

DOMAINS OF DEPENDENCE FOR SUBELLIPTIC WAVE EQUATIONS AND UNIQUE CONTINUATION FOR FRACTIONAL POWERS OF HÖRMANDER'S OPERATORS

NICOLAS BURQ^{1,2,*}  AND CLAUDE ZUILY¹

Abstract. We prove the sharp domain of dependence property for solutions to subelliptic wave equations for sums of squares of vector fields satisfying Hörmander bracket condition. We deduce a unique continuation property for the square root of subelliptic Laplace operators under an additional analyticity condition. Then, with a different, more involved method, we prove the same result of unique continuation for more general s -powers ($0 < s < 1$).

Mathematics Subject Classification. 35A02, 35H20, 35L05.

Received September 16, 2024. Accepted August 29, 2025.

1. INTRODUCTION AND RESULTS

In this article, we are interested in uniqueness properties for *subelliptic wave operators* and fractional powers of such subelliptic operators. In this note, we are interested in unique continuation properties of fractional powers of Hörmander's subelliptic sum of squares operators. In the first part of the paper, we determine the sharp domain of dependence for solutions of subelliptic wave equations in $\mathbf{R} \times \mathbf{R}^d$ with smooth coefficients, a result of independent interest. As an application we give, in the second part, a rather simple proof of unique continuation for square roots of Hörmander's operators with analytic coefficients. Finally, in the third part, we consider the general case of s -powers of these operators ($0 < s < 1$) and we prove a similar result of unique continuation. Let $X = (X_1, \dots, X_r)$ be a set of real vector fields with C^∞ coefficients on \mathbf{R}^d .

Following Hörmander we shall denote by $\mathcal{L}(X_1, \dots, X_r)$ the Lie algebra generated by X_1, \dots, X_r and we shall assume Hörmander's condition, that is,

$$\mathcal{L}(X_1, \dots, X_r) \text{ has maximal rank at every point of } \mathbf{R}^d. \quad (1.1)$$

We shall consider the operator,

$$P = \sum_{j=1}^r X_j^* X_j, \quad (1.2)$$

Keywords and phrases: Subelliptic, waves, domain of dependence, unique continuation.

¹ Université Paris-Saclay, Laboratoire de Mathématique d'Orsay, UMR 8628 du CNRS 91405 Orsay Cedex, France.

² Institut universitaire de France, France.

* Corresponding author: nicolas.burq@universite-paris-saclay.fr

where X_j^* is the adjoint of X_j and denote by H_X^1 the closure of $C_0^\infty(\mathbb{R}^d)$ for the norm

$$\|u\|_{H_X^1}^2 = \sum_j \|X_j u\|_{L^2(\mathbb{R}^d)}^2$$

The quadratic form

$$Q(u, v) = \sum_j (X_j u, X_j v)_{L^2} + (u, v)_{L^2}$$

is continuous on H_X^1 and Riesz representation theorem shows that the operator $P + \text{Id}$ is an isometry from H_X^1 into its dual space H_X^{-1} (which is a distribution space because C_0^∞ is dense in H_X^1). We now recall the definition of the Friedrich's extension of P (still denoted by P) with domain given by,

$$D(P) = \{u \in H_X^1(\mathbf{R}^d) : Pu \in L^2(\mathbf{R}^d)\}. \quad (1.3)$$

It is well known that the operator P is non negative and selfadjoint (we refer to Section 2 for more about Friedrich's extension and the solutions to wave equations)

The functional calculus allows to define for any $U_0 = (u_0, u_1) \in H_X^1 \times L^2$ $u(t) = \cos(\sqrt{P})u_0 + \frac{\sin(\sqrt{P})}{\sqrt{P}}u_1$, the solution to the wave equation $(\partial_t^2 + P)u = 0$, with initial data $(u, \partial_t u)|_{t=0} = (u_0, u_1)$. Our first result (sharp domain of dependence) is the following.

Theorem 1.1. *Let P be defined by (1.2) satisfying condition (1.1).*

Let $U_0 = (u_0, u_1) \in H_X^1 \times L^2$ and consider the solution to the subelliptic wave equation $U = (u, \partial_t u) = S(t)(U_0)$ given by the functional calculus. Let d_P denote the sub-riemmanian distance associated to the operator P (see Sect. 3 for more details), and for $x_0 \in \mathbf{R}^d$, $t_0 > 0$, $\rho > 0$,

$$\begin{aligned} C(t_0) &= \{(t, x) \in [0, t_0] \times \mathbf{R}^d : d_P(x, x_0) < t_0 - t\}, \\ B_P(x_0, \rho) &= \{x \in \mathbf{R}^d : d_P(x, x_0) < \rho\}. \end{aligned} \quad (1.4)$$

Assume that $u_0 = u_1 = 0$ in $B_P(x_0, t_0)$. Then u vanishes within the cone $C(t_0)$.

In the case of operators defined on a compact manifold (instead of \mathbf{R}^d) this result has been proved by Melrose in [1].

Our second result is the following.

Theorem 1.2. *Assume (1.1) and that the coefficients of the vector fields X_j are real analytic. Let $0 < s < 1$. If $u_0 \in D(P^s)$ is such that $u_0 = P^s u_0 = 0$ vanish on a non empty open subset $\omega \subset \mathbf{R}^d$ then $u_0 = 0$ on \mathbf{R}^d .*

Remark 1.3. 1. Unique continuation for square roots of Laplacian with respect to a C^∞ metric has been obtained by Masuda [2]. For more general fractional powers of elliptic operators see for instance, [3], [4], [5] and [6].

2. In the case $s = \frac{1}{2}$ we shall give another proof of Theorem 1.2 based on Theorem 1.1 and on an argument due to Masuda in [2]. This proof is different in essence from that of the case $s \in (0, 1)$ since the latter is based on a uniqueness result by Baouendi-Goulaouic which does not require any ‘‘hyperbolicity’’ condition.
3. Notice that, concerning the unique continuation, for our operators, the case of C^∞ vector fields remain an open question due to the lack of stable uniqueness for the operators ‘‘sum of squares’’ (see [7]).
4. Notice that in the analytic context, some quantitative uniqueness results were recently obtained by Laurent-Léautaud [8] for wave equations associated to hypoelliptic operators. These results appear to be quite different from ours.

Our motivation to prove Theorem 1.2 was to exhibit examples (in sharp contrast with local operators) where the non-local nature of fractional powers of subelliptic operators forced some uniqueness results.

In the following section we first recall some basic facts about Friedrich's extension for sum of squares of vector fields and set up the framework allowing to solve wave equations (Hille Yosida's theorem). We then prove the sharp domain of dependence property for solutions to subelliptic wave equations for sums of squares of vector fields satisfying Hörmander bracket condition in Section 3. In the next section we prove the unique continuation theorem for the square root of Hörmander's operators, relying on a strategy by Masuda and a uniqueness result by Bony (this is for this last result that we need the analyticity assumption). In the last section we extend the result on unique continuation for more general fractional powers relying on works by Stinga-Torrea, Baouendi-Goulaouic and again Bony.

2. FRIEDRICH'S EXTENSION, HILLE-YOSIDA

2.1. Friedrich's extension

Let $X = (X_1, \dots, X_r)$ be a set of real vector fields with C^∞ coefficients on \mathbf{R}^d satisfying condition (1.1). We shall consider the operator,

$$P = \sum_{j=1}^r X_j^* X_j \quad (2.1)$$

where X_j^* is the adjoint of X_j , with domain $C_0^\infty(\mathbf{R}^d) \subset L^2(\mathbf{R}^d)$.

This operator being symmetric and positive on his domain, we shall still denote by P its Friedrichs self adjoint extension.

Definition 2.1. We shall denote by $H_X^1(\mathbf{R}^d)$ the closure of $C_0^\infty(\mathbf{R}^d)$ for the norm,

$$\|v\|_{H_X^1}^2 = \sum_{j=1}^r \|X_j v\|_{L^2(\mathbf{R}^d)}^2 + \|v\|_{L^2(\mathbf{R}^d)}^2.$$

Then the domain of P is given by,

$$D(P) = \{u \in H_X^1(\mathbf{R}^d) : Pu \in L^2(\mathbf{R}^d)\}. \quad (2.2)$$

where, here, P acts in the distribution sense. We endow $D(P)$ with its natural Hilbert norm

$$\|u\|_{D(P)}^2 = \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2.$$

Since $C_0^\infty(\mathbf{R}^d)$ is, by definition, dense in $H_X^1(\mathbf{R}^d)$ we may define,

$$H_X^{-1}(\mathbf{R}^d) = (H_X^1(\mathbf{R}^d))',$$

the dual of $H_X^1(\mathbf{R}^d)$ as a space of distributions. Notice that since $\|u\|_{H_X^1} \geq \|u\|_{L^2}$, we also have

$$L^2 \subset H_X^{-1}, \quad \|u\|_{H_X^{-1}} \leq \|u\|_{L^2}. \quad (2.3)$$

We recall from Riesz theorem.

