ESAIM: COCV 31 (2025) 79
https://doi.org/10.1051/cocv/2025065

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

EVENT-TRIGGERED DAMPING STABILIZATION OF EULER–BERNOULLI BEAM EQUATION

Xueru Fan[1], Cheng-Zhong Xu[2,*], Hua-Cheng Zhou[3], Weiwei Hu[4] and Chunhai Kou[5]

Abstract. This article deals with the distributed event-triggered feedback stabilization for an Euler–Bernoulli beam equation. Novel event-triggered rules with distributed damping control laws are proposed to stabilize the beam system: one rule is built based on the beam energy and the other on the beam state information plus an exponentially time-decreasing term to reduce the switching frequency of actuators. Well-posedness and no Zeno behaviour are completely proven for our event-triggered sampled feedback control systems. Moreover, by developing Lyapunov functionals, sufficient LMI-type conditions on the system parameters are proposed to ensure exponential stabilization of the event-triggered feedback beam system. On the other hand, we show that the exponential stabilization is not achievable by using the periodic sampled feedback control laws (*i.e.*, constant sample and hold) no matter what the sampling period is. Numerical simulations are presented to validate the stabilization performance of our proposed event-triggered control strategy.

Mathematics Subject Classification. 93D23, 35P05, 35Q93, 93C62, 93D15.

Received September 4, 2024. Accepted July 25, 2025.

1. Introduction

In the past four decades, the Euler–Bernoulli beam, as a benchmark vibrating system [1, 2], has attracted wide attention, because it is regarded as a typical model for controlling systems described by partial differential equations (PDEs). The important applications of Euler–Bernoulli beams in flexible link manipulators and outer space antennas [3] have prompted extensive studies in the stability analysis [4, 5]. Many effective methods have been developed to achieve stabilization, notably the backstepping approach [6], the Lyapunov approach combined with energy multipliers [7, 8], the strain control and proportional derivative algorithm [9], the frequency domain approach [10] *etc.* However, most of these approaches were realized by continuous time controls, necessitating uninterrupted communication and feedback and so resulting in an increased burden on control devices. In this article, we focus on the stabilization problem on the digital implementation of the control laws. A natural

Keywords and phrases: Euler–Bernoulli beam, event-triggered control, Lyapunov techniques, spectral analysis, exponential stabilization.

[1] Department of Mathematics, Huzhou University, Huzhou 313000, PR China.
[2] Université Claude Bernard Lyon 1, Department of Mechanics, LAGEPP, CNRS UMR 5007, Villeurbanne F-69622, France.
[3] School of Mathematics and Statistics, Central South University, Changsha 410075, PR China.
[4] Department of Mathematics, University of Georgia, Athens, GA 30602, USA.
[5] Department of Mathematics, Donghua University, Shanghai 201620, PR China.

* Corresponding author: cheng-zhong.xu@univ-lyon1.fr; hczhou@amss.ac.cn

choice is the periodic update of the continuous control laws that is a powerful and classic method for realizing digital continuous-in-time control laws. For finite-dimensional systems, the sampled-and-hold digital version of a stabilizing state feedback (or dynamic) controller yields a stable sampled-data feedback control system provided that the sampling period is sufficiently small (see [11–13]). However, the principle, when extended to infinite-dimensional systems, has limitations, specifically regarding the requirement that the unstable part of the open-loop system must be finite. Otherwise, there are examples in [14, 15] showing that the principle fails in sampled-data stabilization of infinite-dimensional systems.

To overcome the limitations of discontinuous stabilization methods, we introduce an event-based control as a noteworthy alternative to the traditional sampling methods. This approach can effectively reduce the workload and greatly slow down the actuators wear [16]. One of the remarkable features of the event-based control is that sensor sampling and control actuator turn on only when the plant state deviates from the expected value beyond a certain threshold. It is important to note that the implementation of event-triggered control strategies should avoid Zeno behavior. In other words the occurrence of infinite switching within a limited time interval must be avoided. Event-based control applications have been explored in [17, 18] for finite dimensional network control systems.

For infinite-dimensional systems, event-based control strategies have been investigated in the context of first-order hyperbolic systems [19, 20], parabolic systems [21–24] and some coupled PDE-ODE systems [25, 26] as well as in various triggering methods such as self-triggered control [27] and adaptive event-triggered control [28, 29]. Additionally, in [30] event-triggered stabilization for some infinite-dimensional systems was studied for bounded or unbounded control operators. Two types of trigger laws were analyzed to achieve stabilization with a positive minimum inter-event time. However, the model was subject to limitations regarding the finite unstable part of the generator spectrum. In recent works [31, 32] two event-triggered mechanisms of damping controllers have been proposed to exponentially stabilize wave equations. The stability has been proven by using Lyapunov's direct method.

To the best of our knowledge, few studies are devoted to stabilizing the Euler–Bernoulli beam system with infinite unstable spectrum by using event-triggered damping control. Motivated by the above arguments, we consider the following Euler–Bernoulli beam equation:

$$\begin{cases} w_{tt}(x,t)+w_{xxxx}(x,t)=u(x,t),\ x\in(0,1), t>0,\\ w(0,t)=0,\ w_x(0,t)=0,\ t>0,\\ w_{xx}(1,t)=0,\ w_{xxx}(1,t)=0,\ t>0,\\ w(x,0)=w_0(x),\ w_t(x,0)=w_1(x),\ x\in(0,1), \end{cases} \tag{1.1}$$

where $u(x,t)\in L^1(0,+\infty,L^2(0,1))$ is the control input, $(w_0(x),w_1(x))$ is the initial state and $(w(x,t),\ w_t(x,t))$ is the system state. Consider the beam system in the Hilbert space $\mathcal{H}=H_L^2(0,1)\times L^2(0,1)$, where $H_L^2(0,1)=\{f\in H^2(0,1): f(0)=f'(0)=0\}$ with the inner product

$$\langle(f_1,f_2)^\top,(g_1,g_2)^\top\rangle_{\mathcal{H}}=\int_0^1[f_1''(x)\overline{g_1''(x)}+f_2(x)\overline{g_2(x)}]\mathrm{d}x,\ \forall f,g\in\mathcal{H}.$$

Define the operator $A: D(A)(\subset\mathcal{H})\to\mathcal{H}$,

$$\begin{cases} A(\varrho,\varsigma)^\top=(\varsigma,-\varrho'''')^\top,\ \forall(\varrho,\varsigma)^\top\in D(A),\\ D(A)=\big\{(\varrho,\varsigma)^\top\in\mathcal{H}\bigcap(H^4(0,1)\times H_L^2(0,1))|\varrho''(1)=\varrho'''(1)=0\big\}. \end{cases} \tag{1.2}$$

Then the beam equation (1.1) can be rewritten as

$$\dot{\phi}(t)=A\phi(t)+Bu(t), \tag{1.3}$$

where $\phi(t)=(w(\cdot,t),w_t(\cdot,t))^\top,\ t>0$, and the input operator $B=[0,I]^\top\in\mathcal{B}(L^2(0,1),\mathcal{H})$. By the Lemma 4 in [33], we know that A is a skew-adjoint operator with compact resolvent in $\mathcal{H}$ and generates a C_0-group e^{At}. Then the system operator A has infinitely many unstable eigenvalues. According to [11], by analyzing the spectral decomposition, the distributed damping control law

$$u(x,t)=F\phi(t)=-c\,w_t(x,t),\ c>0, \tag{1.4}$$

can exponentially stabilize system (1.1), where feedback operator $F=[0,-cI]\in\mathcal{B}(\mathcal{H},L^2(0,1))$. In practical applications, the distributed damping control strategy enhances the energy dissipation of the beam system, thereby improving its stability and longevity. This strategy is widely employed to mitigate unwanted oscillations in flexible structures such as aerospace components and mechanical systems.

To save the control energy, the purpose of this paper is to exponentially stabilize Euler–Bernoulli equation (1.1) by using the event-triggered damping control with new trigger rules. Precisely, the paper deals with stabilization of the following closed-loop system:

$$\begin{cases} w_{tt}(x,t)+w_{xxxx}(x,t)=-cw_t(x,t_k),\ x\in(0,1),\ t\in[t_k,t_{k+1}),\ \forall k\in\mathbb{N},\\ w(0,t)=w_x(0,t)=0,\ t>0,\\ w_{xx}(1,t)=w_{xxx}(1,t)=0,\ t>0,\\ w(x,0)=w_0(x),\ w_t(x,0)=w_1(x),\ x\in(0,1), \end{cases} \tag{1.5}$$

where the trigger time sequence $(t_k)_{k\in\mathbb{N}}$ is defined by

$$t_{k+1}=\inf\left\{t\geq t_k\ \middle|\ \|w_t(\cdot,t)-w_t(\cdot,t_k)\|^2>2\nu E(t)\right\}, \tag{1.6}$$

or,

$$t_{k+1}=\inf\left\{t\geq t_k\ \middle|\ \|w_t(\cdot,t)-w_t(\cdot,t_k)\|^2>\alpha\left(\|w(\cdot,t)\|^2+\|w_t(\cdot,t)\|^2\right)+E(0)\mathrm{e}^{-\theta t}\right\}, \tag{1.7}$$

where $0<\nu<1$, $0<\alpha<1$ and $\theta>0$ are positive tuning parameters. Note that, throughout the paper,

$$\|\varphi\|^2=\int_0^1\varphi^2(x)\mathrm{d}x\ \ \forall\varphi\in L^2(0,1),\ \ E(t)=\frac{1}{2}(\|w_{xx}\|^2+\|w_t\|^2).$$

The main contributions are fourfold. (i) By analyzing the eigenvalues of the semigroup associated with closed-loop system, we show that periodic sampled data control cannot guarantee exponential stability of the closed-loop system no matter what the sampling period is. (ii) To reduce the workload of the controller, we design an event-triggered control law with two novel trigger rules (1.6)–(1.7) to stabilize Euler–Bernoulli beam equation. (iii) Sufficient LMI-based conditions for exponential stabilization are derived by Lyapunov's direct method. The operator theory is employed to prove the stability of closed-loop system (1.5). (iv) Zeno behavior under two trigger rules is proven to be excluded.

The rest of this paper is organized as follows. Section 2 is devoted to the statement of the stabilization problem and presenting the main results, including the discussion on well-posedness, nonrobustness of closed-loop stability under the periodic sampled control on the one hand, and on the other hand, event-triggered damping stabilization for the beam system without Zeno behavior. Section 3 is devoted to the proof of main results by using the operator theory and Lyapunov's direct method. To validate the theoretical results, numerical simulations are implemented in Section 4 by applying the finite element method. Section 5 is devoted to our conclusions. Appendix A contains our proof of some technical results.

2. Main result

2.1. Nonrobustness of closed-loop stability for beam systems under periodic sampled-data control

In this part, we first discuss the periodic sampled-data control of the system and show that the closed-loop stability for beam systems is not robust with respect to periodic sampling. For finite dimensional systems, applying a periodic sampled-data control law derived from a stabilizing state feedback control law gives us a still stable sampled-data closed-loop feedback system, provided that the sampling period is sufficiently small (see [11]). However, unlike the finite dimensional case, the principle fails to be extended to general infinite-dimensional systems, except that the state feedback law is compact and the input operator is bounded or the given system is analytic.

Consider the infinite-dimensional control system with period sampled feedback control

$$\begin{cases} \dot{\phi}(t) = A\phi(t) + Bu(t), & t \geq 0, \quad \phi(0) = \phi_0, \\ u(t) = F\phi(k\tau), & t \in [k\tau, k(\tau+1)), k \in \mathbb{N}, \end{cases} \tag{2.1}$$

with period time $\tau > 0$.

Theorem 2.1. *[15] Assume that A generates a strongly continuous semigroup $T(t)$ on X, $B \in \mathcal{B}(U, X)$, $F \in \mathcal{B}(X, U)$ is compact, and the semigroup generated by $A + BF$ is exponentially stable. Then, there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$, the solution of (2.1) satisfies $\|x(t)\| \leq M\mathrm{e}^{-\nu t}\|x(0)\|$ for some $M \geq 1$ and $\nu > 0$ and $\forall\ x(0) \in X$ and $t \geq 0$.*

Theorem 2.2. *[14] Assume that A generates an analytic semigroup $T(t)$ on X, $B \in \mathcal{B}(U, X_{-1})$, $F \in \mathcal{B}(X, U)$ is compact. If the semigroup generated by A_{BF}:*

$$A_{BF}\, x = (A + BF)x, \qquad \forall x \in D(A_{BF}) = \{x \in X \mid (A + BF)x \in X\}.$$

is exponentially stable. Then, there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$, the solution of (2.1) satisfies $\|x(t)\| \leq M\mathrm{e}^{-\nu t}\|x(0)\|\ \forall\ x(0) \in X$, $\forall\ t \geq 0$ and for some $M \geq 1$ and $\nu > 0$.

Remark 2.3. Assume that the operator A is the infinitesimal generator of an analytic semigroup (typically arising from a parabolic PDE). From Theorem 2.2, if $B \in \mathcal{B}(U, X_{-1})$, $F \in \mathcal{B}(X, U)$ is compact and A_{BF} is exponentially stable, then there exists a $\tau > 0$ such that the periodic sampled-data control system (2.1) is exponentially stable.

Theorem 2.1 and Theorem 2.2 show that the robustness of stability and stabilization with respect to periodic sampling still holds for infinite-dimensional systems with compact state feedback law where either the input operator is bounded or the state-space semigroup is analytic. Note that the compactness assumption on the state feedback operator F is crucial when the input operator is bounded. Several counterexamples (see references [14, 15]) indicate that the compactness assumption on F cannot be relaxed. In our case, while the input operator B defined in (1.3) is bounded, the feedback operator F defined in (1.4) is noncompact. We will prove that the closed-loop stability of the Euler–Bernoulli beam system is nonrobust under periodic sample. In other words the stability of the closed-loop fails under the periodic sampling no matter what the sampling period is. Precisely, consider the beam system (1.1) with period time $\tau > 0$ and periodic sampling controller:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = -c\ w_t(x, k\tau),\ x \in (0,1),\ t \in [k\tau, (k+1)\tau),\ k \in \mathbb{N}, \\ w(0,t) = 0,\ w_x(0,t) = 0, \\ w_{xx}(1,t) = 0,\ w_{xxx}(1,t) = 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \end{cases} \tag{2.2}$$

where the sampled control law is given by:

$$u(x,t) = -c\ w_t(x,k\tau),\ c>0,\ x\in(0,1), t\in[k\tau,(k+1)\tau)). \tag{2.3}$$

Then the periodic sampling controlled system (2.2) is governed by

$$\dot{\phi}(t) = A\phi(t) + BF\phi(k\tau), \quad t\in[k\tau,(k+1)\tau), \tag{2.4}$$

where $\phi(t) = (w(\cdot,t), w_t(\cdot,t))^\top$ and

$$B = [0,I]^\top \in \mathcal{B}(L^2(0,1),\mathcal{H}), \qquad F = [0,-cI] \in \mathcal{B}(\mathcal{H}, L^2(0,1)). \tag{2.5}$$

Notice that $F = -cB^*$. By the Lemma 4 in [33], we know that A is a skew-adjoint operator with compact resolvent in $\mathcal{H}$ and generates a C_0- group e^{At}. From Theorem 2.9 in [34], p. 109, since $BF\in\mathcal{L}(\mathcal{H})$, the following well-posedness result is obvious.

