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APPROXIMATE CONTROLLABILITY AND IRREDUCIBILITY OF THE 2D ZAKHAROV–KUZNETSOV–BURGERS EQUATION

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Abstract. The present paper is a continuation of [Chen and Rosier, Discrete Continuous Dyn. Syst.-B **25** (2020) 3889–3916] and [Chen, Appl. Math. Optim. **87** (2023) 12]. In this paper, we consider the 2D Zakharov–Kuznetsov–Burgers equation posed on the periodic domain $\mathbb{T}^2$. We show that the 2D Zakharov–Kuznetsov–Burgers equation is globally approximately controllable by a finite dimensional external force, the proof is based on the Agrachev–Sarychev approach in geometric control theory. As an application of the approximate controllability result, we establish the irreducibility of the transition semigroup associated with stochastic Zakharov–Kuznetsov–Burgers equation driven by a degenerate noise.

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1. Introduction

The 2D Zakharov–Kuznetsov–Burgers (ZKB) equation

$$u_t + u_{xxx} + u_{xyy} + uu_x - u_{xx} - u_{yy} = 0$$

appears in the description of ion-acoustic waves in dusty-plasmas (see [1, 2]). Local and global well-posedness for the 2D ZKB equation are considered in [3–8] study the traveling wave solutions of ZKB equation. Symmetry analysis of a generalized 2D ZKB equation with a variable dissipation is considered in [9]. [10] obtains the optimal decay estimate and asymptotic profile for solutions to the generalized ZKB equation. There is few work on the approximate controllability and irreducibility of ZKB equation.

1.1. Approximately controllability

In this paper, we are interested in the approximately controllability of ZKB equation on $\mathbb{T}^2$, namely, let us consider the following deterministic system

$$\begin{cases} u_t + u_{xxx} + u_{xyy} + uu_x - u_{xx} - u_{yy} = h + \eta & \text{in } \mathbb{T}^2 \times (0,T), \\ u(0) = u_0 & \text{in } \mathbb{T}^2, \end{cases} \tag{1.1}$$

where $T > 0$, u_0 is the initial value, $h = h(x,y,t)$ is a given function and $\eta = \eta(x,y,t)$ is a control.

Keywords and phrases: Zakharov–Kuznetsov–Burgers equation, Agrachev–Sarychev approach, approximate controllability, irreducibility, degenerate noise, geometric control theory.

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Let $\mathcal{H}$ be a finite-dimensional space defined by

$$\mathcal{H} := \text{span}\{1, \cos x, \sin x, \cos y, \sin y\}.$$

For any $k \in \mathbb{N}$, $H^k(\mathbb{T}^2)$ denotes the Sobolev space of order k endowed with the usual norm $\|\cdot\|_k$. For simplicity, the norm on $H^0(\mathbb{T}^2) = L^2(\mathbb{T}^2)$ is denoted by $\|\cdot\|$. Now, we are in a position to present the first main result in this paper.

Theorem 1.1. *Let $T > 0, k \in \mathbb{N}$, $k \geq 2$, $u_0 \in H^k(\mathbb{T}^2)$ and $h \in L^2(0,T; H^{k-1}(\mathbb{T}^2))$, for any $\varepsilon > 0$ and any $u_1 \in H^k(\mathbb{T}^2)$, there exists a control $\eta \in L^2(0,T;\mathcal{H})$ such that the solution u of (1.1) satisfies*

$$\|u(T) - u_1\|_k < \varepsilon.$$

We plan to use the Agrachev–Sarychev approach in geometric control theory to prove Theorem 1.1. Approximate controllability of PDEs by additive finite-dimensional forces has been studied by many authors in the recent years. The first results are obtained in [11, 12], they consider the Navier-Stokes and Euler systems on the two dimensional tours. Their approach has been generalized by [13, 14] to the cases of 2D and 3D Navier-Stokes system and by [15] to the Burgers equation.

The proof of Theorem 1.1 is based on a technique of applying large controls on short time intervals, such ideas have been used to study finite-dimensional control system. Then [16] applies this idea to parabolic equation with the nonlinearity which is a smooth function that grows polynomially without any restriction on the degree and on the space dimension. Then the technique is further developed in [17–20], and it is applied to prove approximate controllability for the Kuramoto-Sivashinsky equation, Korteweg-de Vries equation, Benjamin-Bona-Mahony equation and Kadomtsev-Petviashvili I equation, respectively. In this paper, we overcome the difficulties that come from the structure of the ZKB equation and apply this idea to establish its approximate controllability.

1.2. Irreducibility

The irreducibility is a fundamental concept in stochastic dynamic system, and plays a crucial role in the research of ergodic theory, moderate deviation principle. An important reason for the study of the irreducibility property is its relevance in the analysis of the uniqueness and ergodicity of invariant measures. It is well known that the strong Feller property together with irreducibility imply the uniqueness of invariant measures for the transition semigroup.

In rest of this paper, we consider the irreducibility of the stochastic ZKB equation driven by a degenerate noise, namely, let us consider the following stochastic system

$$\begin{cases} u_t + u_{xxx} + u_{xyy} + uu_x - u_{xx} - u_{yy} = \sum_{i=1}^{5} b_i \dot{\beta}_i(t) e_i & \text{in } \mathbb{T}^2 \times (0,T), \\ u(0) = u_0 & \text{in } \mathbb{T}^2, \end{cases} \tag{1.2}$$

where $e_1(x,y) = 1, e_2(x,y) = \cos x, e_3(x,y) = \sin x, e_4(x,y) = \cos y, e_5(x,y) = \sin y$, $\{\beta_i\}_{i=1}^5$ are real valued mutually independent Wiener processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{b_i\}_{i=1}^5$ are real valued numbers. According to the similar theory as in [21] and [22], Theorem 3.4, we can prove that

Let $k \in \mathbb{N}$, $k \geq 2$. If $u_0 \in H^k(\mathbb{T}^2)$, (1.2) admits a unique weak solution $u \in C([0,T]; H^k(\mathbb{T}^2))$ $\mathbb{P}-a.s.$ Moreover, u is a Markov process in $H^k(\mathbb{T}^2)$, which is Feller in $H^k(\mathbb{T}^2)$.

Let $u(t, u_0)$ denote the unique solution to (1.2) for the initial value $u_0 \in H^k(\mathbb{T}^2)$, and we define the *transition probabilities* $\{P(t, u_0, \cdot) : t \in [0,T], u_0 \in H^k(\mathbb{T}^2)\}$ by

$$P(t, u_0, \Gamma) := \mathbb{P}\{u(t, u_0) \in \Gamma\}$$

for all Borel sets Γ of $H^k(\mathbb{T}^2)$ and all $t \in [0,T]$.

Definition 1.2. We say that a family of transition probabilities $\{P(t,u_0,\cdot) : t \in [0,T], u_0 \in H^k(\mathbb{T}^2))\}$ is irreducible in $H^k(\mathbb{T}^2)$ if for every $u_0 \in H^k(\mathbb{T}^2), t \in [0,T], u_1 \in H^k(\mathbb{T}^2)$ and $\varepsilon > 0$, we have $P(t,u_0,B_{H^k(\mathbb{T}^2)}(u_1,\varepsilon)) > 0$.

Now, we are in a position to present the second main result in this paper.

Theorem 1.3. *Let $T > 0, k \in \mathbb{N}$, $k \geq 2$. If $b_i \neq 0$ for $i = 1, 2, \cdots, 5$ then the family $\{P(t,u_0,\cdot) : t \in [0,T], u_0 \in H^k(\mathbb{T}^2)\}$ is irreducible in $H^k(\mathbb{T}^2)$.*

The proof of Theorem 1.3 relies on the approximate controllability of ZKB equation and the continuity of the mapping *noise* $\mapsto$ *solution* along the controllers. It is well known that one usually solves a control problem to prove the irreducibility for stochastic partial differential equations. Control theory is a useful tool to study irreducibility of stochastic dynamical systems. The irreducibility of SPDEs has attracted many authors' attention, (see *e.g.* [23–28] and references within), in all these papers, the noises are nondegenerate. In [18], the author establishs irreducibility for Kuramoto-Sivashinsky equation driven by a degenerate noise. In this paper, we further develop the method in [18] to (1.2) and establish its irreducibility.

The rest of this paper is organized as follows: In Section 2, we give some preliminary results which will be used in this paper. Section 3 is devoted to the saturating property. The asymptotic property is given in Section 4. The proofs of Theorem 1.1 and Theorem 1.3 can be found in Section 5 and Section 6.

