precise, our main goal is to avoid the a.e. injectivity assumption in Ω of the boundary value u_0, replacing it with condition AIB.

We first state a version of the continuity of the degree.

Proposition 4.2. *Let $\{u_k\}_{k\in\mathbb{N}} \subset C(\partial\Omega; \mathbb{R}^d)$ be a sequence such that $u_k \to u$ uniformly on $\partial\Omega$ as $k \to \infty$. Then for every $y \in \mathrm{im_T}(u; \Omega)$ there exists some $k_0 \in \mathbb{N}$ such that $\deg(u_k; \Omega; y) = \deg(u; \Omega; y)$ for all $k \geq k_0$.*

Proof. Let $y \in \operatorname{im_T}(u;\Omega)$, so $y \notin u(\partial\Omega)$. By the uniform convergence of $\{u_k\}_{k\in\mathbb{N}}$ and the continuity of the degree, there exists some $k_0 \in \mathbb{N}$ such that $y \notin u_k(\partial\Omega)$ for every $k \geq k_0$ and $\deg(u_k;\Omega;y) = \deg(u;\Omega;y)$ for every $k \geq k_0$. □

With this, we proceed to prove the theorem.

Theorem 4.3. *Let $p > d-1$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz open set such that $\mathbb{R}^d \setminus \partial\Omega$ has exactly two connected components. Let $u \in \overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ and assume that $\det Du > 0$ a.e. Then u is injective a.e. in Ω and $\operatorname{im_G}(u;\Omega) = \operatorname{im_T}(u;\Omega)$ a.e.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}} \subset C(\partial\Omega;\mathbb{R}^d)$ be the sequence uniformly convergent to u from Definition 2.1. By Theorem 2.7 we have that $\deg(u_k;\Omega;\cdot) = \gamma_k$ in $\operatorname{im_T}(u_k;\Omega)$ for some $\gamma_k \in \{\pm 1\}$ and $k \in \mathbb{N}$.

By Proposition 3.2 and the fact that $\mathcal{N}_u$ is a non-negative function we can see that $\deg(u;\Omega;\cdot) \geq 0$ a.e. in $\mathbb{R}^d \setminus u(\partial\Omega)$. In fact, by the continuity of the degree, $\deg(u;\Omega;\cdot) \geq 0$ everywhere in $\mathbb{R}^d \setminus u(\partial\Omega)$. In particular, $\deg(u;\Omega;\cdot) > 0$ everywhere in $\operatorname{im_T}(u,\Omega)$. By Proposition 4.2, for every $y \in \operatorname{im_T}(u;\Omega)$ there exists some $k_0 \in \mathbb{N}$ such that $\deg(u_k;\Omega;y) = \deg(u;\Omega;y)$ for all $k \geq k_0$, so $y \in \operatorname{im_T}(u_k;\Omega)$ and $\deg(u_k;\Omega;y) = 1$. Therefore,

$$\deg(u;\Omega;\cdot) = 1 \quad \text{in } \operatorname{im_T}(u;\Omega).$$

By Proposition 3.2 again,

$$\mathcal{N}_u = 1 \text{ a.e. in } \operatorname{im_G}(u;\Omega).$$

As $\det Du > 0$ a.e., u satisfies Lusin's N^{-1} condition, *i.e.*, the preimage of a subset of $\mathbb{R}^d$ with measure zero has measure zero (see, *e.g.*, [22], Rem. 2.3 (b)). This implies that u is injective a.e. □

Theorem 4.3 is neither stronger nor weaker than Theorem 4.1. Indeed, Theorem 4.3 does not request an a.e. injective map to coincide on $\partial\Omega$ with u but needs for $\mathbb{R}^d \setminus \partial\Omega$ to have exactly two connected components.

5. Counterexample to global injectivity in $\mathcal{A}_p \cap \mathrm{AIB}$

The family $\overline{\mathcal{A}}_p(\Omega)$, as opposed to $\mathcal{A}_p(\Omega)$ (see [12]), requires certain regularity at the boundary, as shown in [8], Section 5. Similarly, deformations in the family AIB enjoy some regularity at the boundary, as a limit of continuous injective mappings. Therefore, in the class $\overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ two regularity conditions are imposed on the boundary. In this section we show that both have to be assumed, in the sense that the conclusion of Theorem 4.3 does not hold in the class $\mathcal{A}_p(\Omega) \cap \mathrm{AIB}$.

The counterexample that we construct is a variant of [6], Figure 6 (which was also used in [8], Exam. 5.3). Let $\Omega = (-1,1) \times (0,1)$ be reference configuration, which is transformed under several deformations depicted in Figure 1.

The first of these deformations is

$$u_1 : \Omega \to \mathbb{R}^2, \qquad u_1(x) = \frac{|x|_\infty + 3}{4|x|_\infty} x,$$

where $|x|_\infty$ is the max-norm of the vector x. The map u_1 creates a cavity on the boundary of Ω. The second deformation, $u_2 : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$u_2(x_1,x_2) = \begin{cases} (x_1, 1-(1-x_2)(7-8|x_1|)) & \text{if } |x_1| < \frac{3}{4}, \\ (x_1,x_2) & \text{if } |x_1| \geq \frac{3}{4}, \end{cases}$$

grips the material near the surface of the cavity and stretches it down. The third deformation, $u_3 : \mathbb{R}^2 \to \mathbb{R}^2$, closes the cavity leaving some part of the material outside the boundary and then rescales the main part of the

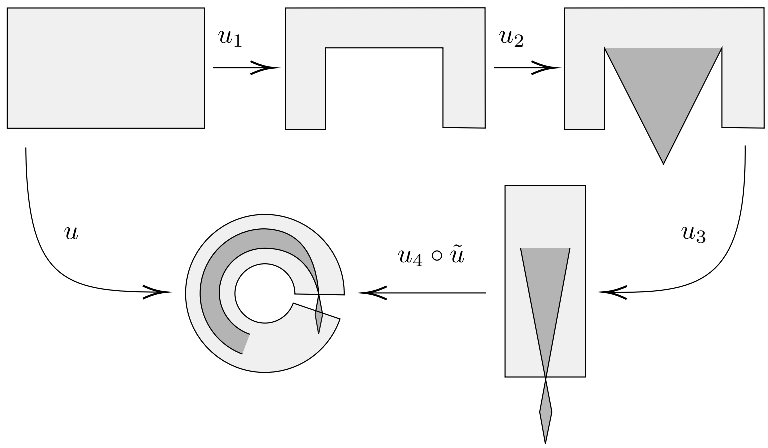

FIGURE 1. Counterexample to a.e. injectivity in $\mathcal{A}_p((-1,1)\times(0,1))\cap\mathrm{AIB}$.

body to fit the rectangle $[1,2]\times[-1,1]$; the leaked part of the material follows the same rescaling. This third deformation is defined by

$$u_3(x_1,x_2) = \begin{cases} (\frac{1}{2}(\operatorname{sign} x_1(1-4(1-|x_1|)(1-x_2))+3), 2x_2-1) & \text{if } 0 \le x_2 < \frac{3}{4},\ \frac{3}{4} < |x_1|, \\ (\frac{4x_1x_2}{4x_2+3}+\frac{3}{2}, 2x_2-1) & \text{if } |x_1| < \frac{4x_2+3}{8},\ 0 \le x_2 < \frac{3}{4}, \\ (\frac{-4x_1x_2}{4x_2+3}+\frac{3}{2}, 2x_2-1) & \text{if } |x_1| < \frac{4x_2+3}{8},\ -\frac{1}{4} \le x_2 < 0, \\ (\frac{x_1+3}{2}, 2x_2-1) & \text{elsewhere.} \end{cases}$$

For the last map of the deformation, we first change to polar coordinates on the right half-plane $\widetilde{u}:(0,\infty)\times\mathbb{R}\to\mathbb{R}^2$ given by $\widetilde{u}(x_1,x_2) = (\sqrt{x_1^2+x_2^2}, \arctan(x_2/x_1))$ and define

$$u_4:\mathbb{R}^2\to\mathbb{R}^2, \qquad u_4(r,\theta) = (r\cos(\alpha\theta), r\sin(\alpha\theta))$$

for some $\frac{4\pi}{5} < \alpha < \pi$. The map $u_4\circ\widetilde{u}$ is a revolution of the "leaking rectangle" around the point $0\in\mathbb{R}^2$ and creates an overlaping surface between the leaked part of the material and the top part of Ω under the previous deformations. Therefore, the deformation $u = u_4\circ\widetilde{u}\circ u_3\circ u_2\circ u_1$ is not injective a.e.

Arguing as in [8], Examples 5.2 and 5.3, one can show that the map u is in $\mathcal{A}_p(\Omega)$ for any $1\le p<2$, but not in $\overline{\mathcal{A}}_p(\Omega)$. In addition, $\det Du > 0$ a.e. Finally, $u_{|\partial\Omega} = u_{0|\partial\Omega}$ for some diffeomorphism $u_0:\overline{\Omega}\to\mathbb{R}^2$, so in particular $u\in\mathrm{AIB}$. Indeed, this was shown in [6, 8] for the map $u_3\circ u_2\circ u_{1|\partial\Omega}$, while the part $u_4\circ\widetilde{u}$ of the deformation maintains the same property.

The key point allowing the loss of injectivity is that the cavitation at the boundary permits a lack of the monotonicity of the degree with respect to the domain; that is, for many open sets $U\subset\Omega$ it is not true that $\deg(u;\Omega;\cdot)\ge\deg(u;U;\cdot)$. Explicit instances of such U are $(-1+\delta,1-\delta)\times(\delta,1-\delta)$ for $\delta>0$ small.

6. Weak continuity of minors of tangential derivatives

The objective of the rest of this article is to show the existence of minimizers of an appropriate functional in the class $\overline{\mathcal{A}}_p(\Omega)\cap\mathrm{AIB}$. For this, we will show the weak continuity of the minors of $D^\tau u$ in $W^{1,p}(\partial\Omega;\mathbb{R}^d)$. The map $D^\tau u$ is the tangential derivative of $u\in W^{1,p}(\partial\Omega;\mathbb{R}^d)$, which sends $\mathcal{H}^{d-1}$-a.e. $x\in\partial\Omega$ to $D^\tau u(x)\in\mathcal{L}(T_x\partial\Omega;\mathbb{R}^d)$. Here, $T_x\partial\Omega$ is the tangent space of $\partial\Omega$ at x. We also denote by $T\partial\Omega = \{(x,v): x\in\partial\Omega,\ v\in T_x\partial\Omega\}$ the tangent bundle of $\partial\Omega$, and define $T^d\partial\Omega \coloneqq \{(x,F): x\in\partial\Omega, F\in(T_x\partial\Omega)^d\}$.

6.1. Minors of linear maps

Let $V \subseteq \mathbb{R}^d$ be an m-dimensional vector space, for some number $1 \leq m \leq d$, and let $1 \leq k \leq m$ be an integer. Let $L \in \mathcal{L}(V;\mathbb{R}^d)$, fix a basis in V and consider the matrix representation of L with respect to that basis in V and the canonical basis in $\mathbb{R}^d$. Given $1 \leq i_1 < \cdots < i_k \leq d$ and $1 \leq j_1 < \cdots < j_k \leq m$, we denote by

$$M_{\substack{i_1 \dots i_k \\ j_1 \dots j_k}}(L)$$

the minor of order k resulting by the choice of rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$ in the matrix representation of L. There are $\binom{m}{k}\binom{d}{k}$ minors of L of order k, and $\sum_{k=1}^{m} \binom{m}{k}\binom{d}{k}$ minors of L of any order. We will denote this last number by ν_m and we will use the convention that $\nu_0 = 0$; this notation does not indicate the dependence on d, since d is fixed throughout the article. Particularly important are ν_d, the number of minors of any $d \times d$ matrix, and ν_{d-1}, the number of minors of any $d \times (d-1)$ matrix. This notation will be of use in Sections 7 and 8.

