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GLOBAL APPROXIMATE CONTROLLABILITY OF THE CAMASSA–HOLM EQUATION BY A FINITE DIMENSIONAL FORCE

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Abstract. In this article, we consider the Camassa–Holm equation posed on the periodic domain $\mathbb{T}$. We show that in any positive time, the Camassa–Holm equation is globally approximately controllable in $H^s(\mathbb{T})$ for $s > \frac{3}{2}$ by a three dimensional external force. The proof is based on Agrachev–Sarychev approach in geometric control theory. In particular, our three dimensional control prevents a finite time blow up of the solution of Camassa–Holm equation in $H^s(\mathbb{T})$ for $s > \frac{3}{2}$.

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1. Introduction

The controllability of non-linear partial differential equations (PDEs) has gained a lot of interest in the literature throughout the years. For instance, see the book of Coron [1] and the reference therein. Numerous local controllability results have been established using linearization techniques, with some extending to semiglobal or even global controllability. More recently, several authors have shown interest in global controllability results from a geometric perspective, see Jurdjevic [2] and Agrachev *et al.* [3, 4], along with the cited references. In this article, we explore this perspective for the Camassa–Holm equation.

1.1. The system under study

We consider a control problem related to the Camassa–Holm equation on the circle $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, given by

$$u_t - u_{txx} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx} \qquad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{T}. \tag{1.1}$$

This equation models the unidirectional propagation of shallow water waves, including the phenomenon of wave breaking. Here, $u(t,x)$ denotes the fluid velocity at time t and position x, while κ is a non-negative constant. The Camassa–Holm equation was first introduced by Fokas and Fuchssteiner [5] as a bi-Hamiltonian system. Later, Camassa and Holm [6] derived it as a model for shallow water waves and showed that it admits peaked

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solitary wave solutions, known as peakons. Furthermore, the equation (1.1) can be written as

$$u_t + uu_x + (I - \partial_x^2)^{-1}\partial_x\big(2\kappa u + u^2 + \frac{1}{2}(u_x)^2\big) = 0,$$

which is a nonlocal and nonlinear perturbation of the inviscid Burgers equation.

1.2. The control problem

We consider the following control system:

$$\begin{cases} u_t - u_{txx} + u_x + 3uu_x = 2u_xu_{xx} + uu_{xxx} + \eta(t,x), & \text{for } (t,x) \in (0,T)\times\mathbb{T}, \\ u(0,x) = u_0(x), & \text{for } x \in \mathbb{T}, \end{cases} \tag{1.2}$$

where $T > 0$, u_0 is a given initial condition, and η is a control. For simplicity in calculations, we choose $\kappa = \frac{1}{2}$, though a similar analysis can be carried out for general κ.

The primary question we address is the following controllability problem: Given u_0, u_1, and $T > 0$, can we find a control η such that the solution $u(t,x)$ of (1.2), starting from u_0, stays arbitrarily close to u_1 at time T?

To formalize this question, we introduce the notion of approximate controllability for the system.

Definition 1.1. Let H be a Hilbert space and $\mathcal{G}$ a vector space. The equation (1.2) is said to be approximately controllable in H using $\mathcal{G}$-valued control if, for any given $T > 0$, $\varepsilon > 0$, and $u_0, u_1 \in H$, there exists a control η with values in $\mathcal{G}$ and a solution u of (1.2) such that

$$\|u(T) - u_1\|_H \le \varepsilon.$$

In this paper, we establish the approximate controllability of (1.2) using controls that belong to a finite-dimensional vector space $\mathcal{G}$. To formulate our main result, we first introduce the relevant function spaces.

1.3. Function spaces and Notations

Let $s \in \mathbb{R}$. We define the pseudo-differential operator $\Lambda^s := (I - \partial_{xx})^{\frac{s}{2}}$ *via* its action on Fourier coefficients:

$$\widehat{\Lambda^s f}(n) = (1+n^2)^{\frac{s}{2}}\,\hat{f}(n), \qquad n \in \mathbb{Z},$$

where $\hat{f}(n)$ denotes the n-th Fourier coefficient of f, given by

$$\hat{f}(n) = \int_{\mathbb{T}} \mathrm{e}^{-inx} f(x)\,\mathrm{d}x.$$

We next introduce the Sobolev spaces $H^s = H^s(\mathbb{T})$ of periodic functions on the torus $\mathbb{T}$, defined for any $s \in \mathbb{R}$ by

$$H^s(\mathbb{T}) = \left\{ f \in \mathcal{D}'(\mathbb{T}) : \|f\|_{H^s} = \|\Lambda^s f\|_{L^2} \simeq \left(\sum_{n\in\mathbb{Z}} (1+n^2)^s |\hat{f}(n)|^2\right)^{\frac{1}{2}} < \infty \right\}.$$

The Cauchy problem for (1.2) is shown to be locally (in time) well-posed in $H^s(\mathbb{T})$ for $s > \frac{3}{2}$. Consequently, our focus is on establishing approximate controllability results in the same functional framework.

Throughout this article, we use the notation $P \lesssim Q$ for positive quantities P and Q to mean there exists a constant $c > 0$ such that $P \le cQ$.

1.4. Main result

We are now in a position to announce the main result of the present work.

Theorem 1.2. *(Global approximate controllability of the Camassa–Holm equation) For $s > \frac{3}{2}$, the equation (1.2) is approximately controllable in $H^s(\mathbb{T})$ by piecewise constant controls taking values in $\mathcal{H}$, where*

$$\mathcal{H} = span\{1, \sin(x), \cos(x)\}. \tag{1.3}$$

Next, we briefly discuss known results on the well-posedness and control for the Camassa–Holm equation.

1.5. Literature on the Camassa–Holm equation

- **Well-posedness:** Constantin and Escher [7] established the local-in-time well-posedness of the periodic Camassa–Holm equation in H^3 and demonstrated that solutions can blow up in finite time. Subsequently, Himonas and Misiolek [8] extended this result by proving the local-in-time well-posedness and finite-time blow-up phenomena for the periodic Camassa–Holm equation in H^s with $s > \frac{3}{2}$. For related results, see also the works of Danchin [9], Himonas and Holliman [10], and Misiolek [11]. The phenomenon of blow-up for the Camassa–Holm equation has been extensively studied by many authors; for further details, refer to [12, 13].
- **Controllability:** Here, we discuss existing controllability results for both the linearized and nonlinear Camassa–Holm equations.

  The linearized systems of the Camassa–Holm equation and the Benjamin–Bona–Mahony (BBM) equation around zero share the same structure. Both exhibit an accumulation point in the spectrum of their associated linearized operators, which makes null controllability unattainable *via* localized interior or boundary controls. This limitation is discussed in Glass [14] (Sect. 1.3) and Micu [15] (Sect. 4). Specifically, Micu [15] demonstrated that the system is not spectrally controllable but is approximately controllable using boundary control in $L^2(0, T)$.

  Glass [14] used the return method and scaling arguments to establish the global exact controllability of the nonlinear Camassa–Holm equation in $H^s(\mathbb{T})$ for $s > \frac{3}{2}$, using localized interior control on the one-dimensional torus. On the other hand, for the viscous Camassa–Holm equation, Mitra [16] utilized the moment method and source term arguments to demonstrate that the system can be driven to a constant steady state *via* localized interior control, provided the initial condition is sufficiently close to the steady state.
- **Stabilizability:** The asymptotic behavior of the Camassa–Holm equation has been studied in detail in [17]. In [14], the global asymptotic stabilizability of the Camassa–Holm equation with periodic boundary conditions was proved using an interior feedback control. Similarly, Perrollaz [18] established global asymptotic stabilization by means of a Dirichlet boundary feedback law.

  The exponential stabilization of parabolic partial differential equations (PDEs) using localized finite-dimensional controls, either applied in the interior or at the boundary, is a well-studied area. Notable contributions in this field include the works of Raymond, Barbu, Liu, and Triggiani [19–24].

1.6. Our approach and achievement of the present work

- **Main contributions of Theorem 1.2**: The global controllability result is very significant for non-linear PDEs, as in this case the control does not impose restrictions on the amplitude of initial and final targets or on the control time T. In [14], Glass established the global exact controllability in H^s for $s > 3/2$ using localized controls in $C([0, T]; H^{s-3}(\mathbb{T}))$, whereas our controls act everywhere and take values in a three-dimensional space spanned by 1, $\sin x$ and $\cos x$. The construction of finite-dimensional controls is particularly advantageous for practical applications in physics and engineering. Roughly speaking, the control obtained by Glass is localized in space, but in our case the control is localized in Fourier modes.

It is well-known that solutions of the Camassa–Holm equation (1.1) can exhibit finite-time blow-up phenomena in $H^s(\mathbb{T})$ for $s > \frac{3}{2}$, due to the presence of nonlinear terms in the equation [12, 25]. However, by introducing a three-dimensional force term into (1.1), we not only guarantee global controllability at any given time T, but also prevent the occurrence of finite-time blow-up in the solution.

- **Agrachev–Sarychev Method.** The Lie algebraic framework in geometric control theory for finite-dimensional nonlinear ODEs has been extensively investigated (*e.g.*, [2, 26]). This methodology has also been extended to various nonlinear PDEs. The first significant application, by Agrachev and Sarychev in [4, 27], demonstrated the approximate controllability of the two-dimensional Navier–Stokes/Euler equations on a torus using finite-dimensional controls over the entire domain.

  Subsequent developments expanded this approach to other equations. For instance, Shirikyan applied it to the three-dimensional Navier–Stokes equations [28, 29], while Nersisyan tackled the compressible and incompressible Euler equations [30, 31]. Shirikyan also explored its application to the viscous Burgers equation [32, 33], and Sarychev extended it to the Schrödinger equation [34]. More recently, Coron, Xiang, and Zhang [35] employed this method to establish small-time approximate controllability for the one-dimensional semiclassical cubic Schrödinger equation.