2. Preliminaries

We consider the following generalisation of system (1.1)

$$\begin{cases} u_t - A(u+\zeta) + (u+\zeta)(u+\zeta)_x = \varphi & \text{in } \mathbb{T}^2 \times (0,T), \\ u(0) = u_0 & \text{in } \mathbb{T}^2, \end{cases} \tag{2.1}$$

where A is an operator defined as follows:

$$Au = -\triangle u_x + \triangle u, \ \forall u \in \mathcal{D}(A),$$

where $\triangle = \partial_x^2 + \partial_y^2$ and $\mathcal{D}(A) = \{u \in L^2(\mathbb{T}^2) \mid Au \in L^2(\mathbb{T}^2)\}$. It is easy to see that for any $k \in \mathbb{N}$, A generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on the space $H^k(\mathbb{T}^2)$.

For any $k \in \mathbb{N}$ and $T > 0$, we define the space

$$X_k^T = C([0,T]; H^k(\mathbb{T}^2)) \cap L^2(0,T; H^{k+1}(\mathbb{T}^2))$$

endowed with its natural Hilbert norm

$$\|u\|_{X_k^T} = \|u\|_{C([0,T];H^k(\mathbb{T}^2))} + \|u\|_{L^2(0,T;H^{k+1}(\mathbb{T}^2))}.$$

We shall need the following lemmas.

Lemma 2.1. *For any $k \in \mathbb{N}$, $T > 0$, $u_0 \in H^k(\mathbb{T}^2)$ and $f \in L^2(0,T;H^{k-1}(\mathbb{T}^2))$, there exists a constant $C = C(k,T) > 0$ such that*

$$\|S(\cdot)u_0\|_{X_k^T} \leq C\|u_0\|_k,$$
$$\|\int_0^{\cdot} S(\cdot - s)f(s)ds\|_{X_k^T} \leq C\|f\|_{L^2(0,T;H^{k-1}(\mathbb{T}^2))}.$$

Proof. It is easy to see that $S(\cdot)u_0$ is the unique solution of the following system

$$\begin{cases} u_t - Au = 0 & \text{in } \mathbb{T}^2 \times (0,T), \\ u(0) = u_0 & \text{in } \mathbb{T}^2. \end{cases} \tag{2.2}$$

Multiplying equation (2.2) by $u + (-\triangle)^k u$ and integrating on $\mathbb{T}^2$, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2 + \|(-\triangle)^{\frac{k}{2}}u\|^2\right) + \|(-\triangle)^{\frac{k+1}{2}}u\|^2 \le 0.$$

It follows from Gronwall inequality that

$$\|u\|_{X_k^T} \le C\|u_0\|_k.$$

For the second estimate, we can see that $\int_0^{\cdot} S(\cdot - s)f(s)ds$ is the unique solution of

$$\begin{cases} u_t - Au = f & \text{in } \mathbb{T}^2 \times (0,T), \\ u(0) = 0 & \text{in } \mathbb{T}^2. \end{cases} \tag{2.3}$$

Multiplying equation (2.3) by $u + (-\triangle)^k u$ and integrating on $\mathbb{T}^2$, for any $\varepsilon > 0$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u\|^2 + \|(-\triangle)^{\frac{k}{2}}u\|^2\right) + \|(-\triangle)^{\frac{1}{2}}u\|^2 + \|(-\triangle)^{\frac{k+1}{2}}u\|^2 \\ \le &\varepsilon\left(\|u\|_1^2 + \|(-\triangle)^{\frac{k+1}{2}}u\|^2\right) + C(\varepsilon)\left(\|f\|_{-1}^2 + \|(-\triangle)^{\frac{k-1}{2}}f\|^2\right) \\ \le &\varepsilon\left(\|(-\triangle)^{\frac{1}{2}}u\|^2 + \|(-\triangle)^{\frac{k+1}{2}}u\|^2\right) + C(\varepsilon)\left(\|u\|^2 + \|f\|_{k-1}^2\right). \end{aligned}$$

For sufficiently small ε, we arrive to

$$\frac{d}{dt}\left(\|u\|^2 + \|(-\triangle)^{\frac{k}{2}}u\|^2\right) + \|(-\triangle)^{\frac{k+1}{2}}u\|^2 \le C\left(\|u\|^2 + \|f\|_{k-1}^2\right).$$

Applying Gronwall inequality, we obtain that

$$\|u\|_{X_k^T} \le C\|f\|_{L^2(0,T;H^{k-1}(\mathbb{T}^2))}.$$

This ends the proof of Lemma 2.1. □

Lemma 2.2. *[19] Let $k \in \mathbb{N}$, $k \ge 2$. For any $u, v \in H^k(\mathbb{T}^2)$, we have*

$$\|uv\|_k \le C(\|u\|_2\|v\|_k + \|u\|_k\|v\|_2).$$

Lemma 2.3. *[29] There exists a constant $C > 0$ such that for any $u \in H^1(\mathbb{T}^2)$,*

$$\|u\|_{L^4(\mathbb{T}^2)}^2 \le C\|u\|\|u\|_1.$$

Now we can obtain the well-posedness of system (2.1).

Proposition 2.4. *Let $T > 0$ and $k \in \mathbb{N}$, $k \ge 2$. For any $u_0 \in H^k(\mathbb{T}^2), \zeta \in H^{k+3}(\mathbb{T}^2)$ and $\varphi \in L^2(0,T;H^{k-1}(\mathbb{T})^2)$, system (2.1) admits a unique solution $u \in X_k^T$.*

Proof. In order to prove Proposition 2.4, we use a fixed point approach to obtain the local existence and uniqueness of the solution, then we extend this local solution to an arbitrary time interval by applying a priori estimate.

Let $\theta \in (0, 1 \wedge T)$ and $v \in X_k^\theta$, set

$$\Phi(v)(t) = S(t)u_0 + \int_0^t S(t-s)(-vv_x - (v\zeta)_x - \zeta\zeta_x + A\zeta + \varphi)(s)ds.$$

According to Lemma 2.1 and the fact that $H^k(\mathbb{T}^2)(k \geq 2)$ is a algebra, we can see that

$$\begin{aligned}
&\|\Phi(v)\|_{X_k^\theta}\\
\leq& C\|u_0\|_k + C\| - vv_x - (v\zeta)_x - \zeta\zeta_x + A\zeta + \varphi\|_{L^2(0,\theta;H^{k-1}(\mathbb{T}^2))}\\
\leq& C\|u_0\|_k + C\Big(\int_0^\theta (\|v^2\|_k^2 + \|v\zeta\|_k^2 + \|\zeta^2\|_k^2 + \|\zeta\|_{k+3}^2 + \|\varphi\|_{k-1}^2)ds\Big)^{\frac{1}{2}}\\
\leq& C(1 + \|u_0\|_k + \|\zeta\|_{k+3}^2 + \|\varphi\|_{L^2(0,T;H^{k-1}(\mathbb{T}^2))}) + C\Big(\int_0^\theta (\|v\|_k^4 + \|\zeta\|_{k+3}^2\|v\|_k^2)ds\Big)^{\frac{1}{2}}\\
\leq& C(1 + \|u_0\|_k + \|\zeta\|_{k+3}^2 + \|\varphi\|_{L^2(0,T;H^{k-1}(\mathbb{T}^2))})\\
&+ C\theta^{\frac{1}{2}}(\|v\|_{C([0,\theta];H^k(\mathbb{T}^2))}^2 + \|\zeta\|_{k+3}\|v\|_{C([0,\theta];H^k(\mathbb{T}^2))})\\
\leq& C_1(1 + \|u_0\|_k + \|\zeta\|_{k+3}^2 + \|\varphi\|_{L^2(0,T;H^{k-1}(\mathbb{T}^2))}) + C_2\theta^{\frac{1}{2}}\|v\|_{X_k^\theta}^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|\Phi(v_1) - \Phi(v_2)\|_{X_k^\theta}\\
\leq& C\|[(v_1 + v_2)(v_1 - v_2)]_x\|_{L^2(0,\theta;H^{k-1}(\mathbb{T}^2))} + C\|[(v_1 - v_2)\zeta]_x\|_{L^2(0,\theta;H^{k-1}(\mathbb{T}^2))}\\
\leq& C\|(v_1 + v_2)(v_1 - v_2)\|_{L^2(0,\theta;H^k(\mathbb{T}^2))} + C\|(v_1 - v_2)\zeta\|_{L^2(0,\theta;H^k(\mathbb{T}^2))}\\
\leq& C\theta^{\frac{1}{2}}(\|v_1\|_{C([0,\theta];H^k(\mathbb{T}^2))} + \|v_2\|_{C([0,\theta];H^k(\mathbb{T}^2))})\|v_1(s) - v_2(s)\|_{C([0,\theta];H^k(\mathbb{T}^2))}\\
&+ C\theta^{\frac{1}{2}}\|\zeta\|_{k+3}\|v_1 - v_2\|_{C([0,\theta];H^k(\mathbb{T}^2))}\\
\leq& C_3\theta^{\frac{1}{2}}(\|v_1\|_{X_k^\theta} + \|v_2\|_{X_k^\theta} + \|\zeta\|_{k+3})\|v_1 - v_2\|_{X_k^\theta}
\end{aligned}$$

where C_1, C_2 and C_3 are independent of θ. Set

$$R = 2C_1(1 + \|u_0\|_k + \|\zeta\|_{k+3}^2 + \|\varphi\|_{L^2(0,\theta;H^{k-1}(\mathbb{T}^2))}).$$

