ESAIM: COCV 31 (2025) 21
https://doi.org/10.1051/cocv/2025008

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

A NOTE ON THE FAILURE OF THE FABER–KRAHN INEQUALITY FOR THE VECTOR LAPLACIAN

David Krejčiřík[1], Pier Domenico Lamberti[2] and Michele Zaccaron[3,*]

Abstract. We consider a natural eigenvalue problem for the vector Laplacian related to stationary Maxwell's equations in a cavity and we prove that an analog of the celebrated Faber–Krahn inequality doesn't hold, regardless of whether a volume or perimeter constraint is applied.

Mathematics Subject Classification. 35P15, 35Q60, 35Q61, 35Q93, 78M50.

Received October 9, 2024. Accepted January 17, 2025.

1. Introduction

In this paper, we consider the eigenvalue problem for the curl curl operator

$$\begin{cases} \operatorname{curl}\operatorname{curl} u = \lambda u & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u \times \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

on bounded domains Ω in $\mathbb{R}^3$ with Lipschitz boundaries, in the unknown vector field $u : \Omega \to \mathbb{R}^3$ (the eigenvector) and the unknown $\lambda \in \mathbb{R}$ (the eigenvalue). Here ν denotes the unit outer normal to $\partial\Omega$. Recall that $\operatorname{curl}\operatorname{curl} u = -\Delta u + \nabla \operatorname{div} u$ hence $\operatorname{curl}\operatorname{curl} u = -\Delta u$ if u is a divergence-free vector field. Note also that the second condition in (1.1) is immediately implied by the first equation if $\lambda \neq 0$. Problem (1.1) admits a sequence of eigenvalues that are ordered in increasing order, taking into account their multiplicity, as follows:

$$0 \leq \lambda_1^\Omega \leq \lambda_2^\Omega \leq \dots \lambda_j^\Omega \leq \dots \nearrow +\infty.$$

We refer to [1] for basic results concerning problem (1.1) and for references.

Keywords and phrases: Maxwell's equations, Hodge Laplacian, electromagnetic cavities, estimates for eigenvalues.

[1] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 12000 Prague 2, Czech Republic.
[2] Dipartimento di Tecnica e Gestione dei Sistemi Industriali (DTG), University of Padova, Stradella S. Nicola 3, 36100 Vicenza, Italy.
[3] Institut Fresnel, Aix-Marseille Université, Faculté des Sciences – Campus Saint Jérôme, Avenue Escadrille Normandie-Niémen, 13397 Marseille CEDEX, France.

* Corresponding author: mic.zaccaron@gmail.com

We assume that the boundary of Ω has only one connected component so that zero is not an eigenvalue and $\lambda_1^\Omega > 0$. Under these assumptions on Ω, we consider the problem of minimizing and maximizing λ_1^Ω under the volume constraint $|\Omega| =$ constant or the perimeter constraint $|\partial\Omega| =$ constant and we prove that

$$\inf_{|\Omega|=\text{const.}} \lambda_1^\Omega = 0, \quad \inf_{|\partial\Omega|=\text{const.}} \lambda_1^\Omega = 0 \tag{1.2}$$

and

$$\sup_{|\Omega|=\text{const.}} \lambda_1^\Omega = +\infty, \quad \sup_{|\partial\Omega|=\text{const.}} \lambda_1^\Omega = +\infty. \tag{1.3}$$

Note that $|\Omega|$ denotes the volume of Ω, while $|\partial\Omega|$ the two-dimensional surface measure of $\partial\Omega$.

The first equality in (1.2) and the equalities in (1.3) will be easily proved by considering suitable families of cuboids, *i.e.*, rectangular parallelepipeds, in which case explicit formulas are known. By using the same formulas, it turns out that

$$\inf_{\substack{\Omega \text{ cuboid} \\ |\partial\Omega|=k}} \lambda_1^\Omega = \frac{4\pi^2}{k}, \tag{1.4}$$

for all $k > 0$, see Theorem 2.2. Thus, the second equality in (1.2) will be proved by considering another family of non-convex domains, namely suitable three dimensional dumbbell domains: in this case, we shall use the full description of the spectrum of problem (1.1) in cross product domains provided by [2].

