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OUTPUT FEEDBACK STABILIZABILITY OF PERIODIC SAMPLED-DATA CONTROL SYSTEMS

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Abstract. In this work, a characterization in terms of weak observability inequalities is built up for output feedback stabilizability of linear periodic sampled-data control systems with a given decay rate. Based on this characterization, we give a verifiable condition to ensure the output feedback stabilizability of a class of periodic sampled-data control systems coupled by constant matrices. Finally, we provide several valuable examples as applications of our main result.

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1. Introduction

1.1. Motivation, formulation of the problem, aim and notations

In the finite dimensional periodic sampled-data control system, the state $y(\cdot)$ is sampled at discrete instants $kT, k = 1, 2, \cdots$, and it produces discrete vectors $y(kT), k = 1, 2, \cdots$ as sampled data, which are digestible by the computer. The sampled-data feedback stabilizability of the system asks for a linear bounded operator F such that the closed-loop system with feedback control $u_k = Fy(kT), k \in \mathbb{N}^+$ is exponentially stable. (The discrete controls enter the continuous-time system through a zero-order-hold device). This can be viewed as a kind of state feedback stabilizability since the data $y(kT), k \in \mathbb{N}^+$ are the full state at the sampling period kT.

However, for most of sampled-data controlled PDEs, the state functions have not only time argument t, but also space argument x. The state functions are usually sampled both in t and x. In this case, the feedback stabilizability based on the sampled data is a kind of output feedback stabilizability. This can be illustrated by the following controlled one-dimensional heat equation

$$\begin{cases}\partial_t y(t,x) = \partial_{xx} y(t,x) + \sum_{k=0}^{\infty} \sum_{j=1}^{2N+1} \chi_{[kT,(k+1)T)}(t)\delta(x-x_j)u_{kj}, \quad (t,x) \in \mathbb{R}^+ \times (0,\pi), \\ y(t,x)|_{x=0} = y(t,x)|_{x=\pi} = 0. \end{cases} \tag{1.1}$$

Keywords and phrases: Observer-based feedback, stabilizability, detectability, sampled-data control, coupled with constant matrices.

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Here $T > 0$, $N \in \mathbb{N}^+$ and $\{x_j\}_{j=1}^{2N+1} \subset (0, \pi)$. The discrete controls $\mathbf{u}_k := (u_{kj})_{j=1}^{2N+1} \in \mathbb{R}^{2N+1}$. One of the usually adopted sampling ways is as follows

$$\mathbf{z}_k = (y(kT, \hat{x}_j))_{j=1}^{2N+1}, \quad k = 1, 2, \cdots, \tag{1.2}$$

where $\{\hat{x}_j\}_{j=1}^{2N+1} \subset (0, \pi)$. The sampled data $(y(kT, \hat{x}_j))_{j=1}^{2N+1}$ are the observations of the state $y(kT)$ at points $\hat{x}_j, j = 1, 2, \cdots, 2N+1$. Whether there is a sampled-data feedback control stabilizing the system (1.1) is therefore a kind of output feedback stabilizability.

Inspired by the above problems for sampled-data controlled PDEs, this work intends to study the output feedback stabilizability for infinite-dimensional linear sampled-data control systems. (The exact definition of output feedback stabilizability will be given afterwards.) The abstract control system we shall consider is written as

$$y'(t) = Ay(t) + \sum_{k=0}^{\infty} \chi_{[kT,(k+1)T)}(t) B u_k, \quad t > 0, \tag{1.3}$$

and the observation system is given by

$$z_k = Cy(kT), \quad k \in \mathbb{N}^+. \tag{1.4}$$

Furthermore, we shall also consider the output feedback stabilizability for the following coupled system

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + S\mathbf{y}(t) + \sum_{k=0}^{\infty} \chi_{[kT,(k+1)T)}(t) D\mathbf{B}\mathbf{u}_k, \tag{1.5}$$

with observation

$$\mathbf{z}_k = Q\mathbf{C}\mathbf{y}(kT), \quad k \in \mathbb{N}^+. \tag{1.6}$$

Here, $T > 0$ is the sampling period, $\chi_{[kT,(k+1)T)}(\cdot)$ is the characteristic function of $[kT, (k+1)T)$, $y(t) \in Y$, $z_k \in Z$, for each $k \in \mathbb{N}^+$; $u_k \in U$, for each $k \in \mathbb{N}$, Y, Z and U are Hilbert spaces which are identified with their dual spaces respectively, $\mathbf{y}(t) \in \mathbf{Y}^n := \underbrace{Y \times Y \cdots \times Y}_{n}$, $\mathbf{u}_k \in \mathbf{U}^m := \underbrace{U \times U \cdots \times U}_{m}$, $\forall k \in \mathbb{N}$, $\mathbf{z}_k \in \mathbf{Z}^p := \underbrace{Z \times Z \cdots \times Z}_{p}$, $\forall k \in \mathbb{N}^+$, $S \in \mathbb{R}^{n\times n}, D \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{p\times n}$ are matrices, $\mathbf{A} := \mathbf{A}_n = \mathrm{diag}(\underbrace{A, \cdots, A}_{n})$, $\mathbf{B} := \mathbf{B}_m = \mathrm{diag}(\underbrace{B, \cdots, B}_{m})$ and $\mathbf{C} := \mathbf{C}_n = \mathrm{diag}(\underbrace{C, \cdots, C}_{n})$. For different subscript, we still use the same notation in case of no confusion caused. For example, $\mathbf{A}_{n_1}$ will be still denoted by $\mathbf{A}$ when $n_1 \neq n$. The same convention applies for $\mathbf{B}$, $\mathbf{C}$ and other operators throughout this paper. The operators A, B and C satisfy the following conditions:

(H_1) The operator $A : D(A) \subset Y \to Y$, where Y is a Hilbert space, generates a strongly continuous semigroup $\mathrm{e}^{At}, t \geq 0$ on Y;

(H_2) The operator B belongs to $\mathcal{L}(U; Y_{-\theta})$ for some $\theta \in [0, 1]$, where $Y_{-\theta}$ is the completion of Y with respect to the norm $\|y\|_{-\theta} = \|(\rho_0 I - A)^{-\theta} y\|_Y$ for some $\rho_0 \in \rho(A)$ large enough;

(H_3) $C \in \mathcal{L}(Y_{\hat{\theta}}; Z)$ and $C\mathrm{e}^{AT} \in \mathcal{L}(Y; Z)$, where $\hat{\theta} \in [0, 1-\theta]$, $Y_{\hat{\theta}} = D((\rho_0 I - A)^{\hat{\theta}})$ with norm $\|z\|_{\hat{\theta}} := \|(\rho_0 I - A)^{\hat{\theta}} z\|_Y$.

For the above systems, we give the following remarks:

Remark 1.1. (i) We denote for simplicity the control system (1.3) by $[A,B]$, the observation system $y'(t) = Ay(t), t \geq 0,\ z_k = Cy(kT)$ by $[C,A]$, and the control system (1.3) with observation (1.4) by system $[A,B,C]$. The control system (1.5) with observation (1.6) can be put into the general framework (1.3) with (1.4), and we can denote it by $[\mathbf{A}+S, D\mathbf{B}, Q\mathbf{C}]$. Nevertheless, it has special coupling structure. The system (1.5) is the coupling of the "scalar" equation (1.3) by constant matrices S and D. The observation is composed of two steps: firstly, observing each component, which produces $\mathbf{Cy}(kT)$; then observing the vector $\mathbf{Cy}(kT)$, which leads to $\mathbf{z}_k$. This work first gives a characterization of output feedback stabilizability for the general system $[A,B,C]$. Then we apply this characterization to study the output feedback stabilizability for $[\mathbf{A}+S, D\mathbf{B}, Q\mathbf{C}]$ based on the independent conditions on the "scalar" system $[A,B,C]$ and the coupled matrices S, D and Q.

(ii) Under Assumptions (H_1) and (H_2), for each $y_0 \in Y$, the control system $[A,B]$, with the initial data $y(0)=y_0$, admits a unique solution in $C(\mathbb{R}^+;Y)$ (see Sect. 2.3 in [1]), which is given by

$$y(t) = \mathrm{e}^{A(t-kT)}y(kT) + \int_{kT}^{t} \mathrm{e}^{\tilde{A}(t-s)}Bu_k \mathrm{d}s, t \in [kT,(k+1)T),\ \ k \in \mathbb{N}. \tag{1.7}$$

Here $\tilde{A} \in \mathcal{L}(Y;Y_{-1})$ is the unique extension of $A \in \mathcal{L}(D(A),Y)$, and $\mathrm{e}^{\tilde{A}t}, t \geq 0$ is the semigroup on Y_{-1} with infinitesimal generator $\tilde{A}$ (see [2]). Moreover, $\int_0^t \mathrm{e}^{\tilde{A}(t-s)}B\mathrm{d}s \in \mathcal{L}(U;Y_{1-\theta})$, for each $t>0$ (see Thm. A.1). This, together with Assumption (H_3) implies that $z_k \in Z, \forall k \in \mathbb{N}^+$, where z_k is given by (1.4). In the rest of this paper, if there is no risk of causing ambiguity, we shall still denote $\tilde{A}$ by A, and $\mathrm{e}^{\tilde{A}t}$ by e^{At} for simplicity.

(iii) The operators B and C are allowed to be unbounded in certain compatible manner. The unboundedness of B can be smoothened by $\int_0^t \mathrm{e}^{As}\mathrm{d}s$. The assumption $C\mathrm{e}^{AT} \in \mathcal{L}(Y;Z)$ given in (H_3) is imposed to assure that $z_k \in Z$. When C is bounded, $C\mathrm{e}^{AT} \in \mathcal{L}(Y;Z)$ automatically. When C is unbounded, the semigroup $\mathrm{e}^{At}, t \geq 0$ needs to smoothen and absorb the unboundedness of C.

(iv) The control system (1.1) with observation (1.2) can be written in the form (1.3) and (1.4) with $Y = L^2(0,\pi)$, $U = \mathbb{R}^{2N+1}, Z = \mathbb{R}^{2N+1}$, $A : D(A)(= H^2(0,\pi) \cap H_0^1(0,\pi)) \to Y$ defined by $Af = \frac{d^2 f}{\mathrm{d}x^2}$, $\forall f \in D(A)$, $B : U \to Y_{-1/2}$ defined by $B\mathbf{v} = \sum_{j=1}^{2N+1} \delta(x-x_j)v_j$, $\forall \mathbf{v} = (v_j)_{j=1}^{2N+1} \in U$, and $C : Y_{1/2} \to Z$ defined by $Cg = (g(\hat{x}_j))_{j=1}^{2N+1}$, $\forall g \in Y_{1/2}$. Moreover, $C\mathrm{e}^{AT} \in \mathcal{L}(Y;Z)$ because the semigroup $\mathrm{e}^{At}, t \geq 0$ is analytic.

The output feedback stabilizability of $[A,B,C]$ with a given decaying rate is defined as follows.

Definition 1.2. Let $\beta > 0$ be given. The system $[A,B,C]$ is β-output feedback stabilizable, if there is $F \in \mathcal{L}(Y;U)$, $L \in \mathcal{L}(Z;Y)$, $\varepsilon > 0$ and $c > 0$, such that the solution to the control system

$$y'(t) = Ay(t) + \sum_{k=0}^{\infty} \chi_{[kT,(k+1)T)}(t)BFw(kT), \tag{1.8}$$

and the solution to the observer

$$\begin{cases} w'(t) = Aw(t) + \sum_{k=0}^{\infty} \chi_{[kT,(k+1)T)}(t)BFw(kT), & t \in \mathbb{R}^+ \setminus \{kT\}_{k\in\mathbb{N}}; \\ w(0)=0,\ w(kT) = w(kT-) + LCw(kT-) - Lz_k, & k \in \mathbb{N}^+, \end{cases} \tag{1.9}$$

satisfy

$$\|y(t)\|_Y + \|w(t)\|_Y \leq c\mathrm{e}^{-(\beta+\varepsilon)t}\|y(0)\|_Y,\ \ \forall t > 0. \tag{1.10}$$

For the above definition, we give the following notes:

- The β-output feedback stabilizability ensures the exponentially stability of the closed-loop system with decaying rate greater than β.
- Substituting A, B, C and $F \in \mathcal{L}(Y; U)$, $L \in \mathcal{L}(Z; Y)$ by $\mathbf{A} + S, D\mathbf{B}, Q\mathbf{C}$ and $\mathbf{F} \in \mathcal{L}(\mathbf{Y}^n; \mathbf{U}^m)$, $\mathbf{L} \in \mathcal{L}(\mathbf{Z}^p; \mathbf{Y}^n)$ respectively in the above, we obtain the definition of the β-output feedback stabilizability for $[\mathbf{A} + S, D\mathbf{B}, Q\mathbf{C}]$.
- It is a kind of observer based output feedback stabilizability. The observer is constructed based on the sampled observation, and the feedback control is designed based on the observer. We briefly explain how the observer is constructed. The observation system $[C, A]$ can be written in a discrete form

$$\begin{cases} y_k = \mathrm{e}^{AT} y_{k-1}, \;\; k \in \mathbb{N}^+, \\ z_k = C\mathrm{e}^{AT} y_{k-1}, \;\; k \in \mathbb{N}^+. \end{cases} \tag{1.11}$$

Here we denote $y_k = y(kT), \forall k \in \mathbb{N}$. The so-called Luenberger observer for the above observation system is given by

$$w_k = \mathrm{e}^{AT} w_{k-1} + LC\mathrm{e}^{AT} w_{k-1} - Lz_k, \;\; k \in \mathbb{N}^+, \tag{1.12}$$

where $L \in \mathcal{L}(Z; Y)$. If the solution to the error equation $\varphi_k = \mathrm{e}^{AT}\varphi_{k-1} + LC\mathrm{e}^{AT}\varphi_{k-1}$ decays exponentially to zero, then $\{w_k\}_{k\in\mathbb{N}}$ asymptotically recovers the state $\{y_k\}_{k\in\mathbb{N}}$. Notice that the discrete form of (1.9) (without the term $\sum_{k=0}^{\infty} \chi_{[kT,(k+1)T)}(t)BFw(kT)$) is exactly (1.12). We write the observer in the form of (1.9) instead of the discrete form because it is more convenient to compare with the state equation (1.3). Similar observers have been proposed for state estimation of systems with discrete observation (see [3] and references therein).

The *aim* of this work consists of two parts:

- Characterizing the β-output feedback stabilizability for system $[A, B, C]$ in terms of some weak observability inequalities.
- Using the characterization of β-output feedback stabilizability to give an answer to the following question: given $\beta > 0$, when B, C, S, D, Q and T meet what conditions (which are easy to be verified and reasonable), the system $[\mathbf{A} + S, D\mathbf{B}, Q\mathbf{C}]$ under the following additional assumption $(A0)$ is β-output feedback stabilizable.

  $(A0)$ Given $\beta > 0$, there is a projection P^β on Y, $\tilde{c} > 0$ and $\tilde{\alpha} > \bar{\sigma}(S) + \beta$ such that ($\bar{\sigma}(S)$ is the maximum of real part of eigenvalues of S)

$$\|(I - P^\beta)\mathrm{e}^{A^* t}\varphi\|_Y \leq \tilde{c}\mathrm{e}^{-\tilde{\alpha}t}\|\varphi\|_Y \;\text{ for any }\; t \in \mathbb{R}^+ \text{and }\; \varphi \in Y. \tag{1.13}$$

  Moreover, $P^\beta \mathrm{e}^{A^* t} = \mathrm{e}^{A^* t} P^\beta, \forall t > 0$, and for each $\varphi \in Y$, $P^\beta\varphi \in D(A)$.

  The above assumption is imposed on the operator A, which amounts to saying that the high-frequency part of the solution $t \to \mathrm{e}^{A^* t}\varphi$ decays exponentially. With this be done, we can apply the result to discuss the stabilizability of some typical parameter distributed sampled-data control systems, including the coupled system of (1.1) with observation (1.2).

We end this subsection with the following notation which will be used in this paper: We write $\mathbb{R}^+ := (0, +\infty)$, $\mathbb{N}^+ := \{1, 2, \ldots\}$, $\mathbb{N} := \mathbb{N}^+ \cup \{0\}$, $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and $\mathbb{C}_\alpha^+ := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq \alpha\}$, with $\alpha \in \mathbb{R}$; Given a Hilbert space X_1, we write $\|\cdot\|_{X_1}$ and $\langle\cdot,\cdot\rangle_{X_1}$ for the norm and the inner product of X_1 respectively; Given Banach spaces X_1 and X_2, we write $\mathcal{L}(X_1; X_2)$ for the space of all linear and bounded operators from X_1 to X_2 and let $\mathcal{L}(X_1) := \mathcal{L}(X_1; X_1)$; We use I to denote the identity operator; Given a linear operator F, we use $\|F\|$, F^* (or $F^\top$ when F is a matrix), $\rho(F)$ and $\sigma(F)$ to denote its operator norm, its adjoint operator (its transpose, when F is a matrix), its resolvent set and its spectrum set, respectively; Given a square matrix R, we write

$\bar{\sigma}(R) := \max\{\mathrm{Re}(\lambda) : \lambda \in \sigma(R)\}$ and $\underline{\sigma}(R) := \min\{\mathrm{Re}(\lambda) : \lambda \in \sigma(R)\}$; We use $\imath$ to denote the unitary imaginary number, *i.e.*, $\imath^2 = -1$; We use $\mathcal{F}$ to denote the Fourier transform $\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \mathrm{e}^{-\imath \xi x} \mathrm{d}x, \forall f \in L^2(\mathbb{R})$, and use $\mathcal{F}^{-1}$ to denote the inverse of the Fourier transform; Given a set E, we let χ_E be its characteristic function; Given $t \in \mathbb{R}^+$, we let $\lfloor t \rfloor := \max\{n \in \mathbb{N} : n \leq t\}$; We use $c(\cdots)$ to denote a positive constant that depends on what is enclosed in the brackets.

1.2. Main results

The first main result gives a characterization of β-output feedback stabilizability of $[A, B, C]$.

Theorem 1.3. *Suppose that assumptions* (H_1)*-*(H_3) *hold. The system* $[A, B, C]$ *is* β*-output feedback stabilizable if and only if there are constants* $N_1, N_2 \in \mathbb{N}^+$, $\delta_1, \delta_2 \in (0, 1)$, *and* $c_1, c_2 \geq 0$ *such that*

$$\|\mathrm{e}^{A^*(N_1 T)} \varphi\|_Y^2 \leq c_1 \sum_{k=0}^{N_1 - 1} \left\| \int_{kT}^{(k+1)T} B^* \mathrm{e}^{A^* t} \varphi \mathrm{d}t \right\|_U^2 + \delta_1 \mathrm{e}^{-2\beta N_1 T} \|\varphi\|_Y^2 \ \ \textit{for any} \ \ \varphi \in Y; \tag{1.14}$$

$$\|\mathrm{e}^{A(N_2 T)} \varphi\|_Y^2 \leq c_2 \sum_{k=1}^{N_2} \|C \mathrm{e}^{AkT} \varphi\|_Z^2 + \delta_2 \mathrm{e}^{-2\beta N_2 T} \|\varphi\|_Y^2 \ \ \textit{for any} \ \ \varphi \in Y. \tag{1.15}$$

Remark 1.4. (i) The characterization is given in the form of weak observability inequalities. We shall show that the first inequality (1.14) is equivalent to β-stabilizability of $[A, B]$ (see Def. 2.3 and Thm. 2.4); The second inequality (1.15) is equivalent to β-detectability of the observation system $[C, A]$ (see Def. 2.5 and Thm. 2.6). Hence, Theorem 1.3 amounts to saying that system $[A, B, C]$ is β-output feedback stabilizable if and only if $[A, B]$ is β-stabilizable and $[C, A]$ is β-detectable.

(ii) The idea to characterize the stabilizability by weak observability inequality was first realized in [4]. Then, it was applied to characterize the periodic stabilizability and complete stabilizability of linear systems (see [5] and [6]). Recently, we built up a criterion in terms of weak observability for stabilizability of sampled-data control system $[A, B]$ in [7]. This work extends [7] in the following aspects: first, we give a criterion for observer-based output feedback stabilizability; second, the control operator is allowed to be unbounded; third, the decay rate can be given in advance.