  Recent applications of this technique include Narsesyan's work on nonlinear parabolic PDEs [36], which generalizes earlier results by Glatt–Holtz, Herzog, and Mattingly [37]. Using similar methods, Chen demonstrated global approximate controllability for the Korteweg–de Vries equation [38], Jellouli addressed the Benjamin–Bona–Mahony equation [39], and Gao investigated the Kuramoto–Sivashinsky equation [40]. To our knowledge, this article provides the first application of the Agrachev–Sarychev method for the Camassa–Holm equation.

  Agrachev [41] posed an interesting open problem on the approximate controllability of the incompressible Navier–Stokes equations using finite-dimensional localized interior controls. Recently Nersesyan and Rissel [42, 43] made significant contributions on this problem.
- **Strategy of the Proof :** The proof of Theorem 1.2 is inspired by recent applications of the Agrachev–Sarychev method, particularly the works of Nersesyan [36] and Jellouli [39]. However, adapting this approach to the Camassa–Holm equation presents significant difficulties due to its nonlocal and nonlinear structure, which involves multiple interacting terms of different types. This complexity introduces substantial complications at various stages of the proof, including the establishment of well-posedness and stability estimates, the implementation of the large-control/small-time scaling limit, and the verification of the saturation property. Overcoming these challenges requires refined tools to manage the intricate interplay between the various nonlinear components.

  Using the saturation property, we reduce the problem of approximate controllability to that of steering the system toward targets lying in an affine finite-dimensional subspace translated by the initial condition. We then show that, over sufficiently small time intervals, the system can be driven from the initial state u_0 to a state arbitrarily close to any point in $u_0 + \mathcal{H}$. By iteratively applying this procedure, we establish global approximate controllability within arbitrarily small time intervals. The core of this approach lies in employing large controls over short time spans, exploiting a specific asymptotic property linked to the nonlinear structure of the equation. Finally, we establish the global approximate controllability at any prescribed time using the continuity of the solution with respect to time and the stability properties of the system.

  This method extensively employs nonlinear dynamics and geometric control theory, offering a robust framework for tackling global controllability challenges in nonlinear PDEs, even in the presence of blow-up phenomena.

We end this section with the outline of the article below.

1.7. Organization of the paper

The rest of this article is organized as follows.

- Section 2 contains some propositions regarding well-posedness (local in time existence and uniqueness), stability, asymptotic property and density of certain space in H^s with the algebraic property of resolvent map, which are very useful ingredients of the proof of our main result.
- Section 3 is devoted to proving the Theorem 1.2.
- Basic properties of the pseudo-differential operator Λ^s and Friedrichs mollifier are studied in Section 4. Then the proof of the Proposition (2.1)–(2.4) on well-posedness, stability and asymptotic property are presented.

2. Existence and properties

In this section, we discuss the well-posedness, semi-global stability, and asymptotic behavior of the solution to the equation (1.2). These properties serve as fundamental tools for establishing approximate controllability. The proofs of most these results will be presented in Section 4.

First we note that the study of (1.2) is equivalent to analyzing the following equation:

$$\begin{cases} u_t + \Lambda^{-2}\partial_x u + \mathcal{N}(u) = \Lambda^{-2}\eta, \\ u(0,x) = u_0(x), \end{cases} \tag{2.1}$$

where

$$\begin{cases} \Lambda^{-2}u = (I - \partial_{xx})^{-1}u, \\ \mathcal{N}(u) := uu_x + \Lambda^{-2}\Big(2uu_x + u_x u_{xx}\Big). \end{cases}$$

To establish our main result, we will focus on a perturbed version of (2.1). Let $\varphi := \varphi(x)$ be a given function defined on $\mathbb{T}$. Consider the perturbed equation:

$$\begin{cases} u_t + \Lambda^{-2}\partial_x(u+\varphi) + \mathcal{N}(u+\varphi) = \Lambda^{-2}f, \\ u(0,x) = u_0(x), \end{cases} \tag{2.2}$$

where u_0 and f are given functions. We now present the well-posedness result for (2.2).

Proposition 2.1 (Well-posedness)**.** *Let $s > \frac{3}{2}$, $u_0 \in H^s(\mathbb{T})$, $\varphi \in H^{s+1}(\mathbb{T})$ and $f \in L^\infty((0,\infty); H^{s-2}(\mathbb{T}))$. Then there exists a time $T_* := T_*(u_0, \varphi, f) > 0$ such that system (2.2) admits a solution $u \in C([0,T_*]; H^s(\mathbb{T}))$.*

Further, we establish the following stability result.

Proposition 2.2 (Semi-global Stability)**.** *Let $s > \frac{3}{2}$, $R > 0$, $\varphi \in H^{s+1}(\mathbb{T})$ and $f \in L^\infty((0,\infty); H^{s-2}(\mathbb{T}))$. Let $u_0, v_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \le R$ and $\|v_0\|_{H^s} \le R$. Then there exists a time $T^* := T^*(R, \varphi, f) > 0$ and a positive constant C_R such that*

$$\|u(t) - v(t)\|_{H^{s-1}} \le C_R \|u_0 - v_0\|_{H^{s-1}}, \quad \text{for all } t \in [0, T^*], \tag{2.3}$$

where $u(t)$ and $v(t)$ are the solutions of (2.2) associated with initial data u_0, v_0 respectively. Furthermore, (2.2) admits a unique solution for initial conditions belonging to H^s.

We define the solution of (2.2) as

$$u(t) := \mathcal{R}_t(u_0, \varphi, f),$$

provided it exists. Based on the uniqueness of the solutions to (2.2) and (2.1), we have the following property:

$$\mathcal{R}_t(u_0, \varphi, f) = \mathcal{R}_t(u_0 + \varphi, 0, f) - \varphi, \tag{2.4}$$

whenever the solution exists. This equivalence highlights that the study of approximate controllability for both (1.2) and (2.2) is the same.

Now, if we choose $u_0, v_0 \in H^{s+1}(\mathbb{T})$ with $\|u_0\|_{H^{s+1}} \leq R$ and $\|v_0\|_{H^{s+1}} \leq R$, the corresponding solutions satisfy the following stability inequality:

$$\|\mathcal{R}_t(u_0, 0, f) - \mathcal{R}_t(v_0, 0, f)\|_{H^s} \leq C_R \|u_0 - v_0\|_{H^s}, \tag{2.5}$$

which follows from (2.3). This stability result plays a crucial role in establishing the approximate controllability of (1.2).

The solution map $\mathcal{R}_t$ satisfies the following flow property:

Lemma 2.3. *Let the assumptions of Proposition 2.1 on s, u_0, and φ hold, and define*

$$f(s) = \begin{cases} f_1(s), & s \in [0, t_1), \\ f_2(s), & s \in [t_1, t_1 + t_2), \\ f_3(s), & s \in [t_1 + t_2, t_1 + t_2 + t_3), \end{cases}$$

for some $t_1, t_2, t_3 > 0$, with $f_i \in H^{s-2}(\mathbb{T})$. Assume $T^* > t_1 + t_2 + t_3$. Then the solution $\mathcal{R}_t(u_0, 0, f)$ of (2.2) satisfies

$$\mathcal{R}_{t_1+t_2+t_3}(u_0, 0, f) = \mathcal{R}_{t_3}(\mathcal{R}_{t_2}(\mathcal{R}_{t_1}(u_0, 0, f_1(\cdot)), 0, f_2(\cdot - t_1)), 0, f_3(\cdot - t_1 - t_2)).$$

Proof. Let $0 < s \leq t$ and $\widehat{\mathcal{R}}(t, s, v, f)$ denote the solution of (2.2) at time t with $\varphi = 0$ and initial condition $\widehat{\mathcal{R}}(s, s, v, f) = v$. This implies that $\mathcal{R}_t(u_0, 0, f) = \widehat{\mathcal{R}}(t, 0, u_0, f)$. By the uniqueness of solutions, we have the following relationship for all $\sigma \in [s, t]$

$$\widehat{\mathcal{R}}(t, \sigma, \widehat{\mathcal{R}}(\sigma, s, u_0, f), f) = \widehat{\mathcal{R}}(t, s, u_0, f). \tag{2.6}$$

Using (2.6), we can write

$$\begin{aligned}
\widehat{\mathcal{R}}(t_1 + t_2 + t_3, 0, u_0, f) &= \widehat{\mathcal{R}}(t_1 + t_2 + t_3, t_1 + t_2, \widehat{\mathcal{R}}(t_1 + t_2, t_1, \widehat{\mathcal{R}}(t_1, 0, u_0, f_1(\cdot)), f_2(\cdot - t_1)), f_3(\cdot - t_2 - t_1)) \\
&= \widehat{\mathcal{R}}(t_3, 0, \widehat{\mathcal{R}}(t_1 + t_2, t_1, \widehat{\mathcal{R}}(t_1, 0, u_0, f_1(\cdot)), f_2(\cdot - t_1)), f_3(\cdot - t_2 - t_1)) \\
&= \mathcal{R}_{t_3}\left(\widehat{\mathcal{R}}(t_1 + t_2, t_1, \widehat{\mathcal{R}}(t_1, 0, u_0, f_1(\cdot)), f_2(\cdot - t_1)), 0, f_3(\cdot - t_2 - t_1)\right) \\
&= \mathcal{R}_{t_3}\left(\mathcal{R}_{t_2}\left(\mathcal{R}_{t_1}(u_0, 0, f_1(\cdot)), 0, f_2(\cdot - t_1)\right), 0, f_3(\cdot - t_2 - t_1)\right).
\end{aligned}$$

This completes the proof. □

Next, we analyze the properties of solutions to the perturbed equation (2.2) when f is constant in time. For $T > 0$, define the set

$$\Theta(u_0, T) = \left\{ (\varphi, f) \in H^{s+1}(\mathbb{T}) \times H^{s-2}(\mathbb{T}) \;\middle|\; \text{the solution of (2.2) exists in } C([0, T]; H^s(\mathbb{T})) \right\}.$$

This set contains all φ and f for which a solution exists on the interval $[0,T]$. In general, a solution may not exist for the entire time interval $[0,T]$, so the set $\Theta(u_0,T)$ could be empty. However, the following results ensure that the solutions exist on $[0,T]$ for any finite positive time T, provided that φ and f are sufficiently large. Let $\delta>0$. Consider the system:

$$\begin{cases} u_t+\Lambda^{-2}\partial_x\left(u+\delta^{-\frac{1}{2}}\varphi\right)+\mathcal{N}\left(u+\delta^{-\frac{1}{2}}\varphi\right)=\delta^{-1}\Lambda^{-2}f, & t>0,\ x\in\mathbb{T},\\ u(0,x)=u_0(x), & x\in\mathbb{T}. \end{cases} \tag{2.7}$$

We now present an asymptotic property for the solution of (2.7).