Lemma 2.2. *The operator $P + \text{Id}$ is an isometry from H_X^1 to H_X^{-1} , and*

$$\forall f, g \in H_X^{-1}, (f, g)_{H_X^{-1}} = \langle (P + \text{Id})^{-1}f, \bar{g} \rangle_{H_X^{-1} \times H_X^1} = \langle (P + \text{Id})^{-1}\bar{g}, f \rangle_{H_X^{-1} \times H_X^1} \quad (2.4)$$

Proof. Indeed, from Riesz theorem, for any $v \in H_X^{-1}$, there exists a unique $u \in H_X^1$ such that

$$\forall w \in H_X^1, (u, w)_{H_X^1} = \langle v, \bar{w} \rangle_{H_X^{-1} \times H_X^1} \quad (2.5)$$

Since $C_0^\infty(\mathbb{R}^d)$ is dense in H_X^1 , we deduce that this is equivalent to

$$\forall w \in C_0^\infty(\mathbb{R}^d), (u, w)_{H_X^1} = \langle (P + \text{Id})u, \bar{w} \rangle = \langle v, \bar{w} \rangle_{H_X^{-1} \times H_X^1}$$

and hence, u is the unique distribution $u \in H_X^1$ solution to

$$(P + \text{Id})u = v.$$

We denote $u = (P + \text{Id})^{-1}v$. Taking $w = u$ in (2.5) we get

$$\|u\|_{H_X^1} \leq \|v\|_{H_X^{-1}},$$

while taking the supremum over all $w \in H_X^1$ with $\|w\|_{H_X^1} = 1$ in (2.5) gives,

$$\|v\|_{H_X^{-1}} \leq \|u\|_{H_X^1}.$$

As a consequence, $P + \text{Id}$ is an isometry from H_X^1 to H_X^{-1} . We deduce (2.4) for $f = g \in H_X^{-1}$ (and hence for all $f \in H_X^{-1}$ by polarization). \square

Lemma 2.3. *The operator $P + \text{Id}$ is a bi-continuous isomorphism from $D(P)$ to L^2 (and consequently the norm $\|(P + \text{Id})u\|_{L^2}$ is equivalent to $\|u\|_{D(P)}$).*

Proof. Indeed, we have

$$\|(P + \text{Id})u\|_{L^2} \leq \|Pu\|_{L^2} + \|u\|_{L^2},$$

and from Lemma 2.2 and (2.3),

$$\|(P + \text{Id})^{-1}u\|_{H_X^1} \leq \|u\|_{H_X^{-1}} \leq \|u\|_{L^2}$$

and

$$\|P((P + \text{Id})^{-1}u)\|_{L^2} = \|u - (P + \text{Id})^{-1}u\|_{L^2} \leq 2\|u\|_{L^2}.$$

\square

2.2. The Hille Yosida theorem

As in the sequel, we shall work with several operators with different domains, and in order to navigate between wave equations defined on these different domains, we need to have a unified framework to study these wave equations. So, rather than using the functional calculus for self-adjoint operators, we shall use the more

flexible Hille Yosida theory below. Notice that (see Lem. 2.7) the functional calculus and Hille-Yosida theory give the same wave solutions (when the operator is self adjoint).

Let us recall how Hille-Yosida theory applies to solve the Cauchy problem for the wave equations with respect the subelliptic operator P ,

$$\begin{aligned} \partial_t^2 u + Pu &= 0 \quad \text{in } \mathbf{R} \times \mathbf{R}^d, \\ u|_{t=0} &= u_0, \quad \partial_t u|_{t=0} = u_1. \end{aligned} \quad (2.6)$$

We consider the two operators,

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ -P & 0 \end{pmatrix}, \quad \tilde{\mathcal{P}} = \begin{pmatrix} 0 & 1 \\ -P & 0 \end{pmatrix},$$

acting respectively on,

$$\mathcal{H} = H_X^1 \times L^2, \quad \tilde{\mathcal{H}} = L^2 \times H_X^{-1}, \quad (2.7)$$

with domains,

$$D(\mathcal{P}) = D(P) \times H_X^1, \quad D(\tilde{\mathcal{P}}) = H_X^1 \times L^2, \quad (2.8)$$

where $D(P)$ has been defined in (1.3). Recall that these domains are actually characterized as follows:

$$D(\mathcal{P}) = \{U \in \mathcal{H} : \mathcal{P}_0 U \in \mathcal{H}\}, \quad D(\tilde{\mathcal{P}}) = \{U \in \tilde{\mathcal{H}} : \mathcal{P}_0 U \in \tilde{\mathcal{H}}\},$$

where \mathcal{P}_0 is the operator $\begin{pmatrix} 0 & 1 \\ -P & 0 \end{pmatrix}$ acting on distributions.

Then setting $U = (u, \partial_t u)$, $U_0 = (u_0, u_1)$, the problem (2.6) is equivalent to,

$$\partial_t U + \mathcal{P}U = 0 \quad \text{in } \mathbf{R} \times \mathbf{R}^d, \quad U|_{t=0} = U_0. \quad (2.9)$$

Recall that an operator T , with domain $D(T)$ defined on a Hilbert H is maximal dissipative (resp. accretive) if it satisfies,

$$\Re(TU, U)_H \leq 0 \quad (\text{resp. } \geq 0), \quad \forall U \in D(T),$$

$$\exists \lambda < 0 \quad (\text{resp. } \lambda > 0) : \forall U \in H \quad \exists V \in D(T) : (T + \lambda)V = U.$$

Lemma 2.4. *The operators $\mathcal{P} - Id$ and $\tilde{\mathcal{P}} - Id$ are maximal dissipative (resp. $\mathcal{P} + Id$ and $\tilde{\mathcal{P}} + Id$ are maximal accretive).*

Proof. Let us prove the result first for \mathcal{P} . For $U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in D(P) \times H_X^1$,

$$(\mathcal{P}U, U)_{H_X^1 \times L^2} = (u_1, u_0)_{H_X^1} - (Pu_0, u_1)_{L^2} = (Xu_1, X_0)_{L^2} - (Xu_0, Xu_1)_{L^2}, \quad (2.10)$$

and consequently

$$\Re(\mathcal{P}U, U)_{H_X^1 \times L^2} = 0.$$

It remains to prove that for $\epsilon = \pm 1$, and all $V = (v_0, v_1) \in H_X^1 \times L^2$, we can find $U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in D(P \times H_X^1)$ such that

$$(\mathcal{P} + \epsilon \text{Id})U = V.$$

This is equivalent to

$$u_1 = v_0 - \epsilon u_0, \quad Pu_0 + u_0 = \epsilon v_0 - v_1 \in L^2$$

which has a solution given by

$$u_0 = (P + \text{Id})^{-1}(\epsilon v_0 - v_1), \quad u_1 = v_0 - \epsilon(P + \text{Id})^{-1}(\epsilon v_0 - v_1).$$

(u_0 is a priori only in H_X^1 , but the equation $Pu_0 + u_0 = \epsilon v_0 - v_1$ shows that actually $u_1 \in D(P)$ and hence $u_0 \in H_X^1$).

Let us now turn to the proof of Lemma 2.4 for $\tilde{\mathcal{P}}$. From (2.4), we have for all $U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$

$$\begin{aligned} \left(\tilde{\mathcal{P}}U, U \right)_{L^2 \times H_X^{-1}} &= (u_1, u_0)_{L^2} - (Pu_0, u_1)_{H_X^{-1}} \\ &= (u_1, u_0)_{L^2} - \langle (P + \text{Id})^{-1}(P + \text{Id} - 1)u_0, \overline{u_1} \rangle_{H_X^{-1} \times H_X^1} \\ &= (u_1, u_0)_{L^2} - \langle u_0, \overline{u_1} \rangle_{H_X^{-1} \times H_X^1} + \langle (P + \text{Id})^{-1}u_0, \overline{u_1} \rangle_{H_X^{-1} \times H_X^1} \\ &= (u_1, u_0)_{L^2} - (u_0, u_1)_{L^2} + \langle (P + \text{Id})^{-1}u_0, \overline{u_1} \rangle_{H_X^{-1} \times H_X^1}. \end{aligned} \tag{2.11}$$

We deduce

$$\left| \Re \left(\tilde{\mathcal{P}}U, U \right)_{L^2 \times H_X^{-1}} \right| \leq \|u_0\|_{H^{-1}} \|u_1\|_{H^{-1}} \leq \|u_0\|_{L^2} \|u_1\|_{H^{-1}} \leq \frac{1}{2} (\|u_0\|_{L^2}^2 + \|u_1\|_{H^{-1}}^2).$$

which implies

$$\Re \left((\tilde{\mathcal{P}} - \text{Id})U, U \right)_{L^2 \times H_X^{-1}} \leq 0 \text{ resp } \Re \left((\tilde{\mathcal{P}} + \text{Id})U, U \right)_{L^2 \times H_X^{-1}} \geq 0.$$