The second main result concerns the sufficient conditions for β-stabilizability of system $[\mathbf{A} + S, D\mathbf{B}, Q\mathbf{C}]$ (system (1.5) with observation (1.6)) under Assumptions $(H_1) - (H_3)$ and $(A0)$. To present the result, we let $\beta > 0$, denote $Y_{P^\beta} := \mathrm{Range}(P^\beta)$, $\lambda_0 := \inf_{t>0}\{\ln \|\mathrm{e}^{At}\|_{\mathcal{L}(Y)}/t\}$, and give the following assumptions:

$(A1)$ There is $\tilde{d} := \tilde{d}(\beta) > 0$ and $\hat{d} := \hat{d}(\beta) > 0$, such that

$$\|P^\beta \varphi\|_Y \leq \tilde{d} \|B^* P^\beta \varphi\|_U \ \ \text{for any} \ \ \varphi \in Y, \tag{1.16}$$

$$\|P^\beta \varphi\|_Y \leq \hat{d} \|C P^\beta \varphi\|_Z \ \ \text{for any} \ \ \varphi \in Y. \tag{1.17}$$

Here P^β is given in Assumption $(A0)$.

$(A2)$ For each $\gamma \in \sigma(S) \cap \mathbb{C}^+_{-\lambda_0 - \beta}$, and for each $g \in Y_{P^\beta}$, there is a unique $f \in Y_{P^\beta}$, such that

$$\int_0^T \mathrm{e}^{(A+\gamma I)t} \mathrm{d}t f = g. \tag{1.18}$$

Moreover, there is $c > 0$ such that $\|(\int_0^T \mathrm{e}^{(A+\gamma I)t} \mathrm{d}t)^{-1}\|_{\mathcal{L}(Y)} \leq c$, where $(\int_0^T \mathrm{e}^{(A+\gamma I)t} \mathrm{d}t)^{-1}$ denotes the map $g \mapsto f$ defined above.

$(A3)$

$$\operatorname{rank}(\gamma I - S, D) = n, \quad \text{when } \ \gamma \in \mathbb{C}^+_{-\lambda_0-\beta}. \tag{1.19}$$

$(A4)$

$$\operatorname{rank}(\gamma I - S^\top, Q^\top) = n, \quad \text{when } \ \gamma \in \mathbb{C}^+_{-\lambda_0-\beta}. \tag{1.20}$$

$(A5)$ The sampling period T does not belong to the following set:

$$\mathcal{P}_{S,\beta} := \{2k\pi/|\mathrm{Im}(\gamma_l - \gamma_j)| : k \in \mathbb{N}^+, \gamma_l, \gamma_j \in \sigma(S) \cap \mathbb{C}^+_{-\lambda_0-\beta}, \mathrm{Re}(\gamma_l) = \mathrm{Re}(\gamma_j)\}. \tag{1.21}$$

The second main result of this paper is stated as follows:

Theorem 1.5. *Let $\beta > 0$. Suppose that (H_1)-(H_3), $(A0)$ and $(A1)$-$(A5)$ hold. Then the system $[\mathbf{A} + S, D\mathbf{B}, Q\mathbf{C}]$ is β-output feedback stabilizable.*

Remark 1.6. Several notes on Theorem 1.5 and the above assumptions are given as follows:

(i) Given $\beta > 0$, Theorem 1.5 provides conditions (depending on β) that the operators B, C, the matrices S, D and the sampling period T should satisfy to ensure the β-output feedback stabilizability for the system (1.5) with observation (1.6). In many cases, these conditions are easy to verify. For example, in Section 4.1.2, we study the system of heat equations (1.1) with (1.2) (coupled by matrices S, D). We apply Theorem 1.5 to find a condition (that N needs to satisfy) to ensure that the system (with $(A3)$, $(A4)$ and $(A5)$) is β-output feedback stabilizable. Another example where Theorem 1.5 can be applied is the system coupled by heat equations over $\mathbb{R}$, with time-spatial sampled-data controls and observations (see Sect. 4.2).

(ii) When $[A, B]$ is n-dimensional sampled-data control system, it is well-known that $[A, B]$ is controllable if and only if (see [8] or [9])

$$\operatorname{rank}[B, \mathrm{e}^{AT}B, \cdots, \mathrm{e}^{(n-1)AT}B] = n \textit{ and } \int_0^T \mathrm{e}^{At}\mathbf{d}t \textit{ is invertible.} \tag{1.22}$$

This can be ensured by the following more explicit condition:

$$\operatorname{rank}(\gamma I - A, B) = n, \forall \gamma \in \sigma(A); \ T \notin \{\frac{2k\pi}{|\gamma_l - \gamma_j|} : k \in \mathbb{N}^+, \gamma_l, \gamma_j \in \sigma(A), \textit{with } \mathrm{Re}(\gamma_l) = \mathrm{Re}(\gamma_j)\},$$

which amounts to saying the corresponding continuous-time control system is controllable and the sampling should not destroy the controllability. When the above explicit condition holds for eigenvalues with nonnegative real part, then the unstable part of system $[A, B]$ is controllable and it is consequently stabilizable (see [10] or [9]). The conditions $(A3)$-$(A5)$ are analogous to the above explicit condition for finite-dimensional system.

(iii) The conditions $(A1)$ and $(A2)$ are given for similar purpose: guaranteeing the "unstable" part (the part whose decaying rate is less than or equal to β) to be controllable (or observable). The condition $(A1)$ ensures the following: there is $\tilde{d}, \hat{d} > 0$, $\tilde{N}, \hat{N} \in \mathbb{N}^+$, such that

$$\begin{cases} \|P^\beta\varphi\|_Y \le \tilde{d} \sum_{k=0}^{\tilde{N}} \|B^* P^\beta \mathrm{e}^{A^* kT}\varphi\|_U \ \text{ for any } \ \varphi \in Y, \\ \|P^\beta\varphi\|_Y \le \hat{d} \sum_{k=0}^{\hat{N}} \|C P^\beta \mathrm{e}^{AkT}\varphi\|_Z \ \text{ for any } \ \varphi \in Y. \end{cases} \tag{1.23}$$

In general, the unstable part of the system $[A, B]$ might not be of finite dimensional. (1.23) is an alternative form of the rank condition in (1.22). It guarantees the "unstable" continuous-time counterpart is controllable (or observable). The condition $(A2)$ is analogous to the second condition in (1.22), which is imposed to avoid destroying the controllability because of sampling.

(iv) The condition $(A1)$ is much easier to be verified than the inequalities in (1.23). When we regard P^β as a spectral projection, condition $(A1)$ corresponds to a spectral inequality, which has been used in the study of the stabilizability for linear system with continuous-time controls (see [11], [6]), and also turns out to be very convenient for constructing feedback controls (see [12]). We shall see in this work that condition $(A1)$ is also effective for studying the stabilizability of sampled-data control systems in many cases (see Sect. 4). However, in some cases, such as the 1-D parabolic equation with boundary control and boundary observation, it might happen that the inequalities in (1.23) hold, but the condition $(A1)$ is not satisfied (see [12]). Using the same arguments in this work, we can show the result in Theorem 1.5 still holds true with $(A1)$ substituted by (1.23) when $n = m = p = 1$. (We shall not go to detail in this work). However, for general cases, we do not find a way to prove it.

(v) In Section 4, we check the condition $(A2)$ for some typical examples by analyzing the spectrum of operator A and the matrix S. It is illustrated that whether the condition $(A2)$ holds or not depends on the relation between the sampling period and the spectrum of $P^\beta(A + \gamma I)$, where $\gamma \in \sigma(S) \cap \mathbb{C}^+_{-\beta-\gamma_0}$. It would be interesting to figure out this relation for general cases, and we leave it as future work.

1.3. Related works

We prefer to mention the following related works:

- Output feedback stabilization based on observer for parabolic sampled-data control system in bounded domain was considered in [13]. A delayed output feedback law has been designed in [14] for stabilizing sampled-data controlled heat equations. The generalized sampled-data stabilization (with time periodic feedback) and dynamic feedback stabilization of well-posed linear infinite-dimensional systems have been considered in [15] and [16] respectively, wherein the control operator and observation operator are assumed to satisfy finite rank condition.
- Sampled-data output feedback controllers for 1-D parabolic PDEs in bounded interval have been proposed in [17–19] by using linear matrix inequalities.
- The paper [20] obtains results that guarantee closed-loop exponential stability for 1-D parabolic equations under Zero-Order-Hold implementations of some continuous-time boundary feedbacks, by applying small-gain arguments.
- A criterion for state feedback stabilizability is obtained in [21] for linear sampled-data control system under the assumption that the system can be decomposed into two decoupled subsystems: one is finite-dim unstable, while another is infinite-dim stable. In [22], it is shown that if B has some compactness, then the linear systems with piecewise polynomial controls are stabilizable if and only if they can be decomposed into two subsystems mentioned above. Recently, [7] provides a criterion (*via* weak observability) for the stabilizability of linear sampled-data control systems, under the condition that $B \in \mathcal{L}(U; Y)$.
- For controllability and stabilizability of coupled system with continuous-time control or impulsive control, we mention the following: First, the work [23] provides a criterion (satisfied by $[S, D]$) on the controllability of one dimensional parabolic systems with continuous-time boundary control; Second, the observability of coupled system with continuous-time control has been investigated in [24] based on the observability of the "scalar" equation and a rank condition of $[S, D]$; Third, the controllability and stabilizability of impulse controlled systems of heat equations coupled by constant matrices have been studied respectively in [25] and [26], which show that the impulse instants realizing the controllability/stabilizability essentially depend on the coupling matrices.

1.4. Novelty

- We provide a new characterization of observer-based output feedback stabilizability with given decaying rate for a quite general class of linear sampled-data control systems.
- Our framework (1.5) with (1.6) is broader than that in [13]: We consider a coupled system; Our system may not be divided into two decoupled subsystems: one is finite-dim unstable while another is infinite-dim stable, which is the case in [13]; The observing operator C is allowed to be unbounded in this work. Our method proving Theorem 1.5 differs from that in [13] (which strongly depends on the above mentioned decomposition of the system). We use the first main result in this work that builds up the equivalence between the stabilizability and some weak observability inequalities.

1.5. The plan of this paper

The rest of this paper is organized as follows: In Section 2, we first present a characterization of β-stabilizability of general linear discrete control systems. Then, we give the characterization for β-stabilizability of $[A, B]$ and β-detectability of $[C, A]$, and finally we give the proof of Theorem 1.3. In Section 3, we apply Theorem 1.3 to prove the second main result Theorem 1.5. Section 4 provides several concrete partial differential systems with sampled-data controls and space-time discrete observations, which can be proven to be output feedback stabilizable by applying Theorem 1.5.

2. Proof of Theorem 1.3

In this section, we first give a characterization on β-stabilizability of general discrete linear systems. Then, we give characterizations in terms of weak observability inequalities for β-stabilizability of $[A, B]$, and β-detectability of $[C, A]$. Based on these we are able to prove our main result, Theorem 1.3.

2.1. Characterization on stabilizability of discrete linear systems

Let $\mathcal{Y}$ and $\mathcal{U}$ be two Hilbert spaces, which are identified with their dual spaces respectively. Let $\Phi \in \mathcal{L}(\mathcal{Y})$, and $\Lambda \in \mathcal{L}(\mathcal{U};\mathcal{Y})$. We consider the following discrete control system

$$y_{k+1} = \Phi y_k + \Lambda u_k, \quad k = 0, 1, \cdots . \tag{2.1}$$

Given $\beta > 0$, we define the β-stabilizability of (2.1) as follows.

Definition 2.1. The system (2.1) is β-stabilizable if there is $F \in \mathcal{L}(\mathcal{Y};\mathcal{U})$, $\varepsilon > 0$ and $c > 0$, such that

$$\|(\Phi + \Lambda F)^k\|_{\mathcal{L}(\mathcal{Y})} \leq c\mathrm{e}^{-(\beta+\varepsilon)k}, \quad \forall k \in \mathbb{N}^+. \tag{2.2}$$

For the β-stabilizability of (2.1), we have the following characterization.

Theorem 2.2. *The system (2.1) is β-stabilizable if and only if there is $N \in \mathbb{N}^+$, $c > 0$, and $\delta \in (0, 1)$, such that*

$$\|(\Phi^*)^N\varphi\|^2_{\mathcal{Y}} \leq c\sum_{k=0}^{N-1}\left\|\Lambda^*(\Phi^*)^k\varphi\right\|^2_{\mathcal{U}} + \delta\mathrm{e}^{-2\beta N}\|\varphi\|^2_{\mathcal{Y}}, \quad \forall \varphi \in \mathcal{Y}. \tag{2.3}$$

Proof. We first prove the necessity. Suppose that (2.1) is β-stabilizable. By Definition 2.1, we see that there is $F \in \mathcal{L}(\mathcal{Y};\mathcal{U})$, $c_0 > 0$ and $\varepsilon \in (0, 1)$, such that the solution to the following equation

$$y_{k+1} = \Phi y_k + \Lambda F y_k, \quad k = 0, 1, \cdots , \tag{2.4}$$

satisfies

$$\|y_k\|_{\mathcal{Y}} \leq c_0 \mathrm{e}^{-(\beta+\varepsilon)k}\|y_0\|_{\mathcal{Y}}, \quad \forall k \in \mathbb{N}^+. \tag{2.5}$$

Take $N = \lfloor \frac{\ln[2(\max\{c_0,1\})^2]}{\varepsilon} \rfloor + 1$. We can write y_N as

$$y_N = \Phi^N y_0 + \sum_{k=0}^{N-1} \Phi^{N-k-1}\Lambda F y_k.$$

Based on above, we find that, for each $\varphi \in \mathcal{Y}$,

$$\begin{aligned}\langle y_N, \varphi\rangle_{\mathcal{Y}} &= \langle \Phi^N y_0, \varphi\rangle_{\mathcal{Y}} + \langle \sum_{k=0}^{N-1} \Phi^{N-k-1}\Lambda F y_k, \varphi\rangle_{\mathcal{Y}} \\ &= \langle y_0, (\Phi^*)^N \varphi\rangle_{\mathcal{Y}} + \sum_{k=0}^{N-1} \langle F y_k, \Lambda^*(\Phi^*)^{N-k-1}\varphi\rangle_{\mathcal{U}}. \end{aligned} \tag{2.6}$$

We obtain from (2.5) and (2.6) that

$$\begin{aligned}\langle y_0, (\Phi^*)^N \varphi\rangle_{\mathcal{Y}} \leq & \sum_{k=0}^{N-1} \|F\|_{\mathcal{L}(\mathcal{Y};\mathcal{U})} c_0 \mathrm{e}^{-(\beta+\varepsilon)k}\|y_0\|_{\mathcal{Y}} \|\Lambda^*(\Phi^*)^{N-k-1}\varphi\|_{\mathcal{U}} \\ & + c_0 \mathrm{e}^{-(\beta+\varepsilon)N}\|y_0\|_{\mathcal{Y}}\|\varphi\|_{\mathcal{Y}} \\ \leq & \|F\|_{\mathcal{L}(\mathcal{Y};\mathcal{U})} c_0 (\sum_{k=0}^{N-1} \mathrm{e}^{-2(\beta+\varepsilon)k})^{\frac{1}{2}} (\sum_{k=0}^{N-1} \|\Lambda^*(\Phi^*)^k \varphi\|_{\mathcal{U}}^2)^{\frac{1}{2}} \|y_0\|_{\mathcal{Y}} \\ & + c_0 \mathrm{e}^{-(\beta+\varepsilon)N}\|y_0\|_{\mathcal{Y}}\|\varphi\|_{\mathcal{Y}}. \end{aligned}$$

Notice that $\delta := 2c_0^2 \mathrm{e}^{-2\varepsilon N} < 1$, and $y_0 \in \mathcal{Y}$ is arbitrarily given. The above leads to (2.3) with

$$c = 2\|F\|_{\mathcal{L}(\mathcal{Y};\mathcal{U})}^2 c_0^2 \sum_{k=0}^{N-1} \mathrm{e}^{-2(\beta+\varepsilon)k}; \quad N = \lfloor \frac{\ln[2(\max\{c_0,1\})^2]}{\varepsilon} \rfloor + 1; \quad \delta = 2c_0^2 \mathrm{e}^{-2\varepsilon N}.$$

Now, we prove the sufficiency. We divide the proof into two steps.

Step 1. We prove that the discrete linear quadratic problem (**d-LQ**) defined as follows is solvable: for any $y_0 \in \mathcal{Y}$, find $v^* = (v_k^*)_{k\in\mathbb{N}} \in l^2(\mathbb{N};\mathcal{U})$ such that

$$J(v^*; y_0) = \inf_{v \in l^2(\mathbb{N};\mathcal{U})} J(v; y_0) := V(y_0).$$

Here the cost functional $J(v; y_0)$ is defined by

$$J(v; y_0) := \sum_{k=0}^{+\infty} (\|z_k(y_0, v)\|_{\mathcal{Y}}^2 + \|v_k\|_{\mathcal{U}}^2) \text{ for any } v = (v_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N};\mathcal{U}) \text{ and } y_0 \in \mathcal{Y},$$

where $(z_k(y_0, v))_{k\in\mathbb{N}}$ is the solution to the following equation:

$$z_{k+1} = \mathrm{e}^{\beta}\Phi z_k + \mathrm{e}^{\beta}\Lambda v_k, \quad \text{with } k \in \mathbb{N}; \quad z_0 = y_0. \tag{2.7}$$

We call the problem (**d-LQ**) solvable if $V(y_0) < +\infty$ for any $y_0 \in \mathcal{Y}$. By [7], Lemma 2.2, it suffices to prove that there is $M > 0$, which is independent on j, such that

$$\inf_{v\in l^2(\mathbb{N};\mathcal{U})} J_{jN}(v;y_0) \le M\|y_0\|^2_{\mathcal{Y}}, \ \ \forall y_0 \in \mathcal{Y} \text{ and } j \in \mathbb{N}^+, \tag{2.8}$$

where $J_{jN}(v;y_0) = \sum_{k=0}^{jN}(\|z_k(y_0,v)\|^2_{\mathcal{Y}} + \|v_k\|^2_{\mathcal{U}})$. To this aim, we write (2.3) as

$$\|\mathrm{e}^{\beta N}(\Phi^*)^N\varphi\|^2_{\mathcal{Y}} \le c_1 \sum_{k=0}^{N-1} \left\|\Lambda^*\mathrm{e}^{(N-k)\beta}(\Phi^*)^{N-k-1}\varphi\right\|^2_{\mathcal{U}} + \delta\|\varphi\|^2_{\mathcal{Y}} \ \text{ for any } \ \varphi \in \mathcal{Y}, \tag{2.9}$$

where $c_1 = c\mathrm{e}^{2(N-1)\beta}$. Let $\mathcal{U}^N := \{u : u = (u_k)_{k=0}^{N-1}, u_k \in \mathcal{U}\}$ with norm $\|u\|_{\mathcal{U}^N} = \{\sum_{k=0}^{N-1}\|u_k\|^2_{\mathcal{U}}\}^{1/2}$. Define the operators $\mathcal{R} : \mathcal{Y} \to \mathcal{Y}$ and $\mathcal{O} : \mathcal{Y} \to \mathcal{U}^N$ by

$$\mathcal{R}\varphi := \mathrm{e}^{\beta N}(\Phi^*)^N\varphi, \ \ \mathcal{O}\varphi := \{\Lambda^*\mathrm{e}^{(N-k)\beta}(\Phi^*)^{N-k-1}\varphi\}_{k=0}^{N-1}, \varphi \in \mathcal{Y}.$$

Then, the inequality (2.9) can be equivalently written as

$$\|\mathcal{R}\varphi\|^2_{\mathcal{Y}} \le c_1\|\mathcal{O}\varphi\|^2_{\mathcal{U}^N} + \delta\|\varphi\|^2_{\mathcal{Y}}. \tag{2.10}$$