Proposition 2.4 (Asymptotic property). *Let $T>0$, $s>\frac{3}{2}$, and $u_0,\varphi\in H^{s+1}(\mathbb{T})$. Then the solution of (2.7) satisfies the following properties.*

1. *For $\eta_0\in H^{s-2}(\mathbb{T})$, there exists $\delta_0>0$, for all $\delta\in(0,\delta_0)$, such that*

$$(\delta^{-\frac{1}{2}}\varphi,\delta^{-1}\eta_0)\in\Theta(u_0,T).$$

2. *For $\eta_0\in H^{s-1}(\mathbb{T})$, the following limit holds at time $t=\delta$*

$$\mathcal{R}_\delta(u_0,\delta^{-\frac{1}{2}}\varphi,\delta^{-1}\eta_0)\to u_0+\Lambda^{-2}\eta_0-\mathcal{N}(\varphi),\quad \text{in}\ \ H^s(\mathbb{T})\ \text{as}\ \ \delta\to0^+.$$

This asymptotic property demonstrates that the controlled trajectory, starting from u_0, can approach any neighborhood of any point in the affine space $u_0+\mathcal{C}$ within a short time. Here, the subspace $\mathcal{C}$ is generated by the nonlinearity of the equation. This result is useful to achieve approximate controllability in a short time using a large control that lies in a three-dimensional subspace. Motivated by this, we introduce the following chain of subsets.

For any set G, we define

$$\mathcal{F}(G):=\left\{\eta-\sum_{i=1}^{d}\mathcal{N}(\varphi_i)\ \middle|\ \eta,\varphi_i\in G,\ \forall\ d\geq1\right\}.$$

Using this definition, we construct a sequence of sets as follows:

$$\mathcal{H}_0:=\mathcal{H}=\operatorname{span}\{1,\sin(x),\cos(x)\},\quad \mathcal{H}_n:=\mathcal{F}(\mathcal{H}_{n-1})\ \text{for}\ n\geq1,\quad \mathcal{H}_\infty:=\bigcup_{n=1}^{\infty}\mathcal{H}_{n-1}.$$

From this construction, it is clear that

$$\mathcal{H}_0\subset\mathcal{H}_1\subset\cdots\subset\mathcal{H}_n\subset\cdots\subset\mathcal{H}_\infty. \tag{2.8}$$

Definition 2.5. A space $\mathcal{H}$ is said to be saturating if $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T})$ for all $s\geq0$.

It is not immediately clear that $\mathcal{H}_N$ forms a vector space. However, the following proposition establishes this fact and provides additional properties.

Proposition 2.6 (Density). *The spaces $\mathcal{H}_N$ satisfy the following properties:*

(i) For $N\geq1$, $\mathcal{H}_N$ is a vector space of dimension $2^{N+1}+1$. Specifically,

$$\mathcal{H}_N=\operatorname{span}\{1,\sin(x),\cos(x),\dots,\sin(2^Nx),\cos(2^Nx)\}. \tag{2.9}$$

(ii) The space $\mathcal{H}$ is saturating, that is, $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T})$.

Proof. First we prove that $\mathcal{H}_N$ is a vector space spanned by $\{\sin(mx), \cos(mx) \mid 0 \le m \le 2^N\}$. We establish this by induction. Before applying induction, we observe that

$$\begin{cases} (I - \partial_{xx})^{-1} \sin(mx) = \frac{1}{(1+m^2)} \sin(mx), \\ (I - \partial_{xx})^{-1} \cos(mx) = \frac{1}{(1+m^2)} \cos(mx). \end{cases}$$

Using above formulas and trigonometric identities, we obtain that for $k, m \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{N}\Big(\sin(mx) + \sin(kx)\Big) &= \mathcal{B}_m \sin(2mx) + \mathcal{B}_k \sin(2kx) \\ &\quad + \mathcal{A}_{m,k} \sin((m+k)x) - \mathcal{A}_{m,-k} \sin((m-k)x), \end{aligned} \tag{2.10}$$

$$\mathcal{N}\Big(\cos(mx) + \cos(kx)\Big) = -\mathcal{N}\Big(\sin(mx) + \sin(kx)\Big), \tag{2.11}$$

$$\begin{aligned} \mathcal{N}\Big(\sin(mx) \pm \cos(kx)\Big) &= \mathcal{B}_m \sin(2mx) - \mathcal{B}_k \sin(2kx) \\ &\quad \pm \mathcal{A}_{m,k} \cos((m+k)x) \pm \mathcal{A}_{m,-k} \cos((m-k)x), \end{aligned} \tag{2.12}$$

where

$$\mathcal{A}_{m,k} = \frac{(k+m)(3+km+k^2+m^2)}{2(1+(m+k)^2)}, \quad \mathcal{B}_m = \frac{3m(1+m^2)}{2(1+4m^2)}.$$

We begin by proving the result for $N = 1$. Setting $m = k = 1$ in (2.10) and (2.11), it follows that

$$\operatorname{span}\{\sin(2x)\} \subseteq \mathcal{H}_1.$$

Similarly, substituting $m = k = 1$ in (2.12), we deduce that

$$\operatorname{span}\{\cos(2x)\} \subseteq \mathcal{H}_1.$$

From the definition of $\mathcal{H}_1$, we therefore have

$$\mathcal{H}_1 = \operatorname{span}\{1, \sin(x), \cos(x), \sin(2x), \cos(2x)\}.$$

Next, we prove the result for $N > 1$ by induction. Assume the result holds for $N-1$, that is,

$$\mathcal{H}_{N-1} = \operatorname{span}\{1, \sin(x), \cos(x), \ldots, \sin(2^{N-1}x), \cos(2^{N-1}x)\}.$$

To show that the result holds for N, we set $m = k = 2^{N-2} + j,\ 1 \le j \le 2^{N-2}$ in (2.10) and (2.11), to obtain

$$\operatorname{span}\{\sin((2^{N-1} + 2j)x)\} \subseteq \mathcal{H}_N, \quad \text{for } j = 1, 2, \ldots, 2^{N-2}. \tag{2.13}$$

Choosing $m = 2^{N-1} + 2j,\ 0 \le j \le 2^{N-1} - 1,\ k = 1$ and using the structure $\mathcal{H}_N = \mathcal{F}(\mathcal{H}_{N-1})$, we see that

$$\operatorname{span}\{\sin((2^{N-1} + 2j - 1)x)\} \subseteq \mathcal{H}_N \quad \text{for } j = 0, 1, 2, \ldots, 2^{N-1} - 1. \tag{2.14}$$

We combine (2.13) and (2.14) to get

$$\operatorname{span}\{\sin((2^{N-1} + j)x)\} \subseteq \mathcal{H}_N \quad \text{for } j = 1, 2, \ldots, 2^{N-1}. \tag{2.15}$$

Similarly choosing m, k properly in (2.12), we get that

$$\text{span}\{\cos((2^{N-1}+j)x)\} \subseteq \mathcal{H}_N \quad \text{for } j = 1, 2, \dots, 2^{N-1}. \tag{2.16}$$

From (2.15), (2.16) using the definition of $\mathcal{H}_N$, we see that

$$\mathcal{H}_N = \text{ span } \{1, \sin(x), \cos(x), \dots, \sin(2^N x), \cos(2^N x)\}.$$

This completes the proof of (i). The proof of (ii) follows from (i). □

Remark 2.7. The above proposition suggests that if $c_1, c_2, \dots, c_M \in \mathbb{R}$ and $\eta, \varphi_1, \varphi_2, \dots, \varphi_M \in \mathcal{H}_{N-1}$, then there exist $\tilde{\eta}, \tilde{\varphi}_1, \tilde{\varphi}_2, \dots \tilde{\varphi}_P \in \mathcal{H}_{N-1}$ such that

$$\eta - \sum_{i=1}^{M} c_i \mathcal{N}(\varphi_i) = \tilde{\eta} - \sum_{s=1}^{P} \mathcal{N}(\tilde{\varphi}_s).$$

The saturation property plays a crucial role in establishing approximate controllability. Specifically, for any given $u_0, u_1 \in H^s(\mathbb{T})$, we use the saturation property to find an integer N and a point $u_N \in \mathcal{H}_N$ such that u_N is arbitrarily close to $u_1 - u_0$. Utilizing the asymptotic property, with the particular choices $\varphi = 0$ and $\eta_0 = \Lambda^2 u_N$, we have

$$\mathcal{R}_\delta(u_0, 0, \delta^{-1}\eta_0) \to u_0 + u_N \quad \text{in } H^s, \text{ as } \delta \to 0^+.$$

This implies that the initial state u_0 can be driven close to a desired target state u_1 within a short time $t_* > 0$ using a control that takes values in $\mathcal{H}_N$. It suggests that approximate controllability can be achieved with a control that lies in a finite-dimensional subspace of $H^s(\mathbb{T})$. However, the dimension of the control space depends on the initial and target states. Then an immediate question arises: Can we find a subspace V such that the equation is approximately controllable with a V-valued control for any pair of initial and target states? The answer is affirmative, and it can be achieved using an $\mathcal{H}$-valued control, where

$$\mathcal{H} = \text{span}\{1, \sin(x), \cos(x)\}.$$

This is the main novelty of this work, and the next section is devoted to establishing this result.