Lemma 2.4. *For each initial condition $(w_0,w_1)\in\mathcal{H}$, there exists a unique strong solution to (2.2) satisfying*

$$w\in C([0,\infty),H_L^2(0,1))\cap C^1([0,\infty),L^2(0,1)). \tag{2.6}$$

Now, let us show the nonrobustness of closed-loop stability under periodic sampled-data control. Define the operator $\widehat{A}\triangleq A+BF: D(\widehat{A})(\subset\mathcal{H})\to\mathcal{H}$:

$$\begin{cases} \widehat{A}(\varrho,\varsigma)^\top = (\varsigma, -\varrho'''' - c\varsigma)^\top,\ \forall (\varrho,\varsigma)^\top\in D(\widehat{A}),\\ D(\widehat{A}) = \big\{(\varrho,\varsigma)^\top\in\mathcal{H}\bigcap(H^4(0,1)\times H_L^2(0,1))|\varrho''(1)=\varrho'''(1)=0\big\}, \end{cases} \tag{2.7}$$

where A, B and F are defined in (1.2) and (2.5). We can rewrite the closed-loop system (2.2) as

$$\dot{\phi}(t) = \widehat{A}\phi(t) + BF[\phi(k\tau)-\phi(t)], \quad t\in[k\tau,(k+1)\tau), \tag{2.8}$$

where $\phi(t) = (w(\cdot,t), w_t(\cdot,t))^\top$. Since A is the generator of a C_0-semigroup of contractions and BF is dissipative and bounded operator, from the perturbation theorem in [34], p. 82, $\widehat{A}$ generates a C_0-semigroup $T_1(t)$ of contractions, *i.e.*, $\|T_1(t)\|\le 1$. By direct computation, we see that the eigenvalues $\{\lambda_n,\bar{\lambda}_n\}$, $n\in\mathbb{N}$, of $\widehat{A}$ are of the form

$$\lambda_n = \frac{1}{2}\left(-c+\sqrt{c^2-4\left(\frac{\pi}{2}+n\pi+\mathcal{O}(\mathrm{e}^{-n})\right)^4}\right), \quad n\to\infty. \tag{2.9}$$

Then the growth bound of C_0-semigroup $T_1(t)$ is $\omega_0 = \sup_{n\in\mathbb{N}} Re(\lambda_n) < 0$. For any $0<\sigma<-\omega_0$, there exists $M\ge 1$ such that $\|T_1(t)\|\le M\mathrm{e}^{-\sigma t}\ \forall\ t\ge 0$, which means that $A+BF$ generates an exponentially stable C_0-semigroup. However, as F is not compact, even though $A+BF$ generates an exponentially stable semigroup, no sampling period $\tau>0$ guarantees the exponential stability of the periodic sampled closed-loop system.

Theorem 2.5. *For any sampled time $\tau>0$, the sampled control closed-loop system (2.2) is not exponentially stable. Moreover, there is no $\tau^*>0$ such that (2.2) is strongly stable for all $\tau\in(0,\tau^*)$.*

Remark 2.6. Our Theorem 2.5 is consistent with Theorem 2.2 and Theorem 2.1. From Theorem 2.1, the infinite-dimensional systems with a compact stabilizing feedback F can be still stabilized by the periodic sampled-data control law, if B is bounded and unstable part of the spectrum of A is finite. Our Theorem 2.5 states that for the skew-adjoint operator A having an infinite number of unstable spectrum points and B, F bounded, the same

periodic sampled-data control law can no longer ensure exponential stabilization no matter what the sampling period τ is, though the semigroup generated by $A+BF$ is exponentially stable.

Remark 2.7. Theorem 2.1 and Theorem 2.2 provide sufficient conditions for the robustness of stability with respect to periodic sampling, for a class of infinite-dimensional control systems. However, in the context of general infinite-dimensional systems the robustness of the closed-loop stability to periodically sampling the control law is still open and interesting question. The Euler–Bernoulli beam system studied in this work demonstrates the lack of robustness in closed-loop stability under periodic sampling.

2.2. Well-posedness of sampled-data controller structure

As the periodic sampling control law does not guarantee stabilization, we propose event-triggered control laws to stabilize the beam system (1.1). Referring to the exponentially stabilizing control law (1.4), we design the event-triggered controller as follows:

$$u(x,t) = -cw_t(x,t_k),\ c>0, x\in(0,1), t\in[t_k,t_{k+1}), \tag{2.10}$$

where $(t_k)_{k\in\mathbb{N}}$ are the trigger instants defined by the event-triggered control mechanisms (1.6) or (1.7) with $t_0=0$. With the control law (2.10), we have the following closed-loop system:

$$\begin{cases} w_{tt}(x,t)+w_{xxxx}(x,t) = -cw_t(x,t_k),\ x\in(0,1), t\in[t_k,t_{k+1}), \forall k\in\mathbb{N},\\ w(0,t)=0,\ w_x(0,t)=0,\ t\geq 0,\\ w_{xx}(1,t)=0,\ w_{xxx}(1,t)=0,\ t\geq 0,\\ w(x,0)=w_0(x),\ w_t(x,0)=w_1(x),\ x\in(0,1). \end{cases} \tag{2.11}$$

Then the closed-loop system (2.11) can be rewritten as:

$$\dot{\phi}(t) = A\phi(t)+BF\phi(t_k),\quad t\in[t_k,t_{k+1}), \forall k\in\mathbb{N}, \tag{2.12}$$

where $\phi(t)=(w(\cdot,t),w_t(\cdot,t))^\top$, and A, B and F are defined respectively in (1.2) and (2.5).

Definition 2.8. [35] An increasing trigger instants $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ is said to have Zeno behavior if and only if

$$\lim_{k\to\infty} t_k = \sum_{k=0}^{\infty}(t_{k+1}-t_k) = T<\infty. \tag{2.13}$$

And then T is called the Zeno time.

Proposition 2.9. *A strictly increasing sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ has no Zeno behavior if and only if $\lim_{k\to\infty} t_k=+\infty$.*

By Definition 2.8 and Proposition 2.9, having no Zeno behavior is necessary to guarantee existence of the solution of beam system (2.11) for all $t\in\mathbb{R}^+$. Therefore the following theorem states the well-posedness of the event-triggered control system (2.11) and confirms the avoidance of Zeno behavior.

Theorem 2.10. *For each initial condition $(w_0,w_1)\in\mathcal{H}$, there exists a unique strong solution to (2.11) satisfying*

$$w\in C([0,\infty),H_L^2(0,1))\cap C^1([0,\infty),L^2(0,1)). \tag{2.14}$$

2.3. Event-triggered control mechanisms and stability of the closed-loop system

For Euler–Bernoulli beam equation (1.1), consider the following energy function

$$E(t)=\frac{1}{2}\int_0^1 [w_{xx}^2(x,t)+w_t^2(x,t)]\mathrm{d}x. \tag{2.15}$$

Define the speed deviation $e_k(x,t)$ as

$$e_k(x,t)=w_t(x,t)-w_t(x,t_k),\ x\in(0,1), t\in[t_k,t_{k+1}), k\in\mathbb{N}. \tag{2.16}$$

Then, we propose two event-triggered control mechanisms as follows:

$$t_{k+1}=\inf\left\{t\geq t_k \ \middle|\ \|e_k(\cdot,t)\|^2>2\nu E(t)\right\}, \tag{2.17}$$

$$t_{k+1}=\inf\left\{t\geq t_k \ \middle|\ \|e_k(\cdot,t)\|^2>\alpha\left(\|w(\cdot,t)\|^2+\|w_t(\cdot,t)\|^2\right)+E(0)\mathrm{e}^{-\theta t}\right\}, \tag{2.18}$$

where $\nu<1, \alpha<1$ and θ are positive tuning parameters, $(t_k)_{k\in\mathbb{N}}$ is the sequence of event-driven instants when the actuator signal is updated. The event triggered mechanism (2.17) is a static rule, which only supervises the energy and the damping deviation and does not contain any internal dynamical variable. The event-triggered mechanism (2.18) is built based on the system state and the initial energy data $E(0)$. The controller (2.10) will be updated when the event-triggered rules (2.17) and (2.18) are met. In other words, $\|e_k(\cdot,t)\|^2\leq 2\nu E(t)$ and $\|e_k(\cdot,t)\|^2\leq\alpha\left(\|w(\cdot,t)\|^2+\|w_t(\cdot,t)\|^2\right)+E(0)\mathrm{e}^{-\theta t}$ hold between two instants t_k and t_{k+1}. Moreover, we will prove that the designed mechanisms (2.17) and (2.18) have no Zeno behavior as stated in Theorem 2.10. It means no occurrence of Zeno behavior [35–37] in the sense that for each $T>0$, there exists $T_{\min}>0$ such that all the event-triggered instants satisfy $t_{k+1}-t_k\geq T_{\min}\ \forall\, t_{k+1},t_k\in[0,T]$ and $\lim_{k\to\infty}t_k=\infty$.

Remark 2.11. The event-triggered mechanism (2.17) follows a static rule that is straightforward to implement. To effectively reduce the frequency of controller updates during the initial stage without compromising subsequent control decisions, we improve the trigger mechanism using $\|e_k(\cdot,t)\|^2\geq 2\nu E(t)+E(0)\mathrm{e}^{-\theta t}$, where θ is a large parameter. Further, to achieve close tracking, we take into account the L^2 norm of position w and velocity w_t, and tuning parameter α (noting that, by Poincaré inequality, $\|w(\cdot,t)\|^2+\|w_t(\cdot,t)\|^2\leq 2E(t)$). In other words, the event-triggered law (2.18) reduces the number of event-triggered updates at initial stage while ensuring close tracking for more accurate control decisions. Therefore, the trigger mechanism (2.18) offers a perfect balance between simplicity and accuracy, reducing the frequency of actuator communication while guaranteeing system stability. In summary, the event-triggered mechanism (2.17) is simpler to implement, while the trigger mechanism (2.18) improves accuracy and efficiency in controller updates.

Remark 2.12. The thresholds ν and α influence the number of event-triggered updates imposed by (2.17) and (2.18) respectively. The smaller ν or α, the more frequent the updates. Moreover, the parameter θ in trigger law (2.18) is designed to reduce the communtation frequency of actuators, *i.e.*, a smaller θ can reduce the frequency of the updates. The influence of the parameters ν and θ on the triggered frequency could be shown by numerical simulations in Section 4.

The asymptotic behavior of the closed-loop system (2.11) under the event-triggered control is described in the following theorem.

Theorem 2.13. *Given the damping parameter $c>0$, there exist positive parameters ν,α and θ such that the event-triggered controller (2.10) with the trigger law (2.17) or (2.18) exponentially stabilizes beam system (1.1), i.e.,*

$$\left\|\begin{pmatrix} w(\cdot,t)\\ w_t(\cdot,t)\end{pmatrix}\right\|_{\mathcal{H}}\leq M\mathrm{e}^{-\vartheta t}\left\|\begin{pmatrix} w_0\\ w_1\end{pmatrix}\right\|_{\mathcal{H}} \tag{2.19}$$

for some positive constants M and ϑ.

3. Proof of the main result

3.1. Periodic sampled-data control system

In this part, we will prove Theorem 2.5, which states the non-robustness of closed-loop stability for beam system (2.2) under periodic sampling control, *i.e.*,

$$\dot{\phi}(t) = A\phi(t) + BF\phi(k\tau), \qquad t \in [k\tau, (k+1)\tau),$$

where $\phi(t) = (w(\cdot,t), w_t(\cdot,t))^\top$, and A, B and F are respectively defined in (1.2) and (2.5). Let ϱ be the eigenvector of A corresponding to the eigenvalue $\lambda = i\mu^2$. Then we have the characteristic equation:

$$\begin{cases} \varrho''''(x) - \mu^4\varrho(x) = 0, \ x \in (0,1), \\ \varrho(0) = \varrho'(0) = \varrho''(1) = \varrho'''(1) = 0. \end{cases} \tag{3.1}$$

Direct calculation gives the eigenvalues $\lambda_n = i\mu_n^2 = i\left[\left(\frac{\pi}{2} + n\pi\right)^2 + \mathcal{O}(\mathrm{e}^{-|n|})\right]$ as $n \to \infty$, with the corresponding eigenvectors $(\varrho_n, \lambda_n\varrho_n)^\top$, where

$$\varrho_n = (\cosh\mu_n + \cos\mu_n)[\cosh(\mu_n x) + \cos(\mu_n x)] + (\sin\mu_n - \sinh\mu_n)[\sinh(\mu_n x) - \sin(\mu_n x)]. \tag{3.2}$$

Denote $e_n = (\varrho_n/\mu_n^2, i\varrho_n)^\top$, $n \in \mathbb{N}$. Hence $\bar{\lambda}_n = -\lambda_n$ and $\bar{e}_n = (\varrho_n/\mu_n^2, -i\varrho_n)^\top$. Then the eigenvectors $\{e_n, \bar{e}_n | n \in \mathbb{N}\}$ of operator A forms a Riesz basis for $\mathcal{H}$. From the spectral mapping theorems in [34], p. 45, the C_0 semigroup e^{tA} has eigenvalues $\mathrm{e}^{\lambda_n t}$, $\mathrm{e}^{-\lambda_n t} \in \sigma(\mathrm{e}^{At})$, $n \in \mathbb{N}$ and the corresponding eigenvectors e_n, $\bar{e}_n$, $n \in \mathbb{N}$. Denote $\phi(t) = (w(\cdot,t), w_t(\cdot,t))^\top$. Then the solution of (2.4) satisfies

$$\begin{aligned} \phi(t_{k+1}) &= \mathrm{e}^{A\tau}\phi(t_k) + \int_{t_k}^{t_k+\tau} \mathrm{e}^{A(t_k+\tau-s)}BF\phi(t_k)\mathrm{d}s \\ &= \left[\mathrm{e}^{A\tau} + A^{-1}(\mathrm{e}^{A\tau} - I)BF\right]\phi(t_k) \ \triangleq \Delta_\tau\phi(t_k), \qquad \forall k \in \mathbb{N}, \end{aligned} \tag{3.3}$$

where $\Delta_\tau : \mathcal{H} \to \mathcal{H}$ is the new operator:

$$\Delta_\tau = \mathrm{e}^{A\tau} + A^{-1}(\mathrm{e}^{A\tau} - I)BF. \tag{3.4}$$

Now Theorem 2.5 will be proven by analyzing the spectrum of operator Δ_τ.

Proof of Theorem 2.5. Note that $\{e_n, \bar{e}_n | n \in \mathbb{N}\}$ is a Riesz basis for $\mathcal{H}$, $\lambda_n = i\mu_n^2$ and $\bar{\lambda}_n = -\lambda_n$. We look for $\Lambda_n \in \mathbb{C}$ such that $\Delta_\tau\varphi_n = \Lambda_n\varphi_n$. We set:

$$\varphi_n = \alpha_1 e_n + \alpha_2\bar{e}_n = \alpha_1 \begin{pmatrix} \varrho_n/\mu_n^2 \\ i\varrho_n \end{pmatrix} + \alpha_2 \begin{pmatrix} \varrho_n/\mu_n^2 \\ -i\varrho_n \end{pmatrix}, \qquad \alpha_1 \neq 0 \text{ or } \alpha_2 \neq 0.$$

By B and F defined in (2.5), it yields that

$$\begin{aligned}\Delta_\tau\varphi_n &= [\mathrm{e}^{A\tau}+A^{-1}(\mathrm{e}^{A\tau}-I)BF](\alpha_1 e_n+\alpha_2\bar{e}_n)\\ &= \alpha_1\mathrm{e}^{\lambda_n\tau}e_n+\alpha_2\mathrm{e}^{\bar{\lambda}_n\tau}\bar{e}_n+c\left(\frac{\alpha_2-\alpha_1}{2}\right)A^{-1}\left[(\mathrm{e}^{A\tau}-I)(e_n-\bar{e}_n)\right]\\ &= \alpha_1\mathrm{e}^{\lambda_n\tau}e_n+\alpha_2\mathrm{e}^{-\lambda_n\tau}\bar{e}_n+c\left(\frac{\alpha_2-\alpha_1}{2}\right)\left(\frac{\mathrm{e}^{\lambda_n\tau}-1}{\lambda_n}e_n-\frac{\mathrm{e}^{\bar{\lambda}_n\tau}-1}{\bar{\lambda}_n}\bar{e}_n\right)\\ &= \left[\alpha_1\mathrm{e}^{\lambda_n\tau}+c\left(\frac{\alpha_2-\alpha_1}{2\lambda_n}\right)(\mathrm{e}^{\lambda_n\tau}-1)\right]e_n+\left[\alpha_2\mathrm{e}^{-\lambda_n\tau}+c\left(\frac{\alpha_2-\alpha_1}{2\lambda_n}\right)(\mathrm{e}^{-\lambda_n\tau}-1)\right]\bar{e}_n\\ &= \Lambda_n\alpha_1 e_n+\Lambda_n\alpha_2\bar{e}_n.\end{aligned}\tag{3.5}$$