According to [27], Lemma 5.1, for each $y_0 \in \mathcal{Y}$, there is a $v^{y_0} \in \mathcal{U}^N$ such that

$$\frac{1}{c_1}\| - v^{y_0}\|^2_{\mathcal{U}^N} + \frac{1}{\delta}\|\mathcal{R}^*y_0 - \mathcal{O}^*(-v^{y_0})\|^2_{\mathcal{Y}} \le \|y_0\|^2_{\mathcal{Y}}. \tag{2.11}$$

By direct calculation, one can see that $\mathcal{R}^*y_0 - \mathcal{O}^*(-v^{y_0}) = z_N(y_0, v^{y_0})$. Moreover,

$$\|z_N(y_0, v^{y_0})\|^2_{\mathcal{Y}} \le \delta\|y_0\|^2_{\mathcal{Y}}; \ \ \|v^{y_0}\|^2_{\mathcal{U}^N} \le c_1\|y_0\|^2_{\mathcal{Y}}.$$

Denote $z_{0N} := y_0, z_{1N} = z_N(y_0, v^{y_0})$. Now, by (2.10) and the same arguments as above, we see that for z_{1N}, there is $v^{z_{1N}} \in \mathcal{U}^N$ such that

$$\|z_N(z_{1N}, v^{z_{1N}})\|^2_{\mathcal{Y}} \le \delta\|z_{1N}\|^2_{\mathcal{Y}}; \ \ \|v^{z_{1N}}\|^2_{\mathcal{U}^N} \le c_1\|z_{1N}\|^2_{\mathcal{Y}}.$$

Repeat the above arguments, we can find a sequence $\{v^{z_{kN}}\}_{k=0}^{j-1} \subset \mathcal{U}^N$ with $z_{(k+1)N} := z_N(z_{kN}, v^{z_{kN}})$, such that, for each $k \in \{0, 1, \cdots, j\}$,

$$\|z_N(z_{kN}, v^{z_{kN}})\|^2_{\mathcal{Y}} \le \delta\|z_{kN}\|^2_{\mathcal{Y}}; \ \ \|v^{z_{kN}}\|^2_{\mathcal{U}^N} \le c_1\|z_{kN}\|^2_{\mathcal{Y}}. \tag{2.12}$$

Then, we define $\hat{v} = (\hat{v}_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N};\mathcal{U})$ as follows

$$\hat{v}_k := \begin{cases} v_k^{z_{0N}}, & \text{if } \ k \in [0, N), \\ \cdots\cdots, & \\ v_{k-(j-1)N}^{z_{(j-1)N}}, & \text{if } \ k \in [(j-1)N, jN), \\ 0, & \text{if } \ k \in [jN, +\infty). \end{cases} \tag{2.13}$$

We see that $z_{kN}(y_0,\hat{v}) = z_{kN}$, for each $k \in \{1, 2, \cdots, j\}$. Moreover, it follows from (2.12) and (2.13) that

$$\sum_{k=0}^{jN} \|\hat{v}_k\|_{\mathcal{U}}^2 = \sum_{i=0}^{j-1} \|v^{z_{iN}}\|_{\mathcal{U}^N}^2 \le c_1 \sum_{i=0}^{j-1} \delta^i \|y_0\|_{\mathcal{Y}}^2 \le \frac{c_1}{1-\delta} \|y_0\|_{\mathcal{Y}}^2. \tag{2.14}$$

Notice that, for $kN < i \le (k+1)N$, where $k \in \{0, 1, \ldots, j-1\}$,

$$\begin{aligned}
\|z_i(y_0,\hat{v})\|_{\mathcal{Y}}^2 &= \|(\mathrm{e}^{\beta}\Phi)^{i-kN} z_{kN}(y_0,\hat{v}) + \sum_{l=kN}^{i-1} (\mathrm{e}^{\beta}\Phi)^{i-1-l}\mathrm{e}^{\beta}\Lambda\hat{v}_l\|_{\mathcal{Y}}^2 \\
&\le 2\|(\mathrm{e}^{\beta}\Phi)^{i-kN} z_{kN}(y_0,\hat{v})\|_{\mathcal{Y}}^2 + 2(i-kN-1)\sum_{l=kN}^{i-1} \|(\mathrm{e}^{\beta}\Phi)^{i-1-l}\mathrm{e}^{\beta}\Lambda\hat{v}_l\|_{\mathcal{Y}}^2 \\
&\le 2\mathrm{e}^{2\beta N}(\|\Phi\|_{\mathcal{L}(\mathcal{Y})}^2)^{i-kN}\|z_{kN}(y_0,\hat{v})\|_{\mathcal{Y}}^2 + 2N\mathrm{e}^{2\beta N}\|\Lambda\|_{\mathcal{L}(\mathcal{U};\mathcal{Y})}^2 \sum_{l=0}^{i-kN-1} \|\Phi\|_{\mathcal{L}(\mathcal{Y})}^{2l} \sum_{l=kN}^{i-1} \|\hat{v}_l\|_{\mathcal{U}}^2 \\
&\le c_2\|z_{kN}(y_0,\hat{v})\|_{\mathcal{Y}}^2 + c_3 \sum_{l=kN}^{(k+1)N-1} \|\hat{v}_l\|_{\mathcal{U}}^2,
\end{aligned}$$

where $c_2 = 2\mathrm{e}^{2\beta N}(1 + \|\Phi\|_{\mathcal{L}(\mathcal{Y})}^2)^N$, and $c_3 = 2N\mathrm{e}^{2\beta N}\|\Lambda\|_{\mathcal{L}(\mathcal{U};\mathcal{Y})}^2 \sum_{l=0}^{N} \|\Phi\|_{\mathcal{L}(\mathcal{Y})}^{2l}$. We obtain from the latter inequality and (2.12), (2.13) and (2.14) that

$$\begin{aligned}
\sum_{i=1}^{jN} \|z_i(y_0,\hat{v})\|_{\mathcal{Y}}^2 &= \sum_{k=0}^{j-1} \sum_{l=kN+1}^{(k+1)N} \|z_l(y_0,\hat{v})\|_{\mathcal{Y}}^2 \\
&\le c_2 N \sum_{k=0}^{j-1} \|z_{kN}(y_0,\hat{v})\|_{\mathcal{Y}}^2 + c_3 N \sum_{k=0}^{jN} \|\hat{v}_k\|_{\mathcal{U}}^2 \\
&\le N\frac{c_2 + c_1 c_3}{1-\delta}\|y_0\|_{\mathcal{Y}}^2.
\end{aligned} \tag{2.15}$$

This proves inequality (2.8) with $M = 1 + \frac{N(c_2+c_1c_3)}{1-\delta} + \frac{c_1}{1-\delta}$. Hence the problem (**d-LQ**) is solvable.

Step 2. We prove that (2.1) is β-stabilizable. Since the problem (**d-LQ**) is solvable, there is $F \in \mathcal{L}(\mathcal{Y};\mathcal{U})$ such that

$$\mathtt{r}(\mathrm{e}^{\beta}\Phi + \mathrm{e}^{\beta}\Lambda F) < 1, \tag{2.16}$$

where $\mathtt{r}(\mathrm{e}^{\beta}\Phi + \mathrm{e}^{\beta}\Lambda F)$ is the spectral radius of $\mathrm{e}^{\beta}\Phi + \mathrm{e}^{\beta}\Lambda F$. (The detailed proof can be found in the proof of Thm. 1.6 in [7].) This implies that, there is $n^* \in \mathbb{N}$ and $r_0 \in (0,1)$, such that the solution to equation (2.7), with the feedback control $\tilde{v} = (\tilde{v}_k)_{k\in\mathbb{N}} := (Fz_k)_{k\in\mathbb{N}}$, satisfies

$$\|z_k(y_0,\tilde{v})\|_{\mathcal{Y}} = \|\mathrm{e}^{\beta k}(\Phi + \Lambda F)^k y_0\|_{\mathcal{Y}} \le r_0^k \|y_0\|_{\mathcal{Y}} \text{ for any } k \ge n^*. \tag{2.17}$$

Let $\tilde{y}_k = \mathrm{e}^{-\beta k} z_k(y_0,\tilde{v})$ and $\tilde{u}_k = \mathrm{e}^{-\beta k}\tilde{v}_k$. Then we have that $\tilde{u}_k = F\tilde{y}_k,\ k \in \mathbb{N}$. Moreover, $(\tilde{y}_k, \tilde{u}_k)_{k\in\mathbb{N}}$ satisfies the equation (2.1), and

$$\|\tilde{y}_k\|_{\mathcal{Y}} \le \mathrm{e}^{-\beta k} r_0^k \|y_0\|_{\mathcal{Y}} \text{ for any } k \ge n^*. \tag{2.18}$$

As a consequence, we infer that

$$\|\tilde{y}_k\|_{\mathcal{Y}} \leq \begin{cases} (1+\|\Phi+\Lambda F\|_{\mathcal{L}(\mathcal{Y})})^{n^*}\mathrm{e}^{(\beta+\varepsilon_0)n^*}\mathrm{e}^{-(\beta+\varepsilon_0)k}\|y_0\|_{\mathcal{Y}}, & \text{if } k < n^*, \\ \mathrm{e}^{-(\beta+\varepsilon_0)k}\|y_0\|_{\mathcal{Y}}, & \text{if } k \geq n^*, \end{cases} \tag{2.19}$$

where $\varepsilon_0 := -\ln r_0 > 0$. This leads to the β-stabilizability of the system (2.1), and completes the proof of Theorem 2.2. □

2.2. Stabilizability of linear sampled-data control system

In this subsection, we apply Theorem 2.2 to study the stabilizability of general linear sampled-data control system $[A, B]$. Throughout this subsection, we suppose A satisfies the condition (H_1), but the operator B is not necessary to satisfy the condition (H_2). Instead, we impose the following more general condition

$(\tilde{H}_2)$ $B \in \mathcal{L}(U; Y_{-1})$.

We give firstly the definition of β-stabilizability of $[A, B]$.

Definition 2.3. The sampled-data control system $[A, B]$ is called β-stabilizable if there is $F \in \mathcal{L}(Y; U)$, $c > 0$ and $\varepsilon > 0$ such that the solution to the closed-loop system

$$y'(t) = Ay(t) + \sum_{k=0}^{\infty} \chi_{[kT,(k+1)T)}(t) BFy(kT), \quad t > 0 \tag{2.20}$$

satisfies

$$\|y(t)\|_Y \leq c\mathrm{e}^{-(\beta+\varepsilon)t}\|y(0)\|_Y, \quad \forall t > 0. \tag{2.21}$$

For the β-stabilizability of the control system $[A, B]$, we have the following characterization.

Theorem 2.4. *Suppose (H_1) and $(\tilde{H}_2)$ hold true. The following statements are equivalent:*

(i) The sampled-data control system $[A, B]$ is β-stabilizable;
(ii) The following discrete control system is βT-stabilizable

$$y_{k+1} = \Phi_T y_k + D_T u_k, \quad k = 0, 1, 2, \cdots, \tag{2.22}$$

where the operators Φ_T and D_T are defined by:

$$\Phi_T := \mathrm{e}^{AT}; \quad D_T := \int_0^T \mathrm{e}^{At} B \mathrm{d}t; \tag{2.23}$$

(iii) There are constants $N \in \mathbb{N}^+$, $\delta \in (0, 1)$, and $c > 0$ such that

$$\|\mathrm{e}^{A^*(NT)}\varphi\|_Y^2 \leq c \sum_{k=0}^{N-1} \Big\| \int_{kT}^{(k+1)T} B^* \mathrm{e}^{A^* t}\varphi \mathrm{d}t \Big\|_U^2 + \delta \mathrm{e}^{-2\beta NT}\|\varphi\|_Y^2 \text{ for any } \varphi \in Y. \tag{2.24}$$

Proof. It is clear that $\Phi_T \in \mathcal{L}(Y)$, and it follows by $(\tilde{H}_2)$ that $D_T \in \mathcal{L}(U; Y)$ (see Sect. 2.3 in [1]). The equivalence between (ii) and (iii) follows from Theorem 2.2. We now show the equivalence between (i) and (ii).

$(i) \Rightarrow (ii)$ Suppose that $[A, B]$ is β-stabilizable. Then there is a feedback operator $F \in \mathcal{L}(Y;U)$, and constants $c > 0, \varepsilon > 0$ such that the solution to the system (2.20) satisfies (2.21). It can be verified that

$$y(kT) = (\Phi_T + D_T F)^k y(0), \quad \forall k \in \mathbb{N}.$$

This together with (2.21) implies that

$$\|(\Phi_T + D_T F)^k\|_{\mathcal{L}(Y)} \le c\mathrm{e}^{-(\beta+\varepsilon)kT}, \quad k \in \mathbb{N}.$$

By Definition 2.1 we see that (2.22) is $\beta T-$ stabilizable.

$(ii) \Rightarrow (i)$ Suppose that the system (2.22) is βT-stabilizable. Then, there is $F \in \mathcal{L}(Y;U)$ and constants $c_0 > 0, \varepsilon > 0$ such that

$$\|(\Phi_T + D_T F)^k\|_{\mathcal{L}(Y)} \le c_0\mathrm{e}^{-(\beta+\varepsilon)kT}, \quad k \in \mathbb{N}. \tag{2.25}$$

Taking $u(t) = Fy(t)$ in system $[A, B]$, and denoting the solution to the closed system with initial data $y(0) = y_0$ by $y_F(t; y_0)$ ($y_0 \in Y$ is arbitrarily given). Then it follows by (2.25) that

$$\|y_F(kT; y_0)\|_Y = \|(\Phi_T + D_T F)^k y_0\|_Y \le c_0\mathrm{e}^{-(\beta+\varepsilon)kT}\|y_0\|_Y, \quad \forall k \in \mathbb{N}. \tag{2.26}$$

For an arbitrarily fixed $m^* \in \mathbb{N}$, we can directly check that when $t \in [m^*T, (m^*+1)T)$,

$$\|y_F(t; y_0)\|_Y = \Big\|\Big(\mathrm{e}^{A(t-m^*T)} + \int_{m^*T}^t \mathrm{e}^{A(t-s)}BF\mathrm{d}s\Big)y_F(m^*T; y_0)\Big\|_Y, \tag{2.27}$$

and by Theorem A.1, we know that there is $c(T) > 0$ such that

$$\sup_{t\in[0,T]} \|\int_0^t \mathrm{e}^{A(t-s)}B\mathrm{d}s\|_{\mathcal{L}(U;Y)} \le c(T). \tag{2.28}$$

It follows by (2.27) and (2.28) that

$$\|y_F(t; y_0)\|_Y \le c(F)\|y_F(m^*T; y_0)\|_Y, \tag{2.29}$$

where $c(F) := \sup_{t\in[0,T]} \|\mathrm{e}^{At}\|_{\mathcal{L}(Y)} + c(T)\|F\|_{\mathcal{L}(Y;U)}$. By (2.26) and (2.29), we obtain that for any $t \in [m^*T, (m^*+1)T)$,

$$\|y_F(t; y_0)\|_Y \le c_0 c(F)\mathrm{e}^{(\beta+\varepsilon)T}\mathrm{e}^{-(\beta+\varepsilon)t}\|y_0\|_Y. \tag{2.30}$$

This leads to the β-stabilizability of the system $[A, B]$. The proof is completed. □

2.3. Detectability of the observation system

In this subsection, we apply Theorem 2.2 to study the detectability of the system $[C, A]$:

$$\begin{cases} y'(t) = Ay(t), & t \in \mathbb{R}^+ \setminus \{kT\}_{k\in\mathbb{N}^+}, \\ z_k = Cy(kT), & k \in \mathbb{N}^+. \end{cases} \tag{2.31}$$

Throughout this subsection, we suppose A satisfies the condition (H_1), but the operator C is not necessary to satisfy the condition (H_3). Instead, we impose the following more general condition
$(\tilde{H}_3)$ $C\mathrm{e}^{AT} \in \mathcal{L}(Y;Z)$.

We first give the definition of β-detectability of the system $[C, A]$.

Definition 2.5. The system $[C, A]$ is called β-detectable if there is $L \in \mathcal{L}(Z;Y)$, $c > 0$ and $\varepsilon > 0$ such that any solution to the following system

$$\begin{cases} z'(t) = Az(t), \quad t \in \mathbb{R}^+ \setminus \{kT\}_{k\in\mathbb{N}^+}, \\ z(kT) = z(kT-) + LCz(kT-), \quad k \in \mathbb{N}^+, \end{cases} \tag{2.32}$$

satisfies

$$\|z(t)\|_Y \le c\mathrm{e}^{-(\beta+\varepsilon)t}\|z(0)\|_Y, \quad \forall t > 0. \tag{2.33}$$

For the β-detectability of $[C, A]$, we have the following result.

Theorem 2.6. *Suppose that (H_1) and $(\tilde{H}_3)$ hold true. The observation system $[C, A]$ is β-detectable if and only if there are constants $N \in \mathbb{N}^+$, $\delta \in (0, 1)$, and $c > 0$ such that*

$$\|\mathrm{e}^{A(NT)}\varphi\|_Y^2 \le c\sum_{k=1}^{N}\|C\mathrm{e}^{AkT}\varphi\|_Z^2 + \delta\mathrm{e}^{-2\beta NT}\|\varphi\|_Y^2 \ \textit{ for any } \ \varphi \in Y. \tag{2.34}$$

Proof. We first prove that the β-detectability of $[C, A]$ is equivalent to the βT-stabilizability of the following discrete control system

$$y_{k+1} = \Phi_T^* y_k + \Phi_T^* C^* u_k. \tag{2.35}$$

Here Φ_T^* is the adjoint operator of Φ_T which is defined in (2.23). The control space is Z, and $u_k \in Z$ for $k \in \mathbb{N}$. By Assumption $(\tilde{H}_3)$, we see that $\Phi_T^* C^* \in \mathcal{L}(Z;Y)$.

Suppose that $[C, A]$ is β-detectable. Then there is $L \in \mathcal{L}(Z;Y)$, $c > 0$ and $\varepsilon > 0$ such that the solution to (2.32) satisfies

$$\|z(kT)\|_Y \le c\mathrm{e}^{-(\beta+\varepsilon)kT}\|z(0)\|_Y, \quad \forall k \in \mathbb{N}. \tag{2.36}$$

Notice that $z(kT) = (\Phi_T + LC\Phi_T)^k z(0)$ for arbitrarily given $z(0) \in Y$, we see from (2.36) that

$$\|(\Phi_T + LC\Phi_T)^k\|_{\mathcal{L}(Y)} \le c\mathrm{e}^{-(\beta+\varepsilon)kT}, \quad \forall k \in \mathbb{N}.$$

This implies that

$$\|(\Phi_T^* + \Phi_T^* C^* L^*)^k\|_{\mathcal{L}(Y)} \le c\mathrm{e}^{-(\beta+\varepsilon)kT}, \quad \forall k \in \mathbb{N}. \tag{2.37}$$

From (2.37) and Definition 2.1 that the system (2.35) is βT-stabilizable with feedback $F = L^*$.

Now, suppose that the system (2.35) is βT-stabilizable. Then there is $F \in \mathcal{L}(Y;Z)$, $c > 0$ and $\varepsilon > 0$ such that

$$\|(\Phi_T^* + \Phi_T^* C^* F)^k\|_{\mathcal{L}(Y)} \le c\mathrm{e}^{-(\beta+\varepsilon)kT}, \quad \forall k \in \mathbb{N}.$$

This implies that

$$\|(\Phi_T + F^* C\Phi_T)^k\|_{\mathcal{L}(Y)} \le c\mathrm{e}^{-(\beta+\varepsilon)kT}, \quad \forall k \in \mathbb{N}. \tag{2.38}$$

Taking $L = F^*$ in (2.32), we can infer that the solution to the equation (2.32) with initial condition $z(0) = z_0$ satisfies ($z_0 \in Y$ is arbitrarily given)

$$\|z(kT)\|_Y = \|(\Phi_T + F^* C\Phi_T)^k z_0\|_Y \leq c\mathrm{e}^{-(\beta+\varepsilon)kT}\|z_0\|_Y, \ \ \forall k \in \mathbb{N}. \tag{2.39}$$

For any $t > 0$, there is $m^* \in \mathbb{N}$, such that $t \in [m^*T, (m^*+1)T)$. One can see that

$$\|z(t)\|_Y = \|\mathrm{e}^{A(t-kT)}z(kT)\|_Y \leq \tilde{c}\mathrm{e}^{-(\beta+\varepsilon)t}\|z_0\|_Y, \tag{2.40}$$

where $\tilde{c} = c\sup_{t\in[0,T]}\|\mathrm{e}^{At}\|_{\mathcal{L}(Y)}\mathrm{e}^{(\beta+\varepsilon)T}$.