To get a solution to

$$(\tilde{\mathcal{P}} \pm \text{Id})U = F,$$

we proceed as in the case \mathcal{P} . □

We deduce that \mathcal{P} and $\tilde{\mathcal{P}}$ are the generators of strongly continuous groups of operators on \mathcal{H} , and $\tilde{\mathcal{H}}, S(t)$ and $\tilde{S}(t)$. which implies

Theorem 2.5 ([9], Thm. 2.10 and Prop. 2.24). *Consider a strongly continuous group of operators $\Sigma(t)$ on a Hilbert space H , and A its infinitesimal generator with domain $D(A)$. Then,*

1. For any $U_0 \in D(A)$, the function $U : t \mapsto U(t) = \Sigma(t)U_0$ is the unique function in $C^1(\mathbf{R}, H) \cap C^0(\mathbf{R}, D(A))$ such that,

$$U(0) = U_0, \quad \frac{d}{dt}U(t) = AU(t).$$

2. For any $U_0 \in H$, the function $U : t \mapsto U(t) = \Sigma(t)U_0$ is the unique function in $C^0(\mathbf{R}, H)$ such that,
 (i) $U(0) = U_0$,
 (ii) for any $\psi \in C_0^1(\mathbf{R})$ we have,

$$\int \psi(t)U(t)dt \in D(A), \quad \text{and} \quad A\left(\int \psi(t)U(t)dt\right) = -\int \psi'(t)U(t)dt,$$

(in other words, U is a distribution in time with values in the domain of A and it satisfies the equation $\partial_t U = AU$ as a distribution in time).

Let us now check that when applying Theorem 2.5 to \mathcal{P} , we can weaken the continuity assumption on U for the uniqueness part in (2), and assume only $U \in C^0(\mathbf{R}, \tilde{\mathcal{H}})$, while still assuming $U_0 \in \mathcal{H}$. Indeed, this follows directly from the following observation.

Lemma 2.6. *Let $U_0 \in \mathcal{H} \subset \tilde{\mathcal{H}}$. According to (2.7) and (2.8) we have, $\mathcal{H} = D(\tilde{\mathcal{P}})$. Applying (1) in Theorem 2.5 with $H = \tilde{\mathcal{H}}$, $A = \tilde{\mathcal{P}}$ and (2) with $H = \mathcal{H}$, $A = \mathcal{P}$ we obtain two solutions $\tilde{U}(t)$ and $U(t)$. Then, these two solutions coincide i.e. we have $\tilde{U}(t) = U(t)$.*

Proof. Indeed, since $U_0 \in D(\tilde{\mathcal{P}})$, we have,

$$\tilde{U}(t) \in C^1(\mathbf{R}, \tilde{\mathcal{H}}) \cap C^0(\mathbf{R}, \mathcal{H}),$$

and it satisfies,

$$\frac{d}{dt}\tilde{U}(t) = \tilde{\mathcal{P}}\tilde{U}(t), \quad \tilde{U}(0) = U_0.$$

This implies,

$$\tilde{\mathcal{P}}\left(\int \psi(t)\tilde{U}(t)dt\right) = \int \psi(t)\tilde{U}'(t)dt = -\int \psi'(t)\tilde{U}(t)dt,$$

where the equality holds in $\tilde{\mathcal{H}}$. We deduce that in the distribution sense,

$$\mathcal{P}_0\left(\int \psi(t)\tilde{U}(t)dt\right) = \int \psi(t)\tilde{U}'(t)dt \in \mathcal{H},$$

which also implies by definition that,

$$\int \psi(t)\tilde{U}(t)dt \in D(\mathcal{P}).$$

This implies by the point (2) in Theorem 2.5 that $\tilde{U}(t) = S(t)U_0 = U(t)$. □

We end this section with an elementary lemma showing that Hille Yosida theory and the functional calculus of self-adjoint operators allow to define the same solutions to wave equations

Lemma 2.7. *Let Q non negative defined on $L^2(\mathbb{R}^d)$ with domain $D(Q)$ be self-adjoint. Then the two solutions of wave equations given respectively by Hille-Yosida Theorem*

$$U(t)U_0 = S_A(t), A = \begin{pmatrix} 0 & 1 \\ -Q & 0 \end{pmatrix},$$

with A defined on $D(Q^{1/2}) \times L^2$ with domain $D(Q) \times D(Q^{1/2})$ and the functional calculus of self-adjoint operators

$$V(t) = (u(t), \partial_t u(t)), \quad u(t) = \cos(t\sqrt{Q})u_0 + \frac{\sin(t\sqrt{Q})}{\sqrt{p}}u_1$$

are equal.

Proof. It is clear that $t \mapsto V(t) \in C(\mathbb{R}; D(Q^{1/2}) \times L^2)$ satisfies conditions i) in Theorem 2.5 and to conclude, it is enough to prove that it satisfies also conditions ii) in Theorem 2.5. We have since $u_0 \in D(Q^{1/2})$ and $u_1 \in L^2$,

$$\begin{aligned} \int \psi(t)u(t)dt &= \int \psi(t) \cos(t\sqrt{Q})u_0 + \frac{\sin(t\sqrt{Q})}{\sqrt{Q}}u_1 dt \\ &= \int \psi(t) \frac{d}{dt} \left(\frac{\sin(t\sqrt{Q})}{\sqrt{Q}}u_0 - \frac{\cos(t\sqrt{Q})}{Q}u_1 \right) dt = - \int \psi'(t) \left(\frac{\sin(t\sqrt{Q})}{\sqrt{Q}}u_0 - \frac{\cos(t\sqrt{Q})}{Q}u_1 \right) dt \in D(Q). \end{aligned} \quad (2.12)$$

Similarly

$$\begin{aligned} \int \psi(t)\partial_t u(t)dt &= \int \psi(t) \cos(t\sqrt{Q})u_1 - \sqrt{Q} \sin(t\sqrt{Q})u_0 dt \\ &= \int \psi(t) \frac{d}{dt} \left(\cos(t\sqrt{Q})u_0 + \frac{\sin(t\sqrt{Q})}{\sqrt{Q}}u_1 \right) dt = - \int \psi'(t) \left(\cos(t\sqrt{Q})u_0 + \frac{\sin(t\sqrt{Q})}{\sqrt{Q}}u_1 \right) dt \in D(Q^{1/2}). \end{aligned} \quad (2.13)$$

and from (2.12), (2.13) we deduce

$$\begin{pmatrix} 0 & 1 \\ -Q & 0 \end{pmatrix} \int \psi(t) \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} dt = - \int \psi'(t) \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} dt$$

□

3. DOMAINS OF DEPENDENCE

3.1. The metrics

We shall denote by d_P the sub-Riemannian distance attached to the operator P as defined in [10] Proposition 3.1. It is defined as follows. If $x, y \in \mathbf{R}^d$ we set,

$$\begin{aligned} \mathcal{C}(x, y) &= \{ \sigma : [0, 1] \rightarrow \mathbf{R}^d, \sigma \text{ is a Lipschitz curve,} \\ &\quad \sigma(0) = x, \sigma(1) = y \text{ and } \dot{\sigma}(t) = \sum_{j=1}^r a_j(t) X_j(\sigma(t)) \text{ for a.e. } t. \}, \end{aligned}$$

Then,

$$d_P(x, y) = \inf \left\{ \left(\sum_{j=1}^r \int_0^1 |a_j(t)|^2 dt \right)^{\frac{1}{2}}, \quad \sigma \in \mathcal{C}(x, y) \right\}. \quad (3.1)$$

Notice that if we denote by Δ_0 the flat Laplacian in \mathbf{R}^d then $d_{\Delta_0}(x, y) = |x - y|$ (the Euclidian distance).

3.2. Domains of dependence

The main result of this section concerns the domain of dependence of the solution of the subelliptic wave problem,

$$\begin{aligned} (\partial_t^2 + P)u &= 0 \text{ in } \mathbf{R} \times \mathbf{R}^d, \\ u|_{t=0} &= u_0, \quad \partial_t u|_{t=0} = u_1. \end{aligned} \tag{3.2}$$

We first begin with a corollary of Theorem 1.1

Corollary 3.1. *Let $U_0 = (u_0, u_1) \in (H^1 \cap H_X^1) \times L^2$ and consider the solution to the subelliptic wave equation $U = (u, \partial_t u) = S(t)U_0$ given by the Hille–Yosida theorem. Assume that $u_0 = u_1 = 0$ in a ball $\{x \in \mathbf{R}^d : |x - x_0| < r_0\}$. Then there exists $\eta > 0$ and $r > 0$ such that $u = 0$ in the set $\{(t, x) : 0 \leq t < \eta, |x - x_0| < r\}$.*

Proof of the Corollary. Indeed, notice that for every compact $K \subset \mathbf{R}^d$ there exist $M > 0, \rho_0 > 0$ such that for every $x_0 \in K$ we have $B_P(x_0, \rho) \subset B_{\Delta_0}(x_0, M\rho)$ for every $0 < \rho < \rho_0$.