It is interesting to observe that if Ω is a ball with surface area equal to k then its first eigenvalue is larger than the lower bound in (1.4), see Remark 2.5. The reader interested in explicit computations of Maxwell's eigenvalues for analogous boundary value problems can find more results and formulas in [3].

The main physical motivation for studying problem (1.1) comes from the analysis of the stationary Maxwell's equations, in which case u plays the role of an electric field in a cavity Ω surrounded by a perfect conductor $\partial\Omega$ that is responsible for the boundary condition. In particular, the study of electromagnetic cavities has applications in designing cavity resonators or shielding structures for electronic circuits. We refer to [4], Chapter 10 for a detailed introduction to this subject and to the extensive monographs [5, 6] on the mathematical theory of electromagnetism.

We note that the energy space naturally associated with problem (1.1) is $X_N(\operatorname{div} 0, \Omega) := H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div} 0, \Omega)$ that is defined as the space of divergence free vector fields u in $L^2(\Omega)^3$ with $\operatorname{curl} u$ in $L^2(\Omega)^3$, satisfying the boundary condition in (1.1) (as usual, the boundary condition is understood in the weak sense by defining $H_0(\operatorname{curl}, \Omega)$ as the closure in $H(\operatorname{curl}, \Omega)$ of the space of smooth vector fields with compact support). In particular it turns out that

$$\lambda_1^\Omega = \min_{\substack{u \in X_N(\operatorname{div} 0, \Omega) \\ u \neq 0}} \frac{\int_\Omega |\operatorname{curl} u|^2 \mathrm{d}x}{\int_\Omega |u|^2 \mathrm{d}x}.$$

It is clear that the eigenvalue problem (1.1) is the natural vectorial version of the eigenvalue problem for the Dirichlet Laplacian in which case the energy space is the usual Sobolev space $H_0^1(\Omega)$. For the Dirichlet Laplacian, the celebrated Faber–Krahn inequality states that the ball minimizes the first eigenvalue under volume constraint, see *e.g.*, [7]. Thus, our observations point out that, although the ball is critical for the elementary symmetric functions of the eigenvalues of (1.1) under both volume and perimeter constraint (*cf.* [1]), the Faber–Krahn inequality doesn't hold, no matter whether the volume or the perimeter constraint is used. On the other hand the lower bound in (1.4) suggests that the problem of minimizing the first eigenvalue under perimeter constraint and additional geometric constraints could be of interest. With regard to this, we observe that it has been proved in [8] that in case of bounded convex domains the first non-zero eigenvalue

$\mu^{\Omega}_{1,\mathcal{N}}$ of the Neumann Laplacian in Ω satisfies the estimate $\mu^{\Omega}_{1,\mathcal{N}} \le \lambda_1^{\Omega}$, and a classical inequality by Payne and Weinberger [9] states that if Ω is convex then $\mu^{\Omega}_{1,\mathcal{N}} \ge \pi^2/d^2$ where d is the diameter of Ω. See also [10] for further references and inequalities between Maxwell's eigenvalues and the eigenvalues of the Dirichlet Laplacian.

We observe that λ_1^{Ω} coincides with the first eigenvalue $\mu_1^{[2]}$ of the Hodge Laplacian acting on 2-forms with so-called absolute boundary conditions, see [11] for notation. By using this non-trivial fact, it is possible to give another proof of the first equality in (1.2) and the equalities in (1.3) based on the general two-sided estimate proved for convex domains in [11], Theorem 1.1. We explain how to do that in Remark 2.4. As far as this approach is concerned, we note that the first equalities in (1.2) and (1.3) for the limiting behavior of $\mu_1^{[2]}$ to zero or infinity under volume constraint, can also be found in [12], Section 5.

Finally, we note that the failure of the Faber–Krahn inequality for a different vectorial eigenvalue problem associated with the Stokes operator has been recently discussed in [13].