3. Proof of Theorem 1.2

We begin this section by introducing the notion of small-time approximate controllability for the control system (1.2). The formal definition is provided below.

Definition 3.1. Let H be a Hilbert space and $\mathcal{G}$ be a vector space. We say that the equation (1.2) is small time approximately controllable in H by a control values in $\mathcal{G}$ if for any $\varepsilon > 0$, $\sigma > 0$ and any $u_0, u_1 \in H$, there exists a $\delta \in (0, \sigma)$ and a control $f \in L^\infty(0, \delta; \mathcal{G})$ such that the solution u of (1.2) satisfies

$$\|\mathcal{R}_\delta(u_0, 0, f) - u_1\|_H \leq \varepsilon.$$

In order to prove the main theorem, first we show that the equation (1.2) is small time approximately controllable. The proof is inspired by [36], which establishes the approximate controllability by a finite dimensional control for a nonlinear parabolic PDEs.

Proposition 3.2. *For $s > \frac{3}{2}$ and $\mathcal{H}$ as (1.3), equation (1.2) is small time approximately controllable by $\mathcal{H}$-valued control.*

Proof. Let $u_0 \in H^{s+1}(\mathbb{T})$, $\varepsilon > 0$, and $\sigma > 0$. We first use the asymptotic property to establish small-time approximate controllability from u_0 to $u_0 + \mathcal{H}_N$. This is proven using induction through the following steps.

STEP 1: **Small time approximate controllability from u_0 to $u_0 + \mathcal{H}_0$.** Let $u_0 \in H^{s+1}$ and $u_1 \in u_0 + \mathcal{H}_0$. Then, there exists a $\eta \in \mathcal{H}_0$ such that $u_1 = u_0 + \eta$. Using Proposition 2.4, we find that

$$\mathcal{R}_\delta(u_0, 0, \delta^{-1}\Lambda^2\eta) \to u_0 + \eta, \quad \text{as } \delta \to 0^+.$$

Thus, there exists a $\delta \in (0, \sigma)$ such that

$$\|\mathcal{R}_\delta(u_0, 0, f) - u_1\|_{H^s} < \varepsilon,$$

where $f(t, \cdot) = \frac{1}{\delta}\Lambda^2\eta(\cdot)$. This shows that the equation (1.2) is small-time approximately controllable from u_0 to $u_0 + \mathcal{H}_0$ using an $\mathcal{H}_0$-valued control.

STEP 2: **Small time approximate controllability from u_0 to $u_0 + \mathcal{H}_N$.** Assume that the equation (1.2) is small-time approximately controllable from u_0 to $u_0 + \mathcal{H}_{N-1}$ using an $\mathcal{H}_0$-valued control. We will show that it is also controllable from u_0 to $u_0 + \mathcal{H}_N$.

Let $u_1 \in u_0 + \mathcal{H}_N$. From the definition of $\mathcal{H}_N$, there exist $\eta, \varphi_1, \varphi_2, \ldots, \varphi_d \in \mathcal{H}_{N-1}$ such that

$$u_1 = u_0 + \eta - \sum_{i=1}^{d} \mathcal{N}(\varphi_i),$$

for some $d \in \mathbb{N}$. We prove the result using induction on d.

Case $d = 1$: Suppose $u_1 = u_0 + \eta - \mathcal{N}(\varphi_1)$. Using the asymptotic property of Proposition 2.4 and the flow property (2.4), we have

$$\mathcal{R}_\delta(u_0 + \delta^{-1/2}\varphi_1, 0, 0) - \delta^{-1/2}\varphi_1 \to u_0 - \mathcal{N}(\varphi_1) \quad \text{in } H^s \text{ as } \delta \to 0^+.$$

Thus, there exists a $\delta_2 \in (0, \sigma/3)$ such that

$$\|\mathcal{R}_{\delta_2}(u_0 + \delta_2^{-1/2}\varphi_1, 0, 0) - (u_0 + \delta_2^{-1/2}\varphi_1 - \mathcal{N}(\varphi_1))\|_{H^s} < \frac{\varepsilon}{3}. \tag{3.1}$$

Since $u_0 + \delta_2^{-1/2}\varphi_1 \in u_0 + \mathcal{H}_{N-1}$, the induction hypothesis implies there exists a $\delta_1 \in (0, \sigma/3)$ and $f_1 \in L^\infty((0, \delta_1); \mathcal{H}_0)$ such that

$$\|\mathcal{R}_{\delta_1}(u_0, 0, f_1) - (u_0 + \delta_2^{-1/2}\varphi_1)\|_{H^s} < \frac{\varepsilon}{3}. \tag{3.2}$$

Define $\hat{u}_0 = u_0 + \delta_2^{-1/2}\varphi_1 - \mathcal{N}(\varphi_1)$, so that

$$u_1 = u_0 + \eta - \mathcal{N}(\varphi_1) = \hat{u}_0 + \eta - \delta_2^{-1/2}\varphi_1 \in \hat{u}_0 + \mathcal{H}_{N-1}.$$

By the induction hypothesis, there exists a $\delta_3 \in (0, \sigma/3)$ and $f_2 \in L^\infty((0, \delta_3); \mathcal{H}_0)$ such that

$$\|\mathcal{R}_{\delta_3}(\hat{u}_0, 0, f_2) - u_1\|_{H^s} < \frac{\varepsilon}{3}. \tag{3.3}$$

Now define the control function as follows:

$$f(t) = \begin{cases} f_1(t), & t \in (0, \delta_1), \\ 0, & t \in (\delta_1, \delta_1 + \delta_2), \\ f_2(t), & t \in (\delta_1 + \delta_2, \delta_1 + \delta_2 + \delta_3). \end{cases}$$

Let $\delta = \delta_1 + \delta_2 + \delta_3$, where $\delta \in (0, \sigma)$. Using the stability property (2.5), the flow property (2.4) and Lemma 2.3 combined with the results from (3.1), (3.2), and (3.3), we deduce that

$$\|\mathcal{R}_\delta(u_0, 0, f) - u_1\|_{H^s} \lesssim \varepsilon.$$

This proves the result for $d = 1$.
Case $d > 1$: Assume the result holds for $d-1$. Let $u_1 = u_0 + \eta - \sum_{i=1}^{d} \mathcal{N}(\varphi_i)$, where $\eta, \varphi_1, \varphi_2, \ldots, \varphi_d \in \mathcal{H}_{N-1}$. By the induction hypothesis, there exists a $\delta_1 \in (0, \sigma/2)$ and $f_1 \in L^\infty((0, \delta_1); \mathcal{H}_0)$ such that

$$\|\mathcal{R}_{\delta_1}(u_0, 0, f_1) - (u_0 + \eta - \sum_{i=1}^{d-1} \mathcal{N}(\varphi_i))\|_{H^s} < \frac{\varepsilon}{2}.$$

Define $\hat{u}_0 = u_0 + \eta - \sum_{i=1}^{d-1} \mathcal{N}(\varphi_i)$. Then $u_1 = \hat{u}_0 - \mathcal{N}(\varphi_d)$. Using the case for $d = 1$ with $\hat{u}_0$, there exists a $\delta_2 \in (0, \sigma/2)$ and $f_2 \in L^\infty((0, \delta_2); \mathcal{H}_0)$ such that

$$\|\mathcal{R}_{\delta_2}(\hat{u}_0, 0, f_2) - u_1\|_{H^s} < \frac{\varepsilon}{2}.$$

Let $\delta = \delta_1 + \delta_2$, where $\delta \in (0, \sigma)$, and consider the control function:

$$f(t) = \begin{cases} f_1(t), & t \in (0, \delta_1), \\ f_2(t), & t \in (\delta_1, \delta_1 + \delta_2). \end{cases}$$

Using the results for $d-1$ and $d = 1$, we conclude that

$$\|\mathcal{R}_\delta(u_0, 0, f) - u_1\|_{H^s} \lesssim \varepsilon.$$

This completes the proof for d.

STEP 3: **Global approximate controllability at small time in** H^s. Let $u_0, u_1 \in H^s$ and $\varepsilon > 0$, $\sigma > 0$. By the density of H^{s+1} in H^s, there exists a $\tilde{u}_0 \in H^{s+1}$ such that

$$\|u_0 - \tilde{u}_0\|_{H^s} < \varepsilon. \tag{3.4}$$

Using the saturating property of Proposition 2.6, there exists $\varphi_N \in \mathcal{H}_N$ such that

$$\|u_1 - (\tilde{u}_0 + \varphi_N)\|_{H^s} < \varepsilon. \tag{3.5}$$

By the result of STEP 2, there exists a $\delta \in (0, \sigma)$ and $f \in L^\infty((0, \delta); \mathcal{H}_0)$ such that

$$\|\mathcal{R}_\delta(\tilde{u}_0, 0, f) - (\tilde{u}_0 + \varphi_N)\|_{H^s} < \varepsilon. \tag{3.6}$$

Combining inequalities (3.4), (3.5), and (3.6), along with the stability result (2.5), we deduce that the system exhibits small-time global approximate controllability in H^s with $\mathcal{H}_0$-valued control. This concludes the proof. □

Now we use the small time global approximate controllability result to prove our main result.