If $v, v_1, v_2 \in B_{X_k^\theta}(R) := \{u \in X_k^\theta;\ \|u\|_{X_k^\theta} \leq R\}$, it is easy to see that

$$\begin{aligned}
&\|\Phi(v)\|_{X_k^\theta} \leq \frac{R}{2} + C_2\theta^{\frac{1}{2}}R^2,\\
&\|\Phi(v_1) - \Phi(v_1)\|_{X_k^\theta} \leq C_3\theta^{\frac{1}{2}}(2R + \|\zeta\|_{k+3})\|v_1 - v_2\|_{X_k^\theta}.
\end{aligned}$$

For θ sufficiently small such that

$$\theta^{\frac{1}{2}} \max\{C_2R, C_3(2R + \|\zeta\|_{k+3})\} \leq \frac{1}{2},$$

we arrive to

$$\|\Phi(v)\|_{X_k^\theta} \le R,$$
$$\|\Phi(v_1) - \Phi(v_1)\|_{X_k^\theta} \le \frac{1}{2}\|v_1 - v_2\|_{X_k^\theta}.$$

This implies that Φ has a unique fixed point $u \in B_{X_k^\theta}(R)$.

In order to extend this local solution to an arbitrary time interval, it now suffices to obtain the corresponding a priori estimate. Multiply equation (2.1) by $2u$, integrate the result with respect to x, y over $\mathbb{T}^2$ to obtain

$$\begin{aligned}\frac{d}{dt}\|u\|^2 + 2(\|u_x\|^2 + \|u_y\|^2) \le& C(\|\zeta\|_3\|u\| + \|\zeta\|_3\|u\|^2 + \|\zeta\|_3^2\|u\| + \|\varphi\|_{-1}\|u\|_1\\ \le& \|u_x\|^2 + \|u_y\|^2 + C(1 + \|\zeta\|_3^4)\|u\|^2 + C(1 + \|\varphi\|_{-1}^2).\end{aligned}$$

With the use of Gronwall inequality, we have

$$\|u\|_{X_0^T} \le C(\|u_0\|, \|\zeta\|_3, \|\varphi\|_{L^2(0,T;H^{-1}(\mathbb{T}^2))}). \tag{2.4}$$

Multiply equation (2.1) by $-2\triangle u$, integrate the result with respect to x, y over $\mathbb{T}^2$ to obtain

$$\begin{aligned}&\frac{d}{dt}(\|u_x\|^2 + \|u_y\|^2) + 2\|\triangle u\|^2\\ \le& C(\|\zeta\|_4\|u\|_1 + \|\zeta\|_4\|u\|_1\|\triangle u\| + \|\zeta\|_4^2\|u\|_1 + \|\varphi\|\|\triangle u\|) - 2\int_{\mathbb{T}^2} uu_x\triangle u\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Taking Lemma 2.3 into consideration, the last term can be estimated as follows,

$$\begin{aligned}-2\int_{\mathbb{T}^2} uu_x\triangle u\mathrm{d}x\mathrm{d}y =& \int_{\mathbb{T}^2}(u_x^3 + u_xu_y^2)\mathrm{d}x\mathrm{d}y\\ \le& C\|u_x\|(\|u_x\|_{L^4(\mathbb{T}^2)}^2 + \|u_y\|_{L^4(\mathbb{T}^2)}^2)\\ \le& C\|u_x\|(\|u_x\|\|u_x\|_1 + \|u_y\|\|u_y\|_1)\\ \le& C\|u\|_1^2\|u\|_2\\ \le& \frac{1}{2}\|\triangle u\|^2 + C(1 + \|u\|_1^4).\end{aligned}$$

Then we can see that

$$\begin{aligned}&\frac{d}{dt}(\|u_x\|^2 + \|u_y\|^2) + 2\|\triangle u\|^2\\ \le& C(1 + \|\zeta\|_4^4 + \|u\|_1^2)\|u\|_1^2 + \|\triangle u\|^2 + C(1 + \|\varphi\|^2)\\ \le& C(1 + \|\zeta\|_4^4 + \|u\|_1^2)(\|u_x\|^2 + \|u_y\|^2) + \|\triangle u\|^2 + C(1 + \|\varphi\|^2 + \|u\|^4 + \|\zeta\|_4^4\|u\|^2).\end{aligned}$$

It is worth noting that

$$\int_0^T \|u(t)\|_1^2 dt \le C(\|u_0\|, \|\zeta\|_3, \|\varphi\|_{L^2(0,T;H^{-1}(\mathbb{T}^2))}),$$

together with Gronwall inequality, we find that

$$\sup_{t\in[0,T]}(\|u_x(t)\|^2+\|u_y(t)\|^2)+\int_0^T\|\triangle u(t)\|^2dt\le C(\|u_0\|_1,\|\zeta\|_4,\|\varphi\|_{L^2(0,T;L^2(\mathbb{T}^2))}).$$

Combining (2.4), we get

$$\|u\|_{X_1^T}\le C(\|u_0\|_1,\|\zeta\|_4,\|\varphi\|_{L^2(0,T;L^2(\mathbb{T}^2))}). \tag{2.5}$$

Multiply equation (2.1) by $2(-\triangle)^k u$, integrate the result with respect to x,y over $\mathbb{T}^2$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\|(-\triangle u)^{\frac{k}{2}}\|^2+2\|(-\triangle)^{\frac{k+1}{2}}u\|^2\\
\le&C(\|\zeta\|_{k+3}\|u\|_k+\|u^2\|_k\|(-\triangle)^{\frac{k+1}{2}}u\|+\|\zeta\|_{k+3}\|u\|_k\|(-\triangle)^{\frac{k+1}{2}}u\|\\
&+\|\zeta\|_{k+3}^2\|u\|_k+\|\varphi\|_{k-1}\|(-\triangle)^{\frac{k+1}{2}}u\|)\\
\le&C(\|\zeta\|_{k+3}\|u\|_k+\|u\|_2\|u\|_k\|(-\triangle)^{\frac{k+1}{2}}u\|+\|\zeta\|_{k+3}\|u\|_k\|(-\triangle)^{\frac{k+1}{2}}u\|\\
&+\|\zeta\|_{k+3}^2\|u\|_k+\|\varphi\|_{k-1}\|(-\triangle)^{\frac{k+1}{2}}u\|)\\
\le&C(1+\|\zeta\|_{k+3}^4+\|u\|_2^2)\|u\|_k^2+\|(-\triangle)^{\frac{k+1}{2}}u\|^2+C(1+\|\varphi\|_{k-1}^2),
\end{aligned}$$

where we have used Lemma 2.2.

Applying Gronwall inequality and (2.5), we have

$$\sup_{t\in[0,T]}\|(-\triangle u)^{\frac{k}{2}}(t)\|^2+\int_0^T\|(-\triangle)^{\frac{k+1}{2}}u(t)\|^2dt\le C(\|u_0\|_k,\|\zeta\|_{k+3},\|\varphi\|_{L^2(0,T;H^{k-1}(\mathbb{T}^2))}).$$

Combining (2.4), we have

$$\|u\|_{X_k^T}\le C(\|u_0\|_k,\|\zeta\|_{k+3},\|\varphi\|_{L^2(0,T;H^{k-1}(\mathbb{T}^2))}).$$

This completes the proof of Proposition 2.4. □

Remark 2.5. By the similar methods as in the proof of Proposition 2.4 with minor changes, we can see that Proposition 2.4 also holds for $\varphi\in L^1(0,T;H^k(\mathbb{T}^2))$.

Let $\mathcal{R}$ be the mapping taking a triple (u_0,ζ,φ) to the solution of (2.1) and $\mathcal{R}_t(u_0,\zeta,\varphi)$ be the restriction of $\mathcal{R}(u_0,\zeta,\varphi)$ at time t. The following result is necessary for the proof of Theorem 1.1.