On the other side, according to [11], the hypothesis (1.1) implies that for every compact $K \subset \mathbf{R}^d$ there exist $M > 0, \rho_0 > 0, \delta > 0$ such that $B_{\Delta_0}(x_0, \rho) \subset B_P(x_0, M\rho^\delta)$ for every $x_0 \in K$.

Then Corollary 3.1 follows easily from Theorem 1.1. \square

Remark 3.2.

1. In the case where \mathbf{R}^d is replaced by a compact manifold a sketch of proof of Theorem 1.1 was given in a work of Melrose [1]. Here we follow his strategy and give a detailed self-contained proof which applies also the non-compact setting.
2. In the case where the vector fields (X_1, \dots, X_r) are of Heisenberg type, Theorem 1.1 has been proved by the second author in [12].

Proof of Theorem 1.1. To prove this result, the idea is to solve first the (elliptic) wave equation associated to the family of operators $P_k = P - \varepsilon_k \Delta_0$, where Δ_0 is the flat Laplacian, with initial data a sequence of smooth initial data $(u_{0,k}, u_{1,k})$ converging to (u_0, u_1) in $H_X^1 \times L^2$, with a proper choice of ε_k . Setting $X_k = (X_1, \dots, X_r, \sqrt{\varepsilon_k} \partial_1, \dots, \sqrt{\varepsilon_k} \partial_d)$ we notice that $H_{X_k}^1 = H^1 \cap H_X^1$ and hence it is possible to define such solution. Then we conclude by using the standard dependency domain results for elliptic wave equations and a limiting process.

In all that follows we take $(u_0, u_1) \in H_X^1(\mathbf{R}^d) \times L^2(\mathbf{R}^d)$ such that,

$$u_0 = u_1 = 0 \text{ in } B_P(x_0, t_0). \tag{3.3}$$

We start with the choice of sequences $(u_{0,k}, u_{1,k})$ and ε_k . From the definition of H_X^1 , there exists $u_{0,k} \in C_0^\infty(\mathbf{R}^d)$ converging to u_0 as $k \rightarrow +\infty$ in H_X^1 . There exists also a sequence $(u_{1,k}) \subset C_0^\infty(\mathbf{R}^d)$ converging to u_1 in $L^2(\mathbf{R}^d)$. We shall set,

$$\varepsilon_k = k^{-1} \left(1 + \max_{1 \leq j \leq k} \|u_{0,j}\|_{H^1}^2 \right)^{-2}, \quad P_k = P - \varepsilon_k \Delta_0. \tag{3.4}$$

Notice that the sequence ε_k is nonincreasing and

$$\varepsilon_k \|u_{0,k}\|_{H^1}^2 \rightarrow 0 \text{ when } k \rightarrow +\infty.$$

Let d_{P_k} be the Riemannian metric attached to this operator, defined as in (3.1) and introduce the sets,

$$\begin{aligned} C_k(t_0) &= \{(t, x) \in [0, t_0) \times \mathbf{R}^d : d_{P_k}(x, x_0) < t_0 - t\}, \\ B_{P_k}(x_0, \rho) &= \{x \in \mathbf{R}^d : d_{P_k}(x, x_0) < \rho\}. \end{aligned} \quad (3.5)$$

Then we have,

Proposition 3.3. ([10], Prop. 3.1) *Let $x_0 \in \mathbf{R}^d$ then for every $x \in \mathbf{R}^d$ we have,*

1. *The sequence $k \mapsto d_{P_k}(x, x_0)$ is non decreasing and bounded by $d_P(x, x_0)$.*
2. *$\lim_{k \rightarrow +\infty} d_{P_k}(x, x_0) = d_P(x, x_0)$.*
3. *The convergence is uniform on every compact.*

Proof. The non-decreasing property and the pointwise convergence have been proved in [10]. The uniformity is a consequence of Dini's theorem since the sequence $k \mapsto d_{P_k}(x, x_0)$ is non decreasing and the functions $x \mapsto d_{P_k}(x, x_0)$ and $x \mapsto d_P(x, x_0)$ are continuous. \square

Now fix $\delta > 0$ small. By Proposition 3.3 one can find $k_0 \geq 1$ such that for $k \geq k_0$ and for x in a large fixed ball around x_0 ,

$$d_P(x, x_0) - \frac{\delta}{2} \leq d_{P_k}(x, x_0) \leq d_P(x, x_0).$$

It follows that,

$$B_{P_k}(x_0, t_0 - \delta) \subset B_P(x_0, t_0 - \frac{\delta}{2}), \quad B_{P_k}(x_0, t_0 - \delta) \subset B_{P_k}(x_0, t_0 - \frac{\delta}{2}) \subset B_P(x_0, t_0). \quad (3.6)$$

Let $\chi \in C^\infty(\mathbf{R}^d)$ be such that,

$$\chi(x) = 0 \text{ in } B_P(x_0, t_0 - \frac{\delta}{2}), \quad \chi(x) = 1 \text{ in } B_P(x_0, t_0)^c$$

and set,

$$v_{j,k}(x) = \chi(x)u_{j,k}(x), \quad j = 0, 1.$$

Then, according to (3.6),

$$\begin{aligned} v_{0,k} &\in C_0^\infty(\mathbf{R}^d), \quad v_{0,k} = 0 \text{ in } B_{P_k}(x_0, t_0 - \delta) \quad \text{and } (v_{0,k}) \rightarrow u_0 \text{ in } H_X^1(\mathbf{R}^d). \\ v_{1,k} &\in C_0^\infty(\mathbf{R}^d), \quad v_{1,k} = 0 \text{ in } B_{P_k}(x_0, t_0 - \delta) \quad \text{and } (v_{1,k}) \rightarrow u_1 \text{ in } L^2(\mathbf{R}^d). \end{aligned} \quad (3.7)$$

Now we set,

$$U_k^0 = (v_{0,k}, v_{1,k}). \quad (3.8)$$

For the limiting process the key point is the following result.

Let us set, with $\varepsilon_k \in (0, 1)$ defined in (3.4),

$$\mathcal{P}_k = \begin{pmatrix} 0 & 1 \\ -P + \varepsilon_k \Delta_0 & 0 \end{pmatrix}.$$

Proposition 3.4. For $U_0 = (u_0, u_1) \in \mathcal{H} := (H^1 \cap H_X^1) \times L^2$, consider the solution,

$$U_k(t) = S_k(t)U_k^0 \in C^0(\mathbf{R}, \mathcal{H})$$

given by Theorem 2.5 with the operator \mathcal{P}_k (with initial data U_k^0). Then, when $k \rightarrow \infty$, the sequence (U_k) converges weakly to,

$$U(t) = S(t)U_0,$$

the solution to the wave equation given by the same theorem with \mathcal{P} and initial data $U_0 = (u_0, u_1)$.

Proof. We start with the energy conservation,

$$\begin{aligned} E_k(t) &= \int (\varepsilon_k |\nabla_x u_k|^2 + \sum_{j=1}^r |X_j u_k|^2 + |\partial_t u_k|^2)(t, x) dx = E_k(0), \\ E_k(0) &= \int (\varepsilon_k |\nabla_x v_{0,k}|^2 + \sum_{j=1}^r |X_j v_{0,k}|^2 + |v_{1,k}|^2)(x) dx. \end{aligned}$$

Moreover, writing $u_k(t, \cdot) = u_k(0, \cdot) + \int_0^t \partial_s u_k(s, \cdot) ds$, we obtain,

$$\|u_k(t)\|_{L^2} \leq \|u_0\|_{L^2} + |t|E_k(0)^{1/2}.$$

From the choice of ε_k given in (3.4), we see easily that,

$$\exists C > 0 : \quad E_k(0) \leq C, \quad \forall k.$$

Since E_k controls uniformly the square of the norm in $H_X^1 \times L^2$, we deduce that we can find a subsequence U_{k_n} such that

$$U_{k_n} \rightarrow U \text{ weakly in } L^\infty(\mathbf{R}, H_X^1 \times L^2), \text{ as } n \rightarrow +\infty \quad (3.9)$$

From the equation (satisfied in the distribution sense),

$$\partial_t U_{k_n} = \begin{pmatrix} 0 & 1 \\ -P + \varepsilon_{k_n} \Delta_0 & 0 \end{pmatrix} U_{k_n}, \quad (3.10)$$

we deduce first that,

$$\partial_t U = \begin{pmatrix} 0 & 1 \\ -P & 0 \end{pmatrix} U, \quad \text{in } \mathcal{D}'(\mathbf{R} \times \mathbf{R}^d). \quad (3.11)$$

Now, using Lemma 2.2 and (3.10), we see that,

$$\partial_t U_k \text{ is bounded in } L^\infty((-1, 1), L^2 \times (H^{-1} + H_X^{-1})). \quad (3.12)$$

Let $\Phi = (\varphi, \psi) \in (C_0^\infty(\mathbf{R}^d))^2$ and set $\theta_k(t) = \langle U_k(t), \Phi \rangle$ where $\langle \cdot, \cdot \rangle$ is the distribution duality. Then $(\theta_k) \subset C^0([-1, 1])$ and it follows from (3.9) that (θ_{k_n}) converges to $\langle U(\cdot), \Phi \rangle$ in $\mathcal{D}'(\mathbf{R})$.