2. Cuboids

In this section we consider cuboids Ω in $\mathbb{R}^3$ with sides of length ℓ_1, ℓ_2, ℓ_3 that are always assumed to be in the order $\ell_1 \ge \ell_2 \ge \ell_3$. Although it is not necessary, the reader may think of using a system of coordinates that allows to represent Ω in the form $\Omega = (0, l_1) \times (0, l_2) \times (0, l_3)$. It is well-known that

$$\lambda_1^{\Omega} = \pi^2 \left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} \right),$$

see *e.g.*, [2]. This formula allows to give an immediate proof of the following theorem concerning the case of the volume constraint.

Theorem 2.1. *For any fixed $k > 0$ we have*

$$\inf_{\substack{\Omega \text{ cuboid} \\ |\Omega| = k}} \lambda_1^{\Omega} = 0, \qquad \sup_{\substack{\Omega \text{ cuboid} \\ |\Omega| = k}} \lambda_1^{\Omega} = +\infty.$$

Proof. We begin by considering cuboids Ω with $|\Omega| = 1$. Fix $0 < \ell \le 1$ and take $\ell_3 = \ell$ and $\ell_1 = \ell_2 = \ell^{-1/2}$. The volume of such a cuboid is equal to 1 and the first Maxwell eigenvalue satisfies

$$\lambda_1^{\Omega} = \pi^2 \left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} \right) = 2\pi^2 \ell \xrightarrow[\ell \to 0^+]{} 0^+. \tag{2.1}$$

If instead we take $\ell_2 = \ell_3 = \ell$ and $\ell_1 = \ell^{-2}$, then we have again that $|\Omega| = 1$ and

$$\lambda_1^{\Omega} = \pi^2 \left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} \right) = \pi^2 \left(\ell^4 + \frac{1}{\ell^2} \right) \xrightarrow[\ell \to 0^+]{} +\infty. \tag{2.2}$$

Thus the theorem is proved for cuboids satisfying $|\Omega| = 1$. The general case of arbitrary fixed volumes, can be proved by observing that $\lambda_1^{\alpha\Omega} = \lambda_1^{\Omega}/\alpha^2$ for any $\alpha > 0$. □

As far as the perimeter constraint is concerned, we can prove the following theorem.

Theorem 2.2. *For any fixed $k > 0$ we have*

$$\inf_{\substack{\Omega \text{ cuboid} \\ |\partial\Omega| = k}} \lambda_1^{\Omega} = \frac{4\pi^2}{k}, \quad \sup_{\substack{\Omega \text{ cuboid} \\ |\partial\Omega| = k}} \lambda_1^{\Omega} = +\infty \tag{2.3}$$

and the infimum is not attained.

Proof. We begin with proving the second equality in (2.3). Possibly rescaling, it is enough to consider the case $|\partial\Omega| = 2$ which means that the lengths ℓ_1, ℓ_2, ℓ_3 of the sides of the cuboid have to satisfy the following constraint

$$\ell_1\ell_2 + \ell_1\ell_3 + \ell_2\ell_3 = 1.$$

Consider $0 < \ell \leq 1/\sqrt{3}$ and take $\ell_2 = \ell_3 = \ell$ and $\ell_1 = (1-\ell^2)/2\ell$. Note that the perimeter constraint is satisfied and that the assumptions on ℓ guarantee that $\ell_1 \geq \ell$. It follows that

$$\lambda_1^\Omega = \pi^2 \left(\frac{4\ell^2}{(1-\ell^2)^2} + \frac{1}{\ell^2} \right) \xrightarrow[\ell\to 0^+]{} +\infty \tag{2.4}$$

and the second equality in (2.3) is proved.

We now prove the first equality in (2.3). By the perimeter constraint $|\partial\Omega| = k$, the lengths ℓ_1, ℓ_2, ℓ_3 of the sides of the cuboid have to satisfy the following constraint

$$|\partial\Omega| = 2(\ell_1\ell_2 + \ell_1\ell_3 + \ell_2\ell_3) = k$$

which implies that

$$\ell_2 = \frac{\frac{k}{2} - \ell_3\ell_1}{\ell_3 + \ell_1}\,. \tag{2.5}$$

Note that assuming that $\ell_3 \leq \ell_2 \leq \ell_1$ is equivalent to requiring that

$$\sqrt{\ell_3^2 + \frac{k}{2}} - \ell_3 \leq \ell_1 \leq \frac{\frac{k}{2} - \ell_3^2}{2\ell_3}, \text{ with } 0 < \ell_3 \leq \sqrt{k/6}. \tag{2.6}$$