From above, we see that the β-detectability of $[C, A]$ is equivalent to the βT-stabilizability of (2.35). By Theorem 2.2, (2.35) is βT-stabilizable if and only if there are constants $N \in \mathbb{N}^+$, $\delta \in (0, 1)$, and $c > 0$ such that (2.34) holds. Hence, the proof is completed. □

2.4. Proof of Theorem 1.3

In this subsection, we give the proof of Theorem 1.3 based on results obtained in the previous two subsections.

Proof of Theorem 1.3. We firstly prove the sufficiency. Suppose that there are constants $N_1, N_2 \in \mathbb{N}^+$, $\delta_1, \delta_2 \in (0, 1)$, and $c_1, c_2 \geq 0$ such that the inequalities (1.14) and (1.15) hold. Then, by Theorem 2.4 and Theorem 2.6, we see that $[A, B]$ is β-stabilizable and $[C, A]$ is β-detectable. Therefore, by Theorem 2.4 and Definition 2.1, there is $F \in \mathcal{L}(Y; U)$, $\tilde{c}_1 > 0$ and $\varepsilon_1 > 0$, such that

$$\|(\Phi_T + D_T F)^k\|_{\mathcal{L}(Y)} \leq \tilde{c}_1\mathrm{e}^{-(\beta+\varepsilon_1)kT}, \ \ \forall k \in \mathbb{N}^+, \tag{2.41}$$

and by Definition 2.5, there is $L \in \mathcal{L}(Z; Y)$, $\tilde{c}_2 > 0$ and $\varepsilon_2 > 0$, such that the solution to the equation (2.32) satisfies

$$\|z(t)\|_Y \leq \tilde{c}_2\mathrm{e}^{-(\beta+\varepsilon_2)t}\|z(0)\|_Y, \ \ \forall t > 0. \tag{2.42}$$

Now, we are going to prove that, there is $c > 0$ and $\varepsilon > 0$, such that, $y(\cdot)$ and $w(\cdot)$, which are the respective solutions to equation (1.8) and (1.9) with the above chosen operators F and L, satisfy (1.10). With this be done, we see from Definition 1.2 that the system $[A, B, C]$ is β-output feedback stabilizable.

Letting $z(t) = y(t) - w(t), t \geq 0$. Then, z satisfies the equation (2.32), and $z(0) = y(0)$. The equation (1.8) can be rewritten as

$$y'(t) = Ay(t) + \sum_{k=0}^{\infty}\chi_{[kT,(k+1)T)}(t)BF[y(kT) - z(kT)]. \tag{2.43}$$

By simple calculation, we see from (2.43) that

$$y(kT) = (\Phi_T + D_T F)^k y(0) - \sum_{j=0}^{k-1}(\Phi_T + D_T F)^j D_T F z((k-1-j)T). \tag{2.44}$$

Letting $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\}/2$. Then, it follows by (2.41), (2.42) and (2.44) that

$$\|y(kT)\|_Y \leq \|(\Phi_T + D_T F)^k\|_{\mathcal{L}(Y)}\|y(0)\|_Y + \sum_{j=0}^{k-1}\|(\Phi_T + D_T F)^j\|_{\mathcal{L}(Y)}\|D_T F\|_{\mathcal{L}(Y)}\|z((k-1-j)T)\|_Y$$

$$
\begin{aligned}
&\le \tilde{c}_1 \mathrm{e}^{-(\beta+\varepsilon_1)kT}\|y(0)\|_Y + \tilde{c}_1\tilde{c}_2\|D_TF\|_{\mathcal{L}(Y)} \sum_{j=0}^{k-1} \mathrm{e}^{-(\beta+2\varepsilon_3)jT}\mathrm{e}^{-(\beta+\varepsilon_3)(k-1-j)T}\|y(0)\|_Y \\
&\le \tilde{c}_1 \mathrm{e}^{-(\beta+\varepsilon_1)kT}\|y(0)\|_Y + \tilde{c}_1\tilde{c}_2\|D_TF\|_{\mathcal{L}(Y)}\mathrm{e}^{-(\beta+\varepsilon_3)(k-1)T} \sum_{j=0}^{k-1} \mathrm{e}^{-\varepsilon_3 jT}\|y(0)\|_Y \\
&\le \left(\tilde{c}_1 + \tilde{c}_1\tilde{c}_2\|D_TF\|_{\mathcal{L}(Y)}\mathrm{e}^{(\beta+\varepsilon_3)T}\frac{1}{1-\mathrm{e}^{-\varepsilon_3 T}}\right)\mathrm{e}^{-(\beta+\varepsilon_3)kT}\|y(0)\|_Y. \qquad (2.45)
\end{aligned}
$$

Notice that $\mathrm{e}^{\varepsilon_3 T} - 1 > \varepsilon_3 T$, we see from above that

$$
\|y(kT)\|_Y \le \tilde{c}_3 \mathrm{e}^{-(\beta+\varepsilon_3)kT}\|y(0)\|_Y, \quad \forall k \in \mathbb{N}. \qquad (2.46)
$$

where $\tilde{c}_3 = \tilde{c}_1 + \frac{1}{\varepsilon_3 T}\mathrm{e}^{(\beta+2\varepsilon_3)T}\tilde{c}_1\tilde{c}_2\|D_TF\|_{\mathcal{L}(Y)}$.

For any given $t > 0$, there is $k^* \in \mathbb{N}$, such that $t \in [k^*T, (k^*+1)T)$, and we can see from (2.43) that

$$
y(t) = \mathrm{e}^{A(t-k^*T)}y(k^*T) + \int_{k^*T}^{t} \mathrm{e}^{A(t-s)}\mathrm{d}sBF[y(k^*T) - z(k^*T)]. \qquad (2.47)
$$

By (2.42), (2.46) and (2.47), we can infer that

$$
\|y(t)\|_Y \le \tilde{c}_4 \mathrm{e}^{-(\beta+\varepsilon_3)t}\|y(0)\|_Y, \qquad (2.48)
$$

where $\tilde{c}_4 = \mathrm{e}^{(\beta+\varepsilon_3)T}(\tilde{c}_3 \sup_{t\in[0,T]}\|\mathrm{e}^{At}\|_{\mathcal{L}(Y)} + (\tilde{c}_2+\tilde{c}_3)\sup_{t\in(0,T]}\|\int_0^t \mathrm{e}^{As}\mathrm{d}sBF\|_{\mathcal{L}(Y)})$. It follows by (2.48) and (2.42) that

$$
\|w(t)\|_Y \le \tilde{c}_5 \mathrm{e}^{-(\beta+\varepsilon_4)t}\|y(0)\|_Y, \qquad (2.49)
$$

where $\tilde{c}_5 = \tilde{c}_4 + \tilde{c}_2$, $\varepsilon_4 = \min\{\varepsilon_2, \varepsilon_3\}$. Finally, we see from (2.48) and (2.49) that the inequality (1.10) holds with $c = \tilde{c}_5 + \tilde{c}_4, \varepsilon = \varepsilon_4$.

The rest of the proof is devoted to show the necessity. Suppose that the system $[A, B, C]$ is β-output feedback stabilizable. By Definition 1.2, we see that there is $c > 0$, $\varepsilon > 0$, $F \in \mathcal{L}(Y; U)$ and $L \in \mathcal{L}(Z; Y)$ such that the inequality (1.10) holds. Let $z(t) = y(t) - w(t), t \ge 0$. Then, z satisfies the equation (2.32). Moreover, it follows by (1.10) that $\|z(t)\|_Y \le c\mathrm{e}^{-(\beta+\varepsilon)t}\|z(0)\|_Y, \forall t > 0$. Hence $[C, A]$ is β-detectable. It follow by Theorem 2.6 that the inequality (1.15) holds with some $N_2 \in \mathbb{N}^+$, $\delta_2 \in (0,1)$, and $c_2 \ge 0$. To show the inequality (1.14), we take $N_1 = \lfloor \frac{\ln[2(\max\{1,c\})^2]}{\varepsilon T} \rfloor + 1$, and write $y(N_1T)$ as

$$
y(N_1T) = \Phi_T^{N_1}y(0) + \sum_{k=0}^{N_1-1}\Phi_T^{N_1-k-1}D_TFw(kT).
$$

Based on above, we find that, for each $\varphi \in Y$,

$$
\begin{aligned}
\langle y(N_1T), \varphi\rangle_Y &= \langle \Phi_T^{N_1}y(0), \varphi\rangle_Y + \langle \sum_{k=0}^{N_1-1}\Phi_T^{N_1-k-1}D_TFw(kT), \varphi\rangle_Y \\
&= \langle y(0), (\Phi_T^*)^{N_1}\varphi\rangle_Y + \sum_{k=0}^{N_1-1}\langle Fw(kT), D_T^*(\Phi_T^*)^{N_1-k-1}\varphi\rangle_U.
\end{aligned}
$$

We obtain from (1.10) that

$$\begin{aligned}\langle y(0),(\Phi_T^*)^{N_1}\varphi\rangle_Y &\le \sum_{k=0}^{N_1-1}\|F\|_{\mathcal{L}(Y;U)}c\mathrm{e}^{-(\beta+\varepsilon)kT}\|y(0)\|_Y\|D_T^*(\Phi_T^*)^{N_1-k-1}\varphi\|_U\\ &\quad+c\mathrm{e}^{-(\beta+\varepsilon)N_1T}\|y(0)\|_Y\|\varphi\|_Y\\ &\le \|F\|_{\mathcal{L}(Y;U)}c(\sum_{k=0}^{N_1-1}\mathrm{e}^{-2(\beta+\varepsilon)kT})^{\frac12}(\sum_{k=0}^{N_1-1}\|D_T^*(\Phi_T^*)^k\varphi\|_U^2)^{\frac12}\|y(0)\|_Y\\ &\quad+c\mathrm{e}^{-(\beta+\varepsilon)N_1T}\|y(0)\|_Y\|\varphi\|_Y.\end{aligned}$$

Notice that $\delta_1 := 2c^2\mathrm{e}^{-2\varepsilon N_1T} < 1$, and $y(0)\in Y$ is arbitrarily given. The above leads to (1.14) with

$$N_1 = \lfloor\frac{\ln[2(\max\{c,1\})^2]}{\varepsilon T}\rfloor+1;\quad c_1 = 2\|F\|^2_{\mathcal{L}(Y;U)}c^2\sum_{k=0}^{N_1-1}\mathrm{e}^{-2(\beta+\varepsilon)kT};\quad \delta_1 = 2c^2\mathrm{e}^{-2\varepsilon N_1T}.$$

□

3. Proof of Theorem 1.5

In this section, we give the proof of Theorem 1.5. In Section 3.1, we present two preliminary results, based on which we are able to apply Theorem 1.3 to prove Theorem 1.5. The proofs of the two preliminary results are given respectively in Section 3.2 and Section 3.3.

3.1. Two preliminary results and the proof of Theorem 1.5

To prove Theorem 1.5, we give firstly the following two results, which will be proven in Section 3.1 and Section 3.2 respectively.

Theorem 3.1. *Let $\beta > 0$. Suppose that $(H_1)-(H_3)$, $(A0),(A1),(A2),(A3)$ and $(A5)$ hold. Then there are constants $N_1\in\mathbb{N}^+$, $\delta_1\in(0,1)$, and $c_1\ge 0$ such that for each $\varphi\in\mathbf{Y}^n$,*

$$\|\mathrm{e}^{(\mathbf{A}^*+S^\top)(N_1T)}\varphi\|^2_{\mathbf{Y}^n}\le c_1\sum_{k=0}^{N_1-1}\Big\|\int_{kT}^{(k+1)T}\mathbf{B}^*D^\top\mathrm{e}^{(\mathbf{A}^*+S^\top)t}\varphi\mathrm{d}t\Big\|^2_{\mathbf{U}^m}+\delta_1\mathrm{e}^{-2\beta N_1T}\|\varphi\|^2_{\mathbf{Y}^n}. \tag{3.1}$$

Theorem 3.2. *Let $\beta > 0$. Suppose that $(H_1)-(H_3)$, $(A0),(A1)$, $(A4)$ and $(A5)$ hold. Then there are constants $N_2\in\mathbb{N}^+$, $\delta_2\in(0,1)$, and $c_2\ge 0$ such that for each $\varphi\in\mathbf{Y}^n$,*

$$\|\mathrm{e}^{(\mathbf{A}+S)(N_2T)}\varphi\|^2_{\mathbf{Y}^n}\le c_2\sum_{k=1}^{N_2}\|Q\mathbf{C}\mathrm{e}^{(\mathbf{A}+S)kT}\varphi\mathrm{d}t\|^2_{\mathbf{Z}^p}+\delta_2\mathrm{e}^{-2\beta N_2T}\|\varphi\|^2_{\mathbf{Y}^n}. \tag{3.2}$$

Remark 3.3. By Theorem 2.4 and Theorem 3.1, we see that if $(H_1)-(H_3)$, $(A0),(A1),(A2),(A3)$ and $(A5)$ hold, then $[\mathbf{A}+S, D\mathbf{B}]$ is β-stabilizable. By Theorem 2.6 and Theorem 3.2, we see that if $(H_1)-(H_3)$, $(A0)$, $(A1)$, $(A4)$ and $(A5)$ hold, then $[Q\mathbf{C},\mathbf{A}+S]$ is β-detectable.

Now, we give the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $\beta > 0$. Suppose that $(H_1)-(H_3)$, $(A0)-(A5)$ hold. By Theorems 3.1, 3.2 and 1.3, we see that $[\mathbf{A}+S, D\mathbf{B}, Q\mathbf{C}]$ is β-output feedback stabilizable. □

3.2. Proof of Theorem 3.1

In this subsection, we shall give the proof of Theorem 3.1. We only prove Theorem 3.1 when $[S, D]$ is not controllable. Another case when $[S, D]$ is controllable can be treated in the same way.

By $(A3)$ we know that, if $[S, D]$ is not controllable, then

$$\sigma(S_3) \subset \mathbb{C} \setminus \mathbb{C}^+_{-\lambda_0-\beta}. \tag{3.3}$$

Here, the matrix $S_3 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$ is given by the Kalman decomposition:

$$J^{-1}SJ = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}; \quad J^{-1}D = \begin{pmatrix} D_1 \\ 0 \end{pmatrix}, \tag{3.4}$$

where $J \in \mathbb{R}^{n\times n}$ is invertible, $S_1 \in \mathbb{R}^{n_1\times n_1}, D_1 \in \mathbb{R}^{n_1\times m}, S_2 \in \mathbb{R}^{n_1\times(n-n_1)}$, and $[S_1, D_1]$ is controllable.

Before giving the proof of Theorem 3.1, we present some preliminary propositions, the first two of which can be directly verified, the third can be found in [8]. We give only the proof of the last proposition.

Proposition 3.4. *Let $R \in \mathbb{R}^{l\times l}$ with $l \in \mathbb{N}^+$. Then, for each $t > 0$,*

$$\mathrm{e}^{\mathbf{A}^*t}R = R\mathrm{e}^{\mathbf{A}^*t} \quad \text{and} \quad \mathrm{e}^{(\mathbf{A}^*+R)t} = \mathrm{e}^{\mathbf{A}^*t}\mathrm{e}^{Rt}. \tag{3.5}$$

Proposition 3.5. *Suppose that $(A0)$ and $(A1)$ hold. Let $d \in \mathbb{N}^+$. Let $\mathbf{P}^\beta : \mathbf{Y}^d \to \mathbf{Y}^d$ be defined by $\mathbf{P}^\beta\mathbf{y} := (P^\beta y_1, P^\beta y_2, \cdots, P^\beta y_d)^\top$ for $\mathbf{y} = (y_1, y_2, \cdots, y_d)^\top \in \mathbf{Y}^d$, where P^β is given in condition $(A0)$. Then $\mathbf{P}^\beta\mathrm{e}^{\mathbf{A}^*t} = \mathrm{e}^{\mathbf{A}^*t}\mathbf{P}^\beta$, and for each $\varphi \in \mathbf{Y}^d$, $\mathbf{P}^\beta\varphi \in (D(A))^d := \underbrace{D(A) \times \cdots \times D(A)}_{d}$. Furthermore, the following conclusions are true:*

(a)

$$\|(I - \mathbf{P}^\beta)\mathrm{e}^{\mathbf{A}^*t}\varphi\|_{\mathbf{Y}^d} \le \tilde{c}\mathrm{e}^{-\tilde{\alpha}t}\|\varphi\|_{\mathbf{Y}^d} \text{ for any } t \in \mathbb{R}^+ \text{and } \varphi \in \mathbf{Y}^d, \tag{3.6}$$

where $\tilde{c} > 0$ and $\tilde{\alpha}$ are the constants given in $(A0)$;

(b)

$$\|\mathbf{P}^\beta\varphi\|_{\mathbf{Y}^d} \le \tilde{d}\|\mathbf{B}^*\mathbf{P}^\beta\varphi\|_{\mathbf{U}^d} \text{ for any } \varphi \in \mathbf{Y}^d, \tag{3.7}$$

$$\|\mathbf{P}^\beta\varphi\|_{Y^d} \le \hat{d}\|\mathbf{C}\mathbf{P}^\beta\varphi\|_{Z^d} \text{ for any } \varphi \in \mathbf{Y}^d, \tag{3.8}$$

where $\tilde{d}$ and $\hat{d}$ are the constants given in $(A1)$.

Proposition 3.6. *Suppose that $(A3)$ and $(A5)$ hold. Let S_1, D_1 and n_1 be given in (3.4). Then the matrix $G := (D_1, \mathrm{e}^{S_1T}D_1, \cdots, \mathrm{e}^{S_1(n_1-1)T}D_1)$ is of full rank.*

Proposition 3.7. *Suppose that $(A2)$ and $(A3)$ hold. Then, for each $\mathbf{g} \in \mathbf{Y}^{n_1}_{P^\beta}$, there is a unique $\mathbf{f} \in \mathbf{Y}^{n_1}_{P^\beta}$ such that*

$$\int_0^T \mathrm{e}^{(\mathbf{A}+S_1^\top)t}\mathrm{d}t\mathbf{f} = \mathbf{g}. \tag{3.9}$$

Moreover, there is $c > 0$, such that $\|(\int_0^T \mathrm{e}^{(\mathbf{A}+S_1^\top)t}\mathrm{d}t)^{-1}\|_{\mathcal{L}(\mathbf{Y}^{n_1})} \le c$, where $(\int_0^T \mathrm{e}^{(\mathbf{A}+S_1^\top)t}\mathrm{d}t)^{-1}$ denotes the map $\mathbf{g} \mapsto \mathbf{f}$ defined above.