Proof of Theorem 1.2. Let $T > 0$ and $\varepsilon > 0$. For given $u_0, u_1 \in H^s$, we aim to demonstrate the existence of a control $\eta \in L^2((0,T);\mathcal{H}_0)$ such that $(0,\eta) \in \Theta(u_0, T)$ and the solution of (1.2) satisfies

$$\|\mathcal{R}_T(u_0, 0, \eta) - u_1\|_{H^s} < \varepsilon.$$

Using the small-time approximate controllability result, starting from u_0, we can move close to u_1 in a short time using a $\mathcal{H}_0$-valued control. Then the strategy is to employ a $\mathcal{H}_0$-valued control to keep the trajectory near u_1 for the remaining time period.

First, due to the time continuity of solutions and the stability property (2.5), we note that a control-free trajectory starting near u_1 remains close to u_1 for a certain duration. More precisely, there exists a $r \in (0, \varepsilon)$ and a time $\tau > 0$ such that

$$\|\mathcal{R}_t(v, 0, 0) - u_1\|_{H^s} < \varepsilon \tag{3.7}$$

for all $t \in [0, \tau]$ and $v \in H^s(\mathbb{T})$ with $\|v - u_1\|_{H^s} \leq r$.

Next, applying the small-time approximate controllability result of Proposition 3.2, we find a $\delta \in (0, T)$ such that

$$\|\mathcal{R}_\delta(u_0, 0, f) - u_1\|_{H^s} < r$$

for some $\mathcal{H}_0$-valued control f. Combining this fact with (3.7), along with the stability property (2.5) and the flow property of Lemma 2.3, we deduce that

$$\|\mathcal{R}_{\delta+t}(u_0, 0, g) - u_1\|_{H^s} \lesssim \varepsilon$$

for all $t \in [0, \tau]$, where the control g is defined by f on the interval $(0, \delta)$ and zero else. If $\delta + \tau > T$, then the proof is complete. Otherwise, repeating this process, we find a $\delta' > 0$ and a $\mathcal{H}_0$-valued control f_1 such that

$$\|\mathcal{R}_{\delta+\tau+\delta'+t}(u_0, 0, f_1) - u_1\|_{H^s} \lesssim \varepsilon$$

for all $t \in [0, \tau]$. If $\delta + \tau + \delta' + \tau > T$, the proof is complete. If not, we iterate the process, and after a finite number of steps, we arrive near u_1 at time T using a $\mathcal{H}_0$-valued control. This completes the proof. □

4. Proof of the Propositions 2.1–2.4

In this section, we study the equation (2.2). More precisely, we prove Propositions 2.1–2.4. In order to do this, we collect some properties of the pseudo-differential operator $\Lambda^s = (I - \partial_{xx})^{\frac{s}{2}}$ and the H^s space, which will be used through out this section.

Lemma 4.1. *The space H^s and the operator Λ^s possess the following properties:*

(i) ***Boundedness of*** Λ^s*: The operator $\Lambda^s : H^r \to H^{r-s}$ is bounded, that is,*

$$\|\Lambda^s f\|_{H^{r-s}} \leq \|f\|_{H^r}, \quad \text{for all } f \in H^r. \tag{4.1}$$

(ii) ***Algebraic property of*** H^s*: Let* $s > \frac{1}{2}$*. The space* H^s *forms an algebra, that is,*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}, \quad \text{for all } f, g \in H^s. \tag{4.2}$$

Moreover,

$$\|fg\|_{H^{s-1}} \lesssim \|f\|_{H^s}\|g\|_{H^{s-1}}, \quad \text{for all } f \in H^s,\ g \in H^{s-1}. \tag{4.3}$$

(iii) ***Commutator estimates****: The commutator* $[\Lambda^s, f]$*, defined as*

$$[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s(g),$$

satisfies the following bound for $s > \frac{3}{2}$*:*

$$\|[\Lambda^s, f]g\|_{L^2} \lesssim \|\Lambda^s f\|_{L^2}\|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty}\|\Lambda^{s-1}g\|_{L^2}, \tag{4.4}$$

for all $f \in H^s$ *and* $g \in H^{s-1}$*.*

(iv) ***Calderón–Coifman–Meyer type commutator estimate****: For* $s > \frac{1}{2}$*, the following inequality holds:*

$$\|[\Lambda^s\partial_x, f]g\|_{L^2} \lesssim \|f\|_{H^{s+1}}\|g\|_{H^s}, \tag{4.5}$$

for all $f \in H^{s+1}$ *and* $g \in H^s$*.*

The proofs of (i) and the first part of (ii) are standard, while the second part of (ii) can be found in Lemma 4 of [44]. For (iii), refer to the appendix of Kato and Ponce [45], and for (iv), see Proposition 4.2 in [46].

Next, we outline some key properties of the Friedrichs mollifiers, which will be used to construct approximate solutions for (2.2).

Lemma 4.2. *Let* $\varepsilon \in (0, 1]$ *and* j *be a Schwartz function on* $\mathbb{R}$ *satisfying* $0 \le \widehat{j}(\xi) \le 1$ *for all* $\xi \in \mathbb{R}$ *and* $\widehat{j}(\xi) = 1$ *for* $|\xi| \le 1$*. We define the periodic function* j_ε *by*

$$j_\varepsilon(x) := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \widehat{j}(\varepsilon n)\mathrm{e}^{inx}. \tag{4.6}$$

For $f \in L^2(\mathbb{T})$*, the Friedrichs mollifier* J_ε *is defined by*

$$J_\varepsilon f := j_\varepsilon * f. \tag{4.7}$$

The operator J_ε *satisfies the following properties:*

- ***Smoothing Property:*** $J_\varepsilon f$ *is a smooth periodic function.*
- ***Commutativity:*** *For* $s > 0$*, the operator* J_ε *commutes with* Λ^s*:*

$$\Lambda^s J_\varepsilon = J_\varepsilon \Lambda^s. \tag{4.8}$$

- ***Self-Adjointness:*** *For all* $f, g \in L^2$*, we have*

$$\langle J_\varepsilon f, g\rangle_{L^2} = \langle f, J_\varepsilon g\rangle_{L^2}. \tag{4.9}$$

- ***Boundedness:*** *The operator J_ε is bounded, that is,*

$$\|J_\varepsilon u\|_{H^s} \leq \|u\|_{H^s} \quad \text{for all } u \in H^s. \tag{4.10}$$

- *If w satisfy $\|\partial_x w\|_{L^\infty} < \infty$. Then, there exists a constant $C > 0$ such that for any $f \in L^2$,*

$$\|[J_\varepsilon, w]\partial_x f\|_{L^2} \leq C\|\partial_x w\|_{L^\infty}\|f\|_{L^2}. \tag{4.11}$$

For detailed properties of mollifiers we refer to the Appendix of the book by J. Bedrossian and V. Vicol [47], while inequality (4.11) can be found in [10]. We now proceed to the proof of Proposition 2.1.

4.1. Proof of Proposition 2.1

Let $u_0 \in H^s$, $\varphi \in H^{s+1}(\mathbb{T})$ and $f \in L^\infty((0,\infty); H^{s-2}(\mathbb{T}))$ be given periodic functions. We rewrite (2.2) as

$$\begin{cases} u_t = -(u+\varphi)\partial_x(u+\varphi) - F(u), \\ u(0,x) = u_0(x), \end{cases} \tag{4.12}$$

where

$$F(u) = \Lambda^{-2}\Big[\partial_x(u+\varphi) + 2(u+\varphi)\partial_x(u+\varphi) + \partial_x(u+\varphi)\partial_{xx}(u+\varphi) - f\Big]. \tag{4.13}$$

To proceed, we first analyze the operator associated with F.

Lemma 4.3. *Let $s > 0$, $\psi \in H^s$ and $g \in H^{s-2}(\mathbb{T})$. Then the map $\mathbb{F} : H^s \to H^s$, defined by,*

$$\mathbb{F}(v) = \Lambda^{-2}\Big[\partial_x(v+\psi) + 2(v+\psi)\partial_x(v+\psi) + \partial_x(v+\psi)\partial_{xx}(v+\psi) - g\Big]$$

satisfies the following estimates:

(i) Well-defined : for all $v \in H^s$ and $s > \frac{3}{2}$,

$$\|\mathbb{F}(v)\|_{H^s} \lesssim \|(v+\psi)\|_{H^s} + \|(v+\psi)\|_{H^s}^2 + \|g\|_{H^{s-2}}. \tag{4.14}$$

(ii) Locally Lipschitz : for any $v_1, v_2 \in H^s$,

$$\|\mathbb{F}(v_1) - \mathbb{F}(v_2)\|_{H^s} \lesssim (1 + \|v_1\|_{H^s} + \|v_2\|_{H^s})\,\|v_1 - v_2\|_{H^s}, \quad \text{for } s > \frac{3}{2}, \tag{4.15}$$

and

$$\|\mathbb{F}(v_1) - \mathbb{F}(v_2)\|_{H^s} \lesssim (1 + \|v_1\|_{H^{s+1}} + \|v_2\|_{H^{s+1}})\,\|v_1 - v_2\|_{H^s}, \quad \text{for } s > \frac{1}{2}. \tag{4.16}$$