Proposition 2.6. *Let $T>0$, $k\in\mathbb{N},k\ge2$. For any $\varphi\in L^2(0,T;H^{k-1}(\mathbb{T}^2))$ and $u_{01},u_{02}\in H^k(\mathbb{T}^2)$, we have*

$$\|\mathcal{R}(u_{01},0,\varphi)-\mathcal{R}(u_{02},0,\varphi)\|_{C([0,T];H^k(\mathbb{T}^2))}\le C\|u_{01}-u_{02}\|_k.$$

Proof. Let $u_1=\mathcal{R}(u_{01},0,\varphi)$, $u_2=\mathcal{R}(u_{02},0,\varphi)$ and $w=u_1-u_2$, then w is a solution of problem

$$\begin{cases}w_t-Aw+\frac{1}{2}[(u_1+u_2)w]_x=0. & \text{in } \mathbb{T}^2\times(0,T),\\ w(0)=w_0=u_{01}-u_{02} & \text{in } \mathbb{T}^2.\end{cases} \tag{2.6}$$

By Proposition 2.4, $u_1,u_2\in C([0,T];H^k(\mathbb{T}^2))$.

Multiplying (2.6) by $2w$ and integrating on $\mathbb{T}^2$, we can deduce that

$$\begin{aligned}\frac{d}{dt}\|w\|^2+2(\|w_x\|^2+\|w_y\|^2)\leq& C\|u_1+u_2\|_{L^\infty(\mathbb{T}^2)}\|w\|\|w_x\|\\ \leq& C(\|u_1\|_2^2+\|u_2\|_2^2)\|w\|^2+\|w_x\|^2.\end{aligned}$$

Applying Gronwall inequality, we have

$$\|w\|_{C([0,T];L^2(\mathbb{T}^2))}\leq C\|w_0\|. \tag{2.7}$$

Then, multiplying (2.6) by $2(-\triangle)^k w$ and integrating on $\mathbb{T}^2$, we can deduce that

$$\begin{aligned}&\frac{d}{dt}\|(-\triangle)^{\frac{k}{2}}w\|^2+2\|(-\triangle)^{\frac{k+1}{2}}w\|^2\\ \leq& C\|(-\triangle)^{\frac{k-1}{2}}\partial_x((u_1+u_2)w)\|\|(-\triangle)^{\frac{k+1}{2}}w\|\\ \leq& C\|(u_1+u_2)w\|_k^2+\|(-\triangle)^{\frac{k+1}{2}}w\|^2\\ \leq& C(\|u_1\|_k^2+\|u_2\|_k^2)\|w\|_k^2+\|(-\triangle)^{\frac{k+1}{2}}w\|^2\\ \leq& C(\|u_1\|_k^2+\|u_2\|_k^2)(\|(-\triangle)^{\frac{k}{2}}w\|^2+\|w\|^2)+\|(-\triangle)^{\frac{k+1}{2}}w\|^2.\end{aligned}$$

Combining Gronwall inequality and (2.7), we have

$$\|w\|_{C([0,T];H^k(\mathbb{T}^2))}\leq C\|w_0\|_k.$$

The proof is complete. □

3. Saturating property

For $I\subset\mathbb{Z}^2$, we set

$$\mathcal{H}(I)=\text{span}\{\cos(px+qy),\sin(px+qy);\ \text{for } (p,q)\in I\},$$

and

$$\mathcal{C}(I)=\Big\{\zeta_0-\sum_{i=1}^{d}\zeta_i\zeta_{ix};\ \zeta_i\in\mathcal{H}(I)\ (0\leq i\leq d),\ d\geq 1\Big\}.$$

Let

$$I_0=\{(p,q)\in\mathbb{Z}^2;\ |p|+|q|\leq 1\},\ I_1=\{(p,q)\in\mathbb{Z}^2;\ |p|\leq 1 \text{ and } |q|\leq 1\},$$

and

$$I_N=\{(p,q)\in\mathbb{Z}^2;\ |p|\leq N \text{ and } |q|\leq (N-1)\},\quad \text{for } N\geq 2.$$

In this section, we investigate the relationship between $\mathcal{H}(I_N)$ and $\mathcal{C}(I_N)$, the idea comes from [17, 19].

Proposition 3.1. *For $N\in\mathbb{N}$, we have*

$$\mathcal{H}(I_{N+1})\subset\mathcal{C}(I_N).$$

Proof. For $N = 0$. Since $I_0 \subset I_1$, to prove $\mathcal{H}(I_1) \subset \mathcal{C}(I_0)$, we only need to show that for any $(p, q) \in I_1 \setminus I_0 = \{(p, q);\ |p| = 1, |q| = 1\}$ and any $\alpha, \beta \in \mathbb{R}$,

$$\alpha \cos(px + qy) + \beta \sin(px + qy) = \zeta_0 - \sum_{i=1}^{d} \zeta_i \zeta_{ix}, \quad \text{where } \zeta_i \in \mathcal{H}(I_0)\ (0 \le i \le d).$$

By symmetry arguments, the above result follows from the following equations,

$$\begin{aligned} &\sin(x + y) = \sin x \cos y + \cos x \sin y, \quad \sin(x - y) = \sin x \cos y - \cos x \sin y, \\ &\cos(x + y) = \cos x \cos y - \sin x \sin y, \quad \cos(x - y) = \cos x \cos y + \sin x \sin y, \end{aligned}$$

and

$$\begin{aligned} \pm \sin x \cos y &= -(\cos x \pm \cos y)(\cos x \pm \cos y)_x - \sin x (\sin x)_x, \\ \pm \cos x \sin y &= -(\sin x \mp \sin y)(\sin x \mp \sin y)_x - \cos x (\cos x)_x, \\ \pm \cos x \sin y &= -(\sin x \mp \cos y)(\sin x \mp \cos y)_x - \cos x (\cos x)_x, \\ \pm \sin x \sin y &= -(\cos x \pm \sin y)(\cos x \pm \sin y)_x - \sin x (\sin x)_x. \end{aligned}$$

For $N = 1$, let $f \in \mathcal{H}(I_2)$. In order to prove that $f \in \mathcal{C}(I_1)$, there remains to show that for any $(p, q) \in I_2 \setminus I_1 = \{(p, q);\ |p| = 2, |q| \le 1\}$ and any $\alpha, \beta \in \mathbb{R}$,

$$\alpha \cos(px + qy) + \beta \sin(px + qy) = \zeta_0 - \sum_{i=1}^{d} \zeta_i \zeta_{ix}, \quad \text{where } \zeta_i \in \mathcal{H}(I_1)\ (0 \le i \le d).$$

By symmetry arguments, we consider just the cases that $(p, q) = (2, 0), (2, 1), (2, -1)$.

If $(p, q) = (2, 0)$. For any $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 \neq 0$, let

$$\zeta_1(x, y) = -\frac{\alpha}{\sqrt{-\beta + \sqrt{\alpha^2 + \beta^2}}} \cos x + \sqrt{-\beta + \sqrt{\alpha^2 + \beta^2}} \sin x.$$

Then we have

$$\alpha \cos(2x) + \beta \sin(2x) = -\zeta_1 \zeta_{1x}.$$

If $(p, q) = (2, 1)$. Let

$$\begin{aligned} \zeta_2(x, y) &= \cos x + \sin x + c \cos(x + y) + d \sin(x + y), \\ \zeta_3(x, y) &= \cos x - \sin x + c \cos(x + y) - d \sin(x + y), \end{aligned}$$

simple calculations give

$$\zeta_2 \zeta_{2x} + \zeta_3 \zeta_{3x} = (d - c) \sin(2x + y) + (d + c) \cos(2x + y).$$

If $(p, q) = (2, -1)$. Let

$$\begin{aligned} \zeta_4(x, y) &= \cos x + \sin x + c \cos(x - y) + d \sin(x - y), \\ \zeta_5(x, y) &= \cos x - \sin x + c \cos(x - y) - d \sin(x - y), \end{aligned}$$

simple calculations give

$$\zeta_4\zeta_{4x}+\zeta_5\zeta_{5x}=(d-c)\sin(2x-y)+(d+c)\cos(2x-y).$$

Since the map $(c,d)\mapsto(d-c,d+c)$ is bijective, we prove that $\mathcal{H}(I_2)\subset\mathcal{C}(I_1)$.
For $N\geq 2$, our task is reduced to show that for any $(p,q)\in I_{N+1}\setminus I_N$ and any $\alpha,\beta\in\mathbb{R}$,

$$\alpha\cos(px+qy)+\beta\sin(px+qy)=\zeta_0-\sum_{i=1}^{d}\zeta_i\zeta_{ix},\quad \text{where } \zeta_i\in\mathcal{H}(I_N)(0\leq i\leq d). \tag{3.1}$$

In fact, let

$$\begin{aligned}\zeta_6(x,y)&=\sin(x+y)+\cos(x+y)+c\sin(mx+ny)+d\cos(mx+ny),\\ \zeta_7(x,y)&=a\sin(x+y)+b\cos(x+y)+d\sin(mx+ny)-c\cos(mx+ny),\end{aligned}$$

and

$$\begin{aligned}\zeta_8(x,y)&=\sin(x-y)+\cos(x-y)+c\sin(mx+ny)+d\cos(mx+ny),\\ \zeta_9(x,y)&=a\sin(x-y)+b\cos(x-y)+d\sin(mx+ny)-c\cos(mx+ny).\end{aligned}$$