On the other hand we have $\partial_t \theta_{k_n}(t) = \langle \partial_t U_{k_n}(t), \Phi \rangle$ so by (3.12) $(\partial_t \theta_{k_n})$ is uniformly bounded in $L^\infty([-1, 1])$. Therefore the set (θ_{k_n}) is equicontinuous, and hence from the Ascoli theorem it is relatively compact in

$C^0([-1, 1])$. But there exist only one possible accumulation point $\langle U(\cdot), \Phi \rangle$ which implies that θ_{k_n} converges to θ in $C^0([-1, 1])$. As a consequence, we have $\theta \in C^0$ and

$$U(0) = \lim_{n \rightarrow +\infty} U_{0, k_n} = U_0, \text{ in } \mathcal{D}'(\mathbf{R}^d).$$

We also have from the characterization of Hille-Yosida solutions given in Theorem 2.5, (writing k instead of k_n)

$$\int \psi(t) U_k(t) dt \in D(\mathcal{P}_k),$$

$$\mathcal{P}_k \left(\int \psi(t) U_k(t) dt \right) = - \int \psi'(t) U_k(t) dt. \quad (3.13)$$

We can now pass to the limit $k \rightarrow +\infty$ in (3.13) and get (in distribution sense),

$$\mathcal{P} \left(\int \psi(t) U(t) dt \right) = - \int \psi'(t) U(t) dt. \quad (3.14)$$

Now, clearly $\int \psi'(t) U(t) dt \in \mathcal{H}$ and from (3.14) we deduce,

$$\mathcal{P} \left(\int \psi(t) U(t) dt \right) \in \mathcal{H} \text{ so } \int \psi(t) U(t) dt \in D(\mathcal{P}).$$

From (3.11), we know that,

$$\partial_t U \in L^\infty(\mathbf{R}, L^2 \times H_X^{-1}),$$

and consequently,

$$U \in C^0(\mathbf{R}, L^2 \times H_X^{-1})$$

while on the other hand we have $U(0) = U_0$ (in \mathcal{D}' but hence also in \mathcal{H}). Now the uniqueness part in Theorem 2.5 for the operator $\tilde{\mathcal{P}}$ implies $U(t) = \tilde{S}(t)U_0$, and Lemma 2.6 imply in turn $U(t) = S(t)U_0$. Eventually, since the limit is unique, the extraction process is unnecessary and the whole family (U_k) converges to $U = S(t)U_0$. \square

Then we have the following result.

Proposition 3.5. (*[13] Thm. 8, chapter 7*) *Let $(u_0, u_1) \in H_X^1(\mathbf{R}^d) \times L^2(\mathbf{R}^d)$ and let u_k be the solution of the problem,*

$$(\partial_t^2 + P_k)u_k = 0 \text{ in } \mathbf{R}^{1+d}, \quad u_k|_{t=0} = v_{0,k}, \quad \partial_t u_k|_{t=0} = v_{1,k}$$

given by Hille Yosida Theorem, where $(v_{j,k})$ for $j = 0, 1$ satisfy (3.7).

Then u_k vanishes within $C_k(t_0 - \delta)$.

Let us now come back to the proof of Theorem 1.1

Assume that (u_0, u_1) vanish in $B_P(x_0, t_0)$. Then, by (3.7) for k large enough, $(v_{0,k}, v_{1,k})$ vanish in $B_{P_k}(x_0, t_0 - \delta)$. Then Proposition 3.5 implies that u_k vanish in $C_k(t_0 - \delta) = \{(t, x) \in [0, t_0 - \delta] \times \mathbf{R}^d : d_{P_k}(x, x_0) < t_0 - \delta - t\}$, thus in the set $C(t_0 - \delta) = \{(t, x) \in [0, t_0 - \delta] \times \mathbf{R}^d : d_P(x, x_0) < t_0 - \delta - t\}$ since $d_{P_k} \leq d_P$. Since (u_k) converges

to u , in the space of distributions, we deduce that u vanishes in the set $C(t_0 - \delta)$. Since this holds for every $\delta > 0$ we deduce that u vanishes in the set $C(t_0)$.

The proof of Theorem 1.1 is complete. \square

4. UNIQUE CONTINUATION OF SQUARE ROOTS OF HÖRMANDER'S OPERATORS

In this section we prove Theorem 1.2 for $s = 1/2$. We assume in this section that the vector fields $X = (X_1, \dots, X_r)$ have analytic coefficients on \mathbf{R}^d .

Since the operator P is a positive Friedrichs self-adjoint extension, we can define its square root \sqrt{P} with domain $H_X^1(\mathbf{R}^d)$ introduced in Definition 2.1. The proof of Theorem 1.2 for $s = 1/2$ combines three arguments. First, our result on the dependency domain for the solutions of the weakly hyperbolic operator $Q = \partial_t^2 + P$, then an argument of holomorphic extension due to Masuda [2], and eventually a result of Bony [14], concerning the unique continuation for P .

4.0.1. Masuda's argument

Let $u_0 \in H^1(\mathbf{R}^d) \cap H_X^1(\mathbf{R}^d)$ be such that $u_0 = \sqrt{P}u_0 = 0$ on an open subset $\omega \subset \mathbf{R}^d$. Consider the problem,

$$(i\partial_t + \sqrt{P})u = 0, \quad u|_{t=0} = u_0. \quad (4.1)$$

Since $\partial_t u|_{t=0} = i\sqrt{P}u_0$ it follows that u is also a solution of the problem (3.2) with $u_0 = u_1 = 0$ in ω . Therefore, by Corollary 3.1 there exists $\delta > 0$ and ω_1 open with $\bar{\omega}_1 \subset \omega$ such that,

$$u = 0 \text{ in } (-\delta, \delta) \times \omega_1. \quad (4.2)$$

Now since \sqrt{P} is a positive self-adjoint operator in $L^2(\mathbf{R}^d)$ by the spectral theorem the solution of (4.1) can be written as,

$$u(t, \cdot) = e^{it\sqrt{P}}u_0 = \int_0^{+\infty} e^{it\lambda} dE(\lambda)u_0. \quad (4.3)$$

The above formula shows that u has a holomorphic extension to the upper half plane $\text{Im } z > 0$ with values in $L^2(\mathbf{R}^d)$ in the sense that the function,

$$z \mapsto \int_0^{+\infty} e^{iz\lambda} \langle dE(\lambda)u_0, \varphi \rangle$$

is holomorphic $\text{Im } z > 0$ for any $\varphi \in L^2(\mathbf{R}^d)$.

We shall still denote by $u(z, \cdot)$ this holomorphic extension.

According to (3.2) this extension satisfies,

$$\partial_z^2 u + Pu = 0, \quad \text{in } \{z \in \mathbf{C} : \text{Im } z > 0, x \in \mathbf{R}^d\}. \quad (4.4)$$

Now for $z \in \mathbf{C}$ such that $|\text{Re } z| < \delta$ we set,

$$U(z, \cdot) = \begin{cases} u(z, \cdot), & \text{Im } z \geq 0, \\ \bar{u}(\bar{z}, \cdot), & \text{Im } z < 0. \end{cases} \quad (4.5)$$

Let $\varphi \in C_0^\infty(\omega_1)$ and set,

$$\theta(z) = \langle U(z, \cdot), \varphi \rangle \quad (4.6)$$

Then θ is holomorphic for $\text{Im } z > 0$, for $\text{Im } z < 0$ and continuous in $\{z : |\text{Re } z| < \delta\}$ by (4.2). By Morera's theorem, θ is holomorphic in the set $\{z : |\text{Re } z| < \delta\}$. Since, by (4.2), it vanishes in the set $\{z : |\text{Re } z| < \delta, \text{Im } z = 0\}$ we have $\theta(z) = 0$ in $\{z : |\text{Re } z| < \delta\}$.