By Young's inequality we get

$$\lambda_1^\Omega = \pi^2 \left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} \right) = \pi^2 \left(\frac{1}{\ell_1^2} + \frac{(\ell_3+\ell_1)^2}{(\frac{k}{2} - \ell_3\ell_1)^2} \right) \geq \frac{2(\ell_3+\ell_1)\pi^2}{\ell_1\left(\frac{k}{2} - \ell_3\ell_1\right)} > \frac{4\pi^2}{k}, \tag{2.7}$$

which proves that

$$\inf_{\substack{\Omega \text{ cuboid} \\ |\partial\Omega| = k}} \lambda_1^\Omega \geq \frac{4\pi^2}{k}$$

and that the infimum cannot be a minimum. Choosing $\ell_1 = \sqrt{\ell_3^2 + \frac{k}{2}} - \ell_3$ and passing to the limit as $\ell_3 \to 0^+$ in the second equality of (2.7) yield

$$\lambda_1^\Omega \xrightarrow[\ell_3\to 0^+]{} \pi^2 \left(\frac{1}{\ell_1^2} + \frac{4\ell_1^2}{k^2} \right) = \frac{4\pi^2}{k}$$

and the proof of the first equality in the statement is proved. □

Remark 2.3. The proof of the previous theorem shows that the infimum in (2.3) is reached by a minimizing sequence of cuboids that degenerate to a square with side $\sqrt{k/2}$.

Remark 2.4. Following the terminology and the notation in [11], consider the Hodge Laplacian acting on p-forms with absolute boundary conditions on a bounded convex domain Ω of $\mathbb{R}^3$, and denote by $\mu_1^{[p]}$ the corresponding first eigenvalue. Here $p = 0, 1, 2, 3$ and $\mu_1^{[0]} \leq \mu_1^{[1]} \leq \mu_1^{[2]} \leq \mu_1^{[3]}$. Recall from [11] that $\mu_1^{[0]} = \mu_1^{[1]}$ is the first positive eigenvalue of the Neumann Laplacian, while $\mu_1^{[3]}$ is the first eigenvalue of the Dirichlet Laplacian. Denote by $\mu_1^{[p]'}$ the first eigenvalue of the Hodge Laplacian acting on exact p-forms. Keeping in mind the standard identification of vector fields u in $\mathbb{R}^3$ with 2-forms ω, the Hodge Laplacian $-\Delta\omega$ acting on 2-forms ω corresponds to $\operatorname{curl}\operatorname{curl} u - \nabla \operatorname{div} u$ and the absolute boundary conditions correspond to the boundary conditions $u \times \nu = 0$ on $\partial\Omega$ in (1.1) together with the boundary condition $\operatorname{div} u = 0$ on $\partial\Omega$ (which in our problem is included in the second condition in (1.1)). On the other hand, restricting the Hodge Laplacian to exact 2-forms corresponds to imposing the second condition $\operatorname{div} u = 0$ in Ω in (1.1). We conclude that $\mu_1^{[2]'} = \lambda_1^\Omega$. Importantly, by the proof of [14], Theorem 2.6 we have that $\mu_1^{[2]'} = \mu_1^{[2]}$, see also [11], p. 1802 (we note that this result could also be deduced by the inequality between Maxwell and Helmholtz eigenvalues proved in [10], Theorem 1.1, for domains that are not necessarily convex).