After standard calculations, we arrive to

$$\begin{aligned}\zeta_6\zeta_{6x}+\zeta_7\zeta_{7x}=&\frac{a^2-b^2}{2}\sin(2x+2y)+(1+ab)\cos(2x+2y)\\ &+\frac{m+1}{2}(c(1+b)+d(-1+a))\sin((m+1)x+(n+1)y)\\ &+\frac{m+1}{2}(c(1-a)+d(1+b))\cos((m+1)x+(n+1)y)\\ &+\frac{m-1}{2}(c(-1+b)+d(-1-a))\sin((m-1)x+(n-1)y)\\ &+\frac{m-1}{2}(c(1+a)+d(-1+b))\cos((m-1)x+(n-1)y),\end{aligned}$$

and

$$\begin{aligned}\zeta_8\zeta_{8x}+\zeta_9\zeta_{9x}=&\frac{a^2-b^2}{2}\sin(2x-2y)+(1+ab)\cos(2x-2y)\\ &+\frac{m+1}{2}(c(1+b)+d(-1+a))\sin((m+1)x+(n-1)y)\\ &+\frac{m+1}{2}(c(1-a)+d(1+b))\cos((m+1)x+(n-1)y)\\ &+\frac{m-1}{2}(c(-1+b)+d(-1-a))\sin((m-1)x+(n+1)y)\\ &+\frac{m-1}{2}(c(1+a)+d(-1+b))\cos((m-1)x+(n+1)y).\end{aligned}$$

It is not difficult to see that

$$I_{N+1}\setminus I_N=\{(p,q);\ |p|=N+1,|q|\leq N\}\cup\{(p,q);\ |p|\leq N,|q|=N\}:=J_1\cup J_2.$$

If $(p,q)\in J_1$, by symmetry arguments, it only remains to prove that (3.1) holds for $(p,q)\in J_{11}\cup J_{12}$, where

$$J_{11}=\{(p,q):\ p=N+1, 0\le q\le N\},\quad J_{12}=\{(p,q);\ p=-N-1, 0\le q\le N\}.$$

For $(p,q)\in J_{11}$, taking $m=N, n=q-1, a=-1$ and $b=1$ in ζ_6 and ζ_7, we can obtain that $(1,1),(m,n)\in I_N$, namely, $\zeta_6,\zeta_7\in\mathcal{C}(I_N)$, and

$$\zeta_6\zeta_{6x}+\zeta_7\zeta_{7x}=(N+1)(c-d)\sin((N+1)x+qy)+(N+1)(c+d)\cos((N+1)x+qy).$$

For $(p,q)\in J_{12}$, taking $m=-N, n=q-1, a=1$ and $b=-1$ in ζ_8 and ζ_9, we can obtain that $(1,-1),(m,n)\in I_N$, namely, $\zeta_6,\zeta_7\in\mathcal{C}(I_N)$, and

$$\zeta_8\zeta_{8x}+\zeta_9\zeta_{9x}=-(N-1)(c+d)\sin((N+1)x+qy)+(N-1)(c-d)\cos((N+1)x+qy).$$

If $(p,q)\in J_2$, by symmetry arguments, it only remains to prove that (3.1) holds for $(p,q)\in J_{21}\cup J_{22}$, where

$$J_{21}=\{(p,q):\ 0\le p\le N, q=N\},\quad J_{22}=\{(p,q);\ -N\le p\le 0, q=N\}.$$

For $(p,q)\in J_{21}$, taking $m=p-1, n=N-1, a=-1$ and $b=1$ in ζ_6 and ζ_7, we can obtain that $(1,1),(m,n)\in I_N$, namely, $\zeta_6,\zeta_7\in\mathcal{C}(I_N)$, and

$$\zeta_6\zeta_{6x}+\zeta_7\zeta_{7x}=(N+1)(c-d)\sin(px+Ny)+(N+1)(c+d)\cos(px+Ny).$$

For $(p,q)\in J_{22}$, taking $m=p+1, n=N-1, a=1$ and $b=-1$ in ζ_8 and ζ_9, we can obtain that $(1,-1),(m,n)\in I_N$, namely, $\zeta_8,\zeta_9\in\mathcal{C}(I_N)$, and

$$\zeta_8\zeta_{8x}+\zeta_9\zeta_{9x}=-(N-1)(c+d)\sin(px+Ny)+(N-1)(c-d)\cos(px+Ny).$$

Since the map $(c,d)\mapsto(k(c-d),k(c+d))$ is bijective for any $k\neq 0$, (3.1) holds for $(p,q)\in I_{N+1}\setminus I_N$ and any $\alpha,\beta\in\mathbb{R}$. Thus the conclusion of Proposition 3.1 follows. □

Define

$$\mathcal{H}_0=\mathcal{H}(I_0)=\mathcal{H},\ \mathcal{H}_N=\mathcal{C}(I_{N-1}),\ N\ge 1,\ \mathcal{H}_\infty=\bigcup_{N=0}^{\infty}\mathcal{H}_N.$$

By Proposition 3.1, we can easily obtain the following corollary.

Corollary 3.2. *For any $k\in\mathbb{N}$, $\mathcal{H}_\infty$ is dense in $H^k(\mathbb{T}^2)$.*

4. Asymptotic property

Now we are in position to prove the asymptotic property which is crucial for the proof of Theorem 1.1.

Proposition 4.1. *Let $T>0$, $k\in\mathbb{N}$ and $k\ge 2$. For any $u_0\in H^{k+3}(\mathbb{T}^2),\eta\in H^{k+3}(\mathbb{T}^2)$, $\zeta\in H^{k+4}(\mathbb{T}^2)$ and $h\in L^2(0,\theta;H^{k-1}(\mathbb{T}^2))$, for any $\delta\in(0,T)$, the following limit holds*

$$\mathcal{R}_\delta(u_0,\delta^{-\frac{1}{2}}\zeta,h+\delta^{-1}\eta)\longrightarrow u_0+\eta-\zeta\zeta_x\quad \text{in } H^k(\mathbb{T}^2)\quad \text{as } \delta\to 0.$$

Proof. Let us consider the equation

$$\begin{cases} u_t - A(u+\delta^{-\frac{1}{2}}\zeta) + (u+\delta^{-\frac{1}{2}}\zeta)(u+\delta^{-\frac{1}{2}}\zeta)_x = h + \delta^{-1}\eta, \\ u(0) = u_0. \end{cases}$$

Make a time substitution and consider the functions

$$\begin{aligned} w(t) &:= u_0 + t(\eta - \zeta\zeta_x), \\ v(t) &:= u(\delta t) - w(t), \end{aligned}$$

where $t \in [0,1]$. Then it is not difficult to see that v is a solution of the following system

$$\begin{cases} v_t - \delta A(v+w+\delta^{-\frac{1}{2}}\zeta) + \delta(v+w+\delta^{-\frac{1}{2}}\zeta)(v+w+\delta^{-\frac{1}{2}}\zeta)_x - \zeta\zeta_x = \delta h, \\ v(0) = 0. \end{cases} \tag{4.1}$$

For $k=0$, taking the scalar product in $L^2(\mathbb{T}^2)$ of equation (4.1) with $2v$, we can obtain that

$$\begin{aligned} \frac{d}{dt}\|v\|^2 + 2\delta(\|v_x\|^2 + \|v_y\|^2) \leq & C\delta\|w\|_3\|v\|^2 + C\delta^{\frac{1}{2}}\|\zeta\|_3\|v\|^2 + C\delta^{\frac{1}{2}}\|w\|_3\|\zeta\|_3\|v\| \\ & + C\delta\|w\|_3^2\|v\| + C\delta\|h\|_{-1}\|v\|_1 \\ \leq & C\delta^{\frac{1}{2}}(1+\|v\|^2) + \delta(\|v_x\|^2 + \|v_y\|^2). \end{aligned}$$