According to (4.5) and (4.6) it follows that $u(z, \cdot) = 0$ for $|\text{Re } z| < \delta$ in $\mathcal{D}'(\omega_1)$ and consequently $u(z, x) = 0$ for almost all x in ω_1 .

Now, for $x \in \mathbf{R}^d$, $|\xi| < \delta$ let us set,

$$v(\xi, \eta, x) = u(\xi + i\eta, x). \quad (4.7)$$

Then, by the above argument and (4.2) we have,

$$v(\xi, \eta, x) = 0 \quad \text{if } |\xi| < \delta, \quad \text{a.a } x \in \omega_1. \quad (4.8)$$

On the other hand, since u is holomorphic in the set $\{z : \text{Im } z > 0\}$, the function v is harmonic with respect to (ξ, η) in the set $\mathcal{O} = \{(\xi, \eta) : |\xi| < \delta, \eta > 0\}$. Therefore,

$$(\partial_\xi^2 + \partial_\eta^2)v(\xi, \eta, x) = 0, \quad \text{in } \mathcal{O} \times \mathbf{R}^d.$$

Now, by (4.4) we have in $\mathcal{O} \times \mathbf{R}^d$,

$$\partial_\eta^2 v(\xi, \eta, x) = -\partial_z^2 u(\xi + i\eta, x) = Pv(\xi, \eta, x).$$

Using the two above equations we find that,

$$(-\partial_\xi^2 v - 2\partial_\eta^2 v + Pv)(\xi, \eta, x) = 0, \quad \text{in } \mathcal{O} \times \mathbf{R}^d. \quad (4.9)$$

Notice that by Hörmander's theorem [15], v is a C^∞ function in $\mathcal{O} \times \mathbf{R}^d$.

4.0.2. Bony's result

In [14] Bony has proved the following.

Theorem 4.1. *Let (Y_1, \dots, Y_m) be a set of vector fields on \mathbf{R}^N with analytic coefficients satisfying condition (1.1).*

Let w be a distribution solution, in \mathbf{R}^N , of the equation $\sum_{k=1}^m Y_k^ Y_k w + \sum_{j=1}^m a_j(x) Y_j w + a_0(x) w = 0$ where the a_j 's are analytic. If w vanishes in any open subset then w vanishes identically in \mathbf{R}^N .*

Coming back to the equation (4.9) we set,

$$Y_1 = \partial_\xi, \quad Y_2 = \sqrt{2}\partial_\eta, \quad Y_{j+2} = X_j, \quad j = 1, \dots, r.$$

Then the function v is a solution of the equation,

$$\sum_{k=1}^{r+2} Y_k^* Y_k v = 0$$

in $\mathcal{O} \times \mathbf{R}^d$. The system (Y_1, \dots, Y_{r+2}) has analytic coefficients and satisfy Hörmander's condition in $\mathcal{O} \times \mathbf{R}^d$. Since, by (4.8), v vanishes on the open set $\{(\xi, \eta, x) : |\xi| < \delta, \eta > 0, x \in \omega_1\}$ it follows from Bony's result that,

$$v \text{ vanishes identically in } \{(\xi, \eta) : |\xi| < \delta, \eta > 0\} \times \mathbf{R}^d.$$

Since v is continuous in $\{(\xi, \eta) : |\xi| < \delta, \eta \geq 0\}$ we deduce from (4.7) that $u_0(x) = v(0, 0, x) = 0$ for all $x \in \mathbf{R}^N$, which completes the proof of Theorem 1.2 for $s = 1/2$.

5. UNIQUE CONTINUATION FOR S-POWERS OF HÖRMANDER'S OPERATORS, $0 < s < 1$.

In this section we consider, as before, a system of vector fields (X_1, \dots, X_r) on \mathbf{R}^d , on which we make, as in the second part, the following assumptions,

- (i) the Lie algebra generated by these vector fields has maximal rank at every point in \mathbf{R}^d ,
- (ii) the coefficients of the X_j 's are analytic in \mathbf{R}^d .

We shall consider the operator, $P = \sum_{j=1}^r X_j^* X_j$, where X_j^* is the adjoint of X_j and denote by H_X^1 the closure of $C_0^\infty(\mathbf{R}^d)$ for the norm,

$$\|u\|_{H_X^1}^2 = \sum_j \|X_j u\|_{L^2(\mathbf{R}^d)}^2$$

We shall work, as before, with the Friedrich's extension of P (still denoted by P) with domain given by,

$$D(P) = \{u \in H_X^1(\mathbf{R}^d) : Pu \in L^2(\mathbf{R}^d)\}, \quad (5.1)$$

and it is well known that the operator P is non negative and selfadjoint.

Then for $0 < s < 1$ we can define, by the functional calculus, the fractional powers P^s of P . It is defined by the formula,

$$P^s \varphi(x) = \int_0^{+\infty} \lambda^s dE(\lambda) \varphi,$$

where $E(\lambda)$ is the spectral decomposition of P with domain,

$$D(P^s) = \{\varphi \in L^2(\mathbf{R}^d) : P^s \varphi \in L^2(\mathbf{R}^d)\}.$$

The goal of this section is to prove Theorem 1.2.

5.1. An extension result

Expressing the fractional powers of Laplace operators as the Dirichlet-Neumann operators of some degenerate elliptic equations goes back to the work of Caffarelli-Silvestre [16]. It has been further generalised to more general settings by Stinga and Torrea [17] (see also [18]). Here we shall use the following version.

Theorem 5.1 ([17], Thm. 1.1). *Let $s \in (0, 1)$ and $\varphi \in L^2(\mathbf{R}^d)$. Consider the following problem in $(0, +\infty) \times \mathbf{R}^d$,*

$$\partial_t^2 u(t, x) + \frac{1-2s}{t} \partial_t u(t, x) - Pu(t, x) = 0, \quad u|_{t=0} = \varphi. \quad (5.2)$$

Then the function $u : (0, +\infty) \times \mathbf{R}^d \rightarrow \mathbf{R}$ given by the formula,

$$u(t, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} P^s e^{-\tau P}(\varphi)(x) e^{-\frac{t^2}{4\tau}} \frac{d\tau}{\tau^{1-s}}, \quad (5.3)$$

where Γ is the Gamma function, is a solution of (5.2) which is in $C^\infty(]0, +\infty[\times \mathbf{R}^d)$ and satisfies $t^k \partial_t^k u \in C([0, +\infty[, L^2(\mathbf{R}^d))$ for every $k \in \mathbf{N}$. Moreover when φ belongs to the domain of P^s we have, in the L^2 sense,

$$\lim_{t \rightarrow 0^+} t^{1-2s} \partial_t u(t, x) = C(s) P^s \varphi(x), \quad (5.4)$$

where $C(s)$ is a (non zero) constant depending only on $s \in (0, 1)$.

For the sake of completeness, we recall a proof of this result in the appendix

5.2. A Baouendi–Goulaouic uniqueness result

We quote here a particular case, adapted to our situation, of [19], Theorem 4.

Theorem 5.2. (Baouendi-Goulaouic [19]) *Let*

$$\mathcal{P} = t^2 \partial_t^2 + a_1 t \partial_t + a_0 + t^2 P(x, \partial_x), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^d,$$

be a second order Fuchs type operator, where a_1, a_2 are real numbers and P is a second order differential operator with analytic coefficients in a neighborhood of a point $x_0 \in \mathbf{R}^d$.

Let λ_1, λ_2 be the two roots of the characteristic equation,

$$\lambda(\lambda - 1) + a_1 \lambda + a_0 = 0,$$

and let $h \in \mathbf{N}$ be such that $\operatorname{Re}(\lambda_j) < h, j = 1, 2$.

Then if $v \in C^0(\mathbf{R}, \mathcal{D}'(\omega_0))$ is such that $t^{-h} v \in C^0(\mathbf{R}, \mathcal{D}'(\omega_0))$, where ω_0 is a neighborhood of x_0 and satisfies,

$$\mathcal{P}u = 0, \quad \partial_t^j u(0, \cdot) = 0, \quad 0 \leq j \leq h - 1,$$

then v vanishes identically near $(0, x_0)$ in $\mathbf{R} \times \mathbf{R}^d$.