Proof. (i). Using the boundedness of

$$\Lambda^{-2}\partial_x : H^{s-1} \to H^s,$$

and the fact that H^{s-1} forms an algebra for $s > \frac{3}{2}$, we deduce that

$$\begin{aligned} \|\Lambda^{-2}\Big((v+\psi)(v+\psi)_x\Big)\|_{H^s} &= \frac{1}{2}\|\Lambda^{-2}\partial_x((v+\psi)^2)\|_{H^s} \\ &\lesssim \|(v+\psi)^2\|_{H^{s-1}} \lesssim \|(v+\psi)\|_{H^{s-1}}^2. \end{aligned}$$

Similarly, we have

$$\|\Lambda^{-2}\Big(\partial_x(v+\psi)\partial_{xx}(v+\psi)\Big)\|_{H^s} \lesssim \|(\partial_x(v+\psi))^2\|_{H^{s-1}} \lesssim \|(v+\psi)\|_{H^s}^2.$$

Combining the above estimates, we obtain the desired estimate (4.14).
(ii). First we estimate the difference of $\mathbb{F}(v_1)$ and $\mathbb{F}(v_2)$ as

$$\begin{aligned}\|\mathbb{F}(v_1)-\mathbb{F}(v_2)\|_{H^s} \lesssim \|\Lambda^{-2}\partial_x(v_1-v_2)\|_{H^s} &+ \|\Lambda^{-2}\partial_x[(v_1+\psi)^2-(v_2+\psi)^2]\|_{H^s}\\ &+ \|\Lambda^{-2}\partial_x[(\partial_x(v_1+\psi))^2-\partial_x(v_2+\psi))^2]\|_{H^s}.\end{aligned}$$

Further using the bound of $\Lambda^{-2}\partial_x$, we have

$$\begin{aligned}\|\mathbb{F}(v_1)-\mathbb{F}(v_2)\|_{H^s} &\lesssim \|v_1-v_2\|_{H^{s-1}} + \|(v_1-v_2)(v_1+v_2+2\psi)\|_{H^{s-1}}\\ &\quad + \|(\partial_x v_1-\partial_x v_2)(\partial_x v_1+\partial_x v_2+2\psi_x)\|_{H^{s-1}}\\ &\lesssim \|v_1-v_2\|_{H^s} + \|(v_1-v_2)(v_1+v_2+2\psi)\|_{H^s}\\ &\quad + \|(\partial_x v_1-\partial_x v_2)(\partial_x v_1+\partial_x v_2+2\psi_x)\|_{H^{s-1}}.\end{aligned}$$

To conclude the proof, we use the algebra of H^s in the second term and the inequality (4.3) with $s>1/2$ in the last term. □

For establishing the existence of a solution to equation (4.12), one natural approach is to treat it as an initial value problem for an ODE in H^s. This requires the right-hand side of (4.12) to lie in H^s. The Lemma 4.3 confirms that $F(u)\in H^s$ for any $u\in H^s$. However, the presence of the Burgers' nonlinearity uu_x poses a difficulty, as this term does not, in general, belong to H^s when $u\in H^s$. To address this, we mollify the Burgers' term and analyze the corresponding ordinary differential equation (ODE). This mollification technique is inspired by the work of Himonas and Holliman [10], which examines the Cauchy problem for a generalized Camassa–Holm equation. For an introduction to the mollification approach, readers may refer to Chapter 2 of the book by Bedrossian and Vicol [47].

The Mollified IVP: Let $\varepsilon\in(0,1]$. We consider the mollified version of the problem (4.12), given by:

$$\begin{cases} u_t = -J_\varepsilon[(J_\varepsilon u)(J_\varepsilon u_x)+u\varphi_x+\varphi u_x+\varphi\varphi_x]-F(u),\\ u(0,x)=u_0(x).\end{cases} \tag{4.17}$$

By utilizing the smoothing property of J_ε, we observe that the mapping

$$u\mapsto J_\varepsilon[(J_\varepsilon u)(J_\varepsilon u_x)]$$

is locally Lipschitz in H^s. Using the Picard theorem for initial value problems (IVPs), we conclude that for each $\varepsilon>0$, equation (4.17) has a unique solution $u_\varepsilon\in C([0,T_\varepsilon);H^s)$, where $T_\varepsilon>0$ is the maximal interval of existence.

To establish the existence of a solution to equation (4.12), we proceed with the following steps:

STEP 1: Prove the existence of a time $T_*>0$ such that u_ε exists on $[0,T_*]$ for all $\varepsilon>0$.
STEP 2: Establish uniform bounds on u_ε.
STEP 3: Show that u_ε converges to a solution u of (4.12).

With these steps outlined, we now proceed to their proofs.

STEP 1 : We begin by estimating the H^s energy of u_ε. To do so, we apply the operator Λ^s to both sides of (4.17), multiply the resulting equation by $\Lambda^s u_\varepsilon$, and integrate over $\mathbb{T}$. This yields the following identity:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon\|_{H^s}^2 &= -\langle \Lambda^s J_\epsilon[(J_\varepsilon u_\varepsilon)(\partial_x J_\varepsilon u_\varepsilon)], \Lambda^s u_\varepsilon\rangle_{L^2} - \langle \Lambda^s J_\epsilon[u_\varepsilon\varphi_x + \varphi\partial_x u_\varepsilon + \varphi\varphi_x], \Lambda^s u_\varepsilon\rangle_{L^2} \\ &\qquad - \langle \Lambda^s F(u_\varepsilon), \Lambda^s u_\varepsilon\rangle_{L^2}, \\ &=: -I_1 - I_2 - I_3. \end{aligned} \tag{4.18}$$

First, we rewrite I_1 by commuting the exterior operator J_ε with Λ^s. Utilize the self-adjoint property of J_ε we deduce

$$\begin{aligned} I_1 &= \int_{\mathbb{T}} \Lambda^s[(J_\varepsilon u_\varepsilon)(\partial_x J_\varepsilon u_\varepsilon)] \cdot \Lambda^s J_\epsilon u_\varepsilon \,\mathrm{d}x \\ &= \int_{\mathbb{T}} [\Lambda^s, J_\varepsilon u_\varepsilon]\partial_x(J_\varepsilon u_\varepsilon) \cdot \Lambda^s J_\varepsilon u_\varepsilon \,\mathrm{d}x + \int_{\mathbb{T}} J_\varepsilon u_\varepsilon \partial_x \Lambda^s(J_\varepsilon u_\varepsilon) \cdot \Lambda^s J_\varepsilon u_\varepsilon \,\mathrm{d}x. \end{aligned} \tag{4.19}$$

Setting $v = J_\varepsilon u_\varepsilon$ and using (4.4) from Lemma 4.1 we can bound the first term of (4.19)

$$\begin{aligned}\left|\int_{\mathbb{T}} [\Lambda^s, v]\partial_x v \cdot \Lambda^s v \,\mathrm{d}x\right| &\lesssim \Big(\|v\|_{H^s}\|\partial_x v\|_{L^\infty} + \|\partial_x v\|_{L^\infty}\|v\|_{H^s}\Big)\|v\|_{H^s} \\ &\lesssim \|u_\varepsilon\|_{H^s}^2 \|\partial_x u_\varepsilon\|_{L^\infty}. \end{aligned}$$

The second term of (4.19) is estimated as

$$\begin{aligned}\left|\int_{\mathbb{T}} v\,\partial_x(\Lambda^s v)\,\Lambda^s v \,\mathrm{d}x\right| = \left|-\frac{1}{2}\int_{\mathbb{T}} \partial_x v\,(\Lambda^s v)^2 \,\mathrm{d}x\right| &\lesssim \|\partial_x v\|_{L^\infty}\left|\int_{\mathbb{T}} (\Lambda^s v)^2 \,\mathrm{d}x\right| \\ &\lesssim \|\partial_x u_\varepsilon\|_{L^\infty}\|u_\varepsilon\|_{H^s}^2. \end{aligned}$$

Combining these estimates, we obtain

$$|I_1| \lesssim \|u_\varepsilon\|_{H^s}^3. \tag{4.20}$$

Next, we estimate I_2. Recall that

$$I_2 = \int_{\mathbb{T}} \Lambda^s\ J_\epsilon(u_\varepsilon\varphi_x) \cdot \Lambda^s u_\varepsilon \,\mathrm{d}x + \int_{\mathbb{T}} \Lambda^s\ J_\epsilon(\varphi\partial_x u_\varepsilon) \cdot \Lambda^s u_\varepsilon \,\mathrm{d}x + \int_{\mathbb{T}} \Lambda^s\ J_\epsilon(\varphi\varphi_x) \cdot \Lambda^s u_\varepsilon \,\mathrm{d}x. \tag{4.21}$$

Using the properties of Λ^s and J_ε, along with the Cauchy–Schwarz inequality, the first term of I_2 is estimated as

$$\left|\int_{\mathbb{T}} \Lambda^s\ J_\epsilon(u_\varepsilon\varphi_x) \cdot \Lambda^s u_\varepsilon \,\mathrm{d}x\right| \le \|u_\varepsilon\varphi_x\|_{H^s}\|u_\varepsilon\|_{H^s} \le \|\varphi_x\|_{H^s}\|u_\varepsilon\|_{H^s}^2.$$

The second term of I_2 can be rewritten as

$$\begin{aligned}&\int_{\mathbb{T}} \Lambda^s(\varphi\partial_x u_\varepsilon) \cdot \Lambda^s J_\epsilon u_\varepsilon \,\mathrm{d}x \\ &= \int_{\mathbb{T}} [\Lambda^s, \varphi]\partial_x u_\varepsilon \cdot \Lambda^s J_\varepsilon u_\varepsilon \,\mathrm{d}x + \int_{\mathbb{T}} \Big([J_\varepsilon, \varphi]\Lambda^s\partial_x u_\varepsilon\Big) \cdot \Lambda^s u_\varepsilon \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}} \partial_x\varphi\Big(J_\varepsilon\Lambda^s u_\varepsilon\Big)\Lambda^s u_\varepsilon \,\mathrm{d}x. \end{aligned}$$