Proof. To prove Proposition 3.7, it suffices to prove that, for each $\mathbf{g} := (g_1, \cdots, g_{n_1}) \in \mathbf{Y}^{n_1}_{P^\beta}$, there is a unique $\mathbf{f} := (f_1, \cdots, f_{n_1}) \in \mathbf{Y}^{n_1}_{P^\beta}$ such that the equation (3.9) holds true, and there is $c > 0$ such that

$$\|\mathbf{f}\|_{\mathbf{Y}^{n_1}} \leq c\|\mathbf{g}\|_{\mathbf{Y}^{n_1}}. \tag{3.10}$$

To this aim, we let $K_1 \in \mathbb{R}^{n_1 \times n_1}$ be the non-singular matrix such that $\hat{S}_1 = K_1^{-1} S_1^\top K_1$ is the Jordan form of $S_1^\top$, *i.e.*,

$$\hat{S}_1 = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix},$$

where each J_k is the Jordan block (of size $m_k \times m_k$) corresponding to eigenvalue μ_k. We write $\hat{\mathbf{f}} := K_1^{-1}\mathbf{f}, \hat{\mathbf{g}} := K_1^{-1}\mathbf{g}$. Then, the equation (3.9) can be written as

$$\int_0^T \mathrm{e}^{(\mathbf{A}+\hat{S}_1)t}\mathrm{d}t\hat{\mathbf{f}} = \hat{\mathbf{g}}. \tag{3.11}$$

Write $\hat{\mathbf{f}} = (\hat{\mathbf{f}}_1, \cdots, \hat{\mathbf{f}}_r)^\top$, $\hat{\mathbf{g}} = (\hat{\mathbf{g}}_1, \cdots, \hat{\mathbf{g}}_r)^\top$, where $\hat{\mathbf{f}}_k, \hat{\mathbf{g}}_k \in \mathbf{Y}^{m_k}_{P^\beta}$ for each $k \in \{1, \cdots, r\}$. Then (3.11) is equivalent to the following sequence of equations

$$\int_0^T \mathrm{e}^{(\mathbf{A}+J_k)t}\mathrm{d}t\hat{\mathbf{f}}_k = \hat{\mathbf{g}}_k, k \in \{1, \cdots, r\}. \tag{3.12}$$

For $k = 1$, the equation (3.12) can be written as

$$\begin{pmatrix} \int_0^T \mathrm{e}^{(A+\mu_1 I)t}\mathrm{d}t & & \cdots & \int_0^T \mathrm{e}^{(A+\mu_1 I)t}\frac{t^{m_1-1}}{(m_1-1)!}\mathrm{d}t \\ & \ddots & & \\ & & \int_0^T \mathrm{e}^{(A+\mu_1 I)t}\mathrm{d}t & \int_0^T \mathrm{e}^{(A+\mu_1 I)t}t\mathrm{d}t \\ & & & \int_0^T \mathrm{e}^{(A+\mu_1 I)t}\mathrm{d}t \end{pmatrix} \begin{pmatrix} \hat{f}_1 \\ \vdots \\ \hat{f}_{m_1-1} \\ \hat{f}_{m_1} \end{pmatrix} = \begin{pmatrix} \hat{g}_1 \\ \vdots \\ \hat{g}_{m_1-1} \\ \hat{g}_{m_1} \end{pmatrix}, \tag{3.13}$$

where $\hat{\mathbf{f}}_1 = (\hat{f}_1, \cdots, \hat{f}_{m_1})^\top, \hat{\mathbf{g}}_1 = (\hat{g}_1, \cdots, \hat{g}_{m_1})^\top$. By Assumption $(A2)$ we know that there is a unique $\hat{f}_{m_1} = (\int_0^T \mathrm{e}^{(A+\mu_1 I)t}\mathrm{d}t)^{-1}\hat{g}_{m_1} \in Y_{P^\beta}$ such that

$$\int_0^T \mathrm{e}^{(A+\mu_1 I)t}\mathrm{d}t\hat{f}_{m_1} = \hat{g}_{m_1}.$$

Then, we can find a unique $\hat{f}_{m_1-1} \in Y_{P^\beta}$ such that

$$\int_0^T \mathrm{e}^{(A+\mu_1 I)t}\mathrm{d}t\hat{f}_{m_1-1} + \int_0^T \mathrm{e}^{(A+\mu_1 I)t}t\mathrm{d}t\hat{f}_{m_1} = \hat{g}_{m_1-1},$$

which can be explicitly written as

$$\hat{f}_{m_1-1} = (\int_0^T \mathrm{e}^{(A+\mu_1 I)t}\mathrm{d}t)^{-1}[\hat{g}_{m_1-1} - \int_0^T \mathrm{e}^{(A+\mu_1 I)t}t\mathrm{d}t(\int_0^T \mathrm{e}^{(A+\mu_1 I)t}\mathrm{d}t)^{-1}\hat{g}_{m_1}].$$

Step by step, we can find $\hat{f}_{m_1}, \cdots, \hat{f}_1$ satisfy (3.13). Moreover, by Assumption $(A2)$, there is $c_1 > 0$ such that

$$\|\hat{\mathbf{f}}_1\|^2_{\mathbf{Y}^{m_1}} = \sum_{k=1}^{m_1} \|\hat{f}_k\|^2_Y \le c_1 \sum_{k=1}^{m_1} \|\hat{g}_k\|^2_Y = c_1 \|\hat{\mathbf{g}}_1\|^2_{\mathbf{Y}^{m_1}}. \tag{3.14}$$

By the same arguments as above, for each $k = 2, \cdots, r$, one can find unique $\hat{\mathbf{f}}_k \in \mathbf{Y}^{m_k}_{P^\beta}$ satisfying (3.12). Moreover, there is $c_k > 0$ such that

$$\|\hat{\mathbf{f}}_k\|^2_{\mathbf{Y}^{m_k}} \le c_k \|\hat{\mathbf{g}}_k\|^2_{\mathbf{Y}^{m_k}}. \tag{3.15}$$

Hence, $\hat{\mathbf{f}} = (\hat{\mathbf{f}}_1, \cdots, \hat{\mathbf{f}}_r)^\top$ satisfies (3.11), and $\mathbf{f} = K_1 \hat{\mathbf{f}}$ satisfies (3.9). Moreover, (3.10) holds true because (3.14) and (3.15). □

Now we are in the position to prove Theorem 3.1.

Proof of Theorem 3.1. In the case when $[S, D]$ is not controllable, we have the Kalman decomposition (3.4). Given $\varphi \in \mathbf{Y}^n$, we write $\hat{\varphi} = J^\top \varphi = (\hat{\varphi}_1, \hat{\varphi}_2)^\top$, where $\hat{\varphi}_1 \in \mathbf{Y}^{n_1}, \hat{\varphi}_2 \in \mathbf{Y}^{n-n_1}$. Let $N_1 \in \mathbb{N}^+ \cap [n_1, +\infty)$, which will be determined later. By Proposition 3.4, we see that

$$\begin{aligned}
\mathrm{e}^{(\mathbf{A}^* + S^\top)(N_1 T)} \varphi &= (J^\top)^{-1} J^\top \mathrm{e}^{\mathbf{A}^* N_1 T} (J^\top)^{-1} J^\top \mathrm{e}^{S^\top N_1 T} (J^\top)^{-1} J^\top \varphi \\
&= (J^\top)^{-1} \mathrm{e}^{\mathbf{A}^* N_1 T} J^\top (J^\top)^{-1} \mathrm{e}^{(J^{-1} S J)^\top N_1 T} J^\top \varphi \\
&= (J^\top)^{-1} \mathrm{e}^{(\mathbf{A}^* + (J^{-1} S J)^\top)(N_1 T)} \hat{\varphi}.
\end{aligned} \tag{3.16}$$

By solving the differential equation $\psi'(t) = (\mathbf{A}^* + (J^{-1} S J)^\top)\psi(t), \psi(0) = \hat{\varphi}$ with (3.4), we see that

$$\mathrm{e}^{(\mathbf{A}^* + (J^{-1} S J)^\top)(N_1 T)} \hat{\varphi} = \begin{pmatrix} \mathrm{e}^{(\mathbf{A}^* + S_1^\top)(N_1 T)} & 0 \\ \int_0^{N_1 T} \mathrm{e}^{(\mathbf{A}^* + S_3^\top)(N_1 T - t)} S_2^\top \mathrm{e}^{(\mathbf{A}^* + S_1^\top) t} \mathrm{d}t & \mathrm{e}^{(\mathbf{A}^* + S_3^\top)(N_1 T)} \end{pmatrix} \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix}. \tag{3.17}$$

It follows by (3.16) and (3.17) that

$$\begin{aligned}
&\|\mathrm{e}^{(\mathbf{A}^* + S^\top)(N_1 T)} \varphi\|^2_{\mathbf{Y}^n} \\
&= \|(J^\top)^{-1} \mathrm{e}^{(\mathbf{A}^* + (J^{-1} S J)^\top)(N_1 T)} \hat{\varphi}\|^2_{\mathbf{Y}^n} \\
&\le \|J^{-1}\|^2 \left(\|\mathrm{e}^{(\mathbf{A}^* + S_1^\top)(N_1 T)} \hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}} + 2 \left\| \int_0^{N_1 T} \mathrm{e}^{(\mathbf{A}^* + S_3^\top)(N_1 T - t)} S_2^\top \mathrm{e}^{(\mathbf{A}^* + S_1^\top) t} \mathrm{d}t \hat{\varphi}_1 \right\|^2_{\mathbf{Y}^{n-n_1}} \right) \\
&\quad + 2\|J^{-1}\|^2 \|\mathrm{e}^{(\mathbf{A}^* + S_3^\top)(N_1 T)} \hat{\varphi}_2\|^2_{\mathbf{Y}^{n-n_1}}.
\end{aligned} \tag{3.18}$$

Next, we will estimate the three terms on the right hand side of (3.18) by the following three steps:

Step 1. We claim that, for each $k \in \mathbb{N}^+ \cap [n_1, +\infty)$, there is $c_1 > 0$, $c_2 > 0$ and $\varepsilon_1 > 0$ (which are independent of k) such that

$$\|\mathrm{e}^{(\mathbf{A}^* + S_1^\top) kT} \hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}} \le c_1 \sum_{j=k-n_1}^{k-1} \left\| \int_{jT}^{(j+1)T} \mathbf{B}^* D_1^\top \mathrm{e}^{(\mathbf{A}^* + S_1^\top) t} \hat{\varphi}_1 \mathrm{d}t \right\|^2_{\mathbf{U}^m} + c_2 \mathrm{e}^{-2(\beta + \varepsilon_1) kT} \|\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}. \tag{3.19}$$

To prove (3.19), we give several facts. Fact One: we have by Proposition 3.4 and (a) in Proposition 3.5 that

$$\begin{aligned}\|\mathrm{e}^{(\mathbf{A}^*+S_1^\top)kT}\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}} &\leq 2\|\mathbf{P}^\beta\mathrm{e}^{(\mathbf{A}^*+S_1^\top)kT}\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}+2\|(I-\mathbf{P}^\beta)\mathrm{e}^{(\mathbf{A}^*+S_1^\top)kT}\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}\\ &\leq 2\|\mathbf{P}^\beta\mathrm{e}^{(\mathbf{A}^*+S_1^\top)kT}\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}+2\tilde{c}\mathrm{e}^{-2\tilde{\alpha}kT}\|\mathrm{e}^{S_1^\top kT}\|^2\|\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}.\end{aligned} \tag{3.20}$$

Fact Two: By Proposition 3.7, we have

$$\begin{aligned}&\|\mathbf{P}^\beta\mathrm{e}^{(\mathbf{A}^*+S_1^\top)kT}\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}\\ \leq\ &\|\mathrm{e}^{S_1^\top n_1T}\|^2\|\mathbf{P}^\beta\mathrm{e}^{\mathbf{A}^*kT+S_1^\top(k-n_1)T}\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}\\ =\ &\|\mathrm{e}^{S_1^\top n_1T}\|^2\Big\|\Big(\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\Big)^{-1}\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\mathbf{P}^\beta\mathrm{e}^{\mathbf{A}^*kT+S_1^\top(k-n_1)T}\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n_1}}\\ \leq\ &M_1\Big\|\mathbf{P}^\beta\mathrm{e}^{\mathbf{A}^*(k-1)T+S_1^\top(k-n_1)T}\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n_1}},\end{aligned} \tag{3.21}$$

where $M_1:=\|\mathrm{e}^{S_1^\top n_1T}\|^2\Big\|\Big(\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\Big)^{-1}\Big\|^2_{\mathcal{L}(\mathbf{Y}^{n_1})}\|\mathrm{e}^{\mathbf{A}^*T}\|^2_{\mathcal{L}(\mathbf{Y}^{n_1})}$.

Fact Three: According to $(A5)$ and Proposition 3.6, the matrix $G:=(D_1,\mathrm{e}^{S_1T}D_1,\cdots,\mathrm{e}^{S_1(n_1-1)T}D_1)$ is of full rank. So the matrix $GG^\top$ is positive definite, which leads to $M_2:=(\underline{\sigma}(GG^\top))^{-1}>0$. Let $M_3:=\sup_{t\in[0,(n_1-1)T]}\|\mathrm{e}^{At}\|^2_{\mathcal{L}(Y)}$. Then, direct computations lead to

$$\begin{aligned}&\Big\|\mathbf{P}^\beta\mathrm{e}^{\mathbf{A}^*(k-1)T+S_1^\top(k-n_1)T}\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n_1}}\\ \leq\ &M_2\Big\|G^\top\mathbf{P}^\beta\mathrm{e}^{\mathbf{A}^*(k-1)T+S_1^\top(k-n_1)T}\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{mn_1}}\\ =\ &M_2\sum_{i=0}^{n_1-1}\Big\|D_1^\top\mathrm{e}^{S_1^\top(iT)}\mathbf{P}^\beta\mathrm{e}^{\mathbf{A}^*(k-1)T+S_1^\top(k-n_1)T}\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^m}\\ \leq\ &M_2M_3\sum_{i=0}^{n_1-1}\Big\|\mathbf{P}^\beta D_1^\top\mathrm{e}^{(\mathbf{A}^*+S_1^\top)(k-n_1+i)T}\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^m}.\end{aligned} \tag{3.22}$$

Fact Four: Notice that $\int_0^T\mathrm{e}^{At}\mathrm{d}t\varphi\in D(A),\forall\varphi\in Y$. We see that $\Phi_j:=D_1^\top\mathrm{e}^{(\mathbf{A}^*+S_1^\top)(k-n_1-1+j)T}\mathrm{e}^{S_1^\top T}\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}dt\hat{\varphi}_1\in(D(A))^m$, $j\in\{0,1,\cdots,n_1-1\}$. We have by (b) in Proposition 3.5 that

$$\begin{aligned}&\Big\|\mathbf{P}^\beta D_1^\top\mathrm{e}^{(\mathbf{A}^*+S_1^\top)(k-n_1+j)T}\int_0^T\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^m}\\ =\ &\|\mathbf{P}^\beta\mathrm{e}^{\mathbf{A}^*T}\Phi_j\|^2_{\mathbf{Y}^m}\\ \leq\ &\tilde{d}\|\mathbf{B}^*\mathbf{P}^\beta\mathrm{e}^{\mathbf{A}^*T}\Phi_j\|^2_{\mathbf{U}^m}\\ \leq\ &2\tilde{d}\|\mathbf{B}^*\mathrm{e}^{\mathbf{A}^*T}\Phi_j\|^2_{\mathbf{U}^m}+2\tilde{d}\|\mathbf{B}^*(I-\mathbf{P}^\beta)\mathrm{e}^{\mathbf{A}^*T}\Phi_j\|^2_{\mathbf{U}^m},\end{aligned}$$

which, together with (a) in Proposition 3.5, implies

$$\begin{aligned}
&\sum_{j=0}^{n_1-1}\Big\|\mathbf{P}^{\beta}D_1^{\top}\mathrm{e}^{(\mathbf{A}^*+S_1^{\top})(k-n_1+j)T}\int_0^T \mathrm{e}^{(\mathbf{A}^*+S_1^{\top})t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^m}\\
&\leq \sum_{j=0}^{n_1-1}[2\tilde{d}\|\mathbf{B}^*\mathrm{e}^{\mathbf{A}^*T}\Phi_j\|^2_{\mathbf{U}^m}+2\tilde{d}\|\mathbf{B}^*(I-\mathbf{P}^{\beta})\mathrm{e}^{\mathbf{A}^*T}\Phi_j\|^2_{\mathbf{U}^m}]\\
&\leq 2\tilde{d}\sum_{j=k-n_1}^{k-1}\Big\|\int_{jT}^{(j+1)T}\mathbf{B}^*D_1^{\top}\mathrm{e}^{(\mathbf{A}^*+S_1^{\top})t}\hat{\varphi}_1\mathrm{d}t\Big\|^2_{\mathbf{U}^m}+M_4\mathrm{e}^{-2\tilde{\alpha}kT}\|\mathrm{e}^{S_1^{\top}kT}\|^2\|\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}},
\end{aligned}\tag{3.23}$$

where $M_4:=2n_1\tilde{c}\tilde{d}\|\int_0^T\mathbf{B}^*\mathrm{e}^{(\mathbf{A}^*+S_1^{\top})t}dt\|^2_{\mathcal{L}(\mathbf{Y}^{n_1};\mathbf{U}^{n_1})}\|D_1\|^2\mathrm{e}^{2\tilde{\alpha}n_1T}\sup_{t\in[-n_1T,-T]}\|\mathrm{e}^{S_1^{\top}t}\|^2$.

Fact Five: For any $r>\bar{\sigma}(S_1^{\top})$, there is $m_r>0$ such that $\|\mathrm{e}^{S_1^{\top}t}\|\leq m_r\mathrm{e}^{rt},\forall t>0$. Taking $r=\bar{\sigma}(S_1^{\top})+\varepsilon_1$, where $\varepsilon_1=\frac{\tilde{\alpha}-\bar{\sigma}(S_1^{\top})-\beta}{2}$. (We know from Asm. $(A0)$ that $\varepsilon_1>0$). Then, there is $c_{\varepsilon_1}>0$ such that

$$\mathrm{e}^{-2\tilde{\alpha}kT}\|\mathrm{e}^{S_1^{\top}kT}\|^2\leq c_{\varepsilon_1}\mathrm{e}^{-2(\beta+\varepsilon_1)kT}.\tag{3.24}$$

Finally, based on (3.20), (3.21), (3.22), (3.23) and (3.24), we obtain (3.19), with $c_1=4M_1M_2M_3\tilde{d}$, and $c_2=2c_{\varepsilon_1}(M_1M_2M_3M_4+\tilde{c})$.

Step 2. We claim that there is $\varepsilon_2>0$ and $c_3>0$ (which are independent of t) such that

$$\|\mathrm{e}^{(\mathbf{A}^*+S_3^{\top})t}\hat{\varphi}_2\|^2_{\mathbf{Y}^{n-n_1}}\leq c_3\mathrm{e}^{-2(\beta+\varepsilon_2)t}\|\hat{\varphi}_2\|^2_{\mathbf{Y}^{n-n_1}},\ \ \forall t>0.\tag{3.25}$$

Indeed, by $(A3)$, we have $\delta_0:=-\lambda_0-\beta-\bar{\sigma}(S_3)>0$. Hence, there is $\tilde{c}_1\geq 1$ and $\tilde{c}_2\geq 1$ such that

$$\|\mathrm{e}^{\mathbf{A}^*t}\|_{\mathcal{L}(\mathbf{Y}^{n-n_1})}\leq\tilde{c}_1\mathrm{e}^{(\lambda_0+\delta_0/4)t},\ \ \|\mathrm{e}^{S_3^{\top}t}\|\leq\tilde{c}_2\mathrm{e}^{(\bar{\sigma}(S_3)+\delta_0/4)t},\ \ \forall t>0.$$

Based on this and Proposition 3.4, we see

$$\|\mathrm{e}^{(\mathbf{A}^*+S_3^{\top})t}\hat{\varphi}_2\|^2_{\mathbf{Y}^{n-n_1}}\leq\tilde{c}_1^2\tilde{c}_2^2\mathrm{e}^{-2(\beta+\delta_0/2)t}\|\hat{\varphi}_2\|^2_{\mathbf{Y}^{n-n_1}},\tag{3.26}$$

which leads to (3.25), with $\varepsilon_2=\delta_0/2$ and $c_3=\tilde{c}_1^2\tilde{c}_2^2$.