Since $\{e_n,\bar{e}_n|n\in\mathbb{N}\}$ is the orthonormal basis for $\mathcal{H}$, it follows from (3.5) that

$$\begin{cases}\left[\Lambda_n-\mathrm{e}^{\lambda_n\tau}+\dfrac{c(\mathrm{e}^{\lambda_n\tau}-1)}{2\lambda_n}\right]\alpha_1-\dfrac{c(\mathrm{e}^{\lambda_n\tau}-1)}{2\lambda_n}\alpha_2=0,\\ \dfrac{c(\mathrm{e}^{-\lambda_n\tau}-1)}{2\lambda_n}\alpha_1+\left[\Lambda_n-\mathrm{e}^{-\lambda_n\tau}-\dfrac{c(\mathrm{e}^{-\lambda_n\tau}-1)}{2\lambda_n}\right]\alpha_2=0.\end{cases}\tag{3.6}$$

Thus the system of homogeneous linear equations (3.6) has a non-zeros solution (α_1,α_2) if and only if the following holds:

$$\Lambda_n^2-\left[\mathrm{e}^{\lambda_n\tau}+\mathrm{e}^{-\lambda_n\tau}-\frac{c(\mathrm{e}^{\lambda_n\tau}-\mathrm{e}^{-\lambda_n\tau})}{2\lambda_n}\right]\Lambda_n+1-\frac{c(\mathrm{e}^{\lambda_n\tau}-\mathrm{e}^{-\lambda_n\tau})}{2\lambda_n}=0.$$

This quadratic equation has two roots in $\mathbb{C}$ given by:

$$\begin{aligned}(\Lambda_n)_{1,2}=&\frac{1}{2}\left[\mathrm{e}^{\lambda_n\tau}+\mathrm{e}^{-\lambda_n\tau}-\frac{c(\mathrm{e}^{\lambda_n\tau}-\mathrm{e}^{-\lambda_n\tau})}{2\lambda_n}\right]\\ &\pm\frac{1}{2}\left[\left(\mathrm{e}^{\lambda_n\tau}+\mathrm{e}^{-\lambda_n\tau}-\frac{c(\mathrm{e}^{\lambda_n\tau}-\mathrm{e}^{-\lambda_n\tau})}{2\lambda_n}\right)^2+\frac{2c(\mathrm{e}^{\lambda_n\tau}-\mathrm{e}^{-\lambda_n\tau})}{\lambda_n}-4\right]^{1/2}.\end{aligned}\tag{3.7}$$

Substituting $\lambda_n=i\mu_n^2$, $\mathrm{e}^{\lambda_n\tau}+\mathrm{e}^{-\lambda_n\tau}=2\cos(\mu_n^2\tau)$ and $\mathrm{e}^{\lambda_n\tau}-\mathrm{e}^{-\lambda_n\tau}=2i\sin(\mu_n^2\tau)$ into (3.7) yields

$$\begin{aligned}(\Lambda_n)_{1,2}=&\cos(\mu_n^2\tau)-\frac{c}{2\mu_n^2}\sin(\mu_n^2\tau)\\ &\pm\left[\left(\cos(\mu_n^2\tau)-\frac{c}{2\mu_n^2}\sin(\mu_n^2\tau)\right)^2+\frac{c}{\mu_n^2}\sin(\mu_n^2\tau)-1\right]^{1/2}.\end{aligned}\tag{3.8}$$

Since $\mu_n^2=\left(\frac{\pi}{2}+n\pi\right)^2+\mathcal{O}(\mathrm{e}^{-|n|})$ tends to infinity as n goes to infinity, from (3.8) we have: $\lim_{n\to\infty}|\Lambda_n|=1$. Therefore, the spectral radius $r(\Delta_\tau)\geq 1$, $\forall\tau>0$. From Gelfand formula in Lemma A.1, we know that $r(\Delta_\tau)=\lim_{k\to\infty}\|\Delta_\tau^k\|^{1/k}$ and Δ_τ is not exponential power stable. By equivalence between exponentially power stability of Δ_τ and exponential stability of system (2.4), it follows from Lemma 2.3 in [14] that the system is not exponentially stable $\forall\tau>0$.

Next, we will show that $\forall\tau^*>0$, there exists a $\tau\in(0,\tau^*)$ such that the system (2.4) is not strongly stable with the sampling period τ.

Indeed, $\forall \tau^* > 0$, there exists a N such that $\mu_n^2 > 2\pi/\tau^*, \forall n \geq N$. Taking two sampling periods $\tau_1 = \tau^* - \pi/\mu_n^2$ and $\tau_2 = \tau^* - 2\pi/\mu_n^2$, then we have $\tau_1, \tau_2 \in (0, \tau^*)$ and $\mu_n^2\tau_1 = \mu_n^2\tau^* - \pi$, $\mu_n^2\tau_2 = \mu_n^2\tau^* - 2\pi$, and

$$\sin(\mu_n^2\tau) \leq 0, \quad \text{for } \tau = \tau_1 \text{ or } \tau_2, \quad \forall n \geq N. \tag{3.9}$$

Then $\forall \tau^* > 0$, there always exists a $\tau \in (0, \tau^*)$ such that $\sin(\mu_n^2\tau) \leq 0$. Taking sampling period τ and a sufficiently large integer n satisfying (3.9), we see that

$$\begin{aligned} &\left[\cos(\mu_n^2\tau) - \frac{c}{2\mu_n^2}\sin(\mu_n^2\tau)\right]^2 + \frac{c}{\mu_n^2}\sin(\mu_n^2\tau) - 1 \\ =&\left(\frac{c^2}{4\mu_n^4} - 1\right)(\sin(\mu_n^2\tau))^2 - \frac{c}{\mu_n^2}\sin(\mu_n^2\tau)\left(\cos(\mu_n^2\tau) - 1\right) \leq 0. \end{aligned} \tag{3.10}$$

Considering the modulus of eigenvalue in (3.8), it follows from (3.9) and (3.10) that

$$|\Lambda_n|^2 = 1 - \frac{c}{\mu_n^2}\sin(\mu_n^2\tau) \geq 1.$$

Hence, for the eigenvalue Λ_n and the corresponding eigenvector φ_n, we have

$$\|\Delta_\tau^k\varphi_n\| = \|\Lambda_n^k\varphi_n\| \geq \|\varphi_n\|, \qquad \lim_{k\to\infty}\|\Delta_\tau^k\varphi_n\| \neq 0.$$

Therefore, the system (2.4) is not strongly stable (see Def. 2.1, [38]). Therefore, for any $\tau^* > 0$, there always exists a sampling period $\tau \in (0, \tau^*)$ such that the system (2.11) is not strongly stable. The proof is complete. □

3.2. Event-triggered control systems

Before proving the well-posedness of event-triggered control system (2.11) and the absence of Zeno phenomenon in Theorem 2.10, we present the following lemma, whose proof is postponed to Appendix.

Lemma 3.1. *Let $(t_k)_{k\in\mathbb{N}}$ be a strictly increasing trigger sequence defined by the event-triggered mechanism (2.17) or (2.18). The energy $E(t)$ of the controlled system satisfies the following inequalities:*

(i) For the first event-triggered mechanism (2.17),

$$\mathrm{e}^{-2\tilde{M}_0 t}E(0) \leq E(t) \leq \mathrm{e}^{2M_0 t}E(0), \quad t \in [0, t_k], \ \forall\, k \in \mathbb{N},$$

where $M_0 = \nu c/4$ and $\tilde{M}_0 = \nu c/4 + 2c$;

(ii) For the second event-triggered mechanism (2.18), on the one hand,

$$E(t) \leq \alpha_2\mathrm{e}^{2M_1 t}E(0), \quad t \in [0, t_k], \ \forall\, k \in \mathbb{N}, \tag{3.11}$$

where $M_1 = \alpha c/4$ and $\alpha_2 = 1 + \frac{c}{2(\alpha c + 2\theta)}$, and on the other hand, by taking $\theta \geq (\alpha + 9)c/2$,

$$E(t) \geq \alpha_1\mathrm{e}^{-2\tilde{M}_1 t}E(0), \quad t \in [0, t_k], \ \forall\, k \in \mathbb{N}, \tag{3.12}$$

where $\tilde{M}_1 = \alpha c/4 + 2c$ and $\alpha_1 = 1/2$.

Consider the following mapping $\mathcal{H} \to C([t_0, \infty), L^2(0, 1))$:

$$Z(t) = B^*(\mathrm{e}^{(t-t_0)A} - I)(I + A^{-1}BF)\phi_0, \quad t_0 \geq 0, \tag{3.13}$$

where A is defined in (1.2) and B, F are defined in (2.5). B^* is the dual operator of B. We consider the orthonormal basis of $\mathcal{H}$ formed by the eigenvectors of A:

$$Ae_n = i\sigma_n e_n \triangleq i\mu_n^2 e_n, \quad e_n = (\varrho_n/\mu_n^2, i\varrho_n)^\top, \quad e_{-n} = \bar{e}_n, \quad \|\varrho_n\|^2 = 1/2, \quad \forall\, n \in \mathbb{N}. \tag{3.14}$$

For each $\phi_0 \in \mathcal{H}$, we write

$$\phi_0 = \sum_{n=1}^{\infty} \left(\alpha_n e_n + \beta_n e_{-n}\right), \tag{3.15}$$

so that $\|\phi_0\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \left(|\alpha_n|^2 + |\beta_n|^2\right)$. Then the following observability inequality holds true:

Lemma 3.2. *Let ϕ_0 and $Z(t)$ be defined in (3.15) and (3.13), respectively. Then there exists some constant $K > 0$ (depending only on c and σ_1) such that the following observability inequality is satisfied:*

$$\frac{T}{2} < \Lambda\phi_0, \phi_0 > -K\|\phi_0\|_{\mathcal{H}}^2 \le \int_0^T \|Z(t)\|^2 \mathrm{d}t \le \frac{T}{2} < \Lambda\phi_0, \phi_0 > +K\|\phi_0\|_{\mathcal{H}}^2, \quad \forall\, T > 0, \tag{3.16}$$

where the operator $\Lambda : \mathcal{H} \to \mathcal{H}$ defined below is positive definite:

$$\Lambda\phi_0 = \sum_{n=1}^{\infty} \left\{ \left(2 + \frac{c^2}{\sigma_n^2}\right)\left(\alpha_n e_n + \beta_n e_{-n}\right) - \left(1 + \frac{ic}{\sigma_n} + \frac{c^2}{2\sigma_n^2}\right)\beta_n e_n - \left(1 - \frac{ic}{\sigma_n} + \frac{c^2}{2\sigma_n^2}\right)\alpha_n e_{-n} \right\}.$$

Moreover, we have

$$< \Lambda\phi_0, \phi_0 >_{\mathcal{H}} \ge \left(\frac{3}{3+K}\right)\|\phi_0\|_{\mathcal{H}}^2.$$

Remark 3.3. The assertion of Lemma 3.2 still holds for the integral $\int_{t_0}^{t_0+T} \|Z(t)\|^2 \mathrm{d}t \;\; \forall\, t_0 \ge 0$.

Based on the above lemmas, the well-posedness of the closed-loop system (2.11) and the absence of Zeno phenomenon can be established, provided that the initial energy $E(0) \neq 0$. If $E(0) = 0$, then the unique solution of the closed-loop system is zero for all positive time.

Lemma 3.4. *For any initial condition $(w_0, w_1) \in \mathcal{H}$ such that $E(0) \neq 0$, there exist tuning parameters α, ν and θ in trigger laws (2.17) and (2.18) such that the event-triggered instants $(t_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence satisfying $\lim_{k\to\infty} t_k = \infty$ and the closed-loop system (2.11) has a unique solution satisfying*

$$w \in C([0,\infty), H_L^2(0,1)) \cap C^1([0,\infty), L^2(0,1)).$$

Proof of Lemma 3.4. This lemma will be proven by two steps. In the first step, we will utilize the induction method to prove the existence of the next instant $t_{k+1} > t_k$ for any $k \in \mathbb{N}$ and the well-posedness within every sampled interval $[t_k, t_{k+1}]$. In the second step, we show that the strictly increasing sequence $(t_k)_{k\in\mathbb{N}}$ diverges to infinity, thereby guaranteeing the well-posedness result for the entire interval $(0,\infty)$.

Set $\phi(t) = (w(\cdot,t), w_t(\cdot,t))^\top$. Then the solution of (2.11) or (2.12) is given by

$$\phi(t) = \mathrm{e}^{A(t-t_k)}\phi(t_k) + \int_{t_k}^{t} \mathrm{e}^{A(t-s)} BF\phi(t_k)\mathrm{d}s, \qquad \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N}, \tag{3.17}$$

where A is the generator defined in (1.2), and B, F are bounded operators defined in (2.5). Recall that the speed deviation is given by

$$e_k(x,t) = w_t(x,t) - w_t(x,t_k) = B^*(\phi(t) - \phi(t_k)) = B^*(\mathrm{e}^{A(t-t_k)} - I)(I + A^{-1}BF)\phi(t_k). \tag{3.18}$$

Base case: On the first time interval $[0, t_1)$, the closed-loop system (2.11) under control $u(x,t) = -cw_t(x,t_0) = -cw_1(x)$, is governed by

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = -cw_1(x),\ x \in (0,1), t \in [0,t_1), \\ w(0,t) = 0,\ w_x(0,t) = 0,\ w_{xx}(1,t) = 0,\ w_{xxx}(1,t) = 0,\ t \geq 0, \end{cases} \tag{3.19}$$

with initial state $\phi_0 = (w_0, w_1)^\top \in \mathcal{H}$. From Theorem 2.9 in [34], p. 109, since the source term is $-cw_1(x) \in L^2(0,1)$ and $\phi_0 \in \mathcal{H}$, system (3.19) has a unique solution $w(x,t)$ satisfying $(w, w_t)^\top \in C([0,t_1], \mathcal{H})$.

Set

$$T_0 = \frac{4K(3+K)}{3}, \qquad \nu = \frac{\mathrm{e}^{-2M_0T_0}}{2(3+K)} \tag{3.20}$$

with K defined in Lemma 3.2. Now, we prove that there is a positive t_1, $t_1 < T_0$, such that the trigger law (2.17) is triggered at t_1. We prove it by absurd. Suppose that the trigger law is not satisfied on the interval $[0, T_0]$.

By (3.18), $e_0(\cdot,t)$ is nothing else than the function $Z(t)$ defined in (3.13): $e_0(\cdot,t) = B^*(\mathrm{e}^{At} - I)(I + A^{-1}BF)\phi_0$. By Lemma 3.2 we have:

$$\int_0^T \|e_0(\cdot,t)\|^2 \mathrm{d}t \geq \frac{T}{2} < \Lambda\phi_0, \phi_0 >_{\mathcal{H}} - K\|\phi_0\|^2_{\mathcal{H}}. \tag{3.21}$$

Since by Lemma 3.2 $< \Lambda\phi_0, \phi_0 >_{\mathcal{H}} \geq \dfrac{3}{3+K}\|\phi_0\|^2_{\mathcal{H}}$, by taking $T = T_0 = \dfrac{4K(3+K)}{3}$ the following holds true:

$$\int_0^T \|e_0(\cdot,t)\|^2 \mathrm{d}t \geq K\|\phi_0\|^2_{\mathcal{H}} = 2KE(0). \tag{3.22}$$

Since $\|e_0(\cdot,t)\|^2$ is continuous in $t \in [0,T]$, there exists $t_0^* \in (0,T]$ such that

$$\|e_0(\cdot,t_0^*)\|^2 = \max_{0 \leq t \leq T} \|e_0(\cdot,t)\|^2.$$

Then from (3.22), it means that

$$\|e_0(\cdot,t_0^*)\|^2 \geq \frac{1}{T}\int_0^T \|e_0(\cdot,t)\|^2 \mathrm{d}t \geq \frac{K}{T}\|\phi_0\|^2_{\mathcal{H}} = \frac{3}{2(3+K)}E(0), \quad t_0^* \in (0,T]. \tag{3.23}$$

By Lemma 3.1 we have $E(0) \geq E(t_0^*)\mathrm{e}^{-2M_0t_0^*}$ and, *a fortiori*,

$$E(0) \geq E(t_0^*)\mathrm{e}^{-2M_0T_0}. \tag{3.24}$$

Substituting (3.24) into (3.23), we get:

$$\|e_0(\cdot,t_0^*)\|^2 > 2\nu E(t_0^*),$$

where

$$\nu < \frac{\mathrm{e}^{-2M_0T_0}}{2(3+K)}. \tag{3.25}$$

Hence the trigger law is triggered at $t_1 = t_0^*$. This is in contradiction with the hypothesis. So the existence of such a t_1 is guaranteed.