Making use of Gronwall inequality, we have

$$\|v(t)\|^2 \leq C\delta^{\frac{1}{2}}, \quad \forall\ t \in [0,1]. \tag{4.2}$$

For $k=1$, taking the scalar product in $L^2(\mathbb{T}^2)$ of equation (4.1) with $-2\triangle v$, we can deduce that

$$\begin{aligned} & \frac{d}{dt}(\|v_x\|^2 + \|v_y\|^2) + 2\delta\|\triangle v\|^2 \\ \leq & C(\delta\|w\|_4\|v\|_1 + \delta^{\frac{1}{2}}\|\zeta\|_4\|v\|_1 + \delta\|h\|\|\triangle v\|) \\ & + 2\delta\int_{\mathbb{T}^2}[vv_x + (wv)_x + \delta^{-\frac{1}{2}}(\zeta v)_x + ww_x + \delta^{-\frac{1}{2}}(w\zeta)_x]\triangle v \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Next we estimate separate terms in the right hand side of the last inequality. Applying Gagliardo-Nirenberg inequality and (4.2), we infer that

$$\begin{aligned} 2\delta\int_{\mathbb{T}^2} vv_x\triangle v \mathrm{d}x\mathrm{d}y \leq & C\delta\|v\|^6_{L^6(\mathbb{T}^2)} + C\delta\|v_x\|^3_{L^3(\mathbb{T}^2)} + \frac{\delta}{4}\|\triangle v\|^2 \\ \leq & C\delta\|v\|^4\|v\|_2^2 + C\delta\|v\|\|v\|_2^2 + \frac{\delta}{4}\|\triangle v\|^2 \\ \leq & C\delta^2\|v\|_2^2 + C\delta^{\frac{5}{4}}\|v\|_2^2 + \frac{\delta}{4}\|\triangle v\|^2. \end{aligned} \tag{4.3}$$

Taking δ sufficiently small, we have

$$2\delta\int_{\mathbb{T}^2} vv_x\triangle v \mathrm{d}x\mathrm{d}y \leq \frac{\delta}{2}\|\triangle v\|^2 + C\delta.$$

Using integration by part, it follows that

$$\begin{aligned}
&2\delta\int_{\mathbb{T}^2}(wv)_x\triangle v\mathrm{d}x\mathrm{d}y\\
=&-\delta\int_{\mathbb{T}^2}(2w_{xx}vv_x+2w_xv_x^2+2w_{xy}vv_y+2w_xv_y^2+w_xv_x^2-w_xv_y^2+2w_yv_xv_y)\mathrm{d}x\mathrm{d}y\\
\leq& C\delta\|w\|_4\|v\|_1^2.
\end{aligned}$$

Similarly, we have

$$2\delta^{\frac{1}{2}}\int_{\mathbb{T}^2}(\zeta v)_x\triangle v\mathrm{d}x\mathrm{d}y\leq C\delta^{\frac{1}{2}}\|\zeta\|_4\|v\|_1^2.$$

Considering the remaining terms, it is not difficult to see that

$$\begin{aligned}
2\delta\int_{\mathbb{T}^2}[ww_x+\delta^{-\frac{1}{2}}(w\zeta)_x]\triangle v\mathrm{d}x\mathrm{d}y\leq& C(\delta\|ww_x\|_1+\delta^{\frac{1}{2}}\|(w\zeta)_x\|_1)\|v\|_1\\
\leq& C\delta^{\frac{1}{2}}(\|w\|_4^2+\|w\|_4\|\zeta\|_4)\|v\|_1.
\end{aligned}$$

Substituting these estimates into (4.3), we can obtain from (4.2) that

$$\frac{d}{dt}(\|v_x\|^2+\|v_y\|^2)+2\delta\|\triangle v\|^2\leq\delta\|\triangle v\|^2+C\delta^{\frac{1}{2}}(1+\|v_x\|^2+\|v_y\|^2).$$

By Gronwall's inequality, we have

$$\|v_x(t)\|^2+\|v_y(t)\|^2\leq C\delta^{\frac{1}{2}},\quad\forall\ t\in[0,1].$$

Combining (4.2), we get

$$\|v(t)\|_1^2\leq C\delta^{\frac{1}{2}},\quad\forall\ t\in[0,1].\tag{4.4}$$

For $k=2$, taking the scalar product in $L^2(\mathbb{T}^2)$ of equation (4.1) with $2\triangle^2 v$, we can deduce that

$$\begin{aligned}
&\frac{d}{dt}\|\triangle v\|^2+2\delta\|(-\triangle)^{\frac{3}{2}}v\|^2\\
\leq& C(\delta\|w\|_5\|v\|_2+\delta^{\frac{1}{2}}\|\zeta\|_5\|v\|_2+\delta\|h\|_1\|(-\triangle)^{\frac{3}{2}}v\|)\\
&-2\delta\int_{\mathbb{T}^2}[vv_x+(wv)_x+\delta^{-\frac{1}{2}}(\zeta v)_x+ww_x+\delta^{-\frac{1}{2}}(w\zeta)_x]\triangle^2 v\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

The last term can be estimated as follows. By the fact that $H^2(\mathbb{T}^2)$ is a algebra and the interpolation inequality $\|v\|_2^2 \le \|v\|_1\|v\|_3$, it follows that

$$\begin{aligned}
-2\delta\int_{\mathbb{T}^2} vv_x\triangle^2 v\mathrm{d}x\mathrm{d}y \le& C\delta\|v\|_2^2\|(-\triangle)^{\frac{3}{2}}v\| \\
\le& C\delta\|v\|_1\|v\|_3\|(-\triangle)^{\frac{3}{2}}v\| \\
\le& C\delta\|v\|_1\|(-\triangle)^{\frac{3}{2}}v\|^2 + C\delta\|v\|_1\|v\|\|(-\triangle)^{\frac{3}{2}}v\| \\
\le& C\delta^{\frac{5}{4}}\|(-\triangle)^{\frac{3}{2}}v\|^2 + C\delta^{\frac{3}{2}}\|(-\triangle)^{\frac{3}{2}}v\| \\
\le& \frac{\delta}{2}\|(-\triangle)^{\frac{3}{2}}v\|^2 + C\delta,
\end{aligned}$$

where we have used (4.2), (4.4) and taken δ sufficiently small.

It follows from some elementary calculations that

$$\begin{aligned}
&-2\delta\int_{\mathbb{T}^2}(wv)_x\triangle^2 v\mathrm{d}x\mathrm{d}y \\
=&-2\delta\int_{\mathbb{T}^2}\Big[\triangle(w_xv)\triangle v + \triangle(wv_x)\triangle v\Big]\mathrm{d}x\mathrm{d}y \\
\le& C\delta\|w_xv\|_2\|v\|_2 - 2\delta\int_{\mathbb{T}^2}[w(\triangle v)_x + 2w_xv_{xx} + 2w_yv_{xy} + (\triangle w)v_x]\triangle v\mathrm{d}x\mathrm{d}y \\
\le& C\delta\|w\|_5\|v\|_2^2 + \delta\int_{\mathbb{T}^2} w_x(\triangle v)^2\mathrm{d}x\mathrm{d}y. \\
\le& C\delta\|\triangle v\|^2 + C\delta.
\end{aligned}$$

Similarly, we have

$$-2\delta^{\frac{1}{2}}\int_{\mathbb{T}^2}(\zeta v)_x\triangle^2 v\mathrm{d}x\mathrm{d}y \le C\delta^{\frac{1}{2}}\|\triangle v\|^2 + C\delta^{\frac{1}{2}}.$$

It is easy to obtain that

$$\begin{aligned}
-2\delta\int_{\mathbb{T}^2}[ww_x + \delta^{-\frac{1}{2}}(w\zeta)_x]\triangle^2 v\mathrm{d}x\mathrm{d}y \le& C\delta^{\frac{1}{2}}(\|w\|_5^2 + \|w\|_5\|\zeta\|_5)\|\triangle v\| \\
\le& C\delta^{\frac{1}{2}}\|\triangle v\|^2 + C\delta^{\frac{1}{2}}.
\end{aligned}$$

Gathering together the above estimates, we have

$$\frac{d}{dt}\|\triangle v\|^2 \le C\delta^{\frac{1}{2}}\|\triangle v\|^2 + C\delta^{\frac{1}{2}}.$$

Applying Gronwall's inequality and (4.2), we can see that

$$\|v(t)\|_2^2 \le C\delta^{\frac{1}{2}}, \quad \forall\ t\in[0,1].$$

For $k \geq 3$, taking the scalar product in $L^2(\mathbb{T}^2)$ of equation (4.1) with $2(-\triangle)^k v$, we can deduce that

$$\begin{aligned}
&\frac{d}{dt}\|(-\triangle)^{\frac{k}{2}}v\|^2 + 2\delta\|(-\triangle)^{\frac{k+1}{2}}v\|^2 \\
\leq& C(\delta\|w\|_{k+3}\|v\|_k + \delta^{\frac{1}{2}}\|\zeta\|_{k+3}\|v\|_k + \delta\|h\|_{k-1}\|(-\triangle)^{\frac{k+1}{2}}v\|) \\
&- 2\delta\int_{\mathbb{T}^2}[vv_x + (wv)_x + \delta^{-\frac{1}{2}}(\zeta v)_x + ww_x + \delta^{-\frac{1}{2}}(w\zeta)_x](-\triangle)^k v\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Taking Lemma 2.2 into account, we have

$$\begin{aligned}
-2\delta\int_{\mathbb{T}^2} vv_x(-\triangle)^k v\mathrm{d}x\mathrm{d}y =& -\delta\int_{\mathbb{T}^2}(-\triangle)^{\frac{k-1}{2}}\partial_x(v^2)(-\triangle)^{\frac{k+1}{2}}v\mathrm{d}x\mathrm{d}y \\
\leq& \delta\|v^2\|_k\|(-\triangle)^{\frac{k+1}{2}}v\| \\
\leq& \delta\|(-\triangle)^{\frac{k+1}{2}}v\|^2 + C\delta\|v\|_2^2\|v\|_k^2 \\
\leq& \delta\|(-\triangle)^{\frac{k+1}{2}}v\|^2 + C\delta^{\frac{3}{2}}\|(-\triangle)^{\frac{k}{2}}v\|^2 + C\delta^{\frac{3}{2}}.
\end{aligned}$$