Let $k \leq m$. We define $M_k(L)$ as the ordered sequence of all minors of order k of L, $M_k^0(L)$ as the ordered sequence of the minors of order k of L not involving the last column of L, and $M_k^1(L)$ as the ordered sequence of the minors of order k of L involving the last column of L, all with respect to the matrix representation of L. Thus,

$$M_k(L) \coloneqq \left(M_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_k}}(L) \right)_{1 \leq j_1 < \cdots < j_k \leq m}^{1 \leq i_1 < \cdots < i_k \leq d},$$

$$M_k^0(L) \coloneqq \left(M_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_k}}(L) \right)_{1 \leq j_1 < \cdots < j_k \leq m-1}^{1 \leq i_1 < \cdots < i_k \leq d} \quad \text{and} \quad M_k^1(L) \coloneqq \left(M_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_{k-1},m}}(L) \right)_{1 \leq j_1 < \cdots < j_{k-1} \leq m-1}^{1 \leq i_1 < \cdots < i_k \leq d}.$$

Moreover, let $\{v_1, \dots, v_d\}$ be a basis of $\mathbb{R}^d$ such that $\{v_1, \dots, v_{d-1}\}$ is a basis of some subspace $W \subset \mathbb{R}^d$ and let $F \in \mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)$; then,

$$M_k^0(F) = M_k(F_{|W}). \tag{6.1}$$

Finally, we define the sequence of all minors of any order of L as

$$M(L) \coloneqq \left(M_1^0(L), \dots, M_m^0(L), M_1^1(L), \dots, M_m^1(L) \right), \tag{6.2}$$

with the same convention. For the sake of notation, we also define

$$M^0(L) \coloneqq \left(M_1^0(L), \dots, M_m^0(L) \right) \qquad \text{and} \qquad M^1(L) \coloneqq \left(M_1^1(L), \dots, M_m^1(L) \right). \tag{6.3}$$

Following the previous notation, we have that $M(L) \in \mathbb{R}^{\nu_m}$. Moreover, when $m = d$, the last component of $M(L)$ is $\det(L)$.

We will use the same notation for the minors if L is a given matrix instead of a linear map.

6.2. Convergence of minors of tangential derivatives

Definition 6.1. We say that $\{v_1, \dots, v_{d-1}\}$ is a measurable basis of $T\partial\Omega$ if $v_i : \partial\Omega \to \mathbb{R}^d$, for $i = 1, \dots, d-1$, is a measurable map and $\{v_1(x), \dots, v_{d-1}(x)\}$ is a basis of $T_x\partial\Omega$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$. The measurable basis is called orthonormal if so is $\{v_1(x), \dots, v_{d-1}(x)\}$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$.

When such basis is fixed we can consider $D^\tau u$ as a map from $\partial\Omega$ to $\mathbb{R}^{d\times(d-1)}$, and $M(D^\tau u)$ as a map from $\partial\Omega$ to $\mathbb{R}^{\nu_{d-1}}$. Moreover, we can choose the map $n : \partial\Omega \to S^{d-1}$ defined as

$$n(x) = \frac{v_1(x) \wedge \cdots \wedge v_{d-1}(x)}{\|v_1(x) \wedge \cdots \wedge v_{d-1}(x)\|}$$

such that the vector $n(x)$ is the outward normal to Ω at x and

$$\{v_1(x), \dots, v_{d-1}(x), n(x)\} \text{ is a basis of } \mathbb{R}^d. \tag{6.4}$$

The following observation calculates the minors M^1 of a type of maps relevant in Sections 7 and 8.

Remark 6.2. Consider the basis (6.4) of $\mathbb{R}^d$. If $L \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$ satisfies $Ln(x) = 0$ then $M^1(L) = 0$.

Definition 6.3. Let $\mathcal{V} = \{v_1, \dots, v_{d-1}\}$ be a measurable basis of $T\partial\Omega$, for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ let $P_x : \mathbb{R}^{d-1} \to T_x\partial\Omega$ with $P_x e_i = v_i(x)$ for each e_i in the canonical basis of $\mathbb{R}^{d-1}$. We say that $\mathcal{V}$ is an L^∞ basis of $T\partial\Omega$ if there exists $\tilde{P}_x : \mathbb{R}^d \to \mathbb{R}^d$ a linear extension of P_x such that $\tilde{P}_x, \tilde{P}_x^{-1} \in L^\infty(\partial\Omega; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d))$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$.

We will use $\mathcal{V} = \{v_1, \dots, v_{d-1}\}$ to refer to a basis of $T\partial\Omega$ and $\mathcal{V}_x = \{v_1(x), \dots, v_{d-1}(x)\}$ for a given $x \in \partial\Omega$ with the subindex notation, to refer to the associated basis of $T_x\partial\Omega$.

We introduce the notation regarding the parametrization of $\partial\Omega$ (see, *e.g.*, [8], Sect. 3). Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be projection on the first $d-1$ coordinates, and $\eta : \mathbb{R}^{d-1} \to \mathbb{R}^d$ the function $\hat{z} \mapsto (\hat{z}, 0)$. As Ω is a Lipschitz domain, there exist $r, \beta > 0$, an integer $m_0 \geq 1$ and bi-Lipschitz maps

$$G_i : [0,r]^{d-1} \times [-\beta, \beta] \to \mathbb{R}^d, \qquad i \in \{1, \dots, m_0\}$$

such that, when one defines $\Gamma_i = G_i((0,r)^{d-1} \times \{0\})$, we have that $\{\Gamma_i\}_{i=1}^{m_0}$ is an open cover of $\partial\Omega$. For each $i \in \{1, \dots, m_0\}$ we define the bi-Lipschitz map $\Psi_i := G_i \circ \eta : [0,r]^{d-1} \to \Gamma_i$. For $\hat{z} \in [0,r]^{d-1}$, we consider the matrix representation of $D\Psi_i(\hat{z})$, with columns $D\Psi_i^{(j)}(\hat{z}) \in \mathbb{R}^d$ for $j = 1, \dots, d-1$, and the basis $\mathcal{B}_{\Psi_i(\hat{z})} := \{D\Psi_i^{(1)}(\hat{z}), \dots, D\Psi_i^{(d-1)}(\hat{z})\}$ of $T_{\Psi_i(\hat{z})}\Gamma_i$. We will use the notation $\mathcal{B}_{\Psi_i}$ whenever we use the basis $\mathcal{B}_{\Psi_i(\hat{z})}$ for every $\hat{z} \in [0,r]^{d-1}$. For any $u : \partial\Omega \to \mathbb{R}^d$, the functions

$$\mathcal{L}_i(u) : \pi(G_i^{-1}(\Gamma_i)) \to \mathbb{R}^d, \quad \mathcal{L}_i(u) := u \circ \Psi_i, \qquad i \in \{1, \dots, m_0\}$$

satisfy the following property (see [8], Lem. 3.3).

Lemma 6.4. *Let $p \geq 1$. For each $n \in \mathbb{N}$,*

(i) let $u_n, u \in W^{1,p}(\partial\Omega; \mathbb{R}^d)$. Then $u_n \rightharpoonup u$ in $W^{1,p}(\partial\Omega; \mathbb{R}^d)$ as $n \to \infty$ if and only if $\mathcal{L}_i(u_n) \rightharpoonup \mathcal{L}_i(u)$ in $W^{1,p}((0,r)^{d-1}; \mathbb{R}^d)$ as $n \to \infty$ for all $i = 1, \dots, m_0$.
(ii) let $u_n, u \in L^p(\partial\Omega; \mathbb{R}^d)$. Then $u_n \rightharpoonup u$ in $L^p(\partial\Omega; \mathbb{R}^d)$ as $n \to \infty$ if and only if $\mathcal{L}_i(u_n) \rightharpoonup \mathcal{L}_i(u)$ in $L^p((0,r)^{d-1}; \mathbb{R}^d)$ as $n \to \infty$ for all $i = 1, \dots, m_0$.

Although there is an intrinsic definition of the spaces $W^{1,p}(\partial\Omega; \mathbb{R}^d)$ and $L^p(\partial\Omega; \mathbb{R}^d)$ and their convergences, we will always use them refering to the result above. We also have the following result regarding the basis $\mathcal{B}_{\Psi_i}$.

Lemma 6.5. *$\mathcal{B}_{\Psi_i}$ is an L^∞ basis of $T\Gamma_i$ for each $i \in \{1, \dots, m_0\}$. Moreover, there exists an L^∞ basis of $T\partial\Omega$.*

Proof. Fix $i \in \{1, \dots, m_0\}$. Observe that $DG_i(x) : \mathbb{R}^d \to \mathbb{R}^d$ extends $D\Psi_i(x) : \mathbb{R}^{d-1} \to T_x\partial\Omega$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ in the sense that $DG_i(x)$ can be seen as a map from $\mathbb{R}^{d-1} \times \{0\}$ to $\mathbb{R}^d$. Since G_i is a bi-Lipschitz map we have that $DG_i : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and its inverse $(DG_i)^{-1} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are essentially bounded.

To construct an L^∞ basis of $T\partial\Omega$ we can join the bases of each $T\Gamma_i$ in the following way: for $x \in \Gamma_1$ we use the basis $\mathcal{B}_{\Psi_1}$, and for $x \in \Gamma_s \setminus \bigcup_{j=1}^{s-1} \Gamma_j$ for some $2 \leq s \leq m_0$ we use $\mathcal{B}_{\Psi_s}$. □

Unlike in Section 6.1, we need to give a precise definition of the convergence of minors of a linear map without the need of fixing bases.

Definition 6.6. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of maps $f_n : T\partial\Omega \to \mathbb{R}^d$ such that $f_n(x,\cdot) : T_x\partial\Omega \to \mathbb{R}^d$ is linear for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$. We say that $Ml(f_n) \rightharpoonup Ml(f)$ in $L^q(\partial\Omega)$ for some $q \geq 1$ if there exists $\mathcal{V}$ an L^∞ basis of $T\partial\Omega$ such that $M(f_n) \rightharpoonup M(f)$ in $L^q(\partial\Omega;\mathbb{R}^{\nu_{d-1}})$ where the matrix representation of each f_n and f is taken with respect to $\mathcal{V}$ and the canonical basis of $\mathbb{R}^d$.

The convergence of minors of a linear map is independent of the choice of the L^∞ basis.

Proposition 6.7. *Let $\mathcal{V}$ and $\mathcal{B}$ be two L^∞ bases of $T\partial\Omega$, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of maps $f_n : T\partial\Omega \to \mathbb{R}^d$ such that $f_n(x,\cdot) : T_x\partial\Omega \to \mathbb{R}^d$ is linear for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and such that $M(f_n) \rightharpoonup M(f)$ in $L^q(\partial\Omega;\mathbb{R}^{\nu_{d-1}})$ for some $q \geq 1$ where the matrix representation of f and f_n are with respect to $\mathcal{V}$ and the canonical basis of $\mathbb{R}^d$. Then $M(f_n) \rightharpoonup M(f)$ in $L^q(\partial\Omega;\mathbb{R}^{\nu_{d-1}})$ where the matrix representation of f and f_n are with respect to $\mathcal{B}$ and the canonical basis of $\mathbb{R}^d$.*

Proof. Let $\mathcal{V} = \{v_1,\dots,v_{d-1}\}$ and $\mathcal{B} = \{b_1,\dots,b_{d-1}\}$. Denote by $\mathcal{V}_{f_n}$ and $\mathcal{B}_{f_n}$ the matrix representations of f_n with respect to $\mathcal{V}$ and $\mathcal{B}$, respectively, and the canonical basis of $\mathbb{R}^d$, and denote by $\mathcal{V}_{f_n(x)}$ and $\mathcal{B}_{f_n(x)}$ the matrix representations of $f_n(x,\cdot)$ with respect to $\mathcal{V}_x$ and $\mathcal{B}_x$, respectively, and the canonical basis of $\mathbb{R}^d$. For $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ there exist measurable maps $\{a_{i,j}\}_{j=1}^{d-1}$ from $\partial\Omega$ to $\mathbb{R}$ such that $v_i(x) = \sum_{j=1}^{d-1} a_{j,i}(x)b_j(x)$ for each $i \in \{1,\dots,d-1\}$, *i.e.*, for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ there exists $A_x = (a_{j,i}(x)) \in \mathbb{R}^{(d-1)\times(d-1)}$ such that $\mathcal{V}_{f_n(x)}A_x^{-1} = \mathcal{B}_{f_n(x)}$. Taking minors we obtain that $M(\mathcal{V}_{f_n(x)}A_x^{-1}) = M(\mathcal{B}_{f_n(x)})$ and since $\mathcal{V}$ and $\mathcal{B}$ are L^∞ bases we have that A_x and A_x^{-1} are bounded. Therefore, by the Cauchy-Binet formula, there exists a linear map $\mathfrak{F}_x : \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}^{\nu_{d-1}}$ such that $\mathfrak{F}_x(M(\mathcal{V}_{f_n(x)})) = M(\mathcal{B}_{f_n(x)})$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$. In the same way, there exists a linear map $\mathfrak{F} : \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}^{\nu_{d-1}}$ such that $\mathfrak{F}(M(\mathcal{V}_{f_n})) = M(\mathcal{B}_{f_n})$ and hence, since $M(\mathcal{V}_{f_n}) \rightharpoonup M(\mathcal{V}_f)$ in $L^q(\partial\Omega;\mathbb{R}^{\nu_{d-1}})$ we also have that $M(\mathcal{B}_{f_n}) \rightharpoonup M(\mathcal{B}_f)$ in $L^q(\partial\Omega;\mathbb{R}^{\nu_{d-1}})$. □

A result on the weak continuity of minors was proved in [23], Proposition 15 using geometric tools. We present a straightforward proof in the next proposition.