For the second trigger law (2.18), we set

$$T_0 = \max\{8K(3+K), 2\theta^{-1}\ln(3+K)\}. \tag{3.26}$$

We prove that there exists $t_1 \leq T_0$ such that the the second trigger law is triggered at t_1. It is proved by absurd. Suppose that there is no triggering in the interval $[0, T_0]$. We will get a contradiction.

Indeed, by Lemma 3.2 and Remark 3.3 we have:

$$\int_{T/2}^{T} \|e_0(\cdot,t)\|^2 \mathrm{d}t \geq \frac{T}{2} < \Lambda\phi_0, \phi_0 > -K\|\phi_0\|^2 - \int_0^{T/2} \|e_0(\cdot,t)\|^2 \mathrm{d}t. \tag{3.27}$$

Application of Lemma 3.2 leads to

$$\int_0^{T/2} \|e_0(\cdot,t)\|^2 \mathrm{d}t \leq \frac{T}{4} < \Lambda\phi_0, \phi_0 > +K\|\phi_0\|^2. \tag{3.28}$$

Substituting (3.28) into (3.27) and application of Lemma 3.2 give:

$$\int_{T/2}^{T} \|e_0(\cdot,t)\|^2 \mathrm{d}t \geq \frac{T}{4} < \Lambda\phi_0, \phi_0 > -2K\|\phi_0\|^2 \geq \left(\frac{3T}{4(3+K)} - 2K\right)\|\phi_0\|^2.$$

As above there exists $t_0^* \in [T/2, T]$ such that

$$\|e_0(\cdot, t_0^*)\|^2 = \max_{T/2 \leq t \leq T} \|e_0(\cdot,t)\|^2.$$

By taking $T = T_0$ in (3.26), it follows from (3.27) that

$$\|e_0(\cdot,t_0^*)\|^2 \geq \frac{2}{T_0}\int_{T_0/2}^{T_0} \|e_0(\cdot,t)\|^2 \mathrm{d}t \geq \left(\frac{1}{3+K}\right)\|\phi_0\|_{\mathcal{H}}^2 = \left(\frac{2}{3+K}\right)E(0), \quad t_0^* \in [T_0/2, T_0]. \tag{3.29}$$

By Lemma 3.1 we have $E(0) \geq \alpha_2^{-1}E(t_0^*)\mathrm{e}^{-2M_1t_0^*}$ and, *a fortiori*,

$$E(0) \geq \alpha_2^{-1}E(t_0^*)\mathrm{e}^{-2M_1T_0}. \tag{3.30}$$

From (3.29), $t_0^* \geq T_0/2$ and $\theta T_0 \geq 2\ln(3+K)$, one has

$$\|e_0(\cdot,t_0^*)\|^2 - \mathrm{e}^{-\theta t_0^*}E(0) \geq \left(\frac{2}{3+K} - \mathrm{e}^{-\theta T_0/2}\right)E(0) \geq \frac{E(0)}{3+K} \geq \frac{\mathrm{e}^{-2M_1T_0}E(0)}{(3+K)\alpha_2}.$$

By Lemma 3.1, the above inequality implies the inequality:

$$\|e_0(\cdot,t_0^*)\|^2 \geq \alpha E(t_0^*) + \mathrm{e}^{-\theta t_0^*}E(0), \tag{3.31}$$

where

$$\alpha < \frac{\mathrm{e}^{-2M_1T_0}}{(3+K)\alpha_2}, \qquad \alpha_2 = 1 + \frac{c}{2\theta + \alpha c}. \tag{3.32}$$

Hence the trigger law is triggered at $t_1 = t_0^*$. This contradiction proves the existence of such a $t_1 > 0$.

Induction step: for any fixed $k \in \mathbb{N}$, assume $(w, w_t)^\top \in C([t_{k-1}, t_k], \mathcal{H})$. Then on the $(k+1)$th time interval $[t_k, t_{k+1})$, from Theorem 2.9 in [34], p. 109 again, the closed-loop system

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = -cw_t(x,t_k),\ x \in (0,1), t \in [t_k, t_{k+1}), \\ w(0,t) = 0,\ w_x(0,t) = 0,\ w_{xx}(1,t) = 0,\ w_{xxx}(1,t) = 0, \end{cases} \tag{3.33}$$

with initial state $(w(x,t_k), w_t(x,t_k))^\top \in \mathcal{H}$ and the source term $-cw_t(x,t_k) \in L^2(0,1)$ has a unique solution $w(x,t)$ satisfying (3.17) and $(w, w_t)^\top \in C([t_k, t_{k+1}], \mathcal{H})$.

Now consider the next trigger instant t_{k+1}. Similarly there exists some $t_1 > 0$, $t_1 \le T_0 = 2K(3+K)/3$, such that the trigger law is triggered at $t = t_k + t_1$. We prove it by absurd. Suppose that there is no triggering in $(t_k, t_k + T_0]$. From (3.33) and (3.2), for each $\phi(t_k) = (w(x,t_k), w_t(x,t_k))^\top \in \mathcal{H}$ we have

$$e_k(\cdot, t) = B^*(\mathrm{e}^{A(t-t_k)} - I)(I + A^{-1}BF)\phi(t_k).$$

Notice that

$$\int_{t_k}^{t_k+T} \|e_k(\cdot,t)\|^2 \mathrm{d}t = \int_0^T \|B^*(\mathrm{e}^{At} - I)(I + A^{-1}BF)\phi(t_k)\|^2 \mathrm{d}t. \tag{3.34}$$

Hence the same reasoning as in the case $k = 0$ is applied by replacing ϕ_0 by $\phi(t_k)$ to prove that the trigger law is triggered at $t = t_k + t_1$ with $0 < t_1 \le 2K(3+K)/3$. This contradiction proves the existence of a triggering time $t_{k+1} \in (t_k, t_k + T_0]$.

For the second trigger law (2.18), we set $T_0 = \max\{8K(3+K),\ 2\theta^{-1}\ln(2(3+K))\}$. By Lemma 3.2, Remark 3.3 and (3.34) we have:

$$\int_{t_k}^{t_k+T} \|e_k(\cdot,t)\|^2 \mathrm{d}t \ge \frac{T}{2} < \Lambda\phi_k, \phi_k >_{\mathcal{H}} - K\|\phi_k\|_{\mathcal{H}}^2 \tag{3.35}$$

and

$$\int_{t_k}^{t_k+T/2} \|e_k(\cdot,t)\|^2 \mathrm{d}t \le \frac{T}{4} < \Lambda\phi_k, \phi_k >_{\mathcal{H}} + K\|\phi_k\|_{\mathcal{H}}^2, \tag{3.36}$$

which leads to

$$\|e_k(\cdot, t_k + t_0^*)\|^2 \ge \frac{2}{T}\int_{t_k+T_0/2}^{t_k+T_0} \|e_k(\cdot,t)\|^2 \mathrm{d}t \ge 2E(t_k)/(3+K),\ t_0^* \in [T_0/2, T_0]. \tag{3.37}$$

Notice that the same reasoning as in the case $k = 0$ is applicable to prove existence of a triggering time $t_{k+1} = t_k + t_0^*$. Indeed, although we do not have exactly the same inequality (3.30), we claim that the similar one still holds as follows:

$$E(t_k) \ge \alpha_2^{-1}\mathrm{e}^{-2M_1T_0}E(t_k + t_0^*) \tag{3.38}$$

where α_2 is defined in (3.11). The inequality (3.12) in Lemma 3.1 gives us:

$$E(0) \leq \alpha_1^{-1} \mathrm{e}^{2\check{M}_1 t_k} E(t_k). \tag{3.39}$$

From (3.37), (3.39) and (3.38) we write immediately:

$$\|e_k(\cdot, t_k + t_0^*)\|^2 - \mathrm{e}^{-\theta(t_k+t_0^*)} E(0) \geq \left(\frac{2}{3+K} - 2\mathrm{e}^{-\theta T_0/2} \right) \tilde{\alpha}_2^{-1} E(t_k + t_0^*) \mathrm{e}^{-\alpha c T_0/2}. \tag{3.40}$$

As $T_0 \geq 2\theta^{-1} \ln(2(3+K))$ and α satisfies (3.32), the inequality (3.40) implies the required triggering one:

$$\|e_k(\cdot, t_k + t_0^*)\|^2 - \mathrm{e}^{-\theta(t_k+t_0^*)} E(0) \geq \alpha E(t_k + t_0^*).$$

We have only to prove the claim. Indeed, integrating (A.4) from t_k to t yields:

$$E(t) \leq \mathrm{e}^{\alpha c(t-t_k)/2} E(t_k) + \frac{cE(0)\mathrm{e}^{-\theta t_k}}{2(2\theta + \alpha c)} \left[\mathrm{e}^{\alpha c(t-t_k)/2} - \mathrm{e}^{-\theta(t-t_k)} \right]. \tag{3.41}$$

By (3.12) in Lemma 3.1 we have

$$E(0) \leq \alpha_1^{-1} \mathrm{e}^{2\check{M}_1 t_k} E(t_k). \tag{3.42}$$

Substituting (3.42) into (3.41) gives us the following:

$$E(t_k + t_0^*) \leq \alpha_2 e^{\frac{\alpha c t_0^*}{2}} E(t_k). \tag{3.43}$$

As $t_0^* \leq T_0$, reversing the above inequality proves the claim.

Conclusion step: by induction, for any $k \in \mathbb{N}$, we get $(w, w_t)^\top \in C([0, t_k], \mathcal{H})$. And this way we define a strictly increasing sequence (t_k) in $\mathbb{R}$. Now we prove that the strictly increasing sequence diverges to infinity, or, $\lim_{k\to\infty} t_k = \infty$. If not, we assume $T = \lim_{k\to\infty} t_k < \infty$.

From the well-posedness of the closed-loop system (2.11) in $[t_k.t_{k+1}]$, $k \in \mathbb{N}$, the solution of (2.11) satisfies $w_t \in C([0,T], L^2(0,1))$, $\forall k \in \mathbb{N}$. Then we have the uniform continuity of the function $t \mapsto e(t)$ from the compact set $[0,T]$ to Hilbert space $L^2(0,1)$ defined by $e(t) = e_k(\cdot, t)$ $\forall\, t \in [t_k, t_{k+1}]$. Hence, $\forall \epsilon > 0$ there exists $\delta > 0$ such that for every $s, t \in [t_k, t_{k+1}]$, $\quad |t-s| < \delta \;\Rightarrow\; \|e_k(\cdot,t) - e_k(\cdot,s)\| < \epsilon$. Then the contrapositive of this uniform continuity says that if, for a $\epsilon > 0$,

$$\|e_k(\cdot,t) - e_k(\cdot,s)\| > \epsilon \;\; \forall\, k \in \mathbb{N}, \tag{3.44}$$

then there exists some $\delta > 0$ such that $|t-s| > \delta$. From event-triggered control mechanism (2.17) and Lemma 3.1, setting $s = t_k$ and $t = t_{k+1}$ results that

$$\begin{aligned} \|e(t_{k+1}) - e(t_k)\| = \|e_k(\cdot, t_{k+1}) - e_k(\cdot, t_k)\| = \|e_k(\cdot, t_{k+1})\| &> \sqrt{2\nu E(t_{k+1})} \\ &\geq \sqrt{2\nu E(0)} \mathrm{e}^{-M_0 t_{k+1}} \geq \sqrt{2\nu E(0)} \mathrm{e}^{-M_0 T} = \epsilon_T, \end{aligned} \tag{3.45}$$

where $M_0 = c(1+\sqrt{\nu})$. For the event-triggered control mechanism (2.18), letting $s = t_k$, $t = t_{k+1}$ yields that

$$\|e_k(\cdot, t_{k+1}) - e_k(\cdot, t_k)\| = \|e_k(\cdot, t_{k+1})\| \geq \sqrt{E(0)} \mathrm{e}^{-\theta t_{k+1}/2} \geq \sqrt{E(0)} \mathrm{e}^{-\theta T/2} = \tilde{\epsilon}_T. \tag{3.46}$$

Note that ϵ_T and $\tilde{\epsilon}_T$ depend only on T, initial energy and tuning parameters c, ν and θ. Thus, there exists a real number $\delta_T > 0$ not depending on k such that

$$|t_{k+1} - t_k| > \delta_T, \tag{3.47}$$

which implies that

$$T = \lim_{k\to\infty} t_k \geq \lim_{k\to\infty} (k-1)\delta_T = \infty.$$

Hence, the contradiction with the assumption $T < \infty$, proves that the sequence $(t_k)_{k\in\mathbb{N}}$ has no finite accumulation point, which means the absence of Zeno behavior in the event-triggered mechanisms (2.17) and (2.18).

From the continuous extension at the instants, we see that the unique solution (w, w_t) of system (2.11) under trigger laws (2.17) or (2.18) is continuous for all $t \in [0, \infty)$, therefore,

$$w \in C([0,\infty), H_L^2(0,1)) \cap C^1([0,\infty), L^2(0,1)).$$

This implies that if $E(0) \neq 0$, then the studied event-triggered control problem for system (2.11) is well-posed and has no Zeno behavior. The proof is complete. □

Remark 3.5. In the proof of Lemma 3.4, the absence of Zeno phenomenon is established by showing that $t_k \to \infty$ as $k \to \infty$. For any $T > 0$, the minimum inter-event time $\delta_T > 0$ depends on the final time T, the initial energy and the tuning parameters of the event-triggered control. Since $k < T/\delta_T < \infty$, the actuator cannot be triggered infinitely many times in any finite time interval. As time progresses, the minimum inter-event time may decrease, and the control may approach a nearly continuous-in-time implementation as $t \to \infty$, which is unachievable by periodic sampling control. Although the trigger frequency may increase with time, event-triggered controllers are feasible and effective in practical applications, because they can stabilize the beam system within a finite time interval with acceptable precision while avoiding the Zeno phenomenon.

According to Lemma 3.4, the well-posedness of closed-loop system (2.11) and the absence of Zeno behavior can be proven by considering only $E(0) = 0$. This proof is provided in Appendix.

To prove the stability of beam system (2.11) under event-triggered control (2.10), we consider the following candidate of Lyapunov functional

$$V(w, w_t) = \frac{1}{2}\left(\|w_{xx}\|^2 + \|w_t\|^2 + \varepsilon c\|w\|^2\right) + \int_0^1 \varepsilon w w_t \mathrm{d}x, \tag{3.48}$$

where $(w, w_t)^\top \in \mathcal{H}$ and $\varepsilon \in (0, 1)$ is a positive constant. We claim that $V(w, w_t)$ is equivalent to the system energy (2.15) for every $0 < \varepsilon < 1$, which can be proven by the following inequalities.

Lemma 3.6. *[39], p. 182 (Wirtinger inequality) For $g \in H^1(0,1)$, if $g(0) = 0$ then*

$$\|g(\cdot)\|^2_{L^2(0,1)} \leq \frac{4}{\pi^2}\|g'(\cdot)\|^2_{L^2(0,1)}.$$

Then, it is not difficult to show the equivalence between the Lyapunov functional (3.48) and the system energy (2.15).

Lemma 3.7. *Let $0 < \varepsilon < 1$ in (3.48). Then, there exist positive constants β_1 and β_2 such that*

$$\beta_1 E(t) \leq V(t) \leq \beta_2 E(t), \ t \geq 0, \tag{3.49}$$

where

$$\beta_1 = 1-\varepsilon > 0,\ \ \beta_2 = 1+\varepsilon\left(1+\frac{16c}{\pi^4}\right) > 0. \tag{3.50}$$

Now, we propose LMI-based conditions to guarantee exponential stability of the closed-loop system (2.11) under the event-triggered mechanism (2.17). By using Lyaponov's direct method and the $\mathcal{S}-$procedure in Lemma A.2, the following results can be proven.