Consider now the John ellipsoid of Ω, *i.e.*, the ellipsoid of maximal volume contained in Ω and denote by D_1, D_2, D_3 the lenghts of its principal axes, ordered by $D_1 \geq D_2 \geq D_3$. It is proved in [11], Theorem 1.1, that

$$\frac{a}{D_2^2} \leq \mu_1^{[2]} \leq \frac{a'}{D_2^2}$$

for some positive constants a, a' independent of Ω. By the previous observations, it follows that $aD_2^{-2} \leq \lambda_1^\Omega \leq a' D_2^{-2}$. Accordingly, if we consider a cuboid Ω with sides ℓ_1, ℓ_2, ℓ_3 in the order $\ell_1 \geq \ell_2 \geq \ell_3$, we have that $D_2 = l_2$ hence

$$\frac{a}{\ell_2^2} \leq \lambda_1^\Omega \leq \frac{a'}{\ell_2^2}.$$

Keeping the volume of the cuboid Ω fixed and letting ℓ_2 go either to ∞ or to 0 we obtain that either $\lambda_1^\Omega \to 0$ or $\lambda_1^\Omega \to \infty$, respectively. In this way, we retrieve in an alternative asymptotic form what is explicitly written in formulas (2.1) and (2.2). Similarly, one can retrieve formula (2.4).

The advantage of this approach lies in the fact that, in principle, one may use not only cuboids but also more general convex domains with suitable John ellipsoids. However, using cuboids and the corresponding explicit formulas allows us to write one-line proofs.

Remark 2.5. The ball B_k of surface area k has radius $R = \sqrt{k/(4\pi)}$, hence

$$\inf_{\substack{\Omega \text{ cuboid} \\ |\partial\Omega| = k}} \lambda_1^\Omega = \frac{4\pi^2}{k} < \lambda_1^{B_k} = \left(\frac{a'_{1,1}}{R}\right)^2 = \frac{4\pi (a'_{1,1})^2}{k} < \frac{12\pi^2}{k} = \lambda_1^{Q_k},$$

where Q_k is the cube with surface area k. Here $a'_{1,1} \approx 2.7437 \pm 0.0001$ is the first positive zero of the derivative ψ_1' of the Riccati–Bessel function ψ_1' defined by $\psi_1(z) = zj_1(z)$ where j_1 is the usual spherical Bessel function of the first kind, see *e.g.*, [1]. Thus, from the point of view of the minimization problem under perimeter constraint, the ball is better than the cube but it is worse than other cuboids.

Moreover, if $\tilde{B}_k$ is a ball and $\tilde{Q}_k$ is a cube both of volume k, then the radius of the ball is $R = \sqrt[3]{\frac{3k}{4\pi}}$ hence

$$\lambda_1^{\tilde{B}_k} = \left(\frac{a'_{1,1}}{R}\right)^2 = (a'_{1,1})^2 \sqrt[3]{\frac{16\pi^2}{9k^2}} < \frac{2\pi^2}{\sqrt[3]{k^2}} = \lambda_1^{\tilde{Q}_k}.$$

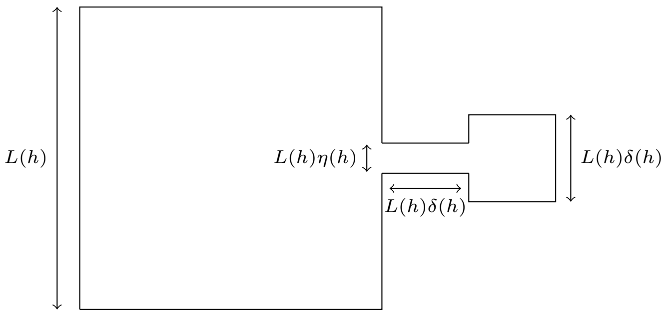

FIGURE 1. The domain ω_h.

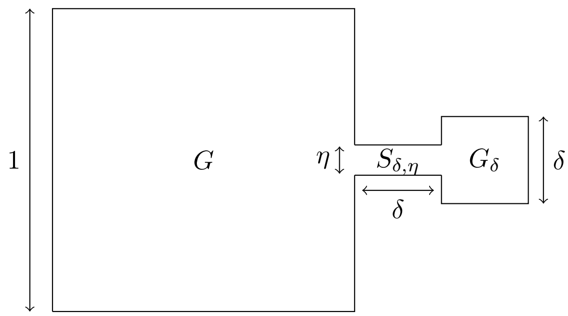

FIGURE 2. The domain $\omega_{\delta,\eta}$.

Thus, from the point of view of the minimization problem under volume constraint, the cube is worse than the ball.