Proposition 6.8. *Let $p > d-1$. Let $u \in W^{1,p}(\partial\Omega;\mathbb{R}^d)$ and let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\partial\Omega;\mathbb{R}^d)$ be such that $u_n \rightharpoonup u$ in $W^{1,p}(\partial\Omega;\mathbb{R}^d)$ as $n \to \infty$. Then $Ml(D^\tau u_n) \rightharpoonup Ml(D^\tau u)$ in $L^1(\partial\Omega)$ as $n \to \infty$.*

Proof. By Lemma 6.4(i) we have that $\mathcal{L}_i(u_n) \rightharpoonup \mathcal{L}_i(u)$ in $W^{1,p}((0,r)^{d-1};\mathbb{R}^d)$ for each $i \in \{1,\dots,m_0\}$. For each $i \in \{1,\dots,m_0\}$ the result of [24], Theorem 8.20 gives us that $M(D\mathcal{L}_i(u_n)) \rightharpoonup M(D\mathcal{L}_i(u))$ in $L^1((0,r)^{d-1};\mathbb{R}^{\nu_{d-1}})$ where both matrix representations are with respect to the canonical bases.

As $\mathcal{L}_i(u) = u \circ \Psi_i$ we have that $D\mathcal{L}_i(u)(\hat{z}) = D^\tau u(\Psi_i(\hat{z}))D\Psi_i(\hat{z}) : \mathbb{R}^{d-1} \to \mathbb{R}^d$ for each $i \in \{1,\dots,m_0\}$ and any $\hat{z} \in (0,r)^{d-1}$. Fix the L^∞ basis $\mathcal{B}_{\Psi_i}$ and observe that for any $\hat{z} \in (0,r)^{d-1}$ we have that $D\Psi_i(\hat{z}) : \mathbb{R}^{d-1} \to T_{\Psi_i(\hat{z})}\partial\Omega$ is defined by $e_j \mapsto D\Psi_i(\hat{z})e_j = D\Psi_i^{(j)}(\hat{z})$ for each e_j in the canonical basis of $\mathbb{R}^{d-1}$; consequently, $D\Psi_i = \mathrm{Id}$. On the other hand, since $D\mathcal{L}_i(u) = D^\tau u(\Psi_i)$ with respect to $\mathcal{B}_{\Psi_i}$ and the canonical basis, we have that $M(D^\tau u_n(\Psi_i)) \rightharpoonup M(D^\tau u(\Psi_i))$ in $L^1((0,r)^{d-1};\mathbb{R}^{\nu_{d-1}})$, where again, the matrices are with respect to $\mathcal{B}_{\Psi_i}$ and the canonical basis of $\mathbb{R}^{d-1}$. By Definition 6.6 this means that

$$Ml(D^\tau u_n(\Psi_i)) \rightharpoonup Ml(D^\tau u(\Psi_i)) \qquad \text{in} \qquad L^1((0,r)^{d-1}). \tag{6.5}$$

Observe that $Ml(D^\tau u(\Psi_i)) = Ml(D^\tau u) \circ \Psi_i = \mathcal{L}_i(Ml(D^\tau u))$ and hence, expression (6.5) means that

$$\mathcal{L}_i(Ml(D^\tau u_n)) \rightharpoonup \mathcal{L}_i(Ml(D^\tau u)) \qquad \text{in} \qquad L^1((0,r)^{d-1};\mathbb{R}^{\nu_{d-1}}).$$

Lemma 6.4(ii) gives us that $Ml(D^\tau u_n) \rightharpoonup Ml(D^\tau u)$ in $L^1(\partial\Omega)$. □

7. Tangential polyconvexity and quasiconvexity

We first give a definition used along the rest of the article (see [24]).

Definition 7.1. Let $V \subset \mathbb{R}^d$ be a m-dimensional vector space for some natural $m \leq d$.

(i) A function $f : \mathbb{R}^{d\times m} \to \mathbb{R}$ is said to be polyconvex if there exists $\varphi : \mathbb{R}^{\nu_m} \to \mathbb{R}$ convex such that $f(F) = \varphi(M(F))$.
(ii) A function $W_0 : \mathcal{L}(V;\mathbb{R}^d) \to \mathbb{R}$ is called polyconvex if there exist $\mathcal{B}_V$ a measurable basis of V and a convex function $\Phi : \mathbb{R}^{\nu_m} \to \mathbb{R}$ such that $W_0(F) = \Phi(M(F))$ for all $F \in \mathcal{L}(V;\mathbb{R}^d)$ in the sense of (i) where $M(F)$ refers to the minors of the matrix representation of F with respect to $\mathcal{B}_V$ and the canonical basis in $\mathbb{R}^d$.

We will use the cases $m = d$ and $m = d - 1$.

We now define the energy functional for which we will prove the existence of minimizers in $\overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$. As natural in the theory of nonlinear elasticity, the functional will be of the form

$$I[u] = \int_\Omega W(x, u(x), Du(x))\mathrm{d}x + \int_{\partial\Omega} U(x, u(x), D^\tau u(x), n(x))\mathrm{d}\mathcal{H}^{d-1}(x), \tag{7.1}$$

where the function W refers to the elastic energy of the deformation u applied on the body occupying Ω in its reference configuration, and U refers to the elastic energy of the deformation u applied to the boundary of the body. The potentials W and U do not usually depend on $u(x)$, but we have included them here since the theory applies also for this case. In fact, external forces depend on $u(x)$. Recall that $D^\tau u$ is the tangential derivative of $u_{|\partial\Omega}$. Proofs of these kind often only take into account the functional over Ω, and follow standard polyconvexity and lower semicontinuity reasonings. However, as we are working in the class AIB, we also need the term over $\partial\Omega$, and, hence, an analogous concept to polyconvexity on the boundary.

Remark 7.2. The domain of W is $\Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_+$, however, the functional U has a more specific domain:

$$\mathcal{D}_U := \{(x, y, F, n) : x \in \partial\Omega,\ y \in \mathbb{R}^d,\ F \in \mathcal{L}(T_x\partial\Omega;\mathbb{R}^d),\ n \in N_x\partial\Omega \cap S^{d-1}\}.$$

Note that $\mathcal{L}(T_x\partial\Omega;\mathbb{R}^d) \simeq (T_x\partial\Omega)^d$.

In order to prove the existence of minimizers of I we need to prove weak lower semicontinuity on the boundary integral of (7.1). In the same way that polyconvexity is sufficient for semicontinuity on the integral over Ω, the following concept will provide a sufficient condition for semicontinuity on $\partial\Omega$.

Definition 7.3. A function $U : T^d\partial\Omega \to \mathbb{R}$ is said to be tangentially polyconvex if there exists a measurable basis of $T\partial\Omega$ and a function $\Phi : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}$ such that $\Phi(x,\cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and $U(x, F) = \Phi(x, M(F))$ for every $F \in \mathcal{L}(T_x\partial\Omega)^d$.

The definition of tangential polyconvexity is independent of the choice of the measurable basis.

Proposition 7.4. *Let $U : T^d\partial\Omega \to \mathbb{R}$ be tangentially polyconvex and let $\mathcal{V} = \{\tilde{v}_1, \ldots, \tilde{v}_{d-1}\}$ be a measurable basis of $T\partial\Omega$. Then there exists $\Phi_\mathcal{V} : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}$ such that $\Phi_\mathcal{V}(x,\cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and such that $U(x, F) = \Phi_\mathcal{V}(x, M(F))$ for every $F \in \mathcal{L}(T_x\partial\Omega;\mathbb{R}^d)$, where the matrix representation of F is with respect to $\mathcal{V}$ and the canonical basis.*

Proof. There exist a measurable basis $\mathcal{B} := \{v_1, \ldots, v_{d-1}\}$ of $T\partial\Omega$ and a map $\Phi : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}$ such that $\Phi(x,\cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and $U(x, F) = \Phi(x, M(\mathcal{B}_F))$ for every $F \in \mathcal{L}(T_x\partial\Omega;\mathbb{R}^d)$ where $\mathcal{B}_F$ refers to the matrix representation of F with respect to $\{v_1(x), \ldots, v_{d-1}(x)\}$ and the canonical basis of $\mathbb{R}^d$. Let $\mathcal{V}_F$ be the matrix representation of F with respect to $\{\tilde{v}_1(x), \ldots, \tilde{v}_{d-1}(x)\}$ and the canonical basis of $\mathbb{R}^d$, there exist measurable maps $\{a_{i,j}\}_{j=1}^{d-1}$ from $\partial\Omega$ to $\mathbb{R}$ such that $\tilde{v}_i(x) = \sum_{j=1}^{d-1} a_{j,i}(x)v_j(x)$ and therefore that there also exists a matrix $A_x = (a_{i,j}(x))_{i,j} \in \mathbb{R}^{(d-1)\times(d-1)}$ such that $\mathcal{V}_F = \mathcal{B}_F A_x^{\mathrm{T}}$. Taking minors we obtain that

$M(\mathcal{B}_F) = M(\mathcal{V}_F A^{-\mathrm{T}})$, and by the Cauchy-Binet formula there exists a linear map $\mathfrak{F}_{A_x} : \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}^{\nu_{d-1}}$ such that $M(\mathcal{B}_F) = \mathfrak{F}_{A_x}(M(\mathcal{V}_F))$. As the composition of a linear map with a convex map is convex, we have that

$$U(x, F) = \Phi(x, M(\mathcal{B}_F)) = \Phi(x, \mathfrak{F}_{A_x}(M(\mathcal{V}_F))) = \Phi_{\mathcal{V}}(x, M(\mathcal{V}_F)).$$

for some convex map $\Phi_{\mathcal{V}} : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}$ defined by $(x, (a_1, \ldots, a_{\nu_{d-1}})) \mapsto \Phi_{\mathcal{B}}(x, \mathfrak{F}_{A_x}((a_1, \ldots, a_{\nu_{d-1}})))$. □

The relationship between tangential polyconvexity and usual polyconvexity is presented in the following proposition.

Proposition 7.5. *The following are properties of tangential polyconvexity.*

(i) Let $\widetilde{U} : \partial\Omega \times \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}$. The map $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^d) \ni A \mapsto \widetilde{U}(x, A_{|T_x\partial\Omega})$ is polyconvex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ if and only if the map $U : T^d\partial\Omega \to \mathbb{R}$ defined as $U := \widetilde{U}_{|T^d\partial\Omega}$ is tangentially polyconvex.

(ii) Let $U : \partial\Omega \times \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}$ be such that $U(x, \cdot)$ is polyconvex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$, then $U_{|T^d\partial\Omega}$ is tangentially polyconvex.

Proof. (i) Assume that the map $U : T^d\partial\Omega \to \mathbb{R}$ is tangentially polyconvex. Then there exists some $\Phi : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}$ such that $\Phi(x; \cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and $U(x, F) = \Phi(x, M(F))$ for all $F \in (T_x\partial\Omega)^d$ with respect to a measurable basis in $T_x\partial\Omega$. We define

$$\begin{aligned} \widetilde{\Phi} : \partial\Omega \times \mathbb{R}^{\nu_d} &\to \mathbb{R} \\ (x, (a_1, \ldots, a_{\nu_d})) &\mapsto \Phi(x, (a_1, \ldots, a_{\nu_{d-1}})). \end{aligned}$$

By (6.1) and (6.3), for any $\tilde{F} \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$ extending F we have that $M^0(\tilde{F}) = M(F)$. Moreover, by (6.2), $M(\tilde{F}) = (M(F), M_1^1(\tilde{F}), \ldots, M_d^1(\tilde{F}))$. As a consequence, we have that $\Phi(x, M(F)) = \widetilde{\Phi}(x, M(\tilde{F}))$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and therefore, that $\widetilde{U}(x, \tilde{F}) = U(x, F)$. Finally $\widetilde{U}(x, \tilde{F}) = \widetilde{\Phi}(x, M(\tilde{F}))$, and since $\widetilde{\Phi}(x; \cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$, we have that $\widetilde{U}(x, \cdot)$ is polyconvex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$.