Applying (4.4) and (4.11), we find

$$\left|\int_{\mathbb{T}} \Lambda^s(\partial_x u_\varepsilon \varphi)\cdot \Lambda^s J_\epsilon u_\varepsilon \,\mathrm{d}x\right| \lesssim \|\varphi\|_{H^s}\|u_\varepsilon\|_{H^s}^2.$$

The final term in (4.21) is bounded by

$$\left|\int_{\mathbb{T}} \Lambda^s J_\epsilon(\varphi\varphi_x)\cdot \Lambda^s u_\varepsilon \,\mathrm{d}x\right| \lesssim \|\varphi\|_{H^s}\|\varphi_x\|_{H^s}\|u_\varepsilon\|_{H^s}.$$

Combining these estimates yields

$$|I_2| \lesssim \Big(\|\varphi\|_{H^{s+1}}\|u_\varepsilon\|_{H^s}^2 + \|\varphi\|_{H^{s+1}}^2\|u_\varepsilon\|_{H^s}\Big). \tag{4.22}$$

For the third term of (4.18), using (4.14), we have

$$\begin{aligned} |I_3| &= \left|\int_{\mathbb{T}} \Lambda^s F(u_\varepsilon)\ .\ \Lambda^s u_\varepsilon \mathrm{d}x\right| \le \|F(u_\varepsilon)\|_{H^s}\|u_\varepsilon\|_{H^s} \\ &\lesssim \Big(\|(u_\varepsilon+\varphi)\|_{H^s} + \|(u_\varepsilon+\varphi)\|_{H^s}^2 + \|f\|_{L^\infty((0,\infty);H^{s-2})}\Big)\|u_\varepsilon\|_{H^s}. \end{aligned} \tag{4.23}$$

Using the estimates (4.20), (4.22) and (4.23) in (4.18), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon\|_{H^s}^2 \le C(1+\|u_\varepsilon\|_{H^s}^2 + \|u_\varepsilon\|_{H^s}^3)$$

where C is a positive constant, depends on $\varphi,\ f$ and s but independent of ε. Further, applying the inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

for non-negative a, b with $\frac{1}{p}+\frac{1}{q}=1$, we derive the following differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon\|_{H^s}^2 \le C(1+\|u_\varepsilon\|_{H^s}^4). \tag{4.24}$$

Solving this, we obtain

$$\|u_\varepsilon(t)\|_{H^s}^2 \le \tan\left(3Ct + \tan^{-1}\left(\|u_0\|_{H^s}^2\right)\right), \quad \text{for all} \quad t\in[0,T_\varepsilon). \tag{4.25}$$

Choose, $T_*(u_0,f,\varphi)>0$, such that

$$0 \le 3CT_* + \tan^{-1}\left(\|u_0\|_{H^s}^2\right) =: C_{s,u_0} < \frac{\pi}{2},$$

which is possible as

$$0 \le \tan^{-1}\left(\|u_0\|_{H^s}^2\right) < \frac{\pi}{2}.$$

This implies that

$$u_\varepsilon \in C([0,T_*];H^s)$$

is a solution of (4.17).

STEP 2: The above analysis establishes that u_ε satisfies the uniform bound

$$\sup_{t\in[0,T_*]} \|u_\varepsilon(t)\|_{H^s} \leq \sqrt{\tan(C_{s,u_0})}, \quad \text{for all } \varepsilon \in (0,1]. \tag{4.26}$$

Using (4.2) and (4.10), we have the following estimates

$$\|J_\varepsilon((J_\varepsilon u_\varepsilon)(\partial_x J_\varepsilon u_\varepsilon))\|_{H^{s-1}} \lesssim \|u_\varepsilon\|^2_{H^s},$$

and

$$\|J_\varepsilon(\varphi \partial_x u_\varepsilon)\|_{H^{s-1}} \lesssim \|\varphi\|_{H^s}\|u_\varepsilon\|_{H^s}.$$

Applying these inequalities and (4.26), we derive the uniform estimate for the time derivative

$$\sup_{t\in[0,T_*]} \|\partial_t u_\varepsilon(t)\|_{H^{s-1}} < M, \quad \text{for all } \varepsilon \in (0,1], \tag{4.27}$$

for some positive constant M.
Using (4.26) and (4.27), we further deduce

$$\|u_\varepsilon\|_{C^{0,\sigma}([0,T_*];H^{s-\sigma})} \leq M, \quad \text{for all } \varepsilon \in (0,1], \tag{4.28}$$

for some positive constant M, where $0<\sigma<1$. Here, we have used the inequality

$$\|w\|_{C^{0,\sigma}([0,T_*];H^{s-\sigma})} \leq \|w\|_{C^0([0,T_*];H^s)} + \|w_t\|_{C^0([0,T_*];H^{s-1})}.$$

Finally, we use (4.26), (4.27), and (4.28), to conclude that the limit of u_ε as $\varepsilon \to 0$ is a solution of (4.12).

STEP 3: We begin by defining the interval $I=[0,T_*]$, where T_* is the common lifespan of all u_ε. Using the bounds established in (4.26), (4.27), and (4.28), we conclude that there exists a subsequence of u_ε, still denoted by u_ε, such that

(i) $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $L^\infty(I;H^s)$,

(ii) $\partial_t u_\varepsilon \overset{*}{\rightharpoonup} \partial_t u$ in $L^\infty(I;H^{s-1})$,

(iii) $u_\varepsilon \to u$ in $C(I;H^{s-1})$, and

(iv) $u_\varepsilon \to u$ in $C(I;H^{s-\sigma})$ for $\sigma \in (0,1)$,

where $u \in L^\infty(I;H^s) \cap \mathrm{Lip}(I;H^{s-1}) \cap C(I;H^{s-1}) \cap C(I;H^{s-\sigma})$. The first two convergence results follow from the Banach–Alaoglu theorem, while the last two are derived using Ascoli's theorem. Next, we show that u is a solution of (4.12). By choosing σ sufficiently small such that

$$H^{s-\sigma}(\mathbb{T}) \hookrightarrow C^1(\mathbb{T}),$$

we deduce that

$$u_\varepsilon \to u \text{ in } C(I;C^1(\mathbb{T})).$$

This implies the following

$$F(u_\varepsilon) \to F(u) \text{ in } C(I;C(\mathbb{T})),$$

and

$$(J_\varepsilon u_\varepsilon)(J_\varepsilon \partial_x u_\varepsilon) \to u u_x \text{ in } C(I;C(\mathbb{T})).$$

Since u_ε satisfies (4.12), we have

$$\partial_t u_\varepsilon \to -(u+\varphi)\partial_x(u+\varphi) - F(u) \text{ in } C(I;C(\mathbb{T})).$$

Using (iii) and the fact that $u \in L^\infty(I;H^s) \cap \mathrm{Lip}(I;H^{s-1})$, we can further conclude that

$$u \in C(I;H^s).$$

(See (2.28), page 44, in [47].) Hence $u \in C(I;H^s(\mathbb{T})) \cap C^1(I;H^{s-1}(\mathbb{T}))$ is a solution of the equation

$$\begin{cases} u_t = -(u+\varphi)\partial_x(u+\varphi) - F(u), \\ u(0,x) = u_0(x). \end{cases}$$

This completes the proof of existence.

Our next objective is to prove Proposition 2.2.

4.2. Proof of Proposition 2.2

Let $s > \frac{3}{2}$, $R > 0$, $\varphi \in H^{s+1}(\mathbb{T})$ and $f \in L^\infty((0,\infty);H^{s-2}(\mathbb{T}))$. Let $u_0, v_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \le R$ and $\|v_0\|_{H^s} \le R$. Then, by the previous proposition, there exists a time

$$T^* := T^*(R,\varphi,f) > 0$$

such that the corresponding solutions

$$u, v \in C([0,T^*];H^s(\mathbb{T}))$$

with initial data $u(0) = u_0$, $v(0) = v_0$ exist. Moreover, the estimate (4.25) ensures that the existence time T^* depends only on R, φ, and f. Thus T^* is uniform for all initial data in the ball of radius R in $H^s(\mathbb{T})$. We define

$$\begin{cases} w(t) := u(t) - v(t), \\ \vartheta(t) := u(t) + v(t), \end{cases}$$

for $t \in [0, T^*]$. Then w is a solution of the equation

$$\begin{cases} w_t = -\frac{1}{2}\partial_x(\vartheta w) - \Big(F(u) - F(v)\Big), \\ w(0,x) = w_0(x) := u_0(x) - v_0(x). \end{cases} \tag{4.29}$$

Consequently, we see that w satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{H^{s-1}} &= -\langle \Lambda^{s-1}\partial_x(\vartheta w), \Lambda^{s-1}w\rangle_{L^2} - 2\langle \Lambda^{s-1}(F(u) - F(v)), \Lambda^{s-1}w\rangle_{L^2} \\ &=: -\mathcal{I}_1 - \mathcal{I}_2. \end{aligned} \tag{4.30}$$

Using the definition of commutator, $\mathcal{I}_1$ can be expressed as

$$\mathcal{I}_1 = \int_{\mathbb{T}} [\Lambda^{s-1}\partial_x, \vartheta]w \ \Lambda^{s-1}w \mathrm{d}x + \int_{\mathbb{T}} \vartheta\partial_x(\Lambda^{s-1}w) \ \Lambda^{s-1}w \mathrm{d}x. \tag{4.31}$$