Step 3. Let $\varepsilon_3=\min\{\varepsilon_1,\varepsilon_2\}$. We claim that there is $c_4>0$ and $c_5>0$, where c_5 is independent of N, such that

$$\begin{aligned}
&\Big\|\int_0^{N_1T}\mathrm{e}^{(\mathbf{A}^*+S_3^{\top})(N_1T-t)}S_2^{\top}\mathrm{e}^{(\mathbf{A}^*+S_1^{\top})t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n-n_1}}\\
&\leq c_4\sum_{j=0}^{N_1-1}\Big\|\int_{jT}^{(j+1)T}\mathbf{B}^*D_1^{\top}\mathrm{e}^{(\mathbf{A}^*+S_1^{\top})t}\hat{\varphi}_1\mathrm{d}t\Big\|^2_{\mathbf{U}^m}+c_5N_1^3\mathrm{e}^{-2(\beta+\varepsilon_3)N_1T}\|\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}.
\end{aligned}\tag{3.27}$$

Indeed, by direct computations, we have

$$\begin{aligned}
\Big\|\int_0^{N_1T}\mathrm{e}^{(\mathbf{A}^*+S_3^{\top})(N_1T-t)}S_2^{\top}\mathrm{e}^{(\mathbf{A}^*+S_1^{\top})t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n-n_1}}&\leq 2\Big\|\int_0^{n_1T}\mathrm{e}^{(\mathbf{A}^*+S_3^{\top})(N_1T-t)}S_2^{\top}\mathrm{e}^{(\mathbf{A}^*+S_1^{\top})t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n-n_1}}\\
&+2\Big\|\int_{n_1T}^{N_1T}\mathrm{e}^{(\mathbf{A}^*+S_3^{\top})(N_1T-t)}S_2^{\top}\mathrm{e}^{(\mathbf{A}^*+S_1^{\top})t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n-n_1}}.
\end{aligned}\tag{3.28}$$

For the first term on the right hand side of (3.28), we infer from (3.25) that

$$\begin{aligned}&\Big\|\int_0^{n_1T}\mathrm{e}^{(\mathbf{A}^*+S_3^\top)(N_1T-t)}S_2^\top\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n-n_1}}\\&\le c_3(n_1T)^2\|S_2^\top\|^2\sup_{t\in[0,n_1T]}\|\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\|^2_{\mathcal{L}(\mathbf{Y}^{n_1})}\mathrm{e}^{-2(\beta+\varepsilon_2)(N_1-n_1)T}\|\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}.\end{aligned}\tag{3.29}$$

For the second term on the right hand side of (3.28), we obtain from (3.25) that

$$\begin{aligned}&\Big\|\int_{n_1T}^{N_1T}\mathrm{e}^{(\mathbf{A}^*+S_3^\top)(N_1T-t)}S_2^\top\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n-n_1}}\\&\le (N_1-n_1)\sum_{k=n_1}^{N_1-1}\Big\|\int_{kT}^{(k+1)T}\mathrm{e}^{(\mathbf{A}^*+S_3^\top)(N_1T-t)}S_2^\top\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n-n_1}}\\&\le c_3(N_1-n_1)^2T^2\|S_2^\top\|^2\sup_{t\in[0,T]}\|\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\|^2_{\mathcal{L}(\mathbf{Y}^{n_1})}\sum_{k=n_1}^{N_1-1}\mathrm{e}^{-2(\beta+\varepsilon_2)[N_1-(k+1)]T}\|\mathrm{e}^{(\mathbf{A}^*+S_1^\top)kT}\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}.\end{aligned}$$

With (3.19), the above implies

$$\begin{aligned}&\Big\|\int_{n_1T}^{N_1T}\mathrm{e}^{(\mathbf{A}^*+S_3^\top)(N_1T-t)}S_2^\top\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\mathrm{d}t\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n-n_1}}\\&\le c_1c_3(N_1-n_1)^3T^2\|S_2^\top\|^2\sup_{t\in[0,T]}\|\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\|^2_{\mathcal{L}(\mathbf{Y}^{n_1})}\sum_{j=0}^{N_1-1}\Big\|\int_{jT}^{(j+1)T}\mathbf{B}^*D_1^\top\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\hat{\varphi}_1\mathrm{d}t\Big\|^2_{\mathbf{U}^m}\\&\quad+c_2c_3(N_1-n_1)^3T^2\|S_2^\top\|^2\sup_{t\in[0,T]}\|\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\|^2_{\mathcal{L}(\mathbf{Y}^{n_1})}\mathrm{e}^{2(\beta+\varepsilon_2)T}\mathrm{e}^{-2(\beta+\varepsilon_3)N_1T}\|\hat{\varphi}_1\|^2_{\mathbf{Y}^{n_1}}.\end{aligned}\tag{3.30}$$

Now, by (3.28)–(3.30), we obtain (3.27), with $c_4=2c_1c_3(N_1-n_1)^3T^2\|S_2^\top\|^2\sup_{t\in[0,T]}\|\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\|^2_{\mathcal{L}(\mathbf{Y}^{n_1})}$, and $c_5=4n_1^2(c_2+1)c_3T^2\|S_2^\top\|^2\sup_{t\in[0,n_1T]}\|\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\|^2_{\mathcal{L}(\mathbf{Y}^{n_1})}\mathrm{e}^{2n_1(\beta+\varepsilon_2)T}$.

Finally, we obtain from (3.18), (3.19), (3.25) and (3.27) that

$$\begin{aligned}&\|\mathrm{e}^{(\mathbf{A}^*+S^\top)(N_1T)}\varphi\|^2_{\mathbf{Y}^n}\\&\le 2\|J^{-1}\|^2(c_1+c_4)\sum_{j=0}^{N_1-1}\Big\|\int_{jT}^{(j+1)T}\mathbf{B}^*D_1^\top\mathrm{e}^{(\mathbf{A}^*+S_1^\top)t}\hat{\varphi}_1\mathrm{d}t\Big\|^2_{\mathbf{U}^m}\\&\quad+2\|J^{-1}\|^2(c_2+c_3+c_5N_1^3)\mathrm{e}^{-2(\beta+\varepsilon_3)N_1T}\|\hat{\varphi}\|^2_{\mathbf{Y}^n}\\&\le 2\|J^{-1}\|^2(c_1+c_4)\sum_{i=0}^{N_1-1}\Big\|\int_{jT}^{(j+1)T}\mathbf{B}^*D^\top\mathrm{e}^{(\mathbf{A}^*+S^\top)t}\varphi\mathrm{d}t\Big\|^2_{\mathbf{U}^m}\\&\quad+2\|J^{-1}\|^2\|J\|^2(c_2+c_3+c_5N_1^3)\mathrm{e}^{-2(\beta+\varepsilon_3)N_1T}\|\varphi\|^2_{\mathbf{Y}^n}.\end{aligned}\tag{3.31}$$

Since the constants c_2, c_3 and c_5 are independent on N_1, we can take N_1 large enough such that $\delta:=2\|J^{-1}\|^2\|J\|^2(c_2+c_3+c_5N_1^3)\mathrm{e}^{-2\varepsilon_3N_1T}<1$. This completes the proof of Theorem 3.1. □

3.3. Proof of Theorem 3.2

Proof of Theorem 3.2. We shall prove this theorem only for the case when $[S^\top, Q^\top]$ is not controllable. The case when $[S^\top, Q^\top]$ is controllable follows by the same arguments. By $(A4)$, if $[S^\top, Q^\top]$ is not controllable, then

$$\sigma(\hat{S}_3) \subset \mathbb{C} \setminus \mathbb{C}^+_{-\lambda_0-\beta}. \tag{3.32}$$

Here, the matrix $\hat{S}_3 \in \mathbb{R}^{(n-\hat{n}_1)\times(n-\hat{n}_1)}$ is given by the Kalman decomposition:

$$\hat{J}^{-1}S^\top \hat{J} = \begin{pmatrix} \hat{S}_1 & \hat{S}_2 \\ 0 & \hat{S}_3 \end{pmatrix}; \quad \hat{J}^{-1}Q^\top = \begin{pmatrix} \hat{Q}_1 \\ 0 \end{pmatrix}, \tag{3.33}$$

where $\hat{J} \in \mathbb{R}^{n\times n}$ is invertible, $\hat{S}_1 \in \mathbb{R}^{\hat{n}_1\times\hat{n}_1}, \hat{Q}_1 \in \mathbb{R}^{\hat{n}_1\times p}, \hat{S}_2 \in \mathbb{R}^{\hat{n}_1\times(n-\hat{n}_1)}$, and $[\hat{S}_1, \hat{Q}_1]$ is controllable.

Given $\varphi \in \mathbf{Y}^n$, we write $\hat{\varphi} = \hat{J}^\top\varphi = (\hat{\varphi}_1, \hat{\varphi}_2)$, where $\hat{\varphi}_1 \in \mathbb{R}^{\hat{n}_1}, \hat{\varphi}_2 \in \mathbb{R}^{n-\hat{n}_1}$. Let $N_2 \in \mathbb{N}^+ \cap [\hat{n}_1, +\infty)$, which will be determined later. By Proposition 3.4, we see that

$$\begin{aligned} \mathrm{e}^{(\mathbf{A}+S)(N_2T)}\varphi &= (\hat{J}^\top)^{-1}\hat{J}^\top \mathrm{e}^{\mathbf{A}N_2T}(\hat{J}^\top)^{-1}\hat{J}^\top \mathrm{e}^{SN_2T}(\hat{J}^\top)^{-1}\hat{J}^\top\varphi \\ &= (\hat{J}^\top)^{-1}\mathrm{e}^{\mathbf{A}N_2T}\hat{J}^\top(\hat{J}^\top)^{-1}\mathrm{e}^{(\hat{J}^{-1}S^\top\hat{J})^\top N_2T}\hat{J}^\top\varphi \\ &= (\hat{J}^\top)^{-1}\mathrm{e}^{(\mathbf{A}+(\hat{J}^{-1}S^\top\hat{J})^\top)(N_2T)}\hat{\varphi}. \end{aligned} \tag{3.34}$$

Based on (3.34) and (3.33), we determine

$$\begin{aligned} &\|\mathrm{e}^{(\mathbf{A}+S)(N_2T)}\varphi\|^2_{\mathbf{Y}^n} \\ =\ &\|(\hat{J}^\top)^{-1}\mathrm{e}^{(\mathbf{A}+(\hat{J}^{-1}S^\top\hat{J})^\top)(N_2T)}\hat{\varphi}\|^2_{\mathbf{Y}^n} \\ \le\ &\|\hat{J}^{-1}\|^2\left(\|\mathrm{e}^{(\mathbf{A}+\hat{S}_1^\top)(N_2T)}\hat{\varphi}_1\|^2_{\mathbf{Y}^{\hat{n}_1}} + 2\left\|\int_0^{N_2T}\mathrm{e}^{(\mathbf{A}+\hat{S}_3^\top)(N_2T-t)}\hat{S}_2^\top\mathrm{e}^{(\mathbf{A}+\hat{S}_1^\top)t}\mathrm{d}t\hat{\varphi}_1\right\|^2_{\mathbf{Y}^{n-\hat{n}_1}}\right) \\ &+2\|\hat{J}^{-1}\|^2\|\mathrm{e}^{(\mathbf{A}+\hat{S}_3^\top)(N_2T)}\hat{\varphi}_2\|^2_{\mathbf{Y}^{n-\hat{n}_1}}. \end{aligned} \tag{3.35}$$

Next, we apply similar arguments as that of Theorem 3.1 to estimate the three terms on the right hand side of (3.19).

Step 1. We claim that, for each $k \in \mathbb{N}^+ \cap [\hat{n}_1+1, +\infty)$, there is $\hat{c}_1 > 0$, $\hat{c}_2 > 0$ and $\hat{\varepsilon}_1 > 0$ (which are independent of k) such that

$$\|\mathrm{e}^{(\mathbf{A}+\hat{S}_1^\top)kT}\hat{\varphi}_1\|^2_{\mathbf{Y}^{\hat{n}_1}} \le \hat{c}_1\sum_{j=k-\hat{n}_1+1}^{k}\|\mathbf{C}\hat{Q}_1^\top\mathrm{e}^{(\mathbf{A}+\hat{S}_1^\top)jT}\hat{\varphi}_1\|^2_{\mathbf{Z}^p} + \hat{c}_2\mathrm{e}^{-2(\beta+\hat{\varepsilon}_1)kT}\|\hat{\varphi}_1\|^2_{\mathbf{Y}^{\hat{n}_1}}. \tag{3.36}$$

To prove (3.36), we give several facts. Fact One: Since $(A4)$, $(A5)$ and (3.33) hold, we can apply Proposition 3.6 to see that, the matrix $\hat{G} := (\hat{Q}_1, \mathrm{e}^{\hat{S}_1T}\hat{Q}_1, \cdots, \mathrm{e}^{\hat{S}_1(\hat{n}_1-1)T}\hat{Q}_1)$ is of full rank. So the matrix $\hat{G}\hat{G}^\top$ is positive definite, which leads to $\hat{M}_1 := (\underline{\sigma}(\hat{G}\hat{G}^\top))^{-1} > 0$. Let $\hat{M}_2 := \|\mathrm{e}^{\hat{S}_1^\top(\hat{n}_1-1)T}\|^2\sup_{t\in[0,(\hat{n}_1-1)T]}\|\mathrm{e}^{At}\|^2_{\mathcal{L}(Y)}$. Then, direct computations lead to

$$\begin{aligned} &\|\mathrm{e}^{(\mathbf{A}+\hat{S}_1^\top)kT}\hat{\varphi}_1\|^2_{\mathbf{Y}^{\hat{n}_1}} \\ \le\ &\|\mathrm{e}^{\hat{S}_1^\top(\hat{n}_1-1)T}\|^2\|\mathrm{e}^{\mathbf{A}kT+\hat{S}_1^\top(k-\hat{n}_1+1)T}\hat{\varphi}_1\|^2_{\mathbf{Y}^{\hat{n}_1}} \\ \le\ &\hat{M}_1\|\mathrm{e}^{\hat{S}_1^\top(\hat{n}_1-1)T}\|^2\|\hat{G}^\top\mathrm{e}^{\mathbf{A}kT+\hat{S}_1^\top(k-\hat{n}_1+1)T}\hat{\varphi}_1\|^2_{\mathbf{Y}^{p\hat{n}_1}} \\ \le\ &\hat{M}_1\hat{M}_2\sum_{j=1}^{\hat{n}_1}\|\hat{Q}_1^\top\mathrm{e}^{(\mathbf{A}+\hat{S}_1^\top)(k-\hat{n}_1+j)T}\hat{\varphi}_1\|^2_{\mathbf{Y}^p}. \end{aligned} \tag{3.37}$$

Fact Two: Let $\Phi_j := \hat{Q}_1^\top e^{(\mathbf{A}+\hat{S}_1^\top)(k-\hat{n}_1-2+j)T} e^{2\hat{S}_1^\top T}\hat{\varphi}_1,\ j\in\{1,\cdots,\hat{n}_1\}$. We have by (b) in Proposition 3.5 that

$$\begin{aligned}
&\|\hat{Q}_1^\top e^{(\mathbf{A}+\hat{S}_1^\top)(k-\hat{n}_1+j)T}\hat{\varphi}_1\|^2_{\mathbf{Y}^p}\\
&= \|e^{2\mathbf{A}T}\Phi_j\|^2_{\mathbf{Y}^p}\\
&\le 2\|\mathbf{P}^\beta e^{2\mathbf{A}T}\Phi_j\|^2_{\mathbf{Y}^p} + 2\|(I-\mathbf{P}^\beta)e^{2\mathbf{A}T}\Phi_j\|^2_{\mathbf{Y}^p}\\
&\le 2\hat{d}\|\mathbf{C}\mathbf{P}^\beta e^{2\mathbf{A}T}\Phi_j\|^2_{\mathbf{Z}^p} + 2\|(I-\mathbf{P}^\beta)e^{2\mathbf{A}T}\Phi_j\|^2_{\mathbf{Y}^p}\\
&\le 4\hat{d}\|\mathbf{C}e^{2\mathbf{A}T}\Phi_j\|^2_{\mathbf{Z}^p} + 4\hat{d}\|\mathbf{C}e^{\mathbf{A}T}(I-\mathbf{P}^\beta)e^{\mathbf{A}T}\Phi_j\|^2_{\mathbf{Z}^p} + 2\|(I-\mathbf{P}^\beta)e^{2\mathbf{A}T}\Phi_j\|^2_{\mathbf{Y}^p}\\
&\le 4\hat{d}\|\mathbf{C}e^{2\mathbf{A}T}\Phi_j\|^2_{\mathbf{Z}^p} + 2(2\hat{d}\|Ce^{AT}\|^2_{\mathcal{L}(Y;Z)} + \|e^{AT}\|^2_{\mathcal{L}(Y)})\|(I-\mathbf{P}^\beta)e^{\mathbf{A}T}\Phi_j\|^2_{\mathbf{Y}^p}.
\end{aligned}$$

The above together with (a) in Proposition 3.5 implies

$$\begin{aligned}
&\sum_{j=1}^{\hat{n}_1}\|\hat{Q}_1^\top e^{(\mathbf{A}+\hat{S}_1^\top)(k-\hat{n}_1+j)T}\hat{\varphi}_1\|^2_{\mathbf{Y}^p}\\
&\le \sum_{j=1}^{\hat{n}_1}[4\hat{d}\|\mathbf{C}e^{2\mathbf{A}T}\Phi_j\|^2_{\mathbf{Z}^p} + 2(2\hat{d}\|Ce^{AT}\|^2_{\mathcal{L}(Y;Z)} + \|e^{AT}\|^2_{\mathcal{L}(Y)})\|(I-\mathbf{P}^\beta)e^{\mathbf{A}T}\Phi_j\|^2_{\mathbf{Y}^p}]\\
&\le 4\hat{d}\sum_{j=k-\hat{n}_1+1}^{k}\|\mathbf{C}\hat{Q}_1^\top e^{(\mathbf{A}+\hat{S}_1^\top)jT}\hat{\varphi}_1\|^2_{\mathbf{Z}^p} + \hat{M}_3 e^{-2\tilde{\alpha}kT}\|e^{\hat{S}_1^\top kT}\|^2\|\hat{\varphi}_1\|^2_{\mathbf{Y}^{\hat{n}_1}},
\end{aligned}\tag{3.38}$$

where $\hat{M}_3 := 2\hat{n}_1\tilde{c}(2\hat{d}\|Ce^{AT}\|^2_{\mathcal{L}(Y;Z)} + \|e^{AT}\|^2_{\mathcal{L}(Y)})\|\hat{Q}_1^\top\|^2 e^{2\tilde{\alpha}(\hat{n}_1+1)T}\sup_{t\in[-(\hat{n}_1-1)T,0]}\|e^{\hat{S}_1^\top t}\|^2$.

Fact Three: For any $r>\bar{\sigma}(\hat{S}_1^\top)$, there is $m_r>0$ such that $\|e^{\hat{S}_1^\top t}\|\le m_r e^{rt}, \forall t>0$. Taking $r=\bar{\sigma}(\hat{S}_1^\top)+\hat{\varepsilon}_1$, where $\hat{\varepsilon}_1=\frac{\tilde{\alpha}-\bar{\sigma}(\hat{S}_1^\top)-\beta}{2}$. (We know from Asm. $(A0)$ that $\hat{\varepsilon}_1>0$.) Then, according to Assumption $(A0)$, there is $\hat{\varepsilon}_1>0$ and $c_{\hat{\varepsilon}_1}>0$ such that

$$e^{-2\tilde{\alpha}kT}\|e^{\hat{S}_1^\top kT}\|^2 \le c_{\hat{\varepsilon}_1}e^{-2(\beta+\hat{\varepsilon}_1)kT}.\tag{3.39}$$

Finally, based on (3.37), (3.38) and (3.39), we obtain (3.36), with $\hat{c}_1=4\hat{M}_1\hat{M}_2\hat{d}$, and $\hat{c}_2=\hat{M}_1\hat{M}_2\hat{M}_3c_{\hat{\varepsilon}_1}$.