Conversely, assume that the map $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^d) \ni A \mapsto \widetilde{U}(x, A_{|T_x\partial\Omega})$ is polyconvex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$, then there exists some $\widetilde{\Phi} : \partial\Omega \times \mathbb{R}^{\nu_d} \to \mathbb{R}$ such that $\widetilde{\Phi}(x, \cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and such that $\widetilde{\Phi}(x, M(\widetilde{F})) = \widetilde{U}(x, \widetilde{F}_{|T_x\partial\Omega})$ for every $\widetilde{F} \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$. We define

$$\begin{aligned} \Phi : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} &\to \mathbb{R} \\ (x, (a_1, \ldots, a_{\nu_{d-1}})) &\mapsto \widetilde{\Phi}(x, (a_1, \ldots, a_{\nu_{d-1}}, 0, \ldots, 0)), \end{aligned}$$

which is convex. For any $\widetilde{F} \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$ we define $F_x \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$ as the extension of $\widetilde{F}_{|T_x\partial\Omega}$ such that $F_x n(x) = 0$. With the basis selected as in (6.4) and by (6.1) and (6.3), we have that $M(\widetilde{F}_{|T_x\partial\Omega}) = M^0(F_x)$. By Remark 6.2 we also have that $M^1(F_x) = 0 \in \mathbb{R}^{\nu_d - \nu_{d-1}}$. Recalling (6.2) we obtain that $M(F_x) = (M(\widetilde{F}_{|T_x\partial\Omega}), 0, \ldots, 0)$. Therefore $\Phi(x, M(\widetilde{F}_{|T_x\partial\Omega})) = \widetilde{\Phi}(x, M(F_x))$ and if we fix $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$, we have that $\widetilde{U}(x, F_{x|T_x\partial\Omega}) = \widetilde{U}(x, \widetilde{F}_{|T_x\partial\Omega}) = \widetilde{U}_{|T^d\partial\Omega}(x, \widetilde{F}_{|T_x\partial\Omega}) = U(x, \widetilde{F}_{|T_x\partial\Omega})$. This leads to $\Phi(x, M(\widetilde{F}_{|T_x\partial\Omega})) = U(x, \widetilde{F}_{|T_x\partial\Omega})$ and since $\Phi(x, \cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ then U is tangentially polyconvex.

(ii) Let $n : \partial\Omega \to \mathbb{R}^d$ be the unit outward normal vector to Ω. Since U is polyconvex, for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ the map $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^d) \ni A \mapsto U(x, A_{|T_x\partial\Omega})$ satisfies that there exist a measurable basis $\mathcal{V}$ and $\Phi : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}$ such that $\Phi(x, \cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and $U(x, A_{|T_x\partial\Omega}) = \Phi(x, M(A_{|T_x\partial\Omega}))$ for each $A \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$, where the matrix is taken with respect to $\mathcal{V}$ and the canonical basis of $\mathbb{R}^d$. In particular, if for $\mathcal{H}^{d-1}$-each $(x, F) \in T^d\partial\Omega$ we denote by $F_x \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$ the extension of F such that $F_x n(x) = 0$, the map $U_{|T^d\partial\Omega} : T^d\partial\Omega \to \mathbb{R}$ satisfies that $U_{|T^d\partial\Omega}(x, F) = U(x, F_{x|T_x\partial\Omega}) = \Phi(x, M(F_{x|T_x\partial\Omega})) = \Phi(x, M(F))$ and therefore $U_{|T^d\partial\Omega}$ is tangentially polyconvex. □

We present the definition of tangentially quasiconvex, due to [17] (see also [25] and [26]).

Definition 7.6. Let $U : T^d\partial\Omega \to [0,\infty)$ be a Borel function. We say that U is tangentially quasiconvex if for all $(z,\xi) \in T^d\partial\Omega$ and all $\varphi \in W^{1,\infty}(B(0;1); T_z\partial\Omega)$ such that $\varphi(y) = \xi y$ on $\partial B(0;1)$ we have that

$$U(z,\xi) \leq \frac{1}{|B(0;1)|}\int_{B(0;1)} U(z, D\varphi(y))\mathrm{d}y.$$

Here $B(0;1)$ is the unit ball in $\mathbb{R}^d$. We are regarding ξ as an $d\times d$ matrix and note that the fact $\varphi \in W^{1,\infty}(B(0;1); T_z\partial\Omega)$ implies that $D\varphi(x) \in (T_z\partial\Omega)^d$ for a.e. $x \in B(0,1)$. In the same way that polyconvexity is sufficient for quasiconvexity (see *e.g.* [24]), the same result holds for their tangential versions.

Proposition 7.7. *Let $U : T^d\partial\Omega \to \mathbb{R}$ be tangentially polyconvex. Then U is tangentially quasiconvex.*

Proof. Let φ be as in Definition 7.6. Let $\Phi : \partial\Omega \times \mathbb{R}^{\nu_d} \to \mathbb{R}$ be such that $\Phi(x,\cdot)$ is convex and $U(x,\xi) = \Phi(x, M(\xi))$. Let $B = B(0;1)$. Then, by Jensen's inequality, for any $(x,\xi) \in T^d\partial\Omega$,

$$\frac{1}{|B|}\int_B U(x, D\varphi(y))\mathrm{d}y = \frac{1}{|B|}\int_B \Phi(x, M(D\varphi(y)))\mathrm{d}y \geq \Phi\left(x, \frac{1}{|B|}\int_B M(D\varphi(y))\mathrm{d}y\right).$$

Now, by standard properties of minors (see, *e.g.*, [24], Lem. 5.5),

$$\frac{1}{|B|}\int_B M(D\varphi(y))\mathrm{d}y = \frac{1}{|B|}\int_B M(\xi)\mathrm{d}y = M(\xi),$$

and hence,

$$\frac{1}{|B|}\int_B U(x, D\varphi(y))\mathrm{d}y \geq \Phi(x, M(\xi)) = U(x,\xi),$$

so proving that U is tangentially quasiconvex. ☐

8. Interface polyconvexity

Given the formulation of the tangential polyconvexity in Definition 7.3, we ought to mention the *interface polyconvexity*, a similar concept developed in [18]. Since the notion of interface polyconvexity is not really used in this article, this section can be skipped in a first reading. Arising in parallel conditions, both notions respond to the need of a convexity property in the stored-energy function for surfaces. While our formulation of tangential polyconvexity considers $T^d\partial\Omega$, the interface polyconvexity is defined for a given $x \in \partial\Omega$ (see [18], Defs. 5.1 and 6.3).

We first state some definitions and facts from multilinear algebra to be used along the rest of the section. For $k \in \mathbb{N}$, the space $\Lambda_k\mathbb{R}^d$ consists of all alternating k-tensors on $\mathbb{R}^d$, *i.e.*, sums of elements of the form $a_1 \wedge \cdots \wedge a_k$ with $a_1, \ldots, a_k \in \mathbb{R}^d$. Here, $\wedge$ denotes the exterior product between vectors in $\mathbb{R}^d$. We will make the natural identifications $\Lambda_0\mathbb{R}^d \simeq \Lambda_d\mathbb{R}^d \simeq \mathbb{R}$ and $\Lambda_1\mathbb{R}^d \simeq \Lambda_{d-1}\mathbb{R}^d \simeq \mathbb{R}^d$. We will repeatedly use that if $\mathcal{V} = \{v_1, \ldots, v_d\}$ is a basis of $\mathbb{R}^d$ then $\mathcal{V}_k := \{v_{i_1} \wedge \cdots \wedge v_{i_k} : 1 \leq i_1 < \cdots < i_k \leq d\}$ is a basis of $\Lambda_k\mathbb{R}^d$.

Let $L \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$. The map $\Lambda_k L : \Lambda_k\mathbb{R}^d \to \Lambda_k\mathbb{R}^d$ is defined as the only linear map such that $(\Lambda_k L)(a_1 \wedge \cdots \wedge a_k) = La_1 \wedge \cdots \wedge La_k$ for $a_1, \ldots, a_k \in \mathbb{R}^d$; in particular, the map $\Lambda_0 L$ is identified with the identity (*i.e.*, multiplication by 1).

The next definition is from [19], Section 1.7.5.

Definition 8.1. Let $m \in \mathbb{N}$ and let $\mathcal{P}_m$ be the set of permutations of $(1, \dots, m)$. The inner product in $\Lambda_m \mathbb{R}^d$, denoted by $\cdot$, is the only bilinear form such that for all $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_m \in \mathbb{R}^d$,

$$(\xi_1 \wedge \cdots \wedge \xi_m) \cdot (\eta_1 \wedge \cdots \wedge \eta_m) = \sum_{\sigma \in \mathcal{P}_m} \operatorname{sign} \sigma \prod_{i=1}^m \xi_{\sigma(i)} \cdot \eta_i,$$

where the inner product in the right-hand side refers to the standard inner product in $\mathbb{R}^d$.

The following result describes the inner product defined above acting on an orthonormal basis.

Lemma 8.2. *Let $\mathcal{V}$ be an orthonormal basis of $\mathbb{R}^d$, let $\xi_1, \dots, \xi_m$ be m different elements of $\mathcal{V}$, let $\eta_1, \dots, \eta_m$ be m different elements of $\mathcal{V}$ and let $\xi = \xi_1 \wedge \cdots \wedge \xi_m$ and $\eta = \eta_1 \wedge \cdots \wedge \eta_m$.*

(i) If $\{\xi_1, \dots, \xi_m\} \neq \{\eta_1, \dots, \eta_m\}$ then $\xi \cdot \eta = 0$.
(ii) If $\{\xi_1, \dots, \xi_m\} = \{\eta_1, \dots, \eta_m\}$ then $\xi \cdot \eta = \operatorname{sign} \tilde{\sigma}$, where $\tilde{\sigma}$ is the only permutation such that $\xi_{\tilde{\sigma}(i)} = \eta_i$ for all $i \in \{1, \dots, m\}$.

Proof. (i) For each $\sigma \in \mathcal{P}_m$ we have that $\xi_{\sigma(i)} \neq \eta_i$ for some $i \in \{1, \dots, m\}$, so $\xi_{\sigma(i)} \cdot \eta_i = 0$. Consequently, $\prod_{i=1}^m \xi_{\sigma(i)} \cdot \eta_i = 0$.

(ii) For each $\sigma \in \mathcal{P}_m \setminus \{\tilde{\sigma}\}$ we have that $\prod_{i=1}^m \xi_{\sigma(i)} \cdot \eta_i = 0$, as in (i). Therefore, $\xi \cdot \eta = \operatorname{sign} \tilde{\sigma} \prod_{i=1}^m \xi_{\tilde{\sigma}(i)} \cdot \eta_i = \operatorname{sign} \tilde{\sigma} \prod_{i=1}^m \eta_i \cdot \eta_i = \operatorname{sign} \tilde{\sigma}$. □

As a consequence of Lemma 8.2(ii), when $m = 1$, the product of Definition 8.1 is the standard inner product in $\Lambda_1 \mathbb{R}^d \simeq \mathbb{R}^d$, and when $m = 0$, it is the product of real numbers in $\Lambda_0 \mathbb{R}^d \simeq \mathbb{R}$.

The next definition is from [18], Appendix C.

Definition 8.3. Let $0 \leq s \leq r$ be natural numbers. Let $\alpha \in \Lambda_r \mathbb{R}^d$ and $\beta \in \Lambda_s \mathbb{R}^d$. We define the contraction $\alpha \llcorner \beta \in \Lambda_{r-s} \mathbb{R}^d$ of α by β as the alternating $(r-s)$-tensor such that $(\alpha \llcorner \beta) \cdot \gamma = \alpha \cdot (\gamma \wedge \beta)$ for each $\gamma \in \Lambda_{r-s} \mathbb{R}^d$.

The following are properties of the contraction.

Lemma 8.4. *Let $\{v_1, \dots, v_d\}$ be an orthonormal basis of $\mathbb{R}^d$.*

(i) If $\alpha, \beta \in \Lambda_r \mathbb{R}^d$ for some $r \in \mathbb{N}$ then $\alpha \llcorner \beta = \alpha \cdot \beta$.
(ii) Let $1 \leq s \leq r$, let $1 \leq i_1 < \cdots < i_r \leq d$ and $1 \leq j_1 < \cdots < j_s \leq d$. If $\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_s\} = \emptyset$ then

$$v_{i_1} \wedge \cdots \wedge v_{i_r} \llcorner v_{j_1} \wedge \cdots \wedge v_{j_s} = 0.$$

(iii) Consider $n = v_d$. If $1 \leq i_1 < \cdots < i_{k+1} \leq d-1$, then

$$v_{i_1} \wedge \cdots \wedge v_{i_{k+1}} \llcorner n = 0. \tag{8.1}$$

If $1 \leq i_1 < \cdots < i_k \leq d-1$, then

$$v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge n \llcorner n = v_{i_1} \wedge \cdots \wedge v_{i_k}. \tag{8.2}$$

Proof. (i) The contraction $\alpha \llcorner \beta$ is the only constant such that $(\alpha \llcorner \beta)\gamma = \alpha \cdot (\gamma \wedge \beta) = \gamma \alpha \cdot \beta$ for each $\gamma \in \Lambda_0 \mathbb{R}^d \simeq \mathbb{R}$.