5.3. Proof of Theorem 1.2

Let u be the solution of (5.2) given by (5.3). Recall that, $u \in C^0([0, +\infty), L^2(\mathbf{R}^d))$. and by the condition in the Theorem,

$$u|_{t=0} = 0 \quad \text{in } \omega. \quad (5.5)$$

Consequently, since P is an operator in x whose coefficients do not depend on t ,

$$Pu|_{t=0} = 0 \quad \text{in } \omega, \quad (5.6)$$

Let us set,

$$A(t, x) = t^{1-2s} \partial_t u(t, x). \quad (5.7)$$

Using (5.4) and the hypothesis in Theorem 1.2 we see that,

$$\lim_{t \rightarrow 0^+} A(t, x) = 0 \quad \text{in } L^2(\omega) \quad (5.8)$$

$$\lim_{t \rightarrow 0^+} \partial_x^\alpha A(t, x) = 0 \quad \text{in } \mathcal{D}'(\omega). \quad (5.9)$$

Now according to (5.5) we can write, for $x \in \omega$,

$$u(t, x) = t \int_0^1 (\partial_t u)(\lambda t) d\lambda = t^{2s} \int_0^1 \lambda^{2s-1} A(\lambda t, x) d\lambda := t^{2s} B(t, x). \quad (5.10)$$

Since $2s - 1 \in]-1, 1[$ we deduce from (5.8) that,

$$\lim_{t \rightarrow 0^+} B(t, x) = 0 \quad \text{in } L^2(\omega). \quad (5.11)$$

Set for $t > 0, x \in \mathbf{R}^d$,

$$v(t, x) = t^{1-2s} u(t, x) = tB(t, x). \quad (5.12)$$

then v is a C^∞ function in $]0, +\infty[\times \mathbf{R}^d$ which belongs to $C^0([0, +\infty[, L^2(\mathbf{R}^d))$ and satisfies,

$$v|_{t=0} = 0 \quad \text{in } L^2(\omega). \quad (5.13)$$

Moreover,

$$\partial_t v = t^{1-2s} \partial_t u + (1-2s)t^{-2s} u = A + (1-2s)B. \quad (5.14)$$

Therefore by (5.8) and (5.11) we deduce that,

$$\lim_{t \rightarrow 0^+} \partial_t v(t, x) = 0, \quad \text{in } L^2(\omega). \quad (5.15)$$

On the other hand, using (5.7), the equation and (5.10), we can write for $t > 0$,

$$\begin{aligned} \partial_t A(t, x) &= t^{-2s} (t \partial_t^2 u + (1-2s) \partial_t u) = -t^{-2s+1} P u, \\ &= t^{-2s+1} P (t^{2s} B). = t P B. \end{aligned}$$

It follows then from (5.11) that,

$$\lim_{t \rightarrow 0^+} \partial_t A(t, x) = 0, \quad \text{in } \mathcal{D}'(\omega). \quad (5.16)$$

From (5.10) we get,

$$\partial_t B = \int_0^1 \lambda^{2s} \partial_t A(\lambda t, x) d\lambda,$$

therefore,

$$\lim_{t \rightarrow 0^+} \partial_t B(t, x) = 0, \text{ in } \mathcal{D}'(\omega). \quad (5.17)$$

Using (5.14) we deduce,

$$\lim_{t \rightarrow 0^+} \partial_t^2 v(t, x) = \lim_{t \rightarrow 0^+} (\partial_t A(t, x) + (1 - 2s)\partial_t B(t, x)) = 0, \text{ in } \mathcal{D}'(\omega). \quad (5.18)$$

Therefore, v can be extended as a function in $C^2([0, +\infty), \mathcal{D}'(\omega))$ with,

$$v|_{t=0} = \partial_t v|_{t=0} = \partial_t^2 v|_{t=0} = 0 \text{ in } \mathcal{D}'(\omega). \quad (5.19)$$

Let us now check the equation satisfied by v on $]0, +\infty[\times \mathbf{R}^d$. Since $u = t^{2s-1}v$ we have,

$$\begin{aligned} \partial_t u &= (2s-1)t^{2s-2}v + t^{2s-1}\partial_t v, \\ \partial_t^2 u &= (2s-1)(2s-2)t^{2s-3}v + 2(2s-1)t^{2s-2}\partial_t v + t^{2s-1}\partial_t^2 v, \\ tPu &= t^{2s}Pv. \end{aligned}$$

It follows that,

$$0 = t\partial_t^2 u + (1-2s)\partial_t u + tPu = t^{2s-2}(t^2\partial_t^2 v + (2s-1)t\partial_t v - (2s-1)v + t^2Pv),$$

so that,

$$t^2\partial_t^2 v + (2s-1)t\partial_t v - (2s-1)v + t^2Pv = 0. \quad (5.20)$$

Setting $\tilde{v} = H(t)v$ where $H(t) = 1$ if $t > 0$, $H(t) = 0$ if $t < 0$ we deduce from (5.19) that \tilde{v} belongs to $C^0(\mathbf{R}, \mathcal{D}'(\omega))$ and is such that $t^{-2}\tilde{v} \in C^0(\mathbf{R}, \mathcal{D}'(\omega))$. Moreover \tilde{v} satisfies the same equation,

$$t^2\partial_t^2 \tilde{v} + (2s-1)t\partial_t \tilde{v} - (2s-1)v + t^2P\tilde{v} = 0, \text{ in } \mathcal{D}'(\mathbf{R} \times \omega). \quad (5.21)$$

We may apply Theorem 5.2. with $a_1 = 2s-1, a_2 = 1-2s$. The characteristic equation is,

$$\lambda(\lambda-1) + (2s-1)\lambda + 1 - 2s = \lambda^2 + 2(s-1)\lambda + 1 - 2s = 0$$

whose roots are $\lambda_1 = 1, \lambda_2 = 1-2s$. We can take $h = 2$ in order that $\lambda_j < h$.

The conclusion of Theorem 5.2 is then that there exist $\delta > 0$ and an open set ω_1 with $\overline{\omega_1} \subset \omega$ such that \tilde{v} vanishes in $] -\delta, \delta[\times \omega_1$. It follows from the definition of \tilde{v} that the function u vanishes identically in $\mathcal{O} :=]0, \delta[\times \omega_1$. But in $]0, +\infty[\times \mathbf{R}^d$ the operator

$$\mathcal{P} = \partial_t^2 + \frac{1-2s}{t}\partial_t - \sum_{j=1}^r X_j^* X_j$$

is an operator ‘‘sum of squares’’ with analytic coefficients and u is a solution of the equation $\mathcal{P}u = 0$ in $]0, +\infty[\times \mathbf{R}^d$ which vanishes in \mathcal{O} . Bony’s theorem (see Thm. 4.1) shows that u vanishes identically in $]0, +\infty[\times \mathbf{R}^d$, therefore $\varphi = u|_{t=0}$ vanishes identically in \mathbf{R}^d .

ACKNOWLEDGMENTS

The research of the first author has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement 101097172 – GEOEDP). We would also like to thank C. Letrouit for enlightenments about R. Melrose's work [1].

DATA AVAILABILITY STATEMENT

REFERENCES

- [1] R. Melrose, Propagation for the wave group of a positive subelliptic second order differential operator. *Hyperbolic Equations and related Topics*. Academic Press (1986) 181–192.
- [2] K. Masuda, Anti-locality of the one-half power of elliptic differential operators. *Publ. Res. Inst. Math. Sci.* **8** (1972) 207–210.
- [3] M.M. Fall, and V. Felli, Unique continuation property and local asymptotics of solutions to fractional elliptic equations. *Commun. PDE* **39** (2014) 354–397.
- [4] A. Rüländ, Unique continuation for fractional Schrödinger equations with rough potentials. *Commun. PDE* **40** (2015) 77–114.
- [5] H. Yu, Unique continuation for fractional orders of elliptic equations. *Ann. PDE* **3** (2017) paper 16.
- [6] T. Gosh, M. Salo and G. Uhlmann, The Calderon problem for the fractional Schrödinger equation. *Anal. PDE* **13** (2020) 455–475.
- [7] H. Bahouri, Non prolongement unique des solutions d'opérateurs "somme de carrés". *Ann. Inst. Fourier* **36** (1986) 137–155.
- [8] C. Laurent and M. Léautaud, Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves. *J. Eur. Math. Soc.* **21** (2019) 957–1069.
- [9] N. Burq, P. Gérard, Contrôle optimal des équations aux dérivées partielles. Ecole Polytechnique, Département de Mathématique (2003), <http://www.math.u-psud.fr/~burq/articles/coursX.pdf>
- [10] D. Jerison and A. Sanchez-Calle, Subelliptic second order differential operators. *Lecture Notes in Mathematics* 1277. Springer Verlag, Berlin (1987) 46–77.
- [11] C. Fefferman and D.H. Phong, Subelliptic eigenvalues problems. *Proceedings of the Conference on Harmonic Analysis in honor of Antony Zygmund*. Wadsworth Math. Series (1981) 590–606.
- [12] C. Zuily, Existence globale de solutions régulières pour l'équation des ondes non linéaire amortie sur le groupe de Heisenberg. *Indiana Univ. Math. J.* **42** (1993) 323–360.
- [13] L. Evans, *Partial Differential Equations*. Graduate Studies in Mathematics, **19** (1998). American Mathematical Society.
- [14] J.M. Bony, Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. *Ann. Inst. Fourier* **19** (1969) 277–304.
- [15] L. Hörmander, Hypoelliptic second order differential equations. *Acta Math.* **119** (1967) 147–171.
- [16] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian. *Commun. PDE* **32** (2007) 1245–1260.
- [17] P. Stinga and J. Torrea, Extension problem and Harnack's inequality for some fractional operators. *Commun. PDE* **35** (2010) 2092–2122.
- [18] D. Chamorro and O. Jarrin, Fractional Laplacians, extension problems and Lie groups. *C. R. Math. Acad. Sci. Paris* **353** (2015) 517–522.
- [19] M.S. Baouendi and C. Goulaouic, Cauchy problems with characteristic initial hypersurface. *Commun. Pure Appl. Math.* **XXVI** (1973) 455–475.