Step 2. By $(A4)$, (3.32), and the same arguments as that in Step 2 of the proof of Theorem 3.1, we can infer that there is $\hat{\varepsilon}_2>0$ and $\hat{c}_3>0$ (which are independent of t) such that

$$\|e^{(\mathbf{A}+\hat{S}_3^\top)t}\hat{\varphi}_2\|^2_{\mathbf{Y}^{n-\hat{n}_1}} \le \hat{c}_3 e^{-2(\beta+\hat{\varepsilon}_2)t}\|\hat{\varphi}_2\|^2_{\mathbf{Y}^{n-\hat{n}_1}},\ \ \forall t>0.\tag{3.40}$$

Step 3. Let $\hat{\varepsilon}_3=\min\{\hat{\varepsilon}_1,\hat{\varepsilon}_2\}$. By (3.36), (3.40), and the same arguments as that in Step 3 of the proof of Theorem 3.1, we can obtain that there is $\hat{c}_4>0$ and $\hat{c}_5>0$, where $\hat{c}_5$ is independent of N, such that

$$\begin{aligned}
&\Big\|\int_0^{NT} e^{(\mathbf{A}+\hat{S}_3^\top)(NT-t)}\hat{S}_2^\top e^{(\mathbf{A}+\hat{S}_1^\top)t}dt\hat{\varphi}_1\Big\|^2_{\mathbf{Y}^{n-\hat{n}_1}}\\
&\le \hat{c}_4\sum_{j=1}^{N}\|\mathbf{C}\hat{Q}_1^\top e^{(\mathbf{A}+\hat{S}_1^\top)jT}\hat{\varphi}_1\|^2_{\mathbf{Z}^p} + \hat{c}_5N^3e^{-2(\beta+\hat{\varepsilon}_3)NT}\|\hat{\varphi}_1\|^2_{\mathbf{Y}^{\hat{n}_1}}.
\end{aligned}\tag{3.41}$$

Finally, we obtain from (3.35), (3.36), (3.40) and (3.41) that

$$\begin{aligned}
&\|e^{(\mathbf{A}+S)(NT)}\varphi\|^2_{\mathbf{Y}^n}\\
\le\ & 2\|\hat{J}^{-1}\|^2(\hat{c}_1+\hat{c}_4)\sum_{j=1}^{N}\|\mathbf{C}\hat{Q}_1^\top e^{(\mathbf{A}+\hat{S}_1^\top)jT}\hat{\varphi}_1\|^2_{\mathbf{Z}^p}\\
&+2\|\hat{J}^{-1}\|^2(\hat{c}_2+\hat{c}_3+\hat{c}_5N^3)e^{-2(\beta+\hat{\varepsilon}_3)NT}\|\hat{\varphi}\|^2_{\mathbf{Y}^{\hat{n}_1}}\\
\le\ & 2\|\hat{J}^{-1}\|^2(\hat{c}_1+\hat{c}_4)\sum_{j=1}^{N}\|\mathbf{C}Qe^{(\mathbf{A}+S)jT}\varphi\|^2_{\mathbf{Z}^p}\\
&+2\|\hat{J}^{-1}\|^2\|\hat{J}\|^2(\hat{c}_2+\hat{c}_3+\hat{c}_5N^3)e^{-2(\beta+\hat{\varepsilon}_3)NT}\|\varphi\|^2_{\mathbf{Y}^n}.
\end{aligned}\tag{3.42}$$

Since the constants $\hat{c}_2$, $\hat{c}_3$ and $\hat{c}_5$ are independent on N, we can take N large enough such that $\delta := 2\|\hat{J}^{-1}\|^2\|\hat{J}\|^2(\hat{c}_2+\hat{c}_3+\hat{c}_5N^3)e^{-2\hat{\varepsilon}_3NT} < 1$. This completes the proof of Theorem 3.2. □

4. Applications

In this section, we shall give some examples for which our criteria can be applied to give the right way to sample and control such that the system is output feedback stabilizable.

4.1. System coupled by sampled-data controlled heat equations over $(0, \pi)$

4.1.1. Piecewise constant control and averaged sampling

Let $S \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{n\times m}$. Let $T > 0$ and $N \in \mathbb{N}^+$. Let

$$\omega_j := [a_j, b_j] \subset (0,\pi), \hat{\omega}_j := [\hat{a}_j, \hat{b}_j] \subset (0,\pi) \ \text{ for all } \ 1 \le j \le 2N+1,$$

where ω_j and $\hat{\omega}_j, j = 1, 2, \cdots, 2N+1$ satisfy

$$\omega_j \cap \omega_k = \emptyset,\ \hat{\omega}_j \cap \hat{\omega}_k = \emptyset,\ b_j - a_j = b_k - a_k,\ \hat{b}_j - \hat{a}_j = \hat{b}_k - \hat{a}_k \ \text{ for all } \ 1 \le j, k \le 2N+1, j \ne k. \tag{4.1}$$

We consider the following sampled-data control system coupled by heat equations over $(0, \pi)$:

$$\begin{cases}\partial_t\mathbf{y}(t,x) = \mathbf{\Delta}_n\mathbf{y}(t,x) + S\mathbf{y}(t,x)\\ \qquad\qquad + \sum_{k=0}^{\infty}\sum_{j=1}^{2N+1}\chi_{[kT,(k+1)T)}(t)\chi_{\omega_j}(x)D\mathbf{u}_{kj}, (t,x) \in \mathbb{R}^+\times(0,\pi),\\ \mathbf{y}(t,x)|_{x=0} = \mathbf{y}(t,x)|_{x=\pi} = 0,\end{cases} \tag{4.2}$$

with observation

$$\mathbf{z}_k = \left(\left(\sum_{i=1}^{n}\int_{\hat{\omega}_j} q_{1i}y_i(kT,x)\mathrm{d}x\right)_{j=1}^{2N+1}, \cdots, \left(\sum_{i=1}^{n}\int_{\hat{\omega}_j} q_{pi}y_i(kT,x)\mathrm{d}x\right)_{j=1}^{2N+1}\right), \tag{4.3}$$

where $\mathbf{y}(t,x) = (y_1(t,x), \cdots, y_n(t,x))^\top \in \mathbb{R}^n$ for each $(t,x) \in \mathbb{R}^+\times(0,\pi)$, $\mathbf{u}_{kj} \in \mathbb{R}^m$ for all $k \in \mathbb{N}$ and $j = 1, \cdots, 2N+1$, $\mathbf{\Delta}_n = \mathrm{diag}(\underbrace{\partial_{xx}, \cdots, \partial_{xx}}_{n})$, χ_{ω_j} is the characteristic function of ω_j, $q_{li} \in \mathbb{R}$, for all $l = 1, \cdots, p$ and $i = 1, \cdots, n$. *Notice that in* (4.3)*, the sensors provide discrete in time and averaged in space measurements of the state. This can be viewed as a kind of time-space sampling.*

We aim to study the β-output feedback stabilizability for the system (4.2) with observation (4.3), with the help of Theorem 1.5. For this purpose, we let $Y := L^2(0,\pi)$, $U = Z := \mathbb{R}^{2N+1}$. We define $A := \partial_{xx}$, with $D(A) := H^2(0,\pi) \cap H_0^1(0,\pi)$; $B\mathbf{u} := \sum_{j=1}^{2N+1} \chi_{\omega_j}(x)u_j$, for all $\mathbf{u} = (u_j)_{1\le j\le 2N+1} \in U$; and $Cy = (\int_{\hat{\omega}_j} y(x)\mathrm{d}x)_{j=1}^{2N+1}$. Then one can directly check that the operator A generates a strongly continuous semigroup on Y, $B \in \mathcal{L}(U;Y)$ and $C \in \mathcal{L}(Y;Z)$. Thus, the systems (4.2) and (4.3) can be put into the framework (1.5) and (1.6) with $\mathbf{A} := \mathbf{A}_n := \mathrm{diag}(\underbrace{A,\cdots,A}_{n})$, $\mathbf{B} := \mathbf{B}_m := \mathrm{diag}(\underbrace{B,\cdots,B}_{m})$, $\mathbf{C} := \mathbf{C}_n := \mathrm{diag}(\underbrace{C,\cdots,C}_{n})$ and $Q = (q_{ij})_{1\le i\le p, 1\le j\le n}$. Moreover, one can directly obtain that the eigenvalues of A are $\lambda_k = -k^2, k = 1,2,\cdots$, and the corresponding eigenfunctions are $\varphi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx), k = 1,2,\cdots$. We let $Y_{P_k} := \mathrm{span}\{\varphi_j, j = 1,\cdots,k\}$ and denote by P_k the orthogonal projection from Y onto Y_{P_k}.

As an application of Theorem 1.5, we have the following:

Theorem 4.1. *Let $\beta > 0$. Suppose that the pairs (S,D) and (S,Q) satisfy $(A3)$ and $(A4)$ respectively, the sampling period T satisfies $(A5)$, and the number N satisfies*

$$N \ge \lfloor \sqrt{\bar{\sigma}(S) + \beta} \rfloor. \tag{4.4}$$

Then the system (4.2) with observation (4.3) is β-output feedback stabilizable.

Remark 4.2. Theorem 4.1 shows that, to guarantee the output feedback stabilizability, the number of actuators and observers should be large enough (depending on S and β). It should be stressed that, in above setting, we do not impose any requirement on the location of the control intervals ω_j and the observing intervals $\hat{\omega}_j$, $j = 1,2,\cdots,2N+1$. The number N might be reduced if one chooses control intervals and observing intervals with appropriate locations (might depend on the nodal points of the "unstable" eigenfunctions). Such kind of problem has been studied for stabilizability of continuous-time control systems (see [28] and [29]). However, we shall not go into this problem in this work.

Before proving Theorem 4.1, we present the next two lemmas whose proofs will be given after the proof of Theorem 4.1.

Lemma 4.3. *Let $\beta > 0$. Then, with the assumptions above, $P^\beta := P_N$ satisfies the conditions $(A0)$ and $(A1)$.*

Lemma 4.4. *Let $\beta > 0$. Then, with the assumptions above, the condition $(A2)$ holds with $P^\beta = P_N$.*

Now, we give the proof of Theorem 4.1.

Proof of Theorem 4.1. By assumptions of the theorem, $(A3)$, $(A4)$ and $(A5)$ hold. Meanwhile, $(A0)$, $(A1)$ and $(A2)$ are true respectively because Lemma 4.3 and Lemma 4.4. Hence, according to Theorem 1.5, the system (4.2) with observation (4.3) is β-output feedback stabilizable. □

We give now the proofs of Lemma 4.3 and Lemma 4.4.

Proof of Lemma 4.3. Since $\{\varphi_j\}_{j=1}^\infty$ is an orthonormal basis of Y, any $\varphi \in Y$ can be expressed as $\varphi = \sum_{j=1}^\infty c_j\varphi_j$ where c_j $(j \in \mathbb{N}^+)$ are the fourier coefficients. Thus, we have

$$(I - P_N)\mathrm{e}^{A^*t}\varphi = \sum_{j=N+1}^{\infty} \mathrm{e}^{-j^2t}c_j\varphi_j, \ \ \forall t > 0; \ \ P_N\varphi = \sum_{j=1}^{N} c_j\varphi_j. \tag{4.5}$$

We see from (4.4) that $(N+1)^2 > \bar{\sigma}(S) + \beta$. Meanwhile, it follows by (4.5) that

$$\|(I - P_N)\mathrm{e}^{A^*t}\varphi\|_Y \le \mathrm{e}^{-(N+1)^2t}\|\varphi\|_Y, \ \ \forall t > 0.$$

It is obvious that $P_N e^{A^* t} = e^{A^* t} P_N, \forall t > 0$, and for each $\varphi \in Y$, $P_N\varphi \in D(A)$. These lead to the condition $(A0)$.

Next, we show (1.16) in $(A1)$. By (4.5) and the definition of B, we find

$$\|P_N\varphi\|_Y^2 = \sum_{k=1}^N c_k^2, \quad \|B^* P_N \varphi\|_U^2 = \|\sum_{k=1}^N c_k \mathbf{v}_k\|_U^2, \tag{4.6}$$

where $\mathbf{v}_k = (\int_{\omega_1} \sin(kx)\mathrm{d}x, \cdots, \int_{\omega_{2N+1}} \sin(kx)\mathrm{d}x)^\top \in U$. Since

$$\sin(kx) = \frac{e^{\imath kx} - e^{-\imath kx}}{2\imath},$$

we can write

$$\mathbf{v}_k = \frac{1}{2\imath}\mathbf{w}_k - \frac{1}{2\imath}\mathbf{w}_{-k}, \quad k = 1, 2, \cdots, N,$$

where $\mathbf{w}_k = (\int_{\omega_1} e^{\imath kx}\mathrm{d}x, \cdots, \int_{\omega_{2N+1}} e^{\imath kx}\mathrm{d}x)^\top, k = -N, \cdots, N$. Thus, by setting $\tilde{U} := \mathbb{C}^{2N+1}$, we obtain

$$\|B^* P_N\varphi\|_U^2 = \|\sum_{k=-N}^N \tilde{c}_k \mathbf{w}_k\|_{\tilde{U}}^2, \tag{4.7}$$

where $\tilde{c}_k = \frac{1}{2\imath}c_k$, if $k = 1, \cdots, N$; $\tilde{c}_k = -\frac{1}{2\imath}c_k$, if $k = -N, \cdots, -1$; and $\tilde{c}_0 = 0$. Since $\sum_{k=-N}^N |\tilde{c}_k|^2 = \frac{1}{2}\sum_{k=1}^N c_k^2$, we see that the condition (1.16) in $(A1)$ is ensured by the linear independence of the vectors $\mathbf{w}_k$, $k = -N, \cdots, N$, in $\tilde{U}$. Thus, it suffices to show the following matrix is invertible:

$$\mathbb{A} = \begin{pmatrix} \frac{1}{-\imath N}e^{-Na_1\imath}\beta_{-N} & \cdots & \frac{1}{-\imath}e^{-a_1\imath}\beta_{-1} & l_0 & \frac{1}{\imath}e^{a_1\imath}\beta_1 & \cdots & \frac{1}{\imath N}e^{Na_1\imath}\beta_N \\ \frac{1}{-\imath N}e^{-Na_2\imath}\beta_{-N} & \cdots & \frac{1}{-\imath}e^{-a_2\imath}\beta_{-1} & l_0 & \frac{1}{\imath}e^{a_2\imath}\beta_1 & \cdots & \frac{1}{\imath N}e^{Na_2\imath}\beta_N \\ \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{-\imath N}e^{-Na_{2N+1}\imath}\beta_{-N} & \cdots & \frac{1}{-\imath}e^{-a_{2N+1}\imath}\beta_{-1} & l_0 & \frac{1}{\imath}e^{a_{2N+1}\imath}\beta_1 & \cdots & \frac{1}{\imath N}e^{Na_{2N+1}\imath}\beta_N \end{pmatrix},$$

where $l_0 = |b_j - a_j|, j = 1, 2, \cdots, 2N+1$, and

$$\beta_k = e^{\imath k l_0} - 1, \quad 0 < |k| \le N. \tag{4.8}$$

To this aim, we compute the determinant of $\mathbb{A}$. Clearly,

$$\det \mathbb{A} = D_1 D_2, \tag{4.9}$$

where

$$D_1 = l_0 \prod_{0<|k|\le N} \frac{1}{\imath k}\beta_k = \frac{l_0}{(N!)^2} \prod_{0<|k|\le N} \beta_k \tag{4.10}$$

and

$$D_2 = \begin{vmatrix} e^{-Na_1\imath} & \cdots & e^{-a_1\imath} & 1 & e^{a_1\imath} & \cdots & e^{Na_1\imath} \\ e^{-Na_2\imath} & \cdots & e^{-a_2\imath} & 1 & e^{a_2\imath} & \cdots & e^{Na_2\imath} \\ \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{-Na_{2N+1}\imath} & \cdots & e^{-a_{2N+1}\imath} & 1 & e^{a_{2N+1}\imath} & \cdots & e^{Na_{2N+1}\imath} \end{vmatrix}.$$

It is clear that D_2 equals to $e^{-N(a_1+a_2+\cdots+a_{2N+1})\imath}$ times the Vandermonde determinant:

$$\begin{vmatrix} 1 & \cdots & e^{(N-1)a_1\imath} & e^{Na_1\imath} & \cdots & e^{2Na_1\imath} \\ 1 & \cdots & e^{(N-1)a_2\imath} & e^{Na_2\imath} & \cdots & e^{2Na_2\imath} \\ \cdots & & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & e^{(N-1)a_{2N+1}\imath} & e^{Na_{2N+1}\imath} & \cdots & e^{2Na_{2N+1}\imath} \end{vmatrix}$$
$$= \prod_{1\le j<k\le 2N+1} (e^{a_j\imath} - e^{a_k\imath}).$$

So

$$D_2 = e^{-N(a_1+a_2+\cdots+a_{2N+1})\imath} \prod_{1\le j<k\le 2N+1} (e^{a_j\imath} - e^{a_k\imath}). \tag{4.11}$$

Since $a_j \neq a_k$ when $j \neq k$, we obtain from (4.11) that $D_2 \neq 0$. Meanwhile, by (4.1) we see that $0 < l_0 < \frac{\pi}{2N+1}$. It follows by (4.8) and (4.10) that $D_1 \neq 0$. Thus, by (4.9), we see that $\det \mathbb{A} \neq 0$. Hence, the matrix $\mathbb{A}$ is invertible.

Similarly as the proof of (1.16) in (A1), we can show that (1.17) in (A1) is true. This completes the proof of Lemma 4.3. □

Proof of Lemma 4.4. To show (*A*2), it suffices to prove that, for each $\gamma \in \sigma(S) \cap \mathbb{C}^+_{-\lambda_0-\beta}$, for each $g \in Y_{P_N}$, there is $f := \sum_{k=1}^N c_k\varphi_k \in Y_{P_N}$, such that,

$$\int_0^T e^{(A+\gamma I)t} dt f = g, \tag{4.12}$$

and there is $c > 0$ such that

$$\|f\|_Y \le c\|g\|_Y. \tag{4.13}$$

We denote $g = \sum_{k=1}^N d_k\varphi_k$ (with $d_k \in \mathbb{R}$). Then $\|g\|_Y = (\sum_{k=1}^N d_k^2)^{1/2}$. The equation (4.12) can be equivalently written as

$$\int_0^T e^{(\lambda_k+\gamma)t} dt c_k = d_k, \quad k = 1, \cdots, N. \tag{4.14}$$

By Assumption (*A*5), we see that, for each $k = 1, \cdots, N$, $\text{Im}(\lambda_k + \gamma) = \text{Im}\gamma \neq \frac{2j\pi}{T}, \forall j \in \mathbb{Z}$. Hence, $\int_0^T e^{(\lambda_k+\gamma)t} dt \neq 0$, for each $k = 1, \cdots, N$. Therefore, each equation in (4.14) has a unique solution c_k, and $f := \sum_{k=1}^N c_k\varphi_k \in Y_{P_N}$ satisfies the equation (4.12). The inequality (4.13) obviously holds true. This completes the proof. □

4.1.2. Pointwise control and pointwise sampling

Let $T>0$ and $N\in\mathbb{N}^+$. Let $\{x_j\}_{j=1}^{2N+1}\subset(0,\pi)$, $\{\hat{x}_j\}_{j=1}^{2N+1}\subset(0,\pi)$, where $x_j,\hat{x}_j, j=1,2,\cdots,2N+1$ satisfy

$$x_j\neq x_k, \hat{x}_j\neq\hat{x}_k \ \text{ for all } \ 1\leq j,k\leq 2N+1. \tag{4.15}$$

We consider the following sampling control system coupled by heat equations over $(0,\pi)$:

$$\begin{cases}\partial_t\mathbf{y}(t,x)=\mathbf{\Delta}_n\mathbf{y}(t,x)+S\mathbf{y}(t,x)\\ \qquad\qquad +\sum_{k=0}^{\infty}\sum_{j=1}^{2N+1}\chi_{[kT,(k+1)T)}(t)\delta(x-x_j)D\mathbf{u}_{kj}, \ \ (t,x)\in\mathbb{R}^+\times(0,\pi),\\ \mathbf{y}(t,x)|_{x=0}=\mathbf{y}(t,x)|_{x=\pi}=0,\end{cases} \tag{4.16}$$

with output data

$$\mathbf{z}_k=((\sum_{i=1}^{n}q_{1i}y_i(kT,\hat{x}_j))_{j=1}^{2N+1},\cdots,(\sum_{i=1}^{n}q_{pi}y_i(kT,\hat{x}_j))_{j=1}^{2N+1}), \tag{4.17}$$

where $\mathbf{y}(t,x)=(y_1(t,x),\cdots,y_n(t,x))^\top\in\mathbb{R}^n$ for each $(t,x)\in\mathbb{R}^+\times(0,\pi)$, $\mathbf{u}_{kj}\in\mathbb{R}^m$ for all $k\in\mathbb{N}$ and $j=1,\cdots,2N+1$, $\mathbf{\Delta}_n=\text{diag}(\underbrace{\partial_{xx},\cdots,\partial_{xx}}_{n})$, $\delta(\cdot)$ is the Dirac function, $q_{li}\in\mathbb{R}$, for all $l=1,\cdots,p$ and $i=1,\cdots,n$. *Notice that in* (4.17)*, the sensors provide discrete in time and point in space measurements of the state. This can be viewed as another kind of time-space sampling.*

We aim to study the β-output feedback stabilizability for the system (4.16) with observation (4.17), with the help of Theorem 1.5. For this purpose, we let $Y:=L^2(0,\pi)$, $U=Z:=\mathbb{R}^{2N+1}$. We define $A:=\partial_{xx}$, with $D(A):=H^2(0,\pi)\cap H_0^1(0,\pi)$; $B\mathbf{u}:=\sum_{j=1}^{2N+1}\delta(x-x_j)u_j$, for all $\mathbf{u}=(u_j)_{1\leq j\leq 2N+1}\in U$; and $Cy=(y(\hat{x}_j))_{j=1}^{2N+1}$, for all $y\in Y_{1/2}$. Then one can directly check that the operator A generates a strongly continuous semigroup on Y, $B\in\mathcal{L}(U;Y_{-1/2})$ and $C\in\mathcal{L}(Y_{1/2};Z)$. Thus, the systems (4.16) and (4.17) can be put into the framework (1.5) and (1.6) with $\mathbf{A}:=\mathbf{A}_n:=\text{diag}(\underbrace{A,\cdots,A}_{n})$, $\mathbf{B}:=\mathbf{B}_m:=\text{diag}(\underbrace{B,\cdots,B}_{m})$, $\mathbf{C}:=\mathbf{C}_n:=\text{diag}(\underbrace{C,\cdots,C}_{n})$ and $Q=(q_{ij})_{1\leq i\leq p,1\leq j\leq n}$. Moreover, one can directly obtain that the eigenvalues of A are $\lambda_k=-k^2, k=1,2,\cdots$, and the corresponding eigenfunctions are $\varphi_k(x)=\sqrt{\frac{2}{\pi}}\sin(kx), k=1,2,\cdots$. We let $Y_k:=\text{span}\{\varphi_j, j=1,\cdots,k\}$ and denote by P_k the orthogonal projection from Y onto Y_k.