(ii) Let $1 \leq l_1 < \cdots < l_{r-s} \leq d$. Then

$$(v_{i_1} \wedge \cdots \wedge v_{i_r} \llcorner v_{j_1} \wedge \cdots \wedge v_{j_s}) \cdot (v_{l_1} \wedge \cdots \wedge v_{l_{r-s}}) = (v_{i_1} \wedge \cdots \wedge v_{i_r}) \cdot (v_{l_1} \wedge \cdots \wedge v_{l_{r-s}} \wedge v_{j_1} \wedge \cdots \wedge v_{j_s}) = 0,$$

where the latter equality is due to Lemma 8.2(i), since $\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_s\} = \emptyset$.

(iii) Equation (8.1) is a direct consequence of (i). As for (8.2), let $1 \le l_1 < \cdots < l_k \le d$ and compute

$$(v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge n \llcorner n) \cdot (v_{l_1} \wedge \cdots \wedge v_{l_k}) = (v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge n) \cdot (v_{l_1} \wedge \cdots \wedge v_{l_k} \wedge n).$$

By Lemma 8.2,

$$(v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge n) \cdot (v_{l_1} \wedge \cdots \wedge v_{l_k} \wedge n) = 0 = (v_{i_1} \wedge \cdots \wedge v_{i_k}) \cdot (v_{l_1} \wedge \cdots \wedge v_{l_k}) \quad \text{if } \{i_1, \dots, i_k\} \neq \{l_1, \dots, l_k\}$$

and

$$(v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge n) \cdot (v_{l_1} \wedge \cdots \wedge v_{l_k} \wedge n) = \operatorname{sign} \sigma = (v_{i_1} \wedge \cdots \wedge v_{i_k}) \cdot (v_{l_1} \wedge \cdots \wedge v_{l_k}) \quad \text{if } \{i_1, \dots, i_k\} = \{l_1, \dots, l_k\},$$

where σ is the only permutation such that $i_{\sigma(j)} = l_j$ for all $1 \le j \le k$. □

The following type of maps are of particular importance in the development of [18].

Definition 8.5. Let $k \in \mathbb{N}$, let $A \in \mathcal{L}(\Lambda_k \mathbb{R}^d; \Lambda_k \mathbb{R}^d)$ and let $\beta \in \mathbb{R}^d$. We define the map $A \wedge \beta \in \mathcal{L}(\Lambda_{k+1} \mathbb{R}^d; \Lambda_k \mathbb{R}^d)$ by $(A \wedge \beta)\alpha \coloneqq A(\alpha \llcorner \beta)$ for each $\alpha \in \Lambda_{k+1} \mathbb{R}^d$.

The following are properties of the map defined above. Recall from Section 6.1 the notation of the minors.

Lemma 8.6. *Let $k \in \mathbb{N}$, let $A \in \mathcal{L}(\Lambda_k \mathbb{R}^d; \Lambda_k \mathbb{R}^d)$ and let $F \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$. Let $\mathcal{V} = \{v_1, \dots, v_d\}$ be an orthonormal basis of $\mathbb{R}^d$ and consider $n = v_d$.*

(i) The map $A \wedge n \in \mathcal{L}(\Lambda_{k+1} \mathbb{R}^d; \Lambda_k \mathbb{R}^d)$ is characterized as follows: for $1 \le j_1 < \cdots < j_{k+1} \le d$,

$$(A \wedge n) v_{j_1} \wedge \cdots \wedge v_{j_{k+1}} = \begin{cases} 0 & \text{if } j_{k+1} < d, \\ A(v_{j_1} \wedge \cdots \wedge v_{j_k}) & \text{if } j_{k+1} = d. \end{cases}$$

(ii) If $\beta \in \mathbb{R}^d$ then $\Lambda_0 F \wedge \beta = \beta$.

(iii) If $1 \le j_1 < \cdots < j_k \le d$ then

$$\Lambda_k F(v_{j_1} \wedge \cdots \wedge v_{j_k}) = \sum_{1 \le i_1 < \cdots < i_k \le d} M_{\substack{i_1 \dots i_k \\ j_1 \dots j_k}}(F) v_{i_1} \wedge \cdots \wedge v_{i_k},$$

where the minors $M_{\substack{i_1 \dots i_k \\ j_1 \dots j_k}}(F)$ are taken with respect to the basis $\mathcal{V}$.

(iv) If $1 \le j_1 < \cdots < j_k \le d - 1$ then

$$(\Lambda_k F \wedge n) v_{j_1} \wedge \cdots \wedge v_{j_k} \wedge n = \sum_{1 \le i_1 < \cdots < i_k \le d} M_{\substack{i_1 \dots i_k \\ j_1 \dots j_k}}(F) v_{i_1} \wedge \cdots \wedge v_{i_k}.$$

Proof. (i) By Lemma 8.4(iii), if $j_{k+1} < d$,

$$(A \wedge n) v_{j_1} \wedge \cdots \wedge v_{j_{k+1}} = A(v_{j_1} \wedge \cdots \wedge v_{j_{k+1}} \llcorner n) = 0$$

while

$$(A \wedge n) v_{j_1} \wedge \cdots \wedge v_{j_k} \wedge n = A(v_{j_1} \wedge \cdots \wedge v_{j_k} \wedge n \llcorner n) = A(v_{j_1} \wedge \cdots \wedge v_{j_k}).$$

(ii) Let $\alpha \in \Lambda_1 \mathbb{R}^d \simeq \mathbb{R}^d$. By Lemma 8.4(i) we have that $(\Lambda_0 F \wedge \beta)\alpha = \Lambda_0 F(\alpha \llcorner \beta) = \alpha \llcorner \beta = \alpha \cdot \beta$.

(iii) Let $f_{ij} = Fv_j \cdot v_i$ for each $1 \leq i, j \leq d$. We compute

$$\begin{aligned}\Lambda_k F(v_{j_1} \wedge \cdots \wedge v_{j_k}) &= Fv_{j_1} \wedge \cdots \wedge Fv_{j_k} \\ &= \sum_{i_1=1}^{d} f_{i_1 j_1} v_{i_1} \wedge \cdots \wedge \sum_{i_k=1}^{d} f_{i_k j_k} v_{i_k} \\ &= \sum_{1 \leq i_1, \ldots, i_k \leq d} f_{i_1 j_1} \cdots f_{i_k j_k} v_{i_1} \wedge \cdots \wedge v_{i_k} \\ &= \sum_{1 \leq i_1 < \cdots < i_k \leq d} M_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_k}}(F) v_{i_1} \wedge \cdots \wedge v_{i_k}.\end{aligned}$$

(iv) We have that

$$(\Lambda_k F \wedge n) v_{j_1} \wedge \cdots \wedge v_{j_k} \wedge n = \Lambda_k F(v_{j_1} \wedge \cdots \wedge v_{j_k}) = \sum_{1 \leq i_1 < \cdots < i_k \leq d} M_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_k}}(F) v_{i_1} \wedge \cdots \wedge v_{i_k}$$

by (i) and (iii). □

As seen in Lemma 8.6, the coefficients of $\Lambda_k F \wedge n$ with respect to $\mathcal{V}_{k+1}$ and $\mathcal{V}_k$ are either zero or the minors of F involving $\{v_1, \ldots, v_{d-1}\}$.

We now relate the tangential polyconvexity from Definition 7.3 with the maps introduced in Definition 8.5. In the rest of the section we will use the set $\mathcal{G} := \{(F, n) \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^m) \times S^{d-1} : Fn = 0\}$. Besides, returning to the notation of the minors from Section 6.1, when the chosen bases have a dependence on some x we will stress this dependence denoting by M_x, $M_{k,x}$, M_x^0, M_x^1, $M_{k,x}^0$ and $M_{k,x}^1$ the sequences of minors M, M_k, M^0, M^1, M_k^0 and M_k^1 in such bases, respectively.

Proposition 8.7. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, let $n : \partial\Omega \to \mathbb{R}^d$ be a measurable map such that $n(x)$ is a unit orthogonal vector to $T_x\partial\Omega$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$. Let $\hat{f} : \mathcal{G} \to \mathbb{R} \cup \{\infty\}$ be such that there exists a convex map $\Psi : \prod_{k=0}^{d-1} \mathcal{L}(\Lambda_{k+1}\mathbb{R}^d; \Lambda_k \mathbb{R}^m) \to \mathbb{R} \cup \{\infty\}$ with*

$$\hat{f}(F, n) = \Psi(\Lambda_0 F \wedge n, \ldots, \Lambda_{d-1} F \wedge n), \qquad (F, n) \in \mathcal{G}.$$

Define $W_0 : T^d\partial\Omega \to \mathbb{R}$ as $W_0(x, F) := \hat{f}(F_x, n(x))$ where F_x is the linear extension of F to $\mathbb{R}^d$ such that $F_x n(x) = 0$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$. Then W_0 is tangentially polyconvex.

Proof. Let $\{v_1, \ldots, v_{d-1}\}$ be an orthonormal measurable basis of $T\partial\Omega$ and consider $v_d = n$, then $\mathcal{V}(x) := \{v_1(x), \ldots, v_d(x)\}$ is an orthonormal basis of $\mathbb{R}^d$, for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$. Let $1 \leq k \leq d-1$ and define $\theta_k = \binom{d}{k}^2$. We number the elements of $\mathbb{R}^{\theta_k}$ as $(a_{j_1, \ldots, j_k}^{i_1, \ldots, i_k})_{1 \leq j_1 < \cdots < j_k \leq d}^{1 \leq i_1 < \cdots < i_k \leq d}$. Define the linear map $\mathfrak{F}_{k,x} : \mathbb{R}^{\theta_k} \to \mathcal{L}(\Lambda_k \mathbb{R}^d; \Lambda_k \mathbb{R}^d)$ as follows: $\mathfrak{F}_{k,x}\left((a_{j_1, \ldots, j_k}^{i_1, \ldots, i_k})_{1 \leq j_1 < \cdots < j_k \leq d}^{1 \leq i_1 < \cdots < i_k \leq d}\right)$ is the only linear map such that for each $1 \leq j_1 < \cdots < j_k \leq d$,

$$\mathfrak{F}_{k,x}\left((a_{j_1, \ldots, j_k}^{i_1, \ldots, i_k})_{1 \leq j_1 < \cdots < j_k \leq d}^{1 \leq i_1 < \cdots < i_k \leq d}\right)(v_{j_1}(x) \wedge \cdots \wedge v_{j_k}(x)) = \sum_{1 \leq i_1 < \cdots < i_k \leq d} a_{j_1, \ldots, j_k}^{i_1, \ldots, i_k} v_{i_1}(x) \wedge \cdots \wedge v_{i_k}(x).$$

Recalling the order of the minors established in Section 6.1 and thanks to Lemma 8.6(iii) we have that $\mathfrak{F}_{k,x}(M_{k,x}(F_x)) = \Lambda_k F_x$. Now define

$$\begin{aligned}&\Phi : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} \to \mathbb{R} \\ &(x, (a_1, \ldots, a_{\nu_{d-1}})) \mapsto \Psi(n(x), \mathfrak{F}_{1,x}(a_1, \ldots, a_{\nu_1}) \wedge n(x), \ldots, \mathfrak{F}_{d-1,x}(a_{\nu_{d-2}+1}, \ldots, a_{\nu_{d-1}}) \wedge n(x)).\end{aligned}$$

Then, $\Phi(x,\cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$, as a composition of a linear with a convex map. Moreover, for $(x,F) \in T^d\partial\Omega$,

$$W_0(x,F) = \hat{f}(F_x, n(x)) = \Psi(\Lambda_0 F_x \wedge n(x), \dots, \Lambda_{d-1} F_x \wedge n(x)), \qquad (x,F) \in T^d\partial\Omega.$$

Now,

$$M_x(F) = \left(M_x^0(F), M_x^1(F)\right) = \left(M_x^0(F_x), M_x^1(F)\right) = \left(M_{1,x}(F_x), \dots, M_{d-1,x}(F_x), M_x^1(F)\right),$$

so, recalling Lemma 8.6(ii) we have that

$$\begin{aligned}\Phi(x, M_x(F)) &= \Phi\left(x, M_{1,x}(F_x), \dots, M_{d-1,x}(F_x), M_x^1(F)\right) \\ &= \Psi(n(x), \mathfrak{F}_{1,x}(M_{1,x}(F_x)) \wedge n(x), \dots, \mathfrak{F}_{d-1,x}(M_{d-1,x}(F_x)) \wedge n(x)) \\ &= \Psi(n(x), \Lambda_1 F_x \wedge n(x), \dots, \Lambda_{d-1} F_x \wedge n(x)) \\ &= \Psi(\Lambda_0 F_x \wedge n(x), \Lambda_1 F_x \wedge n(x), \dots, \Lambda_{d-1} F_x \wedge n(x)),\end{aligned}$$

which proves the result. □

Note that condition $Fn = 0$ in the definition of $\mathcal{G}$ does not play an essential role: in the proof above we pass from a linear map defined in $T_x\partial\Omega$ to a linear extension to $\mathbb{R}^d$, and $Fn = 0$ just fixes a specific extension. A partial converse to the above result also holds.