By (4.5), we get following estimate of the first term in (4.31)

$$\left| \int_{\mathbb{T}} [\Lambda^{s-1}\partial_x, \vartheta]w \ \Lambda^{s-1}w \mathrm{d}x \right| \lesssim \|\vartheta\|_{H^s}\|w\|^2_{H^{s-1}}.$$

Applying integration by parts and Sobolev inequality on the second term of (4.31), we have

$$\left| \int_{\mathbb{T}} \vartheta\partial_x(\Lambda^{s-1}w) \ \Lambda^{s-1}w \mathrm{d}x \right| \lesssim \|\partial_x\vartheta\|_{L^\infty}\|w\|^2_{H^{s-1}} \lesssim \|\vartheta\|_{H^s}\|w\|^2_{H^{s-1}}.$$

Combining the above estimates, we obtain

$$|\mathcal{I}_1| \lesssim \|\vartheta\|_{H^s}\|w\|^2_{H^{s-1}}. \tag{4.32}$$

Considering the second term of (4.30) and using the property (4.16) we have

$$|\mathcal{I}_2| \lesssim (1 + \|\vartheta\|_{H^s})\|w\|^2_{H^{s-1}}. \tag{4.33}$$

Putting the bounds of $\mathcal{I}_1$ and $\mathcal{I}_2$ in (4.30) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^{s-1}} \lesssim \Big(1 + \|\vartheta\|_{H^s}\Big)\|w\|_{H^{s-1}}.$$

Using the bound (4.26), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^{s-1}} \leq K_R\|w\|_{H^{s-1}},$$

for some $K_R > 0$, Applying the Gronwall's inequality, we find

$$\|w(t)\|_{H^{s-1}} \leq C_R \ \|w(0)\|_{H^{s-1}}, \quad \text{for all } t \in [0, T^*], \tag{4.34}$$

where $C_R := \mathrm{e}^{K_R T^*} > 0$. This completes the proof.

4.3. Proof of Proposition 2.4

Let $\delta \in (0,1)$. Since $u_0, \varphi \in H^{s+1}$ and $\eta_0 \in H^{s-2}$, there exists a $T_\delta > 0$ such that $u \in C([0,T_\delta]; H^{s+1})$ is a solution of

$$\begin{cases} u_t = -(1-\partial_{xx})^{-1}\Big[(u+\delta^{-\frac{1}{2}}\varphi)_x + 2(u+\delta^{-\frac{1}{2}}\varphi)(u+\delta^{-\frac{1}{2}}\varphi)_x + (u+\delta^{-\frac{1}{2}}\varphi)_x(u+\delta^{-\frac{1}{2}}\varphi)_{xx} \\ \qquad\qquad -\delta^{-1}\eta_0\Big] - (u+\delta^{-\frac{1}{2}}\varphi)(u+\delta^{-\frac{1}{2}}\varphi)_x, \\ u(0,x) = u_0, \end{cases} \tag{4.35}$$

and we denote this solution as

$$u(t) = \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta_0), \quad \text{for all } t \in [0,T_\delta].$$

We now show that there exists a $\delta_0 > 0$ such that $T_\delta > T$ and $u \in C([0,T]; H^s)$ for all $\delta \in (0,\delta_0)$. To establish (1) of Proposition 2.4, we estimate the H^s-norm of $u(t)$. Following the analysis of STEP 1 in the proof of Proposition 2.1, with $\delta^{-\frac{1}{2}}\varphi$ and $\delta^{-1}\eta_0$ in the place of φ and f, respectively, and $J_\varepsilon = \mathrm{Id}$, we find that $\|u(t)\|_{H^s}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 \lesssim \frac{1}{\delta^2} + \|u\|_{H^s}^4.$$

Applying Gronwall's inequality, we conclude that there exists a $\delta_0 > 0$ such that $0 < \delta_0 < T$ and

$$(\delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta_0) \in \Theta(u_0, T),$$

for all $\delta \in (0,\delta_0)$.

Next, we proceed to prove part (2) of Proposition 2.4. For that we define the functions

$$v(t) := u(\delta t), \quad w(t) := u_0 + t\Big[-\varphi\varphi_x + (1-\partial_{xx})^{-1}(\eta_0 - 2\varphi\varphi_x - \varphi_x\varphi_{xx})\Big],$$

and

$$z(t) := v(t) - w(t), \quad \text{for } t \in [0,1].$$

Then z satisfies the following system

$$\begin{cases} z_t = -\delta\Big\{(1-\partial_{xx})^{-1}\Big[(z+w+\delta^{-\frac{1}{2}}\varphi)_x + 2(z+w)(z+w+\delta^{-\frac{1}{2}}\varphi)_x + 2\delta^{-\frac{1}{2}}\varphi(z+w)_x \\ \qquad +(z+w)_x(z+w+\delta^{-\frac{1}{2}}\varphi)_{xx} + \delta^{-\frac{1}{2}}\varphi_x(z+w)_{xx}\Big] \\ \qquad +(z+w)(z+w+\delta^{-\frac{1}{2}}\varphi)_x + \delta^{-\frac{1}{2}}\varphi(z+w)_x\Big\}, \\ z(0,x) = 0. \end{cases} \tag{4.36}$$

The H^s-norm of $z(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_{H^s(\mathbb{T})}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s z\|_{L^2(\mathbb{T})}^2 = -2\delta(\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4), \tag{4.37}$$

where

$$\mathcal{J}_1 = \Big\langle \Lambda^s(zz_x),\ \Lambda^s z \Big\rangle_{L^2(\mathbb{T})}, \tag{4.38}$$

$$\mathcal{J}_2 = \Big\langle \Lambda^s\partial_x(zw + \delta^{-\frac{1}{2}}\varphi z),\ \Lambda^s z \Big\rangle_{L^2(\mathbb{T})}, \tag{4.39}$$

$$\mathcal{J}_3 = \Big\langle \Lambda^s(ww_x) + \delta^{-\frac{1}{2}}\Lambda^s\partial_x(\varphi w),\ \Lambda^s z \Big\rangle_{L^2(\mathbb{T})}, \tag{4.40}$$

$$\mathcal{J}_4 = \Big\langle \Lambda^s\left(F_1(z) + G_1(z)\right),\ \Lambda^s z \Big\rangle_{L^2(\mathbb{T})}, \tag{4.41}$$

with

$$F_1(z) = \Lambda^{-2}\Big[\partial_x(z+w) + 2(z+w)\partial_x(z+w) + \partial_x(z+w)\partial_{xx}(z+w)\Big],$$

and

$$G_1(z) = \delta^{-\frac{1}{2}}\Lambda^{-2}\partial_x\Big[\varphi + 2(z+w)\varphi + \varphi_x(z+w)_x\Big].$$

Using the commutator property, we estimate $\mathcal{J}_1$ and $\mathcal{J}_2$ as in I_1 and I_2 of the proof of Proposition 2.1, and obtain the following bounds

$$|\mathcal{J}_1| \lesssim \|z\|_{H^s}^3, \tag{4.42}$$

$$|\mathcal{J}_2| \lesssim \|w\|_{H^{s+1}}\|z\|_{H^s}^2 + \delta^{-\frac{1}{2}}\|\varphi\|_{H^{s+1}}\|z\|_{H^s}^2. \tag{4.43}$$

For $\mathcal{J}_3$, we find

$$|\mathcal{J}_3| \lesssim \|w\|_{H^{s+1}}^2\|z\|_{H^s} + \delta^{-\frac{1}{2}}\|\varphi\|_{H^{s+1}}\|w\|_{H^{s+1}}\|z\|_{H^s}. \tag{4.44}$$

To estimate $\mathcal{J}_4$, we first note that

$$\|F_1(z)\|_{H^s} \lesssim \|z+w\|_{H^s} + \|z+w\|_{H^s}^2, \tag{4.45}$$

and

$$\|G_1(z)\|_{H^s} \lesssim \delta^{-\frac{1}{2}}\Big(\|\varphi\|_{H^s} + \|z\|_{H^s}(\|\varphi\|_{H^s} + \|w\|_{H^s}) + \|w\|_{H^s}\|\varphi\|_{H^s}\Big). \tag{4.46}$$

Thus,

$$\begin{aligned}
|\mathcal{J}_4| &\lesssim \|F_1(z)\|_{H^s}\|z\|_{H^s} + \|G_1(z)\|_{H^s}\|z\|_{H^s} \\
&\lesssim \Big(\|z+w\|_{H^s} + \|z+w\|_{H^s}^2\Big)\|z\|_{H^s} \\
&\quad + \delta^{-\frac{1}{2}}\Big(\|\varphi\|_{H^s} + \|z\|_{H^s}(\|\varphi\|_{H^s} + \|w\|_{H^s}) + \|w\|_{H^s}\|\varphi\|_{H^s}\Big)\|z\|_{H^s}.
\end{aligned} \tag{4.47}$$

Combining (4.37), (4.42), (4.43), (4.44), and (4.47), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_{H^s}^2 &\lesssim \delta^{\frac{1}{2}}\Big(\|z\|_{H^s}^3 + \|z\|_{H^s}^2 + \|z\|_{H^s}\Big) \\
&\lesssim \delta^{\frac{1}{2}}\Big(1 + \|z\|_{H^s}^4\Big).
\end{aligned} \tag{4.48}$$

Using Gronwall's inequality, we conclude that

$$\|z(t)\|_{H^s}^2 \lesssim \tan(\sqrt{\delta}t). \tag{4.49}$$

Thus,

$$\|z(1)\|_{H^s}^2 \to 0 \quad \text{as } \delta \to 0^+.$$

This completes the proof.

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Data availability statement

The research data associated with this article are included in the article.

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