As an application of Theorem 1.5, we have the following:

Theorem 4.5. *Let $\beta>0$. Suppose that the pairs (S,D) and (S,Q) satisfy $(A3)$ and $(A4)$ respectively, the sampling period T satisfies $(A5)$, and the number N satisfies*

$$N\geq\lfloor\sqrt{\bar{\sigma}(S)+\beta}\rfloor. \tag{4.18}$$

Then the system (4.16) with observation (4.17) is β-output feedback stabilizable.

Proof of Theorem 4.5. By assumptions of the theorem, $(A3)$, $(A4)$ and $(A5)$ hold. Meanwhile, we can follow almost the same arguments as the proof of Lemma 4.3 to show that $P^\beta:=P_N$ satisfies the conditions $(A0)$ and $(A1)$ for N satisfying the condition (4.18). Moreover, $(A2)$ is true because Lemma 4.4. Hence, according to Theorem 1.5, the system (4.16) with observation (4.17) is β-output feedback stabilizable. □

4.2. System coupled by sampling control heat equations over $\mathbb{R}$

Let $S \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{n\times m}$. Let $T > 0$ and $N \in \mathbb{N}^+$. Let $b, \hat{b} \in (0, 1/2]$ and let

$$\omega_j = \Big[\frac{a_j - b}{N}, \frac{a_j + b}{N}\Big],\;\; \hat{\omega}_j = \Big[\frac{\hat{a}_j - \hat{b}}{N}, \frac{\hat{a}_j + \hat{b}}{N}\Big] \;\text{ for each }\; j \in \mathbb{Z},$$

where $\{a_j\}_{j\in\mathbb{Z}}, \{\hat{a}_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ satisfy

$$\inf_{k,l\in\mathbb{Z}} |a_k - a_l| > 0, \inf_{k,l\in\mathbb{Z}} |\hat{a}_k - \hat{a}_l| > 0; \quad \exists L > 0 \text{ s.t.} |a_j - j| \le L,\; |\hat{a}_j - j| \le L,\;\; \forall\, j \in \mathbb{Z}. \tag{4.19}$$

We consider the following sampling control system coupled by heat equations over $\mathbb{R}$:

$$\partial_t \mathbf{y}(t, x) = \mathbf{\Delta}_n \mathbf{y}(t, x) + S\mathbf{y}(t, x) + \sum_{k=0}^{+\infty}\sum_{j=-\infty}^{+\infty} \chi_{[kT,(k+1)T)}(t)\chi_{\omega_j}(x)D\mathbf{u}_{kj},\;\; (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{4.20}$$

with output data

$$\mathbf{z}_k = ((\sum_{i=1}^{n}\int_{\hat{\omega}_j} q_{1i}y_i(kT, x)\mathrm{d}x)_{j=-\infty}^{+\infty}, \cdots, (\sum_{i=1}^{n}\int_{\hat{\omega}_j} q_{pi}y_i(kT, x)\mathrm{d}x)_{j=-\infty}^{+\infty}). \tag{4.21}$$

Here $\mathbf{y}(t, x) = (y_1(t, x), \cdots, y_n(t, x))^\top \in \mathbb{R}^n$ for each $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, $\mathbf{u}_{kj} \in \mathbb{R}^m$ for all $k \in \mathbb{N}$ and $j \in \mathbb{Z}$, $\mathbf{\Delta}_n = \mathrm{diag}(\underbrace{\partial_{xx}, \cdots, \partial_{xx}}_{n})$, χ_{ω_j} is the characteristic function of ω_j, $q_{li} \in \mathbb{R}$, for all $l = 1, \cdots, p$ and $i = 1, \cdots, n$.

We will study the β-output feedback stabilizability for the system (4.20) with observation (4.21), with the help of Theorem 1.5. For this purpose, we let $Y := L^2(\mathbb{R})$ and $U = Z := l^2(\mathbb{Z}; \mathbb{R})$. We define $A := \partial_{xx}$, with $D(A) := H^2(\mathbb{R})$, $Bu := \sum_{j=-\infty}^{+\infty} \chi_{\omega_j}(x)u_j$, for all $u = (u_j)_{j\in\mathbb{Z}} \in U$, and $Cy = (\int_{\hat{\omega}_j} y(x)\mathrm{d}x)_{j=-\infty}^{+\infty}$, for all $y \in Y$. Then one can directly check that A generates a strongly continuous semigroup on Y, $B \in \mathcal{L}(U; Y)$, and $C \in \mathcal{L}(Y; Z)$. Thus, the systems (4.20) and (4.21) can be put into the framework (1.5) and (1.6) with $\mathbf{A} := \mathrm{diag}(\underbrace{A, \cdots, A}_{n})$, $\mathbf{B} := \mathrm{diag}(\underbrace{B, \cdots, B}_{m})$, $\mathbf{C} := \mathrm{diag}(\underbrace{C, \cdots, C}_{p})$ and $Q = (q_{ij})_{1\le i\le p, 1\le j\le n}$.

Given $h > 0$, we define the subspace $Y_h^{\mathcal{F}} := \{f \in Y \;:\; \mathrm{supp}\mathcal{F}(f) \subset [-h, h]\}$ in Y (where $\mathcal{F}$ is the Fourier transform). Then we define the operator $P_h : Y \to Y_h^{\mathcal{F}}$ by $P_h f = \mathcal{F}^{-1}(\chi_{|\xi|\le h}(\xi)\mathcal{F}(f)),\; f \in Y$. One can easily verify that the operator P_h is an orthogonal projection onto $Y_h^{\mathcal{F}}$.

For the output feedback stabilizability of system (4.20) with observation (4.21), we have the following result.

Theorem 4.6. *Let $\beta > 0$. Suppose that $\{a_k\}_{k\in\mathbb{Z}}$ satisfies (4.19); the pairs (S, D) and (S, Q) satisfy (A3) and (A4) respectively; the sampling period T satisfies (A5); the number N satisfies*

$$N > \frac{\sqrt{\bar{\sigma}(S) + \beta}}{\pi}. \tag{4.22}$$

Then system (4.20) with observation (4.21) is β-output feedback stabilizable.

Before proving it, we give the following lemmas whose proofs will be given later.

Lemma 4.7. *Let $\beta > 0$. Then with the assumptions above, $P^\beta := P_{N\pi}$ satisfies the conditions (A0) and (A1).*

Lemma 4.8. *Let $\beta > 0$. Then with the assumptions above, the condition (A2) holds with $P^\beta = P_{N\pi}$.*

Now, we are in the position proving Theorem 4.6.

Proof of Theorem 4.6. The conditions $(A3)$, $(A4)$ and $(A5)$ are already assumed. By Lemma 4.7, we see that the projection $P^\beta := P_{N\pi}$ with $N \in \mathbb{N}^+$ obeying (4.22) satisfies the conditions $(A0)$ and $(A1)$. By Lemma 4.8, we see that the condition $(A2)$ holds. Thus, according to Theorem 1.5, the system (4.20) with observation (4.21) is β-output feedback stabilizable. This completes the proof of Theorem 4.6. □

We end this subsection with the proofs of Lemma 4.7 and Lemma 4.8.

Proof of Lemma 4.7. Since $\mathcal{F}(\mathrm{e}^{A^*t}\varphi)(\xi) = \mathrm{e}^{-|\xi|^2 t}\mathcal{F}(\varphi)(\xi)$, we see by Plancherel theorem that

$$\|(I - P_{N\pi})\mathrm{e}^{A^*t}\varphi\|_Y = \|\chi_{|\xi|>N\pi}\mathcal{F}(\mathrm{e}^{A^*t}\varphi)(\xi)\|_{L^2_\xi(\mathbb{R})} = \|\chi_{|\xi|>N\pi}\mathrm{e}^{-|\xi|^2 t}\mathcal{F}(\varphi)(\xi)\|_{L^2_\xi(\mathbb{R})} \le \mathrm{e}^{-(N\pi)^2 t}\|\varphi\|_Y.$$

We see from (4.22) that $(N\pi)^2 > \bar{\sigma}(S) + \beta$. It is obvious that $P_{N\pi}\mathrm{e}^{A^*t} = \mathrm{e}^{A^*t}P_{N\pi}, \forall t > 0$, and for each $\varphi \in Y$, $P_{N\pi}\varphi \in D(A)$. These lead to the condition $(A0)$.

Now, we prove (1.16) in $(A1)$. We first claim that there is $c > 0$ such that

$$\int_{\mathbb{R}} |g(x)|^2 \mathrm{d}x \le c \sum_{k\in\mathbb{Z}} |\int_{[a_k-b,a_k+b]} g(x)\mathrm{d}x|^2 \quad \text{for each} \quad g \in Y_\pi^{\mathcal{F}}. \tag{4.23}$$

For this purpose, we arbitrarily fix a $g \in Y_\pi^{\mathcal{F}}$. Since $\mathrm{supp}\mathcal{F}(g) \subset [-\pi, \pi]$, we can use the Fourier inversion formula to get

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\imath x\xi}\mathcal{F}(g)(\xi)\mathrm{d}\xi = \frac{1}{\sqrt{2\pi}} \int_{[-\pi,\pi]} \mathrm{e}^{\imath x\xi}\mathcal{F}(g)(\xi)\mathrm{d}\xi. \tag{4.24}$$

Based on (4.24) and Fubini's theorem, we obtain that for each $k \in \mathbb{Z}$,

$$\begin{aligned}\int_{[a_k-b,a_k+b]} g(x)\mathrm{d}x &= \frac{1}{\sqrt{2\pi}} \int_{[a_k-b,a_k+b]} \int_{[-\pi,\pi]} \mathrm{e}^{\imath x\xi}\mathcal{F}(g)(\xi)\mathrm{d}\xi\mathrm{d}x \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \int_{[-\pi,\pi]} \mathrm{e}^{\imath a_k\xi}\frac{\sin b\xi}{\xi}\mathcal{F}(g)(\xi)\mathrm{d}\xi.\end{aligned} \tag{4.25}$$

Meanwhile, by (4.19), we can apply the classical theorem of Duffin and Schaeffer ([30], see also [31], Thm. 9.8.1, p. 233) to find that the exponentials $\{\mathrm{e}^{\imath a_k\xi}\}$ is a Fourier frame for $L^2[-\pi, \pi]$, *i.e.*, there are constants c_1 and $c_2 > 0$ such that when $h \in L^2(-\pi, \pi)$,

$$c_1 \int_{[-\pi,\pi]} |h(\xi)|^2 \mathrm{d}\xi \le \sum_{k\in\mathbb{Z}} |\int_{[-\pi,\pi]} \mathrm{e}^{\imath a_k\xi}h(\xi)d\xi|^2 \le c_2 \int_{[-\pi,\pi]} |h(\xi)|^2 \mathrm{d}\xi. \tag{4.26}$$

Taking $h(\xi) = \frac{\sin b\xi}{\xi}\mathcal{F}(g)(\xi)$ in (4.26), using (4.25), we obtain

$$\int_{[-\pi,\pi]} |\frac{\sin b\xi}{\xi}\mathcal{F}(g)(\xi)|^2 \mathrm{d}\xi \le \frac{\pi}{2c_1} \sum_{k\in\mathbb{Z}} |\int_{[a_k-b,a_k+b]} g(x)\mathrm{d}x|^2. \tag{4.27}$$

However, since $\sin x \ge \frac{2}{\pi}x$ when $x \in [0, \frac{\pi}{2}]$, and because $b \in (0, \frac{1}{2}]$, we have

$$|\frac{\sin b\xi}{\xi}| \ge \frac{2b}{\pi}, \quad \xi \in [-\pi, \pi].$$

With (4.27), the above yields

$$\int_{[-\pi,\pi]} |\mathcal{F}(g)(\xi)|^2 \mathrm{d}\xi \leq \frac{\pi^3}{8b^2 c_1} \sum_{k\in\mathbb{Z}} |\int_{[a_k-b,a_k+b]} g(x)\mathrm{d}x|^2.$$

This, together with the Plancherel identity, leads to (4.23).

Finally, given $f \in Y^{\mathcal{F}}_{N\pi}$, we let $g(x) = f(\frac{x}{N})$. Since

$$\mathcal{F}(g)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\frac{x}{N})\mathrm{e}^{-\imath x\xi}\mathrm{d}x = \frac{N}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\mathrm{e}^{-\imath Nx\xi}\mathrm{d}x = N\mathcal{F}(f)(N\xi),$$

we see that $g \in Y^{\mathcal{F}}_{\pi}$. Thus, we can use (4.23) to get

$$\int_{\mathbb{R}} |f(x)|^2\mathrm{d}x = \frac{1}{N}\int_{\mathbb{R}} |g(x)|^2\mathrm{d}x \leq \frac{c}{N}\sum_{k\in\mathbb{Z}} |\int_{[a_k-b,a_k+b]} g(x)\mathrm{d}x|^2 = cN\sum_{k\in\mathbb{Z}} |\int_{[\frac{a_k-b}{N},\frac{a_k+b}{N}]} f(x)\mathrm{d}x|^2,$$

i.e., $P_{N\pi}$ satisfies the inequality (1.16) in $(A1)$.

Similarly, we can prove that (1.17) in $(A1)$ holds true. This completes the proof of Lemma 4.7. □

Proof of Lemma 4.8. Let $g \in Y^{\mathcal{F}}_{N\pi}$ and $\gamma \in \sigma(S) \cap \mathbb{C}^+_{-\beta-\gamma_0}$ be arbitrarily given. Then $\mathrm{supp}(\mathcal{F}(g)) \subset [-N\pi, N\pi]$. By Assumption $(A5)$, we see that there is $h(\cdot) \in L^2(\mathbb{R})$ with $\mathrm{supp}(h) \subset [-N\pi, N\pi]$, such that

$$\int_0^T \mathrm{e}^{(-\xi^2+\gamma)t}\mathrm{d}t h(\xi) = (\mathcal{F}g)(\xi) \text{ for each } \xi \in \mathbb{R}. \tag{4.28}$$

Moreover, there is $c > 0$ such that

$$\|h\|_{L^2(\mathbb{R})} \leq c\|\mathcal{F}g\|_{L^2_\xi(\mathbb{R})}. \tag{4.29}$$

Letting $f = \mathcal{F}^{-1}h$. Then $f \in Y^{\mathcal{F}}_{N\pi}$, and by taking Fourier inverse transform on both sides of (4.28), we see that

$$\int_0^T \mathrm{e}^{(A+\gamma I)t}\mathrm{d}t f = g. \tag{4.30}$$

Moreover, it follows from (4.29) that $\|f\|_Y \leq c\|g\|_Y$. This completes the proof.

□

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Appendix A. The well-posedness of $[A, B, C]$

In this section, we show the well-posedness of $[A, B, C]$. It suffices to prove the following theorem.

Theorem A.1. *Suppose that Assumptions* (H_1) *and* (H_2) *hold true. Then, for each* $t > 0$, $\int_0^t e^{\tilde{A}(t-s)} B ds \in \mathcal{L}(U; Y_{1-\theta})$. *Moreover, for any given* $T > 0$, *there is* $c(T) > 0$, *such that*

$$\sup_{t\in[0,T]} \left\| \int_0^t e^{\tilde{A}(t-s)} B ds \right\|_{\mathcal{L}(U;Y_{1-\theta})} \leq c(T). \tag{A.1}$$

Proof. For $\lambda \in \rho(\tilde{A}) (= \rho(A))$, we have

$$Bu = \lambda(\lambda I - \tilde{A})^{-1} Bu - \tilde{A}(\lambda I - \tilde{A})^{-1} Bu, \quad u \in U.$$

Then,

$$\begin{aligned} &\int_0^t e^{\tilde{A}(t-s)} Buds \\ &= \lambda \int_0^t e^{\tilde{A}(t-s)} (\lambda I - \tilde{A})^{-1} Buds - \int_0^t e^{\tilde{A}(t-s)} \tilde{A}(\lambda I - \tilde{A})^{-1} Buds \\ &= \lambda \int_0^t e^{\tilde{A}(t-s)} (\lambda I - \tilde{A})^{-1} Buds - (e^{\tilde{A}t} - I)(\lambda I - \tilde{A})^{-1} Bu \\ &= (\lambda I - \tilde{A})^{-1+\theta} (\lambda \int_0^t e^{\tilde{A}(t-s)} ds - e^{\tilde{A}t} + I)(\lambda I - \tilde{A})^{-\theta} Bu. \end{aligned} \tag{A.2}$$

Since $B \in \mathcal{L}(U; Y_{-\theta})$, and $(\lambda I - \tilde{A})^{-\theta}$ is an isomorphism from $Y_{-\theta}$ to Y, we see that

$$(\lambda \int_0^t e^{\tilde{A}(t-s)} ds - e^{\tilde{A}t} + I)(\lambda I - \tilde{A})^{-\theta} B \in \mathcal{L}(U; Y). \tag{A.3}$$

Notice that $(\lambda I - \tilde{A})^{-1+\theta}$ is an isomorphism from Y to $Y_{1-\theta}$. It follows from (A.2) and (A.3) that $\int_0^t e^{\tilde{A}(t-s)} B ds \in \mathcal{L}(U; Y_{1-\theta})$. Moreover, the inequality (A.1) holds with $c(T)$ given as follows:

$$c(T) = [(\lambda T + 1) \sup_{t\in[0,T]} \|e^{\tilde{A}t}\|_{\mathcal{L}(Y)} + 1] \|(\lambda I - \tilde{A})^{-\theta} B\|_{\mathcal{L}(U;Y)}.$$

□