Proposition 8.8. *Let $\Omega \subset \mathbb{R}^d$ be of class C^1, let $n : \partial\Omega \to S^{d-1}$ be the unit outward normal to Ω. Then there exists a measurable map $S^{d-1} \ni m \mapsto x_m \in \partial\Omega$ such that $n(x_m) = m$ for all $m \in S^{d-1}$. Now, let $U : T^d\partial\Omega \to \mathbb{R}$ be tangentially polyconvex. Define $\hat{f} : \mathcal{G} \to \mathbb{R} \cup \{\infty\}$ as $\hat{f}(F,m) := U(x_m, F_{|T_{x_m}\partial\Omega})$. Then there exists a measurable map $\Psi : S^{d-1} \times \prod_{k=0}^{d-1} \mathcal{L}(\Lambda_{k+1}\mathbb{R}^d; \Lambda_k\mathbb{R}^m) \to \mathbb{R} \cup \{\infty\}$ such that $\Psi(m,\cdot)$ is convex for all $m \in S^{d-1}$ and*

$$\hat{f}(F,n) = \Psi(n, \Lambda_0 F \wedge n, \dots, \Lambda_{d-1} F \wedge n), \qquad (F,n) \in \mathcal{G}.$$

Proof. The normal n is in fact the Gauss map of $\partial\Omega$, which is known to be surjective (see, *e.g.*, [27], Chapter 6). Now let $F : S^{d-1} \to \mathcal{P}(\partial\Omega)$ be the set-valued map defined by $F(m) := n^{-1}(m)$. As n is continuous, $F(m)$ is closed. Moreover, as n is surjective, $F(m)$ is non-empty. Now we show that F is measurable in the sense of [28], Definition 8.1.1: for each relatively open subset $\mathcal{O} \subseteq \partial\Omega$, the set $\{m \in S^{d-1} : F(m) \cap \mathcal{O} \neq \emptyset\}$ must be Borel. To check this, we express

$$\{m \in S^{d-1} : F(m) \cap \mathcal{O} \neq \emptyset\} = n(\mathcal{O})$$

and $\mathcal{O}$ as a countable union of compact sets: $\mathcal{O} = \bigcup_{m=1}^{\infty} K_m$. Since n is continuous, $n(K_m)$ is compact for each $m \in \mathbb{N}$, so $n(\mathcal{O})$ is Borel as a countable union of compact sets. An application of [28], Theorem 8.1.3 concludes that there exists a measurable map $S^{d-1} \ni m \mapsto x_m \in \partial\Omega$ such that $n(x_m) = m$.

There exist $\mathcal{V} = \{v_1, \dots, v_{d-1}\}$ a measurable basis of $T\partial\Omega$ and map $\Phi : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}$ such that $\Phi(x,\cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and $U(x,F) = \Phi(x, M(F))$ for $\mathcal{H}^{d-1}$-a.e. $(x,F) \in T^d\partial\Omega$, where $M(F)$ is taken with respect to the basis $\{v_1(x), \dots, v_{d-1}(x)\}$ and the canonical basis in $\mathbb{R}^d$. Let $v_d = n$ and consider $\tilde{\mathcal{V}} = \{v_1, \dots, v_{d-1}, v_d\}$ as a measurable orthonormal basis of $\mathbb{R}^d$. Let $k \leq d-1$ and $\tilde{\theta}_k = \binom{d-1}{k}\binom{d}{k}$; we number the elements of $\mathbb{R}^{\tilde{\theta}_k}$ as $(a^{i_1,\dots,i_k}_{j_1,\dots,j_k})^{1 \leq i_1 < \dots < i_k \leq d-1}_{1 \leq j_1 < \dots < j_k \leq d}$. Define the linear map

$$\mathfrak{C}_{k,x} : \mathcal{L}(\Lambda_{k+1}\mathbb{R}^d;\Lambda_k\mathbb{R}^d) \to \mathbb{R}^{\tilde{\theta}_k}$$
$$A \mapsto (A(v_{j_1}(x)\wedge\cdots\wedge v_{j_k}(x)\wedge n(x))\cdot(v_{i_1}(x)\wedge\cdots\wedge v_{i_k}(x)))^{1\leq i_1<\cdots<i_k\leq d}_{1\leq j_1<\cdots<j_k\leq d-1}.$$

Note that map $\mathfrak{F}_{k,x}$ of Proposition 8.7 is an isomorphism with inverse

$$\mathfrak{F}^{-1}_{k,x} : \mathcal{L}(\Lambda_k\mathbb{R}^d;\Lambda_k\mathbb{R}^d) \to \mathbb{R}^{\theta_k}$$
$$A \mapsto (A(v_{j_1}(x)\wedge\cdots\wedge v_{j_k}(x))\cdot(v_{i_1}(x)\wedge\cdots\wedge v_{i_k}(x)))^{1\leq i_1<\cdots<i_k\leq d}_{1\leq j_1<\cdots<j_k\leq d}.$$

As a consequence of Lemma 8.6(i), if $B\in\mathcal{L}(\Lambda_k\mathbb{R}^d;\Lambda_k\mathbb{R}^d)$, then

$$\mathfrak{C}_{k,x}(B\wedge n) = \left(\mathfrak{F}^{-1}_{k,x}(B)\right)^{1\leq i_1<\cdots<i_k\leq d}_{1\leq j_1<\cdots<j_k\leq d-1}.$$

Using Lemma 8.6(iii), if $F\in\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)$ then $\mathfrak{F}^{-1}_{k,x}(\Lambda_k F) = M_{k,x}(F)$ with respect to $\tilde{\mathcal{V}}_k(x)$. By Lemma 8.6(iv), $\mathfrak{C}_{k,x}(\Lambda_k F\wedge n) = M^0_{k,x}(F)$. Finally, by (6.1), $M^0_{k,x}(F) = M_{k,x}(F_{|T_x\partial\Omega})$. Altogether,

$$\mathfrak{C}_{k,x}(\Lambda_k F\wedge n) = M_{k,x}(F_{|T_x\partial\Omega}). \tag{8.3}$$

Now for each $x\in\partial\Omega$, define

$$\Psi_x : \prod_{k=0}^{d-1}\mathcal{L}(\Lambda_{k+1}\mathbb{R}^d;\Lambda_k\mathbb{R}^d)\to\mathbb{R}$$
$$(A_0,\ldots,A_{d-1})\mapsto\Phi(x,(\mathfrak{C}_{1,x}(A_1),\ldots,\mathfrak{C}_{d-1,x}(A_{d-1}))),$$

which is convex, as a composition of a convex and a linear map. Thanks to (8.3), for all $F\in\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)$,

$$\Psi_x(\Lambda_0 F\wedge n(x),\ldots,\Lambda_{d-1}F\wedge n(x)) = \Phi(x,(M_{1,x}(F_{|T_x\partial\Omega}),\ldots,M_{d-1,x}(F_{|T_x\partial\Omega}))) = \Phi(x,M(F_{|T_x\partial\Omega})).$$

Then, for all $(F,m)\in\mathcal{G}$,

$$\hat{f}(F,m) = U(x_m,F_{|T_{x_m}\partial\Omega}) = \Phi(x_m,M(F_{|T_{x_m}\partial\Omega})) = \Psi_{x_m}(\Lambda_0 F\wedge m,\ldots,\Lambda_{d-1}F\wedge m).$$

The proof is concluded by defining $\Psi(m,A_0,\ldots,A_{d-1}) \coloneqq \Psi_{x_m}(A_0,\ldots,A_{d-1})$. □

We remarked after Proposition 8.7 that condition $Fn=0$ in the definition of $\mathcal{G}$ is not essential. The following is a precise statement of this fact.

Proposition 8.9. *Let $d,m\in\mathbb{N}$, let $t=\min\{d-1,m\}$ and $\hat{f}:\mathcal{G}\to\mathbb{R}\cup\{\infty\}$. The following statements are equivalent:*

(i) There exists a convex map $\Psi:\prod_{k=0}^t\mathcal{L}(\Lambda_{k+1}\mathbb{R}^d;\Lambda_k\mathbb{R}^m)\to\mathbb{R}\cup\{\infty\}$ such that $\hat{f}(F,n)=\Psi(\Lambda_0F\wedge n,\Lambda_1F\wedge n,\ldots,\Lambda_tF\wedge n)$ for each $(F,n)\in\mathcal{G}$.

(ii) There exist $f:\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m)\times S^{d-1}\to\mathbb{R}\cup\{\infty\}$ an extension of $\hat{f}$ and $\Psi:\prod_{k=0}^t\mathcal{L}(\Lambda_{k+1}\mathbb{R}^d;\Lambda_k\mathbb{R}^m)\to\mathbb{R}\cup\{\infty\}$ convex, such that $f(F,n)=\Psi(\Lambda_0F\wedge n,\Lambda_1F\wedge n,\ldots,\Lambda_tF\wedge n)$ for each $(F,n)\in\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m)\times S^{d-1}$.

Proof. Since $\mathcal{G}\subset\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m)\times S^{d-1}$, statement (i) is a straightforward consequence of (ii). The converse implication is also trivial definining the extension f by $f(F,n)\coloneqq\Psi(\Lambda_0F\wedge n,\Lambda_1F\wedge n,\ldots,\Lambda_tF\wedge n)$ for each $(F,n)\in\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m)\times S^{d-1}$. □

A definition of interface polyconvexity can be given as follows (see [18], Def. 5.1, Thm. 5.3).

Definition 8.10. Let $d, m \in \mathbb{N}$, let $t = \min\{d-1, m\}$ and let $\mathcal{G} \coloneqq \{(F, n) \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^m) \times S^{d-1} : Fn = 0\}$. A map $\hat{f} : \mathcal{G} \to \mathbb{R} \cup \{\infty\}$ is said to be interface polyconvex if there exists a positively 1-homogeneous convex map $\Psi : Y \to \mathbb{R} \cup \{\infty\}$ on $Y \coloneqq \prod_{k=0}^{t} \mathcal{L}(\Lambda_{k+1}\mathbb{R}^d; \Lambda_k \mathbb{R}^m)$ such that

$$\hat{f}(F, n) = \Psi(\Lambda_0 F \wedge n, \Lambda_1 F \wedge n, \dots, \Lambda_t F \wedge n)$$

for each $(F, n) \in \mathcal{G}$.

As seen in Proposition 8.7 and 8.8, the key difference between tangential polyconvexity and interface polyconvexity is that, in the latter, the map Ψ needs to be positively 1-homogeneous. Definition 8.10 comes from a characterization (see [18], Thm. 5.3) more suitable for our framework. The original [18], Definition 5.1 defines a map as interface polyconvex if it is a supremum of a family of null Lagrangians, which, by [18], Theorem 5.2, are linear combinations of maps of the form $\Lambda_k F \wedge n$ with $0 \le k \le d-1$. Because of this, such suprema (and hence, the maps Ψ of interfacial polyconvexity) are convex and positively 1-homogeneous. In contrast, in the case of tangential polyconvexity, the map Ψ only needs to be convex, so it can be expressed as a supremum of a family of affine maps, a property that does not grant the positive 1-homogeneity. In this sense, interface polyconvexity is a more restrictive concept than tangential polyconvexity.

Positive 1-homogeneity is not necessary to achieve the lower semicontinuity of the energy functionals, as shown in [18], Section 6 and in Section 10 below. In [18], positive 1-homogeneity is related to the increase of the area of the so-called competitor interface (in our case, $\partial\Omega$).