3. Dumbbell domains

In this section we consider the eigenvalue problem (1.1) on a three dimensional dumbbell domain obtained as a product of a suitable two dimensional dumbbell domain as in Figure 1 and an interval. Namely, for any $\delta, \eta > 0$ with $\delta < 1$ and $\eta < \delta$, we consider the two dimensional dumbbell domain in Figure 2 defined as follows $\omega_{\delta,\eta} = G \cup S_{\delta,\eta} \cup G_\delta$ where

$$G = (-1, 0) \times \left(\frac{(-1+\eta)}{2}, \frac{1+\eta}{2} \right), \quad S_{\delta,\eta} = [0, \delta] \times (0, \eta)$$

and

$$G_\delta = (\delta, 2\delta) \times \left(\frac{-\delta+\eta}{2}, \frac{\delta+\eta}{2} \right).$$

Then we can prove the following theorem that makes use of Lemma 3.2 below.

Theorem 3.1. *Let $\beta > 0$ be fixed. For any $h > 0$ sufficiently big, let $\delta(h), \eta(h)$ be two positive real numbers depending on h satisfying the following conditions:*

$$\delta(h) = o(h^{-2/\beta}), \quad \text{and} \quad \eta(h) = O(\delta^{3+\beta}(h)), \quad \text{as } h \to +\infty.$$

Moreover, let $\Omega(h)$ *the three dimensional dumbbell domain defined by*

$$\Omega_h := \omega_h \times (0, h),$$

where

$$\omega_h := L(h)\omega_{\delta(h),\eta(h)}$$

and the positive factor $L(h)$ *is defined by the condition* $|\partial\Omega_h| = 1$. *Then*

$$\lim_{h\to+\infty} \lambda_1^{\Omega_h} = 0.$$

Proof. In the sequel, for the sake of simplicity, we shall often omit the dependence on h of the quantities involved.

The surface area of $\partial\Omega_h$ is equal to

$$|\partial\Omega_h| = 2|\omega_h| + h|\partial\omega_h| = 2L^2\left(1 + \delta(\delta + \eta)\right) + 2hL(2 + 3\delta - \eta).$$

Imposing the constraint $|\partial\Omega_h| = 1$ is equivalent to assuming that

$$L = L(h) = \frac{1}{\sqrt{A^2 + B} + A}$$

where

$$A = h(2 + 3\delta - \eta) \qquad \text{and} \qquad B = 2\left(1 + \delta(\delta + \eta)\right).$$

Note that $A \sim 2h$ and $B \sim 2$ as $h \to +\infty$. Therefore $L \sim \frac{1}{4h}$ as $h \to +\infty$. Note also that $\eta < \delta < 1$ for $h > 0$ large enough.

Denoting by $\mu_{1,\mathcal{N}}^U$ the first positive eigenvalue of the Neumann Laplacian on a bounded Lipschitz domain U, by Lemma 3.2 and our assumptions on δ and η it follows that

$$\mu_{1,\mathcal{N}}^{\omega_h} = \frac{\mu_{1,\mathcal{N}}^{\omega_{\delta,\eta}}}{L^2} = \frac{O(\delta^\beta)}{L^2} = h^2 O(\delta^\beta) = h^2 o(h^{-2}) \to 0 \quad \text{as} \quad h \to +\infty\,.$$

We recall that for a product domain $\Omega = \omega \times I$ where ω is a bounded Lipschitz domain in $\mathbb{R}^2$ and $I = (0, h)$ is an interval of length $h > 0$, we have

$$\lambda_1^\Omega = \min\left\{\mu_{1,\mathcal{D}}^\omega,\ \mu_{1,\mathcal{N}}^\omega + \frac{\pi^2}{h^2}\right\}, \tag{3.1}$$

where $\mu_{1,\mathcal{D}}^\omega$ is the first eigenvalue of the Dirichlet Laplacian on ω, see [2], Theorem 1.4. Clearly, since the area of ω^h vanishes as $h \to +\infty$, we have that $\mu_{1,\mathcal{D}}^{\omega_h} \to +\infty$, hence

$$\lambda_1^{\Omega_h} = \mu_1^{\omega_h} + \frac{\pi^2}{h^2} \to 0$$

as $h \to +\infty$. □

The following lemma is about a celebrated example from [15], Ch. VI, Section 2, that shows that the first Neumann eigenvalue on a certain dumbbell domain goes to zero as the channel of the dumbbell shrinks fast enough. Since we need precise information on the decay rate, we include a proof for the convenience of the reader.