Theorem 3.8. *Let the damping parameter $c > 0$. Then there exist positive constants ν, $\delta_1 > 0$ and $M \geq 1$ such that the closed-loop system (2.11) under the event-triggered mechanism (2.17) is exponentially stabilized. More precisely, for every initial state $(w_0, w_1) \in \mathcal{H}$ defined in (1.2), the closed-loop system (2.11) under the event-triggered mechanism (2.17) is exponentially stable with energy decay rate $2\delta_1$ as follows:*

$$E(t) \leq ME(0)\mathrm{e}^{-2\delta_1 t} \quad \forall t \geq 0. \tag{3.51}$$

Remark 3.9. The parameters ν and δ_1 can be explicitly determined by making the following matrix Q positive definite:

$$Q = \begin{pmatrix} a_1-\varepsilon c\delta_1 & -\delta_1\varepsilon & -\varepsilon c/2 & 0 \\ -\delta_1\varepsilon & -a_2\nu-\varepsilon+c-\delta_1 & -c/2 & 0 \\ -\varepsilon c/2 & -c/2 & a_2 & 0 \\ 0 & 0 & 0 & -a_2\nu-16a_1/\pi^4+\varepsilon-\delta_1 \end{pmatrix} > 0, \tag{3.52}$$

where a_1, a_2, ν and δ_1 are real positive constants and $0 < \varepsilon < 1$.

Proof of Theorem 3.8. Differentiating $V(t)$ along with the trajectory $(w(x,t), w_t(x,t))$ of (2.11) and integration by parts yield

$$\begin{aligned} \dot{V}(t) =& \int_0^1 [w_{xx}(x,t)w_{xxt}(x,t) + w_t(x,t)w_{tt}(x,t) + \varepsilon c w_t(x,t)w(x,t) \\ &\qquad +\varepsilon w_t^2(x,t) + \varepsilon w(x,t)w_{tt}(x,t)]\,\mathrm{d}x \\ =& \int_0^1 \left[-cw_t(x,t)w_t(x,t_k) + \varepsilon c w(x,t)w_t(x,t) + \varepsilon w_t^2(x,t)\right. \\ &\qquad \left. -\varepsilon w_{xx}^2(x,t) - \varepsilon c w(x,t)w_t(x,t_k)\right]\mathrm{d}x \\ =&(\varepsilon-c)\|w_t(\cdot,t)\|^2 - \varepsilon\|w_{xx}(\cdot,t)\|^2 \\ &\qquad + c\int_0^1 w_t(x,t)e_k(x,t)\mathrm{d}x + \varepsilon c\int_0^1 w(x,t)e_k(x,t)\mathrm{d}x, \end{aligned} \tag{3.53}$$

where $e_k(x,t)$ is the speed deviation defined in (2.16). For some $\delta_1 > 0$, adding $2\delta_1 V(t)$ to both sides of (3.53) leads to

$$\begin{aligned} &\dot{V}(t) + 2\delta_1 V(t) \\ =&(\varepsilon - c + \delta_1)\|w_t(\cdot,t)\|^2 + (\delta_1-\varepsilon)\|w_{xx}(\cdot,t)\|^2 + \varepsilon c\delta_1\|w(\cdot,t)\|^2 \\ &+\int_0^1 [2\varepsilon\delta_1 w(x,t)w_t(x,t) + cw_t(x,t)e_k(x,t) + \varepsilon c w(x,t)e_k(x,t)]\,\mathrm{d}x \\ \triangleq& \int_0^1 \Phi^\top(x,t)Q_0\Phi(x,t)\mathrm{d}x, \end{aligned} \tag{3.54}$$

with

$$\Phi(x,t)=(w(x,t),w_t(x,t),e_k(x,t),w_{xx}(x,t))^\top \tag{3.55}$$

and

$$Q_0=\begin{pmatrix}\varepsilon c\delta_1 & \delta_1\varepsilon & \varepsilon c/2 & 0\\ \delta_1\varepsilon & \varepsilon-c+\delta_1 & c/2 & 0\\ \varepsilon c/2 & c/2 & 0 & 0\\ 0 & 0 & 0 & \delta_1-\varepsilon\end{pmatrix}. \tag{3.56}$$

It follows from Wirtinger inequality in Lemma 3.6 that

$$\int_0^1 \Phi^\top(x,t)Q_1\Phi(x,t)\mathrm{d}x\geq 0,\quad \forall t>0, \tag{3.57}$$

with $Q_1=\mathrm{diag}(-1,0,0,16/\pi^4)$. According to the event-triggered rule (2.17), we know that $\|e_k(\cdot,t)\|^2\leq 2\nu E(t)$, $\forall t\in[t_k,t_{k+1})$, $\forall k\in\mathbb{N}$, which means

$$\int_0^1 \Phi^\top(x,t)Q_2\Phi(x,t)\mathrm{d}x\geq 0,\quad \forall t\in[t_k,t_{k+1}),\ \forall k\in\mathbb{N},$$

with $Q_2=\mathrm{diag}(0,\nu,-1,\nu)$. Note that although the metrics Q_1 and Q_2 are not positive semi-definite, the integral of their associated bilinear form is non-negative. By using the $\mathcal{S}-$procedure in (3.54), we find that

$$\begin{aligned}&\dot V(t)+2\delta_1V(t)+a_1\int_0^1\Phi^\top(x,t)Q_1\Phi(x,t)\mathrm{d}x+a_2\int_0^1\Phi^\top(x,t)Q_2\Phi(x,t)\mathrm{d}x\\ =&\int_0^1\Phi^\top(x,t)(Q_0+a_1Q_1+a_2Q_2)\Phi(x,t)\mathrm{d}x\\ =&-\int_0^1\Phi^\top(x,t)Q\Phi(x,t)\mathrm{d}x,\quad t\in[t_k,t_{k+1}),\ \forall k\in\mathbb{N},\end{aligned} \tag{3.58}$$

where $Q\triangleq-(Q_0+a_1Q_1+a_2Q_2)$ is given in (3.52).

We claim that there exist positive parameters $a_1,a_2,\delta_1,\varepsilon$ such that the matrix $Q>0$.

In fact, we first consider the case when $\delta_1=0$ and denote the corresponding matrix as Q^*:

$$Q^*=\begin{pmatrix}a_1 & 0 & -\varepsilon c/2 & 0\\ 0 & -\nu^*-\varepsilon+c & -c/2 & 0\\ -\varepsilon c/2 & -c/2 & a_2 & 0\\ 0 & 0 & 0 & -\nu^*-16a_1/\pi^4+\varepsilon\end{pmatrix}, \tag{3.59}$$

where $\nu^*\triangleq a_2\nu$. Then, by Schur complement, it is not difficult to calculate that $Q^*>0$ if and only if

$$a_1>0,\quad -\nu^*-\varepsilon+c>0,\quad -\nu^*-\frac{16a_1}{\pi^4}+\varepsilon>0,\quad a_2-\frac{c^2}{4}\left(\frac{\varepsilon^2}{a_1}+\frac{1}{-\nu^*-\varepsilon+c}\right)>0. \tag{3.60}$$

From $0<\varepsilon<1$, (3.60) is equivalent to $a_1>0$ and

$$a_2>\frac{c^2}{4}\left(\frac{\varepsilon^2}{a_1}+\frac{1}{-\nu^*-\varepsilon+c}\right),\quad \nu^*+\frac{16a_1}{\pi^4}<\varepsilon<\min\{1,c-\nu^*\},\quad \nu^*+\frac{16a_1}{\pi^4}<\min\{1,c-\nu^*\}.$$

Then it is not difficult to know that $Q^*>0$ can be guaranteed by choosing the following parameters

$$\begin{aligned}
&\nu^*\in(0,\min\{1,c/2\}),\\
&a_1\in\left(0,\min\left\{\frac{\pi^4(c-2\nu^*)}{16},\frac{\pi^4(1-\nu^*)}{16}\right\}\right),\\
&\varepsilon\in\left(\nu^*+16a_1/\pi^4,\min\{1,c-\nu^*\}\right),\\
&a_2>\frac{c^2}{4}\left(\frac{\varepsilon^2}{a_1}+\frac{1}{-\nu^*-\varepsilon+c}\right).
\end{aligned}\tag{3.61}$$

Furthermore, the tuning parameter ν is easily recovered from ν^* and a_2, *i.e.*,

$$0<\nu=\frac{\nu^*}{a_2}<\frac{4\min\{1,c/2\}}{c^2\left[\varepsilon^2/a_1+1/(-\nu^*-\varepsilon+c)\right]}.\tag{3.62}$$

Then there exist a_1,a_2,ν and ε such that $Q^*>0$. Next, if $\delta_1\neq 0$, we can write Q as $Q=Q^*-\delta_1 Q_3$ where

$$Q_3=\begin{pmatrix}\varepsilon c & \varepsilon & 0 & 0\\ \varepsilon & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

From (3.61) we have $Q_3\geq 0$. For the parameters a_1,a_2,ν,ε such that $Q^*>0$, there always exists a small enough $\delta_1>0$ such that $Q>0$. The claim is proved.

Then, referring to Chapter 2.6.3 in [40], from (3.58), we obtain $\dot V(t)<-2\delta_1 V(t)$, which means

$$V(t)<\mathrm{e}^{-2\delta_1 t}V(0),\quad t>0.$$

Taking (3.49) into account, we have

$$E(t)<\frac{\beta_2}{\beta_1}E(0)\mathrm{e}^{-2\delta_1 t},\quad t>0.\tag{3.63}$$

The proof is complete. □

From Theorem 3.8, we have proven the exponential stability of closed-loop system (2.11) with event-triggered control mechanism (2.17). This result can be shown by using operator theory. Consider the closed-loop system (2.11), which can be rewritten as:

$$\dot\phi(t)=\widehat{A}\phi(t)+BF[\phi(t_k)-\phi(t)],\quad t\in[t_k,t_{k+1}),\tag{3.64}$$

where $\phi(t)=(w(\cdot,t),w_t(\cdot,t))^\top$, $\widehat{A}$ is defined in (2.7) and B, F are defined in (2.5). Then, the solution of (2.8) satisfies

$$\begin{aligned}\phi(t)&=\mathrm{e}^{\widehat{A}(t-t_k)}\phi(t_k)+\int_{t_k}^{t}\mathrm{e}^{\widehat{A}(t-s)}BF(\phi(t_k)-\phi(s))\mathrm{d}s\\&=\mathrm{e}^{\widehat{A}(t-t_k)}\phi(t_k)+c\int_{t_k}^{t}\mathrm{e}^{\widehat{A}(t-s)}Be_k(\cdot,s)\mathrm{d}s,\quad t\in[t_k,t_{k+1}),\ k\in\mathbb{N}.\end{aligned}\tag{3.65}$$

Now, we will prove the exponential stability of closed-loop system (2.11) directly by using operator theory.

Theorem 3.10. *Let the damping parameter $c>0$. There exists positive parameter $\nu<\sigma^2/(Mc)^2$ such that for any initial condition $(w_0,w_1)^\top\in\mathcal{H}$, the closed-loop system (2.11) under event-triggered control mechanism (2.17) is exponentially stable with decay rate $\vartheta=\sigma-cM\sqrt{\nu}$, i.e.,*

$$\left\|\begin{pmatrix}w(\cdot,t)\\w_t(\cdot,t)\end{pmatrix}\right\|_{\mathcal{H}}\le M\mathrm{e}^{-\vartheta t}\left\|\begin{pmatrix}w_0\\w_1\end{pmatrix}\right\|_{\mathcal{H}}.$$

Proof. Referring to Theorem 5.2 in [34], we introduce the new equivalent norm $|\cdot|$ on $\mathcal{H}$ defined by

$$|x|\triangleq\sup_{t\ge0}\|\mathrm{e}^{\sigma t}T_1(t)x\|_{\mathcal{H}}.\tag{3.66}$$

Then the norm satisfies

$$\|x\|_{\mathcal{H}}\le|x|\le M\|x\|_{\mathcal{H}},\qquad|T_1(t)x|\le\mathrm{e}^{-\sigma t}|x|,\quad\forall x\in\mathcal{H},\ \forall t\ge0.\tag{3.67}$$

Noting that event-triggered mechanism (2.17) guarantees $\|e_k(\cdot,t)\|\le\sqrt{\nu}\|\phi(t)\|_{\mathcal{H}},\quad\forall t\in[t_k,t_{k+1}),\quad\forall k\in\mathbb{N}$, which means that the norm of (3.65) satisfies

$$|\phi(t)|\le\mathrm{e}^{-\sigma(t-t_k)}|\phi(t_k)|+cM\sqrt{\nu}\int_{t_k}^{t}\mathrm{e}^{-\sigma(t-s)}|\phi(s)|\mathrm{d}s.$$

By using the Gronwall's inequality, it yields that $\mathrm{e}^{\sigma t}|\phi(t)|\le\mathrm{e}^{\sigma t_k}\mathrm{e}^{cM\sqrt{\nu}(t-t_k)}|\phi(t_k)|$ or,

$$|\phi(t)|\le\mathrm{e}^{(-\sigma+cM\sqrt{\nu})(t-t_k)}|\phi(t_k)|=\mathrm{e}^{-\vartheta(t-t_k)}|\phi(t_k)|,\quad\forall t\in(t_k,t_{k+1}],\ \forall k\in\mathbb{N},\tag{3.68}$$

where $\vartheta=\sigma-\sqrt{\nu}cM>0$. Using (3.68) recursively gives $|\phi(t)|\le\mathrm{e}^{-\vartheta t}|\phi(0)|,\ \forall\ \phi(0)\in\mathcal{H},\ \forall t\ge0$. Therefore, from (3.67), we obtain

$$\|\phi(t)\|_{\mathcal{H}}\le|\phi(t)|\le\mathrm{e}^{-\vartheta t}|\phi(0)|\le M\mathrm{e}^{-\vartheta t}\|\phi(0)\|_{\mathcal{H}}.$$

The proof is complete. □

Next we consider the asymptotic behavior of the solution under the second event-triggered control mechanism (2.18).

Theorem 3.11. *Let the damping parameter $c>0$. Then there exist positive constants α, θ, $\delta_2>0$, and $M\ge1$ such that the closed-loop system (2.11) under the event-triggered mechanism (2.18) is exponentially stabilized.*

More precisely, for every initial state $(w_0, w_1) \in \mathcal{H}$ *defined in* (1.2)*, the solution of the closed-loop system* (2.11) *under the event-triggered mechanism* (2.18) *converges exponentially to zero with decay rate* δ_2 *as follows:*

$$E(t) \leq ME(0)\mathrm{e}^{-2\delta_2 t}, \quad \forall t \geq 0. \tag{3.69}$$

Remark 3.12. The parameters α, δ_2 and θ can be explicitly determined by making positive definite the following matrix P:

$$P = \begin{pmatrix} b_1 - b_2\alpha & 0 & -\varepsilon c/2 & 0 \\ 0 & -b_2\alpha - \varepsilon + c - \delta_2 & -c/2 & 0 \\ -\varepsilon c/2 & -c/2 & b_2 & 0 \\ 0 & 0 & 0 & -16b_1/\pi^4 + \varepsilon - \delta_2 \end{pmatrix} > 0, \tag{3.70}$$

and $\theta > 2\delta_2/(1-\varepsilon)$, where b_1, b_2, α and δ_2 are real positive constants and $0 < \varepsilon < 1$.