Please help to maintain this journal in open access!

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at <https://edpsciences.org/en/subscribe-to-open-s2o>.

APPENDIX A.

A.1 Proof of Theorem 5.1

Notice first that when $\varphi \in L^2(\mathbf{R}^d)$ the function $e^{-\tau P}\varphi$, for $\tau > 0$, belongs to the domain of P^s (in fact to the domain of P^ρ for every $\rho > 0$), so $P^s e^{-\tau P}(\varphi)$ is well defined. Indeed, since $\lambda^{2s} e^{-2\tau\lambda} \leq c(\tau, s)$ for every $\lambda > 0$ we have,

$$\int_0^{+\infty} \lambda^{2s} e^{-2\tau\lambda} d(E(\lambda)\varphi, \varphi) \leq c(\tau, s) \|\varphi\|_{L^2(\mathbf{R}^d)}^2.$$

Now let us show that for $\varphi \in L^2(\mathbf{R}^d)$ the function u is well defined and belongs to $C^0([0, +\infty), L^2(\mathbf{R}^d))$. Indeed we have,

$$u(t, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \lambda^s \int_0^{+\infty} e^{-\tau\lambda} e^{-\frac{t^2}{4\tau}} \frac{1}{\tau^{1-s}} d\tau dE(\lambda)\varphi.$$

In the integral in τ let us set $\lambda\tau = \mu$. We obtain,

$$u(t, x) = \int_0^{+\infty} \theta(\lambda, t) dE(\lambda)\varphi, \quad \theta(\lambda, t) = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-\mu} e^{-\frac{\lambda t^2}{4\mu}} \frac{d\mu}{\mu^{1-s}} \quad (\text{A.1})$$

Since $\frac{e^{-\mu}}{\mu^{1-s}} \in L^1((0, +\infty))$ Lebesgue's dominated convergence theorem shows that, for fixed λ , the function $t \mapsto \theta(\lambda, t)$ is continuous. Let $t_0 \in [0, +\infty)$ and $(t_j) \subset [0, +\infty)$ a sequence converging to t_0 . Then,

$$\|u(t_j, \cdot) - u(t_0, \cdot)\|_{L^2(\mathbf{R}^d)}^2 = \int_0^{+\infty} |\theta(\lambda, t_j) - \theta(\lambda, t_0)|^2 d(E(\lambda)\varphi, \varphi).$$

Since $\theta(\lambda, t) \leq 1$ and $\varphi \in L^2(\mathbf{R}^d)$ Lebesgue's dominated convergence theorem shows that the right hand side tends to zero when j goes to $+\infty$.

Now, using the fact (proved by induction on $k \in \mathbf{N}$) that, $t^k \partial_t^k e^{-\alpha t^2} = \sum_{j=1}^k c_{jk}(\alpha t^2)^j e^{-\alpha t^2}$, $\alpha > 0$, $c_{jk} \in \mathbf{R}$, we see that,

$$t^k \partial_t^k u(t, x) = \int_0^{+\infty} \theta_k(\lambda, t) dE(\lambda)\varphi, \quad \theta_k(\lambda, t) = \frac{1}{\Gamma(s)} \sum_{j=1}^k d_{jk} \int_0^{+\infty} e^{-\mu} \left(\frac{\lambda t^2}{4\mu}\right)^j e^{-\frac{\lambda t^2}{4\mu}} \frac{d\mu}{\mu^{1-s}}. \quad (\text{A.2})$$

Since $|\theta_k(\lambda, t)| \leq M_k$, $M_k \in \mathbf{R}^+$, the same argument as before shows that $t^k \partial_t^k u \in C^0([0, +\infty[, L^2(\mathbf{R}^d))$. Thus $\partial_t^k u \in C^0(]0, +\infty[, L^2(\mathbf{R}^d))$.

Suppose we have shown that u satisfies equation (5.2), then u is a C^∞ function on $]0, +\infty[\times \mathbf{R}^d$. Indeed the operator appearing in (5.2) is an operator "sum of squares" with C^∞ coefficients, which by hypothesis satisfies Hörmander's condition (1.1). It is then hypoelliptic. So we are left with the proof of (5.2).

According to the computation made before we see that,

$$\begin{aligned} \partial_t^2 u + \frac{1-2s}{t} \partial_t u &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \int_0^{+\infty} e^{-\mu} \lambda \left(-\frac{1-s}{2\mu} + \frac{\lambda t^2}{4\mu^2} \right) e^{-\frac{\lambda t^2}{4\mu}} \frac{1}{\mu^{1-s}} d\mu dE(\lambda) \varphi, \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \int_0^{+\infty} \lambda e^{-\mu} \frac{\partial}{\partial \mu} \left[e^{-\frac{\lambda t^2}{4\mu}} \frac{1}{\mu^{1-s}} \right] d\mu dE(\lambda) \varphi, \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \int_0^{+\infty} \lambda e^{-\frac{\lambda t^2}{4\mu}} \frac{1}{\mu^{1-s}} d\mu dE(\lambda) \varphi. \end{aligned}$$

Setting $\mu = \lambda\tau$ we deduce that,

$$\begin{aligned} \partial_t^2 u + \frac{1-2s}{t} \partial_t u &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \int_0^{+\infty} \lambda^{s+1} e^{-\lambda\tau} e^{-\frac{t^2}{4\tau}} \frac{1}{\tau^{1-s}} d\tau dE(\lambda) \varphi, \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-\tau P} (P^{s+1} \varphi) e^{-\frac{t^2}{4\tau}} \frac{1}{\tau^{1-s}} d\tau = Pu. \end{aligned}$$

Let us prove (5.4). According to (A.1) we have,

$$t^{1-2s} \partial_t u = \frac{1}{\Gamma(s)} \int_0^{+\infty} \int_0^{+\infty} e^{-\mu} \frac{-\lambda t}{2\mu} e^{-\frac{\lambda t^2}{4\mu}} \frac{t^{1-2s}}{\mu^{1-s}} d\mu dE(\lambda) \varphi.$$

Setting $x = \frac{\mu}{\lambda t^2}$ we get,

$$t^{1-2s} \partial_t u = \frac{-1}{2\Gamma(s)} \int_0^{+\infty} \int_0^{+\infty} e^{-\lambda x t^2} \lambda^s \frac{e^{-\frac{1}{4x}}}{x^{2-s}} dx dE(\lambda) \varphi.$$

Since $s < 1$, we have $\frac{e^{-\frac{1}{4x}}}{x^{2-s}} \in L^1((0, +\infty))$. Set $D(s) = \int_0^{+\infty} \frac{e^{-\frac{1}{4x}}}{x^{2-s}} dx$ and $C(s) = -\frac{D(s)}{2\Gamma(s)}$. Now since $P^s \varphi \in L^2(\mathbf{R}^d)$ we can write, $P^s \varphi = \int_0^{+\infty} \lambda^s dE(\lambda) \varphi$ so that,

$$t^{1-2s} \partial_t u - C(s) P^s \varphi = \frac{-1}{2\Gamma(s)} \int_0^{+\infty} \lambda^s \left(\int_0^{+\infty} (e^{-\lambda x t^2} - 1) \frac{e^{-\frac{1}{4x}}}{x^{2-s}} dx \right) dE(\lambda) \varphi.$$

Set $f(\lambda, t) = \int_0^{+\infty} (e^{-\lambda x t^2} - 1) \frac{e^{-\frac{1}{4x}}}{x^{2-s}} dx$. By the dominated convergence theorem we have, for fixed $\lambda \in (0, +\infty)$, $\lim_{t \rightarrow 0^+} |f(\lambda, t)| = 0$ and moreover $|f(\lambda, t)| \leq 2D(s)$. Then,

$$\|t^{1-2s} \partial_t u - C(s) P^s \varphi\|_{L^2(\mathbf{R}^d)}^2 = \frac{1}{4\Gamma(s)^2} \int_0^{+\infty} \lambda^{2s} |f(\lambda, t)|^2 d(E(\lambda) \varphi, \varphi),$$

and, by the dominated convergence theorem, the right hand side tends to zero when t goes to 0^+ .