Lemma 3.2. *Let $\mu_{1,\mathcal{N}}^{\omega_{\delta,\eta}}$ be the first positive eigenvalue of the Neumann Laplacian on the two dimensional domain $\omega_{\delta,\eta}$ with $0<\eta<\delta<1$, see Figure 2. If $\eta=o(\delta^3)$ then $\mu_{1,\mathcal{N}}^{\omega_{\delta,\eta}}\to 0$ as $\delta\to 0$. In particular, if $\eta=O(\delta^{3+\beta})$ with $\beta>0$, then*

$$\mu_{1,\mathcal{N}}^{\omega_{\delta,\eta}}=O(\delta^{\beta})$$

as $\delta\to 0$.

Proof. By the min-max characterization of the eigenvalues we have

$$\mu_1^{\omega_{\delta,\eta}}=\inf_{\substack{u\in H^1(\omega_{\delta,\eta})\\ \int_{\omega_{\delta,\eta}}u=0}}\frac{\int_{\omega_{\delta,\eta}}|\nabla u|^2}{\int_{\omega_{\delta,\eta}}u^2}.$$

Consider the following function

$$u(x,y):=\begin{cases} c, & (x,y)\in G,\\ c-\frac{1}{\delta}\left(\frac{1}{\delta}+c\right)x, & (x,y)\in S_{\delta,\eta},\\ -\frac{1}{\delta}, & (x,y)\in G_\delta,\end{cases}$$

where $c\in\mathbb{R}$ is a constant yet to be determined. Note that u is Lipschitz continuous hence it belongs to $H^1(\omega_{\delta,\eta})$. Since

$$\int_{\omega_{\delta,\eta}}u=c-\delta+\eta\delta c-\frac{\eta}{\delta}\left(\frac{1}{\delta}+c\right)\int_0^\delta x\mathrm{d}x=c-\delta+\frac{1}{2}\eta\delta c-\frac{\eta}{2},$$

in order to guarantee that $\int_{\omega_{\delta,\eta}}u=0$, we set $c=c(\eta,\delta)=\frac{\eta+2\delta}{\eta\delta+2}$. Observe that $\lim_{\delta\to 0}c=0$. Moreover

$$\nabla u(x,y)=\begin{cases}(0,0), & (x,y)\in G,\\ (-\frac{1}{\delta}\left(\frac{1}{\delta}+c\right),0), & (x,y)\in S_{\delta,\eta},\\ (0,0), & (x,y)\in G_\delta.\end{cases}$$

Therefore

$$\int_{\omega_{\delta,h}}|\nabla u|^2=\frac{\eta}{\delta}\left(\frac{1}{\delta}+c\right)^2=\frac{\eta}{\delta}\left(\frac{1}{\delta}+\frac{\eta+2\delta}{\eta\delta+2}\right)^2=O\left(\frac{\eta}{\delta^3}\right)\qquad\text{as }\delta\to 0,$$

while

$$\begin{aligned}\int_{\omega_{\delta,\eta}}u^2&=c^2+1+\eta\int_0^\delta\left(c-\frac{1}{\delta}\left(\frac{1}{\delta}+c\right)x\right)^2\mathrm{d}x\\ &=c^2+1+\frac{\eta}{3\delta}(1-c\delta+c^2\delta^2)\xrightarrow[\delta\to 0]{}1+\frac{1}{3}\lim_{\delta\to 0}\frac{\eta}{\delta}.\end{aligned}$$

Assuming $\eta = o(\delta^3)$ as $\delta \to 0$, we have

$$0 < \mu_1^{\omega_{\delta,\eta}} \leq \frac{\int_{\omega_{\delta,\eta}} |\nabla u|^2 \mathrm{d}x}{\int_{\omega_{\delta,\eta}} u^2 \mathrm{d}x} = O\left(\frac{\eta}{\delta^3}\right) \xrightarrow[\delta \to 0]{} 0.$$