Proof of Theorem 3.11. Differentiating $V(t)$ along with the trajectory $(w(x,t), w_t(x,t))$ of (2.11) leads to

$$\begin{aligned} \dot{V}(t) = (\varepsilon - c)\|w_t(\cdot,t)\|^2 - \varepsilon\|w_{xx}(\cdot,t)\|^2 \\ + c\int_0^1 w_t(x,t)e_k(x,t)\mathrm{d}x + \varepsilon c\int_0^1 w(x,t)e_k(x,t)\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} &\dot{V}(t) + 2\delta_2 E(t) \\ &= (\varepsilon - c + \delta_2)\|w_t(\cdot,t)\|^2 + (\delta_2 - \varepsilon)\|w_{xx}(\cdot,t)\|^2 \\ &\qquad + c\int_0^1 w_t(x,t)e_k(x,t)\mathrm{d}x + \varepsilon c\int_0^1 w(x,t)e_k(x,t)\mathrm{d}x \\ &\triangleq \int_0^1 \Phi^\top(x,t)P_1\Phi(x,t)\mathrm{d}x, \end{aligned} \tag{3.71}$$

for some $\delta_2 > 0$ where $\Phi(x,t)$ is defined in (3.55) and

$$P_1 = \begin{pmatrix} 0 & 0 & \varepsilon c/2 & 0 \\ 0 & \varepsilon - c + \delta_2 & c/2 & 0 \\ \varepsilon c/2 & c/2 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 - \varepsilon \end{pmatrix}. \tag{3.72}$$

From event-triggered mechanism (2.18), we have

$$\|e_k(\cdot,t)\| \leq \alpha\left(\|w(\cdot,t)\|^2 + \|w_t(\cdot,t)\|^2\right) + E(0)\mathrm{e}^{-\theta t}, \quad \forall t \in [t_k, t_{k+1}),\ \forall k \in \mathbb{N},$$

which means

$$\int_0^1 \Phi^\top(x,t)P_2\Phi(x,t)\mathrm{d}x + E(0)\mathrm{e}^{-\theta t} \geq 0, \tag{3.73}$$

with $P_2 = \mathrm{diag}(\alpha, \alpha, -1, 0)$, where the metric P_2 is not positive semi-definite and the associated bilinear form is non-negative. Then by (3.57) and (3.73), it follows that

$$\begin{aligned}
&\dot{V}(t) + 2\delta_2 E(t) \\
\leq &\dot{V}(t) + 2\delta_2 E(t) + b_1 \int_0^1 \Phi^\top(x,t) Q_1 \Phi(x,t) \mathrm{d}x + b_2 \int_0^1 \Phi^\top(x,t) P_2 \Phi(x,t) \mathrm{d}x + b_2 E(0) \mathrm{e}^{-\theta t} \\
= &\int_0^1 \Phi^\top(x,t)(P_1 + b_1 Q_1 + b_2 P_2)\Phi(x,t)\mathrm{d}x + b_2 E(0)\mathrm{e}^{-\theta t} \\
= &-\int_0^1 \Phi^\top(x,t) P \Phi(x,t)\mathrm{d}x + b_2 E(0)\mathrm{e}^{-\theta t}, \qquad t \in [t_k, t_{k+1}),\ \forall k \in \mathbb{N},
\end{aligned} \tag{3.74}$$

where $b_1, b_2 > 0$ and $P \triangleq -(P_1 + b_1 Q_1 + b_2 P_2)$ is given in (3.70). We claim that there exist positive parameters $\alpha, b_1, b_2, \delta_2$ and ε such that the matrix $P > 0$.

In fact, $P > 0$ if and only if

$$b_1 - \alpha^* > 0, \quad c - \varepsilon - \delta_2 - \alpha^* > 0, \quad \varepsilon - \delta_2 - \frac{16 b_1}{\pi^4} > 0, \quad b_2 - \frac{c^2}{4}\left(\frac{\varepsilon^2}{b_1 - \alpha^*} + \frac{1}{c - \varepsilon - \delta_2 - \alpha^*}\right) > 0, \tag{3.75}$$

where $\alpha^* = \alpha b_2$. By $0 < \varepsilon < 1$, it yields that $\alpha^* < b_1$ and

$$\begin{aligned}
&\delta_2 + \frac{16 b_1}{\pi^4} < \varepsilon < \min\{c - \delta_2 - \alpha^*, 1\}, \\
&\delta_2 + \frac{16 b_1}{\pi^4} < \min\{c - \delta_2 - \alpha^*, 1\}, \\
&b_2 > \frac{c^2}{4}\left(\frac{\varepsilon^2}{b_1 - \alpha^*} + \frac{1}{c - \varepsilon - \delta_2 - \alpha^*}\right).
\end{aligned}$$

Then by direct calculation, $P > 0$ can be guaranteed by choosing the parameters

$$\begin{aligned}
&\delta_2 \in (0, \min\{c/2, 1\}), \\
&b_1 \in \left(0, \min\left\{\frac{\pi^4(1-\delta_2)}{16}, \frac{\pi^4(c - 2\delta_2)}{16}\right\}\right), \\
&\alpha^* \in \left(0, \min\{c - 2\delta_2 - 16 b_1/\pi^4, b_1\}\right), \\
&\varepsilon \in \left(\delta_2 + 16 b_1/\pi^4, \min\{1, c - \delta_2 - \alpha^*\}\right), \\
&b_2 > \frac{c^2}{4}\left(\frac{\varepsilon^2}{b_1 - \alpha^*} + \frac{1}{c - \varepsilon - \delta_2 - \alpha^*}\right).
\end{aligned} \tag{3.76}$$

Then we have

$$\theta > 2\delta_2/(1 - \varepsilon), \tag{3.77}$$

and the tuning parameter α is easily recovered from α^* and b_2, *i.e.*,

$$0 < \alpha = \frac{\alpha^*}{b_2} < \frac{16 \min\{c - 2\delta_2 - 16 b_1/\pi^4, b_1\}}{c^2\left[\varepsilon^2/(b_1 - \alpha^*) + 1/(c - \varepsilon - \delta_2 - \alpha^*)\right]}. \tag{3.78}$$

It follows that there exist positive constants $\alpha, b_1, b_2, \varepsilon$ and δ_2 such that $P > 0$. Thereby, (3.74) and $P > 0$ give that

$$\dot{V}(t) \leq -2\delta_2 E(t) + b_2 E(0)\mathrm{e}^{-\theta t}.$$

Because of the equivalent relationship (3.49), it yields that

$$\dot{V}(t) \leq -\frac{2\delta_2}{\beta_1} V(t) + \frac{b_2}{\beta_1} V(0)\mathrm{e}^{-\theta t}, \qquad t \in [0, \infty). \tag{3.79}$$

Integrating (3.79) from $t = 0$ to t gives us the following:

$$V(t) \leq V(0)\mathrm{e}^{-\frac{2\delta_2}{\beta_1}t} + \frac{b_2 V(0)}{\beta_1\theta - 2\delta_2}\left(\mathrm{e}^{-\frac{2\delta_2}{\beta_1}t} - \mathrm{e}^{-\theta t}\right) \leq \left(1 + \frac{b_2}{\beta_1\theta - 2\delta_2}\right)\mathrm{e}^{-\frac{2\delta_2}{\beta_1}t} V(0),$$

where $\theta > \frac{2\delta_2}{\beta_1}$. By (3.49), we have

$$E(t) \leq \frac{\beta_2}{\beta_1}\left(1 + \frac{b_2}{\beta_1\theta - 2\delta_2}\right)\mathrm{e}^{-\frac{2\delta_2}{\beta_1}t} E(0), \tag{3.80}$$

where β_1 and β_2 are given by (3.50). In other words, system (1.1) can be exponentially stabilized with the exponential decay rate $2\delta_2/\beta_1 > 2\delta_2$ by event-triggered control law (2.10) and trigger rule (2.18). □

From Theorem 3.11, we have proven the exponential stability of closed-loop system (2.11) with event-triggered control mechanism (2.18). Similar to Theorem 3.10, this stability result can be proven by utilizing operator theory.

Theorem 3.13. *Let the damping parameter $c > 0$. There exist positive parameters $\alpha < \sigma^2/(Mc)^2$ and $\theta > 2\sigma$ such that for any initial condition $(w_0, w_1)^\top \in \mathcal{H}$, the solution of the closed-loop system (2.11) under event-triggered control mechanism (2.18) converges exponentially to zero with decay rate $\vartheta = \sigma - cM\sqrt{\alpha}$, i.e., for some positive constant $\tilde{M} \geq 1$,*

$$\left\| \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} \leq \tilde{M}\mathrm{e}^{-\vartheta t} \left\| \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right\|_{\mathcal{H}}.$$

Proof. According to the event-triggered mechanism (2.18), one has

$$\|e_k(\cdot,t)\|^2 \leq \alpha\left(\|w(\cdot,t)\|^2 + \|w_t(\cdot,t)\|^2\right) + E(0)\mathrm{e}^{-\theta t}, \quad \forall t \in [t_k, t_{k+1}), \quad \forall k \in \mathbb{N}.$$

From the Poincaré inequality, we see that

$$\|e_k(\cdot,t)\| \leq \sqrt{\alpha}\|\phi(t)\|_{\mathcal{H}} + \sqrt{E(0)}\mathrm{e}^{-\frac{\theta t}{2}} \ \ \forall\, t \in [t_k, t_{k+1}),\ \forall\, k \in \mathbb{N}. \tag{3.81}$$

We define $\tilde{e}(x,t)$ by

$$\tilde{e}(x,t) = e_k(x,t), \ \ \forall\, t \in [t_k, t_{k+1}),\ \forall\, k \in \mathbb{N}. \tag{3.82}$$

From (3.81) and (3.82) it follows that

$$\|\tilde{e}(\cdot,t)\| \leq \sqrt{\alpha}\|\phi(t)\|_{\mathcal{H}} + \sqrt{E(0)}\mathrm{e}^{-\frac{\theta t}{2}}, \ \ \forall\, t \geq 0. \tag{3.83}$$

The solution (3.65) of system (3.64) is written:

$$\phi(t) = \mathrm{e}^{\widehat{A}(t-t_k)}\phi(t_k) + c\int_{t_k}^{t} \mathrm{e}^{\widehat{A}(t-s)}Be_k(\cdot,s)\mathrm{d}s, \quad \forall\, t \in [t_k, t_{k+1}).$$

By continuity and by induction we prove easily that

$$\phi(t) = \mathrm{e}^{\widehat{A}(t-t_k)}\phi(0) + c\int_{0}^{t} \mathrm{e}^{\widehat{A}(t-s)}B\tilde{e}(\cdot,s)\mathrm{d}s, \quad \forall\, t \geq 0. \tag{3.84}$$

Using the exponential stability of $\mathrm{e}^{t\widehat{A}}$ and (3.83) we get:

$$\|\phi(t)\|_{\mathcal{H}} \leq M\mathrm{e}^{-\sigma t}\|\phi(0)\|_{\mathcal{H}} + cM\int_{0}^{t} \mathrm{e}^{-\sigma(t-s)}\left(\sqrt{\alpha}\,\|\phi(s)\|_{\mathcal{H}} + \sqrt{E(0)}\mathrm{e}^{-\frac{\theta s}{2}}\right)\mathrm{d}s.$$

By using the Gronwall's inequality (see Appendix), it yields that

$$\|\phi(t)\|_{\mathcal{H}} \leq \tilde{M}\mathrm{e}^{-\vartheta t}\|\phi(0)\|_{\mathcal{H}},$$

where $\vartheta = \sigma - cM\sqrt{\alpha} > 0$ and $\tilde{M} = M\left(1 + \dfrac{c}{\sqrt{2}(\theta/2-\sigma)}\right)$. The proof is complete. □

Remark 3.14. If the triggering condition (2.18) is replaced by

$$t_{k+1} = \inf\left\{t \geq t_k \geq t_0 \;\middle|\; \|e_k(\cdot,t)\|^2 > \alpha\left(\|w(\cdot,t)\|^2 + \|w_t(\cdot,t)\|^2\right) + E(t_0)\mathrm{e}^{-\theta(t-t_0)}\right\},$$

then the conclusion of Theorem 3.13 holds as follows:

$$\left\|\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t)\end{pmatrix}\right\|_{\mathcal{H}} \leq \tilde{M}\mathrm{e}^{-\vartheta(t-t_0)}\left\|\begin{pmatrix} w(\cdot,t_0) \\ w_t(\cdot,t_0)\end{pmatrix}\right\|_{\mathcal{H}}.$$

By combining Theorem 3.10 and Theorem 3.13 we have proven the main result of this paper, presented in Theorem 2.13, which asserts the exponential stability, or exponential decay of the closed-loop system (2.11) under the event-triggered control mechanisms (2.17) or (2.18).

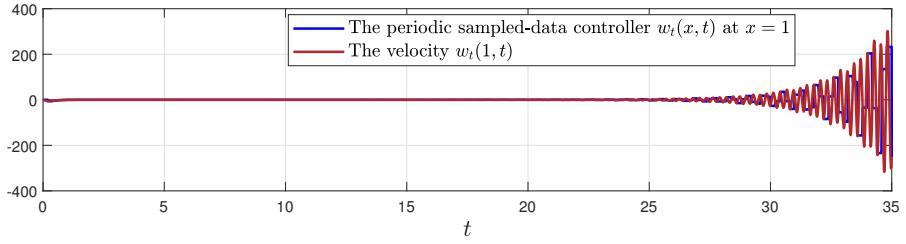
4. Numerical simulations

In this section, we utilize the finite element method to discretize and simulate the Euler–Bernoulli beam system (2.11). The dynamical evolution is presented to show the performance of the designed event-triggered controller (2.10) with the trigger rule (2.17) and (2.18), respectively. For the state $(w, w_t) \in L^{\infty}(0, +\infty; \mathcal{H})$, the initial conditions should be required to satisfy compatibility condition. We choose the initial conditions:

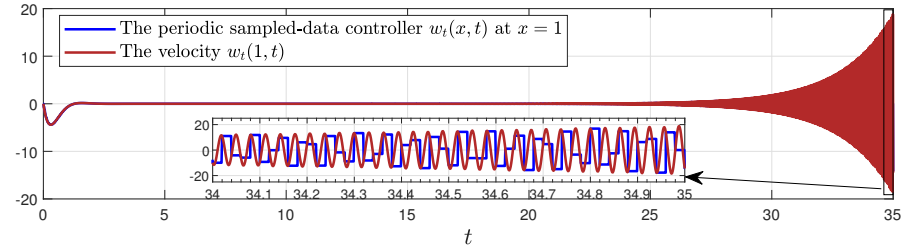
$$w(x,0) = \gamma[\cosh(\rho x) - \cos(\rho x)] + [\sinh(\rho x) - \sin(\rho x)],\ \ w_t(x,0) = 0,$$

where

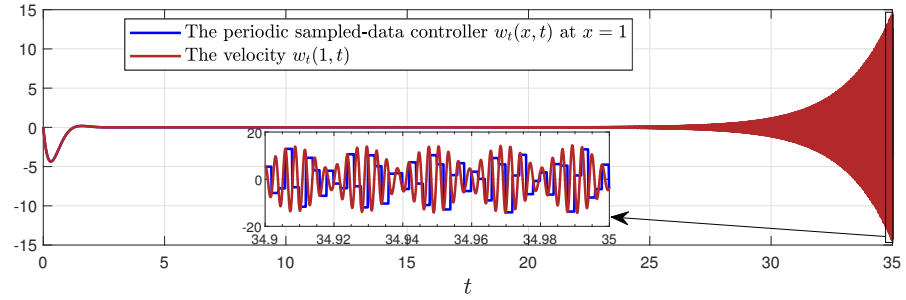
$$\gamma = \frac{\sinh(\rho) + \sin(\rho)}{\cosh(\rho) + \cos(\rho)}$$



(a) Period time $\tau = 0.2$



(b) Period time $\tau = 0.02$



(c) Period time $\tau = 0.002$

FIGURE 1. Sampled-data control input.

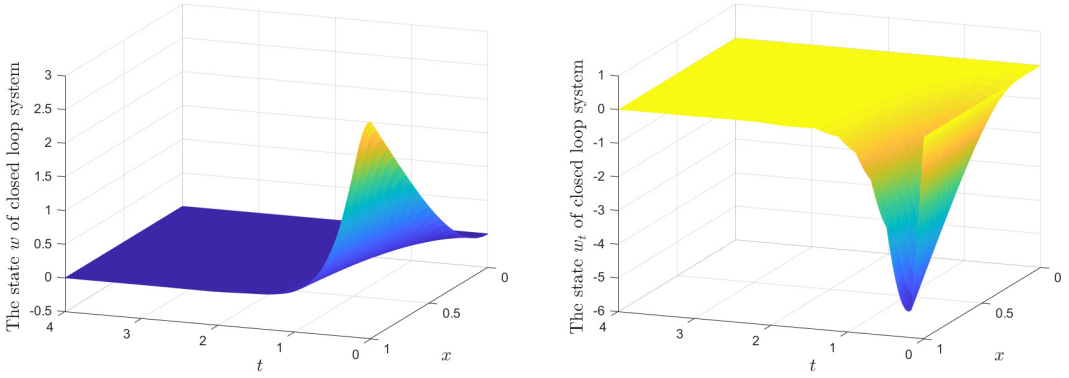
with $\rho \approx 1.8751$ (satisfying $\cosh(\rho)\cos(\rho)+1=0$). With these initial conditions, the dynamic of the open-loop system (1.1), where $u(t)=0$, is a simple harmonic oscillation. Next, we employ a damping control to stabilize the Euler–Bernoulli beam system. The control gain of the controller (2.10) is given as $c=5$. The spatial grid size is set to be $N=20$, by utilizing Hermitian polynomial functions as the basis for discretizing the space of $H^2(0,1)$. We implement the fourth-order Runge-Kutta method with a fixed step of $\mathrm{d}t = 0.0001$ in Matlab.