9. Tangential polyconvexity in surface potentials

Explicit examples of the elastic energy $U : \mathcal{D}_U \to \mathbb{R}$ from (7.1), related to pressure loading and membrane loading on $\partial\Omega$, can be found in [15]. The context of [15] requires maps $u \in C^1(\overline{\Omega}; \mathbb{R}^d)$ and an important role is played by $\operatorname{cof} Du(x)n(x)$ for $x \in \partial\Omega$. In our case we work with maps $u \in W^{1,p}(\partial\Omega; \mathbb{R}^d)$, so we have to give a proper definition of $\operatorname{cof} Du(x)n(x)$.

We first state some facts from multilinear algebra complementing those of Section 8. Assume that $V \subset \mathbb{R}^d$ is a $(d-1)$-dimensional vector subspace of $\mathbb{R}^d$. For $k \in \mathbb{N}$, the space $\Lambda_k V$ consists of all alternating k-tensors on V. We will make the natural identifications $\Lambda_0 V \simeq \Lambda_{d-1} V \simeq \mathbb{R}$ and $\Lambda_1 V \simeq V$. Let $L \in \mathcal{L}(V; \mathbb{R}^d)$. The map $\Lambda_k L : \Lambda_k V \to \Lambda_k \mathbb{R}^d$ is defined as the only linear map such that $(\Lambda_k L)(a_1 \wedge \cdots \wedge a_k) = La_1 \wedge \cdots \wedge La_k$ for $a_1, \dots, a_k \in V$.

Let $\{v_1, \dots, v_{d-1}\}$ be an orthonormal basis of V, let n be a unit normal vector to V and consider $v_d = n$. The space $\Lambda_{d-1} V$ is generated by $\{v_1 \wedge \cdots \wedge v_{d-1}\}$ and can be identified with the subspace of $\mathbb{R}^d$ generated by n. Let $F \in \mathcal{L}(V; \mathbb{R}^d)$. For any $\widetilde{F} \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$ extending F, the vector $\operatorname{cof}(\widetilde{F})n$ does not depend on the extension $\widetilde{F}$, since the map $\Lambda_{d-1} F$ is determined by the value $\Lambda_{d-1} F(n)$ and formula

$$\Lambda_{d-1} F(n) = (\operatorname{cof} \widetilde{F})n \tag{9.1}$$

holds. As in Lemma 8.6(iii), the value of the map $\Lambda_{d-1} F$ can be rewritten in terms of the minors as

$$\Lambda_{d-1} F(v_1 \wedge \cdots \wedge v_{d-1}) = \sum_{1 \le i_1 < \cdots < i_{d-1} \le d} M_{\substack{i_1, \dots, i_{d-1} \\ 1, \dots, d-1}}(F) v_{i_1} \wedge \cdots \wedge v_{i_{d-1}} = \sum_{i=1}^{d} (-1)^{d-i} M_{d-1}(F)_i v_i, \tag{9.2}$$

where the minors are taken with respect to the chosen bases and we have made the identifications $v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_d = (-1)^{d-i} v_i$ for all $i = 1, \dots, d$.

The results of the previous paragraph are now applied to $V = T_x\partial\Omega$ for varying $x \in \partial\Omega$. Let $\{v_1, \dots, v_{d-1}\}$ be an orthonormal measurable basis of $T\partial\Omega$, let $n : \partial\Omega \to \mathbb{R}^d$ be the unit outward normal to Ω and consider

$v_d = n$. Fix $x \in \partial\Omega$. Given $F \in \mathcal{L}(T_x\partial\Omega; \mathbb{R}^d)$ and any extension of it $F_x \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$, by (9.1) and (9.2),

$$(\operatorname{cof} F_x)n(x) = \Lambda_{d-1}F(v_1(x) \wedge \cdots \wedge v_{d-1}(x)) = \sum_{i=1}^{d} (-1)^{d-i} M_{x,d-1}(F)_i v_i(x) \tag{9.3}$$

and we will apply this formula to $F = D^\tau u(x)$.

Some examples in [15] of the boundary energy functional from (7.1) are

$$\int_{\partial\Omega} U_i(x, y, F, n) \mathrm{d}\mathcal{H}^{d-1}(x), \qquad i = 1, 2,$$

with, in their notation,

$$U_1(x, y, F, n) = \pi(y) y \cdot \operatorname{cof} Du(x)n(x) \qquad \text{and} \qquad U_2(x, y, F, n) = \varepsilon_0 |\operatorname{cof} Du(x)n(x)|.$$

The expression of U_1 corresponds to a body having pressure interaction with its environment (see [15], Prop. 5.1) with $\pi : \mathbb{R}^d \to \mathbb{R}_+$ being some pressure function depending on the traction boundary condition. The expression of U_2 corresponds to a body with membrane interaction with its environment (see [15], Prop. 5.3); intuitively, this is a body with an elastic membrane glued to it with $\varepsilon_0 > 0$ being a constant representing the material modulus of the membrane.

These examples can be rewritten with our notation as follows. Using (9.3), the case of pressure interaction has the expression

$$U_1(x, y, F, n) = \pi(y) y \cdot \left(\sum_{i=1}^{d} (-1)^{d-i} M_{x,d-1}(F)_i v_i(x) \right),$$

which is linear with respect to the minors of F, and thus tangentially polyconvex. The case of the membrane interaction has the expression

$$U_2(x, y, F, n) = \varepsilon_0 \left(\sum_{i=1}^{d} \left(M_{x,d-1}(F)_i\right)^2 \right)^{1/2},$$

which is convex with respect to the minors of F, and thus tangentially polyconvex.

Suitable examples of energy functions should be coercive (see Thm. 10.6 below). Neither U_1 nor U_2 satisfy this condition. Nevertheless, if we define the energies as

$$\overline{U_i}(x, y, F, n) \coloneqq U_i(x, y, F, n) + c|F|^p, \qquad i = 1, 2,$$

we achieve the coercivity and retain the tangential polyconvexity, provided $c > 0$ and $p > 1$.

10. Existence of minimizers

In this section, we prove the existence of minimizers of the functional I in (7.1) in the class $\overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ under some natural conditions on the integrands; essentially, polyconvexity of W, tangential polyconvexity of U and standard coercivity assumptions.

The compactness of $\overline{\mathcal{A}}_p(\Omega)$ shown in [8], Proposition 10.2 together with the compactness of AIB given by Lemma 2.2 imply to the following result.

Proposition 10.1. *Let $p > d-1$. Let $\{u_j\}_{j\in\mathbb{N}} \subset \overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ be such that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^d) \cap W^{1,p}(\partial\Omega;\mathbb{R}^d)$ and $\{\det Du_j\}_{j\in\mathbb{N}}$ is equiintegrable. Then there exists $u \in \overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ such that, for a subsequence,*

$$u_j \rightharpoonup u \qquad \text{in } W^{1,p}(\Omega;\mathbb{R}^d) \cap W^{1,p}(\partial\Omega;\mathbb{R}^d) \qquad \text{and} \qquad \det Du_j \rightharpoonup \det Du \qquad \text{in } L^1(\Omega)$$

as $j \to \infty$.

We now state some elementary Poincaré inequalities.

Lemma 10.2. *Let $p \geq 1$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz open set such that $\partial\Omega$ is connected, let $\Gamma \subseteq \partial\Omega$ be a rectifiable set with positive $\mathcal{H}^{d-1}$ measure. Then there exists $C > 0$ such that for all $u \in W^{1,p}(\partial\Omega)$ with $u_{|\Gamma} = 0$, one has*

$$\|u\|_{L^p(\partial\Omega)} \leq C\|D^\tau u\|_{L^p(\partial\Omega)}. \tag{10.1}$$

Lemma 10.3. *Let $p \geq 1$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists $C > 0$ such that for all $u \in W^{1,p}(\Omega) \cap L^p(\partial\Omega)$ with*

$$\int_{\partial\Omega} u(x)\mathrm{d}\mathcal{H}^{d-1}(x) = 0 \tag{10.2}$$

one has

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}.$$

Lemma 10.4. *Let $p \geq 1$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz open set such that $\partial\Omega$ is connected. Then there exists $C > 0$ such that for all $u \in W^{1,p}(\partial\Omega)$ with (10.2), one has (10.1).*

Lower semicontinuity for tangentially quasiconvex integrands was proved in [17], Proposition 2.5. We now prove the lower semicontinuity of the boundary integral of the elastic energy in (7.1) under the assumptions previously stated on the integrand U.

Lemma 10.5. *Let $p > d-1$. Recall $\mathcal{D}_U$ from Remark 7.2 and let $U : \mathcal{D}_U \to \mathbb{R}$ be an $\mathcal{H}^{d-1}_{|\partial\Omega} \times \mathcal{B}^d \times \mathcal{B}^{d\times(d-1)} \times \mathcal{B}_{|S^{d-1}}$-measurable map such that $U(x,\cdot,\cdot,\cdot)$ is lower semicontinuous for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$, such that $U(\cdot,y,\cdot,n)$ is tangentially polyconvex for every $y \in \mathbb{R}^d$ and for every $n \in S^{d-1}$ and such that there exists a constant $c > 0$ and a map $a \in L^1(\partial\Omega)$ with*

$$U(x,y,F,n) \geq a(x) + c|F|^p$$

for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$, all $y \in \mathbb{R}^d$, all $F \in \mathcal{L}(T_x\partial\Omega;\mathbb{R}^d)$ and all $n \in S^{d-1}$. Then for any $\{u_j\}_{j\in\mathbb{N}} \subset W^{1,p}(\partial\Omega;\mathbb{R}^d)$ such that $u_j \rightharpoonup u$ in $W^{1,p}(\partial\Omega;\mathbb{R}^d)$ for some $u \in W^{1,p}(\partial\Omega;\mathbb{R}^d)$ we have that

$$\int_{\partial\Omega} U(x,u(x),D^\tau u(x),n(x))\mathrm{d}\mathcal{H}^{d-1}(x) \leq \liminf_{j\to\infty} \int_{\partial\Omega} U(x,u_j(x),D^\tau u_j(x),n(x))\mathrm{d}\mathcal{H}^{d-1}(x).$$

Proof. By Proposition 6.8 we have that $Ml(D^\tau u_n) \rightharpoonup Ml(D^\tau u)$ in $L^1(\partial\Omega)$ as $n \to \infty$. Since $U(\cdot,y,\cdot,n)$ is tangentially polyconvex for each $y \in \mathbb{R}^d$ and each $n \in S^{d-1}$, let $\Phi : \partial\Omega \times \mathbb{R}^{\nu_{d-1}} \to \mathbb{R}$ be the map such that $\Phi(x,\cdot)$ is convex for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and $U(x,y,F,n) = \Phi(x,M(F))$ for each $F \in \mathcal{L}(T_x\partial\Omega;\mathbb{R}^d)$, each $y \in \mathbb{R}^d$

and each $n \in S^{d-1}$, where F is taken as the matrix representation with respect to some measurable basis, which can be taken as an L^∞ basis thanks to Proposition 7.4, and the canonical basis of $\mathbb{R}^d$. Then we have that

$$\begin{aligned}\int_{\partial\Omega} U(x, u(x), D^\tau u(x), n(x)) \mathrm{d}\mathcal{H}^{d-1}(x) &= \int_{\partial\Omega} \Phi(x, M(D^\tau u(x))) \mathrm{d}\mathcal{H}^{d-1}(x) \\ &\leq \liminf_{j\to\infty} \int_{\partial\Omega} \Phi(x, M(D^\tau u_j(x))) \mathrm{d}\mathcal{H}^{d-1}(x) \\ &= \liminf_{j\to\infty} \int_{\partial\Omega} U(x, u_j(x), D^\tau u_j(x), n(x)) \mathrm{d}\mathcal{H}^{d-1}(x)\end{aligned}$$

thanks to [29], Theorem 7.5 and Definition 6.6. □

We now show the existence of minimizers. As before, we consider $D^\tau u$ as a map from $\partial\Omega$ to $\mathbb{R}^{d\times(d-1)}$ by fixing a measurable basis.