Finally, choosing $\eta = O(\delta^{3+\beta})$ with $\beta > 0$ as $\delta \to 0$, we get that $\mu_1^{\omega_{\delta,\eta}} = O(\delta^\beta)$ as $\delta \to 0$. □

ACKNOWLEDGMENTS

The authors are very thankful to Luigi Provenzano for suggesting [11], Theorem 1.1, and the alternative proof discussed in Remark 2.4. They are also very thankful to Giovanni Franzina for pointing out reference [12] and useful discussions. D.K. was supported by the EXPRO grant No. 20-17749X of the Czech Science Foundation. P.D.L. acknowledges the support from the project "Perturbation problems and asymptotics for elliptic differential equations: variational and potential theoretic methods" funded by the MUR Progetti di Ricerca di Rilevante Interesse Nazionale (PRIN) Bando 2022 grant 2022SENJZ3. P.D.L is member of the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni" (GNAMPA) of the "Istituto Nazionale di Alta Matematica" (INdAM). M.Z. acknowledges financial support by INdAM through their program "Borse di studio per l'estero". The work of M.Z. was partially realized during his stay at the Czech Technical University in Prague with the support of the EXPRO grant No. 20-17749X of the Czech Science Foundation. M.Z. also received support from the French government under the France 2030 investment plan, as part of the Initiative d'Excellence d'Aix-Marseille Université – A*MIDEX AMX-21-RID-012.

Data availability statement

The research data associated with this article are included in the article.

References

[1] P.D. Lamberti and M. Zaccaron, Shape sensitivity analysis for electromagnetic cavities. *Math. Methods Appl. Sci.* **44** (2021) 10477–10500.

[2] M. Costabel and M. Dauge, Maxwell eigenmodes in product domains, in Maxwell's equations—analysis and numeric. *Radon Ser. Comput. Appl. Math.* **24** (2019) 171–198.

[3] F. Ferraresso, P.D. Lamberti and I.G. Stratis, On a Steklov Spectrum in Electromagnetics, in Adventures in Contemporary Electromagnetic Theory, edited by T.G. Mackay and A. Lakhtakia. Springer International Publishing (2023) 195–228.

[4] G.W. Hanson and A.B. Yakovlev, Operator Theory for Electromagnetics. Springer-Verlag, New York (2002).

[5] A. Kirsch and F. Hettlich, The Mathematical Theory of Time-harmonic Maxwell's Equations. Expansion-, Integral-, and Variational Methods. Applied Mathematical Sciences, Vol. 190. Springer, Cham (2015).

[6] G.F. Roach, I.G. Stratis and A.N. Yannacopoulos, Mathematical Analysis of Deterministic and Stochastic Problems in Complex Media Electromagnetics. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ (2012).

[7] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators. Frontiers in Mathematics. Birkhäuser Verlag, Basel (2006).

[8] D. Pauly, On constants in Maxwell inequalities for bounded and convex Domains. *J. Math. Sci. (N.Y.)* **210** (2015) 787–792.

[9] L.E. Payne and H.F. Weinberger, An optimal Poincar´e inequality for convex domains. *Arch. Rational Mech. Anal.* **5** (1960) 286–292.

[10] J. Rohleder, Curl curl versus Dirichlet Laplacian eigenvalues (2024), arXiv:2404.11301.

[11] A. Savo, Hodge–Laplace eigenvalues of convex bodies. *Trans. Amer. Math. Soc.* **363** (2011) 1789–1804.

[12] P. Guerini and A. Savo, The Hodge Laplacian on manifolds with boundary, Séminaire de Théorie Spectrale et Géométrie. Vol. 21. Année 2002–2003, *Sémin. Théor. Spectr. Géom.* **21** (2003) 125–146.

[13] A. Henrot, I. Mazari-Fouquer and Y. Privat, Is the Faber–Krahn inequality true for the Stokes operator? (2024), https://arxiv.org/abs/2401.09801

[14] P. Guerini and A. Savo, Alessandro eigenvalue and gap estimates for the Laplacian acting on p-forms. *Trans. Amer. Math. Soc.* **356** (2004) 319–344.

[15] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 1. Wiley, New York (1989).