We first consider the periodic sampled-data control beam system. Figure 1 illustrates the performance of periodic sampled-data control with sampling time period $\tau = 0.2$, 0.02 and 0.002. It is observed that the closed-loop system under periodic sampled-data control is unstable. As the sampling time period decreases, the time before divergence appears increases.

Next, we use event-triggered controller (2.10) to stabilize the Euler–Bernoulli system (1.1) under two event-triggered mechanisms (2.17) and (2.18), respectively.

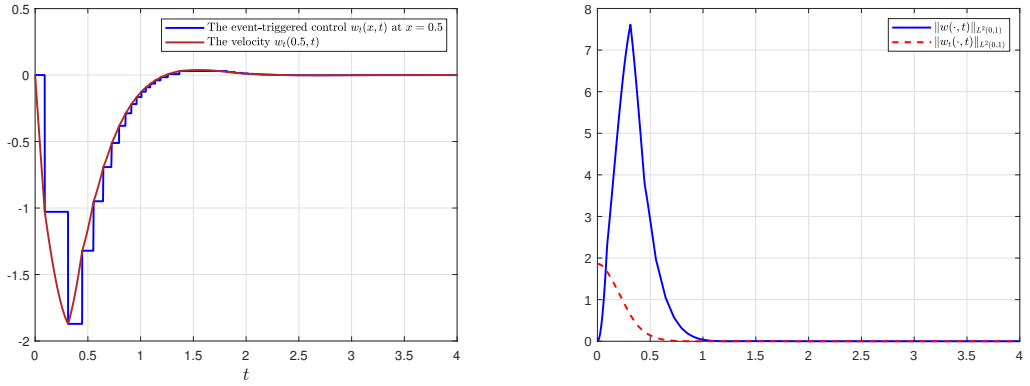
Case 1: For the triggered mechanism (2.17), we take the parameters $\nu^* = 0.5, \varepsilon = 0.92, a_1 = 2.5, a_2 = 5$ and $\nu = 0.1$ satisfying the LMI-type conditions in Theorem 3.8.

Case 2: For the triggered mechanism (2.18), we set the parameters $\alpha = 0.2, \varepsilon = 0.62, b_1 = 2.5, b_2 = 10, \delta = 0.2$ and $\theta = 5$ which satisfy the LMI conditions in Theorem 3.11.



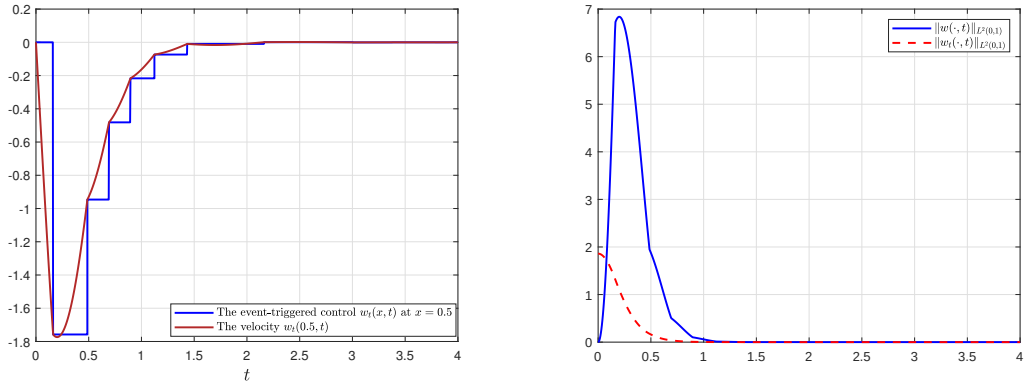
(a) Evolution of displacement $w(x,t)$ (b) Evolution of velocity $w_t(x,t)$

FIGURE 2. The state (w, w_t) of closed-loop system (2.11).



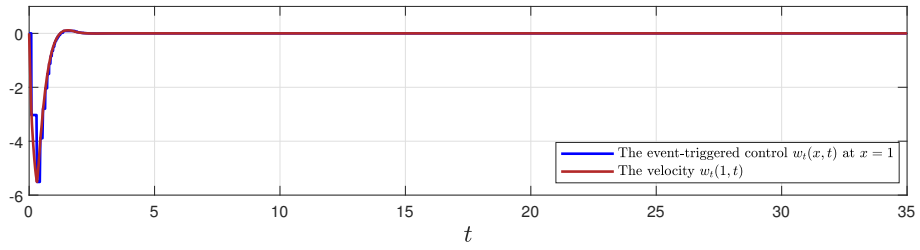
(a) Control input at $x = 0.5$ (b) State norm under trigger rule (2.17)

FIGURE 3. Evolution of the state and event-triggered control input $w_t(1, t_k)$ under trigger rule (2.17).

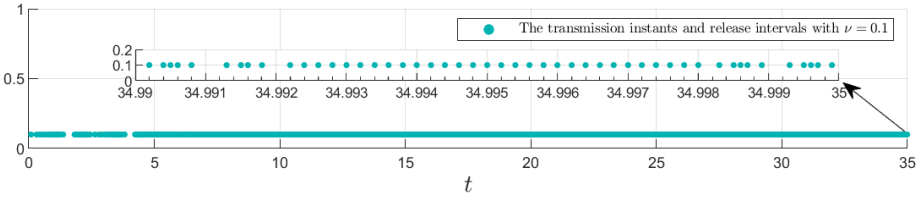


(a) Control input at $x = 0.5$ (b) State norm under trigger rule (2.18)

FIGURE 4. Evolution of the state and event-triggered control input $w_t(1, t_k)$ under trigger rule (2.18).

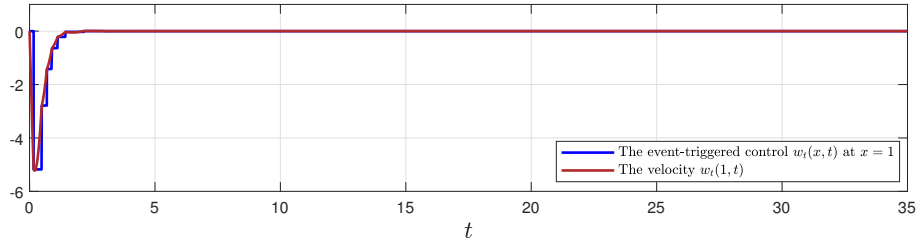


(a) Control input at $x = 1$

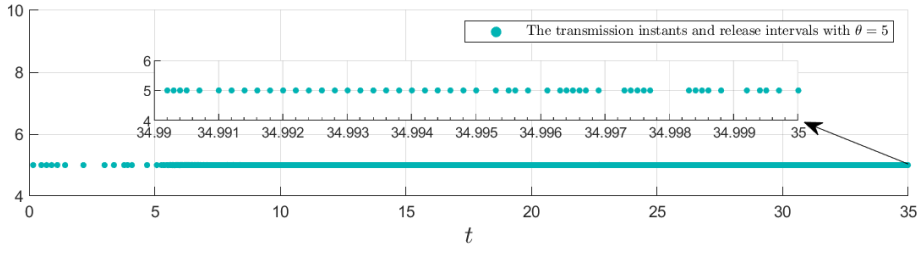


(b) Release instants under trigger rule (2.17)

FIGURE 5. Evolution of the state and the related release instants under trigger rule under trigger rule (2.17).



(a) Control input at $x = 1$



(b) Release instants under trigger rule (2.18)

FIGURE 6. Evolution of the state and the related release instants under trigger rule under trigger rule (2.18).

Figure 2 displays the evolution of the state (w, w_t) under event-triggered control (2.10). It illustrates the exponential convergence of the event-triggered controlled beam system (2.11) to zero as $t \to \infty$. The state exhibits a piecewise behavior induced by the piecewise-constant control signal. Figures 3 and 4 depict the evolution of the beam state $\|w(x,t)\|^2$, $\|w_t(x,t)\|^2$, and the event-triggered controller (2.10) under trigger rules (2.17) and

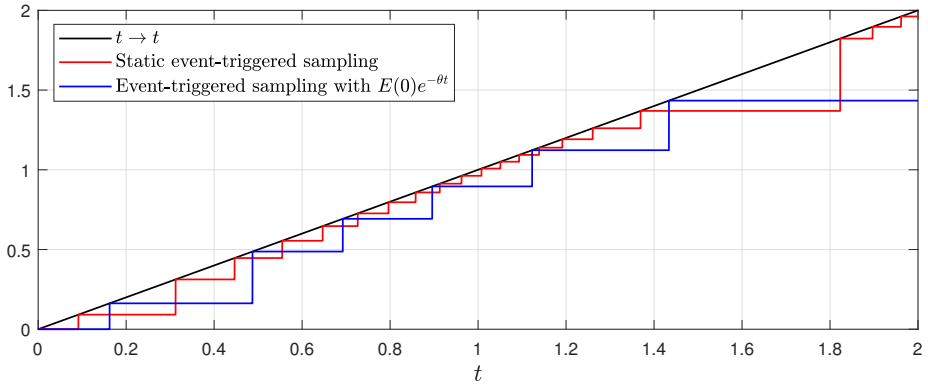
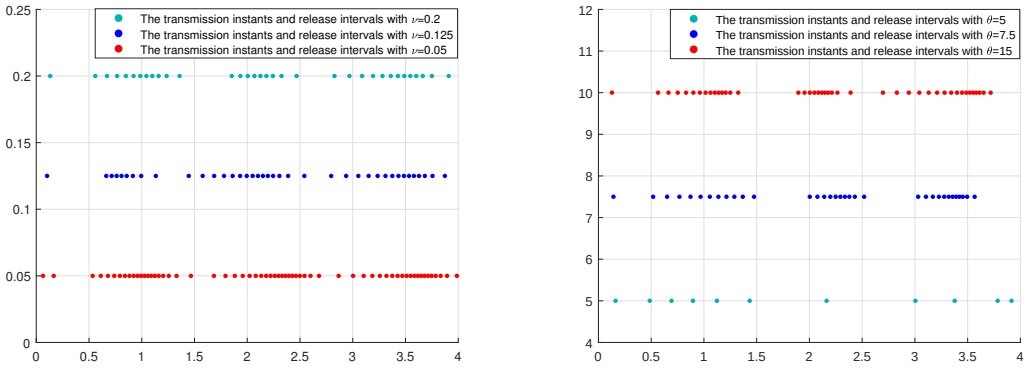


FIGURE 7. Time evolution of the event-triggering mechanisms (2.17) and (2.18).



(a) Release instants under trigger rule (2.17)

(b) Release instants under trigger rule (2.18)

FIGURE 8. Triggered time instants.

(2.18), respectively. It indicates that the event-triggered control strategy (2.18) with an exponential decay term effectively reduces communication frequency and saves control energy compared to the event-triggered control strategy (2.17). Figures 5 and 6 show the system state $w_t(1,t)$ and the corresponding controller release instants under the two event-triggered mechanisms (2.17) and (2.18), respectively. From Figures 5a and 6a, it is observed that the state $w_t(1,t)$ of the event-triggered control system converges under both trigger rules. Furthermore, Figures 5b and 6b confirm that the Zeno phenomenon does not occur. Meanwhile, over time, the triggering time interval decreases but does not reach continuous triggering, as the controller is not activated at every time step. This confirms that the proposed event-triggered controller effectively reduce the frequency of controller updates while ensuring the stabilization of the beam system. This observation agrees with our theoretical understanding that event-triggered control provides a sample-and-hold control strategy within any finite time interval, while gradually approaching nearly continuous-time control as time tends to infinity. Figure 7 shows the sampling time evolution of the event-triggering mechanisms (2.17) and (2.18) for $\nu = 0.1$, $\alpha = 0.2$ and $\theta = 5$. As shown in Figure 7, together with Figures 2–4, the event-triggering mechanisms (2.18) with the term $E(0)\mathrm{e}^{-\theta t}$ effectively reduces the frequency of controller updates during the initial stage and successfully stabilizes the beam system. Figure 8a shows that the release frequency is determined by the parameter ν. The smaller the value of ν, the

faster the triggering. Figure 8b shows that the release frequency is influenced by the parameter θ within the same parameter α: the bigger the value of θ, the faster the triggering.

In summary, as indicated by the simulations, the states of the beam system (2.11) converge to zero under event-triggered control with both trigger mechanisms (2.17) and (2.18), whereas the states of periodic sampled-data control system (2.2) eventually diverges for any sampling time period.

5. Conclusions

In this paper, we have studied the distributed feedback stabilization problem of an Euler–Bernoulli beam equation by event-based control. We have proved that the periodic sampling of the stabilizing control law loses the stability of the closed-loop system, no matter what the sampling time period is. In order to save control energy and reduce the number of updates, the event-triggered control law with two event-triggered mechanisms is proposed to stabilize the beam system. The existence and uniqueness of the solution of the closed-loop system is proven based on the operator semigroup theory. By using an elaborated Lyapunov functional, sufficient LMI-type conditions related to the two trigger rules are established to guarantee exponential stability of the closed-loop system. The absence of Zeno behaviour is also guaranteed. It is worth noting that the exponential decay term in the trigger rule can reduce the frequency of control updates.

Funding

This work is supported by the science and technology innovation Program of Hunan Province (No. 2022RC1188), the National Natural Science Foundation of China (No. 12161141013, No. 62173348) and the French Grant ANR ODISSE (ANR-19-CE48-0004-01), and partially supported (W. Hu) by NSF DMS-2205117 and DMS-2229345.

Data availability statement

The research data associated with this article are included in the article.

References

[1] Z.-H. Luo, N. Kitamura and B.-Z. Guo, Shear force feedback control of flexible robot arms. *IEEE Trans. Robot. Autom.* **11** (1995) 760–765.

[2] Q.C. Nguyen and K.-S. Hong, Transverse vibration control of axially moving membranes by regulation of axial velocity. *IEEE Trans. Control Syst. Technol.* **20** (2011) 1124–1131.

[3] S.M. Han, H. Benaroya and T. Wei, Dynamics of transversely vibrating beams using four engineering theories. *J. Sound Vibr.* **225** (1999) 935–988.

[4] J.-M. Coron, Control and Nonlinearity. No. 136. American Mathematical Society (2007).

[5] R. Dáger and E. Zuazua, Wave Propagation, Observation and Control in 1-D Flexible Multi-structures. No. 50. Springer Science & Business Media (2006).

[6] A. Smyshlyaev, B.-Z. Guo and M. Krstic, Arbitrary decay rate for Euler–Bernoulli beam by backstepping boundary feedback. *IEEE Trans. Autom. Control* **54** (2009) 1134–1140.

[7] G. Chen, M.C. Delfour, A. Krall and G. Payre, Modeling, stabilization and control of serially connected beams. *SIAM J. Control Optim.* **25** (1987) 526–546.

[8] F.-F. Jin and B.-Z. Guo, Lyapunov approach to output feedback stabilization for the Euler–Bernoulli beam equation with boundary input disturbance. *Automatica* **52** (2015) 95–102.

[9] F. Matsuno, T. Ohno and Y.V. Orlov, Proportional derivative and strain (PDS) boundary feedback control of a flexible space structure with a closed-loop chain mechanism. *Automatica* **38** (2002) 1201–1211.

[10] R. Rebarber, Exponential stability of coupled beams with dissipative joints: a frequency domain approach. *SIAM J. Control Optim.* **33** (1995) 1–28.

[11] T. Chen and B.A. Francis, Input–output stability of sampled-data systems. *IEEE Trans. Autom. Control* **36** (1991) 50–58.