Theorem 10.6. *Let $p > d-1$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz open set such that $\mathbb{R}^d \setminus \partial\Omega$ has exactly two connected components. Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_+ \to \mathbb{R}$ and $U : \partial\Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times(d-1)} \times S^{d-1} \to \mathbb{R}$ satisfy the following conditions:*

(i) W is $\mathcal{L}^d \times \mathcal{B}^d \times \mathcal{B}^{d\times d}$-measurable and U is $\mathcal{H}^{d-1}_{|\partial\Omega} \times \mathcal{B}^d \times \mathcal{B}^{d\times(d-1)} \times \mathcal{B}_{|S^{d-1}}$-measurable, where $\mathcal{B}^d$ denotes the Borel σ-algebra in $\mathbb{R}^d$.
(ii) $W(x,\cdot,\cdot)$ and $U(x,\cdot,\cdot,\cdot)$ are lower semicontinuous for a.e. $x \in \Omega$ and $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$, respectively.
(iii) For a.e. $x \in \Omega$ and every $y \in \mathbb{R}^d$, the function $W(x, y, \cdot)$ is polyconvex; and for every $y \in \mathbb{R}^d$ and for every $n \in S^{d-1}$, the function $U(\cdot, y, \cdot, n)$ is tangentially polyconvex.
(iv) There exist constants $c_1, c_2 > 0$, functions $a_1 \in L^1(\Omega)$, $a_2 \in L^1(\partial\Omega)$ and a Borel function $h : (0,\infty) \to [0,\infty)$ such that

$$\lim_{t\searrow 0} h(t) = \lim_{t\to\infty} \frac{h(t)}{t} = \infty,$$

$$W(x,y,F) \geq a_1(x) + c_1|F|^p + h(\det F) \quad \text{for a.e. } x \in \Omega, \text{ all } y \in \mathbb{R}^d \text{ and all } F \in \mathbb{R}^{d\times d}_+$$

and

$$U(x,y,F,n) \geq a_2(x) + c_2|F|^p \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in \partial\Omega, \text{ all } y \in \mathbb{R}^d, \text{ all } F \in \mathcal{L}(T_x\partial\Omega; \mathbb{R}^d) \text{ and all } n \in S^{d-1}.$$

Let I be as in (7.1). Consider the following admissible classes:

1) Let Γ be a rectifiable subset of $\partial\Omega$ with positive $\mathcal{H}^{d-1}$ measure, and let $u_0 : \Gamma \to \mathbb{R}^d$. Define $\mathcal{A}_1$ as the set of $u \in \overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ such that $\det Du > 0$ a.e. and $u_{|\Gamma} = u_{0|\Gamma}$.
2) Define $\mathcal{A}_2$ as the set of $u \in \overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ such that $\det Du > 0$ a.e. and

$$\int_{\partial\Omega} u(x) \mathrm{d}\mathcal{H}^{d-1}(x) = 0.$$

3) Let $K \subset \mathbb{R}^d$ be compact. Define $\mathcal{A}_3$ as the set of $u \in \overline{\mathcal{A}}_p(\Omega) \cap \mathrm{AIB}$ such that $\det Du > 0$ a.e. and $u(x) \in K$ for a.e. $x \in \Omega$.

Fix $i \in \{1,2,3\}$. Assume $\mathcal{A}_i \neq \emptyset$ and I is not identically infinity in $\mathcal{A}_i$. Then there exists a minimizer of I in $\mathcal{A}_i$, and any element of $\mathcal{A}_i$ is injective a.e.

Proof. Fix $i \in \{1,2,3\}$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a minimizing sequence of I in $\mathcal{A}_i$. Assumption (iv) implies that both $\{Du_j\}_{j\in\mathbb{N}}$ and $\{D^\tau u_j\}_{j\in\mathbb{N}}$ are bounded in $L^p(\Omega; \mathbb{R}^{d\times d})$ and $L^p(\partial\Omega; \mathbb{R}^{d\times(d-1)})$, respectively.

Let us see that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^d)\cap W^{1,p}(\partial\Omega;\mathbb{R}^d)$. We will use that Ω and $\partial\Omega$ are connected (Prop. 2.6). In the set $\mathcal{A}_1$, because $u_{j|\Gamma}=u_{0|\Gamma}$ for any $j\in\mathbb{N}$, Poincaré's inequality gives us that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega;\mathbb{R}^d)$, while Lemma 10.2 gives the boundedness of $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\partial\Omega;\mathbb{R}^d)$. In the case of $\mathcal{A}_2$, Lemma 10.3 gives us the boundedness of $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega;\mathbb{R}^d)$, while Lemma 10.4 gives us the boundedness of $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\partial\Omega;\mathbb{R}^d)$. For the set $\mathcal{A}_3$, as K is compact, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega;\mathbb{R}^d)$ and therefore in $W^{1,p}(\Omega;\mathbb{R}^d)$. By continuity of the trace operator, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\partial\Omega;\mathbb{R}^d)$. In the three cases, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^d)\cap W^{1,p}(\partial\Omega;\mathbb{R}^d)$.

Assumption (iv) on h and De la Vallée Poussin's criterion imply that $\{\det Du_j\}_{j\in\mathbb{N}}$ is equiintegrable. By Proposition 10.1, there exists $u\in\overline{\mathcal{A}}_p(\Omega)\cap\mathrm{AIB}$ such that, for a subsequence (not relabelled),

$$u_j\rightharpoonup u \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^d)\cap W^{1,p}(\partial\Omega;\mathbb{R}^d) \quad \text{and} \quad \det Du_j\rightharpoonup \det Du \quad \text{in } L^1(\Omega) \tag{10.3}$$

as $j\to\infty$. As $\det Du_j>0$ a.e., we have that $\det Du\geq 0$ a.e. Thanks to the assumption on h, a standard argument based on Fatou's lemma (see, *e.g.*, [6], Thm. 5.1) shows that $\det Du>0$ a.e.

As $p>d-1$, a standard result on the continuity of minors (*e.g.*, [24], Thm. 8.20) together with (10.3) shows that $M(Du_j)\rightharpoonup M(Du)$ in $L^1(\Omega,\mathbb{R}^{\nu_d})$. By the lower semicontinuity of polyconvex functionals (*e.g.*, [30], Thm. 5.4 or [29], Thm. 7.5),

$$\int_\Omega W(x,u(x),Du(x))\mathrm{d}x\leq\liminf_{j\to\infty}\int_\Omega W(x,u_j(x),Du_j(x))\mathrm{d}x. \tag{10.4}$$

By Lemma 10.5 and equation (10.4) we have that $I[u]\leq\liminf_{j\to\infty}I[u_j]$.

If $u_j\in\mathcal{A}_1$ for all $j\in\mathbb{N}$, then, by continuity of traces, $u_{|\Gamma}=u_{0|\Gamma}$, so $u\in\mathcal{A}_1$ and u is a minimizer of I in $\mathcal{A}_1$. If $u_j\in\mathcal{A}_2$ for all $j\in\mathbb{N}$, then $\int_{\partial\Omega}u\mathrm{d}\mathcal{H}^{d-1}=0$, so $u\in\mathcal{A}_2$ and u is a minimizer of I in $\mathcal{A}_2$. If $u_j\in\mathcal{A}_3$ for all $j\in\mathbb{N}$, then, as K is compact, $u(x)\in K$ for a.e. $x\in\Omega$, so $u\in\mathcal{A}_3$ and u is a minimizer of I in $\mathcal{A}_3$.

The fact that any element of $\mathcal{A}_i$ is injective a.e. in Ω for each $i\in\{1,2,3\}$ is due to Theorem 4.3. □

Note that the particular case of $\mathcal{A}_1$ with $\Gamma=\partial\Omega$ does not need any assumptions in U since, in this case, it is constant. Thanks to [17], Proposition 2.5 we can also assume U to be tangentially quasiconvex instead of tangentially polyconvex.

Funding

Both authors have been supported by the Spanish Agencia Estatal de Investigación through project PID2021-124195NB-C32. C. Mora-Corral has also been supported by the Severo Ochoa Programme CEX2019-000904-S, the ERC Advanced Grant 834728 and by the Madrid Government (Comunidad de Madrid, Spain) under the multiannual Agreement with UAM in the line for the Excellence of the University Research Staff in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

Data availability statement

The research data associated with this article are included in the article.

References

[1] J.M. Ball, Global invertibility of Sobolev functions and the interpenetration of matter. *Proc. Roy. Soc. Edinburgh Sect. A* **88** (1981) 315–328.

[2] P.G. Ciarlet and J. Nečas, Injectivity and self-contact in nonlinear elasticity. *Arch. Rational Mech. Anal.* **97** (1987) 171–188.

[3] D. Henao and C. Mora-Corral, Invertibility and weak continuity of the determinant for the modelling of cavitation and fracture in nonlinear elasticity. *Arch. Rational Mech. Anal* **197** (2010) 619–655.

[4] D. Henao and C. Mora-Corral, Lusin's condition and the distributional determinant for deformations with finite energy. *Adv. Calc. Var.* **5** (2012) 355–409.

[5] S. Müller, T. Qi and B.S. Yan, On a new class of elastic deformations not allowing for cavitation. *Ann. l'I.H.P. Analyse non linéaire* **11** (1994) 217–243.

[6] S. Müller and S.J. Spector, An existence theory for nonlinear elasticity that allows for cavitation. *Arch. Ration. Mech. Anal.* **131** (1995) 1–66.
[7] V. Šverák, Regularity properties of deformations with finite energy. *Arch. Ration. Mech. Anal.* **100** (1998) 105–127.
[8] D. Henao, C. Mora-Corral and M. Oliva, Global invertibility of Sobolev maps. *Adv. Calc. Var.* **14** (2021) 207–230.
[9] S. Krömer, Global invertibility for orientation-preserving Sobolev maps via invertibility on or near the boundary. *Arch. Ration. Mech. Anal.* **238** (2020) 1113–1155.
[10] S. Müller, Det = det. A remark on the distributional determinant. *C. R. Acad. Sci. Paris Sér. I Math.* **311** (1990) 13–17.
[11] S. Müller, Weak continuity of determinants and nonlinear elasticity. *C. R. Acad. Sci. Paris Sér. I Math.* **307** (1988) 501–506.
[12] M. Barchiesi, D. Henao and C. Mora-Corral, Local invertibility in Sobolev spaces with applications to nematic elastomers and magnetoelasticity. *Arch. Ration. Mech. Anal.* **224** (2017) 743–816.
[13] D. Henao and C. Mora-Corral, Fracture surfaces and the regularity of inverses for BV deformations. *Arch. Ration. Mech. Anal.* **201** (2011) 575–629.
[14] P.G. Ciarlet, Mathematical Elasticity. Vol. I. Vol. 20 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam (1988).
[15] P. Podio-Guidugli and G. Vergara Caffarelli, Surface interaction potentials in elasticity. *Arch. Rational Mech. Anal.* **109** (1990) 343–383.
[16] N.C. Owen and P. Sternberg, Gradient flow and front propagation with boundary contact energy. *Proc. Roy. Soc. London Ser. A* **437** (1992) 715–728.
[17] B. Dacorogna, I. Fonseca, J. Malý and K. Trivisa, Manifold constrained variational problems. *Calc. Var. Part. Differ. Equ.* **9** (1999) 185–206.
[18] M. Šilhavý, Equilibrium of phases with interfacial energy: a variational approach. *J. Elast.* **105** (2011) 271–303.
[19] H. Federer, Geometric Measure Theory. Springer-New York (1969).
[20] K. Deimling, Nonlinear Functional Analysis. Springer-Verlag (1985).
[21] A. Czarnecki, M. Kulczycki and W. Lubawski, On the connectedness of boundary and complement for domains. *Ann. Pol. Math.* **103** (2012) 189–191.
[22] M. Bresciani, M. Friedrich and C. Mora-Corral, Variational models with Eulerian–Lagrangian formulation allowing for material failure. Avaliable at: https://arxiv.org/abs/2402.12870v1 (2024).
[23] P. Bernard and U. Bessi, Young measures, Cartesian maps, and polyconvexity. *J. Korean Math. Soc.* **47** (2010) 331–350.
[24] B. Dacorogna, Direct Methods in the Calculus of Variations. Springer-New York (2008).
[25] R. Alicandro and C. Leone, 3D-2D asymptotic analysis for micromagnetic thin films. *ESAIM Control Optim. Calc. Var.* **6** (2001) 489–498.
[26] C. Mora-Corral and M. Oliva, Relaxation of nonlinear elastic energies involving the deformed configuration and applications to nematic elastomers. *ESAIM Control Optim. Calc. Var.* **25** (2019).
[27] J.A. Thorpe, Elementary Topics in Differential Geometry. Springer New York (1979).
[28] J.P. Aubin and H. Frankowska, Set-Valued Analysis. Birkhäuser Boston, MA (2008).
[29] I. Fonseca and G. Leoni, Modern Methods in the Calculus of Variations: L^p Spaces. Springer Monogr. Math. Springer-New York (2007).
[30] J.M. Ball, J.C. Currie and P.J. Olver. Null Lagrangians, weak continuity, and variational problems of arbitrary order. *J. Funct. Anal.* **41** (1991) 135–174.