It is not difficult to obtain that

$$\begin{aligned}
&-2\delta\int_{\mathbb{T}^2}(wv)_x(-\triangle)^k v\mathrm{d}x\mathrm{d}y \\
=& -2\delta\int_{\mathbb{T}^2}(w_x v + wv_x)(-\triangle)^k v\mathrm{d}x\mathrm{d}y \\
=& -2\delta\int_{\mathbb{T}^2}(-\triangle)^{\frac{k}{2}}(w_x v)(-\triangle)^{\frac{k}{2}}v\mathrm{d}x\mathrm{d}y - 2\delta\int_{\mathbb{T}^2} w(-\triangle)^{\frac{k}{2}}(v_x)(-\triangle)^{\frac{k}{2}}v\mathrm{d}x\mathrm{d}y \\
&-2\delta\sum_{i=1}^{k}\int_{\mathbb{T}^2}(-\triangle)^{\frac{i}{2}}w(-\triangle)^{\frac{k-i}{2}}v_x(-\triangle)^{\frac{k}{2}}v\mathrm{d}x\mathrm{d}y \\
\leq& C\delta\|w_x v\|_k\|v\|_k + \delta\int_{\mathbb{T}^2} w_x|(-\triangle)^{\frac{k}{2}}v|^2\mathrm{d}x\mathrm{d}y + C\delta\|w\|_{k+3}\sum_{i=1}^{k}\|v\|_{k-i+1}\|v\|_k \\
\leq& C\delta\|w\|_{k+3}\|v\|_k^2 \\
\leq& C\delta\|(-\triangle)^{\frac{k}{2}}v\|^2 + C\delta^{\frac{3}{2}}.
\end{aligned}$$

Similarly, we have

$$-2\delta^{\frac{1}{2}}\int_{\mathbb{T}^2}(\zeta v)_x(-\triangle)^k v\mathrm{d}x\mathrm{d}y \leq C\delta^{\frac{1}{2}}\|(-\triangle v)^{\frac{k}{2}}\|^2 + C\delta.$$

The remaining terms can be estimated as follows,

$$\begin{aligned}
&-2\delta\int_{\mathbb{T}^2}[ww_x + \delta^{-\frac{1}{2}}(w\zeta)_x](-\triangle)^k v\mathrm{d}x\mathrm{d}y \\
\leq& C\delta^{\frac{1}{2}}(\|w\|_{k+3}^2 + \|w\|_{k+3}\|\zeta\|_{k+3})\|(-\triangle)^{\frac{k}{2}}v\| \\
\leq& C\delta^{\frac{1}{2}}\|(-\triangle)^{\frac{k}{2}}v\|^2 + C\delta^{\frac{1}{2}}.
\end{aligned}$$

Now we conclude that

$$\frac{d}{dt}\|(-\triangle)^{\frac{k}{2}}v\|^2 \le C\delta^{\frac{1}{2}}(1+\|(-\triangle v)^{\frac{k}{2}}\|^2)$$

Applying Gronwall's inequality and (4.2), we have

$$\|v(t)\|_k^2 \le C\delta^{\frac{1}{2}}, \quad \forall\, t\in[0,1].$$

This completes the proof of Proposition 4.1. □

Remark 4.2. In the proof of Proposition 4.1, we have used the same multiplier as in the proof of Proposition 2.4 (a priori estimates), however, the methods to deal with the terms in the right hand side are very different. Since we need to treat the parameter δ more carefully to obtain the estimates of v, the methods in Proposition 2.4 are not enough for this purpose.

5. Proof of Theorem 1.1

In this section, we denote $\mathcal{R}_t(u_0,0,\varphi)$ by $\mathcal{R}_t(u_0,\varphi)$.

Proof of Theorem 1.1. The proof is divided into four steps.

Step 1. Controllability in small time to $u_0+\mathcal{H}_0$. Let us assume for the moment that $u_0\in H^{k+3}(\mathbb{T}^2)$. For any $\hat\eta\in\mathcal{H}_0$, applying Proposition 4.1 for $\zeta=0$ and $\eta=\hat\eta$, we see that

$$\mathcal{R}_\delta(u_0,h+\delta^{-1}\hat\eta)\to u_0+\hat\eta \ \text{ in } H^k(\mathbb{T}^2) \text{ as } \delta\to 0.$$

It follows from the above limit that for any $\varepsilon>0$, there is a time $0<\theta_1<T$ such that $\eta=\theta_1^{-1}\hat\eta\in L^2(0,T;\mathcal{H})$ and

$$\|\mathcal{R}_{\theta_1}(u_0,h+\eta)-u_0-\hat\eta\|_k<\varepsilon.$$

Step 2. Controllability in small time to $u_0+\mathcal{H}_N$. We argue by induction. Assume that the approximate controllability of (1.1) to the set $u_0+\mathcal{H}_{N-1}$ is already proved. Let $\zeta\in\mathcal{H}_N$ be of the form

$$\zeta=\zeta_0-\sum_{i=1}^{d}\zeta_i\zeta_{ix} \tag{5.1}$$

for some integer $d\ge 1$ and vectors $\zeta_0,\zeta_1,\cdots,\zeta_d\in\mathcal{H}(I_{N-1})$. It follows from Proposition 3.1 that

$$\zeta_0,\zeta_1,\cdots,\zeta_d\in\begin{cases}\mathcal{H}(I_0)=\mathcal{H}_0 & N=1,\\ \mathcal{H}(I_{N-1})\subset\mathcal{C}(I_{N-2})=\mathcal{H}_{N-1} & N\ge 2,\end{cases}$$

namely, $\zeta_0,\zeta_1,\cdots,\zeta_d\in\mathcal{H}_{N-1}$.

Applying Proposition 4.1 for the couple $(0,\zeta_1)$, we see that

$$\mathcal{R}_\delta(u_0,\delta^{-\frac{1}{2}}\zeta_1,h)\to u_0-\zeta_1\zeta_{1x} \ \text{ in } H^k(\mathbb{T}^2) \text{ as } \delta\to 0. \tag{5.2}$$

Using the equality

$$\mathcal{R}_\delta(u_0+\delta^{-\frac{1}{2}}\zeta_1,h)=\mathcal{R}_\delta(u_0,\delta^{-\frac{1}{2}}\zeta_1,h)+\delta^{-\frac{1}{2}}\zeta_1$$

and the limit (5.2), we obtain that

$$\|\mathcal{R}_\delta(u_0+\delta^{-\frac{1}{2}}\zeta_1,h)-u_0+\zeta_1\zeta_{1x}-\delta^{-\frac{1}{2}}\zeta_1\|_k\to 0 \ \text{ as } \delta\to 0.$$

Combining this with the induction hypothesis, and Proposition 2.6, we can find a small time $\theta_2>0$ and a control $\eta_1\in L^2(0,T;\mathcal{H})$ such that

$$\|\mathcal{R}_{\theta_2}(u_0,h+\eta_1)-u_0-\eta+\zeta_1\zeta_{1x}\|_k<\varepsilon.$$

Iterating this argument successively for the vectors $\zeta_2,\cdots,\zeta_n$, we construct a small time $\theta>0$ and a control $\eta\in L^2(0,T;\mathcal{H})$ satisfying

$$\|\mathcal{R}_\theta(u_0,h+\eta_1)-u_0-\zeta_0+\zeta_1\zeta_{1x}+\cdots+\zeta_d\zeta_{\mathrm{d}x}\|_k=\|\mathcal{R}_\theta(u_0,h+\eta_1)-u_0-\zeta\|_k<\varepsilon,$$

where we used (5.1). This proves the approximate controllability in small time to any point in $u_0+\mathcal{H}_N$.