[12] J.-M. Coron and L. Rosier, A relation between continuous time-varying and discontinuous feedback stabilization. *J. Math. Syst., Estim. Control* **4** (1994) 67–84.

[13] E.D. Sontag, Mathematical Control Theory: Deterministic Finite Dimensional Systems, vol. 6. Springer Science & Business Media (2013).

[14] H. Logemann, R. Rebarber and S. Townley, Stability of infinite-dimensional sampled-data systems. *Trans. Am. Math. Soc.* **355** (2003) 3301–3328.

[15] R. Rebarber and S. Townley, Nonrobustness of closed-loop stability for infinite-dimensional systems under sample and hold. *IEEE Trans. Autom. Control* **47** (2002) 1381–1385.

[16] M.G. Losada Contributions to Networked and Event-triggered Control of Linear Systems. Springer (2016).

[17] K.-E. Åarzén, A simple event-based PID controller. *IFAC Proc.* **32** (1999) 8687–8692.

[18] K.J. Åström and B. Bernhardsson, Comparison of periodic and event based sampling for first-order stochastic systems. *IFAC Proc.* **32** (1999) 5006–5011.

[19] N. Espitia, A. Girard, N. Marchand C. and Prieur, Event-based control of linear hyperbolic systems of conservation laws. *Automatica* **70** (2016) 275–287.

[20] N. Espitia, A. Girard, N. Marchand and C. Prieur, Event-based boundary control of a linear 2×2 hyperbolic system via backstepping approach. *IEEE Trans. Autom. Control* **63** (2018) 2686–2693.

[21] N. Espitia, I. Karafyllis and M. Krstic, Event-triggered boundary control of constant-parameter reaction-diffusion PDEs: a small-gain approach. *Automatica* **128** (2021) 109562.

[22] F. Koudohode, N. Espitia and M. Krstic, Event-triggered boundary control of an unstable reaction diffusion PDE with input delay. *Syst. Control Lett.* **186** (2024) 105775.

[23] B. Rathnayake, M. Diagne, J. Cortés and M. Krstic, Performance-barrier event-triggered control of a class of reaction-diffusion PDEs. *Automatica* **174** (2025) 112181.

[24] A. Selivanov and E. Fridman, Distributed event-triggered control of diffusion semilinear PDEs. *Automatica* **68** (2016) 344–351.

[25] X. Song, Q. Zhang, S. Song and C.K. Ahn, Event-triggered underactuated control for networked coupled PDE-ODE systems with singular perturbations. *IEEE Syst. J.* **17** (2023) 4474–4484.

[26] J. Wang and M. Krstic, Event-triggered backstepping control of 2×2 hyperbolic PDE-ODE systems. *IFAC-PapersOnLine* **53** (2020) 7551–7556.

[27] B. Rathnayake and M. Diagne, Observer-based periodic event-triggered and self-triggered boundary control of a class of parabolic PDEs. *IEEE Trans. Autom. Control* **69** (2024) 8836–8843.

[28] C. Ji, Z. Zhang and S.S. Ge, Adaptive event-triggered output feedback control for uncertain parabolic PDEs. *Automatica* **171** (2025) 111917.

[29] J. Wang and M. Krstic, Event-triggered adaptive control of a parabolic PDE-ODE cascade with piecewise-constant inputs and identification. *IEEE Trans. Autom. Control* **68** (2022) 5493–5508.

[30] M. Wakaiki and H. Sano, Event-triggered control of infinite-dimensional systems. *SIAM J. Control Optim.* **58** (2020) 605–635.

[31] L. Baudouin, S. Marx and S. Tarbouriech, Event-triggered damping of a linear wave equation. *IFAC-PapersOnLine* **52** (2019) 58–63.

[32] F. Koudohode, L. Baudouin and S. Tarbouriech, Event-based control of a damped linear wave equation. *Automatica* **146** (2022) 110627.

[33] B.-Z. Guo, Riesz basis property and exponential stability of controlled Euler–Bernoulli beam equations with variable coefficients. *SIAM J. Control Optim.* **40** (2002) 1905–1923.

[34] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, vol. 44. Springer Science & Business Media (2012).

[35] A. Lamperski and A.D. Ames, Lyapunov theory for zeno stability. *IEEE Trans. Autom. Control* **58** (2012) 100–112.

[36] A.D. Ames, A. Abate and S. Sastry, Sufficient conditions for the existence of zeno behavior, in *Proceedings of the 44th IEEE Conference on Decision and Control*. IEEE (2005) 696–701.

[37] J. Zhang, K.H. Johansson, J. Lygeros and S. Sastry, Zeno hybrid systems. *Int. J. Robust Nonlinear Control* **11** (2001) 435–451.

[38] T. Eisner, Stability of Operators and Operator Semigroups, vol. 209. Birkhäuser (2019).

[39] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities. Cambridge University Press (1952).

[40] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory. SIAM (1994).

[41] E. Kreyszig, Introductory Functional Analysis with Applications. John Wiley & Sons (1991).

Appendix A.

Proof of Lemma 3.1. Differentiating $E(t)$ along the trajectory $(w(x,t), w_t(x,t))$ of (2.11) leads to:

$$\begin{aligned}\dot{E}(t) &= \int_0^1 [w_{xx}(x,t)w_{xxt}(x,t) + w_t(x,t)w_{tt}(x,t)]\,\mathrm{d}x \\ &= -c\|w_t(\cdot,t)\|^2 + c\int_0^1 w_t(x,t)e_k(x,t)\mathrm{d}x,\ \forall\, t\in[t_k,t_{k+1}).\end{aligned} \tag{A.1}$$

Notice the following inequality:

$$\left|\int_0^1 w_t(x,t)e_k(x,t)\mathrm{d}x\right| \le \|w_t(\cdot,t)\|^2 + \|e_k(\cdot,t)\|^2/4. \tag{A.2}$$

Substituting (A.2) into (A.1) gives us:

$$-2c\|w_t(\cdot,t)\|^2 - \frac{c}{4}\|e_k(\cdot,t)\|^2 \le \dot{E}(t) \le \frac{c}{4}\|e_k(\cdot,t)\|^2,\ \ t\in[t_k,t_{k+1}). \tag{A.3}$$

As by the first trigger law (2.17), $\|e_k(\cdot,t)\|^2 \le 2\nu E(t)\ \ \forall\, t\in[t_k,t_{k+1})$, and $\|w_t(\cdot,t)\|^2 \le 2E(t)$, it follows immediately from (A.3) that

$$-\left(4c+\frac{\nu c}{2}\right)E(t) \le \dot{E}(t) \le \left(\frac{\nu c}{2}\right)E(t),\ \ \forall\, t\in[0,t_k].$$

Integrating the above differential inequality proves the first assertion (i) of the lemma.

Similarly, as by the second trigger law (2.18), $\|e_k(\cdot,t)\|^2 \le \alpha(\|w(\cdot,t)\|^2 + \|w_t(\cdot,t)\|^2) + \mathrm{e}^{-\theta t}E(0)\ \forall\, t\in[t_k,t_{k+1})$, and $\|w(\cdot,t)\|^2 + \|w_t(\cdot,t)\|^2 \le 2E(t)$, it follows immediately from (A.3) that

$$-\left(4c+\frac{\alpha c}{2}\right)E(t) - \left(\frac{c}{4}\right)\mathrm{e}^{-\theta t}E(0) \le \dot{E}(t) \le \left(\frac{\alpha c}{2}\right)E(t) + \left(\frac{c}{4}\right)\mathrm{e}^{-\theta t}E(0),\ \ \forall\, t\in[0,t_k]. \tag{A.4}$$

Integrating the above differential inequality proves the second assertion (ii) of the lemma. Hence the proof of Lemma 3.1 is complete. □

Before proving Lemma 2.10, we state the following useful property related with system energy (2.15).

Proof of Lemma 3.2. For each $\phi_0\in\mathcal{H}$, we write it under the form of (3.15). Then, by direct computation, we have

$$A\phi_0 = \sum_{n\ge1}(i\alpha_n\sigma_n e_n - i\beta_n\sigma_n e_{-n}),$$

and

$$(A+BF)\phi_0 = \frac{1}{2}\sum_{n\geq 1}\left\{\left[2i\sigma_n\alpha_n + c(\beta_n-\alpha_n)\right]e_n + \left[-2i\sigma_n\beta_n + c(\alpha_n-\beta_n)\right]e_{-n}\right\}.$$

Notice that $B^* = [0 \;\; I]$, $\mathrm{e}^{At}e_n = \mathrm{e}^{i\sigma_n t}e_n$ and (3.14), it yields that

$$\int_0^t B^*\mathrm{e}^{A\xi}(A+BF)\phi_0 \mathrm{d}\xi = \frac{1}{2}\sum_{n\geq 1}\left\{\frac{\mathrm{e}^{i\sigma_n t}}{\sigma_n}\left[2i\sigma_n\alpha_n + c(\beta_n-\alpha_n)\right] + \frac{\mathrm{e}^{-i\sigma_n t}}{\sigma_n}\left[-2i\sigma_n\beta_n + c(\alpha_n-\beta_n)\right] + 2i(\beta_n-\alpha_n)\right\}\rho_n,$$

and

$$\left\|\int_0^t B^*\mathrm{e}^{A\xi}(A+BF)\phi_0\mathrm{d}\xi\right\|^2 = \frac{1}{2}\sum_{n\geq 1}\left|(\alpha_n-\beta_n) + \mathrm{e}^{i\sigma_n t}\left[-\alpha_n + \frac{ic(\beta_n-\alpha_n)}{2\sigma_n}\right] + \mathrm{e}^{-i\sigma_n t}\left[\beta_n + \frac{ic(\alpha_n-\beta_n)}{2\sigma_n}\right]\right|^2. \quad (A.5)$$

Set

$$X_n = \alpha_n - \beta_n, \quad Y_n = \frac{ic(\beta_n-\alpha_n)}{2\sigma_n} - \alpha_n, \quad Z_n = \beta_n + \frac{ic(\alpha_n-\beta_n)}{2\sigma_n},$$

which implies that

$$\left|X_n + \mathrm{e}^{i\sigma_n t}Y_n + \mathrm{e}^{-i\sigma_n t}Z_n\right|^2 = |X_n|^2 + |Y_n|^2 + |Z_n|^2 + (X_n\bar{Y}_n + \bar{X}_nZ_n)\mathrm{e}^{-i\sigma_n t} + (X_n\bar{Z}_n + Y_n\bar{X}_n)\mathrm{e}^{i\sigma_n t} + \bar{Z}_nY_n\mathrm{e}^{2i\sigma_n t} + Z_n\bar{Y}_n\mathrm{e}^{-2i\sigma_n t}.$$

Then integrating (A.5) with respect to t gives us

$$\begin{aligned}&\int_0^T\left\|\int_0^t B^*\mathrm{e}^{A\xi}(A+BF)\phi_0\mathrm{d}\xi\right\|^2 dt\\ =&\sum_{n\geq 1}\left\{\frac{T}{2}(|X_n|^2+|Y_n|^2+|Z_n|^2) + \frac{\mathrm{e}^{i\sigma_n T}-1}{2i\sigma_n}(X_n\bar{Z}_n + Y_n\bar{X}_n) + \frac{1-\mathrm{e}^{-i\sigma_n T}}{2i\sigma_n}(X_n\bar{Y}_n + \bar{X}_nZ_n)\right.\\ &\left.+\frac{\mathrm{e}^{2i\sigma_n T}-1}{4i\sigma_n}\bar{Z}_nY_n + \frac{1-\mathrm{e}^{-2i\sigma_n T}}{4i\sigma_n}Z_n\bar{Y}_n\right\}.\end{aligned} \quad (A.6)$$

Firstly, by using the Cauchy inequality, the last four terms can be bounded by $K\sum_{n\geq 1}\left(|\alpha_n|^2+|\beta_n|^2\right)$ for some position constant K depending only on c and σ_1. Secondly, notice that the following linear transformation yields an invertible matrix L^*L:

$$\begin{bmatrix} X_n \\ Y_n \\ Z_n \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -\left(1+\frac{ic}{2\sigma_n}\right) & \frac{ic}{2\sigma_n} \\ \frac{ic}{2\sigma_n} & 1-\frac{ic}{2\sigma_n} \end{bmatrix}\begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \triangleq L\begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix},$$

therefore,

$$L^*L = \begin{bmatrix} 2+\frac{c^2}{2\sigma_n^2} & -1-\frac{ic}{\sigma_n}-\frac{c^2}{2\sigma_n^2} \\ -1+\frac{ic}{\sigma_n}-\frac{c^2}{2\sigma_n^2} & 2+\frac{c^2}{2\sigma_n^2} \end{bmatrix}, \tag{A.7}$$

and

$$|X_n|^2+|Y_n|^2+|Z_n|^2 = \langle L^*L(\alpha_n,\beta_n)^\top,(\alpha_n,\beta_n)^\top\rangle = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}^* L^*L \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}.$$

Denote $\lambda_{min}(L^*L)$ as the smallest eigenvalue of L^*L. Then it follows that

$$|X_n|^2+|Y_n|^2+|Z_n|^2 \geq \lambda_{min}(L^*L)\left\|\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}\right\|^2. \tag{A.8}$$

It is easy to see that $\lambda_{min} \geq \frac{3}{3+K}$ for some positive constant K. Therefore we get

$$|X_n|^2+|Y_n|^2+|Z_n|^2 \geq \frac{3}{3+K}\left(|\alpha_n|^2+|\beta_n|^2\right). \tag{A.9}$$

Substituting (A.8)–(A.9) into (A.5) proves Lemma 3.2. □

Proof of Theorem 2.10. If $E(0)=0$, then the source term is $w_1(x)=0$ and the solution $w(x,t)=0$, $\forall t>0$. Therefore, the two triggering laws are never satisfied for $t>0$, *i.e.*, $t_1=\infty$ and $(w(\cdot,t),w_t(\cdot,t))=(0,0)\ \forall\, t\geq 0$. If $E(0)\neq 0$, from Lemma 3.4, there exist tuning parameters ν, α and θ such that the trigger instants $(t_k)_{k\in\mathbb{N}}$ form a strictly increasing sequence diverging to infinity, *i.e.*, $\lim_{k\to\infty} t_k=\infty$. Then by Lemma 3.4 the closed-loop system (2.11) is well-posed, *i.e.*, there exists a unique solution to (2.11) satisfying (2.14). □

Now, we present two lemmas which are crucial to establishment of the main results of the present paper.

Lemma A.1. *[41] If T is a bounded linear operator on a complex Banach space, then for the spectral radius $r(T)$ of T we have*

$$r(T)=\lim_{n\to\infty}\sqrt[n]{\|T^n\|}. \tag{A.10}$$

Lemma A.2. *[40] Let $F_0,\cdots,F_p$ be quadratic functions of the variable $\xi\in\mathbb{R}^n$:*

$$F_i(\xi)\triangleq \xi^\top P_i\xi+2u_i^\top\xi+v_i,\quad i=0,1,\cdots,p,$$

where $P_i=P_i^\top$ and satisfying $\forall\xi\in\mathbb{R}^n$, $F_i(\xi)\geq 0$, $i=1,\cdots,p$. If there exist $\tau_1\geq 0,\cdots,\tau_p\geq 0$ such that

$$F_0(\xi)-\sum_{i=1}^{p}\tau_iF_i(\xi)\geq 0,\quad \forall\xi\in\mathbb{R}^n,$$

then $F_0(\xi)\geq 0$, $\forall\xi\in\mathbb{R}^n$.

Lemma A.3. *(Grönwall's inequality) Let $y(t)\in C[t_0,T)$, $T\leq+\infty$, satisfying*

$$y(t)\leq h(t)+m(t)\int_{t_0}^{t}k(s)y(s)\mathrm{d}s,\quad t\in[t_0,T), \tag{A.11}$$

where $m(t)$, $k(t)$ and $h(t)$ are nonnegative and continuous in $[t_0,T)$, then the following holds:

$$y(t)\leq h(t)+m(t)\int_{t_0}^{t}k(s)h(s)\mathrm{e}^{\int_s^t m(u)k(u)\mathrm{d}u}\mathrm{d}s. \tag{A.12}$$