Step 3. Global controllability in fixed time T. Let $\varepsilon>0$, $T>0$ and $u_0,u_1\in H^k(\mathbb{T}^2)$ be arbitrary. By Corollary 3.2, there is an integer $N\geq 1$ and point $\zeta\in\mathcal{H}_N$ such that

$$\|R_T(u_0,h)+\zeta-u_1\|_k<\frac{\varepsilon}{2}.$$

Let us denote $\hat{u}_0=R_T(u_0,h)+\zeta$, according to Proposition 2.4 and 2.6, take $t_0>0$ and $r>0$ so small that

$$\|R_t(u,h)-\hat{u}_0\|_k<\frac{\varepsilon}{2},\quad \text{for } t\in[0,t_0] \text{ and } u\in B_{H^k(\mathbb{T}^2)}(\hat{u}_0,r).$$

This implies

$$\|R_t(u,h)-u_1\|_k<\varepsilon,\quad \text{for } t\in[0,t_0] \text{ and } u\in B_{H^k(\mathbb{T}^2)}(\hat{u}_0,r). \tag{5.3}$$

Choosing, if necessary, t_0 smaller, it follows from Proposition 2.4 that

$$\|R_{T-t}(u_0,h)-R_T(u_0,h)\|_k<\frac{r}{2},\quad \text{for } t\in[0,t_0].$$

By the result in Step 2, we can find a time $\tau\in(0,t_0)$ and a control $\eta_2\in L^2(0,\tau;\mathcal{H})$ such that

$$\|R_\tau(R_{T-t_0}(u_0,h),h+\eta_2)-R_{T-t_0}(u_0,h)-\zeta\|_k<\frac{r}{2}.$$

This means that $R_\tau(R_{T-t_0}(u_0,h),h+\eta_2)\in B_{H^k(\mathbb{T}^2)}(\hat{u}_0,r)$. Combining this result and (5.3), if we define $\eta(t)=\chi_{[T-t_0,T-t_0+\tau]}\eta_2(t-T+t_0)$, it holds that

$$\|R_T(u_0,h+\eta)-u_1\|_k<\varepsilon.$$

This completes the proof of Theorem 1.1. □

6. Proof of Theorem 1.3

In order to prove Theorem 1.3, we give some preliminary results which will be used. For $k\in\mathbb{N}$, we define the space $G^k:=\{g\in C([0,T];H^k(\mathbb{T}^2))\mid g(0)=0\}$ with the norm $\|g\|_{G^k}=\sup\limits_{0\leq t\leq T}\|g(t)\|_k$.

Proposition 6.1. *Let $T > 0$, $k \in \mathbb{N}$ and $k \geq 2$, if $u_0 \in H^k(\mathbb{T}^2)$ and $g \in G^{k+3}$, then the equation*

$$u(t) - u_0 + \int_0^t [-Au + uu_x](s)ds = g(t) \tag{6.1}$$

has a unique solution $u \in X_k^T$. Moreover, the map

$$\begin{aligned} F: \ G^{k+3} &\to X_k^T \\ g &\mapsto u \end{aligned}$$

is well-defined and continuous.

Proof. The well-posedness of (6.1) can be established by the classical method. We just prove that the map F is continuous. Motivated by [18] and [30], we make substitution $u = v + g$, then v satisfies the following equation

$$\begin{cases} v_t - Av + (v+g)(v+g)_x = Ag & \text{in } \mathbb{T}^2 \times (0,T), \\ v(0) = u_0 & \text{in } \mathbb{T}^2. \end{cases}$$

Applying the similar methods as in Proposition 2.4, we can obtain that

$$\|v\|_{X_k^T} \leq C(\|u_0\|_k, \|g\|_{G^{k+3}}).$$

This means that

$$\|u\|_{X_k^T} \leq C(\|u_0\|_k, \|g\|_{G^{k+3}}).$$

Let $u_i (i = 1, 2)$ be the solution to (6.1) with $g = g_i$. Define $v_i = u_i - g_i$, $g = g_1 - g_2$ and $v = v_1 - v_2$, then v satisfies the following equation

$$\begin{cases} v_t - Av + (\xi(v+g))_x = Ag & \text{in } \mathbb{T}^2 \times (0,T), \\ v(0) = 0 & \text{in } \mathbb{T}^2, \end{cases} \tag{6.2}$$

where $\xi = \frac{1}{2}(u_1 + u_2)$ and $\|\xi\|_{X_k^T} \leq C(\|u_0\|_k, \|g_1\|_{G^{k+3}}, \|g_2\|_{G^{k+3}})$.

Multiplying (6.2) by $2v$ yields

$$\begin{aligned} \frac{d}{dt}\|v\|^2 + 2\|v_x\|^2 + 2\|v_y\|^2 \leq & C\|\xi\|_2\|v+g\|\|v_x\| + C\|g\|_3\|v\| \\ \leq & \|v_x\|^2 + C(\|\xi\|_2^2 + 1)\|v\|^2 + C(\|\xi\|_2^2 + 1)\|g\|_3^2. \end{aligned}$$

Combining Gronwall inequality, we have

$$\|v\|_{X_0^T} \leq C(\|\xi\|_{X_2^T})\|g\|_{G^3}. \tag{6.3}$$

Multiplying (6.2) by $2(-\triangle)^k v$ yields

$$\begin{aligned}
&\frac{d}{dt}\|(-\triangle)^{\frac{k}{2}}v\|^2+2\|(-\triangle)^{\frac{k+1}{2}}v\|^2\\
\leq&C\|\xi(v+g)\|_k\|(-\triangle)^{\frac{k+1}{2}}v\|+C\|g\|_{k+3}\|(-\triangle)^{\frac{k}{2}}v\|\\
\leq&C\|\xi\|_k(\|v\|_k+\|g\|_k)\|(-\triangle)^{\frac{k+1}{2}}v\|+C\|g\|_{k+3}\|(-\triangle)^{\frac{k}{2}}v\|\\
\leq&\|(-\triangle)^{\frac{k+1}{2}}v\|^2+C(\|\xi\|_k^2+1)\|(-\triangle)^{\frac{k}{2}}v\|^2+C(\|\xi\|_k^2+1)\|g\|_{k+3}^2+C\|\xi\|_k^2\|v\|^2.
\end{aligned}$$

It follows from Gronwall inequality and (6.3) that

$$\|v\|_{X_k^T}\leq C(\|\xi\|_{X_k^T})\|g\|_{G^{k+3}}\leq C(\|u_0\|_k,\|g_1\|_{G^{k+3}},\|g_2\|_{G^{k+3}})\|g\|_{G^{k+3}}.$$

Namely, we have

$$\|u_1-u_2\|_{X_k^T}\leq C(\|u_0\|_k,\|g_1\|_{G^{k+3}},\|g_2\|_{G^{k+3}})\|g_1-g_2\|_{G^{k+3}}.$$

This completes the proof. □

Now, we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We define $\xi(t):=\sum\limits_{i=1}^{5}b_i\beta_i(t)e_i$. Denote $E:=C([0,T];\mathcal{H})$. We know that the norms $\|\cdot\|_E$ and $\|\cdot\|_{G^{k+3}}$ are equivalent for the functions in E.

We denote the solution to (6.1) with u_0 and g by $u^{u_0}(\cdot,g)$, by this notation, the solution of (1.2) can be written as $u^{u_0}(\cdot,\xi)$.

According to Theorem 1.1, (1.1) is approximately controllable by $\mathcal{H}$-valued control, for any $\varepsilon>0$, there exists a control $\eta\in L^2(0,T;\mathcal{H})$ such that

$$\|u^{u_0}(T,\xi_1)-u_1\|_k<\frac{\varepsilon}{2},$$

where $\xi_1(t):=\int_0^t\eta(s)ds$, this leads to the fact

$$\mathbb{P}(\|u^{u_0}(T,\xi)-u_1\|_k<\varepsilon)\geq\mathbb{P}(\|u^{u_0}(T,\xi)-u^{u_0}(T,\xi_1)\|_k<\frac{\varepsilon}{2}).$$

It follows from Proposition 2.6 that there exists a positive constant $\delta>0$ such that when $\|\xi-\xi_1\|_E<\delta$,

$$\|u^{u_0}(T,\xi)-u^{u_0}(T,\xi_1)\|_k<\frac{\varepsilon}{2},$$

thus, we can obtain that

$$\mathbb{P}(\|u^{u_0}(T,\xi)-u_1\|_k<\varepsilon)\geq\mathbb{P}(\|\xi-\xi_1\|_E<\delta).$$

Since the support of $\mathcal{L}(\xi)$ is E, we have

$$\mathbb{P}(\|\xi-\xi_1\|_E<\delta)>0,$$

then,

$$\mathbb{P}(\|u^{u_0}(T,\xi)-u_1\|_k<\varepsilon)>0.$$

The proof of Theorem 1.3 is complete. □

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Conflict of interest

The author declares that there is no conflict of interest.

Data availability statement

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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